Real-valued-measurable cardinals

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Contents:

Introduction
1. Basic theory
2. Solovay’s theorems
3. The Gitik-Shelah theorem
4. The enormity of real-valued-measurable cardinals
5. Combinatorial implications
6. Measure-theoretic implications
7. Partially ordered sets
8. PMEA and NMA
9. Quasi-measurable cardinals

Appendix:
A1. Combinatorics
A2. Measure theory
A3. General topology
A4. Indescribable cardinals

P. Problems
References
Index

Introduction

In these notes I seek to describe the theory of real-valued-measurable cardinals, collecting together various results which are scattered around the published literature, and including a good deal of unpublished material.

Almost immediately after Lebesgue introduced his theory of measure and integration, it was established that (subject to the axiom of choice) not every subset of $[0, 1]$ is Lebesgue measurable. The question naturally arose, is there any extension of Lebesgue measure to a countably-additive measure defined on $\mathcal{P}[0, 1]$? It was early understood that such an extension cannot be translation-invariant. In the seminal paper Ulam 30, S. Ulam showed that a probability space $(X, \mathcal{P}X, \mu)$ in which every subset of $X$ is assigned a probability is either trivial or as complex an object as the set theory of that time could readily envisage. Since then, measure theorists and others have periodically had occasion to wonder whether non-trivial examples do, or can, exist (see, for instance, 6M-6N below), and with less regularity, but with impressive frequency, have turned up some new curiosity concerning them.

The actual definition of ‘real-valued-measurable cardinal’ involves some technical considerations which I prefer to leave to §1 below; here I will say only that there is a real-valued-measurable cardinal if and only if there is a non-trivial probability space $(X, \mathcal{P}X, \mu)$. Ulam’s work already showed, in effect (the terms I use date from later on) that real-valued-measurable cardinals are of two kinds: ‘atomlessly-measurable’ cardinals, less than or equal to the continuum, associated with extensions of Lebesgue measure to $\mathcal{P}\mathbb{R}$,
and ‘two-valued-measurable’ cardinals, much greater than the continuum, associated with ultrafilters closed under countable intersections.

It was observed by Hanf & Scott 61 that a two-valued-measurable cardinal has some extraordinary properties from the point of view of mathematical logic. The mixing of combinatorial and metamathematical intuitions and techniques which followed was wonderfully fertile (see, for instance, Keisler & Tarski 64), and quickly gave two-valued-measurable cardinals a central place in a rapidly growing theory of ‘large cardinals’. I do not propose to describe this theory here; there are accounts in Drake 74, Kanamori & Magidor 78, Jech 78. The relevant measures on a two-valued-measurable cardinal are all purely atomic, and while some of their properties can be described in the language of measure theory, there is little there which is connected with the ordinary concerns of measure theorists or probabilists. I wish rather to look at atomlessly-measurable cardinals, where the deepest concepts of abstract measure theory are both employed and illuminated. Two-valued-measurable cardinals will never be far away, as the work of R.M.Solovay and K.Kunen shows (see Solovay 71 and §2 below); indeed their constructions give general methods for translating ideas about two-valued-measurable cardinals into ideas about atomlessly-measurable cardinals, and many of the results described in this article were suggested in this way.

Recently, the spectacular near-resolution by Gitik & Shelah 89 of the problem of determining the measure algebra of an atomlessly-measurable cardinal has given new ways of applying Solovay’s concept of ‘random real forcing’, and opens up yet another channel through which ideas developed in other contexts may be applied to the theory of atomlessly-measurable cardinals.

It is fair to say that most of the questions in measure theory depending on the existence of real-valued-measurable cardinals are peripheral. (I do not say this of questions in set theory depending on the existence of, or on supposing the consistency of the existence of, two-valued-measurable cardinals.) However, an atomlessly-measurable cardinal, if one can exist (and there is a problem here; see 1Ee below), necessarily has a structure which makes it as remarkable as anything in measure theory. My aim here is to describe this structure. The investigation will take us into a fascinating blend of measure theory, infinitary combinatorics and metamathematics, drawing on deep ideas from all three.

Out of personal taste and prejudice I will play down the metamathematical aspects, seeking wherever possible to find ‘conventional’ expressions of the ideas involved. However, deep results from measure theory and set theory are going to be central to my arguments, and many concepts from various branches of mathematics will be called on at some point; so I have written an Appendix to give definitions, statements of theorems I use, and proofs or references. As fundamental references I will take Jech 78 and Kunen 80 for set theory, Engelking 89 for general topology, and my own books Fremlin 74 and Fremlin 84 for measure theory and miscellaneous material; I will try to indicate any divergences from these texts in notation.

I have been interested in real-valued-measurable cardinals since 1965, as nearly as I can remember, and cannot trust my memory to give a full list of those from whom I have learnt about them. But I recall that it was W.A.J.Luxemburg who gave a seminar more or less on the subject of 6N, and A.R.D.Mathias who showed me the paper Keisler & Tarski 64, the combination being the basis of my first work in this area. My interest was maintained by correspondence with Solovay and conversations with R.G.Haydon in the early seventies. Rather later I learnt of the work of K.Prikry, who introduced me to more of Solovay’s ideas. The actual stimulus for writing these notes came from receiving a preprint of Gitik & Shelah 89 from M.Gitik and from visiting Madison, where Kunen showed me some of his unpublished work. Most recently I have had valuable discussions with Gitik and with S.Shelah. So, pausing for a moment to apologise to those I have missed, I should like to thank all those I have named, for leading me into this garden of delights.

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1. Basic theory

1A The Banach-Ulam problem Can we describe all measure spaces of the form \((X, \mathcal{P}X, \mu)\) in terms enabling us to decide whether \(\mu\) can be an extension of Lebesgue measure?

There certainly exist measure spaces \((X, \mathcal{P}X, \mu)\), constructed as follows. Let \(X\) be any set, \(f : X \to [0, \infty]\) any function, and \(\mathcal{I}\) any \(\sigma\)-ideal of subsets of \(X\). Write
\[
\mu A = \sum_{x \in A} f(x) \text{ if } A \in \mathcal{I}, \\
= \infty \text{ if } A \in \mathcal{P}X \setminus \mathcal{I}.
\]

Then \( \mu : \mathcal{P}X \to [0, \infty] \) is countably additive. For the purposes of these notes, I will call a measure space \((X, \mathcal{P}X, \mu)\) trivial if it can be obtained by this construction. What I will call the ‘Banach-Ulam problem’ therefore becomes: is there a non-trivial measure space \((X, \mathcal{P}X, \mu)\)?

I include the \(\sigma\)-ideal \(\mathcal{I}\) here for the sake of completeness. However, it is clear that a semi-finite trivial measure space (in particular, a trivial probability space) \((X, \mathcal{P}X, \mu)\) must be defined by the function \(f(x) = \mu\{x\}\) alone, and that in this case we may take \(\mathcal{I} = \mathcal{P}X\), \(\mu A = \sum_{x \in A} f(x)\) for every \(A \subseteq X\).

**1B First reduction** Suppose now that there is some non-trivial measure space \((X, \mathcal{P}X, \mu)\). Write

\[
f(x) = \mu\{x\} \text{ if } \mu\{x\} < \infty, \\
= 0 \text{ if } \mu\{x\} = \infty,
\]

and let \(\mathcal{I}\) be the \(\sigma\)-ideal of subsets of \(X\) generated by \(\{A : \mu A < \infty\}\). Let \(\mu'\) be the measure defined from \(f\) and \(\mathcal{I}\) by the method of 1A. Then \(\mu' A \leq \mu A\) for every \(A \subseteq X\). By hypothesis, \(\mu' \neq \mu\); let \(A \subseteq X\) be such that \(\mu' A < \mu A\). Surely \(A \in \mathcal{I}\); let \(\{A_n\}_{n \in \mathbb{N}}\) be an increasing sequence of sets such that \(\mu A_n < \infty\) for every \(n \in \mathbb{N}\) and \(\bigcup_{n \in \mathbb{N}} A_n = A\). Now there must be some \(n \in \mathbb{N}\) such that \(\mu' A_n < \mu A_n\). Set \(\mu'' = \mu - \mu'[A_n]\). Then \((A_n, \mathcal{P}A_n, \mu'')\) is a measure space, with \(0 < \mu'' A_n < \infty\) and \(\mu''\{x\} = 0\) for every \(x \in A_n\). Setting

\[\hat{\mu} = (\mu'' A_n)^{-1}\mu'\]

we obtain a probability space \((A_n, \mathcal{P}A_n, \hat{\mu})\) in which singleton sets are negligible.

Accordingly we can say that there is a positive answer to the Banach-Ulam problem iff there is a probability space \((X, \mathcal{P}X, \mu)\) in which \(\mu\{x\} = 0\) for every \(x \in X\).

**1C Notation** For the next step it will be convenient to introduce some phrases which will dominate these notes. Recall that a measure \(\nu\) is \(\kappa\)-additive if \(\nu(\bigcup \mathcal{E}) = \sum_{E \in \mathcal{E}} \nu E\) whenever \(\mathcal{E}\) is a disjoint family of \(\nu\)-measurable sets and \(#(\mathcal{E}) < \kappa\), and that a filter \(\mathcal{F}\) is \(\kappa\)-complete if \(\bigcap A \in \mathcal{F}\) whenever \(A \subseteq \mathcal{F}\) and \(0 < \#(A) < \kappa\). (See 1A1B, A2C below.) Now:

(a) A cardinal \(\kappa\) is real-valued-measurable if there is a \(\kappa\)-additive probability \(\nu\) with domain \(\mathcal{P}\kappa\) which is zero on singleton sets; in this context I will call such a \(\nu\) a witnessing probability on \(\kappa\).

(b) A cardinal \(\kappa\) is two-valued-measurable (often called just ‘measurable’) if there is a non-principal \(\kappa\)-complete ultrafilter on \(\kappa\).

(c) A cardinal \(\kappa\) is atomlessly-measurable if there is an atomless \(\kappa\)-additive probability \(\nu\) with domain \(\mathcal{P}\kappa\).

**1D Ulam’s Theorem** (a) Let \((X, \mathcal{P}X, \mu)\) be a probability space, with ideal \(\mathcal{N}_\mu\) of negligible sets. Then either it is trivial, and \(\text{add}(\mu) = \text{add}(\mathcal{N}_\mu) = \infty\), or \(\text{add}(\mu) = \text{add}(\mathcal{N}_\mu)\) is a real-valued-measurable cardinal.

(b) A cardinal is real-valued-measurable if it is either atomlessly-measurable or two-valued-measurable.

(c) An atomlessly-measurable cardinal is weakly inaccessible and not greater than \(\kappa\).

(d) A two-valued-measurable cardinal is strongly inaccessible.

(e) There is an extension of Lebesgue measure to a measure defined on every subset of \(\mathbb{R}\) iff there is an atomlessly-measurable cardinal.

**proof** (a) As remarked in A2Cd, \(\text{add}(\mu) = \text{add}(\mathcal{N}_\mu)\). If \((X, \Sigma, \mu)\) is trivial, then \(\mu X = \sum_{x \in X} \mu\{x\}\) and \(\text{add}(\mu) = \infty\). Otherwise, set \(H = \{x : \mu\{x\} = 0\}\); then \(\mu H > 0\) so \(\text{add}(\mathcal{N}_\mu) \leq #(X)\). There must be a disjoint family \(\langle E_\xi \rangle_{\xi < \kappa} \in \mathcal{N}_\mu\) such that \(E = \bigcup_{\xi < \kappa} E_\xi \notin \mathcal{N}_\mu\). Define \(f : E \to \kappa\) by setting \(f(x) = \xi\) if \(x \in E_\xi\). Write

\[
\nu_0 = (\mu|E)^{-1}f^{-1}
\]

(A2Db), so that \(\nu_0\) is a \(\kappa\)-additive measure on \(\kappa\), zero on singletons, and \(\nu_0\kappa = \mu E \in [0, \infty]\). Set \(\nu = (\mu|E)^{-1}\nu_0\); then \(\nu\) witnesses that \(\kappa\) is real-valued-measurable.

(b)(i) If \(\kappa\) is atomlessly-measurable, of course it is real-valued-measurable. If \(\kappa\) is two-valued-measurable, with witnessing filter \(\mathcal{F}\), then we can set

\[
\nu A = 1 \forall A \in \mathcal{F}, \nu A = 0 \forall A \in \mathcal{P}\kappa \setminus \mathcal{F},
\]
and \( \nu \) will witness that \( \kappa \) is real-valued-measurable.

(ii) Now suppose that \( \kappa \) is real-valued-measurable, with witnessing probability \( \nu \). (a) If \( (\kappa, \mathcal{P}\kappa, \nu) \) has an atom \( A \subseteq \kappa \), set

\[
\mathcal{F} = \{ F : F \subseteq \kappa, A \setminus F \in \mathcal{N}_\nu \}.
\]

Then \( \mathcal{F} \) is a \( \kappa \)-complete ultrafilter on \( \kappa \), so \( \kappa \) is two-valued-measurable. (\( \beta \)) Otherwise, \( (\kappa, \mathcal{P}\kappa, \nu) \) is atomless, and \( \nu \) witnesses that \( \kappa \) is atomlessly-measurable.

(c) Let \( \kappa \) be a real-valued-measurable cardinal, with witnessing probability \( \nu \).

(i) \( \kappa \leq \text{add}(\mathcal{N}_\nu) \leq \kappa \), because \( \nu \) is \( \kappa \)-additive and \( \kappa = \bigcup \mathcal{N}_\nu \). So \( \kappa = \text{add}(\mathcal{N}_\nu) \) is regular (A1Ac).

(ii) ? Now suppose, if possible, that \( \kappa = \lambda^+ \) for a cardinal \( \lambda < \kappa \). For each \( \xi < \kappa \), let \( f_\xi : \xi \to \lambda \) be an injective function. For \( \eta < \kappa, \alpha < \lambda \) set

\[
A_{\eta\alpha} = \{ \xi : \eta < \xi < \kappa, f_\xi(\eta) = \alpha \}.
\]

For any fixed \( \eta < \kappa \),

\[
\nu(\bigcup_{\alpha < \lambda} A_{\eta\alpha}) = \nu(\{ \xi : \eta < \xi < \kappa \}) = 1,
\]

and \( \nu \) is \( \lambda^+ \)-additive, so there must be a \( \beta_\eta < \lambda \) such that \( \nu A_{\eta\beta_\eta} > 0 \). Set

\[
B_\beta = \{ \eta : \eta < \kappa, \beta_\eta = \beta \}
\]

for \( \beta < \lambda \). Then \( \nu(\bigcup_{\beta < \lambda} B_\beta) = \nu \kappa = 1 \), so there is a \( \beta < \lambda \) such that \( \nu B_\beta > 0 \); in particular, \( B_\beta \) is uncountable. But now observe that if \( \eta \in B_\beta \) then \( \nu A_{\eta\beta} > 0 \), and that \( \langle A_{\eta\beta} \rangle_{\eta < \kappa} \) is disjoint, because every \( f_\xi \) is injective. So we have

\[
1 = \nu \kappa \geq \sum_{\eta < \kappa} \nu A_{\eta\beta} \geq \sum_{\eta \in B_\beta} \nu A_{\eta\beta} = \infty,
\]

which is absurd. \( \mathbf{X} \)

(iii) Thus \( \kappa \) is weakly inaccessible. The argument so far assumes only that \( \kappa \) is real-valued-measurable. But if in fact \( \kappa \) is atomlessly-measurable, then \( (\kappa, \mathcal{P}\kappa, \nu) \) may be taken to be atomless, in which case there is a function \( f : \kappa \to [0, 1] \) which is inverse-measure-preserving for \( \nu \) and Lebesgue measure on \([0, 1] \) (A2Kc). Now \( \nu f^{-1} \) is a \( \kappa \)-additive measure on \([0, 1] \) which is zero on singletons. Accordingly \( \kappa \leq \#([0, 1]) = \mathfrak{c} \).

(d) Now let \( \kappa \) be a two-valued-measurable cardinal with witnessing ultrafilter \( \mathcal{F} \). By (b) and (c-i) above, \( \kappa \) is an uncountable regular cardinal. ? Suppose, if possible, that \( \kappa \leq 2^\lambda \) for some cardinal \( \lambda < \kappa \). Let \( f : \kappa \to \mathcal{P}\lambda \) be any injection. For \( \alpha < \lambda \) write

\[
A_\alpha = \{ \xi : \xi < \kappa, \alpha \in f(\xi) \}.
\]

Set

\[
B = \{ \alpha : \alpha < \lambda, A_\alpha \in \mathcal{F} \}.
\]

Then \( \kappa \setminus A_\alpha \in \mathcal{F} \) for \( \alpha \in \lambda \setminus B \). Because \( \mathcal{F} \) is \( \kappa \)-complete,

\[
A = \bigcap_{\alpha \in B} A_\alpha \cap \bigcap_{\alpha \in \lambda \setminus B} (\kappa \setminus A_\alpha) \in \mathcal{F}.
\]

Now if \( \xi \in A, \alpha < \lambda \) then \( \alpha \in f(\xi) \) iff \( \alpha \in B \); i.e., \( f(\xi) = B \) for every \( \xi \in A \). But of course \( A \) is not a singleton (because \( \mathcal{F} \) is non-principal), so \( f \) cannot be injective. \( \mathbf{X} \)

Thus \( \kappa \) is strongly inaccessible.

(e) This is now easy. (i) If \( \kappa \) is an atomlessly-measurable cardinal, with witnessing probability \( \nu \), let \( f : \kappa \to [0, 1] \) be inverse-measure-preserving for \( \nu \) and Lebesgue measure on \([0, 1] \), as in (c-iii) above. Define \( \mu : \mathcal{P}\mathfrak{R} \to [0, \infty] \) by writing

\[
\mu A = \sum_{n \in \mathbb{Z}} \nu f^{-1}[A + n] \quad \forall A \subseteq \mathbb{R};
\]

then \( \mu \) is a countably-additive (in fact, \( \kappa \)-additive) extension of Lebesgue measure to \( \mathcal{P}\mathfrak{R} \). (ii) If \( \mu \) is a countably-additive extension of Lebesgue measure to \( \mathcal{P}\mathfrak{R} \), then \( \kappa = \text{add}(\mu[[0, 1]]) \) is real-valued-measurable, by (a) above; but as \( \kappa \leq \mathfrak{c} \), \( \kappa \) cannot be two-valued-measurable, by (d), and (b) tells us that \( \kappa \) is actually atomlessly-measurable.

1E Remarks (a) The theorem above is taken virtually directly from Ulam 30. Evidently (c)-(d) show that the division of real-valued-measurable cardinals into atomlessly-measurable cardinals and two-valued-
measurable cardinals is exclusive; this is ‘Ulam’s Dichotomy’. In §2 below we shall see an extraordinary
reunification of these two phenomena at a higher level.

Readers may recognise the family $\langle A_{\alpha \beta}\rangle_{\alpha \beta \in A^+}$ of part (c-ii) of the proof above as an ‘Ulam matrix’; this
concept has many other applications (see Erdős Hajnal Máthé & Rado 84) and is one of the principal
contributions of the Banach-Ulam problem to mathematics.

(b) It is worth noting explicitly that Ulam’s Dichotomy is sharp. If $\kappa$ is a real-valued-measurable cardinal,
with witnessing probability $\nu$, then either $(\kappa, \mathcal{P}_\kappa, \nu)$ is atomless and $\kappa$ is atomlessly-measurable, or it is purely
atomic and $\kappa$ is two-valued-measurable. For in (b-ii-a) of the proof above we saw that if $(\kappa, \mathcal{P}_\kappa, \nu)$ is not
atomless then $\kappa$ is two-valued-measurable.

While if $(\kappa, \mathcal{P}_\kappa, \nu)$ is not purely atomic, there is an $A \in \mathcal{P}_\kappa \setminus \mathcal{N}_\nu$ such that $(A, \mathcal{P}_A, \nu[A])$ is atomless; if we set $\nu'[B] = \nu(B \setminus A)/\nu A$ for every $B \subseteq \kappa$, then $\nu'$ will witness
that $\kappa$ is atomlessly-measurable. And, as already remarked, no cardinal can be simultaneously atomlessly-
measurable and two-valued-measurable.

(c) The phrase ‘two-valued-measurable’ is used just because there is a natural correspondence between
$\omega_1$-complete filters and complete measures taking exactly the values 0 and 1, as described in the formulae
of part (b) of the proof above. We shall find that this language enables us to unify certain arguments, as
in 1G below. Of course there is not much measure theory to be found in a $\{0,1\}$-valued measure, and
the qualities of two-valued-measurable cardinals and atomlessly-measurable cardinals are rather different. At
the right metamathematical level, they come together again, as the work of Solovay and Kunen shows; one
of the purposes of these notes is to try to describe the combinatorial foundations of this reunification.

(d) Note that Ulam’s theorem, while a large step forward, does not give us a working description of all
measure spaces $(X, \mathcal{P}X, \mu)$, even if we think we understand real-valued-measurable cardinals. Rather, it
gives lower bounds to the possible complexity of a non-trivial $(X, \mathcal{P}X, \mu)$. I will return to this question later
(3L-3M, §8).

(e) Because it is relatively consistent with ZFC to suppose that there are no weakly inaccessible cardinals,
it is relatively consistent to suppose that every measure space $(X, \mathcal{P}X, \mu)$ is trivial in the sense of 1A. It
remains open, in a sense, whether it is relatively consistent with ZFC to suppose that there is a real-
valued-measurable cardinal, and therefore a non-trivial measure space $(X, \mathcal{P}X, \mu)$. However, very much
stronger assertions have been explored systematically in the last two decades, without so far leading to
any contradiction; and at present almost no-one is seriously searching for a proof in ZFC that real-valued-
measurable cardinals don’t exist. The rest of these notes will tacitly assume that there is at least one real-valued-measurable cardinal. Those unhappy with such an assumption may however prefer to regard them as preliminary investigations which might eventually lead to a proof by
contradiction that there are no real-valued-measurable cardinals. I assure you that such a proof would make
you famous enough to justify any effort you put into learning this material.

1F Definition Let $\kappa$ be a regular uncountable cardinal.

(a) An ideal $I$ of $\mathcal{P}_\kappa$ is normal if it is proper, includes $\kappa$ (or I could say, contains every subset of $\kappa$ of
cardinal less than $\kappa$), and whenever $S \in \mathcal{P}_\kappa \setminus I$ and $f : S \to \kappa$ is regressive (that is, $f(\xi) < \xi$ for every
non-zero $\xi \in S$), then there is a $\xi < \kappa$ such that $f^{-1}([\xi]) \notin I$.

(b) A filter $F$ on $\kappa$ is normal if its dual ideal $\{\kappa \setminus F : F \in F\}$ is normal.

(c) A measure $\nu$ on $\kappa$ is normal if the ideal $\mathcal{N}_\nu$ is normal.

Remarks The definition of ‘normal’ given here is adapted to the needs of the next theorem, but is not quite
standard. For the usual definition, and elementary properties of these ideals and filters, see A1E below. It is
worth noting immediately that the intersection of any non-empty family of normal ideals or filters is again
a normal ideal or filter.

1G Theorem (a) Let $\kappa$ be an atomlessly-measurable cardinal. Then there is a Maharam homogeneous
normal atomless $\kappa$-additive probability with domain $\mathcal{P}_\kappa$.

(b) Let $\kappa$ be a two-valued-measurable cardinal. Then there is a normal $\kappa$-complete non-principal ultra-
filter on $\kappa$. 

proof In case (a), start from an atomless \( \kappa \)-additive probability \( \nu \) with domain \( \mathcal{P} \kappa \). In case (b), start from a \( \kappa \)-complete non-principal ultrafilter \( \mathcal{F} \) on \( \kappa \), and construct \( \nu \) from \( \mathcal{F} \) by setting \( \nu A = 1 \) if \( A \in \mathcal{F} \), \( \nu A = 0 \) if \( A \in \mathcal{P} \kappa \setminus \mathcal{F} \). Then in both cases \( \nu \) is a \( \kappa \)-additive probability with domain \( \mathcal{P} \kappa \) which is zero on singletons.

Let \( F \) be the set of all functions \( f : \kappa \rightarrow \kappa \) such that \( \nu f^{-1}[\xi] = 0 \) for every \( \xi < \kappa \). Then there is an \( f_0 \in F \) such that

\[
\nu \{ \xi : f(\xi) < f_0(\xi) \} = 0 \quad \forall \ f \in F.
\]

P Note first that if \( f, g \in F \) then \( f \land g \in F \), where \( (f \land g)(\xi) = \min(f(\xi), g(\xi)) \) for every \( \xi \in \kappa \). \( \square \) If there is no suitable \( f_0 \in F \), then we may define inductively a decreasing family \( \langle g_n \rangle_{\alpha < \kappa} \) in \( F \), as follows. Set \( g_0(\xi) = \xi \) for every \( \xi < \kappa \); then \( g_0 \in F \) because \( \nu \) is \( \kappa \)-additive and zero on singletons. Given \( g_n \in F \), take \( g_{n+1} \in F \) such that \( \nu E_\alpha > 0 \), where \( E_\alpha = \{ \xi : g_\alpha(\xi) < g_{n}(\xi) \} \), and set \( g_{n+1} = g_n \land g_\alpha \). Given \( \langle g_\beta \rangle_{\beta < \kappa} \), where \( \alpha \) is a non-zero countable limit ordinal, set

\[
g_\alpha(\xi) = \min_{\beta < \alpha} g_\beta(\xi) \quad \forall \ \xi < \kappa;
\]

then \( g_\alpha \in F \) because

\[
\nu g_\alpha^{-1}[\xi] = \nu (\bigcup_{\beta < \alpha} g_\beta^{-1}[\xi]) = 0 \quad \forall \ \xi < \kappa.
\]

Now consider the family \( \langle E_\alpha \rangle_{\alpha < \omega_1} \). By A2Mb, there is a \( \xi < \kappa \) such that

\[
A = \{ \alpha : \xi \in E_\alpha \}
\]

is infinite. But if \( \langle \alpha(n) \rangle_{n \in \mathbb{N}} \) is any strictly increasing sequence in \( A \),

\[
g_{\alpha(n)}(\xi) > g_{\alpha(n+1)}(\xi) = g_{\alpha(n+1)}(\xi) \geq g_{\alpha(n+1)}(\xi)
\]

for every \( n \in \mathbb{N} \), and \( \langle g_{\alpha(n)}(\xi) \rangle_{n \in \mathbb{N}} \) is a strictly decreasing sequence of ordinals, which is impossible. \( \Box \)

Now set \( v_0 = \nu f_0^{-1} \). We see that \( v_0 \) is a \( \kappa \)-additive probability defined on \( \mathcal{P} \kappa \), because \( \nu \) is. Because \( f_0 \in F \), \( v_0(\xi) \leq \nu f_0^{-1} [\xi + 1] = 0 \) for every \( \xi < \kappa \), and \( v_0 \) is zero on singletons.

The next step is to show that \( v_0 \) is normal. \( \mathbf{P} \) Take \( S \in \mathcal{P} \kappa \setminus \mathcal{N}_{v_0} \), and a regressive function \( f : S \rightarrow \kappa \). Extend \( f \) to a function \( g : \kappa \rightarrow \kappa \) by setting \( g(\xi) = \xi \) for \( \xi \in \kappa \setminus S \). Consider \( f_1 = g \circ f_0 \). Then \( \nu \{ \xi : f_1(\xi) < f_0(\xi) \} \geq \nu f_0^{-1} [S \setminus \{ 0 \}] = v_0 (S \setminus \{ 0 \}) > 0 \), so \( f_1 \notin F \) and there is a \( \zeta < \kappa \) such that \( \nu f_1^{-1} [\zeta] > 0 \); because \( \nu \) is \( \kappa \)-additive, there is a \( \xi < \zeta \) such that \( 0 < \nu f_1^{-1} [\xi] = v_0 f_1^{-1} [\xi] \), because \( f_1^{-1} [\xi] = f_0^{-1} [f^{-1} [\xi]] \cup f_0^{-1} [\xi] \) for every \( \xi \in S \) and \( f \) are arbitrary, \( v_0 \) is normal. \( \mathbf{Q} \)

By A2Hb, there is an \( \mathcal{E} \subseteq \kappa \) such that \( v_0 \mathcal{E} > 0 \) and \( v_0 | \mathcal{E} \) is Maharam homogeneous. Set \( v_1 A = v_0 (A \cap \mathcal{E})/v_0 \mathcal{E} \) for every \( A \subseteq \kappa \); then it is easy to check that \( v_1 \) is a Maharam homogeneous normal \( \kappa \)-additive probability on \( \kappa \).

Now let us re-examine the two cases (a), (b). In case (a), \( \kappa \leq \epsilon \) so by 1D and 1Eb \( v_1 \) must be atomless, and satisfies the requirements of (a). In case (b), \( \nu \) takes only the values 0 and 1, so \( v_0 \) also takes only these values, and \( \mathcal{P} \kappa \setminus \mathcal{N}_{v_0} \) is an ultrafilter; an elementary check will confirm that it is \( \kappa \)-complete and normal and non-principal, as demanded by (b).

1H Remarks (a) This theorem is due in the first place to Keisler & Tarski 64, who proved it for two-valued-measurable cardinals; the adaptation to atomlessly-measurable cardinals is due independently to Solovay (SOLOVAY 71), R.Jensen and myself. Part of the proof reappears in 2G; an extension of the theorem, due to Solovay, is in 9B.

(b) The original impulse behind this theorem was the question: can the first (weakly) inaccessible cardinal be real-valued-measurable? The answer is spectacularly negative; I give some of the theorems describing how enormously complicated real-valued-measurable cardinals have to be in §4 below.

(c) It will be convenient to use the phrase normal witnessing probability, in such contexts as ‘Let \( \kappa \) be a real-valued-measurable cardinal, with normal witnessing probability \( \nu \), to mean a normal \( \kappa \)-additive probability with domain \( \mathcal{P} \kappa \) which is zero on singletons, as in 1Ga.

(d) For a proper discussion of the combinatorial properties of real-valued-measurable cardinals, see §5 below. It is however worth remarking now that if \( \nu \) is a normal witnessing probability on \( \kappa \) and \( f : \kappa \rightarrow \kappa \) is any function, then there is a countable set \( D \subseteq \kappa \) such that \( \nu \{ \xi : \xi < \kappa, f(\xi) < \xi, f(\xi) \notin D \} = 0 \). \( \mathbf{P} \) Set \( D = \{ \xi : \nu f^{-1} [\xi] > 0 \} \); then \( D \) is countable. Set \( S = \{ \xi : f(\xi) < \xi, f(\xi) \notin D \} \). Then \( f | S \) is regressive, and \( (f | S)^{-1} [\xi] \in \mathcal{N}_\nu \) for every \( \xi < \kappa \), so \( \mathcal{S} \in \mathcal{N}_\nu \), as claimed. \( \mathbf{Q} \)
(e) Theorem 1Ga speaks of ‘Maharam homogeneous’ normal witnessing probabilities. These are homogeneous
in the sense that they have homogeneous measure algebras. But it should be noted that in other senses
they are about as inhomogeneous as can be imagined. If \( \nu \) is a normal probability on \( \kappa \), and \( f : \kappa \to \kappa \) is any
function, set \( A = \{ \xi : f(\xi) < \xi \} \), \( B = \{ \xi : f(\xi) = \xi \} \), \( C = \{ \xi : f(\xi) > \xi \} \). Then there is a countable \( D \)
such that \( \nu(A \setminus f^{-1}[D]) = 0 \) ((d) above). On the other hand, looking at \( f[C] \), we have a function \( g : f[C] \to \kappa \)
given by \( g(\eta) = \min f^{-1}[\eta] \) for \( \eta \in f[C] \); now \( g \) is regressive and injective, so we must have \( \nu f[C] = 0 \).

Thus our arbitrary \( f \) corresponds to a trisection of \( \kappa \) into a negligible piece \( A \setminus f^{-1}[D] \), a piece \( C \cup f^{-1}[D] \)
which is mapped onto a negligible piece, and a piece on which \( f \) is the identity. So if we ask, for instance,
that \( f^{-1}[E] \) should be negligible for every \( E \in \mathcal{N}_\nu \), or that \( \nu f[E] \) should be equal to \( \nu E \) for every \( E \), we
must have \( f(\xi) = \xi \) for almost every \( \xi \). Accordingly none of the many automorphisms of \( \mathcal{P}\kappa/\mathcal{N}_\nu \), other than
the identity, can be represented by functions from \( \kappa \) to itself, and if \( A, B \) are disjoint non-negligible sets
then \( (A, \nu|A) \) and \( (B, \nu|B) \) cannot be isomorphic as measure spaces.

(f) It has been recognised since SOLOVAY 71 that many of the properties of real-valued-measurable
cardinals depend not on their measures but on the presence of suitably saturated ideals. It would be possible
to begin this work with a study of such ideals, later specializing to real-valued-measurable cardinals. However
the general theory remains largely dependent on the special case for its inspiration, so I prefer to relegate it
to §9 below.

11 Rvm filters and ideals (a) If \( \kappa \) is a real-valued-measurable cardinal, consider
\[ \mathcal{W} = \{ W : W \subseteq \kappa, \nu W = 1 \text{ for every normal witnessing probability } \nu \text{ on } \kappa \}. \]
This is an intersection of normal filters, so is a normal filter on \( \kappa \); I will call it the rvm filter of \( \kappa \). Similarly,
its dual ideal
\[ \mathcal{J} = \mathcal{N}_\nu : \nu \text{ is a normal witnessing probability on } \kappa \]
is the rvm ideal of \( \kappa \).

(b) It is perhaps worth noting an elementary fact. If \( \kappa \) is real-valued-measurable and \( Z \in \mathcal{P}\kappa \setminus \mathcal{J} \), then
there is a Maharam homogeneous normal witnessing probability \( \nu \) on \( \kappa \) such that \( \nu Z = 1 \). (See the end
of the proof of 1G.) In particular, if \( \kappa \) is two-valued-measurable and \( Z \in \mathcal{P}\kappa \setminus \mathcal{J} \) then there is a normal
ultrafilter \( \mathcal{F} \) on \( \kappa \) containing \( Z \).

1*1J I give here a well-known theorem concerning two-valued-measurable cardinals because it throws light
on similar results in §4 below, and it is instructive to contrast the techniques of proof.

**Lemma** Let \( \kappa \) be a two-valued-measurable cardinal with normal ultrafilter \( \mathcal{F} \). Suppose that we have a set
\( F \in \mathcal{F} \) and for each \( \alpha \in F \) an \( n \)-place relation \( C_\alpha \) on \( \alpha \). Then there is an \( n \)-place relation \( C \) on \( \kappa \) such that
\[ \{ \alpha : \alpha \in F, C_\alpha = C|\alpha \} \in \mathcal{F}. \]

**proof** For \( \eta_1, \ldots, \eta_n < \kappa \) write
\[ C(\eta_1, \ldots, \eta_n) \iff \{ \alpha : C_\alpha(\eta_1, \ldots, \eta_n) \} \in \mathcal{F}. \]
Now set
\[ E(\eta_1, \ldots, \eta_n) = \{ \alpha : \alpha \in F, C_\alpha(\eta_1, \ldots, \eta_n) \} \text{ if } C(\eta_1, \ldots, \eta_n), \]
\[ = \{ \alpha : \alpha \in F, \neg C_\alpha(\eta_1, \ldots, \eta_n) \} \text{ if } \neg C(\eta_1, \ldots, \eta_n). \]
so that \( E(\eta_1, \ldots, \eta_n) \in \mathcal{F} \). For \( \beta < \kappa \) set
\[ H_\beta = \bigcap\{ E(\eta_1, \ldots, \eta_n) : \eta_1, \ldots, \eta_n < \beta \} \in \mathcal{F}. \]
Then
\[ H = \{ \alpha : \alpha \in F \text{ is a limit ordinal, } \alpha \in H_\beta \forall \beta < \alpha \} \in \mathcal{F}, \]
because \( \mathcal{F} \) is normal (cf. A1E(c-iv)) (cf. A1E(c-iv)). But now \( C|\alpha = C_\alpha \) for every \( \alpha \in H \).
1K Lemma Let \( \phi \) be a \( \Pi^3_1 \) formula in the third-order language \( \mathcal{L}_3 \) of A4K below. Let \( \kappa \) be a two-valued-measurable cardinal with normal ultrafilter \( \mathcal{F} \). Suppose that \( F_0 \in \mathcal{F} \) and that for each \( \alpha \in F_0 \) we are given third-order relations \( A_{\alpha 1}, \ldots, A_{\alpha l} \) on \( \alpha \). For \( i \leq l \) let \( A_i \) be the third-order relation on \( \kappa \) defined by writing

\[
A_i(D_1, \ldots, D_r, \eta_1, \ldots, \eta_s)
\]

\[
\iff \{ \alpha : \eta_j < \alpha \ \forall \ j \leq s, A_{\alpha i}(D_1|\alpha, \ldots, D_r|\alpha, \eta_1, \ldots, \eta_s) \} \in \mathcal{F}
\]

for any strings \( D_1, \ldots, D_r \) of relations on \( \kappa \) and \( \eta_1, \ldots, \eta_s \) of members of \( \kappa \). If \( C_1, \ldots, C_k \) are relations on \( \kappa \) and \( \xi_1, \ldots, \xi_m \), then

\[
(k; A_{11}, \ldots, A_l; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi \iff \{ \alpha : \alpha < \kappa, (\alpha; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1|\alpha, \ldots, C_k|\alpha; \xi_1, \ldots, \xi_m) \models \phi \} \in \mathcal{F}.
\]

proof Induce on the length of \( \phi \).

(a) If \( \phi \) is of the form \( S(R_1, \ldots, x_n) \) or \( R(x_1, \ldots, x_n) \) we have just to observe that

\[
A_i(C_{j_1}, \ldots, \xi_{j_n}) \iff \{ \alpha : A_{\alpha i}(C_{j_1}|\alpha, \ldots, \xi_{j_n}) \} \in \mathcal{F},
\]

\[
C_i(\xi_{j_1}, \ldots, \xi_{j_n}) \iff \{ \alpha : (C_i|\alpha)(\xi_{j_1}, \ldots, \xi_{j_n}) \} \in \mathcal{F}.
\]

(b) If \( \phi \) is one of the forms \( \neg \psi, \psi \land \chi, \psi \lor \chi, \ldots \) the inductive step is easy (using the fact that \( \mathcal{F} \) is an ultrafilter for the case \( \neg \psi \)).

(c) Suppose \( \phi \) is of the form \( \forall S \psi \). Set

\[
F = \{ \alpha : (\alpha; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1|\alpha, \ldots, C_k|\alpha; \xi_1, \ldots, \xi_m) \models \phi \}.
\]

(i) If \( F \in \mathcal{F} \) take any relation \( C \) on \( \alpha \) of the same number of places as the variable \( S \). For each \( \alpha \in F \) we have

\[
(\alpha; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1|\alpha, \ldots, C_k|\alpha; \xi_1, \ldots, \xi_m) \models \psi,
\]

where \( C|\alpha \) is interpolated into the string \( C_1|\alpha, \ldots, C_k|\alpha \) in such a way as to assign it to the variable \( S \) in \( \psi \). By the inductive hypothesis,

\[
(k; A_{11}, \ldots, A_l; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \psi,
\]

so, as \( C \) is arbitrary,

\[
(k; A_{11}, \ldots, A_l; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi.
\]

(ii) If \( F \not\in \mathcal{F} \) then for each \( \alpha \in \kappa \setminus F \) choose a relation \( B_\alpha \) on \( \alpha \), of the same number of places as \( S \), such that

\[
(k; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1|\alpha, \ldots, B_\alpha|\alpha, \ldots, C_k|\alpha; \xi_1, \ldots, \xi_m) \models \neg \psi.
\]

Let \( B \) be the relation on \( \kappa \) derived from \( (B_\alpha)_{\alpha \in \kappa \setminus F} \) as in 1J, so that

\[
E = \{ \alpha : \alpha \in \kappa \setminus F; B_\alpha = B|\alpha \} \in \mathcal{F}.
\]

Then

\[
(\alpha; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1|\alpha, \ldots, B|\alpha, \ldots, C_k|\alpha; \xi_1, \ldots, \xi_m) \not\models \psi,
\]

for every \( \alpha \in E \), so

\[
(k; A_{11}, \ldots, A_l; C_1, \ldots, B, \ldots, C_k; \xi_1, \ldots, \xi_m) \not\models \psi,
\]

by the inductive hypothesis, and

\[
(k; A_{11}, \ldots, A_l; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \not\models \forall S \psi.
\]

This deals with the inductive step to \( \phi = \forall S \psi \).

(d) If \( \phi \) is of the form \( \forall \forall S \psi \) the same procedure works. For the case \( F \in \mathcal{F} \) we copy (c-i) but with a new ordinal \( \zeta \) interpolated into the string \( \xi_1, \ldots, \xi_m \) rather than a new relation \( B \) interpolated into the string.
$C_1, \ldots, C_k$. For the case $F \not\in \mathcal{F}$ we take a witnessing family $(\zeta_\alpha)_{\alpha \in \kappa \setminus F}$ in place of $(\beta_n)_{\alpha \in \kappa \setminus F}$, and note that because $\mathcal{F}$ is normal and $\alpha \mapsto \zeta_\alpha$ is regressive, there is a $\zeta < \kappa$ such that $E = \{\alpha : \zeta_\alpha = \zeta\} \in \mathcal{F}$, so that $(\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \not\in \psi$.

(e) If $\phi$ is of the form $\exists S \psi$ it is logically equivalent to $\neg \forall S \neg \psi$ so (b)-(c) deal with it. Similarly, (b) and (d) deal with the case $\exists x \psi$.

*1L Theorem* If $\kappa$ is a two-valued-measurable cardinal, it is $\Pi^2_1$-indescribable, and its $\Pi^2_1$-filter is included in its rvm filter.

**Proof** Let $\phi$ be a formula of the language $L_3$ (A4K) of the form $\forall R_1 \ldots \forall R_l \psi$, where $\psi$ is a $\Pi^2_1$ formula in $L_3$ in which the only third-order variables are in the list $R_1, \ldots, R_l$, and let $\mathcal{F}$ be a normal ultrafilter on $\kappa$. Let $C_1, \ldots, C_k, \xi_1, \ldots, \xi_m$ be such that

$$(\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi.$$  

Set

$$F = \{\alpha : \alpha < \kappa, (\alpha; C_1, \ldots, C_k) \models \phi\}.$$  

? If $F \not\in \mathcal{F}$ then for $\alpha \in \kappa \setminus F$ let $A_{\alpha 1}, \ldots, A_{\alpha l}$ be third-order relations on $\alpha$ such that

$$(\alpha; A_{\alpha 1}, \ldots, A_{\alpha l}; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \psi.$$  

For each $i \leq l$ let $A_i$ be the third-order relation on $\kappa$ given by saying that

$$A_i(D_1, \ldots, D_r, \eta_1, \ldots, \eta_s)$$  

$$\iff \{\alpha : \eta_j < \alpha \forall j \leq s, A_{\alpha i}(D_1[\alpha, \ldots, D_r], \alpha, \eta_1, \ldots, \eta_s)\} \in \mathcal{F}$$

for all second-order relations $D_1, \ldots, D_r$ on $\kappa$ and ordinals $\eta_1, \ldots, \eta_s < \kappa$. It follows from Lemma 1K that

$$(\kappa; A_1, \ldots, A_l; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \psi$$

and

$$(\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \phi,$$

which is absurd. **X**

Thus $F \in \mathcal{F}$. But sets of the form of $F$ form a base for the $\Pi^2_1$-filter of $\kappa$, so every set in that filter belongs to $\mathcal{F}$; as $\mathcal{F}$ was arbitrary, the $\Pi^2_1$-filter of $\kappa$ is included in its rvm filter.

**Remark** This is due to Hanf & Scott 61. See also Drake 74, §9.3, Jech 78, p. 385, Lemma 32.2 and Kanamori & Magidor 78, §I.4.

**1M Ergodic theory** Going a little deeper into the question considered in 1He above, we have the following, largely due to Zakrzewski 91.

**Proposition** Let $(X, P X, \mu)$ be a probability space.

(a) Let $G$ be the group of measure-preserving bijections of $X$. Then there is a partition $\mathcal{K}$ of $X$ into finite sets such that $G$ is precisely the set of bijections $g : X \to X$ such that $\mu(\bigcup \{K : K \in \mathcal{K}, g[K] \neq K\}) = 0$.

(b) Let $G^*$ be the group of bijections $g : X \to X$ such that $\mathcal{N}_\mu = \{g^{-1}[A] : A \in \mathcal{N}_\mu\}$. Then there is a partition $\mathcal{L}$ of $X$ into countable sets such that $G^*$ is precisely the set of bijections $g : X \to X$ such that $\mu(\bigcup \{L : L \in \mathcal{L}, g[L] \neq L\}) = 0$.

**Proof** (a)(i) Let us note first that if $H \subseteq G$ is a countable subgroup, $E \subseteq X$ is any set, and $f : E \to X$ is an injection such that $f(x) \in \text{Orb}_H(x) = \{h(x) : h \in H\}$ for every $x \in E$, then $\mu f[E] = \mu E$. **P** Let $(h_n)_{n \in \mathbb{N}}$ run over $H$. For each $n \in \mathbb{N}$ set

$$E_n = \{x : x \in E, f(x) = h_n(x), f(x) \neq h_i(x) \forall i < n\}.$$  

Then $\mu f[E_n] = \mu h_n[E_n] = \mu E_n$ for each $n$, so $\mu f[E] = \sum_{n \in \mathbb{N}} \mu f[E_n] = \sum_{n \in \mathbb{N}} \mu E_n = \mu E$. **Q**

(ii) It follows that if $H \subseteq G$ is a countable subgroup then $E = \{x : \text{Orb}_H(x) \text{ is infinite}\}$ is negligible. **P** Of course $h(x) \in E$ for $h \in H, x \in E$. Next, $E$ (if it is not empty) has a partition $(E_n)_{n \in \mathbb{N}}$ such that for each $x \in E, n \in \mathbb{N}$ the intersection $E_n \cap \text{Orb}_H(x)$ consists of a single point. Now for any $n, m \in \mathbb{N}$ we
have a bijection $f : E_m \to E_n$ defined by saying that $f(x)$ is the unique member of $\text{Orb}_H(x) \cap E_n$; by (i), $\mu E_n = \mu E_m$. As $\mu E$ is certainly finite, it must be 0. \hfill \Box

(iii) For any $\epsilon > 0$ there is a finite set $F \subseteq G$ such that $\mu \{ x : g(x) \notin \{ f(x) : f \in F \} \} \leq \epsilon$ for every $g \in G$. \hfill \Box 

Otherwise we can choose $(g_n)_{n \in \mathbb{N}}$ in $G$ inductively so that $\mu E_n \geq \epsilon$ for every $n \in \mathbb{N}$, where $E_n = \{ x : g_n(x) \neq g_i(x) \forall i < n \}$ for each $n$. Let $H$ be the subgroup of $G$ generated by $(g_n)_{n \in \mathbb{N}}$; then $H$ is countable. Set $E = \bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} E_i$; then $\mu E \geq \epsilon$ and $\text{Orb}_H(x)$ is infinite for every $x \in E$, contradicting (ii) just above. \hfill \Box

(iv) So there is a countable subgroup $H$ of $G$ such that $\mu \{ x : g(x) \notin \text{Orb}_H(x) \} = 0$ for every $g \in G$. By (ii) again, $F = \{ x : \text{Orb}_H(x) \text{ is infinite} \}$ is negligible. Set $K = \{ \text{Orb}_H(x) : x \in X \setminus F \} \cup \{ \{ x : x \in F \} \};$ then $K$ is a partition of $X$ into finite sets. Let $(h_n)_{n \in \mathbb{N}}$ run over $H$.

(v) Let $g : X \to X$ be a bijection and set $A = \bigcup \{ K : K \in K, g[K] \neq K \}$. If $\mu A = 0$, then $g(x) \in \text{Orb}_H(x)$ for every $x \in X \setminus A$. So $\mu g[E] = \mu E$ for every $E \subseteq X \setminus A$, by (i). As $g[A] = A \in N_\mu$, $\mu g[E] = \mu E$ for every $E \subseteq X$, i.e., $g \in G$.

If $g \in G$ set $D = \{ x : g(x) \notin \text{Orb}_H(x) \};$ then $\mu D = 0$, by the choice of $H$. Set $C = F \cup \bigcup_{h \in H} h[D]$; then $\mu C = 0$. If $K \in K$ and $g[K] \neq K$ then $K \subseteq C$; thus $A \in N_\mu$, and $K$ is the required partition.

(b)(i) If $H \subseteq G^*$ is a subgroup of cardinal at most $\omega_1$, $E \subseteq X$ is any set, and $g : E \to X$ is an injection such that $g(x) \in \text{Orb}_H(x)$ for every $x \in E$, then $E \in N_\mu$ if $g[E] \in N_\mu$; the argument is the same as for (a)(i) above, but now using the fact that $\mu$ is $\omega_2$-additive (1D).

(ii) Next, if $H \subseteq G^*$ is a subgroup of cardinal at most $\omega_1$, then $\{ x : \text{Orb}_H(x) \text{ is uncountable} \}$ is negligible; the argument is the same as for (a)(ii) above.

(iii) There is a countable subgroup $H$ of $G^*$ such that $\mu \{ x : g(x) \notin \text{Orb}_H(x) \} = 0$ for every $g \in G^*$. \hfill \Box 

If not, we can choose $(g_{\xi})_{\xi < \omega_1}$ in $G^*$ inductively so that $\mu E_{\xi} > 0$ for every $\xi$, where $E_{\xi} = \{ x : g_{\xi}(x) \neq g_{\eta}(x) \forall \eta < \xi \}$. Let $H$ be the subgroup of $G^*$ generated by $(g_{\xi}) : \xi < \omega_1$, and set $E = \bigcap_{\xi < \omega_1} \bigcup_{\eta \geq \xi} E_{\eta}$; then (because $\mu$ is $\omega_2$-additive) $\mu E \geq \limsup_{\xi \to \omega_1} \mu E_{\xi} > 0$, while $\text{Orb}_H(x)$ is uncountable for every $x \in E$, which is impossible. \hfill \Box

(iv) Now take $L$ to be the set of orbits of $H$. Let $g : X \to X$ be a bijection and set $A = \bigcup \{ L : L \in L, g[L] \neq L \}$. If $A \in N_\mu$, then $g(x) \in \text{Orb}_H(x)$ for every $x \in X \setminus A$, while also $g[A] = A$; it follows that $g \in G^*$ as in (a)(v) above.

If $g \in G^*$, set $D = \{ x : g(x) \notin \text{Orb}_H(x) \} \in N_\mu$, $C = \bigcup_{h \in H} h[D]$; then $A \subseteq C \in N_\mu$. Thus $L$ witnesses the truth of (b).

Remark Compare 9E. In this proposition I have tried to give a succinct but adequate description of the structure involved. Many corollaries can be drawn concerning both $G$ and $G^*$ and the corresponding subgroups of $\text{Aut}(P X / N_\mu)$; see ZAKRZEWSKI 91.

Version of 10.12.91

2. Solovay’s Theorems

As remarked in 1Ee, there cannot be a proof in ZFC that real-valued-measurable cardinals exist. It remains just conceivable that there is a proof that two-valued-measurable cardinals or atomlessly-measurable cardinals do not exist. However, if one of these gives a difficulty, so does the other; this is the main result of SOLOVAY 71, covered in 2A-2D. The method of proof in 2C is important because it provides a technique - to date the only technique known - for proving the relative consistency of further propositions with the existence of an atomlessly-measurable cardinal, as in 2I. I therefore spell out some of the properties of the construction relevant to questions considered later in this paper (2H, 2J, 2K).

This material belongs to the ‘metamathematical’ part of the subject, and as I said in the Introduction I seek to avoid reliance on such methods; only the second half of §4 and a few paragraphs elsewhere will
depend on them in a formal sense. Historically, however, these ideas have been dominant in the development of the subject, and they remain an invaluable guide.

**2A Forcing** I try to follow Kunen 80 for the theory and notation of forcing. In particular, a p.o.set is a triple \( (\mathbb{P}, \leq, 1_\mathbb{P}) \) such that

\[
p \leq p \forall p \in \mathbb{P},
\]

if \( p \leq q \) and \( q \leq r \) then \( p \leq r \),

\[1_\mathbb{P} \in \mathbb{P} \text{ and } p \leq 1_\mathbb{P} \forall p \in \mathbb{P};\]

and \( p \leq q \) means that \( p \) is a stronger condition than \( q \). Two elements \( p, q \) of \( \mathbb{P} \) are incompatible if there is no \( r \in \mathbb{P} \) such that \( r \leq p \) and \( r \leq q \). A set \( A \subseteq \mathbb{P} \) is an antichain (‘down-antichain’ in Fremlin 84) if \( p \) and \( q \) are incompatible for all distinct \( p, q \in A \). A set \( D \subseteq \mathbb{P} \) is dense (‘coinitial’ in Fremlin 84) if for any \( p \in \mathbb{P} \) there is a \( q \in D \) such that \( q \leq p \).

If \( \mathbb{P} \) is any p.o.set, it has a natural topology (the ‘down-topology’ of Fremlin 84) generated by sets of the form \( \{ q : q \leq p \} \) as \( p \) runs through \( \mathbb{P} \). Let \( \mathcal{A} \) be the algebra of regular open sets for this topology (Fremlin 84, §12). For \( p \in \mathbb{P} \) write \( p^* \) for the corresponding member of \( \mathcal{A} \), viz.

\[
\text{int}\{ q : q \leq p \} = \{ q : \text{ every } r \leq q \text{ is compatible with } p \}.
\]

The map \( p \mapsto p^* : \mathbb{P} \to \mathcal{A} \setminus \{0\} \) is a dense embedding in the sense of Kunen 80.

**2B Random real p.o.sets** A random real p.o.set is a p.o.set \( \mathbb{P} \) such that there is a functional \( \mu : \mathbb{P} \to [0, 1] \) such that (i) \( \mu 1_\mathbb{P} = 1 \) (ii) if \( p \in \mathbb{P} \) and \( A \) is a maximal antichain in \( \{ q : q \leq p \} \), then \( \mu_p = \sum_{a \in A} \mu_a \).

It is fairly easy to see that \( \mathbb{P} \) is a random real p.o.set in this sense iff the regular open algebra of \( \mathbb{P} \) is a measurable algebra in the sense of A2Fc. Consequently forcing with any random real p.o.set corresponds to forcing with a measurable algebra, which by Maharam’s theorem (A2I) is reducible to some assembly of forcings with the standard homogeneous measure algebras based on powers of \( \{0, 1\} \) (A2G).

The idea of this definition is to unify the two examples with which we shall be concerned: (i) \( \mathbb{P} = \mathcal{A} \setminus \{0\} \), where \( \mathcal{A} \) is a non-zero measurable algebra; (ii) \( \mathbb{P} = \Sigma \setminus \mathcal{N}_\mu \), where \( (X, \Sigma, \mu) \) is a probability space. But it is also worth noting that if \( \mathbb{P} \) is a random real p.o.set and \( \mathcal{Q} \) is a dense subset of \( \mathbb{P} \) containing \( 1_\mathbb{P} \), then \( \mathcal{Q} \) is a random real p.o.set.

**2C Theorem** If \( \kappa \) is a real-valued-measurable cardinal and \( \mathbb{P} \) is a random real p.o.set then

\[\Vdash_{\mathbb{P}} \kappa \text{ is real-valued-measurable.}\]

**proof** Let \( \mathcal{A} \) be the regular open algebra of \( \mathbb{P} \); fix on a \( \hat{\mu} \) such that \( (\mathcal{A}, \hat{\mu}) \) is a probability algebra, and let \( \nu \) be a witnessing probability on \( \kappa \). For each \( \mathbb{P} \)-name \( \sigma \) for a subset of \( \kappa \) let \( \langle a_\xi(\sigma) \rangle_{\xi < \kappa} \) be the family in \( \mathcal{A} \) defined by the formula

\[a_\xi(\sigma) = \sup \{ p^* : p \Vdash \xi \in \sigma \}.
\]

Define \( u_\sigma \in L^\infty(\mathcal{A}) \) by writing

\[\int_a u_\sigma \, d\hat{\mu} = \int \hat{\mu}(a_\xi(\sigma) \cap a) \nu(d\xi)\]

for every \( a \in \mathcal{A} \); this is well-defined because the functional

\[a \mapsto \int \hat{\mu}(a_\xi(\sigma) \cap a) \nu(d\xi)\]

is additive and dominated by \( \hat{\mu} \) (A2Fg). Observe that if \( \sigma, \sigma' \) are both names for subsets of \( \kappa \) and \( p \Vdash \sigma = \sigma' \), then \( p^* \cap a_\xi(\sigma) = p^* \cap a_\xi(\sigma') \) for every \( \xi < \kappa \), so that \( u_\sigma \times \chi(p^*) = u_{\sigma'} \times \chi(p^*) \). We therefore have a \( \mathbb{P} \)-name \( \tilde{\nu} \) for a function from \( \mathcal{P} \kappa \) to \( [0, 1] \) such that, for any rational numbers \( s, t \),

\[p \Vdash \hat{s} \leq \tilde{\nu}(\sigma) \leq \hat{t} \text{ iff } s \chi(p^*) \leq u_\sigma \times \chi(p^*) \leq t \chi(p^*).\]

(The point is that if \( p \Vdash \sigma = \sigma' \), then \( u_\sigma \times \chi(p^*) = u_{\sigma'} \times \chi(p^*) \) so \( p \Vdash \tilde{\nu}(\sigma) = \tilde{\nu}(\sigma') \).)

Now we have to check the properties of \( \tilde{\nu} \).

(i) If \( \sigma_1, \sigma_2 \) and \( \sigma \) are \( \mathbb{P} \)-names for subsets of \( \kappa \) and \( \Vdash \sigma_1 \cap \sigma_2 = \emptyset, \sigma_1 \cup \sigma_2 = \sigma \) then
Thus for every $\lambda \in \mathfrak{A}$ such that $\models \text{GCH}$ and $\beta \in \mathfrak{A}$, this witnesses that $\nu$ with $\chi = \chi(1)$, so that $\nu = \nu(a,1) \neq 0$.

Accordingly $\sum_{\alpha<\lambda} u_{\sigma} = \chi(1)$ in $\mathcal{L}^\infty(\mathfrak{A})$. Now if $p \in \mathbb{P}$, $t < 1$ there must be a non-zero $a \subseteq P^*$ and $\beta(0), \ldots, \beta(n) < \lambda$, $t_0, \ldots, t_n > 0$ such that $\sum_{i \leq n} t_i \geq t$ and $t_i \chi(a) \leq u_{\sigma(i)}$ for each $i \leq n$. There is a $p_1 \in \mathbb{P}$ such that $p_1 \subset p$ such that $p_1 \mathcal{L}^\infty(\mathfrak{A})$.

$$a_\xi(\sigma_1) \cap a_\xi(\sigma_2) = 0 \quad \forall \xi < \kappa,$$

$$a_\xi(\sigma) = a_\xi(\sigma_1) \cup a_\xi(\sigma_2) \quad \forall \xi < \kappa.$$ 

So if $a \in \mathfrak{A}$,

$$\mu(a \cap a_\xi(\sigma)) = \mu(a \cap a_\xi(\sigma_1)) + \mu(a \cap a_\xi(\sigma_2)) \quad \forall \xi < \kappa,$$

where $\mu(a \cap a_\xi(\sigma)) \in \mathcal{P}^\mathfrak{A}$. Accordingly $u_\sigma = u_{\sigma_1} + u_{\sigma_2}$ in $\mathcal{P}^\mathfrak{A}$, and

$$\models_p \tilde{\nu}(\sigma) = \tilde{\nu}(\sigma_1) + \tilde{\nu}(\sigma_2).$$ 

Thus

$$\models_p \tilde{\nu} \text{ is finitely additive.}$$ 

(ii) If $\xi < \kappa$ and $\sigma$ is a $\mathbb{P}$-name for $\{\xi\}$ then $a_\xi(\sigma) = 0$ for every $\xi \neq \xi$ so $\int \mu a_\xi(\sigma) \nu(d\xi) = 0$ and $u_\sigma = 0$ and $\models_p \tilde{\nu}(\sigma) = 0$.

(iii) Similarly, $u_\xi = \chi(1)$ so $\models_p \tilde{\nu}(\kappa) = 1$.

(iv) If $\lambda < \kappa$ and $\langle \sigma_\alpha \rangle_{\alpha<\lambda}$ is a family of $\mathbb{P}$-names for subsets of $\kappa$ with $\models_\mathbb{P} \bigcup_{\alpha<\lambda} \sigma_\alpha = \kappa$, $\sigma_\alpha \cap \sigma_\beta = \emptyset \quad \forall \alpha \neq \beta$, then $a_\xi(\sigma_\alpha) \cap a_\xi(\sigma_\beta) = 0$ whenever $\xi < \kappa$, $\alpha < \beta < \lambda$, and $\sup_{\alpha<\lambda} a_\xi(\sigma_\alpha) = 1$ in $\mathfrak{A}$ for every $\xi < \kappa$. So

$$\bar{\mu} a = \sum_{\alpha<\lambda} \bar{\mu}(a_\xi(\sigma_\alpha) \cap a) \quad \forall \xi < \lambda, \ a \in \mathfrak{A},$$

and (because $\nu$ is $\kappa$-additive and $\lambda < \kappa$)

$$\bar{\mu} a = \sum_{\alpha<\lambda} \int \nu(a_\xi(\sigma_\alpha) \cap a) \nu(d\xi) \quad \forall \ a \in \mathfrak{A}.$$ 

Accordingly $\sum_{\alpha<\lambda} u_{\sigma_\alpha} = \chi(1)$ in $\mathcal{L}^\infty(\mathfrak{A})$. Now if $p \in \mathbb{P}$, $t < 1$ there must be a non-zero $a \subseteq p^*$ and $\beta(0), \ldots, \beta(n) < \lambda$, $t_0, \ldots, t_n > 0$ such that $\sum_{i \leq n} t_i \geq t$ and $t_i \chi(a) \leq u_{\sigma(i)}$ for each $i \leq n$. There is a $p_1 \in \mathbb{P}$ such that $p_1 \subset p$.

$$p_1 \models_p \tilde{\nu}(\sigma_{\beta(i)}) \geq \bar{t}_i$$

for every $i \leq n$, so that

$$p_1 \models_p \sum_{\alpha<\lambda} \tilde{\nu}(\sigma_\alpha) \geq \bar{t}.$$ 

As $p$ and $t$ are arbitrary,

$$\models_p \sum_{\alpha<\lambda} \tilde{\nu}(\sigma_\alpha) \geq 1.$$ 

As $\langle \sigma_\alpha \rangle_{\alpha<\lambda}$ is arbitrary,

$$\models_p \tilde{\nu} \text{ is } \kappa\text{-additive.}$$ 

Thus $\tilde{\nu}$ witnesses that

$$\models_\mathbb{P} \tilde{\kappa} \text{ is real-valued-measurable.}$$

**Remark** This is due to Solovay and Kunen (Solovay 71, Theorem 7.) For a version of the proof which incorporates the relevant part of the Radon-Nikodým theorem, see Jech 78, p. 423, Lemma 34.6.

**2D Theorem** If $\kappa$ is an uncountable cardinal and $\mathcal{I}$ is a proper $\kappa$-saturated $\kappa$-additive ideal of $\mathcal{P}_\kappa$ containing singletons, then

$$L(\mathcal{I}) \models \text{GCH} + \kappa \text{ is two-valued-measurable.}$$

**proof** Solovay 71, Theorem 6, or Jech 78, p. 416, Theorem 82a.

**2E Corollary** The following are equiconsistent:
(a) ‘ZFC + there is a two-valued-measurable cardinal’;
(b) ‘ZFC + there is an atomlessly-measurable cardinal’;
(c) ‘ZFC + there is a real-valued-measurable cardinal’;
(d) ‘ZFC + $\varepsilon$ is atomlessly-measurable’;
(e) ‘ZFC + GCH + there is a two-valued-measurable cardinal’.

**proof** For (a)$\Rightarrow$(d), use 2C with $\mathbb{P} = \mathcal{A}_\kappa \setminus \{0\}$, where $\mathcal{A}_\kappa$ is the measure algebra of $\{0,1\}^\kappa$ with its usual measure. (d)$\Rightarrow$(b)$\Rightarrow$(c) and (e)$\Rightarrow$(a) are trivial. (c)$\Rightarrow$(e) is covered by 2D.

**2F Definition** If $X$ and $Y$ are sets and $\mathcal{I}$ is an ideal of subsets of $X$ then I write $\text{Tr}_\mathcal{I}(X;Y)$ for
$$\sup\{#(F) : F \subseteq Y^X, \{x : f(x) \neq g(x)\} \in \mathcal{I} \text{ whenever } f, g \in F \text{ and } f \neq g\}.$$  

**2G Lemma** Let $(X, \mathcal{P}X, \nu)$ be a probability space and $Y$ any set. Then $\text{Tr}_{\nu}(X;Y)$ is attained, in the sense that there is a set $G \subseteq Y^X$ such that $#(G) = \text{Tr}_{\nu}(X;Y)$ and $\{x : x \in X, g(x) \neq g'(x)\} \subseteq \nu$ for all distinct $g, g' \in G$.

**proof** It is enough to consider the case in which $Y = \lambda$ is a cardinal, and the case of finite $\lambda$ is elementary, so I suppose from now on that $\lambda \geq \omega$. Set $\theta = \text{Tr}_{\nu}(X;\lambda)$.

(a) If $H \subseteq \lambda^X$ is such that
$$F = \{f : f \in \lambda^X, \{x : f(x) \leq h(x)\} \in \nu, \forall h \in H\} \neq \emptyset,$$
then there is an $f_0 \in F$ such that
$$\{x : f(x) < f_0(x)\} \subseteq \nu, \forall f \in F.$$

**P?** If not, choose a family $(f_\xi)_{\xi < \omega_1}$ in $F$ inductively, as follows. $f_0$ is to be any member of $F$. Given $f_\xi$, there is an $f \in F$ such that $\{x : f(x) < f_\xi(x)\} \notin \nu_\omega$; set $f_{\xi+1}(x) = \min(f(x), f_\xi(x))$ for every $x$; then $f_{\xi+1} \in F$. Given that $f_\eta \in F$ for every $\eta < \xi$, where $\xi < \omega_1$ is a non-zero limit ordinal, set $f_\xi(x) = \min_{\eta < \xi} f_\eta(x)$ for each $x$; then for any $h \in H$ we shall have
$$\{x : f_\xi(x) \leq h(x)\} = \bigcup_{\eta < \xi} \{x : f_\eta(x) \leq h(x)\} \in \nu_\omega,$$
so $f_\xi \in F$ and the induction continues.

Now consider
$$E_\xi = \{x : f_{\xi+1}(x) < f_\xi(x)\} \subseteq \mathcal{P}X \setminus \nu_\omega$$
for $\xi < \omega_1$. By A2Mb there is an $x \in X$ such that
$$A = \{\xi : x \in E_\xi\}$$
is infinite. But if $\langle \xi(n) \rangle_{n \in \mathbb{N}}$ is any strictly increasing sequence in $A$, $\langle f_{\xi(n)}(x) \rangle_{n \in \mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(b) We may therefore choose a family $\langle g_\eta \rangle_{\eta < \xi}$ in $\lambda^X$ as follows. Given $\langle g_\eta \rangle_{\eta < \xi}$, set
$$F_\xi = \{f : f \in \lambda^X, \{x : f(x) \leq g_\eta(x)\} \in \nu, \forall \eta < \xi\}.$$  

If $F_\xi = \emptyset$, set $\alpha = \xi$ and stop. If $F_\xi \neq \emptyset$ choose $g_\xi \in F_\xi$ such that $\{x : f(x) < g_\xi(x)\} \subseteq \nu$ for every $f \in F_\xi$, and continue. Note that for $n < \omega$, $g_\eta(x) = n$ for $n$-almost every $x \in X$ (this is a simple induction on $n$), so that $\alpha \geq \omega$.

(c) Because $g_\xi \in F_\xi$, $\{x : g_\xi(x) = g_\eta(x)\} \subseteq \nu$ whenever $\eta < \xi < \alpha$, so $\#(\alpha) \leq \theta$. On the other hand, suppose that $F \subseteq \lambda^X$ is such that $\{x : f(x) < f'(x)\} \subseteq \nu$ for all distinct $f, f' \in F$. For each $f \in F$, set
$$\zeta_f = \min\{\xi : \xi \leq \alpha, f \notin F_\xi\};$$
this must be defined because $F_\alpha = \emptyset$. Also $F_0 = \lambda^X$ and $F_\xi = \bigcap_{\eta < \xi} F_\eta$ if $\xi \leq \alpha$ is a non-zero limit ordinal, so $\zeta_f$ must be a successor ordinal; let $\zeta_f$ be its predecessor. We have $f \in F_{\zeta_f}$ and
$$\{x : f(x) < g_{\zeta_f}(x)\} \subseteq \nu, \{x : f(x) \leq g_{\zeta_f}(x)\} \notin \nu,$$
so that
$$E_f = \{x : f(x) = g_{\zeta_f}(x)\} \notin \nu.$$
If $f, f'$ are distinct members of $F$ and $\zeta_f = \zeta_{f'}$, then $E_f \cap E_{f'} \in \mathcal{N}_\nu$. So
\[
\{ f : f \in F, \zeta_f = \zeta \}
\]
must be countable for every $\zeta < \alpha$, and $\#(F) \leq \max(\omega, \#(\alpha)) = \#(\alpha)$. As $F$ is arbitrary, $\theta \leq \#(\alpha)$.

(d) Accordingly we may take $G = \{ g_\xi : \xi < \alpha \}$.

Remark Compare 1G.

2H Proposition Suppose that $\kappa$ is a two-valued-measurable cardinal and that $\lambda \geq \kappa$. Let $(\mathfrak{A}_\lambda, \mu_\lambda)$ be the measure algebra of $\{0, 1\}^\lambda$ with its usual measure, and set $P = \mathfrak{A}_\lambda \setminus \{0\}$. Let $\nu$ be a $\{0, 1\}$-valued $\kappa$-additive measure with domain $\mathcal{P}_\kappa$, zero on singletons. Write $\mathcal{N}$ for the ideal of Lebesgue negligible subsets of $\mathbb{R}$.

(a) Set $\zeta = \text{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda)$. Construct the $\mathbb{P}$-name $\tilde{\nu}$ for a measure on $\kappa$ as in 2C. Then
\[
\Vdash \tilde{\nu} \text{ is Maharam homogeneous with Maharam type } \tilde{\zeta}.
\]

(b) Set $\alpha = \lambda^\omega$, the cardinal power. Then
\[
\Vdash \text{cov}(\mathbb{R}, \mathcal{N}) = \epsilon = \tilde{\alpha}.
\]

proof (a)(i) By 2G, there is a family $(g_\alpha)_{\alpha < \zeta}$ in $\mathbb{L}^\kappa$ such that $\{ \zeta : g_\alpha(\zeta) = g_\beta(\zeta) \} \subseteq \mathcal{N}_\nu$ whenever $\alpha < \beta < \zeta$. Fix a stochastically independent family $(e_\eta)_{\eta \in \lambda}$ in $\mathfrak{A}_\lambda$ with $\mu_\lambda e_\eta = \frac{1}{2}$ for every $\eta$. For each $\alpha < \zeta$ let $\sigma_\alpha$ be a $\mathbb{P}$-name for a subset of $\kappa$ such that
\[
eq g_{\alpha}(\xi) \Vdash \tilde{\zeta} / \sigma_{\alpha}, \quad 1 \setminus e_{g_{\alpha}(\xi)} \Vdash \tilde{\xi} \notin \sigma_{\alpha}.
\]

Then
\[
\Vdash \tilde{\nu}(\bigcap_{\alpha \in I} \sigma_{\alpha}) = 2^{-\#(I)}
\]
for every non-empty finite $I \subseteq \zeta$. So
\[
\Vdash (\sigma_{\alpha})_{\alpha < \zeta} \text{ is stochastically independent}
\]
and
\[
\Vdash \text{for every } A \in \mathcal{P}_\kappa \setminus \mathcal{N}_\nu \text{ the Maharam type of } \tilde{\nu}[A] \text{ is at least } \tilde{\zeta}.
\]

(ii) Suppose that $p \in \mathbb{P}$ and that $(\langle \sigma_\alpha \rangle)_{\alpha < \theta}$ is a family of $\mathbb{P}$-names for subsets of $\kappa$ such that $p \Vdash \tilde{\nu}(\sigma_\alpha \triangle \sigma_\beta) \geq 3\epsilon > 0$ for all $\alpha < \beta < \theta$. Then, writing $F$ for the filter $\{ A : \nu A = 1 \}$,
\[
\text{lim}_{\xi \to \mathcal{F}} \mu_\lambda(p \cap (a_\xi(\sigma_\alpha) \triangle d_\xi(\sigma_\beta))) \geq 3\epsilon \mu_\lambda p \forall \alpha < \beta < \theta,
\]
defining $a_\xi(\sigma)$ as in 2C, but regarding them as members of $\mathbb{P}$ itself. Take a metrically dense subset $D$ of $\mathfrak{A}_\lambda$ of cardinal $\lambda$; take $d_{\alpha\xi} \in D$ with $\mu_\lambda(d_{\alpha\xi} \triangle d_\xi(\sigma_\alpha)) \leq \epsilon \mu_\lambda p$ for all $\xi < \kappa, \alpha < \theta$; then
\[
\text{lim}_{\xi \to \mathcal{F}} \mu_\lambda(p \cap (d_{\alpha\xi} \triangle d_\xi(\sigma_\alpha))) > 0 \forall \alpha < \beta < \theta.
\]
Consequently $(d_{\alpha\xi})_{\xi < \alpha, \alpha < \theta}$ witness that $\theta \leq \text{Tr}_{\mathcal{N}_\nu}(\kappa; D) = \text{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda) = \zeta$. This shows that
\[
\Vdash \text{the metric density of } \mathcal{P}_\kappa / \mathcal{N}_\nu \text{ is at most } \zeta;
\]
but as remarked in A2Hi, this is just
\[
\Vdash \text{the Maharam type of } \tilde{\nu} \text{ is at most } \tilde{\zeta}.
\]

(b)(i) We have
\[
\Vdash \epsilon \leq \tilde{\alpha}
\]
because $\#(\mathbb{P}) = \alpha = \#(\mathbb{P})^\omega$ (see A2Hb) and $\mathbb{P}$ is ccc; see Jech 78, Lemma 19.4.

(ii) We need to know the following fact: if $\theta < \alpha$ and $(I_\xi)_{\xi < \theta}$ is any family in $[\lambda]^{<\omega}$, there is a $K \in [\lambda]^\omega$ such that $K \cap I_\xi$ is finite for every $\xi < \theta$. $\mathbb{P}$ If $\theta < \lambda$ this is trivial. If $\theta \geq \lambda$, let $\beta$ be the least cardinal such that $\beta^+ > \theta$; then $\epsilon < \beta < \lambda \leq \theta$ and $\beta^+ \geq \max(\beta, \sup_{\gamma < \beta} \gamma^+) = \omega$, so $\text{cf}(\beta) = \omega$. Let $(\beta_n)_{n \in \mathbb{N}}$ be a strictly increasing sequence of infinite cardinals with supremum $\beta$. For each $n \in \mathbb{N}$ let $\phi_n : [n \times \beta_n]^{\omega} \to \beta_{n+1} \setminus \beta_n$ be an injective function, and for $x : \mathbb{N} \to \beta$ set
Identifying \(x\) with a subset of \(\mathbb{N} \times \beta\), then each \(K_x\) is infinite and \(K_x \cap K_y\) is finite whenever \(x \neq y\). Consequently, for any \(\xi < \theta\),

\[
\{x : K_x \cap I_\xi \text{ is infinite}\} = \{x : \exists I \in [I_\xi]^\omega, I \subseteq K_x\}
\]

has cardinal at most \(c\). Because \(\beta^\omega > \max(\xi, \theta)\), there is some \(x\) such that \(K_x \cap I_\xi\) is finite for every \(\xi\), and this will serve for \(K\). \(\Box\)

(iii) The argument of Theorem 3 of Miller 82 now shows that

\[
\|P\| \text{cov}(\mathbb{R}, \mathcal{N}) \geq \alpha.
\]

Because \(\text{cov}(\mathbb{R}, \mathcal{N}) \leq c\), we’re done.

Remark The result (b) is well known if \(\lambda^\omega = \lambda\); see for instance Kunen 84, 3.14 and 3.19. For \(\lambda^\omega > \lambda\) it is less familiar.

Maharam’s theorem (A21) tells us that any random real forcing must correspond to some forcing of the type described in this proposition. Consequently any atomlessly-measurable cardinal constructed by Solovay’s method from a \{0, 1\}-valued measure must have a homogeneous measure algebra.

21 Corollary The following are equiconsistent:

(a) ‘\(\text{ZFC} + \) there is a two-valued-measurable cardinal’;

(b) ‘\(\text{ZFC} + \) there is an atomlessly-measurable cardinal \(\kappa\), with witnessing probability \(\nu\), such that the Maharam type of \(\nu\) is \(\varepsilon = 2^\omega\);

(c) ‘\(\text{ZFC} + \) there is an atomlessly-measurable cardinal \(\kappa\), with witnessing probability \(\nu\), such that the Maharam type of \(\nu\) is \(\kappa^{(\omega)}\), while \(\varepsilon = 2^\omega = \kappa^{(\omega+1)}\).

proof For (b)\(\Rightarrow\)(a) and (c)\(\Rightarrow\)(a) we have 2D-2E. For (a)\(\Rightarrow\)(b), apply 2H, starting from a two-valued-measurable cardinal \(\kappa\), and using \(\lambda = 2^\kappa\). For (a)\(\Rightarrow\)(c), 2E(a)\(\Rightarrow\)(e) tells us that we may assume GCH and take a two-valued-measurable cardinal \(\kappa\) with witnessing probability \(\nu\). If we now take \(\lambda = \kappa^{(\omega)}\), we shall have \(\text{Tr}_{\mathbb{N}_{\kappa}}(\kappa; \lambda) = \lambda\). \(\mathbb{P}\) take \(F \subseteq \lambda^\kappa\) such that \(\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}_\nu\) for all distinct \(f, g \in F\). For each \(n \in \mathbb{N}\) set \(F_n = \{f : f \in F, \nu f^{-1}[\kappa^{(n)}] > 0\}\). Then \(F = \bigcup_{n \in \mathbb{N}} F_n\). If \(f, g\) are distinct members of \(F_n\), then \(f \cap (\kappa \times \kappa^{(n)}) \neq g \cap (\kappa \times \kappa^{(n)})\), and \(\#(\mathcal{P}(\kappa \times \kappa^{(n)})) = \kappa^{(n+1)} \leq \lambda\). So \(\#(F_n) \leq \lambda\) for every \(n\) and \(\#(F) \leq \lambda\). This shows that \(\text{Tr}_{\mathbb{N}_{\kappa}}(\kappa; \lambda) \leq \lambda\), but the reverse inequality is trivial. \(\Box\)

Also \(\lambda < \lambda^\omega < 2^\lambda = \kappa^{(\omega+1)}\); so applying 2H we get

\[
\|P\| \text{ the Maharam type of } \nu \text{ is } \kappa^{(\omega)};
\]

\[
\|P\| \varepsilon = \kappa^{(\omega+1)}.
\]

But because \(\#(\mathbb{P})^\kappa = \kappa^{(\omega+1)}\),

\[
\|P\| 2^\kappa \leq \kappa^{(\omega+1)};
\]

so we have the (relative) consistency of (c).

2J Proposition Let \(\kappa\) be a real-valued-measurable cardinal with rvm ideal \(\mathcal{J}\), and let \(\mathbb{P}\) be a random real p.o.set.

(a) If \(\nu\) is a witnessing probability on \(\kappa\) and \(\hat{\nu}\) the corresponding \(\mathbb{P}\)-name as in 2C, then

(i) for any \(B \subseteq \kappa\),

\[
\|P\| \hat{\nu}B = (\nu B)^\vee.
\]

(ii) \(\|\mathcal{N}_\nu\|\) is the ideal of \(\mathcal{P}\kappa\) generated by \(\hat{\mathcal{N}}\nu\).

(iii) If \(\nu\) is normal, then

\[
\|\mathbb{P}\| \hat{\nu} \text{ is normal}.
\]

(b) If \(\hat{\nu}\) is a \(\mathbb{P}\)-name for a witnessing probability on \(\kappa\), then there is a witnessing probability \(\nu_1\) on \(\kappa\) such that

\[
\mathcal{N}_{\nu_1} = \{B : B \subseteq \kappa, \|\mathbb{P}\| \hat{\nu}B = 0\}.
\]
If moreover \( \vdash \nu \) is normal, then \( \nu_1 \) is normal.

(c) \( \vdash \) the rvm ideal of \( \kappa \) is the ideal of \( \mathcal{P}\kappa \) generated by \( \bar{\mathcal{J}} \).

**proof** Take \( \mathfrak{A} \) and \( \bar{\mu} \) as in 2C.

(a)(i) If \( B \subseteq \kappa \) then \( a_\xi(B) = 1 \) if \( \xi \in B \), \( 0 \) if \( \xi \notin B \); so \( \int_a \bar{\mu}(a_\xi(B)) \nu(d\xi) = \bar{\mu}_a \nu B \) for every \( a \in \mathfrak{A} \) and \( \vdash \nu \bar{B} = (\nu B)^\\bar{\mu} \).

(ii) Let \( \sigma \) be a \( \mathbb{P} \)-name such that \( \vdash \nu \sigma = 0 \). Let \( \langle a_\xi(\sigma) \rangle_{\xi < \kappa} \) be the corresponding family in \( \mathfrak{A} \). Then \( \int \bar{\mu}(a_\xi(\sigma)) \nu(d\xi) = 0 \), so \( B = \{ \xi : \bar{\mu}(a_\xi(\sigma)) > 0 \} \in \mathcal{N}_\nu \). Now \( \vdash \sigma \subseteq \bar{B} \in \mathcal{N}_\nu \).

(iii) Suppose that \( \nu \) is normal. Let \( \langle \sigma_\xi \rangle_{\xi < \kappa} \) be a family of \( \mathbb{P} \)-names for \( \bar{\nu} \)-negligible subsets of \( \kappa \) and let \( \sigma \) be a \( \mathbb{P} \)-name such that

\[
\vdash \sigma = \{ \eta : \exists \xi < \eta, \eta \in \sigma_\xi \}.
\]

For each \( \xi < \kappa \) we have a \( B_\xi \in \mathcal{N}_\nu \) such that \( \vdash \sigma_\xi \subseteq B_\xi \); set \( B = \{ \eta : \exists \xi < \eta, \eta \in B_\xi \} \). Because \( \nu \) is normal, \( B \in \mathcal{N}_\nu \), and \( \vdash \nu \bar{B} = 0 \). But also if \( \eta < \kappa \), \( p \in \mathbb{P} \) and \( \vdash \eta \in \sigma \), then there are \( p' \leq p \) and \( \xi < \eta \) such that \( p' \vdash \eta \in \sigma_\xi \), so that \( \eta \in B_\xi \) and \( \eta \in B \). Thus \( \vdash \sigma \subseteq \bar{B} \) and \( \vdash \nu \sigma = 0 \).

(b) For each \( B \subseteq \kappa \) we have a unique \( u_B \in L^\infty(\mathfrak{A}) \) representing \( \bar{\nu} \bar{B} \) in the sense that for any rational numbers \( s \), \( t \) and any \( p \in \mathbb{P} \),

\[
p \vdash s \leq \nu \bar{B} \leq t \iff s \chi(p^*) \leq u_B \times \chi(p^*) \leq t \chi(p^*).
\]

Set

\[
\nu_1 B = \int u_B d\bar{\mu}.
\]

Then the same computations as in 2C, taken in reverse, show that \( \nu_1 \) is a \( \kappa \)-additive probability, zero on singletons; while \( \nu_1 \bar{B} = 0 \) iff \( u_B = 0 \) iff \( \vdash \nu \bar{B} = 0 \).

If \( \vdash \nu \) is normal, then take any family \( \langle B_\xi \rangle_{\xi < \kappa} \in \mathcal{N}_\nu \) and set \( B = \{ \eta : \exists \xi < \eta, \eta \in B_\xi \} \). Then

\[
\vdash \bar{B} = \{ \eta : \exists \xi < \eta, \eta \in B_\xi \};
\]

so \( \vdash \nu \bar{B} = 0 \) and \( \nu_1 B = 0 \). Thus \( \nu_1 \) is normal.

(c)(i) If \( B \in \mathcal{J} \) and \( \nu \) is a \( \mathbb{P} \)-name for a normal witnessing probability on \( \kappa \), then take \( \nu_1 \) as in (b); we must have \( \nu_1 B = 0 \) so \( \vdash \nu \bar{B} = 0 \).

(ii) If \( \sigma \) is a \( \mathbb{P} \)-name for a member of the rvm filter of \( \kappa \) in \( \mathcal{V}^\mathbb{P} \), let \( \langle a_\xi(\sigma) \rangle_{\xi < \kappa} \) be the corresponding family in \( \mathfrak{A} \), and set \( B = \{ \xi : a_\xi(\sigma) \neq 0 \} \). Then \( \vdash \sigma \subseteq \bar{B} \). If \( \nu \) is any normal witnessing probability on \( \kappa \) and \( \bar{\nu} \) the corresponding \( \mathbb{P} \)-name, then \( \vdash \nu \sigma = 0 \), so as in (a)(ii) above we must have \( \nu \bar{B} = 0 \). As \( \nu \) is arbitrary, \( B \in \mathcal{J} \) and \( \vdash \sigma \subseteq \bar{B} \in \mathcal{J} \).

2K Theorem If \( \kappa \) is a two-valued-measurable cardinal and \( \mathcal{F} \) is a normal ultrafilter on \( \kappa \), then

\[
L(\mathcal{F}) \vdash \kappa \text{ is two-valued-measurable and } \mathcal{F} \cap L(\mathcal{F}) \text{ is the rvm filter of } \kappa.
\]

**proof** Kunen 70, §6, or Jech 78, Theorem 76, p. 373.

2L Corollary The following are equiconsistent:

(a) ‘\( \mathbf{ZFC} + \) there is a two-valued-measurable cardinal’;
(b) ‘\( \mathbf{ZFC} + \) there is a two-valued-measurable cardinal in which the rvm filter is an ultrafilter’;
(c) ‘\( \mathbf{ZFC} + \) there is an atomlessly-measurable cardinal in which the rvm ideal is the ideal of negligible sets for a normal witnessing probability’.

**proof** (a)\( \Rightarrow \) (b) is covered by 2K. For (b)\( \Rightarrow \) (c), start with a two-valued-measurable cardinal in which the rvm filter is an ultrafilter, and use 2C with \( \mathbb{P} = \mathfrak{A}_\kappa \setminus \{0\} \), as in 2E. Then the rvm ideal of \( \kappa \) in \( \mathcal{V}^\mathbb{P} \) is generated by \( \mathcal{J} \), the rvm ideal of \( \kappa \) in \( \mathcal{V} \), by 2Jc. But if \( \bar{\nu} \) is the \( \mathbb{P} \)-name for a witnessing probability on \( \kappa \) derived from the \{0,1\}-valued probability \( \nu \) associated with \( \mathcal{J} \), then the null ideal of \( \bar{\nu} \) in \( \mathcal{V}^\mathbb{P} \) is also generated by \( \mathcal{J} \), by 2J(a(ii)); and therefore coincides with the rvm ideal, as required.

Finally, (c)\( \Rightarrow \) (a) is covered by 2D-2E above.

Version of 22.11.91
3. The Gitik-Shelah theorem

I come now to the most striking development in the theory of real-valued-measurable cardinals since the work of Solovay and Kunen in the late sixties. Recall that Maharam’s theorem (A2I) gives us a complete description of all probability algebras, and that consequently almost the first question to ask of any probability space is what its measure algebra is. In particular, we ask this of \((\kappa, \mathcal{P}_\kappa, \nu)\) when \(\kappa\) is real-valued-measurable and \(\nu\) is a \(\kappa\)-additive probability on \(\kappa\). If \(\kappa\) is two-valued-measurable, \((\kappa, \mathcal{P}_\kappa, \nu)\) is purely atomic and there is nothing of interest to say about its measure algebra. But if \(\kappa\) is atomlessly-measurable, there is a great deal more to it. The Gitik-Shelah theorem (3F-3G) tells us that in this case \(\mathcal{P}_\kappa/\mathcal{N}_\kappa\) is at least of the order of complexity achieved if \(\nu\) is constructed from a \(\kappa\)-complete ultrafilter with Solovay’s random reals.

This chapter is devoted to a proof of this theorem. The original paper GITIK & SHELAH 89 relied heavily on generic-ultrapower techniques, as does its supplement GITIK & SHELAH p91. Here I give what amounts to a translation of their arguments into measure theory. I do not give quite the shortest proof, as many of the intermediate steps seem to be of sufficient interest to be given as separate lemmas in rather greater generality than is needed immediately. A slightly more condensed version may be found in KAMBURELIS N89, from which some of the ideas below are derived.

3A Theorem Suppose that \((Y, \mathcal{P}Y, \nu)\) is a probability space and that \((X, \mathcal{T}, \Sigma, \mu)\) is a quasi-Radon probability space with the topological weight of \((X, \mathcal{T})\) strictly less than the additivity of \(\nu\). Let \(f: X \times Y \to \mathbb{R}\) be any bounded function; then

\[
\int \left( \int f(x, y) \nu(dy) \right) \mu(dx) \leq \int \left( \int f(x, y) \mu(dx) \right) \nu(dy),
\]

writing \(\int\) for the upper integral, as in A2E.

proof Adding a constant function to \(f\) if necessary, we need consider only the case of non-negative \(f\). Set \(\lambda = w(X) < \text{add}(\nu); \) let \(\langle G_\xi, \xi < \lambda \rangle\) enumerate a base for \(\mathcal{T}\). Fix \(\varepsilon > 0\). For each \(y \in Y\), let \(h_y: X \to \mathbb{R}\) be a lower semi-continuous function such that \(f(x, y) \leq h_y(x)\) for every \(x \in X\) and

\[
\int h_y(x) \mu(dx) \leq \varepsilon + \int f(x, y) \mu(dx)
\]

(A2Je). For each \(I \subseteq \lambda\), set

\[
f_I(x, y) = \sup \{s : \exists \xi \in I, x \in G_\xi, h_y(x') \geq s \ \forall \ x' \in G_\xi\}.
\]

Then \(f_I\) is expressible as \(\sup_{\xi \in I, s \in \mathbb{Q}_+} \chi(G_\xi \times B_{\xi, s})\), writing \(\mathbb{Q}_+\) for the set of non-negative rational numbers and \(\chi(G \times B)\) for the characteristic function of \(G \times B \subseteq X \times Y\), and taking \(B_{\xi, s} = \{y : h_y(x') \geq s \ \forall \ x' \in G_\xi\}\). So \(f_I\) is \((\mu \times \nu)\)-measurable for all countable \(I\), and for such \(I\) we shall have

\[
\int \int f_I(x, y) \mu(dx) \nu(dy) = \int \int f_I(x, y) \nu(dy) \mu(dx),
\]

by Fubini’s theorem. Next,

\[
\sup_{I \subseteq \lambda} \int f_I(x, y) = h_y(x)
\]

for all \(x \in X, y \in Y\), because each \(h_y\) is lower semi-continuous, so that

\[
\sup_{I \subseteq \lambda} \int f_I(x, y) \mu(dx) = \int h_y(x) \mu(dx)
\]

for each \(y \in Y\) (A2Jf). Because \(\lambda < \text{add}(\mu)\), it follows that

\[
\sup_{I \subseteq \lambda} \int \int f_I(x, y) \mu(dx) \nu(dy) = \int \int h_y(x) \mu(dx) \nu(dy)
\]

(A2Cf). On the other hand, if we write

\[
g_I(x) = \int f_I(x, y) \nu(dy)
\]

for each \(x \in X, I \subseteq \lambda\), then (at least for finite \(I\)) \(g_I\) is also lower semi-continuous, so \(g = \sup_{I \subseteq \lambda} g_I\) is lower semi-continuous, and

\[
g(x) = \int h_y(x) \nu(dy) \geq \int f(x, y) \nu(dy)
\]

for each \(x \in X, I \subseteq \lambda\).
for every $x \in X$, the equality in this formula again depending on A2Cf. So we have

$$\int \int f(x, y) \nu(dy) \mu(dx) \leq \int g(x) \mu(dx)$$

$$= \sup_{t \in [\lambda]} \int g_t(x) \mu(dx)$$

$$= \sup_{t \in [\lambda]} \int \int f_t(x, y) \nu(dy) \mu(dx)$$

As $\epsilon$ is arbitrary, we have the result.

**Remark** For the case $X = \{0, 1\}^\lambda$, this is due to Kunen. Readers uninterested in general quasi-Radon measure spaces should imagine $X$ here and in 3B to be a subset of $\{0, 1\}^\lambda$ with the subspace measure.

See also 6A below for a similar result, and 6J for an elementary corollary.

**3B Corollary** Let $\kappa$ be a real-valued-measurable cardinal, with witnessing probability $\nu$, and $(X, \Sigma, \Sigma, \mu)$ a quasi-Radon probability space with $\nu(X, \Sigma) < \kappa$.

(a) If $C \subseteq X \times \kappa$ then

$$\int \nu C([x]) \mu(dx) \leq \int \mu^* C^{-1}([\xi]) \nu(d\xi).$$

(b) If $A \subseteq X$ and $\#(A) \leq \kappa$, then there is a $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu^* B = \mu^* A$.

(c) If $(C_\xi)_{\xi < \kappa}$ is a family in $\mathcal{P} X \setminus \mathcal{N}_\mu$ such that $\#(\bigcup_{\xi < \kappa} C_\xi) < \kappa$, then there are distinct $\xi, \eta < \kappa$ such that $\mu^*(C_\xi \cap C_{\eta}) > 0$.

(d) If we have a family $(h_\xi)_{\xi < \kappa}$ of functions such that each $\text{dom}(h_\xi)$ is a non-negligible subset of $X$ and $\#(\bigcup_{\xi < \kappa} h_\xi) < \kappa$ (identifying each $h_\xi$ with its graph), then there are distinct $\xi, \eta < \kappa$ such that

$$\mu^* \{x : x \in \text{dom}(h_\xi) \cap \text{dom}(h_\eta), h_\xi(x) = h_\eta(x)\} > 0.$$

**proof** (a) Apply 3A to $\chi C : X \times \kappa \to \mathbb{R}$.

(b) Suppose, if possible, otherwise. Then surely $\#(A) = \kappa$; let $f : \kappa \to A$ be a bijection. Set

$$C = \{(f(\eta), \xi) : \eta \leq \xi < \kappa\} \subseteq X \times \kappa.$$

If $x \in A$,

$$\nu C([x]) = \nu\{\xi : f^{-1}(x) \leq \xi < \kappa\} = 1,$$

so $\int \nu C([x]) \mu(dx) = \mu^* A$. If $\xi < \kappa$,

$$\mu^* C^{-1}([\xi]) = \mu^* \{f(\eta) : \eta < \xi\} < \mu^* A,$$

so $\int \mu^* C^{-1}([\xi]) \nu(d\xi) < \mu^* A$. But this contradicts (a).

(c) Let $\tilde{\nu}$ be the probability on $\kappa \times \kappa$ defined by writing

$$\tilde{\nu} A = \int \nu A([\xi]) \nu(d\xi) \quad A \subseteq \kappa \times \kappa.$$

Then $\tilde{\nu}$ is $\kappa$-additive. Set

$$C = \{(x, (\xi, \eta)) : \xi, \eta \text{ are distinct members of } \kappa, x \in C_\xi \cap C_{\eta}\} \subseteq X \times (\kappa \times \kappa).$$

Set
proof (a) We can build inductively a family
This will mean that if $\xi < \kappa$ with $C_\xi \cap E = \emptyset$; thus $\mu^*(X \setminus E) > 0$. Now if $x \in X \setminus E$ then
\[
\tilde{\nu}((\xi, \eta) : (x, (\xi, \eta)) \in C) = (\nu(\xi : x \in C_\xi))^2 > 0.
\]
So we have
\[
0 < \int \tilde{\nu}((\xi, \eta) : (x, (\xi, \eta)) \in C) \mu(dx)
\]
\[
\leq \int \mu^*\{x : (x, (\xi, \eta)) \in C\} \tilde{\nu}(d(\xi, \eta)),
\]
by 3A, and there must be distinct $\xi, \eta < \kappa$ such that $\mu^*\{(x : (\xi, \eta)) \in C\} > 0$, as required.

(d) Set $Y = \bigcup_{\xi < \kappa} \text{Fan}(h_\xi)$. Give $X \times Y$ the measure $\tilde{\mu}$ and topology $\Sigma'$ defined as follows. The domain of $\tilde{\mu}$ is to be the family $\tilde{\Sigma}$ of subsets $H$ of $X \times Y$ for which there are $E, E' \in \Sigma$ such that $\mu(E \cap E') = 0$ and $E \times Y \subseteq H \subseteq E' \times Y$; and for such $H$, $\tilde{\mu}H$ is to be $\mu E = \mu E'$. The topology $\Sigma'$ is to be just the family $\{G \times Y : G \in \Sigma \}$. It is easy to check that $(X \times Y, \Sigma', \tilde{\mu})$ is a quasi-Radon probability space of weight less than $\kappa$, and that $\tilde{\mu}^*\nu_\xi = \mu^*(\text{dom}(h_\xi)) > 0$ for each $\xi < \kappa$. So (c) gives the result.

Remark I have taken the idea of 3Bc from KAMBURELIS N89.

3C Lemma Suppose that $\lambda, \zeta, \delta$ are cardinals, with $\delta < \lambda < \text{cf}(\zeta)$, and that $S$ is a stationary subset of $\zeta$.
Let $\langle I_\alpha \rangle_{\alpha \in S}$ be a family in $[\lambda]^\leq \delta$. Then there is a set $M \subseteq \lambda$ such that $\text{cf}(\#(M)) \leq \delta$ and $\{\alpha : \alpha \in S, I_\alpha \subseteq M\}$ is stationary in $\zeta$.

proof For $M \subseteq \lambda$, set $S_M = \{\alpha : \alpha \in S, I_\alpha \subseteq M\}$. Let $M \subseteq \lambda$ be a set of minimal cardinality such that $S_M$ is stationary in $\zeta$. Set $\theta = \#(M)$. If $\text{cf}(\theta) > \delta$, enumerate $M$ as $\langle \gamma_\xi \rangle_{\xi < \theta}$. For each $\alpha \in S_M$, set $\beta_\alpha = \sup\{\xi : \gamma_\xi \in I_\alpha\}$; because $\#(I_\alpha) \leq \delta < \text{cf}(\theta)$, $\beta_\alpha < \theta$. Because $\theta \leq \lambda < \text{cf}(\zeta)$, there is a $\beta < \theta$ such that $S' = \{\alpha : \alpha \in S_M, \beta_\alpha = \beta\}$ is stationary in $\zeta$. Consider $M' = \{\gamma_\xi : \xi \leq \beta\}$; then $\#(M') < \#(M)$ but $S_{M'} \supseteq S'$ so is stationary in $\zeta$, contrary to the choice of $M$. $\blacksquare$

Thus $M$ and $S = S_M$ will serve.

3D Definition Let $\kappa$ be a cardinal. Write $\text{Tr}^{\kappa}(\kappa)$ for
\[
\text{sup}(\#(F) : F \subseteq \kappa^\kappa; \#(f \cap g) < \kappa \text{ for all distinct } f, g \in F).
\]
Thus $\text{Tr}^{\kappa}(\kappa) = \text{Tr}^{\kappa \kappa}(\kappa; \kappa)$ in the notation of 2F.

3E Lemma (a) For any infinite cardinal $\kappa$,
\[
\kappa^+ \leq \text{Tr}^{\kappa}(\kappa) \leq 2^\kappa.
\]
(b) For any infinite cardinal $\kappa$,
\[
\text{max}(\text{Tr}^{\kappa}(\kappa), \text{sup}_{\kappa < \kappa} 2^\delta) \geq \min(2^\kappa, \kappa^{(\omega + \omega)}).
\]
(c) If $\kappa$ is such that $2^\delta \leq \kappa$ for every $\delta < \kappa$, then $\text{Tr}^{\kappa}(\kappa) = 2^\kappa$.

proof (a) We can build inductively a family $\langle f_\alpha \rangle_{\alpha < \kappa^+}$ in $\kappa^\kappa$, as follows. Given $\langle f_\alpha \rangle_{\alpha < \beta}$, where $\beta < \kappa^+$, let $\theta : \beta \rightarrow \kappa$ be any injection. Now choose $f_\beta : \kappa \rightarrow \kappa$ so that
\[
f_\beta(\xi) \neq f_\alpha(\xi) \text{ whenever } \alpha < \beta \text{ and } \theta(\alpha) \leq \xi.
\]
This will mean that if $\alpha < \beta$, then
\[
\{\xi : f_\alpha(\xi) = f_\beta(\xi)\} \subseteq \theta(\alpha)
\]
has cardinal less than $\kappa$. So at the end of the induction, $F = \{f_\alpha : \alpha < \kappa^+\}$ will witness that $\text{Tr}^{\kappa}(\kappa) \geq \kappa^+$. On the other hand, $\text{Tr}^{\kappa}(\kappa) \leq \#(\kappa^+) = 2^\kappa$.

(b) If not, then take $\lambda = \max(\text{Tr}^{\kappa}(\kappa), \text{sup}_{\kappa < \kappa} 2^\delta) < \min(2^\kappa, \kappa^{(\omega + \omega)})$. For each $\xi < \kappa$ take an injective function $\phi_\xi : \mathcal{P}_\xi \rightarrow \lambda$. But $\lambda < 2^\kappa$, we have an injective function $h : \lambda^+ \rightarrow \mathcal{P}_\kappa$. For $\alpha < \lambda^+$ set
$g_\alpha(\xi) = \phi_\xi(h(\alpha) \cap \xi)$ for every $\xi < \kappa$; then $\langle g_\alpha \rangle_{\alpha < \lambda^+}$ is a family in $\lambda^+$ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha \neq \beta$.

Apply 3C with $\delta = \kappa$, $\zeta = \lambda^+$, $\lambda_\alpha = g_\alpha[k]$ to see that there is a set $M \subseteq \lambda$ with $\text{cf}(\#(M)) \leq \kappa$ and $S = \{ \alpha : \alpha < \lambda^+, g_\alpha[k] \subseteq M \}$ stationary in $\lambda^+$. Because $\lambda < \kappa^{(+\omega)}$, we must have $\#(M) \leq \kappa$. If $f : M \to \kappa$ is any injection, $\langle f_{g_\alpha} \rangle_{\alpha \in S}$ will witness that $\text{Tr}(\kappa) \geq \#(S) = \lambda^+$; which is impossible. \(\Box\)

(c) For each $\xi < \kappa$, let $\phi_\xi : \mathcal{P}\xi \to \kappa$ be injective. For $A \subseteq \kappa$, define $f_A \in \kappa^\kappa$ by writing

$$f_A(\xi) = \phi_\xi(A \cap \xi) \forall \xi < \kappa.$$ 

Then $F = \{ f_A : A \subseteq \kappa \}$ witnesses that $\text{Tr}(\kappa) \geq 2^\kappa$.

3F The Gitik-Shelah Theorem Let $\kappa$ be an atomlessly-measurable cardinal, with witnessing probability $\nu$. Then the Maharam type of $(\kappa, \mathcal{P}\kappa, \nu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$.

**proof:** Suppose, if possible, otherwise.

(a) Let $A \subseteq \kappa$ be such that $\nu A > 0$ and $(A, \mathcal{P}A, \nu[A]$ is Maharam homogeneous, of Maharam type $\lambda$ say; surely $\lambda$ is not greater than the Maharam type of $(\kappa, \mathcal{P}\kappa, \nu)$, so we have $\lambda < 2^\kappa$ and $\lambda < \kappa^{(+\omega)}$. Also $\lambda \geq \omega$, because $\nu$ is atomless.

Set

$$\nu_1 B = (\nu A)^{-1}\nu(A \cap B) \forall B \subseteq \kappa,$$

then $(\kappa, \mathcal{P}\kappa, \nu_1)$ is Maharam homogeneous, with Maharam type $\lambda$.

(b) We shall need some fairly elaborate notation. Let $\mu_\lambda$ be the usual Radon probability on $\{0,1\}^\lambda$ (A2G), $\Sigma_\lambda$ its domain, and $\mathfrak{A}_\lambda = \Sigma_\lambda/\mathcal{N}_{\mu_\lambda}$ its measure algebra. For $M \subseteq \lambda$ let $\pi_M : \{0,1\}^\lambda \to \{0,1\}^M$ be the canonical restriction map, and $\mu_M$ the usual Radon probability on $\{0,1\}^M$, so that $\pi_M$ is measure-preserving for $\mu_\lambda$ and $\mu_M$; write $\Sigma_M^\lambda$ for the Baire $\sigma$-algebra of $\{0,1\}^M$.

Now let us return to the argument. We are supposeing that $\mathfrak{A}_\lambda = \mathfrak{A} = \mathcal{P}\kappa/\mathcal{N}_{\nu_1}$, the measure algebra of $(\kappa, \mathcal{P}\kappa, \nu_1)$. Let $\phi : \mathfrak{A}_\lambda \to \mathfrak{A}$ be a measure-preserving isomorphism and $f : \kappa \to \{0,1\}^\lambda$ a corresponding inverse-measure-preserving function, so that

$$f^{-1}[E]^* = \phi(E^*) \in \mathfrak{A} \forall E \in \Sigma_\lambda$$

(A2.Id)

(c) From here on we are going to need a split argument, depending on whether or not $\lambda < \text{Tr}(\kappa)$. Because large parts of the two arguments are the same, I take them together; but readers may prefer to follow through case 1 completely before returning to examine the modifications necessary in case 2.

**case 1** Suppose that $\lambda < \text{Tr}(\kappa)$. Set $\zeta = \max(\kappa^+, \lambda^+)$; then $\zeta \leq \text{Tr}(\kappa)$ is a successor cardinal so there is a family $\langle g_\alpha \rangle_{\alpha < \zeta}$ of functions from $\kappa$ to $\kappa$ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha < \beta < \zeta$. Set $S = \zeta \setminus \kappa$, so that $S$ is a stationary set in $\kappa$. We know that $\kappa < \zeta$; let $h : \kappa \to \{0,1\}^\omega$ be an injection. Set $\delta = \omega$, $\zeta^* = \kappa$, so that we have $g_\alpha : \kappa \to \min(\zeta^*, \alpha)$ for each $\alpha \in S$, and $h : \zeta^* \to \{0,1\}^\delta$ is an injection. Because $\kappa < \text{Tr}(\kappa) \leq \lambda < \kappa^{(+\omega)}$, $\text{cf}(\lambda) > \omega_1$. By Shelah’s lemma A1F-G, we can find a stationary set $S \subseteq \zeta$ and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from $\kappa$ to $\zeta = \zeta^*$ such that (i) $g_\alpha[k] \subseteq \alpha$ for each $\alpha \in S$ (ii) $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct $\alpha, \beta \in S$ (iii) if $\theta$ is a limit ordinal less than $\kappa$, and $\alpha, \beta \in S$ are such that $g_\alpha(\theta) = g_\beta(\theta)$, then $g_\alpha[\theta] = g_\beta[\theta]$.

(d) Now we have a section in which the two arguments run together, if we keep hold of the notations introduced in (c). For each $\alpha \in S$ consider $h g_\alpha : \kappa \to \{0,1\}^\delta$. For $\iota < \delta$ set

$$U_{\alpha \iota} = \{ \xi : \xi < \kappa, (h g_\alpha(\xi))(\iota) = 1 \},$$

and choose $H_{\alpha \iota} \in \Sigma_\lambda^\iota$ such that $\phi^{-1}(U_{\alpha \iota}^*) = H_{\alpha \iota}^* \in \mathfrak{A}_\lambda$. Define $\bar{g}_\alpha : \{0,1\}^\lambda \to \{0,1\}^\delta$ by setting

$$\bar{g}_\alpha(x)(\iota) = 1 \text{ if } x \in H_{\alpha \iota}, = 0 \text{ otherwise}.$$ 

Then
\[\{\xi : \xi < \kappa, \tilde{g}_\alpha f(\xi) \neq h g_\alpha(\xi)\} = \bigcup_{\iota < \delta} \{\xi : (\tilde{g}_\alpha f(\xi))(\iota) \neq (h g_\alpha(\xi))(\iota)\} = \bigcup_{\iota < \delta} U_{\alpha\iota} \Delta f^{-1}[H_{\alpha\iota}] \in N_{\nu_1}\]

because \(\delta < \kappa = \text{add}(N_{\nu_1})\). Set \(V_{\alpha} = \{\xi : \tilde{g}_\alpha f(\xi) = h g_\alpha(\xi)\}\), so that \(\nu_1 V_{\alpha} = 1\), for each \(\alpha \in S\).

(e) Because every \(H_{\alpha\iota}\) is a Baire set, there is for each \(\alpha \in S\) a set \(I_{\alpha} \subseteq \lambda\) such that \(#(I_{\alpha}) \leq \delta\) and \(H_{\alpha\iota} = \pi_{I_{\alpha}}^{-1}[\pi_{I_{\alpha}}[H_{\alpha\iota}]] \forall \iota < \delta\).

By Lemma 3C there is an \(M \subseteq \lambda\) such that

\[S_1 = \{\alpha : \alpha \in S, I_{\alpha} \subseteq M\}\]

is stationary in \(\zeta\) and \(\text{cf}(#(M)) \leq \delta\); because \(\lambda < \kappa^{(\omega)}\) and \(\text{cf}(\kappa) > \delta\), \(#(M) < \kappa\). Write \(f_M = \pi_M f\), so that \(f_M : \kappa \to \{0, 1\}^M\) is inverse-measure -preserving for \(\nu_1\) and \(\mu_M\).

(f) For each \(\alpha \in S_1\), there is a \(\theta_{\alpha} < \kappa\) such that \(\mu_M^*(f_M[V_{\alpha} \cap \theta_{\alpha}]) = 1\). Apply 3Bb to \(f_M[V_{\alpha}] \subseteq \{0, 1\}^M\). There must be a set \(B \subseteq f_M[V_{\alpha}]\) such that \(\text{cf}(B) < \kappa\) and \(\mu_M^* B = \mu_M^*(f_M[V_{\alpha}])\); because \(\kappa\) is regular, there is a \(\theta_{\alpha} < \kappa\) such that \(B \subseteq f_M[V_{\alpha} \cap \theta_{\alpha}]\). On the other hand, because \(f_M\) is inverse-measure -preserving, \(\mu_M^*(f_M[V_{\alpha}]) \geq \nu_1 V_{\alpha} = 1\). Q

Evidently we may take it that every \(\theta_{\alpha}\) is a non-zero limit ordinal.

(g) Because \(\zeta = \text{cf}(\zeta) > \kappa\), there is a \(\theta < \kappa\) such that

\[S_2 = \{\alpha : \alpha \in S_1, \theta_{\alpha} = \theta\}\]

is stationary in \(\zeta\). At this point the two cases diverge briefly.

**case 1** For \(\alpha \in S_2\), \(g_\alpha[\theta]\) is bounded in \(\kappa\); let \(\theta' < \kappa\) be such that

\[S_3 = \{\alpha : \alpha \in S_2, g_\alpha[\theta] \subseteq \theta'\}\]

is stationary in \(\zeta\); write \(Y = \theta'\).

**case 2** For \(\alpha \in S_2\), \(g_\alpha(\theta) < \alpha\); by the pressing-down lemma there is a \(\theta' < \zeta\) such that

\[S_3 = \{\alpha : \alpha \in S_2, g_\alpha(\theta) = \theta'\}\]

is stationary in \(\zeta\). Then \(g_\alpha[\theta] = g_{\beta}[\theta]\) for all \(\alpha, \beta \in S_3\); take \(Y\) to be the common value of \(g_\alpha[\theta]\) for \(\alpha \in S_3\).

(h) Thus in both cases we have \(Y \subseteq \zeta^*, \#(Y) < \kappa\) and \(g_\alpha[\theta] \subseteq Y\) for all \(\alpha \in S_3\). For each \(\alpha \in S_3\), set

\[Q_\alpha = f_M[V_{\alpha} \cap \theta]\]

so that \(\mu_M^* Q_\alpha = 1\). Now \(I_{\alpha} \subseteq M\), so we can express \(\tilde{g}_\alpha\) as \(g_\alpha^* \pi_M\), where \(g_\alpha^* : \{0, 1\}^M \to \{0, 1\}^\delta\) is Baire measurable in each coordinate. If \(y \in Q_\alpha\), take \(\xi \in V_{\alpha} \cap \theta\) such that \(f_M(\xi) = y\); then

\[g_\alpha^*(y) = g_\alpha^* \pi_M f(\xi) = \tilde{g}_\alpha f(\xi) = h g_\alpha(\xi) \in h[Y]\]

Thus \(g_\alpha^*(Q_\alpha) \subseteq f_M[\theta] \times h[Y]\) for every \(\alpha \in S_3\), and we may apply Corollary 3Bd to \(X = \{0, 1\}^M, \mu = \mu_M\) and the family \(\{g_\alpha^*[Q_\alpha]\}_{\alpha \in S_3}\), where \(S' \subseteq S_3\) is a set of cardinal \(\kappa\), to see that there are distinct \(\alpha, \beta \in S_3\) such that \(\mu_M^*(\{y : y \in Q_\alpha \cap Q_\beta, g_\alpha^*(y) = g_\beta^*(y)\}) > 0\). Now, however, consider

\[E = \{y : y \in \{0, 1\}^M, g_\alpha^*(y)(\iota) = g_\beta^*(y)(\iota)\} \in \Sigma^*_M\]

Then \(E = \bigcap_{\iota < \delta} E_\iota\), where

\[E_\iota = \{y : y \in \{0, 1\}^M, g_\alpha^*(y)(\iota) = g_\beta^*(y)(\iota)\} \in \Sigma^*_M\]

for \(\iota < \delta\). Because \(\delta < \kappa\),
\[ \nu_1 f_M^{-1}[E] = \nu_1 \left( \bigcap_{i \in I} f_M^{-1}[E_i] \right) \]
\[ = \inf_{I \in \mathcal{I}} \nu_1 \left( \bigcap_{i \in I} f_M^{-1}[E_i] \right) \]
\[ = \inf_{I \in \mathcal{I}} \mu_M \left( \bigcap_{i \in I} E_i \right) \geq \mu_M^* E > 0. \]

Consequently
\[ 0 < \nu_1 f_M^{-1}[E] \]
\[ = \nu_1 \{ \xi : g_\alpha^* \pi_M f(\xi) = g_\beta^* \pi_M f(\xi) \} \]
\[ = \nu_1 \{ \xi : \tilde{g}_\alpha f(\xi) = \tilde{g}_\beta f(\xi) \} \]
\[ = \nu_1 \{ \xi : \tilde{g}_\alpha(\xi) = \tilde{g}_\beta(\xi) \} \]

(because \( h \) is injective). But this is absurd, because in (d) above (whether in case 1 or in case 2) we chose \( g_\alpha, g_\beta \) in such a way that \( \{ \xi : g_\alpha(\xi) = g_\beta(\xi) \} \) would be bounded in \( \kappa \). \( \mathbf{X} \) This contradiction completes the proof.

**3G Theorem** Let \((X, \mathcal{P}X, \mu)\) be an atomless probability space. Write \( \kappa = \text{add}(\mu) \). Then the Maharam type of \((X, \mathcal{P}X, \mu)\) is at least \( \min(\kappa^{(\omega)}), 2^\kappa \), and in particular is greater than \( \kappa \).

**proof** As in the proof of 1Da above, there is a disjoint family \( \{E_\xi\}_{\xi \ll \kappa} \in \mathcal{N}_\mu \) such that \( E = \bigcup_{\xi \ll \kappa} E_\xi \notin \mathcal{N}_\mu \).

Let \( f : E \to \kappa \) be the corresponding function, and set \( \nu = (\mu E)^{-1}((\mu [E] f)^{-1}) \). Then \( \nu \) is a witnessing probability on \( \kappa \). Now observe that if \( \mathfrak{A}, \mathfrak{B} \) are the measure algebras of \((X, \mathcal{P}X, \mu)\) and \((\kappa, \mathcal{P} \kappa, \nu)\) respectively, the map
\[ \mathfrak{A}^* \mapsto (f^{-1}[A])^* : \mathfrak{B} \to \mathfrak{A} \]

is an injective order-continuous Boolean homomorphism from \( \mathfrak{B} \) to the principal ideal of \( \mathfrak{A} \) generated by \( E^* \). Its range therefore has Maharam type equal to that of \( \mathfrak{B} \); it follows easily that the Maharam type of \( \mathfrak{A} \) is at least that of \( \mathfrak{B} \) (A2Hc-d), and must be at least \( \min(2^\kappa, \kappa^{(\omega)}) \), by Theorem 3F.

**3H Corollary** Let \((X, \mathcal{P}X, \nu)\) be an atomless probability space, and \( \kappa = \text{add}(\nu) \). Let \((Z, \Sigma, \Sigma, \mu)\) be a Radon probability space of Maharam type \( \lambda \leq \min(2^\kappa, \kappa^{(\omega)}) \) (e.g., \( Z = \{0, 1\}^\lambda \)). Then there is an inverse-measure-preserving function \( f : X \to Z \).

**proof** By 3G, the Maharam type of \((C, \mathcal{P}C, \nu[C])\) is at least \( \lambda \) whenever \( C \subseteq X \) and \( \mu C > 0 \); that is, \( \tau(\mathfrak{A}[c]) \geq \lambda \) whenever \( c \) is a non-zero element of the measure algebra \( \mathfrak{A} \) of \((X, \mathcal{P}X, \nu)\) and \( \mathfrak{A}[c] \) is the principal ideal of \( \mathfrak{A} \) generated by \( c \). Now it follows by A2Ib that there is a measure-preserving homomorphism \( \phi : \Sigma/\mathcal{N}_\mu \to \mathfrak{A} \), which by A2Jd corresponds to an inverse-measure-preserving \( f : X \to Z \).

**3I Corollary** If \( \kappa \) is an atomlessly-measurable cardinal, and \((Z, \mu)\) is a Radon probability space of Maharam type at most \( \min(2^\kappa, \kappa^{(\omega)}) \), then there is an extension of \( \mu \) to a \( \kappa \)-additive measure defined on \( \mathcal{P}Z \).

**proof** Let \( \nu \) be a witnessing probability on \( \kappa \); by 3H, there is an inverse-measure-preserving function \( f : X \to Z \); now \( \nu f^{-1} \) extends \( \mu \) to \( \mathcal{P}Z \).

**3J Corollary** If \( \kappa \) is an atomlessly-measurable cardinal, with witnessing probability \( \nu \), and \( 2^\kappa < \kappa^{(\omega)} \), then \((\kappa, \mathcal{P} \kappa, \nu)\) is Maharam homogeneous, with Maharam type \( 2^\kappa \).

**proof** If \( C \in \mathcal{P} \kappa \setminus \mathcal{N}_\mu \), then the Maharam type of \( \nu[C] \) is at least \( 2^\kappa \), by 3F-3G; but also it cannot be greater than \( \#(\mathcal{P}C) = 2^\kappa \).
3K Remarks  The original theorem of Gitik & Shelah 89 was that the Maharam type $\lambda$ of $(\kappa, \mathcal{P}\kappa, \nu)$ is at least $\kappa^+$ for any atomlessly-measurable cardinal $\kappa$ and witnessing probability $\nu$. Elementary modifications of their arguments showed that it is at least $\min(\text{Tr}(\kappa), \kappa^{(+\omega)})$ (case 1 of 3F above). The extra ideas in case 2 of 3F come from Gitik & Shelah P91.

If there can be measurable cardinals at all, it is consistent to suppose that $\lambda = \kappa^{(+\omega)} < 2^\kappa$ (2Ie above). So there is no obvious sharpening of the result to look for. But it may be that a more delicate combinatorial analysis would give a closer description of $\lambda$, and, in particular, determine whether $(\kappa, \mathcal{P}\kappa, \nu)$ is always Maharam homogeneous. See also 7Qb, for a scrap of further information, and P2, for a discussion of the outstanding questions.

For more in the direction of 3I, see 8A.

3L Proposition  Let $\kappa$ be an atomlessly-measurable cardinal.

(a) The following are equivalent: (i) every witnessing probability $\nu$ on $\kappa$ is Maharam homogeneous; (ii) if $\nu_1$ and $\nu_2$ are witnessing probabilities on $\kappa$ they have the same Maharam type.

(b) The following are equivalent: (i) every normal witnessing probability $\nu$ on $\kappa$ is Maharam homogeneous; (ii) if $\nu_1$ and $\nu_2$ are normal witnessing probabilities on $\kappa$ they have the same Maharam type.

proof (a) Suppose, if possible, that (ii) is true and (i) is false. Let $\nu$ be a witnessing probability on $\kappa$ which is not Maharam homogeneous. Then there is an $E \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ such that $\nu|E$ has strictly smaller Maharam type than $\nu$. But now if we set $\nu_1 = \nu(A \cap E)/\nu E$ for every $A \subseteq \kappa$, we obtain a witnessing probability $\nu_1$ with the same Maharam type as $\nu|E$, and different from that of $\nu$; contradicting (ii).

(b) Argue as for (a); we need note only that if $\nu_1$ and $\nu_2$ are normal so is $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ (because $\mathcal{N}'_{\nu} = \mathcal{N}_{\nu_1} \cap \mathcal{N}_{\nu_2}$), and if $\nu$ is normal and $\nu_1$ is constructed as in (ii) above then $\nu_1$ is normal (because $\mathcal{N}_{\nu_1} = \{A : A \cap E \in \mathcal{N}_{\nu}\}$).

3M Proposition  Let $\kappa$ be an atomlessly-measurable cardinal.

(a) If $\nu$ is a Maharam homogeneous witnessing probability on $\kappa$ with Maharam type $\lambda$, then there is a Maharam homogeneous normal witnessing probability $\nu_1$ on $\kappa$ with Maharam type $\lambda_1 \leq \lambda$.

(b) If $\nu$ and $\nu'$ are Maharam homogeneous witnessing probabilities on $\kappa$ with Maharam types $\lambda$, $\lambda'$, then there is a Maharam homogeneous witnessing probability $\nu''$ on $\kappa$ with Maharam type $\lambda'' \geq \text{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda')$.

proof (a) Construct an essentially minimal $f_0 : \kappa \rightarrow \kappa$ as in the proof of 1G, and set $\nu_0 = \nu f_0^{-1}$. Then $\nu_0$ is a normal witnessing probability on $\kappa$, as observed in 1G; moreover, $f_0$ is inverse-measure-preserving for $\nu$ and $\nu_0$ so induces an embedding of the measure algebra $\mathfrak{A}_0 = \mathcal{P}\kappa/\mathcal{N}_{\nu_0}$ in $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{N}_{\nu}$. Accordingly $\tau(\mathfrak{A}_0)$, the Maharam type of $\nu_0$, is at most $\tau(\mathfrak{A}) = \lambda$ (A2Hd). I do not know whether $\nu_0$ itself must be Maharam homogeneous, but there is surely an $E \in \mathcal{P}\kappa \setminus \mathcal{N}_{\nu_0}$ such that $\nu_0|E$ is Maharam homogeneous, and now setting $\nu_1 = \nu_0(A \cap E)/\nu_0 E$ for $A \subseteq \kappa$ we obtain a Maharam homogeneous normal $\nu_1$ of Maharam type less than or equal to $\lambda$.

(b) Let $\nu_1$ be the $\kappa$-additive probability on $\kappa \times \kappa$ given by

$$\nu_1 C = \int \nu'|C'(\xi) \nu(d\xi) \quad \forall C \subseteq \kappa \times \kappa.$$ 

Set $\theta = \text{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda')$. By 2G there is a family $F \subseteq \lambda^{\kappa}$ such that $\#(F) = \theta$ and $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}_\nu$ for all distinct $f, g \in F$. Let $\langle E_\xi \rangle_{\xi \in \lambda'}$ be a $\nu'$-stochastically independent family of subsets of $\kappa$ of $\nu'$-measure 1/2. For each $f \in F$ set

$$C_f = \{(\xi, \eta) : \xi < \kappa, \eta \in E_{f(\xi)}\}.$$ 

Then for any non-empty finite subset $I$ of $F$, $\nu'((\bigcap_{f \in I} E_{f(\xi)}) = 2^{-\#(I)}$ for $\nu$-almost every $\xi$, so that

$$\nu_1((\bigcap_{f \in I} C_f) = 2^{-\#(I)}.$$ 

Thus $\langle C_f \rangle_{f \in F}$ is stochastically independent for $\nu_1$, and the Maharam type of $\nu_1|C$ is at least $\#(F) = \theta$ whenever $\nu_1 C > 0$. Once again, take $\nu_2 C = \nu_1(C \cap D)/\nu_1 D$ for some $D$ for which $\nu_1 D$ is Maharam
homogeneous, to obtain a Maharam homogeneous $\kappa$-additive probability $\nu_2$ of Maharam type at least $\theta$. Finally, of course, $\nu_2$ can be copied onto a probability $\nu''$ on $\kappa$, as asked for.

Remark Evidently the arguments for (b) have extensions to the case in which we have two real-valued-measurable cardinals $\kappa, \kappa'$ with witnessing probabilities $\nu, \nu'$.

Version of 16.5.91

4. The enormity of real-valued-measurable cardinals.

Under this title I collect together results of the form ‘if $\kappa$ is real-valued-measurable, there are many complex cardinals below it’. Ulam’s theorem that a real-valued-measurable cardinal must be weakly inaccessible (1Dc-d) is the first step: if $\kappa$ is real-valued-measurable, there are $\kappa$ cardinals below it. But enormously more can be said. To develop these ideas, we need labels for some of the intermediate stages. first ‘weakly Mahlo’ and ‘greatly Mahlo’ cardinals (4A), and then ‘weakly $\Pi^1_2$-indescribable’ cardinals (A4C, 4D). Up to the weakly $\Pi^1_2$-indescribable cardinals, we can use ordinary infinitary combinatorics (4A-4L); but thereafter we shall need the apparatus of (elementary) model theory from §4A and forcing from §2. The culminating result is Theorem 4P: if $\kappa$ is real-valued-measurable, there are many weakly $\Pi^1_2$-indescribable cardinals below it. The proof of this theorem includes essentially everything required to prove another remarkable fact: if $\kappa$ is atomlessly-measurable, and $I$ any structure of cardinal less than $\kappa$, then the first- and second-order properties of $I$ are unaffected by random real forcing (Corollary 4Oa). The same arguments provide a general method for proving results of the form ‘if $\kappa$ is real-valued-measurable, there are many $\alpha < \kappa$ such that $\alpha \vDash \phi'$ when we have found a proof that $\kappa \vDash \phi$ for every real-valued-measurable cardinal $\kappa$ (4Ob).

4A Definitions (a) A cardinal $\kappa$ is a weakly Mahlo cardinal if it is weakly inaccessible and the set of weakly inaccessible cardinals below $\kappa$ is stationary in $\kappa$.

(b) If $A$ is any set of ordinals, write $\text{Mh}(A)$ for

$$\{\xi : \xi < \sup A, \text{cf}(\xi) > \omega, A \cap \xi \text{ is stationary in } \xi\}.$$ 

Following Baumgartner Taylor & Wagon 77, I will call $\text{Mh}$ Mahlo’s operation.

(This is close to the operation $H$ of Lévy 71, and in a sense dual to the ‘Mahlo operation’ $M$ of Keisler & Tarski 64.)

(c) A cardinal $\kappa$ is greatly Mahlo if it is uncountable and regular and there is a normal filter $\mathcal{F}$ on $\kappa$ such that $\text{Mh}(A) \subseteq \mathcal{F}$ for every $A \subseteq \mathcal{F}$.

In this case (because the intersection of any non-empty family of normal filters is a normal filter) there is a minimal normal filter $\mathcal{W}$ on $\kappa$ which is closed under $\text{Mh}$; I will call $\mathcal{W}$ the greatly Mahlo filter of $\kappa$.

4B Theorem If $\kappa$ is greatly Mahlo, it is weakly Mahlo and the set of weakly Mahlo cardinals below $\kappa$ belongs to the greatly Mahlo filter of $\kappa$.

proof This is essentially elementary. Let $\kappa$ be a greatly Mahlo cardinal and $\mathcal{W}$ its greatly Mahlo filter. For each $\eta < \kappa$, let $W_\eta$ be the set of ordinals less than $\kappa$ with cofinality at least $\eta$. Then $W_{\eta+1} \supseteq \text{Mh}(W_\eta)$ for every $\eta$, so (because $\mathcal{W}$ is $\kappa$-complete) $W_\eta \subseteq \mathcal{W}$ for every $\eta < \kappa$. Because $\mathcal{W}$ is normal, $W = \{\xi : \omega < \xi < \kappa, \xi \in W_\eta \forall \eta < \xi\}$ belongs to $\mathcal{W}$; but $\mathcal{W}$ is just the set of regular uncountable cardinals below $\kappa$. This forces $\kappa$ to be a weakly inaccessible cardinal. Next, consider the set $\mathcal{W}'$ of weakly inaccessible cardinals below $\kappa$; $\mathcal{W}' \supseteq \mathcal{W} \cap \text{Mh}(\mathcal{W})$, so $\mathcal{W}' \subseteq \mathcal{W}$ and must be stationary in $\kappa$; thus $\kappa$ is weakly Mahlo. Finally, $W'' = \mathcal{W}' \cap \text{Mh}(\mathcal{W}')$ is the set of weakly Mahlo cardinals below $\kappa$, and again belongs to $\mathcal{W}$.

Remark The alternation of diagonal intersection and the Mahlo operation $\text{Mh}$ can be used to describe an indefinitely complex hierarchy of cardinals; near the bottom, the ‘$(\alpha+1)$-Mahlo cardinals’ are those in which the smaller $\alpha$-Mahlo cardinals form a stationary set, taking the weakly inaccessible cardinals to be the ‘0-Mahlo’ cardinals. The ‘greatly Mahlo’ cardinals form a natural staging post well along this progression.

4C Lemma Let $\kappa$ be a regular uncountable cardinal. Then $\kappa$ is greatly Mahlo iff whenever $\xi$ is a well-ordering of $\kappa$ and $\langle S_\xi \rangle_{\xi < \kappa}$ is a family of subsets of $\kappa$ such that
a closed unbounded subset $C$ of $M$.

**Proof (a)** If $\kappa$ is strongly Mahlo, it carries a normal ultrafilter $F$ which is closed under Mahlo's operation; now an easy induction on the $\simeq$-position of $\kappa$ shows that under the conditions described every $S_\zeta$ must belong to $F$, so is stationary.

(b) Now suppose that $\kappa$ is not strongly Mahlo. Recall that if $\text{NS}_\kappa$ is the ideal of non-stationary subsets of $\kappa$ and $\mathfrak{A}$ is the quotient algebra $\mathcal{P}_\kappa/\text{NS}_\kappa$, then any subset of $\mathfrak{A}$ of cardinal $\kappa$ or less has a greatest lower bound in $\mathfrak{A}$. (See 1Fd-e.) Observe also that we have an operation $M : \mathfrak{A} \to \mathfrak{A}$ defined by writing $M(A^*) = (\text{Mh}(A))^*$ for every $A \subseteq \kappa$; the point being that if $A, A'$ are subsets of $\kappa$ and $A \cap C = A' \cap C$ for a closed unbounded subset $C$ of $\kappa$, then $\text{Mh}(A) \cap C' = \text{Mh}(A') \cap C'$, where $C'$ is the set of accumulation points of $C$ in $\kappa$.

We may therefore define a family $\langle a_\alpha \rangle_{\alpha < \kappa^+}$ in $\mathfrak{A}$ by setting

\[
a_0 = 1,
\]

\[
a_\alpha = \inf_{\beta < \alpha} M(a_\beta) \text{ if } 0 < \alpha < \kappa^+.
\]

A simple induction shows that $M(a_\alpha) \subseteq a_\alpha \subseteq a_\beta$ whenever $\beta \leq \alpha < \kappa^+$. \(\Box\) If no $a_\alpha$ is zero, consider

\[
F = \{ A : A \subseteq \kappa, \exists \alpha < \kappa^+ \text{ such that } a_\alpha \subseteq A^* \in \mathfrak{A} \}.
\]

Then it is easy to check that $F$ is a normal filter on $\kappa$ which is closed under Mahlo's operation; which is supposed to be impossible. \(\Box\)

There must therefore be a $\gamma < \kappa^+$ such that $a_\gamma = 0$; we may suppose that $\gamma \geq \kappa$. Let $h : \gamma + 1 \to \kappa$ be any bijection, and $\lessdot$ the well-ordering on $\kappa$ corresponding to that of $\gamma + 1$, as transferred to $\kappa$ by $h$. If we now define $\langle S_\zeta \rangle_{\zeta < \kappa}$ by the formula

\[S_\zeta = \{ \xi : \xi < \kappa, \xi \in \text{Mh}(S_\eta) \text{ whenever } \eta < \xi \text{ and } \eta < \zeta \},\]

we find that $a_\alpha = S_{h(\alpha)}^*$ for every $\alpha \leq \gamma$, so that $S_{h(\gamma)}^* = 0$, i.e., $S_{h(\gamma)}$ is stationary. Thus $\lessdot$, $\langle S_\zeta \rangle_{\zeta < \kappa}$ witness the condition fails.

**Remark** This is due to Baumgartner Taylor & Wagon 77 (p. 212).

**4D Definitions** Let $\kappa$ be a regular infinite cardinal.

(a) Write $\text{Regr}(\kappa)$ for the set of regressive functions from $\kappa$ to itself. For $F \subseteq \text{Regr}(\kappa)$, write $U_F$ for the set of uniform ultrafilters $F$ on $\kappa$ such that $\lim_{\alpha \to F} f(\alpha)$ exists for every $f \in F$, that is to say, every $f \in F$ is constant on some member of $F$.

(b) A set $A \subseteq \kappa$ is $\Pi^1_1$-fully stationary in $\kappa$ if for every $F \in [\text{Regr}(\kappa)]^{\leq \kappa}$ there is an $F \in U_F$ such that $A \in F$; that is, there is a uniform ultrafilter $F$ on $\kappa$ such that $A \in F$ and $\lim_{\alpha \to F} f(\alpha)$ exists for every $f \in F$.

(c) From A4H below we see that $\kappa$ is weakly $\Pi^1_1$-indescribable iff $\kappa$ is $\Pi^1_1$-fully stationary in itself, and that in this case its $\Pi^1_1$-filter is precisely the set

\[
W = \{ W : W \subseteq \kappa, W \cap A = \emptyset \text{ for every } A \text{ $\Pi^1_1$-fully stationary } A \subseteq \kappa \}
\]

\[
= \{ W : \kappa \setminus W \text{ is not } \Pi^1_1 \text{-fully stationary} \}
\]

\[
= \{ W : \exists F \in [\text{Regr}(\kappa)]^{\leq \kappa}, \kappa \setminus W \notin F \forall F \in U_F \}
\]

\[
= \{ W : \exists F \in [\text{Regr}(\kappa)]^{\leq \kappa}, W \subseteq \bigcap U_F \}.
\]

Down to 4K below I will use this characterization as if it were the definition of ‘weakly $\Pi^1_1$-indescribable’ and ‘$\Pi^1_1$-filter’. Of course this is an inefficient procedure in some ways, and many readers will prefer the original proofs of 4F, 4G and 4H as given in Lévy 71 and Baumgartner Taylor & Wagon 77, while 4K is covered by 4P. I take the trouble to spell out the combinatorial arguments because I believe that this alternative route can provide different insights into the relationships between the various concepts here.
4E Lemma Let $\kappa$ be an infinite cardinal and $\{C_\xi\}_{\xi<\kappa}$ a family of subsets of $\kappa$.

(a) If $C_\xi \cap \xi$ is cofinal with $\xi$ for every $\xi < \kappa$, there is an $F \in [\text{Regr}(\kappa)]^{\leq \kappa}$ such that $\lim_{\xi \to F} C_\xi = \{\eta : \{\xi : \eta \in C_\xi \in F\} \text{ is cofinal with } \kappa \text{ for every } F \in U_F\}$.

(b) If $C_\xi \cap \xi$ is relatively closed in $\xi$ for every $\xi < \kappa$, there is an $F \in [\text{Regr}(\kappa)]^{\leq \kappa}$ such that $\lim_{\xi \to F} C_\xi$ is closed in $\kappa$ for every $F \in U_F$.

(c) If $\xi < \kappa$ and $C_\xi \cap \xi$ is cofinal with $\xi$ for every $\xi < \kappa$, there is an $F \in [\text{Regr}(\kappa)]^{\leq \kappa}$ such that $\xi \cap \lim_{\xi \to F} C_\xi$ is cofinal with $\xi$ for every $F \in U_F$.

**proof (a)** For $\alpha, \xi < \kappa$ set

$$f_\alpha(\xi) = \min(C_\xi \cap \xi \setminus \alpha) \text{ if } C_\xi \cap \xi \setminus \alpha \neq \emptyset,$$

$$= 0 \text{ otherwise.}$$

Set $F = \{f_\alpha : \alpha < \kappa\} \in [\text{Regr}(\kappa)]^{\leq \kappa}$. If $F \in U_F$, then

$$\gamma_\alpha = \lim_{\xi \to F} f_\alpha(\xi)$$

is defined and belongs to $C \cup \{0\}$, where $C = \lim_{\xi \to F} C_\xi$.

Because each $C_\xi \cap \xi$ is cofinal with $\xi$, we have $\alpha \leq f_\alpha(\xi)$ whenever $\xi > \alpha$, so that $\alpha \leq \gamma_\alpha$ for every $\alpha < \kappa$, and $C$ must be unbounded in $\kappa$, as required.

(b) For $\alpha, \xi < \kappa$ set

$$f_\alpha(\xi) = \sup(\alpha \cap C_\xi) \text{ if } \alpha < \xi,$$

$$= 0 \text{ otherwise.}$$

Set $F = \{f_\alpha : \alpha < \kappa\} \in [\text{Regr}(\kappa)]^{\leq \kappa}$. Let $F \in U_F$ and consider $C = \lim_{\xi \to F} C_\xi$. Suppose that $\gamma \in \kappa \setminus C$ is a non-zero limit ordinal. Set $\delta = \lim_{\xi \to F} f_\gamma(\xi) < \kappa$. Since $f_\gamma(\xi) \in (C_\xi \cap \gamma) \cup \{0\}$ whenever $\xi < \gamma$, $\delta \in C \cup \{0\}$; but also $\delta \leq \gamma$ and $\gamma \notin C$, so $\delta < \gamma$. Now if $\delta < \beta < \gamma$,

$$\{\xi : \beta \notin C_\xi \supseteq \{\xi : \xi > \gamma, f_\gamma(\xi) = \delta\} \in F,$$

so $\beta \notin C$. Thus $C$ does not meet $[\delta, \gamma]$. Because $\gamma$ was arbitrary, this shows that $C$ is closed.

(c) Take $F, \mathcal{F}$ and $C$ as in (a). This time we see that $\alpha \leq \gamma_\alpha < \zeta$ whenever $\alpha < \zeta$, so that $C \cap \xi$ must be cofinal with $\zeta$. (If $\zeta = 1$ then we have $0 \in C_\xi$ for every $\xi$ so that $0 \in C$.)

4F Theorem Let $\kappa$ be a weakly $\Pi^1_1$-indescribable cardinal, with $\Pi^1_1$-filter $\mathcal{W}$. Then $\mathcal{W}$ is normal and $\text{Mh}(\mathcal{A}) \in \mathcal{W}$ for every stationary $\mathcal{A} \subseteq \kappa$.

**proof** For $F \subseteq \text{Regr}(\kappa)$, write $W_F$ for $\bigcap U_F$, where $U_F$ is defined as in 4Da. Then 4Dc shows that $\mathcal{W} = \bigcup\{W_F : F \in [\text{Regr}(\kappa)]^{\leq \kappa}\}$.

(a) Suppose that $(W_\xi)_{\xi<\kappa}$ is any family in $\mathcal{W}$, and that $W$ is its diagonal intersection. Then $W \in \mathcal{W}$.

**P** For each $\xi < \kappa$, let $F(\xi) \in [\text{Regr}(\kappa)]^{\leq \kappa}$ be such that $W_\xi \in W_{F(\xi)}$. Define $g \in \text{Regr}(\kappa)$ by setting

$$g(\xi) = \min(\eta : \xi \notin W_\eta) \text{ if } \xi \in \kappa \setminus W,$$

$$= 0 \text{ if } \xi \in W.$$

Set $G = \{g\} \cup \bigcup_{\xi<\kappa} F(\xi) \in [\text{Regr}(\kappa)]^{\leq \kappa}$, and consider $W_G$. If $F \in U_F$, set $\gamma = \lim_{\alpha \to F} g(\alpha) < \kappa$. Then $F \in U_{F(\gamma)}$, so $W_\gamma \in \mathcal{F}$ and

$$W \supseteq \{\xi : \xi \in W_\gamma, g(\xi) = \gamma\} \in \mathcal{F}.$$

As $\mathcal{F}$ is arbitrary, $W \in W_G \subseteq \mathcal{W}$.

(b) Because $\kappa \setminus \zeta \in W_\zeta \subseteq W$ for every $\zeta < \kappa$, it follows that $\kappa$ is uncountable and regular and that $\mathcal{W}$ is normal.

(c) Now let $A \subseteq \kappa$ be stationary. Then $\text{Mh}(A) \in \mathcal{W}$. Define $(C_\xi)_{\xi<\kappa}$ as follows. For $\xi \in \kappa \setminus \text{Mh}(A)$, $\text{cf}(\xi) > \omega$ let $C_\xi \subseteq \xi \setminus A$ be a relatively closed cofinal subset of $\xi$; for $\xi \in \text{Mh}(A)$, set $C_\xi = \xi$; for $\xi < \kappa$, $\text{cf}(\xi) = \omega$ let $C_\xi$ be a relatively closed cofinal subset of $\xi$ consisting solely of successor ordinals; for $\xi < \kappa$,...
from 4G we know that \( \kappa \lesssim \) well-ordering \( \lim_{\rightarrow} \) proof

4F and 4Ac together. Let \( \eta \) whenever \( \eta \not\lesssim \) and therefore less than \( h \) for \( \kappa \) it follows that \( \#(\zeta)^+ \) and therefore less than \( \xi \). For \( \eta, \zeta < \kappa \) set

\[
\begin{align*}
 f_{\xi\eta}(\xi) &= g_{\xi\zeta}(\eta) & \text{if } \eta < \zeta < \xi \in C, \\
 &= 0 \text{ otherwise;}
\end{align*}
\]

\[
\begin{align*}
 f(\xi) &= \sup(C \cap \xi) & \text{if } \xi \in \kappa \setminus C, \\
 &= 0 \text{ otherwise.}
\end{align*}
\]

Write \( F = \{f\} \cup \{f_{\xi\eta} : \eta, \zeta < \kappa \} \in [\text{Regr}(\kappa)]^{\leq \kappa}. \)

Now suppose that \( F \in U_F \). Then \( C \in F \) (because \( f^{-1}[\{\gamma\}] \setminus C \) is bounded above in \( \kappa \) for every \( \gamma \)). Set \( \preceq = \lim_{\rightarrow} \lesssim \). To see that \( \preceq \) is a total ordering of \( \kappa \) we need note only that for distinct \( \eta, \eta', \eta'' < \kappa \) the set

\[
\{\xi : \eta \preceq \zeta \leq \eta, \\
\eta \preceq \eta' \text{ or } \eta'' \preceq \eta, \\
\eta \not\preceq \eta' \text{ or } \eta'' \not\preceq \eta, \\
\eta \preceq \eta'' \text{ or } \eta \not\preceq \eta' \text{ or } \eta'' \not\preceq \eta''\}
\]

belongs to \( F \). To see that \( \preceq \) is a well-ordering of \( \kappa \), take any \( \zeta < \kappa \) and consider \( h : \zeta \to \kappa \) defined by setting

\[
h(\eta) = \lim_{\rightarrow} f_{\xi\eta}(\xi)
\]

for \( \eta < \zeta \); then

\[
h(\eta) = \lim_{\rightarrow} g_{\xi\zeta}(\eta)
\]

because \( \{\xi : \zeta < \xi < C\} \in F \). But now, if \( \eta, \eta' < \zeta \) and \( \eta < \eta' \), the set

\[
\{\xi : \zeta < \xi < C, h(\eta) = g_{\xi\zeta}(\eta), h(\eta') = g_{\xi\zeta}(\eta'), \eta \preceq \zeta \}
\]

belong to \( F \), so \( h(\eta) < h(\eta') \). This means that \( \preceq \zeta \) is a well-ordering for every \( \zeta < \kappa \). Because \( \text{cf}(\kappa) > \omega \), it follows that \( \preceq \) is a well-ordering of \( \kappa \).

4I Theorem If \( \kappa \) is weakly \( \Pi_1^1 \)-indescribable, it is greatly Mahlo and the set of greatly Mahlo cardinals below \( \kappa \) belongs to the \( \Pi_1 \)-filter \( W \) of \( \kappa \).

proof I have already remarked (4G) that \( \kappa \) is greatly Mahlo; moreover, the set of weakly inaccessible cardinals less than \( \kappa \) belongs to its greatly Mahlo filter (4B) and therefore to \( W \) (4G).
Suppose, if possible, that the set $B$ of greatly Mahlo cardinals less than $\kappa$ does not belong to $W$. Let $D$ be the set of weakly inaccessible cardinals belonging to $\kappa \setminus B$, so that $D$ is $\Pi^1_3$-fully stationary in $\kappa$.

By Lemma 4C, we may choose for each $\delta \in D$ a well-ordering $\preceq_\delta$ of $\delta$ and a family $(S_{\delta,\zeta})_{\zeta < \delta}$ of subsets of $\delta$ such that
\[ S_{\delta,\zeta} \supseteq \{ \xi : \xi < \alpha, \xi \in \text{Mh}(S_{\delta,\eta}) \text{ whenever } \eta < \xi \} \]
for every $\zeta < \delta$, but not every $S_{\delta,\zeta}$ is stationary in $\delta$. Take a closed unbounded set $C_\delta \subseteq \delta$ and an $h(\delta) < \delta$ such that $S_{\delta, h(\delta)} \cap C_\delta = \emptyset$. Next, for $\zeta < \delta \in D$ and $\xi \in \delta \setminus S_{\delta,\zeta}$ choose $g(\delta, \zeta, \xi) < \xi$ such that $g(\delta, \zeta, \xi) \preceq_\delta \zeta$ and $\xi \notin \text{Mh}(S_{\delta, g(\delta, \zeta, \xi)})$; if $\text{cf}(\xi) > \omega$, choose a relatively closed unbounded set $C_{\delta, \zeta} \subseteq \xi$ such that $C_{\delta, \zeta} \cap S_{\delta, g(\delta, \zeta, \xi)} = \emptyset$.

Fill in these definitions by setting
\[ C_\alpha = \alpha, \preceq_\alpha = \lim_\alpha \preceq \xi, h(\alpha) = 0, g(\alpha, \zeta, \xi) = 0, C_{\alpha, \zeta} = \xi, S_{\alpha, \zeta} = \emptyset \]
if $\alpha, \zeta, \xi < \kappa$ and we do not have $\zeta < \alpha \in D$.

By 4E and 4H there is an $F \in \{ \text{Regr}(\kappa) \}_{\preceq \kappa}$ such that whenever $\mathcal{F} \in \mathcal{U}_F$
\[ \lim_\alpha \neg C_\alpha = \text{a closed unbounded set in } \kappa, \]
\[ \lim_\alpha \neg \preceq_\alpha = \text{a well-ordering of } \kappa, \]
\[ \lim_\alpha \neg C_{\alpha, \zeta} = \text{cofinal with } \xi, \]
\[ \lim_\alpha \neg (C_{\alpha, \zeta} \cup \{ \xi \}) = \text{is closed in } \kappa \]
for all $\zeta, \xi < \kappa$. We may suppose also that $h \in F$ and that $f_{\zeta, \xi} \in F$ for all $\zeta, \xi < \kappa$, where $f_{\zeta, \xi}(\alpha) = g(\alpha, \zeta, \xi)$ for $\alpha, \zeta, \xi < \kappa$.

Because $D$ is supposed to be $\Pi^1_3$-fully stationary, there is an $\mathcal{F} \in \mathcal{U}_F$ such that $D \in \mathcal{F}$. Set
\[ C = \lim_\alpha \neg C_\alpha, \preceq = \lim_\alpha \neg \preceq_\alpha, \epsilon = \lim_\alpha \neg h(\alpha), \]
\[ C_{\zeta, \xi} = \lim_\alpha \neg C_{\alpha, \zeta}, \gamma(\zeta, \xi) = \lim_\alpha \neg f_{\zeta, \xi}(\alpha), \]
\[ S_\zeta = \lim_\alpha \neg S_{\alpha, \zeta} \]
for $\zeta, \xi < \kappa$. Then $\preceq$ is a well-ordering of $\kappa$ and $(S_{\zeta})_{\zeta < \kappa}$ is a family of subsets of $\kappa$.

Suppose that $\zeta < \kappa$, that $\xi \in \kappa \setminus S_\zeta$ and that $\text{cf}(\xi) > \omega$. Set $\eta = \gamma(\zeta, \xi)$. Then
\[ E = \{ \delta : \delta \in D, \zeta < \delta, \xi \in \delta \setminus S_{\delta, \zeta}, g(\delta, \zeta, \xi) = \eta \} \]
belongs to $\mathcal{F}$. If $\delta \in E$ then $C_{\delta, \zeta} \cap S_{\delta, \eta} = \emptyset$; it follows that $C_{\delta, \zeta} \cap S_{\delta, \eta} = \emptyset$. So $\xi \notin \text{Mh}(S_{\delta, \eta})$. Also $\eta < \xi$ and $\eta < \zeta$ because $g(\delta, \zeta, \xi) < \xi$ and $g(\delta, \zeta, \xi) \preceq_\delta \zeta$ for every $\delta \in E$. But this means that
\[ S_\zeta \supseteq \{ \xi : \xi < \kappa, \xi \in \text{Mh}(S_{\delta, \eta}) \text{ whenever } \eta < \xi, \eta \preceq \xi \}, \]
for every $\zeta < \kappa$.

On the other hand, examine $C$ and $S_\zeta$. $C$ is a closed unbounded set in $\kappa$. Also
\[ \{ \alpha : h(\alpha) = \epsilon \} \in \mathcal{F}, \]
so
\[ S_\zeta = \lim_\alpha \neg S_{\alpha, \zeta} \]
and
\[ C \cap S_\zeta = \lim_\alpha \neg C_{\alpha} \cap S_{\alpha, h(\alpha)} = \emptyset. \]

Thus $S_\zeta$ is not stationary. From 4C we conclude that $\kappa$ cannot be greatly Mahlo. But we know very well that it is. $\mathbf{X}$

So $B \notin \mathcal{W}$, as claimed.

**Remark** This theorem is given in Baumgartner, Taylor & Wagon 77; the treatment here is taken from Fremlin & Kunen n87.

**4J Lemma** Let $\kappa$ be a real-valued-measurable cardinal, with normal witnessing probability $\nu$. Suppose that for each $\xi < \kappa$ we are given a cofinal relatively closed subset $C_\xi$ of $\xi$. Then
is a closed unbounded set in $\kappa$.

**proof** For $\alpha, \xi < \kappa$ set

$$f_\alpha(\xi) = \min(C_{\xi} \setminus \alpha) \text{ if } \alpha < \xi,$$

$$= 0 \text{ otherwise}.$$ 

Then $f_\alpha$ is regressive, so there is a $\zeta_\alpha < \kappa$ such that $\nu f_\alpha^{-1}[\zeta_\alpha] = 1$; that is,

$$\nu\{\xi : C_{\xi} \cap \zeta_\alpha \setminus \alpha \neq \emptyset\} = 1.$$ 

Set $C' = \{\beta : \beta < \kappa, \zeta_\alpha < \beta \forall \alpha < \beta\}$. Then $C'$ is a closed unbounded set in $\kappa$. But if $\beta \in C'$, then for every $\alpha < \beta$ the set

$$\{\xi : C_{\xi} \cap \beta \setminus \alpha \neq \emptyset\}$$

has measure 1, so

$$E = \{\xi : \beta < \xi < \kappa, C_{\xi} \cap \beta \setminus \alpha \neq \emptyset \forall \alpha < \beta\}$$

has measure 1, because $\nu$ is $\kappa$-additive. But of course $\beta \in C_{\xi}$ for every $\xi \in E$, because each $C_{\xi}$ is closed in $\xi$. Accordingly $\beta \in C$ and we have $C' \subseteq C$.

This means that $C$ is unbounded. But if $\beta$ is any cluster point of $C$ in $\kappa$, the same arguments show that $\beta \in C$, so that $C$ is closed.

**4K Theorem**: Let $\kappa$ be a real-valued-measurable cardinal, with rvm filter $\mathcal{W}$, rvm ideal $\mathcal{J}$. Then

(a) $\kappa$ is greatly Mahlo and its greatly Mahlo filter is included in $\mathcal{W}$;

(b) the set of weakly $\Pi^1_1$-indescribable cardinals below $\kappa$ belongs to $\mathcal{W}$;

(c) if $Z \subseteq \kappa$ and $Z \notin \mathcal{J}$, then

$$\{\lambda : \lambda < \kappa \text{ is a cardinal and } Z \cap \lambda \text{ is } \Pi^1_1\text{-fully stationary in } \lambda\} \in \mathcal{W}.$$

**proof** (a) Let $\nu$ be a normal witnessing probability on $\kappa$, and set

$$\mathcal{F}_\nu = \{A : A \subseteq \kappa, \nu A = 1\};$$

$\mathcal{F}_\nu$ is a normal filter on $\kappa$. Let $D$ be the set of ordinals less than $\kappa$ with uncountable cofinality.

(i) We have $D \in \mathcal{F}_\nu$. ¶ For $\xi \in \kappa \setminus (D \cup \{0\})$, let $(f_n(\xi))_{n \in \mathbb{N}}$ run over a cofinal subset of $\xi$; for $\xi \in D \cup \{0\}$, set $f_n(\xi) = 0$ for every $n \in \mathbb{N}$. Because every $f_n$ is regressive and $\mathcal{F}_\nu$ is normal and $\omega_1$-complete, there is a $\zeta < \kappa$ such that $F = \bigcap_{n \in \mathbb{N}} f_n^{-1}[\zeta] \in \mathcal{F}_\nu$; but now $D \supseteq F \setminus (\zeta + 1) \in \mathcal{F}_\nu$. ¶

(ii) Next, $\text{Mh}(A) \in \mathcal{F}_\nu$ for every stationary $A \subseteq \kappa$. ¶ For $\xi \in D \setminus \text{Mh}(A)$, let $C_{\xi}$ be a relatively closed and unbounded subset of $\xi$ such that $C_{\xi} \cap A = \emptyset$. For $\xi \in D \setminus \text{Mh}(A)$, take $C_{\xi} = \xi$. ¶ 4J tells us that

$$C = \{\alpha : \xi \in C_{\xi} \in \mathcal{F}_\nu\}$$

is a closed unbounded set in $\kappa$. So there is an $\alpha \in A \cap C$. But now $F = \{\xi : \alpha \in C_{\xi}\} \in \mathcal{F}_\nu$ and $\text{Mh}(A) \supseteq F \cap D \in \mathcal{F}_\nu$. ¶

(iii) Thus $\mathcal{F}_\nu$ is closed under $\text{Mh}$. Because $\nu$ is arbitrary, $\mathcal{W}$ is closed under $\text{Mh}$, and $\kappa$ is greatly Mahlo, with its greatly Mahlo filter included in $\mathcal{W}$.

(b)-(c) Of course (b) is a consequence of (c) (taking $Z = \kappa$) so I prove (c). I begin with the harder case, in which $\kappa$ is atomlessly-measurable.

(i) Take a normal witnessing probability $\nu_1$ on $\kappa$ such that $\nu_1 Z = 1$, and let $H$ be the set of $\xi < \kappa$ such that either $\xi$ is not a weakly $\Pi^1_1$-indescribable cardinal or $\xi$ is weakly $\Pi^1_1$-indescribable but $Z \cap \xi$ is not $\Pi^1_1$-fully stationary in $\xi$. ¶ Suppose, if possible, that $H \notin \mathcal{J}$. Let $\nu_2$ be a normal witnessing probability on $\kappa$ such that $\nu_2 H = 1$. Let $H_0$ be the set of regular infinite cardinals belonging to $H$; then $\nu_2 H_0 = 1$ (using part (a)). For each $\xi \in H_0$, let $(f_\eta)_{\eta \leq \xi}$ be a family of regressive functions from $\xi$ to itself such that there is no uniform ultrafilter $\mathcal{F}$ on $\xi$ containing $Z \cap \xi$ for which $\lim_{\alpha \to \mathcal{F}} f_\eta(\alpha)$ exists for every $\eta < \xi$. For $\eta < \xi \in \kappa \setminus H_0$, set $f_\eta(\alpha) = 0$ for every $\alpha < \xi$. 

$$C = \{\alpha : \alpha < \kappa, \nu_1 \{\xi : \alpha \in C_{\xi}\} = 1\}$$
(ii) Let $\nu$ be the probability on $\kappa \times \kappa$ defined by the formula
\[ \nu C = \int \nu_2(\xi : (\alpha, \xi) \in C) \, d\alpha \quad \forall \ C \subseteq \kappa \times \kappa. \]
Let $\mathfrak{A}(\nu) = \mathcal{P}(\kappa \times \kappa)/N_\nu$ be the corresponding measure algebra. For $\eta, \beta < \kappa$ write
\[ C_{\eta, \beta} = \{ (\alpha, \xi) : \max(\eta, \alpha) < \xi < \kappa, f_\xi(\alpha) = \beta \}. \]
If $\eta < \kappa$, then $\{ C_{\eta, \beta} \}_{\beta < \kappa}$ is disjoint, so
\[ D_\eta = \{ \beta : \beta < \kappa, \nu C_{\eta, \beta} > 0 \} \]
is countable. But also
\[ \nu(\bigcup_{\beta \in D_\eta} C_{\eta, \beta}) = 1. \]
\( \mathbf{P} \) For fixed $\eta, \alpha$ the map $\xi \mapsto f_\xi(\alpha) : \kappa \to \alpha$ is regressive, therefore $\nu_2$-essentially countably valued (1Hd), therefore $\nu_2$-essentially bounded below $\alpha$ whenever $c(\alpha) > \omega$, that is, for $\nu_1$-almost all $\alpha$, by (a-i) above. Because $\nu_1$ is normal, there is an $\zeta < \kappa$ such that, for $\nu_1$-almost all $\alpha$, $f_\xi(\alpha) < \zeta$ for $\nu_2$-almost all $\xi$; that is, $\nu(\bigcup_{\zeta < \kappa} C_{\eta, \beta}) = 0$. Now the result follows, with $D_\eta \subseteq \zeta$, because $\nu$ is $\kappa$-additive. \( \mathbf{Q} \)

(iii) Let $\Sigma_0$ be the subalgebra of $\mathcal{P}(\kappa \times \kappa)$ generated by
\[ \{ C_{\eta, \beta} : \eta, \beta < \kappa \} \cup \{ V_\zeta : \zeta < \kappa \}, \]
where $V_\zeta = Z \times (\kappa \setminus \zeta)$; note that $\nu V_\zeta = 1$ for each $\zeta < \kappa$. Let $\Sigma_1$ be the subalgebra of $\mathcal{P}(\kappa \times \kappa)$ generated by
\[ \Sigma_0 \cup \{ \pi_2^{-1}(\pi_2[E]) : E \in \Sigma_0 \}, \]
writing $\pi_2(\alpha, \xi) = \xi$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}(\nu)$ generated by $\{ E^* : E \in \Sigma_1 \}$. Then $\tau(\mathfrak{B}) \leq \kappa$. Let $\mathfrak{B}_0$ be
\[ \{ b : b \in \mathfrak{B}, \exists F \subseteq \kappa \text{ such that } b = (\kappa \times F)^* \}. \]
Then $\mathfrak{B}_0$ is an order-closed subalgebra of $\mathfrak{B}$. Because $\pi_2$ is inverse-measure-preserving for $\nu$ and $\nu_2$, we have a measure-preserving Boolean homomorphism $\phi_0 : \mathfrak{B}_0 \to \mathfrak{A}(\nu_2) = \mathcal{P}(\kappa)/N_{\nu_2}$ defined by writing
\[ \phi_0((\kappa \times F)^*) = F^* \]
whenever $F \subseteq \kappa$ and $(\kappa \times F)^* \in \mathfrak{B}$.

By the Gitik-Shelah theorem (3F), every non-zero principal ideal of $\mathfrak{A}(\nu_2)$ has $\tau$-weight greater than $\kappa$. So there is a measure-preserving homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}(\nu_2)$ extending $\phi_0$ (A2Ib). For each $\eta < \kappa$ choose a function $h_\eta : \kappa \to D_\eta$ such that
\[ \{ \xi : f_\xi(\eta) = \beta \}^* = \phi(C_{\eta, \beta}) \quad \forall \beta \in D_\eta; \]
this is possible because $(C_{\eta, \beta})_{\beta \in D_\eta}$ and $(\phi(C_{\eta, \beta}))_{\beta \in D_\eta}$ are partitions of unity in $\mathfrak{B}$, $\mathfrak{A}(\nu_2)$ respectively. Now if $\xi \in H_0$, there can be no uniform ultrafilter on $\xi$ containing $Z \cap \xi$ and all the sets $\{ \alpha : \alpha < \xi, f_\xi(\alpha) = h_\eta(\xi) \}$ as $\eta$ runs over $\xi$. So there must be a $\zeta_\xi < \xi$ and a finite $I_\xi \subseteq \zeta_\xi$ such that
\[ \forall \alpha \in \xi \cap Z \setminus \zeta_\xi \exists \eta \in I_\xi \text{ such that } f_\xi(\alpha) \neq h_\eta(\xi). \]

(iv) Because $\nu_2$ is normal, there are $\zeta < \kappa, I \in [\kappa]^{<\omega}$ such that $\nu_2(H_1) > 0$, where
\[ H_1 = \{ \xi : \xi \in H_0, \zeta_\xi = \zeta, I_\xi = I \}. \]
Now $h_\eta(\xi) \in D_\eta$ for every $\xi < \kappa, \eta \in I$, and every $D_\eta$ is countable, so there is a $g : I \to \kappa$ such that $\nu_2(H_2) > 0$, where
\[ H_2 = \{ \xi : \xi \in H_1, h_\eta(\xi) = g(\eta) \quad \forall \eta \in I \}. \]
But now observe that if $\xi \in H_2$ and $\alpha \in \xi \cap Z \setminus \zeta$ then there is an $\eta \in I$ such that $f_\xi(\alpha) \neq g(\eta)$, i.e., $(\alpha, \xi) \notin E^*$, where
\[ E^* = \{ (\alpha, \xi) : \xi \leq \alpha < \xi, \alpha \in Z, f_\xi(\alpha) = g(\eta) \quad \forall \eta \in I \} \in \Sigma_0. \]
Thus $H_2 \cap \pi_2[E^*] = \emptyset$. However,
\[ H^2_2 = H^1_1 \cap \inf_{\eta \in I} \phi(C_{\eta,g(0)}) \]
\[ \subseteq \phi(\bigcap_{\eta \in I} C_{\eta,g(0)})^* \]
\[ = \phi(\{((\eta, \alpha) : f_{\xi_\eta}(\alpha) = g(\eta) \ \forall \ \eta \in I\})^*) \]
\[ = \phi(E^*)^* \]
\[ \subseteq \phi(\pi_2^{-1}[\pi_2[E^*]]^*) \]
\[ = \phi_0(\pi_2^{-1}[\pi_2[E^*]]^*) \]
\[ = \pi_2[E^*]^* \]

because \( E^* \in \Sigma_0 \) so \( \pi_2^{-1}[\pi_2[E^*]] \in \Sigma_1 \) and \( \pi_2^{-1}[\pi_2[E^*]]^* \in \mathcal{B}_0 \).

Of course this is impossible. \( \square \)

This proves (b)-(c) in the case of atomlessly-measurable \( \kappa \).

(v) Now let us turn to the case of a two-valued-measurable cardinal \( \kappa \). Let \( \mathcal{F}_1, \mathcal{F}_2 \) be normal ultrafilters on \( \kappa \), with \( Z \in \mathcal{F}_1 \), and \( \mu_1, \mu_2 \) the associated \( \{0, 1\} \)-valued probabilities. Follow the ideas of (b), but noting that there are dramatic simplifications; for instance, \( \nu \) is also two-valued, and each \( D_\eta \) is a singleton. All the discussion of \( \mathcal{B}, \mathcal{B}_0 \) collapses because \( \mathcal{A}(\nu) \) and \( \mathcal{A}(\nu_2) \) are both of the form \( \{0, 1\} \). So \( \phi_0 \) and \( \phi \) are both the trivial homomorphism, and there is no call for any measure theory at this point; the functions \( h_\eta \) are all constant. The last string of inequalities also boils down to saying that \( \nu_2 H_2 = \nu E^* = 1 \) and that this is impossible.

This completes the proof.

4L Remarks
(a) This (in the atomlessly-measurable case) is due to Kunen (see 4P, A4L below).

(b) Note that a real-valued-measurable cardinal \( \kappa \) need not itself be weakly \( \Pi^1_1 \)-indescribable. For (if there can be real-valued-measurable cardinals at all) \( \epsilon \) can be real-valued-measurable (2Ed). But \( \epsilon \) is never weakly \( \Pi^1_1 \)-indescribable (A4Db).

(c) It is worth pausing over 4Ka, as some elementary corollaries will be used repeatedly below. For instance, if \( \nu \) is a normal witnessing probability on \( \kappa \), and \( \theta < \kappa \), then \( \nu[\xi : \xi < \kappa, \text{cf}(\xi) \geq \theta] = 1 \). (This is because the set \( W \) of weakly inaccessible cardinals below \( \kappa \) belongs to the greatly Mahlo filter of \( \kappa \) (4B), so \( \nu W = 1 \) (4Ka) and \( \nu(W \setminus \theta) = 1 \), while \( \text{cf}(\xi) \geq \theta \) for every \( \xi \in W \setminus \theta \).) Similarly trivial applications occur frequently.

4M Lemma Let \( \kappa \) be a real-valued-measurable cardinal with normal witnessing probability \( \nu \). Let \( \mathcal{A} \) be the measure algebra \( \mathcal{P}\kappa/\mathcal{N}_\nu \) and \( \mathcal{P} \) the p.o.set \( \mathcal{A} \setminus \{0\} \); for \( a \in \mathcal{P}\kappa/\mathcal{N}_\nu \), write \( a^* \) for the corresponding element of \( \mathcal{P} \).

(a) Suppose that \( \langle \xi_\alpha \rangle_{\alpha < \kappa} \) is a family of ordinals such that \( \xi_\alpha < \alpha \) for \( \nu \)-almost every \( \alpha < \kappa \). Then we have a \( \mathcal{P} \)-name \( \dot{\xi} \) for a member of \( \kappa \) defined by saying that
\[ a^* \forces_{\mathcal{P}} \dot{\xi} = \check{\xi} \iff \xi_\alpha = \xi \text{ for } \nu\text{-almost every } \alpha < a \]
whenever \( \xi < \kappa, a \in \mathcal{P}\kappa/\mathcal{N}_\nu \).

(b) Suppose that \( n \in \mathbb{N} \) and that for \( \nu \)-almost every \( \alpha < \kappa \) we are given an \( n \)-place relation \( C_\alpha \) on \( \alpha \). Then we have a \( \mathcal{P} \)-name \( \dot{C} \) for an \( n \)-place relation on \( \kappa \) defined by saying that
\[ a^* \forces_{\mathcal{P}} \dot{C}(\check{\xi}_1, \ldots, \check{\xi}_n) \iff C_\alpha(\xi_1, \ldots, \xi_n) \text{ for } \nu\text{-almost every } \alpha < a \]
whenever \( \xi_1, \ldots, \xi_n < \kappa \) and \( a \in \mathcal{P}\kappa/\mathcal{N}_\nu \).

(c) Conversely, given any \( \mathcal{P} \)-name \( \dot{\xi} \) and \( \dot{C} \) for a member of \( \kappa \) and a relation on \( \kappa \), we can find corresponding families \( \langle \xi_\alpha \rangle_{\alpha < \kappa}, \langle (C_\alpha)_{\alpha < \kappa} \rangle \) for which the formulae of (a) and (b) will be valid.

(d) Now suppose that we have a formula \( \phi \) of the second-order language \( \mathcal{L} \) of \( \S A4 \), and families \( \langle \xi_\alpha \rangle_{\alpha < \kappa}, \ldots, \langle C_\alpha \rangle_{\alpha < \kappa}, \langle \xi_{(a)} \rangle_{\alpha < \kappa}, \ldots, \langle \xi_{(m)} \rangle_{\alpha < \kappa} \) of relations and ordinals matching the free variables of \( \phi \); suppose that for each \( j \) we have \( \xi_{(j)} < \alpha \) for \( \nu \)-almost all \( \alpha \). Let \( \dot{C}_1, \ldots, \dot{C}_k, \dot{\xi}_1, \ldots, \dot{\xi}_m \) be the corresponding \( \mathcal{P} \)-names. Then for any \( a \in \mathcal{P}\kappa/\mathcal{N}_\nu, \beta \leq \kappa \),
\[ a^* \forces_{\mathcal{P}} (\check{\beta}; \dot{C}_1, \ldots, \dot{C}_k; \check{\xi}_1, \ldots, \check{\xi}_m) \models \phi \]
if and only if

\[(\min(\alpha, \beta); C_{\alpha 1}, \ldots, C_{\alpha k}; \xi_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \phi \] for \(\nu\)-almost every \(\alpha < \kappa\).

**proof (a)** The point is that \(\alpha \mapsto \xi_\alpha\) is essentially regressive and \(\nu\) is normal. So given \(a \in \mathcal{P}_\kappa \setminus \mathcal{N}_\nu\) we shall be able to find \(\zeta < \kappa\) such that \(b = \{\alpha : \alpha \in a, \xi_\alpha = \zeta\} \notin \mathcal{N}_\nu\); and now our formula tells us that \(b^* \models_p \dot{\xi} = \dot{\zeta}\). This means that we have a name \(\dot{\xi}\) such that

\[\|_p \exists \zeta \in \kappa, \dot{\zeta} = \zeta,\]

so that \(\dot{\zeta}\) is indeed a name for a member of \(\kappa\).

**(b)** is elementary.

**(c)** Because \(\mathbb{P}\) is ccc and \(\mathcal{N}_\nu\) is a \(\sigma\)-ideal, every maximal antichain in \(\mathbb{P}\) corresponds to a partition of \(\kappa\) into non-negligible sets; consequently every \(\mathbb{P}\)-name \(\xi\) for a member of \(\kappa\) corresponds to a family \(\langle \xi_\alpha \rangle_{\alpha < \kappa}\) which takes only countably many values, so that \(\xi_\alpha < \alpha\) for almost every \(\alpha\).

As in (a)-(b), the corresponding result for relations is really simpler than the result for points. Given the name \(\check{C}\) for an \(n\)-place relation, then for each \(\zeta_1, \ldots, \zeta_n < \kappa\) choose \(a(\zeta_1, \ldots, \zeta_n) \subseteq \kappa\) such that

\[a(\zeta_1, \ldots, \zeta_n)^* \models_p \check{C}(\zeta_1, \ldots, \zeta_n),\]

and for \(\alpha < \kappa\) write

\[C_\alpha(\zeta_1, \ldots, \zeta_n)\] for \(\langle \alpha \in a(\zeta_1, \ldots, \zeta_n)\rangle^*\).

It is easy to check that \(\langle C_\alpha \rangle_{\alpha < \kappa}\) now represents \(\check{C}\).

**(d)** Induce on the length of \(\phi\).

**(i)** If \(\phi\) is of the form \(R_1(x_1, \ldots, x_n)\) and we are given \(C_\alpha, \xi_{\alpha j}\) for \(\alpha < \kappa, j \leq n\) then note first that

\[a^* \models_p \dot{\xi}_j < \dot{\beta}\]

if and only if

\[\xi_{\alpha j} < \min(\alpha, \beta)\] for almost every \(\alpha \in a\).

Next,

\[a^* \models_p \check{C}(\dot{\xi}_1, \ldots, \dot{\xi}_n)\]

\[\iff \text{whenever } b \subseteq a, \zeta_1, \ldots, \zeta_n < \kappa, b^* \models_p \check{\zeta}_i = \dot{\zeta}_i \forall i \leq n\]

\[\text{then } b^* \models_p \check{C}(\dot{\xi}_1, \ldots, \dot{\xi}_n)\]

\[\iff \text{whenever } b \subseteq a, \zeta_1, \ldots, \zeta_n < \kappa, \xi_{\alpha i} = \zeta_i \forall \alpha \in b, i \leq n\]

\[\text{then } C_\alpha(\zeta_1, \ldots, \zeta_n) \text{ for almost every } \alpha \in b\]

\[\iff C_\alpha(\xi_1, \ldots, \xi_n) \text{ for almost every } \alpha \in a.\]

**(ii)** If \(\phi\) is of the form \(\psi \wedge \chi\) or \(\psi \vee \chi\) the inductive step to \(\phi\) is elementary. If \(\phi\) is of the form \(\neg \psi\) we need the translation

\[a^* \models_p (\dot{\beta}; \check{C}_1, \ldots, \check{C}_m) \models \phi\]

\[\iff b^* \models_p (\dot{\beta}; \check{C}_1, \ldots, \check{C}_m) \models \psi \text{ for every } b \in \mathcal{P}a \setminus \mathcal{N}_\nu\]

\[\iff (\nu(a : \alpha \in a, (\min(\alpha, \beta); C_{\alpha 1}, \ldots, C_{\alpha m}) \models \psi)) = 0\]

\[\iff (\min(\alpha, \beta); C_{\alpha 1}, \ldots, C_{\alpha m}) \models \phi \text{ for almost all } \alpha \in a.\]

**(iii)** If \(\phi\) is of the form \(\exists S\psi\) and

\[a^* \models_p (\dot{\beta}; \check{C}_1, \ldots, \check{C}_m) \models \phi,\]
then there must be a \( P \)-name \( C \) such that
\[
a^* \Vdash \langle \beta; C_1, \ldots, C_m \rangle \models \psi.
\]
Let \( \langle C_{\alpha} \rangle_{\alpha < \kappa} \) be a corresponding family of relations. Then by the inductive hypothesis
\[
(\min(\alpha, \beta); C_{\alpha 1}, \ldots, C_{\alpha m}) \models \psi
\]
for almost every \( \alpha \in a \), so that
\[
(\min(\alpha, \beta); C_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \phi
\]
for almost every \( \alpha \in a \).

(iv) If \( \phi \) is of the form \( \exists \nu \psi \) and \( a \in P_{\kappa} \setminus N_\nu \) is such that
\[
(\min(\alpha, \beta); C_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \phi
\]
for almost every \( \alpha \in a \), then we can find relations \( C_{\alpha} \) such that
\[
(\min(\alpha, \beta); C_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \psi
\]
for almost every \( \alpha \in a \). By the inductive hypothesis, taking \( \check{C} \) from \( \langle C_{\alpha} \rangle_{\alpha < \kappa} \) as usual, we have
\[
a^* \Vdash \langle \beta; C_1, \ldots, \check{C}_m \rangle \models \psi, \quad a^* \Vdash \langle \beta; \check{C}_1, \ldots, \check{C}_m \rangle \models \phi.
\]

(v) (iii)-(iv) together deal with the inductive step to \( \exists \nu \psi \). If \( \phi \) is of the form \( \exists \nu \psi \) it is logically equivalent to \( \neg \exists \nu \neg \psi \) so that (ii)-(iv) cover it. Finally, if \( \phi \) is of the form \( \exists \nu \psi \) or \( \forall \nu \psi \) the same ideas suffice.

4N Proposition Let \( \kappa \) be an atomlessly-measurable cardinal, with rvm filter \( \mathcal{W} \) and rvm ideal \( \mathcal{J} \), and \( \lambda > \kappa \) another cardinal. Let \( \mathfrak{A}_{\lambda} \) be the measure algebra of \( \{0,1\}^\lambda \) and let \( P_\lambda \) be the p.o.set \( \mathfrak{A}_{\lambda} \setminus \{0\} \). Let \( \phi \) be a formula of the second-order language \( L \) of \( \mathfrak{A} \) and \( C_1, \ldots, C_k \) relations on \( \kappa, \xi_1, \ldots, \xi_m \) ordinals less than \( \kappa \). Let \( \beta < \kappa \). Then the following are equivalent:

(i) \( \Vdash_{P_\lambda} \langle \beta; C_1, \ldots, \xi_m \rangle \models \phi \).

(ii) \( \{ \alpha : \alpha < \kappa, (\min(\alpha, \beta); C_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \phi \} \in \mathcal{W} \);

(iii) \( \{ \alpha : \alpha < \kappa, (\min(\alpha, \beta); C_{\alpha 1}, \ldots, \xi_{\alpha m}) \models \phi \} \notin \mathcal{J} \).

Proof Set \( A = \{ \alpha : (\min(\alpha, \beta); C_{\alpha 1}, \ldots) \models \phi \} \).

(i) \( \Rightarrow \) (ii) Assume (i). Let \( \nu \) be a Maharam homogeneous normal witnessing probability on \( \kappa \); let \( \theta \) be the Maharam type of \( \nu \). We know by the Gitik-Shelah theorem (3F) that \( \theta > \kappa \). Set \( \mathfrak{A} = P_{\kappa}/N_{\nu}, \mathcal{P} = \mathfrak{A} \setminus \{0\} \); then \( \mathfrak{A} \) is isomorphic to the measure algebra of \( \{0,1\}^\theta \), so Theorem A4I tells us that
\[
\Vdash_{P} \langle \beta; C_1, \ldots \rangle \models \phi.
\]
Now observe that each P-name \( \check{C}_{\alpha}, \check{\xi}_{\alpha} \) can be identified with the P-names \( \check{C}, \check{\xi} \) derived from families \( \langle C_{\alpha \beta} \rangle_{\alpha < \kappa}, \langle \xi_{\alpha \beta} \rangle_{\alpha < \kappa} \), setting \( C_{\alpha \beta} = C_{\alpha}, \xi_{\alpha \beta} = \xi_{\alpha} \) for every \( \alpha \). So Lemma 4Md tells us that
\[
(\min(\alpha, \beta); C_{\alpha 1}, \ldots) \models \phi
\]
for \( \nu \)-almost every \( \alpha < \kappa \), that is, that \( \nu A = 1 \). As \( \nu \) is arbitrary, \( A \in \mathcal{W} \).

(ii) \( \Rightarrow \) (iii) is trivial.

(iii) \( \Rightarrow \) (i) I reverse the argument of (i) \( \Rightarrow \) (ii). Let \( \nu \) be a Maharam homogeneous normal witnessing probability on \( \kappa \) such that \( \nu A = 1 \); let \( \theta > \kappa \) be the Maharam type of \( \nu \), and \( \mathcal{P} = (P_{\kappa}/N_{\nu}) \setminus \{0\} \) the corresponding p.o.set. This time 4Md tells us that
\[
\Vdash_{\mathcal{P}} \langle \beta; C_1, \ldots \rangle \models \phi
\]
and A4I that
\[
\Vdash_{\mathcal{P}} \langle \beta; C_1, \ldots \rangle \models \phi,
\]
as required.

4O Corollary (a) Let \( \kappa \) be an atomlessly-measurable cardinal and \( I \) a set of cardinal less than \( \kappa \). Let \( \phi \) be a formula of the second-order language \( L \), and \( C_1, \ldots, C_k \) relations on \( I, \xi_1, \ldots, \xi_m \) members of \( I \). Then for any random real p.o.set \( \mathcal{P} \),
(\(I; C_1, \ldots, \xi_m\)) \models \phi \iff \|P_1^P (\tilde{I}; \tilde{C}_1, \ldots, \xi_m) \models \phi.

(b) Let \(C_1, \ldots, C_n\) be relations and \(\xi_1, \ldots, \xi_m\) ordinals, all with definitions which are absolute for random real forcing. Let \(\phi\) be a second-order formula such that

\[
\text{ZFC} \vdash \text{for every atomlessly-measurable cardinal } \kappa > \max i \leq m \xi_i, \quad (\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi'.
\]

Then for every atomlessly-measurable cardinal \(\kappa > \max i \leq m \xi_i, \quad \{\alpha : \alpha < \kappa, (\alpha; C_1, \ldots, \xi_m) \models \phi\}\)

belongs to the rvm filter of \(\kappa\).

**Proof (a)** Of course we may take it that \(I = \beta\) is an ordinal less than \(\kappa\). Also, as remarked in 2B, it is enough to consider the case in which \(P = P_\lambda = A_\lambda \setminus \{0\}\) for some cardinal \(\lambda\), taking \(A_\lambda\) to be the measure algebra of \(\{0, 1\}\) as usual.

Set \(\theta = \max (\lambda, \kappa^+\) and let \(P_\theta = A_\theta \setminus \{0\}\) be the corresponding p.o.set. Then

\[
(\beta; C_1, \ldots) \models \phi \iff \|P_\theta (\tilde{\beta}; \tilde{C}_1, \ldots) \models \phi
\]

by 4N. But now note that

\[
\|P_1^P \tilde{\kappa} \text{ is real-valued-measurable}
\]

(2C), so we can repeat the argument in \(V^P\) to see that

\[
\|P_1^P (\tilde{\beta}; \tilde{C}_1, \ldots) \models \phi \iff \|P_\theta (\tilde{\beta}; \tilde{C}_1, \ldots) \models \phi.
\]

The iteration \(P \ast P_\theta = P_\lambda \ast P_\theta\) is isomorphic to \(P_\theta\) (KUNEN 84, 3.13), so we have

\[
\|P_1^P (\tilde{\beta}; \tilde{C}_1, \ldots) \models \phi \iff \|P_\theta (\tilde{\beta}; \tilde{C}_1, \ldots) \models \phi \iff (\beta; C_1, \ldots) \models \phi.
\]

(b) Let \(\nu\) be a normal witnessing probability on \(\kappa\) and \(P\) the p.o.set \((P\kappa/N_\nu) \setminus \{0\}\). Then

\[
\|P_1^P \kappa \text{ is real-valued-measurable}
\]

by 2C, and also of course

\[
\|P_1^P \kappa \geq \kappa > \max i \leq m \xi_i,
\]

so, repeating the ZFC proof of \((\kappa; \ldots) \models \phi\) in \(V^P\),

\[
\|P_1^P (\kappa; \tilde{C}_1, \ldots, \xi_m) \models \phi.
\]

But now, by 4Md,

\[
\nu\{\alpha : \alpha < \kappa, (\alpha; C_1, \ldots, \xi_m) \models \phi\} = 1.
\]

As \(\nu\) is arbitrary, we have the result.

**Remark** Part (a) means that almost any fact about random real forcing is likely to have implications in the presence of an atomlessly-measurable cardinal.

Part (b) is a kind of reflection principle, corresponding to the theorem of Hanf and Scott (A4I) for two-valued-measurable cardinals. The requirements ‘ZFC \(\vdash \ldots\) ’ and ‘with definitions absolute for random real forcing' are more restrictive than is absolutely necessary, but we do have to take care, when applying this method, that whatever argument we have used to prove that \((\kappa; \ldots) \models \phi\) will survive the move to \(V^P\).

**4P Theorem** Let \(\kappa\) be a real-valued-measurable cardinal with rvm filter \(W\) and rvm ideal \(\mathcal{J}\). Then for every \(Z \in P_\kappa \setminus \mathcal{J}\),

\[
\{\alpha : \alpha < \kappa, Z \cap \alpha \text{ belongs to the } \Pi^0_\beta\text{-ideal of } \alpha\}
\]

belongs to \(\mathcal{J}\). In particular, setting \(Z = \kappa\), the set of weakly \(\Pi^0_\beta\)-indescribable cardinals less than \(\kappa\) belongs to \(W\).
Proof For two-valued-measurable cardinals this (and much more) is essentially covered by A4I; so henceforth I shall assume that \( \kappa \) is atomlessly-measurable. I will as usual write \( \mathfrak{A}_I \) for the measure algebra of \( \{0,1\}^I \), \( \mathbb{P}_I \) for the p.o.set \( \mathfrak{A}_I \setminus \{0\} \), for any set \( I \). Write
\[
H = \{ \alpha : 0 < \alpha < \kappa, Z \cap \alpha \text{ belongs to the } \Pi_2^0 \text{-ideal of } \alpha \}.
\]
Suppose, if possible, that \( H \notin \mathcal{F} \). For each \( \alpha \in H \) we have a formula \( \phi_\alpha \) of the language \( \mathcal{L} \) of \( \S A4 \), integers \( k_\alpha, m_\alpha \geq 0 \), relations \( C_{\alpha 1}, \ldots, C_{\alpha n} \) on \( \alpha \), and ordinals \( \xi_{\alpha 1}, \ldots, \xi_{\alpha n} < \alpha \) such that
\[
(\alpha; C_{\alpha 1}, \ldots, \xi_{\alpha n}) \models \phi_\alpha,
\]
\[
(\beta; C_{\alpha 1}, \ldots, \xi_{\alpha n}) \not\models \phi_\alpha \text{ for every } \beta < \alpha.
\]
Because there are only countably many formulae in \( \mathcal{L} \), there must be \( \phi, k, m \) such that \( H_1 \notin \mathcal{F} \), where \( H_1 = \{ \alpha : \alpha \in H, \phi_\alpha = \phi, k_\alpha = k, m_\alpha = m \} \). Let \( \nu_2 \) be a Maharam homogeneous normal witnessing probability on \( \kappa \) such that \( \nu_2 H_1 = 1 \). Let \( \mathbb{P} \) be the p.o.set \( (\mathcal{P}_\kappa/\mathcal{N}_{\nu_2}) \setminus \{0\} \). Let \( \check{C}_1, \ldots, \check{C}_k, \check{\xi}_1, \ldots, \check{\xi}_m \) be the \( \mathbb{P} \)-names for relations and ordinals corresponding to the families \( \langle C_{\alpha 1} \rangle_{\alpha < \kappa}, \langle \xi_{\alpha} \rangle_{\alpha < \kappa} \) as in Lemma 4M. Then we have
\[
\models_{\mathbb{P}} (\check{k}; \check{C}_1, \ldots, \check{\xi}_m) \models \phi
\]
by 4Md. But now observe that the Maharam type \( \lambda \) of \( \nu_2 \) is greater than \( \kappa \), by the Gitik-Shelah theorem again (3F). If we identify \( \mathbb{P} \) with \( \mathbb{P}_\lambda \), we see that there is a set \( J \subseteq \lambda \), of cardinal at most \( \kappa \), such that all the \( \mathbb{P} \)-names \( \check{C}_1, \ldots, \check{\xi}_m \) can be represented by \( \mathbb{P}_J \)-names. Let \( J \) be a subset of \( \lambda \), including \( I \), such that \( J \setminus I \) and \( \lambda \setminus J \) both have cardinal \( \lambda \). Now we can regard \( \mathbb{P} \equiv \mathbb{P}_\lambda \) as an iteration \( \mathbb{P}_J \star \mathbb{P}_{\lambda \setminus J} \) (see Kunen 84, 3.13), so that we have
\[
\models_{\mathbb{P}_J} (\models_{\mathbb{P}_{\lambda \setminus J}} (\check{k}; \check{C}_1, \ldots, \check{\xi}_m) \models \phi).
\]
But from the standpoint of \( V^{\mathbb{P}_J} \), \( \check{C}_1, \ldots, \check{\xi}_m \) are fixed relations and ordinals. Also we have
\[
\models_{\mathbb{P}_J} \check{\kappa} \text{ is real-valued-measurable
}
\]
(Theorem 2C) and
\[
\models_{\mathbb{P}_J} \check{Z} \text{ is not in the rvm ideal of } \check{\kappa}
\]
by 2Jc. So we may use Proposition 4N in \( V^{\mathbb{P}_J} \) to see that
\[
\models_{\mathbb{P}_J} \exists \beta \in \check{Z}, (\beta; \check{C}_1, \ldots, \check{\xi}_m) \models \phi.
\]
Now there is a \( \mathbb{P}_J \)-name \( \check{\beta} \) for a member of \( Z \) such that
\[
\models_{\mathbb{P}_J} (\check{\beta}; \check{C}_1, \ldots, \check{\xi}_m) \models \phi.
\]
Let \( K \subseteq J \) be a countable set such that the \( \mathbb{P}_J \)-name \( \check{\beta} \) can be represented by a \( \mathbb{P}_K \)-name. Regarding \( \mathbb{P}_J \) as an iteration \( \mathbb{P}_{J \setminus K} \star \mathbb{P}_{I \setminus (J \setminus K)} \) and \( \mathbb{P} \) as an iteration \( \mathbb{P}_{I \setminus K} \star \mathbb{P}_{\lambda \setminus (I \setminus K)} \), and observing that \( J \setminus (I \cup K) \) and \( \lambda \setminus (I \cup K) \) both have cardinal \( \lambda > \kappa \), we can use A4I in \( V^{\mathbb{P}_{J \setminus K}} \) to see that
\[
\models_{\mathbb{P}} (\check{\beta}; \check{C}_1, \ldots, \check{\xi}_m) \models \phi.
\]
Now there must be \( \zeta \in Z \) and \( a \in \mathcal{P}_\kappa \setminus \mathcal{N}_{\nu_2} \) such that \( a^* \models_{\mathbb{P}} \check{\beta} = \check{\zeta} \), so that
\[
a^* \models_{\mathbb{P}} (\check{\zeta}; \check{C}_1, \ldots, \check{\xi}_m) \models \phi.
\]
But now 4Md tells us that
\[
(\zeta; C_{\alpha 1}, \ldots, \xi_{\alpha n}) \models \phi
\]
for \( \nu_2 \)-almost every \( \alpha \in a \). In particular, there is an \( \alpha \in a \cap H_1 \) such that \( \alpha \succ \zeta \) and \( (\zeta; C_{\alpha 1}, \ldots) \models \phi \); but \( \phi \models_{\alpha} \phi_\alpha \); so this contradicts the choice of \( C_{\alpha 1}, \ldots \) and \( \phi_\alpha \). \( \mathbf{X} \)

4Q Proposition Suppose that \( \kappa, \kappa' \) are two atomlessly-measurable cardinals with \( \kappa < \kappa' \). Then \( \kappa \) is weakly \( \Pi_2^0 \)-indescribable, and its \( \Pi_2^0 \)-filter is included in its rvm filter.

Proof Let \( \nu \) be a Maharam homogeneous normal witnessing probability on \( \kappa \); let \( \mathbb{P} = (\mathcal{P}_\kappa/\mathcal{N}_\nu) \setminus \{0\} \) be the corresponding p.o.set.
Suppose that $\phi$ is a formula of the language $L$ of §A4 and that $C_1, \ldots, C_k$ are relations on $\kappa, \xi_1, \ldots, \xi_m$ are ordinals less than $\kappa$ such that

$$(\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \vDash \phi.$$ 

By 4Oa, because there is an atomlessly-measurable cardinal greater than $\kappa$,

$\| p (\kappa; C_1, \ldots, C_k) \| \vDash \phi.$

So 4Md tells us that

$$\langle \alpha; C_1, \ldots \rangle \vDash \phi$$

for $\nu$-almost every $\alpha < \kappa$. Because $\nu$ is arbitrary,

$$A = \{ \alpha : \alpha < \kappa, (\alpha; C_1, \ldots) \not\vDash \phi \}$$

belongs to the rvm filter of $\kappa$. As $\phi, C_1, \ldots, \xi_m$ are arbitrary, we have the result.

4R Proposition Let $\kappa$ be a real-valued-measurable cardinal and $\lambda$ any larger cardinal. Let $\mathfrak{A}$ be the measure algebra of $\{0, 1\}^\lambda$ and $\mathbb{P}$ the p.o.set $\mathfrak{A}_\lambda \setminus \{0\}$. Then

$$\| p \kappa \| \text{ is weakly } \Pi^2_0 \text{-indescribable and }$$

the rvm filter of $\kappa$ includes the $\Pi^2_0$-filter of $\kappa$.

proof We can use the same ideas as in the last three results. Let $\phi$ be a formula of the language $L$, and let $\hat{C}_1, \ldots, \hat{C}_k, \hat{\xi}_1, \ldots, \hat{\xi}_m, \sigma$ be $\mathbb{P}_\lambda$-names such that

$$\| p (\kappa; \hat{C}_1, \ldots) \| \vDash \phi,$$

$$\| p \sigma = \{ \alpha : \alpha < \kappa, (\alpha; \hat{C}_1, \ldots) \not\vDash \phi \}.$$ 

Then there is a set $I \subseteq \lambda$ such that $\lambda \setminus I$ has cardinal $\lambda$ and $\sigma$ and every $\hat{C}_i, \hat{\xi}_j$ can be represented by a $\mathbb{P}_I$-name, taking $\mathbb{P}_I$ to be the p.o.set associated with the measure algebra of $\{0, 1\}^I$ as usual. Now we have

$$\| p_I \kappa \| \text{ is real-valued-measurable }$$

(2C) and

$$\| p_I (\| p_{\lambda \setminus I} (\kappa; \hat{C}_1, \ldots) \| \vDash \phi),$$

$$\| p_I (\| p_{\lambda \setminus I} \sigma = \{ \alpha : \alpha < \kappa, (\alpha; \hat{C}_1, \ldots) \not\vDash \phi \}).$$

In $V^{p_I}$, all of $\sigma, \hat{C}_1, \ldots$ are definite sets, so we can apply 4N to see that

$$\| p_I \sigma \| \text{ belongs to the rvm ideal of } \kappa.$$ 

But now 2Jc tells us that

$$\| p_I (\| p_{\lambda \setminus I} \sigma \| \text{ belongs to the rvm ideal of } \kappa),$$

taking the liberty of using the same symbol $\kappa$ for the new name for $\kappa$. Thus

$$\| p \sigma \| \text{ belongs to the rvm ideal of } \kappa.$$ 

As $\phi, \hat{C}_1, \ldots$ are arbitrary we have the result.

Remark This result shows that if an atomlessly-measurable cardinal $\kappa$ is constructed by the method of 2C, then

$$\kappa \text{ is weakly } \Pi^1_1 \text{-indescribable}$$

$$\iff \kappa < \mathfrak{c}$$

$$\iff \kappa \text{ is weakly } \Pi^2_0 \text{-indescribable}$$

and the rvm filter of $\kappa$ includes the $\Pi^2_0$-filter of $\kappa$.

See 6J and P3 below.
Proposition
Let $\kappa$ be an atomlessly-measurable cardinal with normal witnessing probability $\nu$. Let $\phi$ be a $\Sigma_1^1$ formula in the language $\mathcal{L}$ of §A4, $C_1, \ldots, C_k$ relations on $\kappa$ and $\xi_1, \ldots, \xi_m$ ordinals less than $\kappa$. Set
$$A = \{ \alpha : \alpha < \kappa, (\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi \}.$$ 
If $A$ is not in the rvm ideal of $\kappa$ then there is a $B \subseteq \kappa$ such that $\beta \subseteq B$, $\nu B = 1$ and 
$$(B; C_1, \ldots, \xi_m) \models \phi.$$ 
\textbf{proof} (a) By 4N, $\nu A = 1$. Of course, we may suppose that $\xi_j < \beta$ for every $j$. Next, we may take it that $\phi$ is of the form
$$\exists R_{k+1} \ldots \exists R_{k+r} \forall x_{m+1} \exists x_{m+2} \ldots \forall x_{m+2s-1} \exists x_{m+2s} \psi$$
where $\psi$ has no quantifiers, since $\phi$ is surely logically equivalent to such a formula. For every $\alpha \in A$ we have relations $D_{\alpha_1}, \ldots, D_{\alpha_r}$ and functions $h_{\alpha_1} : \alpha \rightarrow \alpha$, $h_{\alpha_2} : \alpha^2 \rightarrow \alpha$, $\ldots$, $h_{\alpha_s} : \alpha^s \rightarrow \alpha$ such that 
$$(\alpha; C_1, \ldots, C_k, D_{\alpha_1}, \ldots, D_{\alpha_r}, \xi_1, \ldots, \xi_m, \eta_1, h_{\alpha_1}(\eta_1), \ldots, \eta_s, h_{\alpha_s}(\eta_1, \ldots, \eta_s)) \models \psi$$
for all $\eta_1, \ldots, \eta_s < \alpha$. For each $\tilde{\eta} = (\eta_1, \ldots, \eta_j) \in \bigcup_{\eta \leq \kappa} \mathcal{F}$ set
$$M(\tilde{\eta}) = \{ \xi : \nu(\alpha : h_{\alpha j}(\tilde{\eta}) = \xi) > 0 \};$$
then $M(\tilde{\eta})$ is countable. For $I \subseteq [\kappa]^{<\omega}$ let $K(I)$ be the smallest subset of $\kappa$ such that $I \subseteq K(I)$ and $M(\tilde{\eta}) \subseteq K(I)$ whenever $\tilde{\eta} \in \bigcup_{\eta \leq \kappa} K(I)$. Then $K(I)$ is countable.

(b) Let $(Z, \tilde{\nu})$ be the hyperstonian space of $(\kappa, P, \kappa, \nu)$, and for $a \subseteq \kappa$ let $a^*$ be the corresponding open-and-closed subset of $Z$, so that $a \mapsto a^*$ is a Boolean homomorphism and $\tilde{\nu} a^* = \nu a$ for every $a \subseteq \kappa$. For $j \leq s$, $\tilde{\eta} \in \kappa$ set
$$Q(\tilde{\eta}) = \{ \alpha : h_{\alpha j}(\tilde{\eta}) = \xi^* : \xi \in M(\tilde{\eta}) \},$$
so that $\tilde{\nu}(Z \setminus \bigcup Q(\tilde{\eta})) = 0$. Set
$$f(I) = \bigcup\{ Z \setminus \bigcup Q(\tilde{\eta}) : \eta \in \bigcup_{\eta \leq \kappa} K(I)^j \} \in \mathcal{N}_\kappa$$
for each $I \subseteq [\kappa]^{<\omega}$.

By Proposition 6E, there is a set $B_0 \subseteq \kappa$ such that $\beta \subseteq B_0$, $\nu B_0 = 1$ and $\bigcup\{ f(I) : I \subseteq [B_0]^{<\omega} \} \neq Z$. Take $z \in Z \setminus \bigcup\{ f(I) : I \subseteq [B_0]^{<\omega} \}$ and set $B = \bigcup\{ K(I) : I \subseteq [B_0]^{<\omega} \}$.

If $j \leq s$, we may define $h_j : B^j \rightarrow B$ as follows. For each $\tilde{\eta} \in B^j$ there is an $I \subseteq [B_0]^{<\omega}$ such that $\tilde{\eta} \in K(I)^j$ (because the function $I \mapsto K(I)$ is increasing), and now $z \in \bigcup Q(\tilde{\eta})$, so there is a (unique) $\xi$ such that $z \in \{ \alpha : h_{\alpha j}(\tilde{\eta}) = \xi^* \}$. We have $\xi \in M(\tilde{\eta}) \subseteq K(I) \subseteq B$; take this $\xi$ for $h_j(\tilde{\eta})$.

Next, define relations $D_1, \ldots, D_r$ on $\kappa$ by writing
$$D_i(\eta_1, \ldots, \eta_s) \iff z \in \{ \alpha : D_{\alpha 1}(\eta_1, \ldots, \eta_s) \}^*.$$ 
Then an easy induction on the length of $\chi$ shows that whenever $\chi$ is a formula of $\mathcal{L}$ without quantifiers, and $\eta_1, \ldots, \eta_s \in B$,
$$(B; C_1, \ldots, C_k, D_1, \ldots, D_r; \xi_1, \ldots, \xi_m, \eta_1, h_1(\eta_1), \ldots, h_s(\eta_1, \ldots, \eta_s)) \models \chi$$
at least when $z \in b(\chi; \eta_1, \ldots, \eta_s)^*$, where $b(\chi; \eta_1, \ldots, \eta_s)$ is the set of those $\alpha < \kappa$ such that
$$(\alpha; C_1, \ldots, C_k, D_{\alpha 1}, \ldots, D_{\alpha r}, \eta_1, h_{\alpha 1}(\eta_1), \ldots, \eta_s, h_{\alpha s}(\eta_1, \ldots, \eta_s)) \models \chi.$$ 
In particular, this is true when $\chi = \psi$. But of course $z \in Z = b(\psi; \eta_1, \ldots, \eta_s)^*$ for all $\eta_1, \ldots, \eta_s < \kappa$, so
$$(B; C_1, \ldots, C_k, D_1, \ldots, D_r; \xi_1, \ldots, \xi_m, \eta_1, h_1(\eta_1), \ldots, h_s(\eta_1, \ldots, \eta_s)) \models \psi$$
for all $\eta_1, \ldots, \eta_s \in B$, and
$$(B; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi,$$
as required.
4T Corollary Let $\kappa$ be an atomlessly-measurable cardinal. Let $\phi$ be a $\Pi_1^1$ formula of the language $\mathcal{L}$, and $\xi_1, \ldots, \xi_m$ ordinals less than $\kappa$. If $(\kappa; <; \xi_1, \ldots, \xi_m) \models \phi$ then there is an $\alpha < \kappa$ such that $(\alpha; <; \xi_1, \ldots, \xi_m) \models \phi$.

**proof** Suppose, if possible, otherwise. Applying 4S to (a $\Sigma_1^1$ formula logically equivalent to) $\neg \phi$, with $\beta$ greater than any $\xi_1$, we obtain a $B \subseteq \kappa$ such that $\beta \subseteq B$, $\#(B) = \kappa$ and 

$$(B; <; \xi_1, \ldots, \xi_m) \models \neg \phi.$$ 

But $(B; <; \xi_1, \ldots, \xi_m)$ is isomorphic to $(\kappa; <; \xi_1, \ldots, \xi_m)$, so this is impossible. $\blacksquare$

**Remark** 4N-4T are due to Kunen. Most of the ideas are in KUNEN N70.

5. Combinatorial implications

I turn now to some of the combinatorial consequences of supposing that there is an atomlessly-measurable cardinal. I begin with technical but very useful results on set-valued functions defined on $[\kappa]^{<\omega}$ (5A-5B), with fairly straightforward corollaries (5C-5D), and Prikry’s theorem on cardinal powers when $\kappa$ is atomlessly-measurable (5E). 5F deals with $\kappa$-Aronszajn trees. The rest of the section is devoted to theorems of Kunen concerning ultrafilters on $\mathbb{N}$ (5G), algebras generated by rectangles (5H-5J) with corollaries on families in $\mathcal{P}\mathcal{N}$ and $\mathcal{N}^\mathcal{N}$ (5K-5L), $\diamondsuit$ ($\kappa$) and a partition relation (50-5P).

5A Lemma Let $(X, \mathcal{P}X, \mu)$ be a probability space and $Y$ a set of cardinal less than the additivity of $\mu$. Let $\theta$ be any cardinal and $f : X \rightarrow [Y]<\theta$ a function.

(a) There is an $M \in [Y]<\theta$ such that $\mu\{x : f(x) \in M\} > 0$.

(b) If $\mathrm{cf}(\theta) > \omega$ then there is an $M \in [Y]<\theta$ such that $\mu\{x : f(x) \subseteq M\} = 1$.

**proof** (a) If $\theta > \#(Y)$ this is trivial; suppose that $\theta \leq \#(Y) < \text{add}(\mu)$. Because $[Y]<\theta = \bigcup_{\alpha < \theta} [Y]<\alpha$, there is an $\alpha < \theta$ such that $\mu X_0 > 0$, where $X_0 = \{x : \#(f(x)) \leq \alpha\}$. If $\alpha < \omega$ then $\#([Y]<\alpha) \leq \max(\omega, \#(Y)) < \text{add}(\mu)$ so there is an $M \in [Y]<\alpha$ such that $\mu\{x : f(x) = M\} > 0$. If $\alpha \geq \omega$, then for each $x \in X_0$ let $\langle h_\xi(x) \rangle_{\xi<\alpha}$ run over a set including $f(x)$. For each $\xi < \alpha$,

$$Y_\xi = \{y : \mu h_\xi^{-1}\{y\}] > 0\}$$

is countable, and because $\#(Y) < \text{add}(\mu)$, $\mu h_\xi^{-1}[Y \setminus Y_\xi] = 0$. Set $M = \bigcup_{\xi<\alpha} Y_\xi \subseteq [Y]<\alpha \subseteq [Y]<\theta$. Because $\alpha < \text{add}(\mu)$,

$$\mu\{x : f(x) \subseteq M\} \geq \mu(X_0 \setminus \bigcup_{\xi<\theta} h_\xi^{-1}[Y \setminus Y_\xi]) > 0,$$

as required.

(b) If $\mathrm{cf}(\theta) > \omega$, we can take $\alpha$ such that $\mu X_0 = 1$, so that $\mu\{x : f(x) \subseteq M\} = 1$.

5B Theorem Let $\kappa$ be a real-valued-measurable cardinal and $\nu$ a normal witnessing probability on $\kappa$. Let $\theta < \kappa$ be a cardinal of uncountable cofinality, and $f : [\kappa]<\omega \rightarrow [\kappa]<\theta$ any function. Then there are $C \subseteq \kappa$ and $f^* : [C]<\omega \rightarrow [\kappa]<\theta$ such that $\nu C = 1$ and $f(I \cap \eta) \subseteq f^*(I \cap \eta)$ whenever $I \in [C]<\omega$ and $\eta < \kappa$.

**proof** (a) I show by induction on $n \in \mathbb{N}$ that if $g : [\kappa]^{<n} \rightarrow [\kappa]<\theta$ is a function then there are $A \subseteq \kappa$, $g^* : [A]<\omega \rightarrow [\kappa]<\theta$ such that $\nu A = 1$ and $g(I \cap \eta) \subseteq g^*(I \cap \eta)$ for every $I \in [A]<\omega$, $\eta < \kappa$. P (i) If $n = 0$ this is trivial; take $A = \kappa$, $g^*(\emptyset) = g(\emptyset)$. (ii) For the inductive step to $n+1$, given $g : [\kappa]^{<n+1} \rightarrow [\kappa]<\theta$, then for each $\xi < \kappa$ define $g^*_\xi : [\kappa]^{<n} \rightarrow [\kappa]<\theta$ by setting $g^*_\xi(J) = g(J \cup \{\xi\})$ for every $J \in [\kappa]^{<n}$. Set

$$D = \{\xi : \xi < \kappa, \mathrm{cf}(\xi) \geq \theta\};$$

then $\nu D = 1$ (4Lc). For $\xi \in D$, $J \in [\kappa]^{<n}$ set $\zeta_{\xi J} = \sup(\xi \cap g_{\xi J}(J)) < \xi$. Then for each $J \in [\kappa]^{<n}$ the function $\xi \mapsto \zeta_{\xi J}$ is regressive, so there is a $\zeta_J^* < \kappa$ such that $\zeta_{\xi J} < \zeta_J^*$ for almost every $\xi < \kappa$. Now by 5Ab we see that there is an $h(J) \in [\kappa]<\theta$ such that $\zeta_J^* < \zeta_J$ for almost every $\xi < \kappa$. By the inductive hypothesis, there are $B \subseteq \kappa$, $h^* : [B]<\omega \rightarrow [\kappa]<\theta$ such that $\nu B = 1$ and $h(J) \cap \eta \subseteq h^*(J \cap \eta)$ for every $J \in [B]^{<n}$ and $\eta < \kappa$.

Try setting
Then \( A_J = \{ \xi : \xi \cap g(J) \subseteq h(J) \} \) for \( J \in [\kappa]^{\leq n} \),

\[
A = B \cap \{ \xi : \xi \in A_J \forall J \in [\kappa]^{\leq n} \},
\]

\( g^*(I) = g(I) \) if \( I \in [\lambda]^{n+1} \), \( g^*(I) = g(I) \cup h^*(I) \) if \( I \in [\lambda]^n \).

Then \( \nu A_J \) is always, by the choice of \( h(J) \), so \( \nu A = 1 \), by A1E(c-iv); while \( g^*(I) \in [\kappa]^{<\theta} \) for every \( I \in [\lambda]^{\leq n+1} \). If \( \eta < \kappa \) then of course \( g(0) \cap \eta \subseteq g^*(0) \) for every \( \eta < \kappa \). If \( I \in [\lambda]^{\leq n+1} \setminus \{ \emptyset \} \) and \( \eta < \kappa \), set \( \xi = \max I \cup \{ \xi \} \); then \( \xi \in A_J \). If \( \eta > \xi \) then \( g(I) \cap \eta \subseteq g(I) \subseteq g^*(I) = g^*(I \cap \eta) \). If \( \eta \leq \xi \) then

\( g(I) \cap \eta = g(\xi)(J) \cap \xi \cap \eta \subseteq h(J \cap \eta) \subseteq h^*(I \cap \eta) = h^*(I \cap \eta) \subseteq g^*(I \cap \eta). \)

Thus the induction continues.

(b) Now applying (a) to \( f([\kappa]^{\leq n}) \) we obtain sets \( C_n \subseteq \kappa \), functions \( f_n^* : [C_n]^{\leq n} \rightarrow [\kappa]^{<\theta} \) such that \( \nu C_n = 1 \) and \( f(I) \cap \eta \subseteq f_n^*(I \cap \eta) \) for every \( I \subseteq [C_n]^{\leq n} \) for every \( \eta < \kappa \). Set \( C = \bigcap_n C_n \) and \( f(I) = \bigcup_{l \in \#(I)} f_l^*(I) \) for each \( I \subseteq [C]^{<\omega} \). Then \( \nu C = 1 \). If \( I \subseteq [C]^{<\omega} \) and \( \eta < \kappa \), set \( n = \#(I) \); then \( I \subseteq [C_n]^n \) so \( f(I) \cap \eta \subseteq f_n^*(I \cap \eta) \), as required.

5C Corollary

Let \( \kappa \) be a real-valued-measurable cardinal and \( \nu \) a normal witnessing probability on \( \kappa \). Let \( \theta < \kappa \) be a cardinal of uncountable cofinality.

(a) If \( Y \) is a set of cardinal less than \( \kappa \) and \( f : [\kappa]^{<\omega} \rightarrow [\kappa]^{<\theta} \) is a function, then there are sets \( C \subseteq \kappa \), \( M \subseteq Y \) such that \( \nu C = 1 \), \#(\( M \)) < \( \theta \) and \( f(I) \subseteq M \) for every \( I \subseteq [C]^{<\omega} \).

(b) If \( Y \) is any set and \( g : [\kappa]^{<\theta} \rightarrow [\kappa]^{<\omega} \) any function there are sets \( C \subseteq \kappa \), \( M \subseteq [\kappa]^{<\theta} \) such that \( \nu C = 1 \) and \( g(\xi) \cap g(\eta) \subseteq M \) for all distinct \( \xi, \eta \in C \).

(c) If \( f : [\kappa]^{<\omega} \rightarrow \kappa \) is a function such that \( f(I) < \min I \) for every non-empty finite \( I \subseteq \kappa \), then there are sets \( C \subseteq \kappa \) such that \( \nu C = 1 \), \( M \subseteq C \) is countable and \( f(I) \subseteq M \) for every \( I \subseteq [C]^{<\omega} \).

(b) As above, we may suppose that \( \bigcup_{\xi \in C} g(\xi) \subseteq \kappa \). Set \( f(I) = \bigcup_{\xi \in I} g(\xi) \in [\kappa]^{<\theta} \) for \( I \subseteq [\kappa]^{<\omega} \); take \( C_0 \subseteq \kappa \), \( f^* : [C_0]^{<\omega} \rightarrow [\kappa]^{<\theta} \) from 5B, and set \( C = \bigcup f^*(\emptyset) \). Write

\[
\nu C = \{ \xi : \xi < \kappa, g(\eta) \subseteq \xi \forall \eta < \xi \},
\]

so that \( C_1 \) is a closed unbounded set and \( \nu C = 1 \), \( C = C_0 \cap C_1 \). If \( \xi, \eta \in C \) and \( \xi < \eta \), then

\[
\nu (\xi \cap g(\eta) \subseteq \nu (\xi \cap g(\eta)) \subseteq \nu (f(\xi)) \subseteq \nu f^*(\xi) \cap g(\eta) = f^*(\emptyset),
\]

so \( g(\xi) \cap g(\eta) \subseteq M, \) as required.

(c) By 5B we have \( f^* : [\kappa]^{<\omega} \rightarrow [\kappa]^{<\omega} \), \( C \subseteq \kappa \) such that \( \nu C = 1 \) and \( \{ f(I) \} \cap g(\eta) = f^*(I \cap \eta) \) whenever \( I \subseteq [C]^{<\omega} \) and \( \eta < \kappa \). We may suppose that \( 0 \notin C \). If \( I \subseteq [C]^{<\omega} \) and \( \emptyset \neq I \) then \( f(I) = f(\emptyset) \cup \{ f(\emptyset) \} \) have the result.

5D Corollary

Let \( \kappa \) be a real-valued-measurable cardinal.

(a) Suppose that \( Y \) is a set of cardinal less than \( \kappa \) and \( \theta \) is a cardinal less than \( \kappa \) of uncountable cofinality.

Then

(i) \( \kappa \) is a precaliber of the Boolean algebra \( \mathcal{P}Y/[Y]^{<\theta} \);

(ii) \( \{ \lambda : \lambda < \kappa \) is a precaliber of \( \mathcal{P}Y/[Y]^{<\theta} \} \) belongs to the rvm filter of \( \kappa \);

(iii) the Souslin number of \( \mathcal{P}Y/[Y]^{<\theta} \) is less than \( \kappa \);

(iv) \( \text{Tr}_{[Y]^{<\theta}}(Z) < \kappa \) whenever \( \#(Z) < \kappa \) (definition: 2F).

(b) If \( \delta \) is a cardinal less than \( \kappa \), and \( h : [\kappa]^{<\omega} \rightarrow [\kappa]^{<\theta} \) is a function, there is a set \( A \subseteq \kappa \) such that \( \#(A) = \kappa \) and \( h(I) \subseteq A \) for every \( I \subseteq [\lambda]^{<\omega} \).

(c) In particular, there is no Jonsson algebra on \( \kappa \).

(d) If \( f : [\kappa]^{<\omega} \rightarrow \omega_1 \) is any function there are \( C \subseteq [\kappa]^{<\omega} \), \( \zeta < \omega_1 \) such that \( f(I) \neq \zeta \) for every \( I \subseteq [\lambda]^{<\omega} \).

\[
\text{proof (a) Write } \mathfrak{A} = \mathcal{P}Y/[Y]^{<\theta}.
\]
(i) Let $\langle a_\xi \rangle _{\xi < \kappa}$ be a family in $\mathfrak{A} \setminus \{0\}$. For $\xi < \kappa$ choose $A_\xi \subseteq Y$ such that $A_\xi^* = a_\xi$. For $I \in [\kappa]^{< \omega}$ set $f(I) = \bigcap_{\xi \in I} A_\xi$ if the intersection has cardinal less than $\theta$, $\emptyset$ otherwise. By 5Ca there is a $C \in [\kappa]^\kappa$ such that $M = \bigcup \{f(I) : I \in [C]^{< \omega}\}$ has cardinal less than $\theta$. Now there is a $y \in Y \setminus M$ such that $D = \{\xi : \xi \in C, y \in A_\xi\}$ has cardinal $\kappa$, and we must have $\#(\bigcap_{\xi \in I} A_\xi) \geq \theta$ for every $I \in [D]^{< \omega}$, so that $\langle a_\xi \rangle _{\xi < \kappa}$ is centered in $\mathfrak{A}$.

(ii) Let $W$ be the set of cardinals less than $\kappa$ which are not precalibers of $\mathfrak{A}$. For each $\alpha \in W$ choose a family $\langle a_\alpha \xi \rangle _{\xi < \alpha}$ in $\mathfrak{A} \setminus \{0\}$ with no centered family of cardinal $\alpha$, and for $\xi < \alpha \in W$ choose $A_\alpha \xi \subseteq Y$ such that $A_\alpha \xi^* = a_\alpha$. For $I \in [\kappa]^{< \omega}$ set $f(I) = \bigcap_{\xi \in I} A_\alpha \xi$ if $\#(I) \geq 2$ and max $I = \alpha \in W$ and $\#(\bigcap_{\xi \in I} A_\xi) < \theta$, $\emptyset$ otherwise. If $W$ is not in the rvm ideal of $\alpha$, let $\nu$ be a normal witnessing probability on $\kappa$ with $\nu W = 1$. By 5Ca there is a $C \subseteq \kappa$ such that $\nu C = 1$ and $M = \bigcup \{f(I) : I \in [C]^{< \omega}\}$ has cardinal less than $\theta$. Let $\alpha \in C \cap W$ be such that $\#(C \cap \alpha) = \alpha$ and $\text{cf}(\alpha) > \#(Y)$ (using 4Lc). For each $\xi \in C \cap \alpha$ there is a $y \in A_\alpha \xi \setminus M$, so there is a $y \in Y$ such that $D = \{\xi : \xi < \alpha, y \in A_\alpha \xi\}$ has cardinal $\alpha$. But now $\langle a_\alpha \xi \rangle _{\xi < \alpha}$ is centered, contrary to the choice of $\langle a_\alpha \xi \rangle _{\xi < \alpha}$. Thus $W$ belongs to the rvm ideal of $\kappa$, and

$$V = \{\alpha : \alpha < \kappa \text{ is a cardinal}\} \setminus W$$

belongs to the rvm filter of $\kappa$. But every member of $V$ is a precaliber of $\mathfrak{A}$.

(iii) If $\alpha$ is any infinite cardinal which is a precaliber of $\mathfrak{A}$, then the Souslin number of $\mathfrak{A}$ is at most $\alpha$.

(iv) $\text{Tr}_{|Y|^{< \omega}}(Y ; Z) \leq S(\mathcal{P}(Y \times Z)/|Y \times Z|)^{< \theta}$.

(b)-(c) For $I \in [\kappa]^{< \omega}$ let $g(I)$ be the smallest set such that $I \subseteq g(I)$ and $h(J) \subseteq g(I)$ for every $J \in [g(I)]^{< \omega}$. Then $\#(g(I)) \leq \delta$ for every $I$ and $h(g(I)) \subseteq g(I')$ whenever $I \subseteq I' \in [\kappa]^{< \omega}$.

Applying 5Ca with $Y = \delta^+ = \theta$ and $f(I) = g(I) \cap \theta$, we see that there are sets $C \subseteq \kappa, M \subseteq \theta$ such that $\#(C) = \kappa, \#(M) \leq \delta$ and $g(I) \cap \theta \subseteq M$ for every $I \in [C]^{< \omega}$. Set

$$A = \bigcup \{g(I) : I \in [C]^{< \omega}\} \subseteq \kappa.$$

Then $C \subseteq A$, so $\#(A) = \kappa$, and $A \cap \theta \subseteq M$, so $A \neq \kappa$. Finally, $\{g(I) : I \in [C]^{< \omega}\}$ is upwards-directed, so if $J \in [A]^{< \omega}$ there is an $I \in [C]^{< \omega}$ such that $J \subseteq g(I)$, in which case $h(J) \subseteq g(I) \subseteq A$.

(d) In fact there are $C \in [\kappa]^\kappa, \zeta < \omega_1$ such that $f(I) < \zeta$ for every $I \in [C]^{< \omega}$.

Remarks (a) The ideas here go back to Lemma 14 of SOLOVAY 71 and Lemma 5 of KUNEN N70. 5Da(ii) is a strengthening of Theorem 2d of Prikry 75. 5Dc is due to Shelah. 5Dd corresponds to formula (7) of §53 in Erdős Hajnal Máté & Rado 84, p. 330.

(b) The results above will more often than not be used with successor cardinals $\theta$, so that $|Y|^{< \theta} = |Y|^{< \delta}$ for some infinite cardinal $\delta < \kappa$. Of course 5Bc-5C also give information about functions $f : [\kappa]^{< \omega} \to Y$, taking $\theta = \omega_1$ and replacing $f$ by $I \mapsto \{f(I) : [\kappa]^{< \omega} \to [Y]^{< \omega}\}$.

(c) For possible strengthenings of 5Ab, 5Ca and 5Da see P9 below.

5E Theorem If $\kappa$ is atomlessly-measurable, then $2^\lambda \leq \kappa$ for every cardinal $\lambda < \kappa$.

proof The proof is by induction on $\lambda$. It starts with the trivial case $\lambda \leq \omega$.

(a) For the inductive step to $\lambda$, where $\omega < \lambda < \kappa$, consider first the case $\text{cf}(\lambda) > \omega$. For each $\xi < \lambda$, let $\theta_\xi : \mathcal{P}_\kappa \to \kappa$ be an injective function. For each $A \subseteq \lambda$, there must be a $\gamma_A < \kappa$ such that $\theta_\xi(A \cap \xi) \leq \gamma_A$ for every $\xi < \lambda$ because $\text{cf}(\kappa) = \kappa$.

(? Suppose, if possible, that $2^\lambda > \kappa$. Then there must be a $\gamma < \kappa$ such that $A = \{A : A \subseteq \lambda, \gamma_A = \gamma\}$ has cardinal greater than $\kappa$. Let $h : \kappa \to A$ be injective, and define $f : [\kappa]^2 \to \lambda$ by setting

$$f((\alpha, \beta)) = \min\{\xi : h(\alpha) \cap \xi \neq h(\beta) \cap \xi\}$$

whenever $\alpha, \beta < \kappa$ are distinct. By Corollary 5Ca, with $\theta = \omega_1, Y = \lambda$, there is a set $C \subseteq \kappa$ such that $\#(C) = \kappa$ and $M = f([\kappa]^2)$ is countable. Because $\text{cf}(\lambda) > \omega$, $\xi = \sup M < \lambda$. But now we see that $h(\alpha) \cap \xi \neq h(\beta) \cap \xi$ whenever $\alpha, \beta$ are distinct members of $C$. So $\theta_\xi(h(\alpha) \cap \xi) \neq \theta_\xi(h(\beta) \cap \xi)$ for all distinct $\alpha, \beta \in C$. But $h(\alpha) \in A$ whenever $\alpha < \kappa$, so $\theta_\xi(h(\alpha) \cap \xi) \leq \gamma$ for every $\alpha \in C$; which is impossible, because $\#(C) > \#(\gamma)$. X
So in this case $2^\lambda \leq \kappa$.

(b) Now if we have $\omega = \text{cf}(\lambda) < \lambda < \kappa$, then there is an increasing sequence $(\lambda_n)_{n \in \mathbb{N}}$ of cardinals cofinal with $\lambda$, so that $2^\lambda \leq \#(\bigcap_{n \in \mathbb{N}} \mathcal{P}(\lambda_n)) \leq \#(2^\lambda) = \kappa$. So in this case also the induction proceeds.

**Remark** This is due to Prikry 75. See also 7Q below.

5F Proposition If $\kappa$ is a real-valued-measurable cardinal, then there is no $\kappa$-Aronszajn tree.

**proof** Let $T$ be a tree of height $\kappa$ in which every level has cardinal strictly less than $\kappa$. Then $\#(T) = \kappa$. Let $\nu$ be a $\kappa$-additive probability with domain $\mathcal{P}T$ which is zero on singletons. For each $t \in T$ write $T^{(t)}$ for the set of elements of $T$ comparable with $t$, and for $\xi < \kappa$ let $T_\xi$ be the set of elements of $T$ of rank $\xi$. Set $S = \{ t : \nu(T^{(t)}) > 0 \}$. If $s \leq t \in S$ then $T^{(s)} \supseteq T^{(t)}$ so $s \in S$; for each $\xi < \kappa$, we have $\#(T_\xi) < \kappa$ and $T = \bigcup_{t \in T_\xi} T^{(t)}$, so $S \cap T_\xi \neq \emptyset$; and $\nu(T^{(s)} \cap T^{(t)}) = 0$ if $s$, $t$ are distinct members of $T_\xi$, so $S \cap T_\xi$ is countable. Thus $S$ is a tree of height $\kappa$ in which every level is countable. Because $\text{cf}(\kappa) > \omega_1$ there must be an $s \in S$ such that $S^{(s)} = S \cap T^{(s)}$ is a branch of length $\kappa$. Thus $T$ is not a $\kappa$-Aronszajn tree.

**Remarks** (a) This is due to Silver 70, Theorem 1.16.

(b) It is also the case that

$$\{ \lambda : \lambda < \kappa \text{ is a cardinal and there is a } \lambda\text{-Aronszajn tree} \}$$

belongs to the rvm ideal of $\kappa$. This may be deduced from the result above using 4Ob (because it is easy to check that there is a second-order formula $\phi$ such that, for cardinals $\lambda$, $(\lambda < \kappa) \models \phi$ if there is a $\lambda$-Aronszajn tree), or from 4Kb, because if $\lambda$ is weakly $\Pi^1_2$-indescribable there is no $\lambda$-Aronszajn tree (Fremlin & Kunen n87, 2N; compare Keisler & Tarski 64, 4.31).

5G Proposition If there is an atomlessly-measurable cardinal, there is no rapid $p$-point ultrafilter on $\mathbb{N}$.

**proof** Let $\kappa$ be an atomlessly-measurable cardinal, with witnessing probability $\nu$.

Suppose, if possible, that $\mathcal{F}$ is a rapid $p$-point ultrafilter on $\mathbb{N}$. Let $\langle C_{kn} \rangle_{k,n \in \mathbb{N}}$ be a stochastically independent double sequence of subsets of $\kappa$ with $\nu C_{kn} = \frac{1}{2}$ for all $k, n$. Set

$$C_k = \lim_{n \to \mathcal{F}} C_{kn} \subseteq \kappa,$$

$$A_k = \{ (\xi, n) : \xi \in C_k \& \xi \in C_{kn} \text{ or } \xi \notin C_k \& \xi \notin C_{kn} \} \subseteq \kappa \times \mathbb{N}$$

for each $k \in \mathbb{N}$. Then, for $k \in \mathbb{N}$, $\xi < \kappa$,

$$\{ n : (\xi, n) \in A_k \} = \{ n : \xi \in C_{kn} \} \in \mathcal{F} \text{ if } \xi \in C_k,$$

$$= \{ n : \xi \notin C_{kn} \} \in \mathcal{F} \text{ if } \xi \notin C_k.$$

Because $\mathcal{F}$ is a $p$-point ultrafilter, there is a set $A \subseteq \kappa \times \mathbb{N}$ such that

$$\{ n : (\xi, n) \in A \} \in \mathcal{F},$$

$$\{ n : (\xi, n) \in A \setminus A_k \} \text{ is finite}$$

for all $k \in \mathbb{N}$, $\xi < \kappa$.

Now observe that if we write

$$B_{kn} = \{ \xi : \xi < \kappa, (\xi, n) \in A, \forall i < k \},$$

then $\lim_{n \to \mathcal{F}} \nu(D \cap B_{kn}) = 2^{-k} \nu D$ for every $k \in \mathbb{N}$, $D \subseteq \kappa$. Induce on $k$. For $k = 0$, this is trivial. For the inductive step to $k + 1$,

$$B_{k+1,n} = \{ \xi : \xi \in B_{kn}, \xi \in C_k \iff \xi \in C_{kn} \} = (B_{kn} \cap C_k \cap C_{kn}) \cup ((B_{kn} \setminus C_k) \setminus C_{kn}).$$

Now if we write $\mathfrak{A}_k$ for the subalgebra of $\mathcal{P}\kappa$ generated by $\{ C_{ij} : i < k, j \in \mathbb{N} \}$, we see that for any $E \subseteq \kappa$, sequence $\langle D_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{A}_k$ we have
The set is finite for every \( \xi < \kappa \), whenever the right-hand-side is defined, just because the \( C_{kn} \) are stochastically independent of each other and the \( D_n \) and all have measure \( \frac{1}{2} \). (See A2N.) It follows that if we write \( \mathfrak{A}_k' \) for the subalgebra of \( \mathcal{P} \kappa \) generated by \( \mathfrak{A}_k \cup \{ C_i : i \leq k \} \cup \{ D \} \) then

\[
\lim_{n \to \infty} \nu(D_n \cap C_{kn}) = \frac{1}{2} \lim_{n \to \infty} \nu(D_n)
\]

for every sequence \( (D_n)_{n \in \mathbb{N}} \) in \( \mathfrak{A}_k' \) for which the right-hand limit exists. Applying this with \( D_n = B_{kn} \cap D \cap C_k \) and with \( D_n = B_{kn} \cap D \setminus C_k \) we get, for any \( D \subseteq \kappa \),

\[
\lim_{n \to \infty} \nu(B_{k+1,n} \cap D) = \lim_{n \to \infty} \nu(B_{kn} \cap D \cap C_k \cap C_{kn}) + \lim_{n \to \infty} \nu(B_{kn} \cap D \setminus C_k) - \lim_{n \to \infty} \nu((B_{kn} \cap D \setminus C_k) \cap C_{kn})
\]

\[
= \frac{1}{2} \lim_{n \to \infty} \nu(B_{kn} \cap D \cap C_k) + \frac{1}{2} \lim_{n \to \infty} \nu(B_{kn} \cap D \setminus C_k)
\]

\[
= \frac{1}{2} \lim_{n \to \infty} \nu(B_{kn} \cap D) = \frac{1}{2} 2^{-k} \nu D = 2^{-k-1} \nu D,
\]

as required. \( \Box \) In particular, \( \lim_{n \to \infty} \nu B_{kn} = 2^{-k} \) for every \( k \in \mathbb{N} \).

If we now set

\( B_n = \{ \xi : (\xi, n) \in A \} \),

the set

\( \{ n : n \in \mathbb{N}, \xi \in B_n \setminus B_{kn} \} \)

is finite for every \( \xi < \kappa, k \in \mathbb{N} \), and consequently

\[
\lim_{n \to \infty} \nu(B_n \setminus B_{kn}) = 0 \quad \forall \ k \in \mathbb{N},
\]

so that

\[
\lim_{n \to \infty} \nu B_n = 0.
\]

Because \( \mathcal{F} \) is rapid, there is an \( F \in \mathcal{F} \) such that \( \sum_{n \in F} \nu B_n < \infty \). There is therefore some \( \xi < \kappa \) such that \( F' = \{ n : n \in F, (\xi, n) \in A \} \) is finite. But \( F' \in F \) for the subalgebra of \( \mathfrak{A}_k' \) for the second-countable norm topology on \( E \), for every \( E \in \mathcal{E} \). \( \Box \)

**Remark** This is due to Kunen; compare Jech 78, §38.

**5H Proposition** Let \( (X, \mu) \) and \( (Y, \nu) \) be probability spaces and \( H \) a Hilbert space. Suppose that \( x \mapsto u_x : X \to H \) and \( y \mapsto v_y : Y \to H \) are bounded \( H \)-scalarly measurable functions. Then

\[
\int \int \langle u_x | v_y \rangle \mu(dx) \nu(dy) = \int \int \langle u_x | v_y \rangle \nu(dy) \mu(dx)
\]

(and both repeated integrals exist).

**proof (a)** Let \( \mathcal{E} \) be the set of closed separable linear subspaces of \( H \). For each \( E \in \mathcal{E} \), let \( P_E \) be the orthogonal projection onto \( E \). Observe that (because \( E \) is separable) the map \( x \mapsto P_E(u_x) : X \to E \) will be measurable in the sense of A2Ae for the second-countable norm topology on \( E \), for every \( E \in \mathcal{E} \).

(b) There is an \( E \in \mathcal{E} \) such that

\( P_E(u_x) = P_{F}(u_x) \) \( \mu \text{-a.e.} \) (x)

whenever \( E \subseteq F \in \mathcal{E} \). \( \Box \) For suppose, if possible, otherwise. Then we can choose inductively an increasing family \( \{ E_\xi \}_{\xi \in \omega_1} \) in \( \mathcal{E} \) such that

\[
\mu \{ x : P_{E_{\xi+1}}(u_x) \neq P_{E_{\xi}}(u_x) \} > 0 \quad \forall \ \xi < \omega_1,
\]

\( E_\xi = \bigcup_{n<\xi} E_n \) whenever \( \xi \) is a non-zero countable limit ordinal.

(The set of \( x \) for which \( P_{E_{\xi+1}}(u_x) \neq P_{E_{\xi}}(u_x) \) is measurable because \( E_\xi \) and \( E_{\xi+1} \) are separable, as remarked above.) Now there must be a rational \( \delta > 0 \) such that
is infinite, where

\[ U_\xi = \{ x : \| P_{E_{\xi,1}}(u_x) - P_{E_\xi}(u_x) \| \geq \delta \} \]

for each \( \xi < \omega_1 \). But in this case there must be an \( x \in X \) such that

\[ A' = \{ \xi : \xi \in A, x \in U_\xi \} \]

is infinite (A2Mb). Let \( \zeta \) be any cluster point of \( A' \) in \( \omega_1 \). Then

\[ P_{E_\xi}(u_x) = \lim_{\xi \uparrow \zeta} P_{E_\xi}(u_x), \]

which is impossible. \( \textbf{XQ} \)

(c) Applying (b) to both \((u_x)_{x \in X} \) and \((v_y)_{y \in Y} \), we see that there is an \( E \in \mathcal{E} \) such that

\[ P_E(u_x) = P_F(u_x) \text{-} \mu\text{-a.e.}(x), \quad P_E(v_y) = P_F(v_y) \text{-} \nu\text{-a.e.}(y) \]

whenever \( E \subseteq F \in \mathcal{E} \). Now observe that if \( y \in Y \) there is an \( F \in \mathcal{E} \) such that \( E \subseteq F \) and \( v_y \in F \). So

\[ \int (u_x|v_y)\mu(dx) = \int (u_x|P_E(v_y))\mu(dx) \]

Similarly

\[ \int (u_x|v_y)\nu(dy) = \int (P_E(u_x)|v_y)\mu(dx)\nu(dy). \]

This is true for every \( y \). So

\[ \int \int (u_x|v_y)\mu(dx)\nu(dy) = \int \int (P_E(u_x)|P_E(v_y))\mu(dx)\nu(dy). \]

But also, because \( x \mapsto P_E(u_x) \) and \( y \mapsto P_E(v_y) \) are measurable maps to the second-countable space \( E \), and \( (u,v) \mapsto (u|v) : E \times E \to \mathbb{R} \) is continuous, \( (x,y) \mapsto (P_E(u_x)|P_E(v_y)) \) is measurable for the product measure \( \mu \times \nu \), and

\[ \int \int (P_E(u_x)|P_E(v_y))\mu(dx)\nu(dy) = \int \int (P_E(u_x)|P_E(v_y))\nu(dy)\mu(dx) \]

by Fubini’s theorem. Putting these together, we have the result.

5I Corollary Let \((X,\mu), (Y,\nu)\) and \((Z,\Sigma,\sigma)\) be probability spaces. Let \( x \mapsto A_x : X \to \Sigma \) and \( y \mapsto B_y : Y \to \Sigma \) be functions such that

\[ x \mapsto \sigma(A_x \cap C), \quad y \mapsto \sigma(B_y \cap C) \]

are measurable for every \( C \in \Sigma \). Then

\[ \int \int \sigma(A_x \cap B_y)\mu(dx)\nu(dy) = \int \int \sigma(A_x \cap B_y)\nu(dy)\mu(dx). \]

Proof: Apply 5H with \( H = L^2(\sigma) \), \( u_x = \chi(A_x)^* \) (the equivalence class in \( L^2 \) of the characteristic function of \( A_x \)), \( v_y = \chi(B_y)^* \).

5J Theorem Let \( \kappa \) be a real-valued-measurable cardinal. For cardinals \( \lambda \leq \kappa \) let \( \Sigma_\lambda \) be the smallest subalgebra of \( \mathcal{P}(\lambda \times \lambda) \) containing all sets of the form \( E \times F \), where \( E, F \subseteq \lambda \), and closed under unions of fewer than \( \lambda \) of its members. Set \( D_\lambda = \{ (\eta,\zeta) : \eta \leq \zeta < \lambda \} \subseteq \lambda \times \lambda \). Then \( \{ \lambda : \lambda < \kappa \text{ is a cardinal, } D_\lambda \in \Sigma_\lambda \} \)

belongs to the rvm ideal of \( \kappa \), and \( D_\kappa \notin \Sigma_\kappa \).

Proof: The following notation will be useful: If \( \theta \) is a cardinal and \( Z \) is a set, a \( \theta \)-subalgebra of \( \mathcal{P}Z \) will be a subalgebra \( \Sigma \) of \( \mathcal{P}Z \) such that \( \bigcup A \in \Sigma \) whenever \( A \subseteq \Sigma \) and \( \#(A) \leq \theta \).

Set
\[ A = \{ \lambda : \lambda < \kappa \text{ is a cardinal}, \ D_\lambda \in \Sigma_\lambda \}. \]

(a) Suppose, if possible, that \( A \) is not in the rvm ideal of \( \kappa \). Let \( \nu \) be a normal witnessing probability on \( \kappa \) such that \( \nu A = 1 \). Set
\[
A_1 = \{ \lambda : \lambda \in A, \ \lambda \text{ is uncountable and regular}; \]
then \( \nu A_1 = \nu A = 1 \), by 4Ka.

For cardinals \( \lambda, \theta \) let \( \Sigma_\lambda \) be the smallest \( \theta \)-subalgebra of \( P(\lambda \times \lambda) \) containing all rectangles \( E \times F \) for \( E, F \subseteq \lambda \). If \( \lambda \in A_1 \) it is regular, so \( \Sigma_\lambda = \bigcup_{\theta < \lambda} \Sigma_{\lambda^\theta} \), and there is an infinite \( \theta < \lambda \) such that \( D_\lambda \in \Sigma_{\lambda^\theta} \). Because \( \nu \) is normal, there is a \( \theta < \kappa \) such that \( \nu A_2 = 1 \) where
\[
A_2 = \{ \lambda : \lambda \in A_1, \ D_\lambda \in \Sigma_{\lambda^\theta} \}.
\]

(b) For each \( \lambda \in A_2 \) there is a family \( \langle E_{\lambda^\xi} \rangle_{\xi < \theta} \) of subsets of \( \lambda \) such that \( D_\lambda \) belongs to the smallest \( \theta \)-subalgebra \( \Sigma_\lambda \) of \( P(\lambda \times \lambda) \) which contains \( E_{\lambda^\xi} \times E_{\lambda^\eta} \) for all \( \xi, \eta < \theta \). (For the union of all such subalgebras \( \Sigma_\lambda \) is a \( \theta \)-subalgebra of \( P(\lambda \times \lambda) \) and must be \( \Sigma_{\lambda^\theta} \).) Set \( X = \{0, 1\}^\theta \) and define \( f_\lambda : \lambda \to X \) by setting
\[
f_\lambda(\eta)(\xi) = 1 \text{ if } \eta \in E_{\lambda^\xi}, \ 0 \text{ otherwise.}
\]

Let \( \Sigma \) be the smallest \( \theta \)-subalgebra of \( P(X \times X) \) containing all the open-and-closed sets. Then every member of \( \Sigma_\lambda \) is of the form
\[
\{ (\eta, \zeta) : (f_\lambda(\eta), f_\lambda(\zeta)) \in R \}
\]
for some \( R \in \Sigma \) (because sets of this form comprise a \( \theta \)-subalgebra of \( P(\lambda \times \lambda) \) containing all the sets \( E_{\lambda^\xi} \times E_{\lambda^\eta} \)); in particular, there is a set \( R_\lambda \in \Sigma \) such that
\[
D_\lambda = \{ (\eta, \zeta) : (f_\lambda(\eta), f_\lambda(\zeta)) \in R_\lambda \}.
\]

(c) Let \( T \) be the smallest \( \theta^+ \)-subalgebra of \( P(\kappa \times X \times X) \) containing all sets of the form \( B \times U \times V \), where \( B \subseteq \kappa \) and \( U, V \subseteq X \). Then
\[
\{ (\lambda, x, y) : \lambda \in A_2, \ (x, y) \in R_\lambda \}
\]
belongs to \( T \). Define \( H_\gamma \) inductively, for ordinals \( \gamma \leq \theta^+ \), by taking \( H_0 \) to be the family of open-and-closed subsets of \( X \times X \),
if \( \gamma < \theta^+ \) is an even ordinal then
\[
H_{\gamma+1} = \{ \bigcup_{\xi < \theta} H_\xi : H_\xi \in H_\gamma, \ \forall \ \xi < \theta \};
\]
if \( \gamma < \theta^+ \) is an odd ordinal then
\[
H_{\gamma+1} = \{ (X \times X) \setminus H : H \in H_\gamma \};
\]
if \( \gamma \leq \theta^+ \) is a non-zero limit ordinal then
\[
H_\gamma = \bigcup_{\delta < \gamma} H_\delta.
\]
Every \( H_\gamma \) is closed under finite unions and intersections, and \( H_\gamma \subseteq H_{\gamma+2} \cap H_{\gamma+3} \), so \( H_\gamma \) is a subalgebra of \( P(X \times X) \) for limit ordinals \( \gamma \), and \( H_{\theta^+} = \Sigma \).

Next, an easy induction on \( \gamma \) shows that if \( H_\xi \in H_\gamma \) for every \( \xi < \kappa \) then \( \{ (\xi, x, y) : (x, y) \in H_\xi \} \in T \). The induction starts with the observation that any open-and-closed set \( H \subseteq X \times X \) is a finite union of rectangles, so that \( B \times H \in T \) for every \( B \subseteq \kappa \). Now \( H_0 \) has cardinal \( \theta \), so if \( H_\xi \in H_0 \) for every \( \xi < \kappa \) we get
\[
\{ (\xi, x, y) : (x, y) \in H_\xi \} = \bigcup_{H \in H_0} \{ \xi : H_\xi = H \} \times H \in T.
\]
For the inductive step to \( \gamma + 1 \), where \( \gamma \) is even, we have a family \( \{ H_\xi \}_{\xi < \kappa} \) such that each \( H_\xi \) is expressible as \( \bigcup_{H \in H_0} H_\xi \) with every \( H_\xi \) in \( H_\gamma \); now
\[
\{ (\xi, x, y) : (x, y) \in H_\xi \} = \bigcup_{\eta < \theta} \{ (\xi, x, y) : (x, y) \in H_\xi \} \in T,
\]
using the inductive hypothesis and the fact that \( T \) is a \( \theta \)-subalgebra. Similarly, the inductive step to \( \gamma + 1 \), where \( \gamma \) is odd, needs only the fact that \( T \) is closed under complements. Finally, for the inductive step to a
Substituting in the formulae above, we get
\[
S
\]
for all
\[
\eta
\]
so that
\[
S
\in
S
P
\]
subalgebra of
\[
S
\].
Of course every intersection, so
\[
\theta
\]
using the inductive hypothesis and the fact that
\[
T
\]
is a \(\theta^+\)-subalgebra.
Observing that
\[
R_\lambda \in \Sigma = \mathcal{H}_{\theta^+}
\]
for every \(\lambda \in A_2\), we have the result. \(\Box\)

(d) Define two probabilities \(\phi, \psi\) on \(\mathcal{P}(\kappa \times X \times X)\) by setting
\[
\phi(W) = \int \int \nu(\lambda : \lambda \in A_2, \lambda > \max(\eta, \zeta), (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W) \nu(d\eta) \nu(d\zeta),
\]
\[
\psi(W) = \int \int \nu(\lambda : \lambda \in A_2, \lambda > \max(\eta, \zeta), (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W) \nu(d\zeta) \nu(d\eta)
\]
for all \(W \subseteq \kappa \times X \times X\). Then both \(\phi\) and \(\psi\) are \(\kappa\)-additive, and \(\phi(\kappa \times X \times X) = \psi(\kappa \times X \times X) = 1\). Set
\[
\mathcal{S} = \{S : S \subseteq \kappa \times X \times X, \phi(S) = \psi(S)\}.
\]
Then \(\mathcal{S}\) is closed under monotonic and disjoint unions of length less than \(\kappa\), and also under complements.

(e) The key to the proof is the following fact: If \(B \subseteq \kappa\) and \(U, V \subseteq X\) then \(B \times U \times V \in \mathcal{S}\). \(\mathcal{P}\)
Substituting in the formulae above, we get
\[
\phi(B \times U \times V) = \int \int \nu(U_\eta \cap V_\zeta) \nu(d\eta) \nu(d\zeta),
\]
\[
\psi(B \times U \times V) = \int \int \nu(U_\eta \cap V_\zeta) \nu(d\zeta) \nu(d\eta)
\]
where
\[
U_\eta = \{\lambda : \lambda \in B \cap A_2, \eta < \lambda, f_\lambda(\eta) \in U\},
\]
\[
V_\zeta = \{\lambda : \lambda \in A_2, \zeta < \lambda, f_\lambda(\zeta) \in V\}
\]
for \(\eta, \zeta < \kappa\). Now the result is immediate from 5I. \(\Box\)

(f) It follows that \(T \subseteq \mathcal{S}\). \(\mathcal{P}\)
Define \(\mathcal{S}_\gamma\) inductively, for ordinals \(\gamma \leq \theta^+\), as follows. Take \(\mathcal{S}_0\) to be the subalgebra of \(\mathcal{P}(\kappa \times X \times X)\) generated by \(\{B \times U \times V : B \subseteq \kappa, U \subseteq X, V \subseteq X\}\). Then each member of \(\mathcal{S}_0\) is a finite disjoint union of cuboids \(B \times U \times V\), which by (e) above all belong to \(\mathcal{S}\), and \(\mathcal{S}_0 \subseteq \mathcal{S}\). Given \(\mathcal{S}_\gamma\), for an even ordinal \(\gamma\), set
\[
\mathcal{S}_{\gamma+1} = \{\bigcup_{\xi < \alpha} \mathcal{S}_\xi : \alpha \leq \theta^+, \{\mathcal{S}_\xi\}_{\xi < \alpha}\) is an increasing family in \(\mathcal{S}_\gamma\}\};
\]
for odd ordinals \(\gamma\) set
\[
\mathcal{S}_{\gamma+1} = \{(\kappa \times X \times X) \setminus S : S \in \mathcal{S}_\gamma\};
\]
for non-zero limit ordinals \(\gamma\) set
\[
\mathcal{S}_\gamma = \bigcup_{\xi < \gamma} \mathcal{S}_\xi.
\]
Of course every \(\mathcal{S}_\gamma\) is included in \(\mathcal{S}\), because \(\theta^+ < \kappa\). Next, every \(\mathcal{S}_\gamma\) is closed under finite unions and intersections, so \(\mathcal{S}_\gamma\) is a subalgebra of \(\mathcal{P}(\kappa \times X \times X)\) for every limit ordinal \(\gamma\); and finally \(\mathcal{S}_{\theta^+}\) is a subalgebra of \(\mathcal{P}(\kappa \times X \times X)\) such that \(\bigcup_{\xi < \alpha} \mathcal{S}_\xi \in \mathcal{S}_{\theta^+}\), whenever \(\alpha \leq \theta^+\) and \(\{\mathcal{S}_\xi\}_{\xi < \alpha}\) is an increasing family in \(\mathcal{S}_{\theta^+}\). Inducing on \(\alpha\) we see that \(\bigcup_{\xi < \alpha} \mathcal{S}_\xi \in \mathcal{S}_{\theta^+}\), whenever \(\alpha \leq \theta^+\) and \(\{\mathcal{S}_\xi\}_{\xi < \alpha}\) is any family in \(\mathcal{S}_{\theta^+}\); so that \(\mathcal{S}_{\theta^+}\) is a \(\theta^+\)-subalgebra of \(\mathcal{P}(\kappa \times X \times X)\), and
\[
T \subseteq \mathcal{S}_{\theta^+} \subseteq \mathcal{S},
\]
as required. \(\Box\)

(g) Putting (c) and (f) together,
\[
W^* = \{\lambda, x, y) : \lambda \in A_2, (x, y) \in R_\lambda\}
\]
belongs to \(\mathcal{S}\), and \(\phi(W^*) = \psi(W^*)\). But if we set out to compute these numbers we have
\[\nu\{\lambda : \lambda \in A_2, \lambda > \max(\eta, \zeta), (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W^*\}\]
\[= \nu\{\lambda : \lambda \in A_2, \lambda > \max(\eta, \zeta), (f_\lambda(\eta), f_\lambda(\zeta)) \in R_\lambda\}\]
\[= \nu\{\lambda : \lambda \in A_2, \lambda > \max(\eta, \zeta), (\eta, \zeta) \in D_\lambda\}\]
\[= 1\] if \(\eta \leq \zeta\), \(= 0\) otherwise.

So \(\phi(W^*) = 0\) and \(\psi(W^*) = \nu A_2 = 1\); which is impossible. \(\Box\)

(b) This contradiction shows that \(A\) belongs to the rvm ideal of \(\kappa\), which is the first part of the theorem. But the second part now follows. \(?\) For if \(D_\kappa \in \Sigma_\kappa\) then (in the notation of (a)) there is a \(\theta < \kappa\) such that \(D_\kappa \in \Sigma_{\alpha, \theta}\); now as \(D_\lambda = D_\kappa \cap (\lambda \times \lambda)\) and \(\Sigma_{\alpha, \theta} = \{E \cap (\lambda \times \lambda) : E \in \Sigma_{\alpha, \theta}\}\) for every \(\lambda < \kappa\), we get \(D_\lambda \in \Sigma_\lambda\) whenever \(\theta < \lambda < \kappa\), which cannot be so. \(\Box\) Thus \(D_\kappa \notin \Sigma_\kappa\) and we are done.

5K Corollary Let \(\kappa\) be a real-valued-measurable cardinal. Then there is a cardinal \(\lambda < \kappa\) such that there is no family \(\langle a_\eta \rangle_{\eta < \lambda}\) in \(\mathcal{P}\mathcal{N}\) for which \(a_\eta \setminus a_\zeta\) is finite for \(\eta \leq \zeta\), infinite for \(\eta \geq \zeta\).

Proof For if \(\lambda\) is an uncountable cardinal and \(\langle a_\eta \rangle_{\eta < \lambda}\) is such a family, set \(B_n = \{\eta : n \in a_\eta\} \subseteq \lambda\) for each \(n \in \mathbb{N}\). Then, in the notation of 5J,
\[D_\lambda = \bigcup_{n \in \mathbb{N}} \bigcap_{m \geq n} ((\lambda \times B_m) \cup ((\lambda \setminus B_m) \times \lambda)) \in \Sigma_\lambda.
\]
So we need only to take an uncountable \(\lambda < \kappa\) such that \(D_\lambda \notin \Sigma_\lambda\), as provided in abundance by 5J.

5L Remarks (a) 5J-5K are due to Kunen. A version of his original proof (adapted to 5K) is given in Fremlin \& Kunen 91. Seeking a non-forcing alternative be and I independently (late 1989) devised forms of the argument above. I do not know whether 5H was known earlier.

(b) Note that the cardinal \(\lambda\) of 5K also has the property that there can be no family \(\langle f_\xi \rangle_{\xi < \lambda}\) in \(\mathbb{N}^\mathbb{N}\) such that \(\{n : f_\xi(n) < f_\xi(n)\}\) is finite whenever \(\xi \leq \eta < \lambda\), infinite when \(\xi > \eta\); apply 5K with \(a_\xi = \{(i, n) : i \leq f_\xi(n)\} \subseteq \mathbb{N} \times \mathbb{N}\) for each \(\xi < \lambda\). See Fremlin \& Kunen 91.

Evidently there are many results along these lines concerning increasing families in ordered topological spaces of small weight when the ordering has a simple relation to the topology.

(c) In the same way, again taking \(\lambda\) from 5K, there is no \(p(\lambda)\)-point ultrafilter on \(\mathbb{N}\).

Version of 18.9.92

5M Proposition Let \(\kappa\) be a real-valued-measurable cardinal with normal witnessing probability \(\nu\), and \(X\) any set. Let \(\langle f_\xi \rangle_{\xi < \kappa}\) be any family in \(X^\kappa\) such that for every countable \(I \subseteq \kappa\) there is a \(g \in X^I\) such that \(\nu\{\xi : f_\xi[I = g]\} > 0\). Then there is an \(h \in X^\kappa\) such that \(\nu\{\xi : \xi < \kappa, f_\xi[\xi = h] > 0\} > 0\).

Proof (a) Set
\[G_0 = \bigcup\{X^I : I \subseteq [\kappa]^{\omega}\}.\]
For \(g \in G_0\), set \(E(g) = \{\xi : f_\xi \supseteq g\}\); write
\[G = \{g : g \in G_0, \nu E(g) > 0\}.
\]

(b) There is a \(g \in G\) such that for every \(\eta < \kappa\) there is an \(x \in X\) such that \(E(g) \setminus E(g \cup \{(\eta, x)\}) \in \mathcal{N}_\nu\).

If not, choose \(\langle I_\alpha \rangle_{\alpha < \omega_1}, \langle H_\alpha \rangle_{\alpha < \omega_1}\) as follows. Start with \(I_0 = \emptyset\). Given that \(I_\alpha\) is countable, set
\[H_\alpha = \{g : g \in G, \text{dom}(g) = I_\alpha\}.\]
Then \(H_\alpha\) is non-empty, by the hypothesis of the lemma, and countable, because \(E(g) \cap E(g') = \emptyset\) for distinct \(g, g' \in H_\alpha\). Now there must be a countable set \(I_{\alpha + 1} \supseteq I_\alpha\) such that
\[\forall g \in H_\alpha \exists \eta \in I_{\alpha + 1}\] such that \(E(g) \setminus E(g \cup \{(\eta, x)\}) \notin \mathcal{N}_\nu\) for every \(x \in X\).

For limit ordinals \(\alpha < \omega_1\) set \(I_\alpha = \bigcup_{\beta < \alpha} I_\beta\).

For each \(\alpha < \omega_1\), take \(g_\alpha \in H_\alpha\). By A2Mc there is a \(\zeta < \kappa\) such that \(D = \{\alpha : \zeta \in E(g_\alpha)\}\) is uncountable. Now for each \(\alpha \in D\) there is some \(\eta_\alpha \in I_{\alpha + 1}\) such that
\[F_\alpha = E(g_\alpha) \setminus E(g_\alpha \cup \{(\eta_\alpha, f_\alpha(\eta_\alpha))\}) \notin \mathcal{N}_\nu.\]
However, if $\beta \in D$ and $\beta > \alpha$, then $g_\beta \mid I_{\alpha+1} \subseteq g_\beta \mid I_\beta \subseteq f_\gamma$, so $F_\beta \cap F_\alpha = \emptyset$. Thus $(F_\alpha)_{\alpha \in D}$ is an uncountable disjoint family of non-negligible sets, which is impossible. **XQ**

(c) Now for any $\eta < \kappa$ choose $h(\eta) \in X$ such that $F_\eta = E(g) \setminus E(g \cup \{(\eta, h(\eta))\}) \in \mathcal{N}_\nu$. Then we have

$$F = \{\xi : \exists \eta < \xi, \xi \in F_\eta\} \in \mathcal{N}_\nu,$$

because $\nu$ is normal. Set $E = E(g) \setminus F$; then $\nu E > 0$ and $f_\xi(\xi) = h(\xi) \forall \xi \in E$.

5N Theorem Suppose that $\epsilon$ is atomlessly-measurable. Then $\diamondsuit_\epsilon$ is true, that is, there is a family $(A_\xi)_{\xi < \epsilon}$ such that $\{\xi : A \cap \xi = A_\xi\}$ is stationary in $\epsilon$ for every $A \subseteq \epsilon$.

**proof** (a) Let $\nu$ be a normal witnessing probability on $\epsilon$. Let $(\xi_1)_{\xi < \epsilon}, (\xi_2)_{\xi < \epsilon}$ be enumerations of $\bigcup \{0, 1\}^I : I \in [\omega]^{\omega}$, respectively. Define inductively a family $(f_\xi)_{\xi < \epsilon}$ as follows. Given $(f_\eta)_{\eta < \xi}$, let $f_\xi$ be the set of those functions $f : \xi \to \{0, 1\}$ such that

(i) $f | \eta \neq f_\eta \forall \eta < \xi$,

(ii) for every $\eta < \xi$ there is a $\zeta < \xi$ such that $f | I_\eta = p_\zeta$.

Now if $F_\xi \neq 0$ take $f_\xi \in F_\xi$; otherwise take $f_\xi$ to be any function from $\xi$ to $\{0, 1\}$. Set $A_\xi = f_\xi^{-1}\{1\}$ for each $\xi < \epsilon$.

(b) If $A \subseteq \epsilon$ is any set, then there is a non-zero limit ordinal $\eta < \xi$ such that $A \cap \xi = A_\xi$. **P?** Suppose, if possible, otherwise. Set $g(\xi) = 1$ for $\xi \in A$, $0$ for $\xi \in \epsilon \setminus A$. Let $C$ be the set of non-zero limit ordinals $\xi < \epsilon$ such that for every $\eta < \xi$ there is a $\zeta < \xi$ such that $g[I_\eta \cap \xi] = p_\zeta$. Then $C$ is a closed unbounded set in $\epsilon$, and $g[\xi] \in F_\xi$ for every $\xi \in C$, so $F_\xi \subseteq F_\xi$ for every $\xi \in C$.

For any set $J = \{\zeta, \eta \in [\epsilon]^2, \min \{\zeta : f_\zeta(\xi) \neq f_\eta(\zeta)\} \}$, then $h(J) < \min J$ because $\eta < \xi$ then $f_\zeta(\eta) = f_\xi(\eta)$. By 5Cc above there are $D \subseteq C, M \subseteq \epsilon$ such that $\nu D = 1, M$ is countable and $h(J) \subseteq M$ for all $J \subseteq [\epsilon]^2$. Let $\eta < \epsilon$ be such that $M = I_\eta$; set $B = \{\xi : M \cup \eta \subseteq \xi \subseteq D\}$, so that $\nu B = 1$. Then for every $\xi \in B$ we shall have a $\zeta(\xi) < \xi$ such that $p(\zeta(\xi)) = f_\xi[M]$. But if $\zeta, \zeta' \in B$ we must have $h((\zeta, \zeta')) \subseteq M$ so $f_\xi[M] \neq f_{\zeta'}[M]$; thus $\xi \mapsto \zeta(\xi) : B \to \kappa$ is an injective regressive function, which is impossible. **XQ**

(c) Of course the family $(A_\xi)_{\xi < \epsilon}$ constructed in (a) is not necessarily a true $\diamondsuit$-sequence as called for in the statement of the theorem. But if we set $A_\xi = \{\eta : 2\eta \in A_\xi\}$ for each $\xi$, we obtain such a sequence (see DEVLIN 84, Ex. III.3A).

**Remark** 5M-5N are due to Kunen. Of course $\diamondsuit_\kappa$ is true for every two-valued-measurable cardinal $\kappa$, by the same argument. See also 9N.

5O Theorem Let $\kappa$ be a real-valued-measurable cardinal. Then

(a) $\kappa \to (\kappa, \gamma)^2$ for every ordinal $\gamma < \omega_1$;

(b) the set

$$\{\alpha : \alpha < \kappa, \alpha \to (\alpha, \gamma)^2 \text{ for every countable ordinal } \gamma\}$$

belongs to the rvm filter of $\kappa$.

**Notation** See A15 for the definition of $\alpha \to (\alpha, \beta)^2$.

**proof** (a) Let $\gamma$ be a countable ordinal and $S \subseteq [\kappa]^2$ any set. Suppose that there is no set $B \subseteq \kappa$ of order type $\kappa$ such that $[B]^2 \cap S = \emptyset$; I seek a set $C$ of order type $\gamma$ such that $[C]^2 \subseteq S$. For each $\xi < \kappa$ set $S_\xi = \{\eta : \eta < \xi, \{\xi, \eta\} \in S\}$. Fix a normal witnessing probability $\nu$ on $\kappa$.

(i) If $E \subseteq \kappa$ then $\nu(E \cap S_\xi) > 0$ for $\nu$-almost every $\xi \in E$. **P** Set

$$E' = \{\xi : \xi \in E, \nu(E \cap S_\xi) = 0\},$$

$$A = \{\xi : \xi < \kappa, \xi \notin E \cap S_\eta \forall \eta \in \xi \cap E'\};$$

then $\nu A = 1$ (because $\nu$ is normal), and also $[A \cap E']^2 \cap S = \emptyset$, so $\nu(A \cap E') = 0$ and $\nu E' = 0$. **Q**

(ii) For each $n \geq 1$ define a $\kappa$-additive probability $\nu_n$ on $[\kappa]^n$ by setting

$$\nu_n V = \int \cdots \int \chi V(\xi_1, \ldots, \xi_n) \nu(d\xi_n) \cdots \nu(d\xi_1)$$

where $V : \kappa \to [0, 1]$ is a given function that is countably additive.
for every $V \subseteq \kappa^n$. Observe that if $V \subseteq \kappa^{n+1}$ then
\[ \nu_{n+1} V = \int \nu_n \{ t : \xi^- t \in V \} \nu(d\xi), \]
writing $\xi^-(\xi_1, \ldots, \xi_n) = (\xi, \xi_1, \ldots, \xi_n)$.

(iii) (The key) If $D, F \subseteq \kappa$ and $n \geq 1$ and
\[ \nu_{n+1}(S_{\kappa}^{n+1} \setminus \bigcup_{\eta \in D} S_{\eta}^{n+1}) = 0 \quad \forall \xi \in F, \]
then for $\nu$-almost every $\xi \in F$
\[ \nu_n(S_{\xi}^n \setminus \bigcup \{ S_{\eta}^n : \eta \in D, \xi \in S_{\eta} \}) = 0. \]

**P**? If not, there is a $\delta > 0$ such that $\nu E > \delta$, where
\[ E = \{ \xi : \xi \in F, \nu_n(S_{\xi}^n \setminus \bigcup \{ S_{\eta}^n : \eta \in D, \xi \in S_{\eta} \}) \geq \delta \}. \]
Choose a sequence $\langle \xi_i \rangle_{i \in \mathbb{N}}$ as follows. Start with any $\xi_0 \in E$ such that $\nu(E \cap S_{\xi_0}) > 0$; such exists by (i) above. Given $\xi_0, \ldots, \xi_k$ in $F$ such that $\nu(E \cap \bigcap_{i \leq k} S_{\xi_i}) > 0$, then
\[ \nu_{n+1}(\bigcup_{i \leq k} S_{\xi_i}^{n+1} \setminus \bigcup_{\eta \in D} S_{\eta}^{n+1}) = 0, \]
so for $\nu$-almost all $\xi$
\[ \nu_n(\bigcup \{ S_{\xi_i}^n : i \leq k, \xi \in S_{\xi_i} \} \setminus \bigcup \{ S_{\eta}^n : \eta \in D, \xi \in S_{\eta} \}) = 0. \]
Now for $\nu$-almost all $\xi \in \bigcap_{i \leq k} S_{\xi_i}$,
\[ \nu_n(\bigcup_{i \leq k} S_{\xi_i}^n \setminus \bigcup \{ S_{\eta}^n : \eta \in D, \xi \in S_{\eta} \}) = 0, \]
and for $\nu$-almost all $\xi \in E \cap \bigcap_{i \leq k} S_{\xi_i}$, we have
\[ \nu_n(S_{\xi}^n \setminus \bigcup_{i \leq k} S_{\xi_i}^n) \geq \delta. \]
We can therefore find a $\xi_{k+1} \in E \cap \bigcap_{i \leq k} S_{\xi_i}$ such that
\[ \nu(S_{\xi_{k+1}} \cap E \cap \bigcap_{i \leq k} S_{\xi_i}) > 0, \]
\[ \nu_n(S_{\xi_{k+1}}^{n+1} \setminus \bigcup_{i \leq k} S_{\xi_i}^n) \geq \delta \]
(using (i) again). Continue.

But we now have a disjoint sequence $\langle S_{\xi_i}^n \setminus \bigcup_{i \leq k} S_{\xi_i}^n \rangle_{i \in \mathbb{N}}$ of subsets of $\kappa^n$ all of measure at least $\delta$, which is impossible. **XQ**

(iv) If $n \geq 1$ and $V \subseteq \kappa^n$ is $\nu_n$-negligible, then there is a set $A \subseteq \kappa$ such that $\nu A = 1$ and $(\xi_1, \ldots, \xi_n) \notin V$ whenever $\xi_1, \ldots, \xi_n \in A$ and $\xi_1 < \ldots < \xi_n$. **P** Induce on $n$. For $n = 1$ we may take $A = \kappa \setminus V$. For the inductive step to $n + 1 \geq 2$, set
\[ E = \{ \xi : \nu_n \{ t : \xi^- t \in V \} > 0 \}, \]
so that $\nu E = 0$. For each $\xi \in \kappa \setminus E$, set $V_\xi = \{ t : \xi^- t \in V \}$; then $\nu_n V_\xi = 0$ so by the inductive hypothesis there is a set $A_\xi \subseteq \kappa$ such that $\nu A_\xi = 1$ and no strictly increasing sequence in $A_\xi$ can belong to $V_\xi$. Set
\[ A = \{ \xi : \xi \in \kappa \setminus E, \xi \in A_\eta \quad \forall \eta \in \xi \setminus E \}. \]
Then $\nu A = 1$ and no strictly increasing family in $A$ can belong to $V$. **Q**

(v) For each $I \in [\kappa]^<\omega$ set
\[ R_I = \kappa \cap \bigcap_{\xi \in I} S_\xi, \quad R'_I = \{ \xi : \xi \in R_I, \nu(R_I \cap S_\xi) = 0 \}, \]
so that $\nu R'_I = 0$, by (i) once again. Set
\[ A = \{ \xi : \xi < \kappa, \xi \notin R'_I \quad \forall \ I \in [\kappa]^<\omega \}; \]
then $\nu A = 1$ (see A1E(c-iv)). And an easy induction on $\sup I$ shows that $\nu R_I > 0$ whenever $I \in [A]^<\omega$ and $R_I \neq \emptyset$. **Q**
such that 

\[ F \subseteq A \cap R_I. \]

Let \( D \subseteq A \) be a countable set such that

\[ \nu_n(\bigcup_{\xi \in D \cap \xi} S^n_{\xi}) = \sup\{\nu_n(\bigcup_{\xi \in D \subseteq F} S^\omega_{\xi}) : D \subseteq F \} \]

for every \( I \in [D']^\omega \), \( n \geq 1 \). Then we shall have

\[ \nu_n(S^n_{\xi} \setminus \bigcup_{\eta \in D \cap F} S^\omega_n) = 0 \]

whenever \( n \geq 1 \), \( I \in [D']^\omega \) and \( \xi \in F_{\xi I} \). Consequently, by (iii) above, if \( I \in [D']^\omega \) and \( n \geq 1 \), then

\[ \nu_n(S^n_{\xi} \setminus \bigcup_{\eta \in D \cap F_{\xi I}, \xi \in S_n}) = 0 \]

for almost every \( \xi \in F_{\xi I} \). Using (iv) we can now find a set \( A_{\xi+1} \subseteq A_\xi \) such that

\[ \nu A_{\xi+1} = 1, \]

\[ D_\xi \subseteq \min A_{\xi+1}, \]

\[ \nu_n(S^n_{\xi} \setminus \bigcup_{\eta \in D \cap F_{\xi I}, \xi \in S_n}) = 0 \]

whenever \( I \in [D']^\omega \), \( n \geq 1 \), \( \xi \in F_{\xi I} \cap A_{\xi+1}, \)

\[ (\xi_1, \ldots, \xi_n) \notin S^n_{\xi_0} \setminus \bigcup_{\eta \in F_{\xi I} \cap D_\xi, \xi_0 \in S_\eta} \]

whenever \( I \in [D']^\omega \), \( n \geq 1 \) and \( \xi_0, \ldots, \xi_n \) is a strictly increasing family in \( A_{\xi+1} \), with \( \xi_0 \in F_{\xi I} \).

This deals with \( D_\xi, A_{\xi+1} \). For non-zero countable limit ordinals \( \zeta \) set \( A_\zeta = \bigcap_{\eta < \zeta} A_\eta \).

(vii) On completing the induction set \( D^* = \bigcup_{\xi < \omega_1} D_\xi \); observe that \( D_\zeta = D^* \cap \alpha_{\zeta+1} \setminus \alpha_\zeta \) and that \( D^* \setminus \alpha_\zeta \subseteq A_\xi \) for each \( \zeta \). Now suppose that \( J \in [D^*]^\omega \) is such that \( [J]^2 \subseteq S \). Then for any \( \zeta < \omega_1 \) there is an \( \eta \in D_\zeta \) such that \( [J \cup \{\eta\}]^2 \subseteq S \). Of course we may suppose that \( J \setminus \alpha_\zeta = \emptyset \). Moreover, because \( J \subseteq A, \nu R_J > 0 \); so if \( J \subseteq D_\zeta \) then \( \nu F_{\xi I} > 0 \) so there is an \( \eta \in F_{\xi I} \) with \( \nu S_n > 0 \) and there must be an \( \eta' \in D_\zeta \cap F_{\xi I}, \) in which case \( [J \cup \{\eta'\}]^2 \subseteq S \). We may therefore suppose that \( J \setminus \alpha_\zeta = J \setminus \alpha_{\zeta+1} \) has at least two members. Set \( I = J \setminus \alpha_\zeta \) and enumerate \( J \setminus I \) in ascending order as \( \langle \xi_i \rangle_{i \leq n} \), where \( n \geq 1 \) and \( \xi_i \in A_{\xi+1} \) for each \( i \). Now \( \xi_0 \in F_{\xi I} \) and

\[ (\xi_1, \ldots, \xi_n) \notin S^n_{\xi_0} \setminus \bigcup_{\eta \in F_{\xi I} \cap D_\xi, \xi_0 \in S_\eta} \]

But as certainly \( (\xi_1, \ldots, \xi_n) \in S^n_{\xi_0} \), there must be some \( \eta \in F_{\xi I} \cap D_\zeta \) such that \( (\xi_0, \ldots, \xi_n) \in S^n_\eta \); and this means that \( [J \cup \{\eta\}]^2 \subseteq S \), as required. \( \Box \)

(viii) Now, at last, turn to look at \( \gamma \). Enumerate it as \( \langle \xi_n \rangle_{n \in \mathbb{N}} \) (the case of finite \( \gamma \) is trivial). Choose \( \langle \xi_n \rangle_{n \in \mathbb{N}} \) inductively so that \( \xi_n \in D_{\xi_n} \) and \( [\xi_i : i \leq n]]^2 \subseteq S \) for each \( n \). Set \( C = \langle \xi_n : n \in \mathbb{N} \rangle \); then \( [C]^2 \subseteq S \) and the order type of \( C \) is \( \gamma \).

As \( S \) and \( \gamma \) are arbitrary, this proves (a).

(b) This now follows. The statement

\[ \forall \alpha < \gamma \wedge < \omega \gamma \]

can readily be expressed in the form

\[ \forall \alpha < \gamma \wedge < \omega \gamma \]

where \( \phi \) is a \( \Pi_1^1 \) formula. So 4Ob gives the result.

5P Remarks This is due to Kunen; I heard of it first from S.Todorčević.

For two-valued-measurable cardinals enormously more can be said. In fact, if \( \kappa \) is two-valued-measurable, then \( \kappa \rightarrow (\kappa, \kappa)^2 \) and \( \langle \alpha : \alpha < \kappa, \alpha \rightarrow (\alpha, \alpha)^2 \rangle \) belongs to the rvm filter of \( \kappa \); this is because, for \( \alpha > \omega, \alpha \rightarrow (\alpha, \alpha)^2 \) if \( \alpha \) is a strongly inaccessible weakly \( \Pi_1^1 \)-indescribable cardinal (ERDŐS HAJNAL MÁTÉ & RADO, 30.3).

I do not know whether (b) above can be strengthened by describing more exactly those \( \alpha < \kappa \) for which \( \alpha \rightarrow (\alpha, \gamma)^2 \) for every \( \gamma < \omega_1 \).

In this section I discuss the consequences in measure theory of supposing that there is a real-valued-measurable cardinal. Naturally many of these involve the supposed cardinal and its witnessing measure, and they are most interesting if the cardinal is atomlessly-measurable. A theorem which would be here if it had not already been needed is 3A. I start with a similar result on reversing the order of integration (6A).

An elaboration of the same techniques gives some results analogous to those of 5A-5B, but for set-valued functions whose values are small in a different sense (6D-6E). In 6B-6C I look at covering numbers for null ideals, and in 6F-6G I look at small non-negligible sets; the latter analysis leads to a version of Shipman’s theorem on changing the order of integration in a multiply-repeated integral (6I) and to a stronger result on repeated integrals of functions with measurable sections (6K). A deeper look at covering numbers gives a description of weakly $\Pi^1_1$-indescribable atomlessly-measurable cardinals (6L). I conclude with descriptions of the way in which real-valued-measurable cardinals appear in the theories of metric measure spaces and vector lattices (6M-6N).

6A Theorem Let $\kappa$ be a real-valued-measurable cardinal and $\nu$ a normal witnessing probability on $\kappa$; let $(X, \mu)$ be a Radon probability space and $f : X \times \kappa \to \mathbb{R}$ a bounded function. Then

$$\int (\int f(x, \xi)\nu(d\xi))\mu(dx) \leq \int (\int f(x, \xi)\mu(dx))\nu(d\xi).$$

proof Suppose, if possible, otherwise. Adding a constant function to $f$, if necessary, we may suppose that $f(x, \xi) \geq 0$ for all $x, \xi$.

(a) We are supposing that there is a $\mu$-integrable function $g : X \to \mathbb{R}$ such that $0 \leq g(x) \leq \int f(x, \xi)\nu(d\xi)$ for every $x \in X$ and

$$\int g(x)\mu(dx) > \int \int f(x, \xi)\mu(dx)\nu(d\xi).$$

Let $F_0, \ldots, F_n$ be disjoint non-empty measurable subsets of $X$ such that

$$\sum_{i\leq n} t_i \mu F_i > \int \int f(x, \xi)\mu(dx)\nu(d\xi),$$

writing $t_i = \inf_{x \in F_i} g(x)$ for each $i \leq n$. Then

$$\int f(x, \xi)\mu(dx) \geq \sum_{i\leq n} \int F_i f(x, \xi)\mu(dx) \quad \forall \xi < \kappa,$$

so there must be some $i \leq n$ such that

$$t_i \mu F_i > \int F_i f(x, \xi)\mu(dx)\nu(d\xi).$$

Set $Y = F_i$, $\tilde{\mu} = (\mu F_i)^{-1}\mu[F_i, t = t_i]$; then $(Y, \tilde{\mu})$ is a Radon probability space and

$$\int \int f(y, \xi)\tilde{\mu}(dy)\nu(d\xi) < t,$$

$$t \leq \int f(y, \xi)\nu(d\xi) \text{ for every } y \in Y.$$

(b) For each $\xi < \kappa$ choose a $\tilde{\mu}$-measurable function $h_\xi : Y \to \mathbb{R}$ such that $f(y, \xi) \leq h_\xi(y)$ for every $y \in Y$ and

$$\int h_\xi(y)\tilde{\mu}(dy) = \int f(y, \xi)\tilde{\mu}(dy).$$

By A2Kb, applied to the family of sets of the form $\{y : h_\xi(y) \geq s\}$ for $\xi < \kappa$ and rational $s$, there is a function $\phi : \{0, 1\}^\kappa \to Y$ such that

$$\int h_\xi(\phi(v))\mu(v)(dv)$$

exists and is equal to $\int h_\xi(y)\tilde{\mu}(dy)$ for every $\xi < \kappa$, taking $\mu$ to be the usual measure on $\{0, 1\}^\kappa$. So setting $f_1(v, \xi) = f(\phi(v), \xi)$ for $v \in \{0, 1\}^\kappa$, $\xi < \kappa$ we have
\[ \int h_\xi(v,\xi) \mu_\kappa(dv) \nu(d\xi) \leq \int h_\xi(v) \mu_\kappa(dv) \nu(d\xi) \]
\[ = \int h_\xi(y) \mu_\gamma(dy) \nu(d\xi) \]
\[ = \int f(y,\xi) \mu_\gamma(dy) \nu(d\xi) < t, \]

while

\[ t \leq \int f(\phi(v),\xi) \nu(d\xi) = \int f_1(v,\xi) \nu(d\xi) \]

for every \( v \in \{0,1\}^\kappa \).

**c** We may choose for each \( \xi < \kappa \) a Baire measurable function \( h'_\xi : \{0,1\}^\kappa \to \mathbb{R} \) such that \( f_1(v,\xi) \leq h'_\xi(v) \) for each \( v \in \{0,1\}^\kappa \) and

\[ \int h'_\xi(v) \mu_\kappa(dv) = \int f_1(v,\xi) \mu_\kappa(dv) \]

(A2Gf). Now there is a countable set \( I_\xi \subseteq \kappa \) such that \( h'_\xi \) factors through \( \{0,1\}^{I_\xi} \), that is, \( h'_\xi(v) = h'_\xi(v') \) whenever \( v|I_\xi = v'|I_\xi \).

By 5Cb, there are \( \Gamma \subseteq \kappa, \gamma < \kappa \) such that \( \nu \Gamma = 1 \) and \( I_\xi \cap I_\eta \subseteq \gamma \) whenever \( \xi, \eta \) are distinct members of \( \Gamma \). Set \( \gamma = \min \Gamma \).

**d** Set

\[ f'_1(u,\xi) = \int h'_\xi(v,u') \mu_{\kappa\chi \gamma}(dv) \]

for \( u \in \{0,1\}^\gamma, \xi < \kappa \). Then, applying Fubini’s theorem to \( \{0,1\}^\kappa \equiv \{0,1\}^\gamma \times \{0,1\}^{\kappa\gamma} \), we have

\[ \int f'_1(u,\xi) \mu_\gamma(du) = \int h'_\xi(v) \mu_\kappa(dv) \]

for every \( \xi \), so that

\[ \int \int f'_1(u,\xi) \mu_\gamma(du) \nu(d\xi) = \int \int f_1(v,\xi) \mu_\kappa(dv) \nu(d\xi) < t, \]

and

\[ \int \int f'_1(u,\xi) \nu(du) \mu_\gamma(d\xi) < t \]

by Theorem 3A. Accordingly there is a \( u \in \{0,1\}^\gamma \) such that

\[ \int f'_1(u,\xi) \nu(d\xi) < t. \]

**e** For each \( \xi \in \Gamma \) take \( u'_\xi \in \{0,1\}^{\kappa\gamma} \) such that \( h'_\xi(u\wedge u'_\xi) \leq f'_1(u,\xi) \). Let \( w \in \{0,1\}^\kappa \) be such that

\[ w|\gamma = u, w|I_\xi = (u\wedge u'_\xi)|I_\xi \quad \forall \xi \in \Gamma; \]

such a \( w \) exists because if \( \xi, \eta \in \Gamma \) and \( \xi < \eta \) then \( I_\xi \cap I_\eta \subseteq \gamma \). Now

\[ f_1(w,\xi) \leq h'_\xi(w) = h'_\xi(u\wedge u'_\xi) \leq f'_1(u,\xi) \]

for every \( \xi \in \Gamma \), so

\[ \int f_1(w,\xi) \nu(d\xi) \leq \int f'_1(u,\xi) \nu(d\xi) < t, \]

contradicting the last sentence of (b) above. \( \blacksquare \)

This completes the proof.

**Remark** This result was inspired by Lemma 5 of Kunen N70; compare 6E below.

**6B Proposition** Let \( \kappa \) be an atomlessly-measurable cardinal. If \( (X,\mu) \) is any Radon measure space with \( \mu X > 0 \), then \( \text{cov}(X,N_\mu) \geq \kappa \).
proof By A2Pb, it is enough to show that \( \text{cov}\{\{0,1\}^\kappa, N_{\mu_\kappa}\} \geq \kappa \), where \( \mu_\kappa \) is the usual measure on \( \{0,1\}^\kappa \). Fix on an atomless \( \kappa \)-additive probability \( \nu \) with domain \( P\kappa \). By 3H, there is an inverse-measure-preserving function \( f: \kappa \to \{0,1\}^\kappa \). Now if \( A \) is any cover of \( \{0,1\}^\kappa \) by \( \mu_\kappa \)-negligible sets, \( \{f^{-1}[A]: A \in A\} \) is a cover of \( \kappa \) by \( \nu \)-negligible sets, so must have cardinal at least \( \kappa \), and \( \#(A) \geq \kappa \), as required.

6C Corollary If \( \kappa \) is an atomlessly-measurable cardinal and \( \lambda \leq \kappa \) is a cardinal of uncountable cardinality, then \( \lambda \) is a precaliber of every probability algebra.

proof If \( \lambda < \kappa \) this is a corollary of 6B and A2Ua. If \( \lambda = \kappa \), we can use 6A and A2Ub. For let \( (X, \Sigma, \mu) \) be a Radon probability space and \( (E_\xi)_{\xi < \kappa} \) an increasing family in \( N_\mu \) with union \( E \in \Sigma \). Set
\[
C = \{(x, \xi): \xi < \kappa, x \in E_\xi\} \subseteq X \times \kappa.
\]
Then
\[
\int \nu C([x]) \mu(dx) = \mu E, \int \mu^* C^{-1}(\xi) \nu(d\xi) = 0,
\]
so 6A, applied to the characteristic function of \( C \), tells us that \( \mu E = 0 \); now A2Ub tells us that \( \kappa \) is a precaliber of the measure algebra of \( X \). But as every probability algebra is (isomorphic to) the measure algebra of some Radon probability space (A2La), we have the result.

6D Lemma Let \( \kappa \) be a real-valued-measurable cardinal and \( \nu \) a normal witnessing probability on \( \kappa \). If \( (X, \mu) \) is any quasi-Radon probability space of weight\(^1\) strictly less than \( \kappa \), and \( f: [\kappa]^{<\omega} \to N_\mu \) is any function, then
\[
\bigcap_{\nu \subseteq \kappa, \nu_\mu \nu} \bigcup_{\nu_\mu \nu} f(I) \in N_\mu.
\]

proof Let \( \mathcal{F} \) be the filter \( \{A: A \subseteq \kappa, \nu A = 1\} \).

(a) I show by induction on \( n \in \mathbb{N} \) that if \( g: [\kappa]^{<n} \to N_\mu \) is any function, then
\[
E(g) = \bigcap_{\nu_\mu \nu} \bigcup_{\nu_\mu \nu} f(I) \in N_\mu.
\]

(i) For \( n = 0 \) this is trivial: \( E(g) = g(\emptyset) \in N_\mu \). (ii) For the inductive step to \( n+1 \), given \( g: [\kappa]^{<n+1} \to N_\mu \), then for each \( \xi < \kappa \) define \( g_\xi: [\kappa]^{<n} \to N_\mu \) by setting \( g_\xi(I) = g(I \cup \{\xi\}) \) for each \( I \in [\kappa]^{<n} \). By the inductive hypothesis, \( E(g_\xi) \in N_\mu \). Set
\[
C = \{(x, \xi): x \in E(g_\xi)\} \subseteq X \times \kappa.
\]
Then
\[
\int \mu^* C^{-1}(\xi) \nu(d\xi) = \int \mu^* E(g_\xi) \nu(d\xi) = 0,
\]
so by 3A
\[
\int \nu C([x]) \mu(dx) = 0,
\]
and \( \mu D = 0 \), where \( D = \{g(\emptyset)\} \cup \{x: \nu C([x]) > 0\} \).

Take any \( x \in X \setminus D \) and set \( W = \kappa \setminus C([x]) \in \mathcal{F} \). For each \( \xi \in W \), \( x \notin E(g_\xi) \), so there is a \( V_\xi \in \mathcal{F} \) such that \( \nu V_\xi = 1 \) and \( x \notin g_\xi(I) \) for every \( I \in [V_\xi]^{<n} \). Set
\[
V = \{\xi: \xi \in W, \xi \in V_\eta, \forall \eta \in W \cap \xi\}.
\]
Then \( V \in \mathcal{F} \). If \( I \in [V]^{<n+1} \), either \( I = \emptyset \) and \( x \notin g(I) \), or there is a least element \( \xi \) of \( I \); in the latter case, \( \xi \in W \) so \( I = I \setminus \{\xi\} \subseteq V_\xi \) and \( x \notin g_\xi(J) = g(I) \). So \( x \notin \bigcup\{g(I): I \in [V]^{<n+1}\} \). As \( x \) is arbitrary, \( E(g) \subseteq D \in N_\mu \) and the induction proceeds. Q \( \Box \)

(b) Now consider
\[
G = \bigcup_{n \in \mathbb{N}} E(f)[[\kappa]^{<n}] \subseteq N_\mu.
\]
If \( x \in X \setminus G \) then for each \( n \in \mathbb{N} \) there is a \( V_n \in \mathcal{F} \) such that \( x \notin \bigcup\{f(I): I \in [V_n]^{<n}\} \). Set \( V = \bigcap_{n \in \mathbb{N}} V_n \in \mathcal{F} \); then \( x \notin \bigcup\{f(I): I \in [V]^{<\omega}\} \). As \( x \) is arbitrary,

\(^1\)In the published version 'Maharam type' was given instead of 'weight'. I do not know if the result is true with this hypothesis.
Let $\kappa$ be a real-valued-measurable cardinal with a normal witnessing probability $\nu$. If $(X, \mu)$ is a Radon probability space, $f : [\kappa]^{<\omega} \to \mathcal{N}_\mu$ is a function and $\beta < \kappa$, then there is a $V \subset \kappa$ such that $\beta \subset V$ and $\nu V = 1$ and $\bigcup \{ f(I) : I \in [V]^{<\omega} \} \neq X$.

**proof** (a) Consider first the case $(X, \mu) = ([0,1]^k, \mu_k)$, where $\mu_k$ is the usual measure on $[0,1]^k$. For any $L \subset \kappa$ let $\mu_L$ be the usual measure on $[0,1]^L$, and $\pi_L : [0,1]^k \to [0,1]^L$ the canonical map. Write $\mathcal{F} = \{ V : V \subset \kappa, \nu V = 1 \}$.

(i) For each $I \in [\kappa]^{<\omega}$, there is a countable set $g(I) \subset \kappa$ such that $\mu_{g(I)}(\pi_{g(I)}[f(I)]) = 0$ (see A2Gc); enlarging $f(I)$ if necessary, we may suppose that $f(I) = \pi_{g(I)}^{-1}[\pi_{g(I)}[f(I)]]$. Set $\theta = \max(\omega, \#(\beta))$ and $g^*(I) = \bigcup \{ g(I \cap K) : K \in [\beta]^{<\omega} \}$ for each $I \in [\kappa]^{<\omega}$. By 5C there is a set $C \in \mathcal{F}$ and a function $h : [\kappa]^{<\omega} \to [\kappa]^{\leq \theta}$ such that $g^*(I) \cap \eta \subset h(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$. Set

$$\Gamma = \{ \gamma : \beta \leq \gamma < \kappa, h(I) \subset \gamma \cap I \in [\gamma]^{<\omega} \} \cup \{ 0 \};$$

then $\Gamma$ is a closed unbounded set in $\kappa$, because $cl(\kappa) > \theta$. Let $(\gamma_{n})_{n<\kappa}$ be the increasing enumeration of $\Gamma$; note that $\gamma_{0} = 0$ and $\gamma_{1} \geq \beta$.

(ii) For $\eta < \kappa$, set $M(\eta) = \kappa \setminus \gamma_{\eta}$ and $L(\eta) = \gamma_{\eta+1} \setminus \gamma_{\eta}$; then $\mu_{M(\eta)}$ can be identified with the product measure $\mu_{L(\eta)} \times \mu_{M(\eta+1)}$. Choose $u_{\eta} \in \{ 0,1 \}^{\gamma_{\eta}}$, $v_{\eta} \subset \kappa$ inductively, as follows. $u_{0} = \emptyset$. Given $u_{\eta}$, then for each $I \in [\kappa]^{<\omega}$ set

$$f_{\eta}^0(I) = \{ v : v \in [0,1]^{L(\eta)}, \mu_{M(\eta+1)}\{ w : u_{\eta}^w \cdot v \cdot w \in f(I) \} > 0 \},$$

and

$$f_{\eta}(I) = f_{\eta}^0(I) \text{ if } \mu_{L(\eta)}(f_{\eta}^0(I)) = 0,$$

$$= \emptyset \text{ otherwise.}$$

By 6D, we can find for each $K \in [\gamma_{n+1}]^{<\omega}$ a set $E_{nK} \subset [0,1]^{L(n)}$ such that $\mu_{L(n)}E_{nK} = 1$ and for every $v \in E_{nK}$ there is a set $V \in \mathcal{F}$ such that $v \notin f_{\eta}(K \cup J)$ for any $J \in [V]^{<\omega}$. Choose $v_{\eta} \in \bigcap \{ E_{nK} : K \in [\gamma_{n+1}]^{<\omega} \}$ (using 6D); for $K \in [\gamma_{n+1}]^{<\omega}$ choose $V_{nK} \in \mathcal{F}$ such that $v_{\eta} \notin f_{\eta}(K \cup J)$ for any $J \in [V_{nK}]^{<\omega}$. Set $V_{0} = \bigcap \{ V_{nK} : K \in [\gamma_{1}]^{<\omega} \} \in \mathcal{F}$ and $u_{\eta+1} = u_{\eta} v_{\eta} \in [0,1]^{\gamma_{n+1}}$.

(iii) Now consider $u = u_{\kappa} \in \{ 0,1 \}^{\kappa}$ and $V = \beta \cup \{ \xi : \xi \in C, \xi \in V_{\eta} \forall \eta < \xi \} \in \mathcal{F}$.

If $I \in [V]^{<\omega}$ then

$$\mu_{M(\eta)}\{ w : u_{\kappa}^w \cdot w \in f(I) \} = 0$$

for every $\eta < \kappa$. P Induce on $\eta$. For $\eta = 0$ this says just that $\mu_{\kappa} f(I) = 0$, which was our hypothesis on $f$. For the inductive step to $\eta + 1$, we have

$$\mu_{M(\eta)}\{ w : u_{\kappa}^w \cdot w \in f(I) \} = 0$$

by the inductive hypothesis, so Fubini’s theorem tells us that

$$\mu_{L(\eta)}\{ v : \mu_{M(\eta+1)}\{ w : u_{\kappa}^w \cdot v \cdot w \in f(I) \} > 0 \} = 0,$$

that is, $\mu_{L(\eta)}f_{\eta}^1(I) = 0$, so that $f_{\eta}(I) = f_{\eta}^0(I)$. Now setting $K = I \cap \gamma_{\eta+1}$, $J = I \setminus \gamma_{\eta+1}$ we see that $J \subset V_{\eta}$ (because of course $\eta < \gamma_{\eta+1}$, while $\beta \subset \gamma_{\eta+1}$), therefore $J \subset V_{nK}$ and $v_{\eta} \notin f_{\eta}(K \cup J) = f_{\eta}^0(I)$; but this says just that

$$\mu_{M(\eta+1)}\{ w : u_{\kappa}^w v_{\eta} w \in f(I) \} = 0,$$

that is, that

$$E(f) \subset G \in \mathcal{N}_\mu,$$

as required.

**Remark** This is Lemma 2 of Kunen N70.
so that the induction continues.

For the inductive step to a non-zero limit ordinal $\eta \leq \kappa$, there is a non-zero $\zeta < \eta$ such that $I \cap \gamma_\eta \subseteq \gamma_\zeta$. Set $J = I \setminus \beta$, $K = I \cap \beta$. Then $J \subseteq C$ so $g(I) = g(J \cup K) \subseteq g^*(J)$ and $g(I) \cap \gamma_\eta \subseteq g^*(J) \cap \gamma_\eta \subseteq h(J \cap \gamma_\eta) = h(J \cap \gamma_\zeta) \subseteq \gamma_\zeta$, by the choice of $\Gamma$. But this means that

$$\{w : w \in \{0,1\}^{M(\zeta)}, \ w^\zeta \subseteq f(I)\} = \{0,1\}^{\gamma_\zeta \setminus \gamma_\zeta} \times \{w : w \in \{0,1\}^{M(\eta)}, \ u^\eta w \subseteq f(I)\}.$$  

By the inductive hypothesis,

$$\mu_{M(\zeta)}\{w : u^\zeta w \subseteq f(I)\} = 0,$$

so that

$$\mu_{M(\eta)}\{w : u^\eta w \subseteq f(I)\} = 0,$$

and the induction continues. $Q$ 

(iv) But now, given $I \in [V]^{<\omega}$, there is surely some $\eta < \kappa$ such that $g(I) \subseteq \gamma_\eta$, and in this case

$$\mu_{M(\eta)}\{w : u^\eta w \subseteq f(I)\} = 0$$

implies that $u \notin f(I)$.

Thus we have a point $u \notin \bigcup f(I) : I \in [V]^{<\omega}$, as required.

(b) For the general case, we have a function $\phi : \{0,1\}^\kappa \to X$ such that $\mu_{\kappa,\phi^{-1}[f(I)]} = 0$ for every $I \in [\kappa]^{<\omega}$, by A2Kb. Now by (a) there are $u \in \{0,1\}^\kappa$, $V \subseteq \kappa$ such that $\beta \subseteq V$, $\nu V = 1$ and $u \notin \phi^{-1}[f(I)]$ for every $I \in [V]^{<\omega}$; in which case $x = \phi(u) \notin f(I)$ for every $I \in [V]^{<\omega}$ and $\bigcup f(I) : I \in [V]^{<\omega} \neq X$.

Remark 6E is implicit in Lemma 5 of Kunen N70. The clause ‘$\beta \subseteq V$’ is a refinement of a type in which I have generally not indulged; but it is useful here for an application in §4 above.

6F Proposition If $\kappa$ is an atomlessly-measurable cardinal and $\lambda$, $\theta$ are infinite cardinals less than $\kappa$, then there is a set $A \subseteq \{0,1\}^\theta$ such that $\#(A) = \lambda$ and no uncountable subset of $A$ is negligible for $\mu_\theta$, the usual measure on $\{0,1\}^\theta$.

Proof Let $\nu$ be an atomless $\kappa$-additive probability defined on $\mathcal{P}\kappa$. By 3H there is a function $f : \kappa \to (\{0,1\}^\theta)^\lambda$ which is inverse-measure-preserving for $\nu$ and the usual measure of $(\{0,1\}^\theta)^\lambda$, identified with $\{0,1\}^{\theta \times \lambda}$. For $\xi < \kappa$, set

$$A_\xi = \{f(\xi)(\eta) : \eta < \lambda\} \subseteq \{0,1\}^\theta.$$}

Suppose, if possible, that for every $\xi < \kappa$ there is a set $J_\xi \subseteq \lambda$ such that $\#(J_\xi) = \omega_1$ but $E_\xi = f(\xi)[J_\xi]$ is $\mu_\theta$-negligible. For each $\xi$ choose a countable set $I_\xi \subseteq \theta$ such that

$$E'_\xi = \{x : x \in \{0,1\}^\theta, \exists x' \in E_\xi, x|I_\xi = x'|I_\xi\}$$

is $\mu_\theta$-negligible. By 5Ab, there is a countable $I \subseteq \theta$ such that $\nu V = 1$, where $V = \{\xi : I_\xi \subseteq I\}$. For $\xi \in \theta$ set

$$E'^*\xi = \{x|I : x \in E_\xi\} \subseteq \{0,1\}^I,$$

so that $\mu_I E'^\xi_\xi = 0$, where $\mu_I$ is the usual measure on $\{0,1\}^I$. Fix a sequence $\langle U_m \rangle_{m \in \mathbb{N}}$ running over the open-and-closed subsets of $\{0,1\}^I$, and for each $\xi \in V$, $n \in \mathbb{N}$ choose an open set $G_n \subseteq \{0,1\}^I$ such that $E'^\xi_\xi \subseteq G_n$ and $\mu_I(G_n) \leq 2^{-n}$. For $m, n \in \mathbb{N}$ set

$$D_{nm} = \{\xi : \xi \in V, U_m \subseteq G_n\}.$$  

For each $\alpha < \lambda$, set $f_\alpha(\xi) = f(\xi)(\alpha)|I \in \{0,1\}^I$ for $\xi < \kappa$; then the functions $f_\alpha$ are all stochastically independent. Consequently, there is for each $\xi < \kappa$ an $\alpha(\xi) \in J_\xi$ such that $f_\alpha(\xi)$ is stochastically independent from the countable family $\{D_{nm} : n, m \in \mathbb{N}\} \subseteq \mathcal{P}\kappa$. Because $\lambda < \kappa$ and $\nu$ is $\kappa$-additive, there is a $\gamma < \lambda$ such that $B = \{\xi : \alpha(\xi) = \gamma\}$ has $\nu B > 0$. Take $n \in \mathbb{N}$ such that $\nu(B) > 2^{-n}$, and examine

$$C = \bigcup_{m \in \mathbb{N}}(D_{nm} \cap f_\gamma^{-1}[U_m]).$$
Accordingly
\[ \nu C = f \]
The argument above is due to Solovay and Prikry; the form here is lifted from Remarks (a) A p89d κ measure spaces of Maharam type less than
Suppose that with #\( \{ \lambda \} \) be Radon probability spaces of Maharam type less than
proof (a) non(\( \#(\lambda) \)) is negligible (6F). Now choose
\[ \theta \]
Now an elementary induction on \( \theta \), using (a) when \( \theta < \kappa \), shows that non(\( \lambda \)) ≤ \( \theta \) whenever \( \omega_1 \leq \theta < \kappa^{(\omega)} \).

(See FREMLIN 89, 6.17.)

**Remarks (a)** The argument above is due to Solovay and Prikry; the form here is lifted from FREMLIN P89d.

(b) Using A2Ka this result can easily be converted into a formally more general result about Radon measure spaces of Maharam type less than \( \kappa \).

(c) I do not know whether, under the hypothesis of this proposition, there is always a set \( A \subseteq \{0,1\}^\theta \) with \( \#(A) = \kappa \) and no uncountable subset of \( A \) negligible for \( \mu_\kappa \); see P4d.

**6G Corollary** Let \( \kappa \) be an atomlessly-measurable cardinal. Writing \( N_{\mu_\kappa} \) for the ideal of negligible subsets of \( \{0,1\}^\theta \),
\[ \begin{align*}
(\text{a}) & \quad \text{non}(\{0,1\}^\theta, N_{\mu_\kappa}) = \omega_1 \text{ for } \omega \leq \theta < \kappa, \\
(\text{b}) & \quad \text{non}(\{0,1\}^\theta, N_{\mu_\kappa}) \leq \kappa \text{ for } \kappa \leq \theta \leq \min(2^\kappa, \kappa^{(\omega)}), \\
(\text{c}) & \quad \text{non}(\{0,1\}^\theta, N_{\mu_\kappa}) \leq \theta \text{ for } 2^\omega \leq \theta < \kappa^{(\omega)}.
\end{align*} \]

**proof (a)** Immediate from 6F.

(b) If \( \nu \) is any witnessing probability on \( \kappa \) then we have an inverse-measure -preserving function \( f : \kappa \to \{0,1\}^\theta \) (3H); now \( f[\kappa] \) witnesses that non(\( \{0,1\}^\theta, N_{\mu_\kappa} \)) ≤ \( \kappa \).

(c) The point is that if \( \text{cf}(\theta) > \omega \) then non(\( N_{\mu_\kappa} \)) ≤ \( \max(\theta, \sup_{\alpha < \theta} \text{non}(N_{\mu_\kappa})) \). If \( \delta = \max(\theta, \sup_{\alpha < \theta} \text{non}(N_{\mu_\kappa})) \). For each ordinal \( \alpha < \theta \) choose \( A_\alpha \subseteq \{0,1\}^\alpha \) such that \( A_\alpha \notin N_{\mu_\kappa} \) and \( \#(A_\alpha) = \text{non}(N_{\mu_\kappa}) \). Choose \( A \subseteq \{0,1\}^\theta \) such that \( \#(A) \leq \delta \) and \( \pi_\alpha[A] \supseteq A_\alpha \) for each \( \alpha \), taking \( \pi_\alpha : \{0,1\}^\theta \to \{0,1\}^\alpha \) to be the canonical map. If \( A \notin N_{\mu_\kappa} \), then there is an \( H \) belonging to the Baire \( \sigma \)-algebra of \( \{0,1\}^\theta \) such that \( A \subseteq H \in N_{\mu_\kappa} \) (A2Gc); now, because \( \text{cf}(\theta) > \omega \), there is an \( \alpha < \theta \) such that \( H \) is expressible as \( \pi_\alpha\{H\} \), where \( H \subseteq \{0,1\}^\alpha \). In this case \( H' \subseteq N_{\mu_\kappa} \), so \( A_\alpha \not\subseteq H' \) and \( A \not\subseteq H \). So \( A \) witnesses that non(\( N_{\mu_\kappa} \)) ≤ \( \delta \).

Now an elementary induction on \( \theta \), using (a) when \( \theta < \kappa \), shows that non(\( \lambda \)) ≤ \( \theta \) whenever \( \omega_1 \leq \theta < \kappa^{(\omega)} \).

(See FREMLIN 89, 6.17.)

**6H Lemma** Let \( \kappa \) be an atomlessly-measurable cardinal, \( \theta < \kappa \) a cardinal, and \( m \geq 1 \) an integer. Set \( Z = \{0,1\}^\theta \) with its usual measure, and suppose that for \( i < m \), \( u \in Z^{m\setminus\{i\}} \) we are given a negligible set \( E(u) \subseteq Z \). Then there is a \( t = \langle t_i \rangle_{i < m} \in Z^m \) such that \( t_i \notin E(t) \setminus \{i\} \) for every \( i < m \).

**proof** For each \( i < m \) we can find a set \( A_i \subseteq Z \) of cardinal \( \omega_{m+1-i} \) such that no uncountable subset of \( A_i \) is negligible (6F). Now choose \( t_0, t_1, \ldots, t_{m-1} \) in such a way that \( t_j \in A_j, t_j \notin E(u) \cup u \in \Pi_{j \leq i} \{t_i\} \times \Pi_{j < i < m} A_i \) for each \( j < m \); this is possible because \( A_j \) cannot be covered by \( \omega_{m-j} \) or fewer negligible sets, while \( \#(\Pi_{j < i} A_i) \leq \omega_{m-j} \) for each \( j \). Now \( t = \langle t_i \rangle_{i < m} \) works.

**6J Proposition** Let \( \kappa \) be an atomlessly-measurable cardinal and \( m \geq 1 \) an integer. Let \( X_0, \ldots, X_{m-1} \) be Radon probability spaces of Maharam type less than \( \kappa \), and \( f : \Pi_{i < m} X_i \to R \) a bounded function. Suppose that \( \sigma : m \to m \) is a permutation such that the two repeated integrals
\[ I = \int (\ldots \int f(x_0, \ldots, x_{m-1}) dx_{m-1}) \ldots dx_0, \]
\[ I' = \int (\ldots \int f(x_0, \ldots, x_{m-1}) dx_{(m-1)}) \ldots dx_{(0)} \]
both exist. Then \( I = I' \).

**Remark** The integrals above may all be taken as existing ‘in the wide sense’, that is, each function being integrated may fail to be defined on a set of measure zero.

**proof (a)** To begin with, let us suppose that every \( X_i \) is \( \{0,1\}^\theta \) for some \( \theta < \kappa \), with its usual measure. Set \( Z = (\{0,1\}^\theta)^N \); then \( Z \) also has a natural measure, identifying \( Z \) with \( \{0,1\}^{\theta \times N} \).

Define \( D_0, \ldots, D_m \) as follows. \( D_0 = \{\emptyset\} = Z^0 \). For \( 0 < j \leq m \), let \( D_j \) be the set of those \((t_0, \ldots, t_j-1) \in Z^j \) such that
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} \int (\ldots \int f(t_{0}, \ldots, t_{j-1}, x_j, \ldots, x_{m-1}) dx_{m-1}) \ldots dx_j
\]
exists and is equal to \( I \), taking \( t_i = (t_{0i})_{i \in N} \) for each \( l < j \). For \( j < m \), \( u = (u_l)_{l \neq j} \in Z^{m \setminus \{j\}} \) set
\[
E(u) = \emptyset \text{ if } (u_0, \ldots, u_{j-1}) \notin D_j,
\]
\[
E(u) = \{t : t \in Z, (u_0, \ldots, u_{j-1}, t) \notin D_{j+1}\}
\]
if \((u_0, \ldots, u_{j-1}) \in D_j \). Then \( E(u) \) is negligible. \( \mathbf{P} \) We need consider only the case \((u_0, \ldots, u_{j-1}) \in D_j \). Express \( u_l \) as \((u_{li})_{i \in N} \) for each \( l < j \), and for \( i \in N \) define \( h_i : \{0,1\}^\Theta \to R \) by setting
\[
h_i(x) = \int (\ldots \int f(u_0, \ldots, u_{j-1}, x, x_{j+1}, \ldots, x_{m-1}) dx_{m-1}) \ldots dx_{j+1}
\]
for each \( x \in \{0,1\}^\Theta \). Then, because \((u_0, \ldots, u_{j-1}) \in D_j \),
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} \int h_i(x) dx \text{ exists = } I.
\]
Also, because \( f \) is bounded, the functions \( h_i \) are uniformly bounded. By A2X,
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} h_i(t_i) \text{ exists and is equal to } I \text{ for almost all } t = (t_{0i})_{i \in N} \in Z \text{; that is, } (u_0, \ldots, u_{j-1}, t) \in D_{j+1} \text{ and } t \notin E(u) \text{ for almost all } t \in Z. \quad \mathbf{Q}
\]

Now suppose that \( t = (t_j)_{j \leq m} = (t_{j1})_{i \in N} \in Z^m \) and that \( t_j \notin E(t|m \setminus \{j\}) \) for each \( j < m \). Then
\[
(t_0, \ldots, t_{j-1}) \in D_j
\]
for each \( j \leq m \), so that \( t \in D_m \) and
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} f(t_0, \ldots, t_{m-1}, i) = I.
\]
In the same way, we can find for each \( u \in \bigcup_{j < m} Z^{m \setminus \{j\}} \) a negligible set \( E'(u) \) such that if \( t \in Z^m \) and \( t_j \notin E'(t|m \setminus \{j\}) \) for every \( j < m \) then
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} f(t_0, \ldots, t_{m-1}, i) = I'.
\]
But by Lemma 6H there is a \( t \in Z^m \) such that \( t_j \notin E(t|m \setminus \{j\}) \cup E'(t|m \setminus \{j\}) \) for every \( j < m \), and now
\[
I = \lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} f(t_0, \ldots, t_{m-1}, i) = I',
\]
as required.

(b) For the general case, there are inverse-measure -preserving functions \( g_i : \{0,1\}^\Theta \to X_i \), where \( \theta < \kappa \) is the maximum of \( \omega \) and the Maharam types of \( X_i \) (A2Ku). Applying (a) to \( F : (\{0,1\}^\Theta)^m \to R \), where \( F(y_0, \ldots, y_{m-1}) = f(y_0(y_0), \ldots, y_m(y_m-1)) \), we obtain the result.

**Remark** This comes from Theorem 1 of Shipman 90. Compare Zakrzewski p91.

6J The following is an elementary corollary of Theorem 3A.

**Proposition** Let \((X, \mathcal{F}, \Sigma, \mu)\) be a totally finite quasi-Radon measure space and \((Y, \mathcal{P}Y, \nu)\) a probability space; suppose that \( w(X) < \text{add}(\nu) \). Let \( f : X \times Y \to R \) be a bounded function such that all the sections \( x \mapsto f(x, y) : X \to R \) are measurable. Then
\[
\int \int f(x,y)\nu(dy)\mu(dx) = \int \int f(x,y)\mu(dx)\nu(dy).
\]

**proof** If \(\mu X = 0\) this is trivial; otherwise, re-scaling \(\mu\) if necessary, we may suppose that \(\mu X = 1\). By 3A,

\[
\int \int f(x,y)\nu(dy)\mu(dx) \leq \int \int f(x,y)\mu(dx)\nu(dy) = \int \int f(x,y)\mu(dx)\nu(dy).
\]

Similarly

\[
\int \int (-f(x,y))\nu(dy)\mu(dx) \leq \int \int (-f(x,y))\mu(dx)\nu(dy),
\]

so that

\[
\int \int f(x,y)\nu(dy)\mu(dx) \geq \int \int f(x,y)\mu(dx)\nu(dy).
\]

Putting these together we have the result.

**6K Proposition** Let \(\kappa\) be an atomlessly-measurable cardinal and \((X, \mathcal{X}, \Sigma, \mu)\) and \((Y, \mathcal{Y}, T, \nu)\) Radon probability spaces of weights less than \(\kappa\). Let \(f : X \times Y \to \mathbb{R}\) be a function such that all its horizontal and vertical sections

\[
x \mapsto f(x, y^*) : X \to \mathbb{R}, \quad y \mapsto f(x^*, y) : Y \to \mathbb{R}
\]

are measurable. Then

(a) if \(f\) is bounded, the repeated integrals

\[
\int \int f(x,y)\mu(dx)\nu(dy), \quad \int \int f(x,y)\nu(dy)\mu(dx)
\]

exist and are equal;

(b) in any case, there is a function \(g : X \times Y \to \mathbb{R}\), measurable for the (ordinary) product measure \(\mu \times \nu\), such that all the sets \(\{x : g(x, y^*) \neq f(x, y^*)\}\), \(\{y : g(x^*, y) \neq f(x^*, y)\}\) are negligible.

**proof (a)** By 3I there is a \(\kappa\)-additive measure \(\tilde{\nu}\) on \(Y\), with domain \(\mathcal{Y}\), extending \(\nu\). Now 6J tells us, among other things, that the function

\[
x \mapsto \int f(x,y)\nu(dy) = \int f(x,y)\tilde{\nu}(dy) : X \to \mathbb{R}
\]

is \(\mu\)-measurable. Similarly, \(y \mapsto \int f(x,y)\mu(dx)\) is \(\nu\)-measurable. So returning to 6J we get

\[
\int \int f(x,y)\mu(dx)\nu(dy) = \int \int f(x,y)\nu(dy)\tilde{\nu}(dy)
\]

\[
= \int \int f(x,y)\tilde{\nu}(dy)\mu(dx) = \int \int f(x,y)\nu(dy)\mu(dx).
\]

(b) Suppose first that \(f\) is bounded. By (a), we can define a measure \(\theta\) on \(X \times Y\) by saying that

\[
\theta G = \int \nu G([x])\mu(dx) = \int \mu G^{-1}([y])\nu(dy)
\]

whenever \(G \subseteq X \times Y\) is such that \(G([x]) \in \mathcal{T}\) for almost every \(x \in X\) and \(G^{-1}([y]) \in \Sigma\) for almost every \(y \in Y\). This \(\theta\) extends the ordinary product measure \(\mu \times \nu\); writing \(\Omega\) for the \(\sigma\)-algebra of subsets of \(X \times Y\) generated by \(\{E \times F : E \in \Sigma, F \in \mathcal{T}\}\), the Radon-Nikodým theorem (ROYDEN 63, chap. 11, §5) tells us that there is an \(\Omega\)-measurable function \(h : X \times Y \to \mathbb{R}\) such that \(\int_G f(x,y)\theta(dx dy) = \int_G h(x,y)\theta(dx dy)\) for every \(G \in \Omega\).

Let \(U\) be a base for the topology \(\mathcal{X}\), with \(#(\mathcal{U}) < \kappa\). For any \(U \in \mathcal{U}\) consider

\[
V_U = \{y : \int_U f(x,y)\mu(dx) > \int_U h(x,y)\mu(dx)\}.
\]

The argument of (a) shows that \(y \mapsto \int_U f(x,y)\mu(dx)\) is measurable, so \(V_U \in \mathcal{T}\), and
so 

Because \( \#(U) < \kappa \), and no non-negligible measurable set in \( Y \) can be covered by fewer than \( \kappa \) negligible sets (6B), we must have

But because \( U \) is a base for the topology of \( X \), we see that

But as (again using (a)) the repeated integral \( \int \int |f(x,y) - h(x,y)|\mu(dx)\nu(dy) \) exists, it must be 0. Thus

Similarly,

But now, changing \( h \) on a set of the form \( (E \times Y) \cup (X \times F) \) where \( \mu E = \nu F = 0 \), we can get a function \( g \), still \( (\mu \times \nu) \)-measurable, such that \( \{ (x,y) : f(x,y) \neq g(x,y) \} \) has all its horizontal and vertical sections negligible.

This deals with bounded \( f \). But for general \( f \) we can look at the truncates \( (x,y) \mapsto \max(-n, \min(n,f(x,y))) \) for each \( n \) to get a sequence \( (g_n)_{n \in \mathbb{N}} \) of functions which will converge at an adequate number of points to provide a suitable \( g \).

**Remark** (a) above arose in the course of correspondence with P.Zakrzewski. I first learnt of strong Fubini theorems of this type, in the context of random real models, from H.Woodin. See ZAKRZEWSKI p91 for further results along these lines.

I have given these results in a general form, allowing the spaces involved to have relatively large Maharam types; but of course they are chiefly interesting in the case in which each factor is \([0,1]\) with Lebesgue measure.

**6L Theorem** Let \( \kappa \) be an atomlessly-measurable cardinal with normal witnessing probability \( \nu \). Then the following are equivalent:

(i) \( \kappa \) is weakly \( \Pi^1_1 \)-indescribable;

(ii) \( \kappa \) is weakly \( \Pi^1_1 \)-indescribable and the rvm filter of \( \kappa \) includes the \( \Pi^1_1 \)-filter of \( \kappa \);

(iii) \( \text{cov}(\{0,1\}^\kappa, N_{\mu_{\kappa}}) > \kappa \), where \( \mu_{\kappa} \) is the usual Radon probability on \( \{0,1\}^\kappa \);

(iv) \( \text{cov}(X, N_{\mu}) > \kappa \) whenever \((X, \mu)\) is a Radon measure space and \( \mu X > 0 \).

**proof** (i) \( \Rightarrow \) (iii) Let \( \nu \) be a normal witnessing probability on \( \kappa \). Let \( \langle A_\alpha \rangle_{\alpha < \kappa} \) be a family in \( N_{\mu_{\kappa}} \). For each \( \alpha < \kappa \) let \( \langle F_{\alpha n} \rangle_{n \in \mathbb{N}} \) be a disjoint sequence of compact subsets of \( \{0,1\}^\kappa \setminus A_\alpha \) such that \( \mu_{\kappa}(\bigcup_{n \in \mathbb{N}} F_{\alpha n}) = 1 \).

By 3H there is a function \( h : \kappa \rightarrow \{0,1\}^\kappa \) which is inverse-measure -preserving for \( \nu \) and \( \mu_{\kappa} \). Set \( H_\alpha = h^{-1}(\bigcup_{n \in \mathbb{N}} F_{\alpha n}) \); then \( \nu H_\alpha = 1 \). Let \( H \) be the diagonal intersection of \( \langle H_\alpha \rangle_{\alpha < \kappa} \), so that \( \nu H = 1 \). Let \( \langle \gamma_\xi \rangle_{\xi < \kappa} \) be the increasing enumeration of \( H \).

For \( \alpha, \xi < \kappa \) set

\[
f_\alpha(\xi) = n \text{ if } n < \xi, \quad h(\gamma_\xi) \in F_{\alpha n},
\]

\[
= 0 \text{ otherwise}.
\]

Then each \( f_\alpha : \kappa \rightarrow \kappa \) is regressive, so there is a uniform ultrafilter \( \mathcal{F} \) on \( \kappa \) such that \( m(\alpha) = \lim_{\xi \rightarrow \mathcal{F}} f_\alpha(\xi) \) exists for each \( \alpha < \kappa \). Now observe that for any \( \alpha < \kappa \) we have \( H \setminus H_\alpha \subseteq \alpha + 1 \), so that \( \{ \xi : \gamma_\xi \notin H_\alpha \} \) is bounded above in \( \kappa \) and cannot belong to \( \mathcal{F} \). Consequently \( \{ \xi : h(\gamma_\xi) \in F_{\alpha,m(\alpha)} \} \in \mathcal{F} \). But this implies
at once that \( \langle F_{\alpha,m}(\alpha) \rangle_{\alpha<\kappa} \) has the finite intersection property; because all the \( F_{\alpha,n} \) are compact, there is a \( y \in \bigcap_{\alpha<\kappa} F_{\alpha,m}(\alpha) \), and now \( y \notin \bigcup_{\alpha<\kappa} A_{\alpha} \).

Because \( \langle A_{\alpha} \rangle_{\alpha<\kappa} \) was arbitrary, \( \text{cov}(\{0,1\}^\kappa,\mathcal{N}_\kappa) > \kappa \).

(iii)\( \Rightarrow \) (iv) This is standard; see A2Ph.

(iv)\( \Rightarrow \) (ii) Let \( \nu \) be any normal witnessing probability on \( \kappa \), and let \( (Z,\nu) \) be the hyperstanion space of \( (\kappa,\mathcal{P}_\kappa,\nu) \); for \( A \subseteq \kappa \) let \( A^* \) be the open-and-closed subset of \( Z \) corresponding to the image \( A^* \) of \( A \) in \( \mathfrak{A} = \mathcal{P}_\kappa/\mathcal{N}_\nu \) (see A2L).

Now let \( \langle f_{\alpha} \rangle_{\alpha<\kappa} \) be a family of regressive functions on \( \kappa \) and \( A \subseteq \kappa \) any set with \( \nu A > 0 \). Because \( \nu \) is normal and \( f_{\alpha} \) is regressive, there is for each \( \alpha < \kappa \) a countable set \( D(\alpha) \subseteq \kappa \) such that \( \nu f_{\alpha}^{-1}[D(\alpha)] = 1 \) (1He). For \( \alpha, \eta < \kappa \) set \( A_{\alpha\eta} = f_{\alpha}^{-1}(\{\eta\}) \); then \( \nu(\bigcup_{\eta \in D(\alpha)} A_{\alpha\eta}) = 1 \) so \( \nu(\bigcup_{\eta \in D(\alpha)} A_{\alpha\eta}) = 1 \) and \( E_{\alpha} = Z \setminus \bigcup_{\eta<\kappa} A_{\alpha\eta} \in \mathcal{N}_\nu \). By hypothesis (iv), \( A^* \not\subseteq \bigcup_{\alpha<\kappa} E_{\alpha} \); take \( z \in A^* \setminus \bigcup_{\alpha<\kappa} E_{\alpha} \). Then for every \( \alpha < \kappa \) there must be a \( \gamma(\alpha) < \kappa \) such that \( z \in A_{\alpha,\gamma(\alpha)}^* \). But this implies that
\[
\{A^* \} \cup \{A_{\alpha,\gamma(\alpha)}^* : \alpha < \kappa \}
\]
is a centered family of open subsets of \( Z \). It follows that \( \{A^* \} \cup \{A_{\alpha,\gamma(\alpha)}^* : \alpha < \kappa \} \) is centered in \( \mathfrak{A} \). What this means is that if \( I \in [\kappa]^{<\omega} \) then
\[
V_I = A \cap \bigcap_{\alpha \in I} f_{\alpha}^{-1}(\gamma(\alpha))
\]
does not belong to \( \mathcal{X}_\nu \), and therefore is unbounded in \( \kappa \). But now of course we can find a uniform ultrafilter \( \mathcal{F} \) on \( \kappa \) containing every \( V_I \), so that \( A \in \mathcal{F} \), while \( \lim_{\alpha \to \kappa} f_{\alpha}(\xi) = \gamma(\alpha) \) for every \( \alpha < \kappa \).

Look back at where \( A, \langle f_{\alpha} \rangle_{\alpha<\kappa} \) came from. Taking \( A = \kappa \) to begin with, we see that \( \kappa \) is indeed weakly \( \Pi^1_1 \)-indescribable. But also, letting \( A \) vary, we see that any such \( A \) must be \( \Pi^1_1 \)-fully stationary, that is, its complement cannot belong to the \( \Pi^1_1 \)-filter \( W \) of \( \kappa \); turning this round, we see that \( \nu W = 1 \) for every \( W \in \mathcal{W} \), as demanded by (ii).

(ii)\( \Rightarrow \) (i) is trivial.

6M I now leave these questions in set theory and logic and turn to two more of the problems in abstract measure theory to which real-valued-measurable cardinals are relevant.

**Theorem** Let \( (X,\rho) \) be a metric space.

(a) \( X \) is Borel measure-complete iff there is no real-valued-measurable cardinal less than or equal to \( d(X) \).

(b) If \( X \) is complete (as metric space!) then it is Radon iff there is no real-valued-measurable cardinal less than or equal to \( d(X) \).

**proof** (a)(i) If \( \kappa \leq d(X) \) is real-valued-measurable, let \( \nu \) be a witnessing probability on \( \kappa \). Let \( \{x_\xi : \xi<\kappa\} \) be a discrete family in \( X \) (see A3Fa). Let \( \mu \) be the Borel measure on \( X \) such that \( \mu E = \nu \{\xi : x_\xi \in E\} \) for every Borel set \( E \subseteq X \). Let \( \mathcal{G} \) be
\[
\{G : G \subseteq X \text{ is open}, \{\xi : x_\xi \in G\} \text{ is finite}\}.
\]
Then \( \mathcal{G} \) is an upwards-directed family of open sets in \( X \) with union \( X \), so \( \mu(\bigcup \mathcal{G}) = \mu X = 1 > 0 = \sup_{G \in \mathcal{G}} \mu G \), and \( X \) is not Borel measure-complete.

(ii) If \( X \) is not Borel measure-complete, let \( \mu \) be a totally finite Borel measure on \( X \) and \( \mathcal{G} \) an upwards-directed family of open subsets of \( X \) such that \( \mu G^* > \sup_{G \in \mathcal{G}} \mu G \), writing \( G^* = \bigcup \mathcal{G} \). Let \( \langle U_n \rangle_{n \in \mathbb{N}} \) be a sequence of discrete families of open sets in \( X \) such that \( \bigcup_{n \in \mathbb{N}} U_n \) is a base for the topology of \( X \) (A3Fb).

For each Borel set \( E \subseteq X \) set
\[
\mu_1 E = \mu(E \cap G^*) - \sup_{G \in \mathcal{G}} \mu(E \cap G);
\]
then \( \mu_1 \) is a Borel measure on \( X \) and \( \mu_1(G^*) > 0 = \sup_{G \in \mathcal{G}} \mu_1 G \). For each \( n \in \mathbb{N} \) set
\[
\mathcal{V}_n = \{ U : U \subseteq U_n, \exists G \in \mathcal{G}, U \subseteq G \}, \quad \mathcal{V}_n^* = \bigcup \mathcal{V}_n;
\]
then \( G^* = \bigcup_{n \in \mathbb{N}} V_n^* \), so there is an \( n \in \mathbb{N} \) such that \( \mu_1(V_n^*) > 0 \). For \( E \subseteq \mathcal{V}_n \), set \( \nu E = \mu_1(\bigcup E) \); then \( (\mathcal{V}_n,\mathcal{P}\mathcal{V}_n,\nu) \) is a non-trivial measure space, so its additivity \( \kappa \) is a real-valued-measurable cardinal (1D).

But of course \( \kappa \leq \#(\mathcal{V}_n) \leq d(X) \), because any dense subset of \( X \) must meet every non-empty member of \( \mathcal{V}_n \), and \( \mathcal{V}_n \) is disjoint.
(b) Now (b) follows from (a) by A2Wb.

Remark For an investigation of the exact properties of metric spaces involved in the arguments used here, see Gardner & Pfeffer 84.

6N Riesz spaces My own introduction to the Banach-Ulam problem was through the following. (For definitions see Fremlin 74. For an account of the elementary theory of Riesz spaces see also Luxemburg & Zaanen 71.)

Theorem Let $E$ be a Dedekind complete Riesz space (= vector lattice) with a sequentially order-continuous positive linear functional on $E$ which is not order-continuous. Then there are a real-valued-measurable cardinal $\kappa$ and an order-bounded disjoint set $A \subseteq E^+$, the positive cone of $E$, of cardinal $\kappa$.

proof Suppose that $f : E \to \mathbb{R}$ is a positive linear functional which is sequentially order-continuous but not order-continuous. Let $D \subseteq E^+$ be a non-empty, downwards-directed set with infimum 0 such that $\inf_{x \in D} f(x) > 0$. Let $\langle d_n \rangle_{n \in \mathbb{N}}$ be a decreasing sequence in $D$ such that $\lim_{n \to \infty} f(d_n) = \inf_{x \in D} f(x)$; set $d^* = \inf_{n \in \mathbb{N}} d_n$. Then (because $f$ is sequentially order-continuous) $f(d^*) = \inf_{d \in D} f(d)$, and if $d \in D$ then $f(d \wedge d^*) = \lim_{n \to \infty} f(d \wedge d_n) = f(d^*)$ (using the distributivity of Riesz spaces; Fremlin 74, 14D). Let $C$ be the set

$$\{x : x \in E, x > 0, \exists d \in D, x \leq d^* \wedge (d^* \wedge d)\};$$

then $f(x) = 0$ for every $x \in C$, $C$ is upwards-directed, and $\sup C = d^*$. Now let $A \subseteq C$ be a maximal set such that $x \wedge y = 0$ for all distinct $x, y \in A$, and set $e = \sup A$. Then $d^* = \sup_{n \in \mathbb{N}} (d^* \wedge n e)$. Because $f$ is sequentially order-continuous, there must be some $n \in \mathbb{N}$ such that $f(d^* \wedge n e) > 0$, and $f(e) > 0$. For each $B \subseteq A$ set $e_B = \sup B$, $\nu B = f(e_B)$. Again because $f$ is sequentially order-continuous, $(A, \mathcal{P} A, \nu)$ is a non-trivial totally finite measure space and $\#(A) \geq \text{add}(\nu)$, which is a real-valued-measurable cardinal.

Remark This comes from Luxemburg 67. Note that if $(X, \mathcal{P} X, \mu)$ is any non-trivial probability space then we can set $E = \ell^\infty(X)$, the Dedekind complete Riesz space of all bounded real-valued functions on $X$, and $f : E \to \mathbb{R}$ will be a sequentially order-continuous positive linear functional on $E$ which is not order-continuous.

Version of 16.6.91

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7. Partially ordered sets

I collect here a variety of facts concerning the impact of real-valued-measurable cardinals on partially ordered sets. In 7A-7D I show that if $\kappa$ is an atomlessly-measurable cardinal there are ccc partially ordered sets $P$ and $Q$ such that $P \times Q$ does not satisfy the $\kappa$-chain condition. A similar method shows that there are large ‘entangled’ subsets of $\mathbb{R}$ (7E-7F). In 7G-7N I discuss the cofinalities of certain partially ordered sets, in particular, of reduced products of families of cardinals. I end the section with an application of these ideas to cardinal exponentiation (7O-7Q).

7A Definition Let $a, b \subseteq \mathbb{N}$. Approximately following Todorčević 86, I write

$$\Delta(a, b) = \min(a \Delta b)$$

$$= \infty$$

if $a = b$.

We have the following elementary lemma.

7B Lemma Let $n, l \in \mathbb{N}$ and suppose that $m \geq 3(n^2 l)!$. Let $\langle a_{ri} \rangle_{r < m, i < n}$ be any family in $\mathcal{P} \mathbb{N}$, and for $r, s < m$ set $D_{rs} = \{\Delta(a_{ri}, a_{sj}) : i < n\} \cap \mathbb{N}$. Then there are $u(0), \ldots, u(l), v(0), \ldots, v(l) < m$ such that $u(r) \neq v(r)$ for $r \leq l$ and $D_{u(j), v(j)} \cap D_{u(k), v(k)} = \emptyset$ whenever $j < k \leq l$.

proof (a) Given finite sets $X$ and $L$ with $\#(X) \geq 3(\#(L))!$ and any function $\phi : [X]^2 \to L$, there is a $J \in [X]^3$ such that $\phi$ is constant on $[J]^2$. P Induce on $\#(L)$. If $\#(L) \leq 1$ this is trivial. For the inductive step to $\#(L) = n + 1$, take any $x \in X$. For each $l \in L$ set $X_l = \{y : y \in X \setminus \{x\}, \phi(x, y) = l\}$. Then there must be some $l \in L$ such that $\#(X_l) \geq 3n!$, because $(n + 1)(3n! - 1) + 1 < 3(n + 1)! \leq \#(X)$. If there are
distinct $y, z \in X_I$ such that $\phi \{y, z\} = l$, take $J = \{x, y, z\}$. Otherwise, consider $\phi | [X_I]^2 : [X_I]^2 \to L \setminus \{\}\}$. By the inductive hypothesis, there is a $J \in [X_I]^3$ such that $\phi$ is constant on $[J]^2$, and we're done. \textbf{Q}

(b) Now examine the given family $\{a_i\}_{r \in m, i  \in n}$ in $\mathcal{P}$. If $L \in [\mathbb{N}]^{<\omega}$ and $m \geq 3(n\#(L))!$, there are distinct $r, s < m$ such that $D_{rs} \cap L = \emptyset$. Otherwise, we can choose a function $\phi : [m]^2 \to n \times \mathbb{L}$ such that $\Delta(a_{ri}, a_{si}) = j$ whenever $r, s < m$ and distinct and $\phi(\{r, s\}) = (i, j)$. By (a), there is a $J \in [m]^3$ such that $\phi$ is constant on $[J]^2$; suppose that $J = \{r, s, t\}$ and $\phi(\{r, s\}) = \phi(\{s, t\}) = \phi(\{r, t\}) = (i, j)$. Now $\Delta(a_{ri}, a_{si}) = \Delta(a_{ri}, a_{ti}) = \Delta(a_{si}, a_{ti}) = j$. But this means that for each pair from $a_{ri}, a_{si}, a_{ti}$ there is exactly one member of the pair containing $j$; which is ridiculous. \textbf{XQ}

(c) Consequently we can choose $u(0), v(0), \ldots, u(l), v(l)$ inductively so that

$$u(r) \neq v(r), D_u(r), v(r) \cap \bigcup_{s < r} D_u(s), v(s) = \emptyset$$

for every $r \leq l$; and these will serve.

7C Lemma Let $\kappa$ be an atomlessly-measurable cardinal and $R, S$ two upwards-ccc partially ordered sets, both of size strictly less than $\kappa$. Let $\lambda < \kappa$ be any cardinal. Then there are partially ordered sets $P, \overline{Q}$ such that

$$\#(P) \leq \max(\omega, \lambda), \#(\overline{Q}) \leq \max(\omega, \lambda),$$

$$S(P \times R) \leq \omega_1, S(Q \times S) \leq \omega_1, S(P \times Q) > \lambda,$$

writing $S(P \times R)$ for the Souslin number of $P \times R$, as in A1P.

**proof (a)** Let $\nu$ be a $\kappa$-additive extension of the usual measure $\mu$ on $[0, 1]^N$ to every subset of $[0, 1]^N$ (1De, A2Gb). Let $A \subseteq \{0, 1\}^N$ be a set of cardinal $\lambda$. For each $z \in \{0, 1\}^N$ set

$$P_z = \{I : I \in [A]^\omega, z(\Delta(a, b)) = 1 \text{ for all distinct } a, b \in I\},$$

$$Q_z = \{I : I \in [A]^\omega, z(\Delta(a, b)) = 0 \text{ for all distinct } a, b \in I\},$$

ordering both by $\subseteq$. Then $S(P_z \times Q_z) > \lambda$ because $\{(\{a\}, \{a\}) : a \in A\}$ is an up-antichain in $P_z \times Q_z$. Also $\#(P_z) \leq \max(\omega, \lambda)$ and $\#(Q_z) \leq \max(\omega, \lambda)$.

(b) I shall therefore be done if I can find a $z \in \{0, 1\}^N$ such that $P_z \times R$ and $Q_z \times S$ are both upwards-ccc. In fact I show that $P_z \times R$ is upwards-ccc for $\nu$-almost every $z$; as the same argument will work for $Q_z \times S$, that will be more than enough.

Set

$$H_0 = \{z : z \in \{0, 1\}^N, S(P_z \times R) > \omega_1\}.$$ 

For each $z \in H_0$ there is a family $\langle (I_\alpha(z), r_\alpha(z)) \rangle_{\alpha < \omega_1}$ enumerating an uncountable up-antichain in $P_z \times R$. There is an $n_\alpha \in \mathbb{N}$ such that $B_z = \{\alpha : \#(I_\alpha(z)) = n_\alpha\}$ is uncountable. Let $\langle (I_\alpha(z), r_\alpha(z)) \rangle_{\alpha < \omega_1}$ be a re-enumeration of the up-antichain $\langle (I_\alpha(z), r_\alpha(z)) \rangle_{\alpha \in B_z}$ in $P_z \times R$.

(c) Suppose now that $\nu H_0 > 0$. In this case there is an $n \in \mathbb{N}$ such that $\nu H_1 > 0$, where

$$H_1 = \{z : z \in H_0, n_\alpha = n\}.$$ 

Because $A$ and $R$ both have cardinal less than $\kappa$, we can find for each $\alpha < \omega_1$ a finite set $J^*_\alpha \subseteq A$ and an $s^*_\alpha \in R$ such that $\nu E_\alpha > 0$, where

$$E_\alpha = \{z : z \in H_1, J_\alpha(z) = J^*_\alpha, s_\alpha(z) = s^*_\alpha\}.$$ 

Enumerate $J^*_{\alpha i}$ as $(a_{\alpha i})_{i < n}$ for each $i < n$. For $\alpha, \beta < \omega_1$ set $D_{\alpha \beta} = \{\Delta(a_{\alpha i}, a_{\beta i}) : i < n\} \cap \mathbb{N}$; note that $\#(D_{\alpha \beta}) \leq n$.

(d) Let $\gamma > 0$ be such that

$$T = \{\alpha : \alpha < \omega_1, \nu E_\alpha \geq \gamma\}$$

is uncountable. For each $\alpha \in T$ set

$$m_\alpha = \sup \{\Delta(a_{\alpha i}, a_{\alpha j}) : i < j < n\} + 1.$$ 

Let $m$ be such that
is uncountable. Because there are only finitely many possibilities for the family \( (a_{oi} \cap m)_{i<n} \) (identifying \( m \) with the set of its predecessors), there is a family \( (b_i)_{i<n} \) in \( \mathcal{P} m \) such that
\[
V = \{ \alpha : \alpha \in U, \ a_{oi} \cap m = b_i \ \forall \ i < n \}
\]
is uncountable.

(e) If \( \alpha, \beta \) are distinct members of \( V \), \( z \in E_\alpha \cap E_\beta \), and \( s^*_\alpha, s^*_\beta \) are upwards-compatible in \( R \), there is a \( k \in D_{\alpha,\beta} \) such that \( z(k) = 0 \). \( P \) \( (J^*_\alpha, s^*_\alpha) = (J_\alpha(z), s_\alpha(z)) \) and \( (J^*_\beta, s^*_\beta) \) are upwards-incompatible in \( P_z \times R \); we are supposing that \( s^*_\alpha \) and \( s^*_\beta \) are upwards-compatible in \( R \), so \( J^*_\alpha \cup J^*_\beta \notin P_z \). In this case, there must be \( a, b \in J^*_\alpha \cup J^*_\beta \) such that \( z(\Delta(a,b)) = 0 \). But \( J^*_\alpha \) and \( J^*_\beta \) do belong to \( P_z \), so neither can contain both \( a \) and \( b \), and we may take it that \( a \in J^*_\alpha \) and \( b \in J^*_\beta \). Thus there must be \( i, j < n \) such that \( z(\Delta(a_{oi}, a_{\beta j})) = 0 \).

If \( i \neq j \), then
\[
\Delta(a_{oi}, a_{\beta j}) = \Delta(b_i, b_j) = \Delta(a_{oi}, a_{\beta j}),
\]
so
\[
z(\Delta(a_{oi}, a_{\beta j})) = z(\Delta(a_{oi}, a_{\beta j})) = 1. \ X
\]
Thus \( i = j \) and \( z(\Delta(a_{oi}, a_{\beta j})) = 0 \), where \( \Delta(a_{oi}, a_{\beta j}) \in D_{\alpha,\beta} \). \ Q

(f) Let \( k \) be so large that \( (1 - 2^{-n})^{k+1} \leq \frac{2}{3} \gamma \); set \( l \equiv 3(n^2 k)! \). By A22, there is an uncountable set \( \mathcal{W} \subseteq \mathcal{V} \) such that \( \mu^*(\bigcap_{\alpha \in \mathcal{E}_\alpha} E_\alpha) \geq \frac{2}{3} \gamma \) whenever \( L \in [\mathcal{W}]^1 \). By A11, there is an \( L \in [\mathcal{W}]^1 \) such that \( \{ s^*_\alpha : \alpha \in L \} \) is bounded above. By 7B, there are \( \alpha(0), \ldots, \alpha(k), \beta(0), \ldots, \beta(k) \) in \( L \) such that \( \alpha(j) \neq \beta(j) \) for \( j \leq k \) and \( (D_{\alpha(j), \beta(j)})_{i \leq k} \) is disjoint. Now however observe that if \( z \in \bigcap_{\alpha \in L} E_\alpha \) and \( i \leq k \) then \( s^*_\alpha(i) \) and \( s^*_\beta(i) \) are upwards-compatible in \( R \), so there is a \( j \in D_{\alpha(i), \beta(i)} \) such that \( z(j) = 0 \), by (e) above. Thus \( \bigcap_{\alpha \in L} E_\alpha \subseteq F \) where
\[
F = \{ z : \forall i \leq k \ \exists j \in D_{\alpha(i), \beta(i)} \text{ such that } z(j) = 0 \}.
\]
But, because \( \langle D_{\alpha(i), \beta(i)} \rangle_{i \leq k} \) is disjoint and \( \#(D_{\alpha(i), \beta(i)}) \leq n \) for each \( i \),
\[
\mu F = \prod_{i \leq k} \mu \{ z : \exists j \in D_{\alpha(i), \beta(i)}, \ z(j) = 0 \}
\]
\[
\leq \prod_{i \leq k} (1 - 2^{-n}) = (1 - 2^{-n})^{k+1} \leq \frac{2}{3} \gamma / 3.
\]
So \( \frac{2}{3} \gamma \leq \mu^*(\bigcap_{\alpha \in L} E_\alpha) \leq \frac{4}{3} \gamma \). \ X

This contradiction shows that \( S(P_z \times R) \leq \omega_1 \) for \( \nu \)-almost all \( z \), as required.

7D Theorem Let \( \kappa \) be an atomlessly-measurable cardinal. Then there are ccc partially ordered sets \( P, Q \) such that \( S(P \times Q) = \kappa \).

Proof Construct \( (P_\xi)_{\xi < \kappa}, (Q_\xi)_{\xi < \kappa}, (P^*_\xi)_{\xi < \kappa}, (Q^*_\xi)_{\xi < \kappa} \) as follows. Given \( (P_\eta)_{\eta < \xi} \) and \( (Q_\eta)_{\eta < \xi} \), then \( P^*_\xi \) and \( Q^*_\xi \) are to be the finite-support products of these families. Given that \( P^*_\xi \) and \( Q^*_\xi \) are ccc and that their cardinals are at most \( \max(\omega, \#(\xi)) \), for \( \xi < \kappa \), then \( P_\xi \) and \( Q_\xi \) are to be partially ordered sets of size at most \( \max(\omega, \#(\xi)) \) such that \( S(P_\xi \times Q_\xi) \leq \omega_1, S(Q_\xi \times Q^*_\xi) \leq \omega_1, \) and \( S(P_\xi \times Q^*_\xi) > \#(\xi) \); Lemma 7C tells us that such can be found. The induction continues because at successor stages \( P_\xi \times P^*_\xi \) and \( Q_\xi \times Q^*_\xi \) can be embedded as cofinal subsets of \( P_{\xi+1} \) and \( Q_{\xi+1} \) respectively, so that \( P^*_\xi \) and \( Q^*_\xi \) will be ccc if \( P^*_\xi \) and \( Q^*_\xi \) are, while for limit ordinals \( \xi > 0, P^*_\xi = \bigcup_{\eta < \xi} P^*_\eta \) will be ccc because all the preceding \( P^*_\eta \) are, by the remarks in A11.

On completing the induction, set \( P = P^*_\kappa, Q = Q^*_\kappa \). Observe that the finite-support product \( R \)
\[
(\mathcal{P}_\xi \times Q_\xi)_{\xi < \kappa}
\]
can be embedded as a cofinal subset of \( P \times Q \), so that \( S(P \times Q) = S(R) \). Also \( \sup \{ S(\prod_{\xi \in J} P_\xi \times Q_\xi) : J \in [\kappa]^{< \omega} = \kappa \), because \( S(P_\xi \times Q_\xi) > \#(\xi) \) for every \( \xi < \kappa \), while \( \#(\prod_{\xi \in J} P_\xi \times Q_\xi) < \kappa \) for every \( J \in [\kappa]^{< \omega} \). Because \( \kappa \) is regular, A11 tells us that \( S(R) = \kappa \), so that \( S(P \times Q) = \kappa \), as claimed.

Remark Note that (if it is consistent to suppose that there is a two-valued-measurable cardinal) it is consistent to suppose that there is an atomlessly-measurable cardinal and that Souslin’s hypothesis is true; see Laver 87.
7E Definition Let \((X, \leq)\) be a totally ordered set. Let \(\leq_1\) be the original order \(\leq\) and \(\leq_{-1}\) the reverse total ordering \(\geq\). We say that a set \(A \subseteq X\) is \(\omega_1\)-entangled if for any \(n \in \mathbb{N}\), \(\epsilon \in \{-1,1\}^n\) and every family \((x_{\xi})_{\xi \in \omega_1, i < n}\) of distinct elements of \(A\) there are distinct \(\xi, \eta < \omega_1\) such that \(x_{\xi(i)} \leq x_{\eta(i)}\) for every \(i < n\).

7F Proposition Let \(\kappa\) be an atomlessly-measurable cardinal. Then for any \(\lambda < \kappa\) there is an \(\omega_1\)-entangled subset of \(\mathbb{R}\) of cardinal \(\lambda\).

proof (a) Let \(X\) be \((-1,1)^\mathbb{N}\) with its lexicographic total ordering, so that \(x < y\) if \(x(\Delta(x,y)) = -1\), \(y(\Delta(x,y)) = 1\), writing \(\Delta(x,y) = \min\{m : x(m) \neq y(m)\}\) (cf. 7A). The map \(x \mapsto \sum_{m \in \mathbb{N}} 3^{-m}x(m) : X \to \mathbb{R}\) is strictly increasing, so that any \(\omega_1\)-entangled subset of \(X\) has an \(\omega_1\)-entangled direct image in \(\mathbb{R}\), and it will be enough to find an \(\omega_1\)-entangled subset of \(X\) of cardinal \(\lambda\).

(b) Set \(S = \bigcup_{n \in \mathbb{N}} \{-1,1\}^n\), \(Z = \{-1,1\}^\mathbb{N}\); let \(\mu\) be the usual measure on \(Z\) (as in A2G), and \(\nu\) a \(\kappa\)-additive extension of \(\mu\) to \(\mathcal{P}Z\) (see 1De, A2Gb). For \(z \in Z\) define \(h_z : X \to X\) by writing

\[
h_x(m) = x(m) \times z(m) \quad \forall \ m \in \mathbb{N}, \ x \in X;
\]

then \(h_z\) is a bijection. Observe that if \(x, y \in X\) and \(\Delta(x,y) = m\) then \(h_x(x) < y, h_y(y)\) if \(z(\Delta(x,y)) = y(m) \times \epsilon\).

(c) Let \(B \subseteq X\) be any set of cardinal \(\lambda\), and for \(z \in Z\) write \(A_z = h_z[B] \in [X]^\lambda\). I aim to show that \(A_z\) is \(\omega_1\)-entangled for some, in fact for \(\nu\)-almost every, \(z \in Z\).

Let \(E\) be \(\{z : z \in Z, A_z \text{ is not } \omega_1\text{-entangled}\}\). For each \(z \in E\) we can find \(n \in \mathbb{N}\), \(\epsilon \in \{-1,1\}^{m^2}\) and a family \((x_{\xi})_{\xi \in \omega_1, i < n}\) of distinct elements of \(B\) such that

\[
\forall \ \xi < \eta < \omega_1 \exists \ i < n, h_x(x_{\xi(i)}) \not\leq_{\epsilon(i)} h_x(x_{\eta(i)}).
\]

Because all the \(x_{\xi(i)}\) are distinct, there will be a family \((u_i)_{i < n}\) of \(\{\pm 1\}\) such that \(h_x(x_{\xi(i)}) \not\leq_{\epsilon(i)} h_x(x_{\eta(i)})\) for every \(i < n\).

(d) Suppose, if possible, that \(\nu E > 0\). Then there are \(n, k \in \mathbb{N}\), \(\epsilon \in \{-1,1\}^n\) and a family \((u_i)_{i < n}\) in \((-1,1)^k\) such that \(\nu F > 0\), where

\[
F = \{z : z \in E, n_z = n, k_z = k, \epsilon_z = \epsilon, u_z = u_i \quad \forall \ i < n\}.
\]

Note that the \(u_i\) must all be different. Next, because \(#(B^n) < \kappa\), we can find for each \(\xi < \omega_1\) a family \((y_{\xi})_{i < n}\) in \(B\) such that \(\nu F_{\xi} > 0\), where

\[
F_{\xi} = \{z : z \in F, y_{\xi(i)} = y_{\xi(i)} \quad \forall \ i < n\}.
\]

Thus

\[
y_{\xi(i)}k = u_i \quad \forall \ i < n, \ \xi < \omega_1,
\]

and if \(z \in F_{\xi} \cap F_{\eta}, \) where \(\xi < \eta < \omega_1\), there must be an \(i < n\) such that

\[
h_x(y_{\xi(i)}) \not\leq_{\epsilon(i)} h_x(y_{\eta(i)}).
\]

(e) Let \(\gamma > 0\) be such that \(U = \{\xi : \nu F_{\xi} \geq \gamma\}\) is uncountable. Let \(l \geq 1\) be such that \((1 - 2^{-n})^l \leq \frac{1}{4}\gamma\). By A2S, there is an uncountable \(V \subseteq U\) such that

\[
\mu^*(\bigcap_{\xi \in L} F_{\xi}) \geq \frac{3}{4}\gamma \quad \text{for every } L \in [V]^l+1.
\]

Let \((y_i)_{i < n}\) be a cluster point of \((y_{\xi(i)})_{i < n}\) in \(V\) in the sense that

\[
\forall \ k \in \mathbb{N}, \ \xi < \omega_1 \exists \ \xi \in V \ \text{such that } y_{\xi(i)}k = y_i k \quad \forall \ i < n.
\]
Note that because \( F_\xi \cap F_\eta \neq \emptyset \) for \( \xi, \eta \in V \), all the \( y_{\xi,i} \), for \( \xi \in V \) and \( i < n \), are distinct; so that there is a cofinite \( W \subseteq V \) such that \( y_{\xi,i} \neq y_i \) for \( \xi \in W, i < n \).

**Remark**

A. for every cofinite \( r \) so that \( V \subseteq \) Deduction using Corollary 4Oa.

\[ y_{\xi,i} | r_j = y_i | r_j \quad \forall \quad i < n, \]

\[ y_{\xi,i} | r_{j+1} \neq y_i | r_{j+1} \quad \forall \quad i < n \]

for every \( j \in \mathbb{N} \). For \( j \in \mathbb{N}, i < n \) set

\[ m_{ji} = \Delta(y_{\xi,i}, y_i), \]

so that \( r_j \leq m_{ji} < r_{j+1} \) and also

\[ m_{ji} = \Delta(y_{\xi+1,i}, y_{\xi,i}). \]

Set

\[ s_{ji} = y_i | m_{ji} \]

\[ = y_{\xi,i} | m_{ji} = y_{\xi+1,i} | m_{ji} \in S. \]

Observe that all the \( s_{ji} \) are distinct; this is because \( r_j \leq m_{ji} < r_{j+1} \) for all \( i, j \), so that \( s_{ji} \neq s_{j'i} \) if \( j \neq j' \), while \( r_0 = k \), so that \( s_{ji} | k = u_i \) for all \( i, j \) and \( s_{ji} \neq s_{j'i} \) if \( i \neq i' \).

(g) If \( z \in Z, j \in \mathbb{N}, i < n \) then

\[ h_z(y_{\xi,i}) \leq \epsilon(i) h_z(y_{\xi,i+1}) \iff z(s_{ji}) = y_i(m_{ji}) \times \epsilon(i). \]

So if we set \( G = \bigcap_{j < i} F_{\xi,i} \), then for any \( z \in G \) we have \( y_{\xi,i} = y_{z,i} \), for every \( j < l, i < n \), so for each \( j < l \) there must be an \( i < n \) such that

\[ h_z(y_{\xi,i}) \not\leq \epsilon(i) h_z(y_{\xi,i+1}), \]

that is,

\[ z(s_{ji}) \neq y_i(m_{ji}) \times \epsilon(i). \]

Accordingly \( G \subseteq H \), where

\[ H = \{ z : \forall \quad j < l \exists \quad i < n \text{ such that } z(s_{ji}) \neq y_i(m_{ji}) \times \epsilon(i) \}. \]

Because all the \( s_{ji} \) are distinct, \( \mu H = (1 - 2^{-n})^l \leq \frac{1}{2^\gamma} \), so \( \mu^* G \leq \frac{1}{2^\gamma} \). But by the choice of \( V, \mu^* G \geq \frac{1}{2^\gamma}. \)

This contradiction shows that \( \nu E = 0. \) So we can take a \( z \in Z \setminus E \) to yield the required entangled set \( A_z \subseteq X. \)

**Remark** 7F is due to S.Todorčević; it corresponds to Theorem 2 of Todorčević 85, from which it may be deduced using Corollary 4Oa.

Version of 18.9.92

**7G** \( \kappa \)-measure-bounded partially ordered sets (a) Let \( P \) be a partially ordered set and \( \kappa \) a real-valued-measurable cardinal. I will say that \( P \) is \( \kappa \)-measure-bounded (upwards) if for every \( \kappa \)-additive probability \( \mu \) on \( P \) with domain \( \mathcal{P}P \) there is a \( p \in P \) such that \( \mu\{p' : p' \leq p\} > 0. \)

(b) Let \( P \) and \( Q \) be partially ordered sets. I say that a function \( f : P \rightarrow Q \) is an \( \omega \)-Tukey function if for every \( q \in Q \) there is a countable set \( D \subseteq P \) such that whenever \( p \in P \) and \( f(p) \leq q \) there is a \( d \in D \) such that \( p \leq d. \)

(For a systematic discussion of this concept see Fremlin P90.)

**7H** Elementary facts Let \( \kappa \) be a real-valued-measurable cardinal.

(a) If \( P \) and \( Q \) are partially ordered sets, \( f : P \rightarrow Q \) is an \( \omega \)-Tukey function, and \( Q \) is \( \kappa \)-measure-bounded, then \( P \) is \( \kappa \)-measure-bounded. \( \textbf{P} \) Let \( \mu \) be a \( \kappa \)-additive probability with domain \( \mathcal{P}P. \) Let \( \nu = \mu f^{-1} : \mathcal{P}Q \rightarrow [0,1] \). Then \( \nu \) is \( \kappa \)-additive, so there is a \( q \in Q \) such that \( \nu\{q' : q' \leq q\} > 0. \) Let \( D \subseteq P \) be a countable set such that whenever \( f(p) \leq q \) there is a \( d \in D \) such that \( p \leq d. \) Then

\[ 0 < \nu\{q' : q' \leq q\} = \mu\{p : f(p) \leq q\} \leq \sum_{d \in D} \mu\{p : p \leq d\}, \]
so there is a $d \in D$ such that $\mu\{p : p \leq d\} > 0$. As $\mu$ is arbitrary, $P$ is $\kappa$-measure-bounded.  \(Q\)

(b) If $P$ is a partially ordered set and $Q \subseteq P$ is cofinal with $P$, then $P$ is $\kappa$-measure-bounded iff $Q$ is.

The identity function from $Q$ to $P$ is $\omega$-Tukey, and any function $f$ from $P$ to $Q$ such that $f(p) \leq p$ for every $p$ is also $\omega$-Tukey.  \(Q\)

(c) If $P$ is any partially ordered set such that $\text{cf}(P) < \kappa$ then $P$ is $\kappa$-measure-bounded.

(d) Suppose that $P$ is a $\kappa$-measure-bounded partially ordered set and that $\mu$ is a $\kappa$-additive probability on $P$ with domain $\mathcal{P}P$. (i) If $P$ is upwards-directed then $\sup_{\mu \in P} \mu\{p' : p' \leq p\} = 1$. (ii) If $\text{add}(P) > \omega$ ($P$ is ‘countably closed’) then there is a $p \in P$ such that $\mu\{p' : p' \leq p\} = 1$.  \(Q\)

(i) For $A \subseteq P$ set

$$\nu A = \mu A - \sup_{\mu \in P} \mu\{p' : p' \in A, p' \leq p\}.$$  

Then $\nu$ is a $\kappa$-additive measure on $P$ and $\nu\{p' : p' \leq p\} = 0$ for every $p \in P$. Because $P$ is $\kappa$-measure-bounded, $\nu P$ must be 0, that is, $\sup_{\mu \in P} \mu\{p' : p' \leq p\} = 1$. (ii) If $\text{add}(P) > \omega$, take a sequence $(p_n)_{n \in \mathbb{N}}$ in $P$ such that $\lim_{n \to \infty} \mu\{p' : p' \leq p_n\} = 1$; there is a $p \in P$ such that $p_n \leq p$ for every $n$, and now $\mu\{p' : p' \leq p\} = 1$.  \(Q\)

(e) (i) If $(P_n)_{n \in \mathbb{N}}$ is a sequence of upwards-directed $\kappa$-measure-bounded partially ordered sets then $P = \prod_{n \in \mathbb{N}} P_n$ is $\kappa$-measure-bounded. (ii) If $(P_{\xi})_{\xi < \lambda}$ is a family of $\kappa$-measure-bounded partially ordered sets such that $\lambda < \kappa$ and $\text{add}(P_{\xi}) > \omega$ for every $\xi < \kappa$, then $P = \prod_{\xi < \lambda} P_{\xi}$ is $\kappa$-measure-bounded.  \(P\)

(i) Let $\mu$ be a $\kappa$-additive probability with domain $\mathcal{P}P$. For each $n \in \mathbb{N}$ set $\mu_n = \mu^{-1}_{\pi_n} : \mathcal{P}P_n \to [0,1]$, where $\pi_n : P \to P_n$ is the canonical map. By (d)-(i) there is a $p_n \in P_n$ such that $\mu_n\{q : q \in P_n, q \leq p_n\} \geq 1 - 2^{-n-2}$. Now if we set $p = (p_n)_{n \in \mathbb{N}} \in P$, we see that $\mu\{p' : p' \leq p\} \geq 1/2$. (ii) Argue as in (i), but taking $p_\xi \in P_\xi$ such that $\mu_\xi\{q : q \leq p_\xi\} = 1$, so that $\mu\{p' : p' \leq p\} = 1$, because $\lambda < \kappa$.  \(Q\)

(f) If $P$ is a $\kappa$-measure-bounded partially ordered set, $Q$ is another partially ordered set, and $f : P \to Q$ is an order-preserving surjection, then $Q$ is $\kappa$-measure-bounded.  \(P\)

Take any $g : Q \to P$ such that $f(g(q)) = q$ for every $q \in Q$; then $g$ is $\omega$-Tukey.  \(Q\)

7I Examples  (a) $\mathbb{N}^\omega$ is $\kappa$-measure-bounded for any real-valued-measurable cardinal $\kappa$. (Use 7H(e-i).)

(b) $\omega_1^\omega$ is $\kappa$-measure-bounded for any real-valued-measurable cardinal $\kappa$. (Use 7H(e-ii).)

(c) $\mathbb{N}^\omega / \mathcal{F}$ is $\kappa$-measure-bounded for any real-valued-measurable cardinal $\kappa$, any filter $\mathcal{F}$ on $\mathbb{N}$. (Use 7Hf.)

(d) If $\lambda$ and $\theta$ are cardinals then $|\lambda|^{< \theta}$ is $\kappa$-measure-bounded for any real-valued-measurable cardinal $\kappa \geq \lambda$.  \(P\)

If $\mu$ is a $\kappa$-additive probability with domain $\mathcal{P}|\lambda|^{< \theta}$ we may apply 5Aa with $X = |\lambda|^{< \theta}, Y = \lambda$ and $f$ the identity map to find an $M \in X$ such that $\mu(\mathcal{P}M) > 0$.  \(Q\)

(e) Let $\mathcal{E}$ be the family of closed Lebesgue negligible subsets of $[0,1]$, ordered by $\subseteq$. Then $\mathcal{E}$ is not $\kappa$-measure-bounded for any atomlessly-measurable cardinal $\kappa$.  \(P\)

If $\kappa$ is atomlessly-measurable, there is a $\kappa$-additive extension $\mu$ of Lebesgue measure to $\mathcal{P}[0,1]$. Define $f : [0,1] \to \mathcal{E}$ by setting $f(a) = \{a\}$ for every $a \in [0,1]$, and set $\nu = \mu f^{-1} : \mathcal{P}\mathcal{E} \to [0,1]$. Then $\nu(\mathcal{E} : \mathcal{E} \subseteq F) = \mu F = 0$ for every $F \in \mathcal{E}$, so $\nu$ witnesses that $\mathcal{E}$ is not $\kappa$-measure-bounded.  \(Q\)

Remark I include (e) because (as explained in FREMLIN p90) many of the natural partially ordered sets $P$ of analysis allow $\omega$-Tukey functions from $\mathcal{E}$ to $P$, so cannot be $\kappa$-measure-bounded.

7J Proposition Suppose that $\kappa$ is a real-valued-measurable cardinal and that $P$ is a $\kappa$-measure-bounded partially ordered set.

(a) If $(Q_\xi)_{\xi < \kappa}$ is an increasing family of subsets of $P$ and $Q = \bigcup_{\xi < \kappa} Q_\xi$ is cofinal with $P$, then there is a $\xi < \kappa$ such that $Q_\xi$ is cofinal with $P$.

(b) $\text{cf}(\text{cf}(P)) \neq \kappa$.

proof (a) If no $Q_\xi$ is cofinal with $P$, then we may choose a function $f : \kappa \to P$ such that $f(\xi) \notin q$ whenever $\xi < \kappa$, $q \in Q_\xi$. Let $\nu$ be a witnessing probability on the real-valued-measurable cardinal $\kappa$, and set $\mu = \nu f^{-1} : \mathcal{P}P \to [0,1]$. Then there should be a $p \in P$ such that $\mu\{p' : p' \leq p\} > 0$. But now there are a $q \in Q$ such that $p \leq q$ and a $\xi < \kappa$ such that $q \in Q_\xi$, in which case

$$0 < \mu\{p' : p' \leq p\} \leq \nu\{q : f(q) \leq q\} \leq \nu\{q : q < \xi\},$$

so there is a $d \in D$ such that $\mu\{p : p \leq d\} > 0$. As $\mu$ is arbitrary, $P$ is $\kappa$-measure-bounded.  \(Q\)
which is impossible.  

(b) Set $\lambda = \text{cf}(P)$.  ? If $\text{cf}(\lambda) = \kappa$, take a cofinal set $Q \subseteq P$ of cardinal $\lambda$. Then there is an increasing family $\langle Q_\xi \rangle_{\xi < \kappa}$ of subsets of $Q$, all of cardinal strictly less than $\lambda$, therefore not cofinal with $P$, but with union $Q$, contradicting (a).

7K Corollaries Let $\kappa$ be a real-valued-measurable cardinal.

(a) If $\mathcal{F}$ is any filter on $\mathbb{N}$, then $\text{cf}(\text{cf}(\mathbb{N}/\mathcal{F})) \neq \kappa$; in particular, $\text{cf}(\emptyset) \neq \kappa$.

(b) If $\lambda$ is a cardinal and $\langle \theta_\xi \rangle_{\xi < \lambda}$ is a family of regular cardinals all greater than $\lambda$ and less than $\kappa$, then $\text{cf}(\prod_{\xi < \lambda} \theta_\xi) < \kappa$.

(c) If $\alpha$ and $\gamma$ are cardinals less than $\kappa$ then $\Theta(\alpha, \gamma)$ (see A1Jb) is less than $\kappa$.

(d) If $\alpha$, $\beta$, $\gamma$ and $\delta$ are cardinals, with $\delta \geq \omega_1$, $\gamma \leq \beta$, $\alpha < \kappa$, then $\text{cov}_{\mathcal{S}_\mathcal{H}\mathcal{B}}(\alpha, \beta, \gamma, \delta)$ (see A1Ja) is less than $\kappa$.

(e) $\Theta(\kappa, \kappa) = \kappa$ and

\[ \{ \alpha : \alpha < \kappa \text{ is a cardinal, } \Theta(\alpha, \alpha) = \alpha \} \]

belongs to the rvm filter of $\kappa$.

(f) If $\kappa = \kappa$, $\lambda < \kappa$ and $\langle P_\xi \rangle_{\xi < \lambda}$ is a family of partially ordered sets such that $\omega < \text{add}(P_\xi)$, $\text{cf}(P_\xi) < \kappa$ for every $\xi < \kappa$, then $\text{cf}(\prod_{\xi < \lambda} P_\xi) < \kappa$.

proof (a) Use 71c and 7Jb.

(b) ? If $\xi < \delta = \text{cf}(\prod_{\xi < \lambda} \theta_\xi)$, then by A11c there is an ultrafilter $\mathcal{F}$ on $\lambda$ such that $\delta = \text{cf}(\prod_{\xi < \lambda} \theta_\xi/\mathcal{F})$, and by A11d there is a family $\langle \theta'_\xi \rangle_{\xi < \lambda}$ such that $\theta'_\xi \leq \theta_\xi$ for every $\xi < \kappa$ and $\kappa = \text{cf}(\prod_{\xi < \lambda} \theta'_\xi/\mathcal{F})$. But by 7H(d-ii) and 7Hf, $\prod_{\xi < \lambda} \theta'_\xi/\mathcal{F}$ is $\kappa$-measure-bounded and its cofinality cannot be $\kappa$, by 7Jb.

(c) ? Suppose, if possible, otherwise. Then for each $\xi < \kappa$ there must be a family $\langle \theta_{\xi\xi} \rangle_{\xi < \lambda}$ of regular cardinals less than $\alpha$ such that $\lambda_\xi < \theta_{\xi\xi}$ for every $\xi < \lambda$ and $\text{cf}(\prod_{\xi < \lambda} \theta_{\xi\xi}) \geq \xi$. Because $\alpha < \kappa$ there is a cardinal $\lambda < \kappa$ such that

\[ A = \{ \xi : \xi < \kappa, \lambda_\xi = \lambda \} \]

is unbounded in $\kappa$. Now by 5Ab there is a set $M \subseteq \alpha$, of cardinal at most $\lambda$, such that

\[ B = \{ \xi : \xi \in A, \theta_\xi \in M \forall \xi < \lambda \} \]

is unbounded in $\kappa$. Let $\langle \theta_\xi \rangle_{\xi < \lambda'}$ be any enumeration of $M$. By (b), there is a cofinal set $F \subseteq \prod_{\xi < \lambda'} \theta_\xi$ with $\#(F) < \kappa$. Let $\xi < \lambda$ be such that $\xi > \#(F)$. For each $f \in F$ define $g_f \in \prod_{\xi < \lambda} \theta_{\xi\xi}$ by setting

\[ g_f(\xi') = f(\xi') \text{ whenever } \theta_{\xi\xi} = \theta_\xi. \]

Then $\{g_f : f \in F\}$ is cofinal with $\prod_{\xi < \lambda} \theta_{\xi\xi}$, because if $h \in \prod_{\xi < \lambda} \theta_{\xi\xi}$ there is an $f \in F$ such that

\[ f(\xi') \geq \sup \{ h(\xi) : \xi < \lambda, \theta_{\xi\xi} = \theta_\xi \}. \]

for every $\xi' < \lambda'$, and in this case $h \leq g_f$. So

\[ \#(F) < \xi \leq \text{cf}(\prod_{\xi < \lambda} \theta_{\xi\xi}) \leq \#(F), \]

which is absurd.

(d) This is trivial if any of the cardinals $\alpha$, $\beta$, $\gamma$ is finite; let us take it that they are all infinite. In this case $\text{cov}_{\mathcal{S}_\mathcal{H}\mathcal{B}}(\alpha, \beta, \gamma, \delta) \leq \text{cov}_{\mathcal{S}_\mathcal{H}}(\alpha, \beta, \gamma, \omega_1)$, and $\text{cov}_{\mathcal{S}_\mathcal{B}}(\alpha, \gamma, \gamma, \omega_1) \leq \Theta(\alpha, \gamma) < \kappa$, by A1K and (c) above.

(e) From (d) we see that

\[ C = \{ \alpha : \alpha < \kappa \text{ is a cardinal, } \Theta(\beta, \beta) \leq \alpha \forall \beta < \alpha \} \]

is a closed unbounded set in $\kappa$, so $D$ belongs to the rvm filter of $\kappa$, where $D$ is the set of regular cardinals in $C$ (using a fraction of 4K). But if $\alpha \in D$, and $\langle \theta_\xi \rangle_{\xi < \lambda}$ is a family of regular cardinals with $\lambda < \theta_\xi < \alpha$ for every $\xi < \lambda$, set $\beta = \sup_{\xi \in I} \theta_\xi$; then

\[ \text{cf}(\prod_{\xi < \lambda} \theta_\xi) \leq \Theta(\beta, \beta) \leq \alpha. \]

Accordingly $\Theta(\alpha, \alpha) = \alpha$ for every $\alpha \in D$. In the same way $\Theta(\kappa, \kappa) = \kappa$. 

(f) By 5E, we have $\text{cf}(\prod_{\xi<\lambda} P_\xi) \leq \kappa$; by 7H(d-ii) and 7Jb, $\text{cf}(\prod_{\xi<\lambda} P_\xi) \neq \kappa$.

**Remarks** Part (f) comes from Prikry 75, Theorem 2c. Parts (b)-(d) come from Gitik & Shelah p91.

**7L Cofinalities II: Proposition** Suppose that $\kappa$ is an atomlessly-measurable cardinal. Let $X$ be a set of cardinal at most $\min(2^\kappa, \kappa^{(\omega)})$ and $\mathcal{F}$ a filter on $X$ which has a base of cardinal less than $\kappa$ and contains no countable set. Then $\text{cf}(\mathbb{N}^X/\mathcal{F}) \geq \kappa$.

**proof** Let $\nu$ be a witnessing probability on $\kappa$. By the Gitik-Shelah theorem (3F-3H), there is a stochastically independent family $\langle E_{xn} \rangle_{x \in X, n \in \mathbb{N}}$ of subsets of $\kappa$ of measure $\frac{1}{2}$. Define $\phi : \kappa \rightarrow \mathbb{N}^X$ by

$$\phi(\xi)(x) = \min\{n : \xi \in E_{xn}\} \text{ if } \xi \in \bigcup_{n \in \mathbb{N}} E_{xn},$$

$$= 0 \text{ otherwise.}$$

Now let $A \subseteq \mathcal{F}$ be a base of cardinal less than $\kappa$. For $f \in \mathbb{N}^X$, $A \subseteq \mathcal{F}$ set

$$W_{fA} = \{\xi : \xi < \kappa, \phi(\xi)(x) \leq f(x) \ \forall \ x \in A\}.$$

Then

$$\nu W_{fA} = \prod_{x \in A} (1 - 2^{-f(x)-1}) = 0$$

because $A$ is uncountable. If $F \subseteq \mathbb{N}^X$ is any set of cardinal less than $\kappa$, then

$$\{\xi : \exists \ f \in F, \ \phi(\xi) \leq \mathcal{F} f\} = \bigcup_{f \in F, A \subseteq \mathcal{F}} W_{fA} \in \mathcal{N}_\nu,$$

so there is a $\xi < \kappa$ such that $\phi(\xi) \leq \mathcal{F} f$ for every $f \in F$, and $\{f^* : f \in F\}$ is not cofinal with $\mathbb{N}^X/\mathcal{F}$. As $F$ is arbitrary, $\text{cf}(\mathbb{N}^X/\mathcal{F}) \geq \kappa$.

**7M Corollary** Let $\kappa$ be an atomlessly-measurable cardinal.

(a) $\text{cf}(\mathbb{N}^\lambda) \geq \kappa$ for every $\lambda > \omega$. (For $\text{cf}(\mathbb{N}^\lambda) \geq \text{cf}(\mathbb{N}^{\omega_1})$.)

(b) If $\lambda < \kappa$ is a regular uncountable cardinal, and $\mathcal{F} = \{F : F \subseteq \lambda, \#(\lambda \setminus F) < \lambda\}$ then $\text{cf}(\mathbb{N}^\lambda/\mathcal{F}) \geq \kappa$.

(c) Suppose that $\kappa$ is atomlessly-measurable. Let $X$ be a set of cardinal less than $\kappa$ and $\mathcal{F}$ a filter on $X$ with a base of cardinal less than $\kappa$, not containing any countable set. Then $\text{cf}(\mathbb{N}^X/\mathcal{F}) = \kappa$. (Use 5E to see that $\#(\mathbb{N}^X) = \kappa$.)

**Remark 7L-7M come from Jech & Prikry 84.**

**7N Remark** The results 7Kb-e are interesting; the theory in which they are embedded, partially described in A1H-A1J, is astonishing. But it may well be that the last word has not been said. From 7Kb we see, for instance, that if $\kappa$ is real-valued-measurable then $\text{cf}(\prod_{n \in \mathbb{N}} \omega_n) < \kappa$. But in fact $\text{cf}(\prod_{n \in \mathbb{N}} \omega_n) < \omega_{\omega_3}$ (Burke & Magidor 90, Theorem 6.1). We can hope that further inequalities of this type will swallow up 7Kb-e completely.

**7O Lemma** Let $\kappa$ be an atomlessly-measurable cardinal, and $\gamma$, $\delta$ cardinals such that $\omega < \gamma < \delta < \kappa$, $2^\delta = 2^\gamma$ for $\gamma < \beta < \delta$, but $2^\delta > 2^\gamma$. Then $\delta$ is regular and $2^\delta = \text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \delta) = \text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) = \text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1)$.

**proof** $\delta$ is regular because $2^\delta$ is at most the cardinal power $\max_{\beta < \delta} 2^{\beta^{(\delta)}}$. Of course

$$\text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \leq \text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) \leq \text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, 2) \leq \#([2^\gamma]^{(\delta)}) \leq 2^\delta.$$

For the reverse inequality, let $E \subseteq [2^\gamma]^{<\kappa}$ be a set of cardinal $\text{cov}_{\text{Sh}}(2^\gamma, \kappa, \delta^+, \delta)$ such that every member of $[2^\gamma]^{<\delta}$ is covered by fewer than $\delta$ members of $E$. For each ordinal $\xi < \delta$ let $\phi_\xi : \mathcal{P}\xi \rightarrow 2^\gamma$ be an injective function. For $A \subseteq \delta$ define $f_A : \delta \rightarrow 2^\gamma$ by

$$f_A(\xi) = \phi_\xi(A \cap \xi) \ \forall \ \xi < \delta.$$

Choose $E_A \in E$ such that $f_A^{-1}[E_A]$ is cofinal with $\delta$; such must exist because $\delta$ is regular and $f_A[\delta]$ can be covered by fewer than $\delta$ members of $E$.

If $2^\delta > \#(E)$ then there must be an $E \in E$ and an $A \subseteq \mathcal{P}\delta$ such that $\#(A) = \kappa$ and $E_A = E$ for every $A \in A$. For each pair $A$, $B$ of distinct members of $A$ set $\xi_{AB} = \min(A \Delta B) < \delta$. By 5B, there is a set $B \subseteq A$, of cardinal $\kappa$, such that $M = \{\xi_{AB} : A, B \in B, A \neq B\}$ is countable. Set $\zeta = \sup M < \delta$. Next,
for each $A \in \mathcal{B}$, take $\eta_A > \zeta$ such that $f_A(\eta_A) \in E$. Let $\eta < \delta$ be such that $\mathcal{C} = \{ A : A \in \mathcal{B}, \eta_A = \eta \}$ has cardinal $\kappa$. Then we have a map

$$A \mapsto f_A(\eta) = \phi_A(A \cap \eta) : \mathcal{C} \to E$$

which is injective, because if $A, B$ are distinct members of $\mathcal{C}$ then $\xi_{AB} \leq \zeta < \eta$, so $A \cap \eta \neq B \cap \eta$. So $\#(E) \geq \kappa$; but $E \in \mathcal{E} \subseteq [2^\gamma]^{< \kappa}$. \hfill \Box

7P Theorem If $\kappa$ is an atomlessly-measurable cardinal,

$$\{ 2^\gamma : 0 \leq \gamma < \kappa \}$$

is finite.

proof Suppose, if possible, otherwise.

(a) Define a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of cardinals by setting

$$\gamma_0 = \omega, \gamma_{n+1} = \min \{ \gamma : 2^\gamma > 2^{\gamma_n} \} \quad \forall \ n \in \mathbb{N}.$$ 

Then we are supposing that $\gamma_n < \kappa$ for every $n$, so by Lemma 7O every $\gamma_n$ is regular and

$$2^{\gamma_{n+1}} = \text{cov}\_\text{SH}(2^{\gamma_n}, \kappa, \gamma_{n+1}^{\pm}, \omega_1) \quad \forall \ n \in \mathbb{N},$$

and by Theorem A1K

$$2^{\gamma_{n+1}} \leq \Theta(2^{\gamma_n}, \gamma_{n+1}^{\pm});$$

also, of course, $\Theta(2^{\gamma_n}, \gamma_{n+1}^{\pm}) \leq 2^{\gamma_{n+1}}$, for every $n$. (For the definition of $\Theta(\alpha, \beta)$, see A1J-A1K.) Thus

$$2^{\gamma_{n+1}} = \Theta(2^{\gamma_n}, \gamma_{n+1}^{\pm})$$

for every $n$.

(b) Now $\Theta(2^{\gamma_n}, \gamma) = \Theta(\gamma, \gamma)$ whenever $n \in \mathbb{N}$ and $\gamma$ is a regular cardinal with $\gamma_n < \gamma < \kappa$. \hfill \Box

Induce on $n$. For $n = 0$ we have $\epsilon = 2^{\gamma_0}$. For the inductive step to $n + 1$, if $\gamma$ is regular and $\gamma_{n+1} < \gamma < \kappa$, then $\epsilon \geq \kappa \geq \Theta(\gamma, \gamma)$ (7Kb-c), so

$$\Theta(2^{\gamma_{n+1}}, \gamma) = \Theta(\Theta(2^{\gamma_n}, \gamma_{n+1}^{\pm}), \gamma) \leq \Theta(\Theta(2^{\gamma_n}, \gamma), \gamma)$$

because $\gamma \geq \gamma_{n+1}^{\pm}$

$$= \Theta(\Theta(\gamma, \gamma), \gamma)$$

by the inductive hypothesis

$$= \Theta(\gamma, \gamma)$$

by Lemma A1L

$$\leq \Theta(2^{\gamma_{n+1}}, \gamma)$$

because $2^{\gamma_{n+1}} \geq \epsilon$. \hfill \Box

In particular, $2^{\gamma_n} = \Theta(\epsilon, \gamma_n^{\pm})$ for every $n \geq 1$ (as well as for $n = 0$).

(c) For each $n \in \mathbb{N}$, let $\alpha_n$ be the least cardinal such that $\Theta(\alpha_n, \gamma_n^{\pm}) > \epsilon$. Then $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is non-increasing and $\alpha_1 \leq \epsilon$, so there are $n \geq 1$, $\alpha \leq \epsilon$ such that $\alpha_m = \alpha$ for every $m \geq n$. Now for $m \geq n$ we have

$$\epsilon < \Theta(\alpha, \gamma_m^{\pm}) \leq \max(\alpha, (\sup_{\alpha' < \alpha} \Theta(\alpha', \gamma_m^{\pm}))^{\text{cf}(\alpha)})$$

$$\leq \max(\alpha, \epsilon^{\text{cf}(\alpha)}) = 2^{\epsilon^{\text{cf}(\alpha)}},$$

using A1M. Also we still have $\epsilon > \alpha \geq \kappa > \Theta(\gamma_m^{\pm}, \gamma_m^{\pm})$ because $\Theta(\alpha', \gamma_m^{\pm}) < \kappa$ for every $\alpha' < \kappa$. So

$$\Theta(\alpha, \gamma_m^{\pm}) = \Theta(\Theta(\alpha, \gamma_m^{\pm}), \gamma_m^{\pm}) \geq \Theta(\epsilon, \gamma_m^{\pm}) = 2^{\gamma_m}$$

for every $m \geq n$; consequently

$$2^{\gamma_m} < 2^{\gamma_{m+1}} \leq 2^{\epsilon^{\text{cf}(\alpha)}}$$

and $\text{cf}(\alpha) > \gamma_m$ for every $m$. But this means that
\[ \Theta(\alpha, \gamma_m) = \sup_{\alpha' < \alpha} \Theta(\alpha', \gamma_m^+) \leq \epsilon \]

for each \( m \), which is absurd. \( \Box \)

This contradiction proves the theorem.

**Remark** This comes from Gitik & Shelah p91.

**7Q Corollary** Let \( \kappa \) be an atomlessly-measurable cardinal.

(a) There is a least cardinal \( \gamma < \kappa \) such that \( 2\gamma^+ = 2^\delta \) for \( \gamma \leq \delta < \kappa \).

(b) If \( \nu \) is a witnessing probability on \( \kappa \) and \( \lambda \) is the Maharam type of \( (\kappa, \mathcal{P}_\kappa, \nu) \) then the cardinal power \( \lambda^\gamma \) is \( 2^\kappa \).

(c) If \( \epsilon < \kappa^{(+\omega)} \) then \( \gamma = \omega \), that is, \( 2^\delta = \epsilon \) for \( \omega \leq \delta < \kappa \).

**proof (a)** Immediate from 7P.

(b) Let \( \mathfrak{A} \) be the measure algebra \( \mathcal{P}_\kappa/\mathcal{N}_\nu \). For \( \xi < \kappa \) let \( \phi_\xi : \mathcal{P}_\xi \rightarrow \mathcal{P}_\gamma \) be an injective function. For \( \eta < \gamma \), \( A \subseteq \kappa \) set

\[ d_{A\eta} = \{ \xi : \eta \in \phi_\xi(A \cap \xi) \} \in \mathfrak{A}. \]

If \( A, B \subseteq \kappa \) are distinct then there is a \( \zeta < \kappa \) such that \( \phi_\xi(A \cap \xi) \neq \phi_\xi(B \cap \xi) \) for \( \zeta \leq \xi < \kappa \), so that (because \( \gamma < \kappa \)) there is an \( \eta < \gamma \) such that \( d_{A\eta} \neq d_{B\eta} \). Thus \( 2^\kappa \leq \#(\mathfrak{A})^\gamma \leq (\lambda^\gamma)^\gamma = \lambda^\gamma \), using A2Hb. On the other hand, \( \lambda \leq 2^\kappa \) and \( \gamma < \kappa \), so \( \lambda^\gamma \leq 2^{\kappa\gamma} \).

(c)(i) We need the following elementary fact: if \( \alpha, \beta, \gamma \) are infinite cardinals, with \( \delta \) regular and \( \text{cf}(\alpha) \neq \delta \), then

\[ \text{cov}_{\mathfrak{Sh}}(\alpha, \beta, \delta^+, \delta) \leq \max(\alpha, \sup_{\alpha' < \alpha} \text{cov}_{\mathfrak{Sh}}(\alpha', \beta, \delta^+, \delta)). \]

**P** Set \( \theta = \max(\alpha, \sup_{\alpha' < \alpha} \text{cov}_{\mathfrak{Sh}}(\alpha', \beta, \delta^+, \delta)) \). For each \( \xi < \alpha \) choose \( E_\xi \subseteq [\xi]^{<\beta} \) such that \( \#(E_\xi) = \text{cov}_{\mathfrak{Sh}}([\xi]^{<\beta}, \beta, \delta^+, \delta) \) and every member of \( [\xi]^{<\delta} \) can be covered by fewer than \( \delta \) members of \( E_\xi \). Set \( \mathcal{E} = \bigcup_{\xi < \alpha} E_\xi \); then \( \#(\mathcal{E}) \leq \theta \). Take any \( A \in [\alpha]^{<\delta} \). If \( \text{cf}(\alpha) > \delta \) then \( A \subseteq \xi \) for some \( \xi < \alpha \) so \( A \) is covered by fewer than \( \delta \) members of \( E_\xi \subseteq \mathcal{E} \). If \( \text{cf}(\alpha) < \delta \), take a cofinal set \( C \subseteq \alpha \) of cardinal less than \( \delta \); then for each \( \xi \in C \) there is an \( A_\xi \in [E_\xi]^{<\delta} \) covering \( A \cap \xi \), so \( A = \bigcup_{\xi \in C} A_\xi \in [\mathcal{E}]^{<\delta} \) (because \( \delta \) is regular), and \( A \subseteq \bigcup \mathcal{A} \).

As \( A \) is arbitrary, \( \mathcal{E} \) witnesses that \( \text{cov}_{\mathfrak{Sh}}(\alpha, \beta, \delta^+, \delta) \leq \theta \), as required. \( \square \)

(ii) ? Now suppose that \( \gamma > \omega \), that is, that \( 2^\gamma > \epsilon \). Let \( \gamma_1 \) be the least cardinal such that \( 2^{\gamma_1} > \epsilon \). Then an easy induction on \( \alpha \), using (i) just above, shows that \( \text{cov}_{\mathfrak{Sh}}(\alpha, \kappa, \gamma_1^+, \gamma_1) \leq \alpha \) whenever \( 1 \leq \alpha < \kappa^{(+\gamma_1)} \).

In particular, \( \text{cov}_{\mathfrak{Sh}}(\epsilon, \kappa, \gamma_1^+, \gamma_1) \leq \epsilon \). But in 7Q we saw that \( \text{cov}_{\mathfrak{Sh}}(\epsilon, \kappa, \gamma_1^+, \gamma_1) = 2^{\gamma_1} \). \( \Box \)

**Remark** The case \( \kappa = \epsilon \) of (b) is due to Prikry 75.

**8. PMEA and NMA**

In this section I give a brief description of two axioms, both much stronger than the assertion ‘\( \epsilon \) is real-valued-measurable’ but nevertheless apparently consistent, with some of their consequences in set theory and general topology.

**8A Theorem** If one of the following statements is true, so are the others:

(i) for every cardinal \( \lambda \), there is a probability space \( (X, \mathcal{P}X, \mu) \) of Maharam type at least \( \lambda \), and with \( \text{add}(\mu) = \epsilon \);

(ii) for every cardinal \( \lambda \), there is a \( \epsilon \)-additive probability with domain \( \mathcal{P}(\{0,1\}^\lambda) \) which extends the usual measure \( \mu_\lambda \) of \( \{0,1\}^\lambda \);

(iii) for every Radon measure space \( (X, \mathcal{F}, \Sigma, \mu) \), there is a \( \epsilon \)-additive probability with domain \( \mathcal{P}X \) which extends \( \mu \).

**proof (a)(i)\(\Rightarrow\)(ii)** Assume (i). Let \( \lambda \) be any cardinal; of course (ii) is surely true for finite \( \lambda \), so we may take it that \( \lambda \geq \omega \). Let \( (X, \mathcal{P}X, \mu) \) be a probability space of Maharam type greater than \( \lambda \) and with \( \text{add}(\mu) = \epsilon \).

Let \( E \in \mathcal{P}X \setminus \mathcal{N}_\mu \) be such that \( (E, \mathcal{P}E, \mu(E)) \) is Maharam homogeneous with Maharam type at least \( \lambda \) (A2Hh). Setting \( \mu' A = \mu A/\mu E \) for \( A \subseteq E \), \( (E, \mathcal{P}E, \mu') \) is a Maharam homogeneous probability space of
Maharam type at least \( \lambda \), and \( \text{add}(\mu') \geq \kappa \). By A2Ka, there is an inverse-measure-preserving function \( f : E \to \{0,1\}^\lambda \). Now \( \nu = \mu f^{-1} \) (A2Db) is a \( \lambda \)-additive extension of \( \mu_A \) to \( \mathcal{P}(\{0,1\}^\lambda) \).

(b)(ii)\(\Rightarrow\)(iii) Assume (ii). Let \((X, \mathcal{S}, \mu)\) be any Radon measure space.

\[ \text{proof} \]

\[ \text{case 1} \] Suppose that \( \mu_X = 1 \) and that \( \mu \) has Maharam type \( \lambda \). Then there is an inverse-measure-preserving function \( f : \{0,1\}^\lambda \to X \) (A2Ka). If \( \nu \) is a \( \lambda \)-additive extension of \( \mu_X \) to \( \mathcal{P}(\{0,1\}^\lambda) \), then \( \nu f^{-1} \) is a \( \lambda \)-additive extension of \( \mu \) to \( \mathcal{P}X \).

\[ \text{case 2} \] In general, there is a partition \((X_i)_{i \in I}\) of \( X \) into measurable sets of finite measure such that \( \mu E = \sum_{i \in I} \mu(E \cap X_i) \) for any \( E \in \mathcal{S} \) (A2Ja). By case 1, applied to a suitable normalization of \( \mu[X_i] \) (the case \( \mu[X_i] = 0 \) being trivial), there is an extension of \( \mu[X_i] \) to a \( \lambda \)-additive measure \( \nu_i \) with domain \( \mathcal{P}X_i \), for each \( i \in I \). Now setting \( \nu A = \sum_{i \in I} \nu_i(A \cap X_i) \) for each \( A \subseteq X \), we get a \( \lambda \)-additive extension of \( \mu \) to \( \mathcal{P}X \).

(c)(iii)\(\Rightarrow\)(ii) and (ii)\(\Rightarrow\)(i) are trivial.

8B Definition PMEA (the ‘product measure extension axiom’) is the assertion that the statements (i)-(iii) of 8A are true.

8C Theorem [Kumen] If ‘ZFC + there is a strongly compact cardinal’ is consistent, so is ‘ZFC + PMEA’.

\[ \text{proof} \] See Fleissner 84, Theorem 3.4.

8D Proposition Assume PMEA. Then \( \kappa \) is atomlessly-measurable.

\[ \text{proof} \] Use 8A(ii), with \( \lambda = \omega \), and 1D.

8E Lemma Assume PMEA. Let \( X \) be a normal topological space and \( \langle F_i \rangle_{i \in I} \) a discrete family of subsets of \( X \). Suppose that for each \( x \in X \) we are given a downwards-directed family \( \mathcal{U}_x \) of sets such that \#(\mathcal{U}_x) < \kappa \) and whenever \( G \subseteq X \) is open, \( i \in I \), and \( x \in F_i \subseteq G \) there is a \( U \in \mathcal{U}_x \) such that \( U \subseteq G \). Then there is a family \( \langle U_{x,y} \rangle_{x \in X} \) such that \( U_{x,y} \subseteq U_x \) for every \( x \in X \) and whenever \( i, j \in I \) are distinct, \( x \in F_i \) and \( y \in F_j \) then \( U_x \cap U_y = \emptyset \).

\[ \text{proof} \] For each \( z \in \{0,1\}^I \) let \( G_z, H_z \) be disjoint open sets such that \( F_i \subseteq G_z \) whenever \( z(i) = 1 \) and \( F_i \subseteq H_z \) whenever \( z(i) = 0 \). Let \( \nu \) be a \( \kappa \)-additive probability on \( \{0,1\}^I \) extending the usual measure on \( \{0,1\}^I \). Write \( F = \bigcup_{i \in I} F_i \).

For each \( x \in F \), \( U \in \mathcal{U}_x \) write

\[ A(x, U) = \{ z : z \in \{0,1\}^I, \text{ either } z(i) = 1 \text{ and } U \subseteq G_z \text{ or } z(i) = 0 \text{ and } U \subseteq H_z \}, \]

where \( i \) is that member of \( I \) such that \( x \in F_i \). For each \( x \in F \), \( \{A(x, U) : U \in \mathcal{U}_x \} \) is an upwards-directed family in \( \mathcal{P}(\{0,1\}^I) \), of cardinal less than \( \kappa \), covering \( \{0,1\}^I \). Consequently there is a \( U_x \in \mathcal{U}_x \) such that \( \nu A(x, U_x) > \frac{1}{2} \) (A2Co).

If now \( x \in F_i, y \in F_j \) where \( i \neq j \), then

\[ \nu \{ z : z \in A(x, U_x) \cap A(y, U_y), z(i) \neq z(j) \} > 0. \]

Take any \( z \in A(x, U_x) \cap A(y, U_y) \) such that \( z(i) \neq z(j) \). Then one of \( U_x, U_y \) is included in \( G_z \) and the other in \( H_z \), so \( U_x \cap U_y = \emptyset \), as required.

8F Theorem Assume PMEA. Let \( X \) be a normal topological space in which \( \chi(x, X) < \kappa \) for every \( x \in X \). Then \( X \) is collectionwise normal.

\[ \text{proof} \] Let \( \langle F_i \rangle_{i \in I} \) be a discrete family of subsets of \( X \). For each \( x \in X \) let \( \mathcal{U}_x \) be a base of neighbourhoods of \( x \) with \#(\mathcal{U}_x) < \kappa \). By 8E there is a family \( \langle U_x \rangle_{x \in X} \) such that \( U_x \in \mathcal{U}_x \) for each \( x \in X \) and \( U_x \cap U_y = \emptyset \) whenever \( x, y \in X \) belong to different \( F_i \). Set \( G_i = \bigcup_{U_x : x \in F_i} \) for each \( i \in I \); then \( \langle G_i \rangle_{i \in I} \) is a disjoint family of open sets and \( F_i \subseteq G_i \) for every \( i \). As \( \langle F_i \rangle_{i \in I} \) is arbitrary, \( X \) is collectionwise normal.

8G Corollary Assume PMEA. Then every normal Moore space is metrizable.

\[ \text{proof} \] Moore spaces are first-countable; by 8F, assuming PMEA, normal Moore spaces are collectionwise normal. By A3D, a collectionwise normal Moore space is metrizable. So we have the result.

Remarks (a) 8G is due to Nyikos 80; 8F, as stated, is due to Junnila 83.

(b) Of course we can do something with the ideas of 8E-8F without the full strength of PMEA. If we have an extension of the usual measure on \( \{0,1\}^I \) to a \( \kappa \)-additive measure with domain \( \mathcal{P}(\{0,1\}^I) \), then
we can make 8E work if \#(U_x) < \kappa for every x \in X. (Compare Kulesza Levy & Nyikos 91, §4.) Thus if \kappa is atomlessly-measurable, we shall have every normal Moore space of weight at most \min(2^\omega,\kappa^{(+\omega)}) metrizable, using the Gitik-Shelah theorem (3F, 3I).

(c) Under NMA, stronger results my be obtained by putting 8M below together with 8E-8G.

8H Definition NMA (the 'normal measure axiom') is the assertion

For every set I there is a \epsilon-additive probability \nu on S = [I]^{<\epsilon}, with domain \mathcal{P}S, such that

(i) \nu\{i : s \in S\} = 1 for every i \in I;

(ii) if A \subseteq S and \nu A > 0 and f : A \rightarrow I is such that f(s) \in s for every s \in A, then there is an i \in I such that \nu\{s : s \in A, f(s) = i\} > 0.

Remarks (a) If we read the above statement as 'NMA(I)', with the set I as a parameter, then we see that the truth of NMA(I) depends only on \#(I), so that if seeking to prove NMA we need consider only NMA(\lambda) for cardinals \lambda. More importantly, we see that NMA(I) implies NMA(J) for every J \subseteq I; for if \nu is a probability on [J]^{<\epsilon} as above, and we define f : [I]^{<\epsilon} \rightarrow [J]^{<\epsilon} by setting f(s) = s \cap J for every s \in [I]^{<\epsilon}, then \nu f^{-1} will witness NMA(J). Consequently NMA will be true iff NMA(\lambda) is true for arbitrarily large cardinals \lambda; e.g. for all regular \lambda \geq \epsilon.

(b) Observe that the condition (ii) of the statement above can be replaced with

(ii)' if A \subseteq S and f : A \rightarrow I is such that f(s) \in s for every s \in A, then there is a countable D \subseteq I such that f(s) \in D for \nu-almost all s \in A,

or with

(ii)'' if A \subseteq S and f : A \rightarrow I is such that f(s) \in s for every s \in A, then for every \delta > 0 there is a finite D \subseteq I such that \nu(A \setminus f^{-1}[D]) \leq \delta.

8I Theorem If 'ZFC + there is a supercompact cardinal' is consistent, so is 'ZFC + NMA'.

proof Prikry 75, Fleissner 89.

8J Proposition NMA implies PMEA.

proof Assume NMA. Let \lambda be any cardinal. Let \theta be a regular cardinal greater than the cardinal power \lambda^\omega. Let \nu be a probability on [\theta]^{<\epsilon} as in 8H. For \xi < \theta define f_\xi : [\theta]^{<\epsilon} \rightarrow \epsilon by setting

f_\xi(s) = \text{otp}(s \cap \xi) \quad \forall s \in [\theta]^{<\epsilon}.

Then if \xi < \eta < \theta we have f_\xi(s) < f_\eta(s) whenever \xi \in s, that is, for \nu-almost all s.

Let g : \epsilon \rightarrow \mathcal{P}N be any injection. For \xi < \theta, n \in \mathbb{N} let a_{\xi n} be the equivalence class of \{s : n \in g(f_\xi(s))\} in the measure algebra \mathfrak{A} of ([\theta]^{<\epsilon}, \mathcal{P}([\theta]^{<\epsilon}), \nu). Then for any \xi < \eta < \theta there must be an n \in \mathbb{N} such that a_{\xi n} \neq a_{\eta n}, so \#(\mathfrak{A}) \geq \theta > \lambda^\omega. Now \#(\mathfrak{A}) \leq \tau(\mathfrak{A})^{\omega} (A2Hb), so \tau(\mathfrak{A}) > \lambda.

Thus we have (i) of 8A, and PMEA is true.

Remark This result is due to Kunen; the argument above is taken from Fleissner 89.

8K Theorem Assume NMA. Then the singular cardinals hypothesis is true.

proof (a) As noted in A1N, it will be more than enough if I can show that \#([\lambda]^{<\epsilon}) \leq \lambda for every regular cardinal \lambda \geq \epsilon. Let \nu be a measure on S = [\lambda]^{<\epsilon} as described in 8H. For each \xi < \lambda choose a set c_\xi \subseteq \xi, cofinal with \xi, of cardinality \text{cf}(\xi).

(b) If \xi < \lambda there is a \zeta' < \lambda such that
\nu\{s : s \in S, c_{\sup s} \cap \zeta' \neq \emptyset\} = 1.

P Set E = \{s : \zeta + 1 \in S\}; then \nu E = 1. For each s \in E, \zeta < \sup s so there is an f_0(s) \in c_{\sup s} such that \zeta \leq f_0(s); now f_0(s) < \sup s so there is an f(s) \in s such that f_0(s) < f(s). As remarked in 8H, there is now a countable D \subseteq \lambda such that \nu\{s : s \in E, f(s) \in D\} = 1; as \lambda is regular, there is a \zeta' < \lambda such that D \subseteq \zeta'. Then f_0(s) \in c_{\sup s} \cap \zeta' \setminus \zeta whenever s \in E and f(s) \in D, so \zeta' serves. Q

(c) So (again because \lambda is regular) there is a family \{\zeta_\xi\}_\xi\in\lambda such that
\[ \nu\{s : c_{\sup s} \cap \zeta_{i+1} \setminus \zeta_i \neq \emptyset\} = 1 \ \forall \ \xi < \lambda, \]
\[ \zeta_i = \sup_{\eta < \xi} \zeta_\eta \] for limit ordinals \( \xi < \lambda \).

Set \( a_\xi = \{\eta : c_\xi \cap \zeta_{\eta + 1} \setminus \zeta_\eta \neq \emptyset\} \) for each \( \xi < \lambda \); then \( \#(a_\xi) \leq \#(c_\xi) = \text{cf}(\xi) \) for each \( \xi \). Now
\[ \nu\{s : \eta \in a_{\sup s}\} = \nu\{s : c_{\sup s} \cap \zeta_{\eta + 1} \setminus \zeta_\eta \neq \emptyset\} = 1 \]
for each \( \eta < \lambda \).

**(d)** Set
\[ A = \{a : \exists \ \xi < \lambda, \text{cf}(\xi) < \varsigma, a \subseteq a_\xi\}. \]
Then \( \#(A) \leq \max(\lambda, \sup_{\delta < \varsigma} 2^\delta) = \lambda \), by 5E. Also, for any \( a \in S \),
\[ \nu\{s : a \subseteq a_{\sup s}\} = 1; \]
but if \( s \in S \) then \( \text{cf}(\sup s) < \varsigma \), so we have \( a \in A \). Thus \( A = S \) and \( \#(S) \leq \lambda \), as required.

**Remark** This is Theorem 3(a-b) of Prikry 75. Compare the corresponding theorem concerning strongly compact cardinals (DRAKE 74, Theorem 3.6 and Corollary 3.8, or SOLOVAY 74, Theorem 1).

**8L Theorem** Suppose that \( I \) is a set and that \( \nu \) is a measure on \( S = [I]^{<\omega} \) as in 8H. Then

**(a)** If \( \langle A_i \rangle_{i \in I} \) is any family of subsets of \( S \) with \( \nu A_i = 1 \) for each \( i \in I \), then
\[ \nu\{s : s \in S, s \in A_i \ \forall \ i \in s\} = 1. \]

**(b)** If \( \theta \) is a cardinal, \( \omega < \text{cf}(\theta) \leq \theta < \varsigma \), and \( f : S \rightarrow [I]^{<\theta} \) is any function, then there is an \( M \in [I]^{<\theta} \) such that
\[ \nu\{s : s \in S, f(s) \cap s \subseteq M\} = 1. \]

**(c)** If \( \theta < \varsigma \) is a cardinal and \( C \subseteq S \) is cofinal with \( S \) then
\[ \nu\{s : s \in S, \exists D \subseteq C, \text{add}(D) > \theta, \bigcup D = s\} = 1. \]

(Here add\( (D) \) is the additivity of the partially ordered set \( (D, \subseteq) \), as in A1Ac.)

**(d)** If \( f : [I]^{<\omega} \rightarrow S \) is any function then
\[ \nu\{s : s \in S, f(J) \subseteq s \ \forall \ J \in [s]^{<\omega}\} = 1. \]

**proof (a)** Set
\[ A = \{s : s \in A_i \ \forall \ i \in s\}. \]
For each \( s \in S \setminus A \) choose \( f(s) \in s \) such that \( s \not\subseteq A_{f(s)} \). If \( \nu A < 1 \) then \( \nu(S \setminus A) > 0 \) so there is an \( i \in I \) such that \( \nu\{s : s \in S \setminus A, f(s) = i\} > 0 \). But now \( \nu A_i \leq 1 - \nu\{s : f(s) = i\} < 1 \). \( \checkmark \) (Compare A1Ea.)

**(b)** As \( \omega < \text{cf}(\theta) < \varsigma \), there is an infinite cardinal \( \delta < \theta \) such that \( \nu A = 1 \), where \( A = \{s : s \in S, \#(f(s)) \leq \delta\} \). For \( s \in A \) let \( \langle f_\xi(s) \rangle_{\xi < \delta} \) run over \( f(s) \cup \{0\} \). Set
\[ M = \{i : i \in I, \exists \ \xi < \delta, \nu\{s : f_\xi(s) = i\} > 0\}; \]
then \( \#(M) \leq \max(\omega, \delta) < \theta \). Set
\[ B = \{s : s \in A, f(s) \cap s \not\subseteq M\}. \]
For each \( s \in B \) choose \( g(s) \in f(s) \cap s \setminus M \). If \( \nu B > 0 \) there is an \( i \in I \) such that \( \nu\{s : s \in B, g(s) = i\} > 0 \); but now there is an \( \eta < \delta \) such that \( \nu B_\eta > 0 \), where
\[ B_\eta = \{s : s \in B, f_\eta(s) = i\}, \]
so that \( i \in M \); which is absurd. \( \checkmark \)

Thus \( \nu B = 0 \) and \( \nu\{s : f(s) \cap s \subseteq M\} = 1 \). (Compare 5A.)

**(c)** We need consider only the case \( \theta > \omega \). If \( \#(I) < \varsigma \) the result is trivial (since \( I \in S \) and \( \nu(I) = 1 \)), so let us take it that \( \#(I) \geq \varsigma \); for convenience, let us suppose that \( \varsigma \subseteq I \). For \( s \in S \) write \( s^* = s \cup \{\#(s)\} \); as \( \#(s) \in \varsigma \subseteq I \), \( s^* \in S \). For each \( i \in I \) choose \( c_i \in C \) such that \( i \in c_i \), and set
\[ A_i = \{ s : c_i^* \subseteq s \leq S \} \].

Then \( \nu A_i = 1 \), because \( \#(c_i^*) < c = \text{add}(\nu) \). Set

\[ A = \{ s : s \in S, s \in A_i \forall \ i \in s \} \],
	hen \( \nu A = 1 \), by (a) above. For \( s \in A \) set

\[ D_s = \{ c : c \leq C, c^* \subseteq s \} \].

Then \( s = \bigcup D_s \) because \( i \in c_i \leq D_s \) for every \( i \in s \).

Set

\[ B = \{ s : s \in A, \text{add}(D_s) \leq \theta \} \].

For each \( s \in B \) choose a family \( D'_s \subseteq D_s \) with \( \#(D'_s) \leq \theta \) and with no upper bound in \( D_s \); set \( f(s) = \{ \#(d) : d \in D'_s \} \leq s \cap c \). By (b), there is an \( M \subseteq c \) such that \( \#(M) < \theta^\ast \) and \( \nu B = \nu B_1 \), where

\[ B_1 = \{ s : s \in B, f(s) \subseteq M \} \].

As \( \theta < \epsilon = \text{cf}(c), \delta = \text{sup}(M \cup \{ \theta \}) < c \). Now, for \( s \in B_1 \), \( d_s = \bigcup D'_s \) has cardinal at most \( \delta \). So by (b) again there is an \( N \subseteq I \) such that \( \#(N) < \delta^\ast \) and \( \nu B_2 = \nu B_1 \) where

\[ B_2 = \{ s : s \in B_1, d_s \cap s \subseteq N \} \].

Let \( c \in C \) be such that \( N \subseteq c \). Then \( \bar{E} = \{ s : c^* \subseteq s \} \) has measure 1. But if \( s \in B_2 \) then \( D'_s \) has no upper bound in \( D_s \), while \( \bigcup D'_s \subseteq N \subseteq c \), so \( c \notin D_s \) and \( c^* \not\subseteq s \) and \( s \notin \bar{E} \). Thus \( E \cap B_2 = \emptyset \) and \( \nu B = \nu B_2 = 0 \).

Accordingly \( \nu (A \setminus B) = 1 \). But if \( s \in A \setminus B \) then \( D_s \subseteq C \) has \( \text{add}(D_s) > \theta \) and \( \bigcup D_s = s \), so we are done.

(d) Apply (c) with \( \theta = \omega \),

\[ C = \{ s : f(J) \subseteq s \forall J \in [s]^{<\omega} \} \],

then

\[ C \supseteq \{ s : \exists D \subseteq C, \bigcup D = s, \text{add}(D) \geq \omega \} \],

so \( \nu C = 1 \), as required.

Remark These results may be found, implicitly or explicitly, in Fleissner 89.

8M Proposition Assume NMA. Let \( X \) be a Hausdorff space such that \( \chi(x, X) < c \) for every \( x \in X \) and every closed set of cardinal at most \( c \) is normal. Then \( X \) is normal.

proof Let \( E, F \) be disjoint closed sets in \( X \). For each \( x \in X \) let \( U_x \) be a base of neighbourhoods of \( x \) of cardinal less than \( c \). Let \( \nu \) be a measure on \( S = [X]^{<c} \) as in 8H.

For each \( s \in S \), its closure \( \bar{s} \) has cardinal at most \( c \), because if \( x, y \) are distinct points of \( \bar{s} \) then \( \{ U \cap s : U \in U_x \} \) and \( \{ U \cap s : U \in U_y \} \) are distinct members of \( [P_S]^{<c} \), and \( \#([P_S]^{<c}) \leq c \), by 5E. So there are disjoint relatively open sets \( G_s, H_s \subseteq \bar{s} \) such that \( E \cap \bar{s} \subseteq G_s, F \cap \bar{s} \subseteq H_s \). For \( U \subseteq X \) set

\[ A(U) = \{ s : U \cap \bar{s} \subseteq G_s \}, B(U) = \{ s : U \cap \bar{s} \subseteq H_s \} \].

Then for each \( x \in E, y \in F \) there are \( U_x \in U_x, V_y \in U_y \) such that \( \nu A(U_x) > 1/2, \nu B(V_y) > 1/2 \). But in this case \( U_x \cap V_y = \emptyset \). Otherwise, take \( z \in U_x \cap V_y \); there is an \( s \) such that \( U_x \cap \bar{s} \subseteq G_s, V_y \cap \bar{s} \subseteq H_s, z \in s \); but in this case \( z \in G_s \cap H_s \). XQ

So \( G = \bigcup \{ U_x : x \in E \} \) and \( H = \bigcup \{ V_y : y \in F \} \) are open sets separating \( E \) from \( F \).

Remark This is due to W.G.Fleissner and I.Juhász (see Juhász 89). I am grateful to P.J.Nyikos for the reference.

Version of 27.11.91

8N Theorem Assume NMA. Then any locally compact normal space is collectionwise normal.

proof (a) Let \( (X, \mathcal{T}) \) be a locally compact normal Hausdorff space; without loss of generality we may suppose that \( X \cap \mathcal{T} = \emptyset \). Let \( \langle F_i \rangle \) be a discrete family of subsets of \( X \). Let \( \nu \) be a measure on \( S = [X \cup \mathcal{T}]^{<c} \) as in 8H.

For \( s \in S \) set
for each $x \in X$, 
\[ Y_i(s) = \bigcup \{B_x(s) : x \in F_i \cap s\} \]
for each $i \in I$.

(b) Set 
\[ A_0 = \{s : s \in S, \bigcap_{j \neq i} Y_j(s) = \emptyset \quad \forall \ i \in I\}; \]
then $\nu A_0 = 1$. P For each $x \in X$ choose $G_x$, $H_x \in \mathcal{F}$ in such a way that if $x \in F_i$ then 
\[ F_i \subseteq G_x \subseteq H_x \subseteq \overline{\mathcal{F}}_x \subseteq X \setminus \bigcup_{j \neq i} F_j; \]
this is possible because $F_i \cap \bigcup_{j \neq i} F_j = \emptyset$ and $X$ is normal. Set 
\[ A'_0 = \{s : s \in S, \{G_x, X \setminus \overline{\mathcal{F}}_x\} \cap s = \emptyset \ \forall \ x \in s \cap X\}; \]
by 8Ld, $\nu A'_0 = 1$. But if $s \in A'_0$ and $Y_i(s) \neq \emptyset$, take $x \in s \cap F_i$; then $Y_i(s) \subseteq \overline{G}_x$ and $\bigcap_{j \neq i} Y_j(s) \subseteq X \setminus H_x$, so $\overline{Y}_i(s) \cap \bigcup_{j \neq i} Y_j(s) = \emptyset$. Thus $A'_0 \subseteq A_0$ and $\nu A_0 = 1$. Q

(c) Set 
\[ A_1 = \{s : s \in S, \langle Y_i(s) \rangle_{i \in I} \text{ is discrete}\}. \]
Then $A_1$ is cofinal with $S$. P Take any $s \in S$. Because $\nu A_0 > 0$, there is an $s' \in A_0$ such that $s' \supseteq s$. Set 
\[ \mathcal{H} = \{H : H \in \mathcal{F}, \#(\{i : H \cap Y_i(s') = \emptyset\}) \leq 1\}. \]
Then $\bigcup_{i \in I} Y_i(s') \subseteq \bigcup \mathcal{H}$, because $s' \in A_0$, so $F = \bigcup_{i \in I} F_i \cap s' \subseteq \bigcup \mathcal{H}$; but $F$ is closed, because $\langle F_i \rangle_{i \in I}$ is locally finite. Take $G \in \mathcal{F}$ such that $F \subseteq G \subseteq \overline{G} \subseteq \bigcup \mathcal{H}$, and set $s'' = s' \cup \{G\} \in S$. Then $s'' \cap F_i = s' \cap F_i \subseteq G$ for each $i$ (this is where I use the requirement $\langle X \setminus \overline{\mathcal{F}} \rangle = \emptyset$), so $Y_i(s'') \subseteq Y_i(s') \cap \overline{G}$ for every $i$. Now we see that $H$ meets at most one $Y_i(s'')$ for every $H \in \mathcal{H}$ and also for $H = X \setminus \overline{G}$, so that $s'' \in A_1$. Q

(d) Set 
\[ A_2 = \{s : s \in S, \langle Y_i(s) \rangle_{i \in I} \text{ is locally finite}\}; \]
then $\nu A_2 = 1$. P Set 
\[ A'_2 = \{s : \exists D \subseteq A_2, \text{add}(D) > \omega, \bigcup D = s\}. \]
By 8Lc, $\nu A'_2 = 1$. P If $A'_2 \nsubseteq A_2$, take $s \in A'_2 \setminus A_2$. Let $x \in X$ be such that $\{i : U \cap Y_i(s) \neq \emptyset\}$ is infinite for every neighbourhood $U$ of $x$. Let $K$ be a compact neighbourhood of $x$, and $J \subseteq I$ a countably infinite set such that $K \cap Y_i(s) \neq \emptyset$ for every $i \in J$. For each $i \in J$ there is an $x(i) \in F_i \cap s$ such that $K \cap B_{x(i)}(s) \neq \emptyset$. Now there is a $D \subseteq A_1$ such that $\text{add}(D) > \omega$ and $s = \bigcup D$; so there is a $d \in D$ such that $x(i) \in d$ for every $i \in J$. In this case $B_{x(i)}(d) \supseteq B_{x(i)}(s)$ for each $i$, so $K \cap Y_i(d) \neq \emptyset$ for every $i \in J$. However $\langle Y_i(d) \rangle_{i \in I}$ is supposed to be discrete, and $K$ is supposed to be compact, so this is impossible. X

Thus $A'_2 \subseteq A_2$ and $\nu A_2 = 1$, as claimed. Q

(e) Consequently $\nu A_1 = 1$, as $A_1 = A_0 \cap A_2$.

(f) Let $A_3$ be the set of those $s \in S$ for which there is a family $\langle W_x \rangle_{x \in X}$ in $s \cap \mathcal{F}$ such that $x \in W_x$ for every $x \in s \cap X$ and $i, j$ are distinct members of $I$, $x \in F_i$ and $y \in F_j$ then $W_x \cap W_y = \emptyset$. Then $\nu A_3 = 1$. P For $x \in X$ take a compact neighbourhood $K_x$ of $x$. Set 
\[ A'_3 = \{s : s \in S, \emptyset \in s, U \cup V \in s \setminus U, V \in s \cap \mathcal{F}, \text{int}(K_x) \subseteq s \setminus U \cap x \in s \cap X\}. \]
Then $\nu A'_3 = 1$ by 8Ld.

Take $s \in A'_3 \cap A_1$. Note that $\langle Y_i(s) \rangle_{i \in I}$ is disjoint, $s \cap Y_i(s) \cap \bigcup_{i \in I} F_i = s \cap F_i$ for every $i \in I$. For each $x \in s \cap \bigcup_{i \in I} F_i$, set $U_x = \{U : x \in U \in s \cap \mathcal{F}\}$; for $x \in X \setminus (s \cap \bigcup_{i \in I} F_i)$, set $U_x = \{\emptyset\}$. Then $U_x$ is downwards-directed and $\#(U_x) < c$ for every $x$. If $i \in I$ and $x \in Y_i(s)$ and $G$ is an open neighbourhood of $Y_i(s)$, then either $x \notin s \cap F_i$ and $\emptyset \in U_x, \emptyset \in G$, or $x \in s \cap F_i$; in the latter case, $\bigcap \{U : U \in U_x\} = B_x(s) \subseteq G$; but as $U_x$ is downwards-directed and contains the relatively compact set $\text{int}(K_x)$, there must be a member of $U_x$ included in $G$. 

We may therefore apply 8E to $X$, $(Y_i(s))_{i \in I}$ and $(U_x)_{x \in X}$ to see that there is a family $(W_x)_{x \in X}$ such that $W_x \in U_x$ for every $x \in X$ and if $x \in Y_i(s)$, $y \in Y_j(s)$ with $i \neq j$ then $W_x \cap W_y = \emptyset$. Evidently $(W_x)_{x \in X}$ witnesses that $s \in A_3$.

Thus $A'_3 \cap A_1 \subseteq A_3$ and $\nu A_3 = 1$. Q

(g) For each $s \in A_3$ take a family $(W_x(s))_{x \in X}$ in $s \cap \mathcal{S}$ as in (f). Now, given any $x \in X$,

$$\nu \{ s : x \in A_3 \} = 1,$$

so there is a finite set $V_x$ of open neighbourhoods of $x$ such that

$$\nu \{ s : x \in A_3, W_x(s) \in V_x \} > \frac{1}{3}$$

(see (ii)$''$ of Remark (b) in 8H). Set $V_x = \bigcap V_x$; then $V_x$ is an open neighbourhood of $x$ for each $x \in X$.

If $i, j \in I$ are distinct and $x \in F_i$, $y \in F_j$ then there is an $s \in A_3$ such that

$$x \in y, y \in s, W_x(s) \in V_x, W_y(s) \in V_y.$$

Now $V_x \cap V_y \subseteq W_x(s) \cap W_y(s) = \emptyset$. So if we set

$$H_i = \bigcup \{ V_x : x \in F_i \} \quad \forall \ i \in I,$$

$(H_i)_{i \in I}$ is a disjoint family of open sets and $F_i \subseteq H_i$ for each $i \in I$.

(h) As $(F_i)_{i \in I}$ is arbitrary, $X$ is collectionwise normal, as claimed.

Remark This comes from Fleissner 89. Some of the ideas were originally developed in Balogh 91.

80 Definition Let $s$ be a set of ordinals.

(a) A set $C \subseteq s$ is relatively order-closed if it is closed in the intrinsic order topology of $s$, that is, if

$$\min \{ \xi : \xi \in s, \eta \leq \xi \land \eta \in C \} \in C$$

whenever $C' \subseteq C$ is non-empty and bounded above in $s$.

(b) A set $A \subseteq s$ is relatively stationary in $s$ if it meets every cofinal set $C \subseteq s$ which is relatively order-closed in $s$.

Observe that these definitions correspond to the ordinary notions of ‘closed’ and ‘stationary’ set in the ordinal otp($s$).

8P Theorem Let $\alpha$ be an ordinal and $\nu$ a measure on $S = [\alpha]^{<\xi}$ as in 8H.

(a) If $\beta < \alpha$ is an ordinal, then otp$(s \cap \beta) < \text{otp}(s)$ for $\nu$-almost every $s \in S$.

(b) If $\alpha$ is a decomposable ordinal then otp$(s)$ is a decomposable ordinal for $\nu$-almost every $s \in S$.

(c) If $\alpha$ is an indecomposable ordinal then otp$(s)$ is an indecomposable ordinal for $\nu$-almost every $s \in S$.

(d) $\#(s) = \text{otp}(s \cap \#(s))$ for $\nu$-almost every $s \in S$ (ii) if $\alpha$ is not a cardinal then otp$(s)$ is not a cardinal, for $\nu$-almost every $s \in S$.

(e) If $\alpha = \lambda^+$ is a successor cardinal then $\#(s) = (\#(s \cap \lambda))^+$ for $\nu$-almost every $s \in S$.

(f) $\text{cf}(s) = \text{cf}(s \cap \text{cf}(s))$ for $\nu$-almost every $s \in S$.

(g) If $\alpha$ is a regular infinite cardinal then $\#(s)$ is a regular infinite cardinal for $\nu$-almost every $s \in S$.

(h) If $\alpha$ is a weakly inaccessible cardinal then $\#(s)$ is a weakly inaccessible cardinal for $\nu$-almost every $s \in S$.

(i) If $C \subseteq \alpha$ is closed then $s \cap C$ is relatively order-closed in $s$ for $\nu$-almost every $s \in S$.

(j) If $C \subseteq \alpha$ is cofinal with $\alpha$ then $s \cap C$ is cofinal with $s$ for $\nu$-almost every $s \in S$.

(k) If $A \subseteq \alpha$ is stationary in $\alpha$ then $s \cap A$ is relatively stationary in $s$ for $\nu$-almost every $s \in S$.

(l) If $\alpha$ is a weak Mahlo cardinal then $\#(s)$ is a weak Mahlo cardinal for $\nu$-almost every $s \in S$.

(m) If $\alpha$ is a non-weakly-Mahlo cardinal then $\#(s)$ is a non-weakly-Mahlo cardinal for $\nu$-almost every $s \in S$.

proof (a) $\beta \in s$ for almost all $s$.

(b) Take $\beta, \gamma < \alpha$ such that $\alpha = \beta + \gamma$. Let $h : \alpha \setminus \beta \to \gamma$ be an order-isomorphism. By 8Ld, with $f(J) = h[J] \cup h^{-1}[J]$ for each $J \in [\alpha]^{<\omega}$, $h[s \setminus \beta] = s \cap \gamma$ for almost every $s \in S$. So otp$(s \setminus \beta) = \text{otp}(s \cap \gamma) < \text{otp}(s)$ for almost every $s$, using (a), and otp$(s)$ is decomposable for almost every $s$. 
(c) Set $A = \{s : s \in S, \operatorname{otp}(s) \text{ is decomposable}\}$. For $s \in A$ take $g(s), h(s) \in s$ and an order-isomorphism $f_s : s \setminus g(s) \to s \cap h(s)$. If $\nu A > 0$, there are $\beta, \gamma < \alpha$ such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, g(s) = \beta, h(s) = \gamma\}$. So $\operatorname{otp}(s \setminus \beta) > \operatorname{otp}(s \cap \gamma)$ for every $s \in A_1$. Let $\phi : \gamma \to (\beta + \gamma) \setminus \beta$ be an order-isomorphism. Then there is an $s \in A_1$ such that $\phi[s \cap \gamma] = s \cap (\beta + \gamma) \setminus \beta$ (using $8Ld$ again) and also $\beta + \gamma \in s$, because $\beta + \gamma < \alpha$. But now

$$\operatorname{otp}(s \setminus \beta) > \operatorname{otp}(s \cap (\beta + \gamma) \setminus \beta) = \operatorname{otp}(s \cap \gamma),$$

which is impossible. \(\blacksquare\)

(d) If $\alpha < \omega_1$ this is trivial; suppose that $\alpha$ is uncountable. Set $A = \{s : \#(s) < \operatorname{otp}(s \cap \#(\alpha))\}$. For $s \in A$, take a surjective function $f_s : s \cap g(s) \to s$, where $g(s) \in s \cap \#(\alpha)$. If $\nu A > 0$, let $\zeta < \#(\alpha)$ be such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, g(s) = \zeta\}$. By $8Lb$, with $f(s) = f_s[s \cap \zeta]$, $\theta = \max(\omega, \#(\zeta))^{+}$ there is a set $M \subseteq \alpha$ with $\#(M) \leq \max(\omega, \#(\zeta))$ and $f_s[s \cap \zeta] \subseteq M$ for almost every $s$. But also $f_s[s \cap \zeta] = s \nsubseteq M$ for almost all $s \in A_1$. \(\blacksquare\) Thus $\#(s) \geq \operatorname{otp}(s \cap \#(\alpha))$ for almost all $s$.

Let $g : \alpha \to \#(\alpha)$ be any bijection. Then $g[s] = s \cap \#(\alpha)$ for $\nu$-almost every $s \in S$ (using $8Ld$ with $f(J) = g[J] \cup g^{-1}[J]$ for $J \in [\alpha]^{\omega}$). So $\#(s) = \#(s \cap \#(\alpha)) \leq \operatorname{otp}(s \cap \#(\alpha))$ for almost all $s$.

(e) For $\lambda < \omega$ this is trivial; take $\lambda \geq \omega$. By (a) and (d), $\#(s) = \operatorname{otp}(s \cap \lambda) \geq \#(s \cap \lambda)$ for almost all $s$. Set $A = \{s : \#(s) > \#(s \cap \lambda)^{+}\}$. For $s \in A$ take $f(s) \in s$ such that $\#(s \cap f(s)) \geq \#(s \cap \lambda)^{+}$. If $\nu A > 0$, let $\zeta < \alpha$ be such that $\zeta \geq \lambda$ and $\nu A_1 > 0$, where $A_1 = \{s : s \in A, f(s) \subseteq \zeta\}$. Let $h : \zeta \to \lambda$ be any bijection. Then $s \cap \zeta = h^{-1}[s \cap \lambda]$ for almost all $s \in A_1$. But of course if $s \in A_1$ then $\#(s \cap \zeta) > \#(s \cap \lambda)$. \(\blacksquare\) So $\nu A = 0$ and $\#(s) \leq \#(s \cap \lambda)^{+}$ for almost all $s$.

(f) If $\alpha$ is a successor ordinal, or $0$, this is trivial; let us take it that $\operatorname{cf}(\alpha) \geq \omega$. Let $C$ be a closed cofinal subset of $\alpha$ of order type $\operatorname{cf}(\alpha)$, and let $h : \operatorname{cf}(\alpha) \to C$ be the increasing enumeration of $C$. (Then $8Ld$ again)

$$B = \{s : \forall \xi < \operatorname{cf}(\alpha), \xi \in s \iff s \cap h(\xi + 1) \setminus h(\xi) \neq \emptyset \iff \xi + 1 \in s\}$$

has $\nu B = 1$. And $\operatorname{cf}(s \cap \alpha) = \operatorname{cf}(s \cap \operatorname{cf}(\alpha))$ for every $s \in B$.

(g) For $\alpha = \omega$ this is trivial; take $\alpha > \omega$. Set $A = \{s : \operatorname{cf}(s) < \#(s)\}$. For $s \in A$ choose $f(s) \in s$ such that $\operatorname{cf}(s) = \operatorname{otp}(s \cap f(s))$. Let $\zeta < \alpha$ be such that $f(s) \subseteq \zeta$ for almost all $s \in A$; say $A_1 = \{s : s \in A, f(s) \subseteq \zeta\}$. For $s \in A_1$ let $g_s : s \cap \zeta \to s$ be a function with range cofinal in $s$. By $8Lb$, there is a $M \subseteq \alpha$, with $\#(M) < \alpha$, such that $g_s[s \cap \zeta] \subseteq M$ for almost all $s \in A_1$. However $s \cap M$ is not cofinal in $s$, for almost all $s$, because $\sup M < \alpha$. Thus $\nu A_1 = 0$ and $\nu A = 0$. Finally, $\operatorname{cf}(s) = \operatorname{cf}(\#(s))$ for almost all $s$, by (d-i) above; so $\operatorname{cf}(\#(s)) = \#(s)$ and $\#(s)$ is regular, for almost all $s$.

(h) By (g), $\#(s)$ is a regular infinite cardinal for almost all $s$. Set

$$A = \{s : \#(s) \text{ is a successor cardinal}\}.$$ 

For $s \in A$ choose $f(s) \in s$ such that $\#(s) = \#(s \cap f(s))^{+}$. If $\nu A > 0$, there is a $\zeta < \alpha$ such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, f(s) = \zeta\}$. Now consider $\delta = \#(\zeta)^{+}$. Recall that, as remarked in $8H$, we have a measure $\nu_1 = \nu \circ \phi^{-1}$ on $[\delta]^{<\omega}$, where $\phi(s) = s \cap \delta$ for each $s \in S$, with the same properties as $\nu$. So we may apply (d) to $\delta, \nu_1$ to see that $\#(s \cap \zeta) < \#(s \cap \delta)$ for almost all $s$. Also, by (d) as written, $\#(s \cap \delta) < \#(s)$ for almost all $s$. So we have $\#(s \cap \zeta)^{+} < \#(s)$ for almost all $s$, and $\nu A_1 = 0$; which is absurd. \(\blacksquare\)

(i) For each $\zeta < \alpha$ set $h(\zeta) = \sup(C \cap \zeta) \in C \cup \{0\}$. Set $A = \{s : h(\zeta) \in s \forall \zeta \in s\}$; then $\nu A = 1$. If $s \in A$ and $C^\prime \subseteq C \cap s$ is non-empty and bounded above in $s$, set $\zeta = \min\{\xi : \xi \in s, \eta \leq \xi \forall \eta \in C^\prime\}$. Then $h(\zeta) \in s$; but of course $h(\zeta)$ is also an upper bound for $C^\prime$, so $h(\zeta) = \zeta$ and $\zeta \in C$. Thus $s \cap C$ is relatively order-closed in $s$ for every $s \in A$.

(j) Let $h : \alpha \to \alpha$ be such that $\xi \leq h(\xi) \in C$ for every $\xi < \alpha$. Then $h[s] \subseteq s$ for almost all $s$; and $C \cap s$ is cofinal with $s$ whenever $h[s] \subseteq s$.

(k)(i) If $\operatorname{cf}(\alpha) \leq \omega$ then there is a $\zeta < \alpha$ such that $\alpha \setminus \zeta \subseteq A$; now $s \cap A$ is stationary in $s$ whenever $\zeta \in s$. (ii) If $\operatorname{cf}(\alpha) > \omega$, set $D = \{s : s \cap A$ is not relatively stationary in $s\}$. For $s \in D$ let $C_s$ be a cofinal relatively order-closed subset of $s$ disjoint from $A$; for $s \in S \setminus D$, set $C_s = s$. Set
Then $C$ is closed and unbounded in $\alpha$. \(\mathbf{P}\) (a) If $\zeta < \alpha$, then $C_\zeta \subseteq \zeta$ for almost all $s$, so there is a countable set $K \subseteq \alpha \setminus \zeta$ such that $C_\zeta \cap K \neq \emptyset$ for almost all $s$. Because $cf(\alpha) > \omega$, we can find an increasing sequence $(\zeta_n)_{n \in \mathbb{N}}$, starting with $\zeta_0 = \zeta$, such that $C_{\zeta} \cap \zeta_{n+1} \setminus \zeta_n \neq \emptyset$ for almost all $s$, for every $n \in \mathbb{N}$. Take $\zeta^* = \sup_{n \in \mathbb{N}} \zeta_n$; then $\zeta^* \in C_\zeta$ for almost all $s$, that is, $\zeta^* \in C$, and $\zeta^* \geq \zeta$. This shows that $C$ is unbounded in $\alpha$. (b) If $\zeta \in \alpha \setminus (C \cup \{0\})$, then $\nu E > 0$, where $E = \{s : \zeta \notin C_s\}$. For each $s \in E$, there is a $\beta \in s \cap \zeta^*$ such that $C_s \cap \zeta \subseteq \beta$. So there is a $\beta < \alpha$ such that $\nu E_1 > 0$, where $E_1 = \{s : C_s \cap \zeta \subseteq \beta\}$. Now $C \cap \zeta = \emptyset$. This shows that $C$ is closed in $\alpha$. \(\mathbf{Q}\)

Accordingly there is a $\zeta \in A \cap C$. But now $\nu D \leq \nu \{s : \zeta \notin C_s\} = 0$. (i) Let $A$ be the set of weakly inaccessible cardinals less than $\alpha$; then $A$ is stationary in $\alpha$. For each $\lambda \in A$, $\text{otp}(s \cap \lambda)$ is a weakly inaccessible cardinal for almost all $s$, applying (d) and (h) in $[\lambda]^{<\gamma}$. So $\nu D = 1$, where

$$D = \{s : \text{otp}(s \cap \lambda) \text{ is weakly inaccessible for every } \lambda \in s \cap A\}.$$ 

On the other hand, we also have $s \cap A$ relatively stationary in $s$ for almost all $s$, by (i). Copying these facts into $\text{otp}(s)$ for each $s$, we see that $\#(s)$ is a weakly Mahlo cardinal for almost all $s$. (m) Let $C$ be a closed unbounded set in $\alpha$ consisting of ordinals which are not weakly inaccessible cardinals. For each $\gamma \in C$, $\text{otp}(s \cap \gamma)$ is not a weakly inaccessible cardinal, for almost all $s$, by (e) and (f). So $\nu D = 1$, where

$$D = \{s : \text{otp}(s \cap \gamma) \text{ is not a weakly inaccessible cardinal for every } \gamma \in s \cap C\}.$$ 

But also $C \cap s$ is relatively order-closed and cofinal with $s$ for almost all $s$, by (i) and (j), so $\text{otp}(s)$ is not a weakly Mahlo cardinal, for almost all $s$. On the other hand, $\#(s) = \text{otp}(s)$ for almost all $s$, so $\#(s)$ is non-weakly-Mahlo for almost all $s$.

8Q I have given ‘elementary’ proofs of the results in 8P. They are of course just reflection properties, and all can be reached by means of the following.

**Theorem (a)** Let $I$ be a set and $\nu$ a measure on $S = [I]^{<\gamma}$ as in 8H. Let $\mathcal{P}$ be the random real p.o.set $\mathcal{P} \setminus \mathcal{N}_\nu$. Let $\phi$ be a formula of the second-order language $\mathcal{L}$ of $\S$A4; let $C_1, \ldots, C_k$ be relations on $I$ and $\xi_1, \ldots, \xi_m$ members of $I$. Then the following are equivalent:

(i) $\langle I, C_1, \ldots, C_k \rangle \models \phi$;

(ii) $\{s : s \in S, (s; C_1, \ldots, C_k) \not\models \phi\} \in \mathcal{N}_\nu$.

(b) Suppose that $\phi$ is a second-order formula and that $\alpha$ is an ordinal such that $\models (\alpha; <) \models \phi$ for every random real p.o.set $\mathcal{P}$. Then for any measure $\nu$ on $S = [\alpha]^{<\gamma}$ as in 8H, we shall have $(\text{otp}(s); <) \models \phi$ for $\nu$-almost every $s \in S$.

**proof (a)** This is a matter of re-writing the proof of 4M. For instance, the step corresponding to 4Ma becomes: Suppose that $(\xi_s)_{s \in S}$ is a family in $I$ such that $\xi_s \in s$ for almost all $s \in S$. Then we have a $\mathcal{P}$-name $\xi$ for a member of $I$ given by

$$p \models \check{\xi} = \check{\zeta} \text{ iff } \xi_s = \zeta \text{ for almost every } s \in p.$$ 

(b) This now follows at once, because $(\text{otp}(s); <)$ is isomorphic to $(s; <)$.

8R I have concentrated here on properties of well-ordered sets. But 8H-8I allow much more general contexts. For instance, if $I$ is a simple group, then $s$ will be a simple subgroup of $I$ for almost every $s \in [I]^{<\gamma}$. As a further example depending on the presence of a measure $\nu$ (rather than just on the ideal $\mathcal{N}_\nu$) I give the following.

**Proposition** Assume NMA. Let $\mathfrak{A}$ be a Dedekind $\sigma$-complete Boolean algebra such that whenever $\mathfrak{B} \subseteq \mathfrak{A}$ is a subalgebra of cardinal less than $\check{c}$ there is a functional $\mu : \mathfrak{B} \to [0, 1]$ such that

(i) $\mu b = \sum_{n \in \mathbb{N}} \mu b_n$ whenever $(b_n)_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$ and $b \in \mathfrak{B}$ is the supremum of $\{b_n : n \in \mathbb{N}\}$ in $\mathfrak{A}$;

(ii) $\mu b > 0$ for every $b \in \mathfrak{B} \setminus \{0\}$.

Then $\mathfrak{A}$ is a measurable algebra.
proof Let $\nu$ be a measure on $[\mathfrak{A}]^{<\iota}$ as in 8H. For each $s \in [\mathfrak{A}]^{<\iota}$ let $\mathfrak{B}_s$ be the subalgebra of $\mathfrak{A}$ generated by $s$, so that $\#(\mathfrak{B}_s) \leq \max(\omega, \#(s)) < \iota$, and let $\mu_s : \mathfrak{B}_s \to [0,1]$ be a functional satisfying (i)-(ii) of the hypothesis above. For $a \in \mathfrak{A}$ set

$$\tilde{\mu} a = \int \mu_s(a) \nu(ds);$$

note that $a \in s \subseteq \mathfrak{B}_s$ for almost all $s$, so that the integral is well-defined. Then we have

(i) whenever $\langle b_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum $b \in \mathfrak{A}$, then $\nu E = 1$, where

$$E = \{ s : s \in [\mathfrak{A}]^{<\iota}, b \in s, b_n \in s \forall n \in \mathbb{N} \},$$

and $\mu_s b = \sum_{n \in \mathbb{N}} \mu_s b_n$ for every $s \in E$, so by B. Levi's theorem

$$\tilde{\mu} b = \sum_{n \in \mathbb{N}} \tilde{\mu} b_n.$$

(ii) If $a \in (\mathfrak{A} \setminus \{0\})$ then $\mu_s a > 0$ for almost all $s$ so $\tilde{\mu} a > 0$.

Now $(\mathfrak{A}, \tilde{\mu})$ is a measure algebra and $\mathfrak{A}$ is a measurable algebra.

8S Remark For further topological consequences of PMEA and NMA see Fleissner Hansell & Junnila 82 (with Fleissner 79 and Fremlin Hansell & Junnila 83), Junnila 83, Burke 84, Fleissner 84b, Tall 84, Fleissner 89, Kulesza Levy & Nyikos 91.

9. Quasi-measurable cardinals

Many of the ideas above can be generalized to contexts outside ordinary measure theory. Here I discuss some of these extensions, concentrating on those which do not involve new concepts. With a few exceptions, the proofs are straightforward adaptations of arguments above, and I therefore omit many of the details.

9A Lemma Let $X$ be a set, $\lambda$ a cardinal of uncountable cofinality and $\mathcal{I}$ a $\lambda$-saturated $\sigma$-ideal of $\mathcal{P}X$.

(a) If $E \subseteq \mathcal{P}X$ there is a set $E' \subseteq E$ such that $\#(E') < \lambda$ and $E \setminus \bigcup E' \in \mathcal{I}$ for every $E \in E$.

(b) If $\langle E_\xi \rangle_{\xi < \lambda}$ is any family in $\mathcal{P}X \setminus \mathcal{I}$ then there is an $x \in X$ such that $\{ \xi : \xi < \lambda, x \notin E_\xi \}$ is infinite.

proof (a) Suppose, if possible, otherwise. Then we can choose inductively a family $\langle F_\xi \rangle_{\xi < \lambda}$ in $E$ such that $G_\xi = F_\xi \setminus \bigcup_{\eta < \xi} F_\eta \notin \mathcal{I}$ for every $\xi < \lambda$. But now $\langle G_\xi \rangle_{\xi < \lambda}$ is a disjoint family in $\mathcal{P} \lambda \setminus \mathcal{I}$, which is impossible, because $\mathcal{I}$ is supposed to be $\lambda$-saturated.

(b) Applying (a) to families of the form

$$E = \{ E_\eta : \beta(\eta) \leq \eta < \lambda \},$$

we can build inductively a strictly increasing sequence $\langle \beta(\eta) \rangle_{n \in \mathbb{N}}$ in $\lambda$ such that

$$E_\xi \setminus \bigcup_{\beta(\eta) \leq \eta < \beta(n+1)} E_\eta \in \mathcal{I} \forall n \in \mathbb{N}, \xi \in \lambda \setminus \beta(n).$$

Set $\beta^* = \sup_{n \in \mathbb{N}} \beta(n) < \lambda$, and consider

$$G_n = E_{\beta^*} \setminus \bigcup_{\beta(\eta) \leq \eta < \beta(n+1)} E_\eta \in \mathcal{I}$$

for each $n \in \mathbb{N}$. Because $\mathcal{I}$ is a $\sigma$-ideal and $E_{\beta^*} \notin \mathcal{I}$, there is an $x \in E_{\beta^*} \setminus \bigcup_{n \in \mathbb{N}} G_n$, and now $\{ \eta : x \in E_\eta \}$ meets $\beta(n+1) \setminus \beta(n)$ for every $n \in \mathbb{N}$, so is infinite.

Remark For a slightly different expression of the same idea, see Fremlin 87, Lemma 1E.

9B Theorem Suppose that $\kappa$ and $\lambda$ are cardinals, with $\omega < \text{cf}(\lambda) \leq \lambda \leq \kappa$, and that there is a proper $\kappa$-additive $\lambda$-saturated ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ which contains all singletons. Then

(a) $\kappa$ is weakly inaccessible;

(b) $\mathcal{P}\kappa$ has a $\lambda$-saturated normal ideal.

proof The argument amounts to extracting the essential ideas from 1D and 1G above.

(a) $\kappa = \text{add} (\mathcal{I})$ is regular. If $\kappa = \theta^+$, choose an injective function $f_\xi : \xi \to \theta$ for each $\xi < \kappa$. Set

$$A_{\eta \alpha} = \{ \xi : \eta < \xi < \kappa, f_\xi(\eta) = \alpha \}$$

for $\eta < \kappa$, $\alpha < \theta$. Because $\mathcal{I}$ is $\kappa$-additive, there is for each $\eta < \kappa$ a $\beta_\eta < \theta$ such that $A_{\eta, \beta_\eta} \notin \mathcal{I}$. Set
for $\beta < \theta$. Again because $\mathcal{I}$ is $\kappa$-additive, there is a $\beta < \theta$ such that $B_\beta \notin \mathcal{I}$. Now $\#(B_\beta) = \kappa \geq \lambda$ and $(A_{\eta,\beta})_{\eta \in B_\beta}$ is a disjoint family in $\mathcal{P}_\kappa \setminus \mathcal{I}$, which is impossible, because $\mathcal{I}$ is $\lambda$-saturated. \xmark

This shows that $\kappa$ is weakly inaccessible.

(b) Let $F$ be the family of functions $f : \kappa \to \kappa$ such that $f^{-1}[\xi] \in \mathcal{I}$ for every $\xi < \kappa$. Then there is an $f_0 \in F$ such that $\{\xi : \xi < \kappa, f(\xi) < f_0(\xi)\} \in \mathcal{I}$ for every $f \in F$. \xmark If not, we can find a decreasing family $(g_\alpha)_{\alpha < \lambda}$ in $F$ such that

$$E_\alpha = \{\xi : g_{\alpha+1}(\xi) < g_\alpha(\xi)\} \notin \mathcal{I} \quad \forall \alpha < \lambda,$$

just as in 1G; we use the $\kappa$-additivity of $\mathcal{I}$ to be sure that $g_\alpha$, defined as $\inf_{\beta < \alpha} g_\beta$, belongs to $F$ for every non-zero limit ordinal $\alpha < \kappa$. By Lemma 9Bb there is a $\xi < \kappa$ such that $A = \{\alpha : \alpha < \lambda, \xi \in E_\alpha\}$ is infinite. But now we can find a strictly increasing sequence $\langle \alpha(n) \rangle_{n \in \mathbb{N}}$ in $A$, and $\langle g_{\alpha(n)}(\xi) \rangle_{n \in \mathbb{N}}$ will be a strictly decreasing sequence of ordinals, which is impossible. \xmark

Set $\mathcal{J} = \{A : A \subseteq \kappa, f_0^{-1}[A] \in \mathcal{I}\}$. Then $\mathcal{J}$ is a proper $\lambda$-saturated $\kappa$-additive ideal of $\mathcal{P}_\kappa$ because $\mathcal{I}$ is. $\mathcal{J}$ contains all singletons because $f_0 \in F$. Finally, let $f : \kappa \to \kappa$ be any function. Because $\mathcal{J}$ is $\kappa$-saturated and $\kappa$ is regular, 9Aa tells us that there is an $\alpha < \kappa$ such that $f^{-1}[\xi \setminus \alpha] \in \mathcal{J}$ for every $\xi < \kappa$. Set $E = \{\xi : \alpha < f(\xi) < \xi < \kappa\}$. Define $f_1 : \kappa \to \kappa$ by setting

$$f_1(\xi) = f(f_0(\xi)) \text{ if } f_0(\xi) \in E,$$

$$= f_0(\xi) \text{ otherwise.}$$

Then $f_1 \in F$ so

$$f_0^{-1}[E] = \{\xi : f_1(\xi) < f_0(\xi)\} \in \mathcal{I},$$

and $E \in \mathcal{J}$. Because $f$ is arbitrary, $\mathcal{J}$ is normal (A1Ed).

Thus $\mathcal{J}$ is an ideal of the required kind.

Remarks The results of 9A-9B above are due to SOLOVAY 71.

9C Definition 1 will call a cardinal $\kappa$ quasi-measurable if it is uncountable and there is a proper $\omega_1$-saturated $\kappa$-additive ideal of $\mathcal{P}_\kappa$ containing singletons; such an ideal being a witnessing ideal.

Observe that every real-valued-measurable cardinal is quasi-measurable, and that a quasi-measurable cardinal $\kappa$ carries a normal $\omega_1$-saturated ideal.

In the spirit of 11 and 4Ac, we can speak of the quasi ideal of a quasi-measurable cardinal $\kappa$, being the intersection of all its normal witnessing ideals, and the dual filter, the quasi filter of $\kappa$.

For results involving quasi-measurable cardinals, see FRELMIN 75C, JECH 78, FRELMIN HANSELL & JUNNILA 83, FRELMIN & JASIŃSKI 86, FRELMIN 87, KAMBURELIS p89 and GLÓWECZSKI 91.

9D Theorem If $\kappa$ is a quasi-measurable cardinal, with witnessing ideal $\mathcal{I} \subset \mathcal{P}_\kappa$, then either $\kappa \leq \tau$ and $\mathcal{P}_\kappa/\mathcal{I}$ is atomless, or $\kappa$ is two-valued-measurable and $\mathcal{P}_\kappa/\mathcal{I}$ is purely atomic.

proof (a) If $\mathcal{P}_\kappa/\mathcal{I}$ has an atom $a$, take $A \subseteq \kappa$ such that $A^* = a$, and see that

$$\{F : F \subseteq \kappa, A \setminus F \in \mathcal{I}\}$$

is a $\kappa$-complete ultrafilter on $\kappa$, so that $\kappa$ is two-valued-measurable.

(b) If $\mathcal{P}_\kappa/\mathcal{I}$ is not purely atomic, take $A \in \mathcal{P}_\kappa \setminus \mathcal{I}$ such that no atom of $\mathcal{P}_\kappa/\mathcal{I}$ is included in $A^*$. Choose $(A_\xi)_{\xi < \omega_1}$ inductively, as follows. $A_0 = \{A\}$. Given that $A_\xi$ is a disjoint family in $\mathcal{P}A \setminus \mathcal{I}$, then for each $B \in A_\xi$ choose disjoint $B', B'' \subseteq B$ such that $B \cup B'' = B$ and neither belongs to $\mathcal{I}$; this is possible because $B^*$ is not an atom in $\mathcal{P}_\kappa/\mathcal{I}$. Given $(A_\eta)_{\eta < \xi}$, where $\xi < \omega_1$ is a non-zero limit ordinal, let $A_\xi$ be a maximal disjoint family in $\mathcal{P}A \setminus \mathcal{I}$ such that for every $B \in A_\xi$, $\eta < \xi$ there is a $C \in A_\eta$ such that $B \subseteq C$. Continue.

? If $\bigcap_{\xi < \omega_1} \bigcup A_\xi \neq \emptyset$, take $x \in \bigcap_{\xi < \omega_1} \bigcup A_\xi$ and for each $\xi < \omega_1$ take $A_\xi \in A_\xi$ such that $x \in A_\xi$. Let $B_\xi \in A_{\xi+1}$ be whichever of $A_\xi$, $A''_\xi$ does not contain $x$. Then $(B_\xi)_{\xi < \omega_1}$ is a disjoint family in $\mathcal{P}_\kappa \setminus \mathcal{I}$; which is impossible. \xmark

So $\bigcap_{\xi < \omega_1} \bigcup A_\xi = \emptyset$, and there is a first $\xi < \omega_1$ such that $D = A \setminus \bigcup A_\xi \notin \mathcal{I}$. Of course $\xi$ cannot be a successor, because $\bigcup A_{\eta+1} = \bigcup A_\eta$ for every $\eta$. Now $E = D \cap \bigcap_{\eta < \xi} \bigcup A_\eta \notin \mathcal{I}$. On $E$ consider the equivalence relation $\sim$ given by
Because every $A_\eta$ is countable, and $\xi < \omega_1$, there are at most $\epsilon$ equivalence classes for $\sim$. Also, every equivalence class must belong to $I$, since otherwise it would have been a candidate for membership of $A_\xi$. So $E$ is covered by at most $\epsilon$ members of $I$, and $\kappa = \text{add}(I) \leq \epsilon$.

(c) Because any two-valued-measurable cardinal is greater than $\epsilon$, this completes the proof.

9E Proposition Let $X$ be a set and $I$ an $\omega_1$-saturated $\sigma$-ideal of $\mathcal{P}X$. Let $G^*$ be the set of bijections $g : X \rightarrow X$ such that $I = \{A : A \subseteq X, g^{-1}[A] \in I\}$. Then there is a partition $\mathcal{L}$ of $X$ into countable sets such that $G^*$ is precisely the set of bijections $g : X \rightarrow X$ such that $\bigcup\{L : L \in \mathcal{L}, g[L] \neq L\} \in I$.

proof As 1Mb.

Remark See Zakrzewski 91.

9F Theorem If $\kappa$ is a quasi-measurable cardinal and $\mathbb{P}$ is a ccc p.o.set, then $\Vdash \check{\kappa}$ is quasi-measurable.

proof Let $\mathcal{I}$ be a proper $\kappa$-additive $\omega_1$-saturated ideal of $\mathcal{P}\kappa$ containing singletons. In $V^\mathbb{P}$, let $\mathcal{J}$ be the ideal of $\mathcal{P}\kappa$ generated by $\mathcal{I}$, that is, if $\sigma$ is a $\mathbb{P}$-name for a subset of $\kappa$,

$$\Vdash (\sigma \in \mathcal{J} \iff \exists A \in \mathcal{I}, \sigma \subseteq A);$$

or, $\Vdash \sigma \in \mathcal{J}$ iff for every $\rho \leq p$ there are $\rho' \leq \rho$ and $A \in \mathcal{I}$ such that $\rho'' \Vdash \sigma \subseteq A$. But as $\mathbb{P}$ is ccc and $\mathcal{I}$ is a $\sigma$-ideal,

$$\rho \Vdash \sigma \in \mathcal{J} \iff \exists A \in \mathcal{I} \text{ such that } \rho \Vdash \sigma \subseteq A.$$ 

Clearly,

$$\rho \Vdash \sigma \subseteq \tau \in \mathcal{J} \Rightarrow \sigma \in \mathcal{J}.$$ 

Because singleton subsets of $\kappa$ all belong to $\mathcal{I}$, we have

$$\rho \Vdash \{\xi\} \in \mathcal{J} \forall \xi < \check{\kappa}.$$ 

Now suppose that $\lambda < \kappa$ and that $\langle \sigma_\alpha \rangle_{\alpha < \lambda}$ is a family of $\mathbb{P}$-names for subsets of $\kappa$. If

$$\rho \Vdash \sigma_\alpha \in \mathcal{J} \forall \alpha < \lambda,$$

then for each $\alpha < \lambda$ we can find an $A_\alpha \in \mathcal{I}$ such that

$$\rho \Vdash \sigma_\alpha \subseteq \check{A}_\alpha.$$ 

But now $A = \bigcup_{\alpha < \lambda} A_\alpha \in \mathcal{I}$, because $\mathcal{I}$ is $\kappa$-additive, and

$$\rho \Vdash \bigcup_{\alpha < \check{\kappa}} \sigma_\alpha \subseteq \check{A},$$

so that $\Vdash \bigcup_{\alpha < \check{\kappa}} \sigma_\alpha \in \mathcal{J}$. Thus $\Vdash \mathcal{J}$ is $\kappa$-additive.

Finally, suppose that $\langle \sigma_\alpha \rangle_{\alpha < \omega_1}$ is a family of $\mathbb{P}$-names for subsets of $\kappa$, and that

$$\rho \Vdash \sigma_\alpha \cap \sigma_\beta = \emptyset \forall \alpha \neq \beta.$$ 

For each $\xi < \kappa$ we may write

$$a_{\alpha\xi} = \sup\{p^* : p \Vdash \check{\xi} \in \sigma_\alpha\} \in \mathfrak{A},$$

where $\mathfrak{A}$ is the regular open algebra of $\mathbb{P}$. In this case we shall have $a_{\alpha\xi} \cap a_{\beta\xi} = \emptyset$ whenever $\alpha \neq \beta$ and $\xi < \kappa$. Because $\mathbb{P}$ is ccc, there is for each $\xi < \kappa$ a $\gamma_\xi < \omega_1$ such that $a_{\alpha\xi} = \emptyset$ whenever $\gamma_\xi < \alpha < \omega_1$. Because $\mathcal{I}$ is $\omega_1$-saturated, there is a $\gamma < \omega_1$ such that $\langle \xi : \gamma_\xi = \gamma' \rangle \in \mathcal{I}$ for every $\gamma' \geq \gamma$; because $\mathcal{I}$ is $\omega_2$-additive,

$$A = \{\xi : \gamma_\xi \geq \gamma\} \in \mathcal{I}.$$

Now $a_{\alpha\xi} = \emptyset$ for all $\xi \in \kappa \setminus A$, $\alpha \geq \gamma$. So

$$\rho \Vdash \sigma_\alpha \subseteq \check{A} \in \mathcal{I}$$

for every $\alpha \geq \gamma$. Because $\langle \sigma_\alpha \rangle_{\alpha < \omega_1}$ is arbitrary,

$$\Vdash \mathcal{J} \text{ is } \check{\omega}_1\text{-saturated},$$

as required.
Remark This is due to Prikry (see SOLOVAY 71, Theorem 8).

9G Proposition The following are equiconsistent:
(a) \( \text{ZFC} + \) there is a two-valued-measurable cardinal\';
(b) \( \text{ZFC} + \) there is a quasi-measurable cardinal\';
(c) \( \text{ZFC} + \) Martin's Axiom + there is a quasi-measurable cardinal \( \kappa < c \);
(d) \( \text{ZFC} + \) Martin's Axiom + \( c \) is quasi-measurable'.

proof (a)\( \Rightarrow \) (c) Start with a two-valued-measurable cardinal \( \kappa \). Let \( P \) be a ccc p.o.set such that
\[ \Vdash P \vDash \kappa = \kappa^+. \]
Let \( \hat{Q} \) be a \( P \)-name for a ccc p.o.set of cardinal \( \kappa^+ \) such that
\[ \Vdash_{\hat{P}} (\Vdash_{\hat{Q}} \text{MA}) \]
(KUNEN 80, §VIII.6). Then \( P \ast \hat{Q} \) is ccc so \( \Vdash_{\hat{P}} \kappa \) is quasi-measurable.
(a)\( \Rightarrow \) (d) As above, reading \( \kappa \) for \( \kappa^+ \) at each opportunity.
(c)\( \Rightarrow \) (b) and (d)\( \Rightarrow \) (b) are trivial, and (b)\( \Rightarrow \) (a) is covered by 2D.

9H Algebras \( P/X/I \) The Gitik-Shelah theorem (3F) may be regarded as an attack on the problem: which measurable algebras \( A \) are isomorphic to quotient algebras of the form \( P/X/I \) where \( X \) is a set and \( I \) is a \( \sigma \)-ideal of \( P/X \)? (For if \( (P/X/I, \mu) \) is a probability algebra, then it will be the measure algebra of \( (X, P/X, \mu) \), where \( \mu A = \mu(A^*) \) for every \( A \subseteq X \).) The Gitik-Shelah theorem tells us that if \( A \) is an atomless non-zero measurable algebra isomorphic to \( P/X/I \), then \( \tau(A) \geq \min(2^\kappa, \kappa^{(\kappa^+)}) \) for some atomlessly-measurable cardinal \( \kappa \). The product measure extension axiom (8A-8B) is the assertion that there are such algebras of arbitrarily large size.

Suppose now we extend the question, and seek to describe the class \( \text{Quot}_\tau \) of Boolean algebras isomorphic to algebras of the form \( P/X/I \) where \( I \) is a \( \sigma \)-ideal of \( P/X \). If \( P/X/I \) is non-zero, ccc and atomless then \( I \) is \( \omega_1 \)-saturated and \( \text{add}(I) \) is quasi-measurable. In GITIK & SHELAH 89 and GITIK & SHELAH p91 a variety of special types of algebra are considered; for instance, writing \( G_\lambda \) for the regular open algebra of \( \{0, 1\}^\lambda \), then if \( G_\lambda \in \text{Quot}_\tau \), there is a quasi-measurable cardinal less than \( \lambda \). On the other hand, GŁÓWczyński 91 points out that (if there can be measurable cardinals) \( \text{Quot}_\tau \) can contain countably-generated ccc algebras. Such algebras are always quotients of the algebra \( B \) of Borel subsets of \( \mathbb{R} \) (FREMLIN 84, 12F). But the ideals involved in Główczyński's construction are necessarily very irregular, and we may reasonably ask if any naturally arising ideal \( I \) of \( B \) can have \( B/I \) in \( \text{Quot}_\tau \). Similarly, we may ask whether the regular open algebra of a 'simple' partially ordered set \( (P, \leq) \) (e.g., one in which \( P \) is an analytic set in a Polish space, and \( \leq \) is an analytic, or Borel, subset of \( P \times P \)) can belong to \( \text{Quot}_\tau \).

I ought perhaps to mention in passing that any Dedekind complete Boolean algebra is isomorphic to a quotient \( P/X/I \) for some set \( X \) and some ideal \( I \); the problems here depend on the insistence on taking \( \sigma \)-ideals.

9I Theorem Let \( \kappa \) be a regular uncountable cardinal, \( F \) a normal filter on \( \kappa \), \( I \) the dual ideal; suppose that \( I \) is \( \kappa^+ \)-saturated. Let \( P \) be either the p.o.set \( (P\kappa/I) \setminus \{0\} \) or the p.o.set \( P\kappa \setminus I \). Let \( \phi \) be a formula of the second-order language \( \mathcal{L} \) of §A4 and \( C_1, \ldots , C_\kappa \) relations on \( \kappa \), \( \xi_1, \ldots , \xi_m \) ordinals less than \( \kappa \). Let \( \beta \leq \kappa \). Then the following are equivalent:

(i) \( \Vdash_P (\beta; C_1, \ldots , \xi_m) \models \phi \)
(ii) \( \{ \alpha : \alpha < \kappa, (\min(\alpha, \beta); C_1, \ldots , \xi_m) \models \phi \} \in F \).

In particular, if \( \beta < \kappa \) then
\[ (\beta; C_1, \ldots , \xi_m) \models \phi \iff \Vdash_P (\beta; C_1, \ldots , \xi_m) \models \phi. \]

proof As in 4M-4N.

9J Theorem Let \( \kappa \) be a quasi-measurable cardinal with \( \text{qm} \) filter \( W \). Then
(i) \( \kappa \) is greatly Mahlo;
(ii) \( \text{Mh}(A) \in W \) for every stationary \( A \subseteq \kappa \) – in particular, \( W \) is closed under the operation \( \text{Mh} \);
(iii) the set of greatly Mahlo cardinals below \( \kappa \) belongs to \( W \).
proof The arguments of 4J-4Ka cover (i) and (ii). Now let $A$ be the set of greatly Mahlo cardinals below $\kappa$. If $\mathcal{I}$ is any normal witnessing ideal for $\kappa$, with dual filter $\mathcal{F}$, then $\mathcal{P} = \mathcal{P}_\kappa \setminus \mathcal{I}$ is ccc; so by 9F

$$\models \mathcal{P} \kappa$$

and

$$\models \mathcal{P} \kappa$$

Next, it is evident from 4C that there is a second-order formula $\phi$ such that for any ordinal $\alpha$

$$\alpha < \kappa \models \phi \iff \alpha$$

is a greatly Mahlo cardinal.

So 9I tells us that $A \in \mathcal{F}$. As $\mathcal{I}$, $\mathcal{F}$ are arbitrary, $A \in \mathcal{W}$.

9K Theorem Let $\kappa$ be a quasi-measurable cardinal, with normal witnessing ideal $\mathcal{I}$ and dual filter $\mathcal{F}$; let $\theta < \kappa$ be a cardinal of uncountable cofinality. Then for any function $f : [\kappa]^{<\omega} \to [\kappa]^{<\theta}$ there are $C \in \mathcal{F}$, $f^* : [C]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$.

proof As in 5A-5B.

9L Corollaries (a) If $\kappa$ is quasi-measurable then there is no Jónsson algebra on $\kappa$.

(b) If $\epsilon$ is quasi-measurable then $2^\lambda = \epsilon$ for $\omega \leq \lambda < \epsilon$.

proof As 5D-5E.

9M Proposition If $\kappa$ is a quasi-measurable cardinal with qm ideal $\mathcal{J}$, then there is no $\kappa$-Aronszajn tree, and moreover $A = \{ \theta : \theta < \kappa$, there is a $\theta$-Aronszajn tree $\}$ belongs to $\mathcal{J}$.

proof The arguments of 5F show that there is no $\kappa$-Aronszajn tree. But it is easy to find a second-order formula $\phi$ such that, for any cardinal $\theta$, $(\theta; <;) \models \phi$ iff there is a $\theta$-Aronszajn tree. So the arguments of 9J show that $A \in \mathcal{J}$.

9N Proposition If $\epsilon$ is quasi-measurable, then $\diamondsuit_\epsilon$ is true.

proof As 5N.

Remark This is due to Kunen.

9O Theorem Let $\kappa$ be a quasi-measurable cardinal.

(a) If $\mathcal{F}$ is any filter on $\mathbb{N}$ then $\text{cf}(\text{cf}(\mathbb{N}/\mathcal{F})) \neq \kappa$.

(b) $\Theta(\alpha, \gamma) < \kappa$ for all cardinals $\alpha$, $\gamma < \kappa$.

(c) $\text{cov}_{\text{SH}}(\alpha, \beta, \gamma, \delta) < \kappa$ whenever $\alpha < \kappa$, $\gamma \leq \beta$ and $\delta \geq \omega_1$.

(d) If $\kappa \leq \epsilon$ then $\{ \gamma : \omega \leq \gamma < \kappa \}$ is finite.

proof Use the ideas of 7G-7K and 7O-7Q, but replacing ‘$\kappa$-measure-bounded’ with the property

if $\mathcal{I} \not\prec \mathcal{P} \mathcal{P}$ is an $\omega_1$-saturated $\kappa$-additive ideal then there is a $p \in \mathcal{P}$ such that $\{ p' : p' \leq p \} \not\subseteq \mathcal{I}$.

9P Remarks 9Od is given in Gitik & Shelah p91.

Many of the results in 9J-9O have generalizations to cardinals $\kappa$ carrying non-trivial $\kappa$-additive ideals which are $\lambda$-saturated for some $\lambda < \kappa$.

Version of 18.9.92

Appendix: Useful Facts

In this appendix I seek to support the main text by giving definitions and theorems which may not be universally familiar, with some proofs.

A1. Combinatorics

I begin with material in (infinitary) combinatorics and set theory.

A1A Partially ordered sets (a) Recall that a partially ordered set is a set $P$ together with a relation $\leq$ such that, for $p, q, r \in P$,
(b) For a partially ordered set $P$, say that a subset $Q$ of $P$ is cofinal with $P$ if for every $p \in P$ there is a $q \in Q$ such that $p \leq q$. Now write 
\[ \text{cf}(P) = \min\{\#(Q) : Q \subseteq P \text{ is cofinal with } P\}, \]
the cofinality of $P$.

If $P$ is totally ordered then $P$ has a well-ordered cofinal subset of order type $\text{cf}(P)$, which in this case is either $0$ (if $P = \emptyset$) or $1$ (if $P$ has a greatest element) or an infinite cardinal which is regular, that is, equal to its own cofinality. (But the cofinalities of general partially ordered sets need not be regular.)

(c) For a partially ordered set $P$ with no greatest element, write
\[ \text{add}(P) = \min\{\#(A) : A \subseteq P \text{ has no upper bound in } P\}, \]
the additivity of $P$. Then $\text{add}(P)$ is either $0$ (if $P = \emptyset$) or $2$ (if $P$ is not upwards-directed) or a regular infinite cardinal. In the last case, there is a family $\{p_\xi : \xi < \text{add}(P)\}$ in $P$ such that $p_\xi \leq p_\eta$ whenever $\xi < \eta < \text{add}(P)$.

If $P$ has a greatest element I will write $\text{add}(P) = \infty$.

If $\kappa$ is a cardinal less than or equal to $\text{add}(P)$ (allowing $\kappa = \infty = \text{add}(P)$) we say that $P$ is $\kappa$-additive; that is, $P$ is $\kappa$-additive iff every subset of $P$ of cardinal less than $\kappa$ has an upper bound in $P$.

If $\text{add}(P) \neq \infty$ then $\text{add}(P) \leq \text{cf}(P)$. If $P$ is totally ordered and has no greatest element then $\text{add}(P) = \text{cf}(P)$.

**A1B Filters and ideals**

(a) If $X$ is a set and $\mathcal{I}$ is an ideal of subsets of $X$, we may think of $\mathcal{I}$ as partially ordered by $\subseteq$, and discuss its additivity (and cofinality) as in A1A. I allow $\mathcal{P}X$ as an (improper) ideal of itself.

(b) If $X$ is a set, $\kappa$ is a cardinal and $\mathcal{F}$ is a filter on $X$, then $\mathcal{F}$ is $\kappa$-complete if $\bigcap A \in \mathcal{F}$ whenever $A \subseteq \mathcal{F}$ and $0 < \#(A) < \kappa$; that is, if the dual ideal $\{X \setminus F : F \in \mathcal{F}\}$ is $\kappa$-additive.

(c) Let $X$ be a set and $\mathcal{F}$ a filter on $X$. Then $\mathcal{F}$ is uniform if $X \setminus A \in \mathcal{F}$ whenever $A \subseteq X$ and $\#(A) < \#(X)$.

(d) Let $X$ and $Y$ be sets, $\mathcal{F}$ a filter on $X$, and $f : X \to Y$ a function. I write $f[[\mathcal{F}]]$ for the filter $\{G : G \subseteq Y, f^{-1}[G] \in \mathcal{F}\}$, that is, the filter on $Y$ generated by $\{f[F] : F \in \mathcal{F}\}$.

(e) If $X$ is a set and $\mathcal{A}$ is any family of sets, I write
\[ \text{non}(X, \mathcal{A}) = \min\{\#(Y) : Y \subseteq X, Y \not\subseteq A \forall A \in \mathcal{A}\}, \]
allowing non$(X, \mathcal{A}) = \infty$ if $X \subseteq A \in \mathcal{A}$.

(f) If $X$ is a set, $\mathcal{I}$ is an ideal of subsets of $X$, and $\kappa$ is a cardinal, then $\mathcal{I}$ is $\kappa$-saturated if there is no family $\{A_\xi : \xi < \kappa\}$ in $\mathcal{P}X \setminus \mathcal{I}$ such that $A_\xi \cap A_\eta \in \mathcal{I}$ whenever $\xi < \eta < \kappa$; that is, if there is no disjoint family of size $\kappa$ in $\mathcal{P}X \setminus \mathcal{I}$. If $\mathcal{I}$ is $\kappa$-additive, then it is $\kappa$-saturated iff there is no disjoint family of size $\kappa$ in $\mathcal{P}X \setminus \mathcal{I}$.

**(A1C Filters on $\mathbb{N}$)**

(a) A $p$-point filter on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that for every sequence $(F_n)_{n \in \mathbb{N}}$ in $\mathcal{F}$ there is an $F \in \mathcal{F}$ such that $F \setminus F_n$ is finite for every $n \in \mathbb{N}$.

(b) A rapid filter on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that for every sequence $(t_n)_{n \in \mathbb{N}}$ of non-negative real numbers converging to $0$ there is an $F \in \mathcal{F}$ such that $\sum_{n \in F} t_n < \infty$.

(c) A selective ultrafilter on $\mathbb{N}$ is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that whenever $(A_n)_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathcal{P} \mathbb{N} \setminus \mathcal{F}$ there is an $F \in \mathcal{F}$ such that $\#(F \cap A_n) \leq 1$ for every $n \in \mathbb{N}$.

(d) If $A$ is a cardinal, a $p(\lambda)$-point filter on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that whenever $A \subseteq F$ and $\#(A) < \lambda$ there is an $F \in \mathcal{F}$ such that $F \setminus A$ is finite for every $A \in \mathcal{A}$. (Thus a $p$-point filter is a $p(\omega_1)$-point filter.)
A1D Cardinals and ordinals (a) Let $\kappa$ be a cardinal. (i) $\kappa$ is weakly inaccessible if it is uncountable and regular and $\lambda^+ < \kappa$ for every cardinal $\lambda < \kappa$. (ii) $\kappa$ is strongly inaccessible if it is uncountable and regular and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$. 

(b) Let $S$ be any class of ordinals and $f$ an ordinal-valued function defined on $S$; then $f$ is regressive if $f(\xi) < \xi$ for every $\xi \in S \setminus \{0\}$.

(c) If $\xi$ is any ordinal and $A \subseteq \xi$, then $A$ is stationary in $\xi$ if $A \cap C \neq \emptyset$ for every set $C \subseteq \xi$ which is cofinal with $\xi$ and closed (for the order-topology of $\xi$).

(d) An ordinal $\alpha$ is decomposable if there are smaller ordinals $\beta, \gamma$ such that $\alpha = \beta + \gamma$; that is, if there is an ordinal $\beta < \alpha$ such that $\text{otp}(\alpha \setminus \beta) < \alpha$. Otherwise $\alpha$ is indecomposable.

A1E Lemma Let $\kappa$ be a regular uncountable cardinal, $\mathcal{F}$ a uniform filter on $\kappa$, $\mathcal{I}$ the dual ideal $\{\kappa \setminus F : F \in \mathcal{F}\}$.

(a) $\mathcal{F}$ and $\mathcal{I}$ are normal (see 1F) iff for every family $\langle F_\xi \rangle_{\xi<\kappa}$ in $\mathcal{F}$ its diagonal intersection $\{\xi : \xi < \kappa, \xi \in F_\eta \forall \eta < \xi\}$ belongs to $\mathcal{F}$.

(b) The ‘club filter’ on $\kappa$, generated by the closed unbounded sets in $\kappa$, is normal; the ‘non-stationary ideal’, consisting of the non-stationary subsets of $\kappa$, is normal.

(c) If $\mathcal{F}$ and $\mathcal{I}$ are normal, then

(i) $\mathcal{F}$ is $\kappa$-complete, $\mathcal{I}$ is $\kappa$-additive;

(ii) every closed unbounded subset of $\kappa$ belongs to $\mathcal{F}$;

(iii) if $\mathcal{H}$ is a subset of the quotient algebra $\mathfrak{A} = \mathcal{P}_\kappa / \mathcal{I}$ and $\#(\mathcal{H}) \leq \kappa$, then $\sup \mathcal{H}$ and $\inf \mathcal{H}$ are defined in $\mathfrak{A}$;

(iv) if $\langle F_I \rangle_{I \in [\kappa]^{<\omega}}$ is any family in $\mathcal{F}$, then $\{\xi : \xi < \kappa, \xi \in F_I \forall I \in [\xi]^{<\omega}\}$ belongs to $\mathcal{F}$.

(d) The following are equivalent:

(i) $\mathcal{I}$ is normal and $\kappa$-saturated in $\mathcal{P}_\kappa$;

(ii) $\mathcal{I}$ is $\kappa$-additive and for every function $f : \kappa \rightarrow \kappa$ there is a $\zeta < \kappa$ such that $\{\xi : \xi < \kappa, \xi \in F_I \forall I \in [\zeta]^{<\omega}\}$ belongs to $\mathcal{F}$.

Proof (a)(i) Suppose that $\mathcal{F}$ is normal, and that $\langle F_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$ with diagonal intersection $F$. Set $S = \kappa \setminus F$, and for $\xi \in S$ take $f(\xi) < \xi$ such that $\xi \notin F_{f(\xi)}$. Then $f$ is regressive, but $f^{-1}(\{\xi\}) \subseteq \kappa \setminus F_\xi \in \mathcal{I}$ for every $\xi < \kappa$; so $S \in \mathcal{I}$ and $F \in \mathcal{F}$.

(ii) Now suppose that $\mathcal{F}$ is closed under intersections, and that we are given $S \in \mathcal{P}_\kappa \setminus \mathcal{I}$ and a regressive function $f : S \rightarrow \kappa$. If $A_\kappa = f^{-1}(\{\xi\}) \in \mathcal{I}$ for every $\xi < \kappa$, set $F_\xi = A_\kappa \setminus A_\xi \in \mathcal{F}$ for each $\xi$, and take $F$ to be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$, so that $F \in \mathcal{F}$. Then there must be a $\xi \in F \cap S \setminus \emptyset$, so that $f(\xi) < \xi$ and $\xi \in A_{f(\xi)}$ and $\xi \notin F_{f(\xi)}$ and $\xi \notin F$, which is absurd.

(b) It is easy to check that if $\langle C_\xi \rangle_{\xi<\kappa}$ is any family of closed unbounded sets in $\kappa$, then its diagonal intersection is closed and also (because $\kappa$ is regular) unbounded. So the club filter is normal and its dual ideal, the non-stationary ideal, is normal.

(Of course the definition of ‘normal’ ideal in 1F above corresponds to the ‘pushing-down lemma’; see KUNEN 80, II.6.15, or JECH 78, Theorem 22.)

(c)(i) If $\lambda < \kappa$ and $\langle F_\xi \rangle_{\xi<\lambda}$ is any family in $\mathcal{F}$, set $F_\xi = \kappa$ for $\xi \in \kappa \setminus \lambda$, and consider the diagonal intersection $F$ of $\langle F_\xi \rangle_{\xi<\kappa}$. Then $F$ is $\kappa$-complete, and $\mathcal{I}$ is $\kappa$-additive.

(ii) Now take any closed unbounded set $C \subseteq \kappa$. For each $\xi < \kappa$ set $\gamma_\xi = \min(C \setminus (\xi+1))$, $F_\xi = \kappa \setminus \gamma_\xi \in \mathcal{F}$; let $F$ be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$. Then $F \in \mathcal{F}$ and $F \setminus C \subseteq \kappa$, so $C \subseteq \mathcal{F}$.

(iii) If $H = \emptyset$ then $\inf H = 1$. Otherwise, let $\langle A_\xi \rangle_{\xi<\kappa}$ run over $H$. For each $\xi < \kappa$ choose $A_\xi \subseteq \kappa$ such that $A_\xi^* = A_\xi$ in $\mathfrak{A}$. Let $A$ be the diagonal intersection of $\langle A_\xi \rangle_{\xi<\kappa}$, and consider $a = A^*$. Because $A \setminus A_\xi \subseteq \xi + 1 \in \mathcal{I}$, $a \subseteq a_\xi$ for each $\xi < \kappa$, and $a$ is a lower bound for $H$. Now let $b$ be any other lower bound for $H$ in $\mathfrak{A}$, and take $B \subseteq \kappa$ such that $B^* = b$. For each $\xi < \kappa$, we have $b \subseteq a_\xi$, that is, $B \setminus A_\xi \in \mathcal{I}$; set $F_\xi = \kappa \setminus (B \setminus A_\xi) \in \mathcal{F}$. Let $F$ be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$, so that $F \in \mathcal{F}$. If there is a $\xi \in (B \setminus A) \cap F$, then there must be an $\eta < \xi$ such that $\xi \notin A_\eta$; but now $\xi \in B \setminus A_\eta$ so $\xi \notin F_\eta$ and $\xi \notin F$. Thus $B \setminus A \subseteq \kappa \setminus F \in \mathcal{I}$, and $b \subseteq a$ in $\mathfrak{A}$. As $b$ is arbitrary, $a = \inf H$ in $\mathfrak{A}$. 

\[ \tag*{\mathfrak{A}} \]
Thus $H$ has an infimum in $\mathfrak{A}$. But applying the argument above to $\{1 \setminus a : a \in H\}$ we see that $H$ also has a supremum.

(iv) Set $E_\xi = \cap_{f \in \mathcal{E}} F_1$ for each $\xi < \kappa$; by (i), $E_\xi \in \mathcal{F}$. Let $C$ be the closed unbounded set of limit ordinals less than $\kappa$; set

$$F = \{\xi : \xi \in C, \xi \in E_\eta \ \forall \ \eta < \xi\}. $$

Then $\{\xi : \xi \in F_1 \ \forall \ I \in [\xi]^{\omega}\} \supseteq F \in \mathcal{F}$.

(d)(i)⇒(ii) Of course $\mathcal{I}$ is $\kappa$-additive, by (c-i). Now let $f : \kappa \to \kappa$ be any function. Because $\mathcal{I}$ is $\kappa$-saturated, the set $A = \{\alpha : f^{-1}(\{\alpha\}) \notin \mathcal{I}\}$ has cardinal less than $\kappa$; because $\kappa$ is regular, $A$ is bounded above in $\kappa$—say $A \subseteq \zeta < \kappa$. For $\alpha \in A$ set $F_\alpha = \kappa$; for $\alpha \in \kappa \setminus A$ set $F_\alpha = \kappa \setminus f^{-1}(\{\alpha\})$. Then

$$F = \{\xi : \xi \in F_\alpha \ \forall \ \alpha < \xi\}$$

belongs to $\mathcal{F}$. If $\xi \leq f(\xi) < \kappa < \kappa$ then $f(\xi) \notin A$, so $\xi \notin f(\xi) \notin F_\xi$ and $\xi \notin F$; thus $\{\xi : \xi \leq f(\xi) < \kappa \subseteq \kappa \setminus F \in \mathcal{I}\}$.

(ii)⇒(iii) is elementary.

(iii)⇒(i) Of course $\mathcal{I}$ is $\kappa$-additive. If it is not $\kappa$-saturated, choose a disjoint family $\langle A_\xi \rangle_{\xi < \kappa}$ in $\mathcal{P}_\kappa \setminus \mathcal{I}$; set $f(\eta) = \xi$ for $\eta \in A_\xi \setminus (\xi + 1)$, $f(\eta) = 0$ for $\eta \in \kappa \setminus \bigcup_{\xi < \kappa} (A_\xi \setminus (\xi + 1))$. Then $f$ is regressive, and $f^{-1}[\xi]$ does not meet $A_\xi \setminus (\xi + 1)$, so it does not belong to $\mathcal{F}$, for any $\xi < \kappa$. X

Now suppose that $\langle F_\xi \rangle_{\xi < \kappa}$ is any family in $\mathcal{F}$. Let $F$ be its diagonal intersection, and define $f : \kappa \to \kappa$ by setting $f(\xi) = 0$ if $\xi \in F$, $f(\xi) = \min(\eta : \xi \notin F_\eta)$ if $\xi \in \kappa \setminus F$. Let $\xi < \kappa$ be such that $f^{-1}[\xi] \in \mathcal{F}$. Then $F \supseteq f^{-1}[\xi] \cap \bigcap_{\eta < \xi} F_\eta \in \mathcal{F}$. This shows that $\mathcal{F}$ is normal.

Remark For another version of this material, see Baumgartner Taylor & Wagon 82.

A1F Almost-square-sequences I give here one of Shelah’s lemmas (SHELAH #351, Lemma 4.4; BURKE & MAGIDOR 90, 7.7) in a form appropriate to Theorem 3F.

Lemma Let $\kappa$, $\lambda$ be infinite cardinals, with $\kappa$ regular and $\lambda > \kappa$, cf($\lambda$) > $\omega_1$. Then we can find a stationary set $S \subseteq \lambda^+$ and a family $(C_\alpha)_{\alpha < \xi}$ of sets such that (i) for each $\alpha \in S$, $C_\alpha$ is a closed unbounded set in $\alpha$ of order type $\kappa$ (ii) if $\alpha, \beta \in S$ and $\gamma$ is a limit point of both $C_\alpha$ and $C_\beta$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

proof For each $\gamma < \lambda^+$ fix an injection $f_\gamma : \gamma \to \lambda$. Let $S_0$ be the set of ordinals $\alpha < \lambda^+$ of cofinality $\kappa$; then $S_0$ is stationary in $\lambda^+$. For each $\alpha \in S_0$ choose an increasing family $\langle C_\alpha \rangle_{\delta < \lambda}$ of subsets of $\lambda^+$ such that

(1) $\langle N_{\alpha, \delta} \rangle$ is a cofinal subset of $\alpha$ of cardinal $\delta$;

(2) $\langle \delta : \delta < \lambda \rangle$ is stationary in $\lambda^+$.

(taking the closure $\mathcal{N}_\alpha$ in the order topology of $\lambda^+$):

(1) $\gamma < \lambda$ is a non-zero limit ordinal then $N_{\alpha, \delta} = \bigcup_{\delta < \gamma} N_{\alpha, \delta'}$.

Then $\#(N_{\alpha, \delta}) \leq \max(\kappa, \#(\delta)) < \lambda$ for each $\beta < \lambda$. Now observe that $\{\delta : \delta < \lambda, N_{\alpha, \delta} \cap \lambda = \delta\}$ is a closed unbounded set in $\lambda$, and in particular contains an ordinal of cofinality $\omega_1$, for every $\alpha \in S_0$. Let $\delta < \lambda$ be such that $\text{cf}(\delta) = \omega_1$ and

$$S_1 = \{\alpha : \alpha \in S_0, N_{\alpha, \delta} \cap \lambda = \delta\}$$

is stationary in $\lambda^+$. For $\alpha \in S_1$, set $C_\alpha = \alpha \cap \mathcal{N}_{\alpha, \delta}$; then $C_\alpha$ is a closed unbounded set in $\alpha$ and $\#(C_\alpha) < \lambda$ so opt($C_\alpha$) < $\lambda$. Let $\zeta < \lambda$ be such that

$$S = \{\alpha : \alpha \in S_1, \text{opt}(C_\alpha) = \zeta\}$$

is stationary in $\lambda^+$. Observe that as $\text{cf}(C_\alpha) = \text{cf}(\alpha) = \kappa$ for each $\alpha \in S$, $\text{cf}(\zeta) = \kappa$.

Take any closed unbounded set $C \subseteq \zeta$ of order type $\kappa$ and for each $\alpha \in S$ let $C_\alpha$ be the image of $C$ in $C_\alpha$ under the order-isomorphism between $\zeta$ and $C_\alpha$. Then $C_\alpha$ will be a closed unbounded subset of $\alpha$ of order type $\kappa$.

I claim that if $\alpha, \beta \in S$ and $\gamma$ is a common limit point of $C_\alpha, C_\beta$ then $C_\alpha \cap \gamma = C_\beta \cap \gamma$.

P case 1 Suppose $\kappa = \omega$. In this case the only limit point of $C_\alpha$ will be $\alpha$ itself, and similarly for $\beta$, so that in this case we have $\alpha = \beta$ and there is nothing more to do.

case 2 Suppose $\text{cf}(\gamma) = \omega < \kappa$. Then $\gamma$ is a limit point of $C_\alpha \subseteq N_{\alpha, \delta}$, so there is an increasing sequence in $N_{\alpha, \delta}$ with supremum $\gamma$; as $N_{\alpha, \delta} \subseteq \bigcup_{\delta < \gamma} N_{\alpha, \delta'}$ and $\text{cf}(\delta) = \omega_1$, this sequence lies entirely within $N_{\alpha, \delta'}$ for
some $\delta' < \delta$, and $\gamma \in \bigcap_{\beta < \gamma} N_{\alpha, \delta' + 1}$. Now, for $\delta' + 1 \leq \xi < \delta$, $N_{\alpha, \xi + 1} \supseteq f^{-1}_\gamma [N_{\alpha, \xi} \cap \gamma] = f^{-1}_\gamma [\delta]$. Similarly, $N_{\alpha, \xi} \cap \gamma = f^{-1}_\gamma [\delta]$. Now

$$C_\alpha \cap \gamma = \bigcap_{\alpha < \gamma} N_{\alpha, \delta} \cap \gamma = f^{-1}_\gamma [\delta] \cap \gamma = C_\beta \cap \gamma.$$  

Thus $C_\alpha \cap \gamma$ and $C_\beta \cap \gamma$ must be equal.

**Case 3** Suppose that $\alpha = \sup(C_\alpha \cap \gamma) = \sup(C_\beta \cap \gamma)$, $D = \{\gamma' : \gamma' < \gamma, \text{ cf}(\gamma') = \omega\}$ is cofinal with $\gamma$, and $C_\alpha \cap \gamma = \bigcup_{\gamma' \in D} C_\alpha \cap \gamma' = C_\beta \cap \gamma$, using case 2. $Q$

**A1G Corollary** Let $\kappa$, $\lambda$ be infinite cardinals with $\kappa$ regular and $\lambda > \kappa$, $\text{cf}(\lambda) > \omega$. Then we can find a stationary subset $S$ of $\lambda^+$ and a family $\{g_\alpha\}_{\alpha \in S}$ of functions from $\kappa$ to $\lambda^+$ such that, for all distinct $\alpha$, $\beta \in S$, (i) $g_\alpha[\alpha] \subseteq \alpha$ for each $\alpha \in S$ (ii) $\#(g_\alpha \cap g_\beta) < \kappa$ for each $\alpha$ (iii) if $\theta < \kappa$ is a limit ordinal and $\alpha$, $\beta \in S$ and $g_\alpha(\theta) = g_\beta(\theta)$ then $g_\alpha(\theta) = g_\beta(\theta)$.

**proof** Take $(C_\alpha)_{\alpha \in S}$ from A1F above and let $g_\alpha$ be the increasing enumeration of $C_\alpha$.

**A1H Products of partially ordered sets (a)** Let $(P_i)_{i \in I}$ be a family of partially ordered sets. Then $X = \prod_{i \in I} P_i$ is a partially ordered set, if we say that $x \leq y$ iff $x(i) \leq y(i)$ for every $i \in I$.

(b) Now suppose that $F$ is a filter on $I$. Then we have an equivalence relation $\equiv_F$ on $X$, given by saying that $f \equiv_F g$ if $i : f(i) = g(i) \in F$. I write $X/\equiv_F$ for the set of equivalence classes under this relation, the ‘reduced product’ of $(P_i)_{i \in I}$ modulo $F$. Now $X/\equiv_F$ is again a partially ordered set, writing

$$f' \leq g' \iff f \leq_F g \iff (i : f(i) \leq g(i)) \in F.$$  

Observe that if every $P_i$ is totally ordered and $F$ is an ultrafilter, then $X/\equiv_F$ is totally ordered.

(c) For any filter $F$ on $I$ we have

$$\text{add}(X) \leq \text{sup}_{F \in \mathcal{F}} \text{add}(\prod_{i \in F} P_i) = \text{sup}_{F \in \mathcal{F}} \text{min}_{i \in F} \text{add}(P_i) \leq \text{add}(X/\equiv_F),$$

$$\text{cf}(X/\equiv_F) \leq \text{min}_{i \in F} \text{cf}(\prod_{i \in F} P_i) \leq \text{cf}(X).$$

**A11 Scraps of pcf** In §7 I need to call on certain results from Shelah’s pcf theory. An admirable exposition of some of this extraordinary development may be found in BURKE & MAGIDOR 90, from which most of the ideas here are drawn; but for the reader’s convenience I extract and reproduce the material I wish to use.

**Theorem** [Shelah] Let $\lambda > 0$ be a cardinal and $(\theta_\zeta)_{\zeta < \lambda}$ a family of regular infinite cardinals, all greater than $\lambda$. Let $X = \prod_{\zeta < \lambda} \theta_\zeta$, ordered as in A1H. For any filter $F$ on $\lambda$, let $\pi_F : X \rightarrow X/\equiv_F$ be the canonical map. For any cardinal $\delta$ set

$$\mathcal{F}_\delta = \{F : F \text{ is an ultrafilter on } \lambda, \text{ cf}(X/\equiv_F) = \delta\},$$

$$\mathcal{F}_\delta^* = \bigcup_{\delta' \geq \delta} \mathcal{F}_\delta^*.$$  

if $\mathcal{F}_\delta^* \neq 0$, let $\mathcal{G}_\delta$ be the filter $\bigcap \mathcal{F}_\delta^*$. Now

(a) if $\mathcal{F}_\delta^* \neq 0$, then $\mathcal{G}_\delta \geq \delta$ (BURKE & MAGIDOR 90, 1.1);

(b) if $\mathcal{F}_\delta \neq 0$ then there is a set $F \subseteq X$ such that $\#(F) \leq \delta$ and $\pi_F[F]$ is cofinal with $X/\equiv_F$ for every $F \in \mathcal{F}_\delta$ (BURKE & MAGIDOR 90, 7.3);

(c) $\mathcal{F}_{\text{cf}(X)} \neq 0$ (BURKE & MAGIDOR 90, 7.10);

(d) if $F$ is an ultrafilter on $\lambda$ and $\kappa$ is a regular cardinal with $\lambda < \kappa \leq \text{cf}(X/\equiv_F)$ then there is a family $(\theta'_\zeta)_{\zeta < \lambda}$ of regular cardinals such that $\lambda < \theta'_\zeta \leq \theta_\zeta$ for every $\zeta < \lambda$ and $\text{cf}(X'/\equiv_F) = \kappa$, where $X' = \prod_{\zeta < \lambda} \theta'_\zeta$ (BURKE & MAGIDOR 90, 2.1).

**proof** The case of finite $\lambda$ is trivial throughout, as then

Version of 13.11.91
\[ cf(X) = \max_{\xi < \lambda} \theta_{\xi}, \]
\[ \mathfrak{F}_\delta = \{ \mathcal{F} : \exists \xi \ , \ \xi \in \mathcal{F} \}, \]
\[ \mathfrak{F}_\xi = \{ \mathcal{F} : \{ \xi : \theta_{\xi} \geq \delta \} \subseteq \mathcal{F} \}, \]
\[ G_\delta = \{ G : \{ \xi : \theta_{\xi} \geq \delta \} \subseteq G \subseteq \lambda \}. \]

So henceforth let us take it that \( \lambda \) is infinite.

For any filter \( \mathcal{F} \) on \( \lambda \), write \( f \leq g \) if \( f, g \in X \) and \( \{ \xi : f(\xi) \leq g(\xi) \} \in \mathcal{F} \), that is, if \( \pi_X f \leq \pi_X g \) in \( X/\mathcal{F} \).

Write \( L = \{ \xi : \theta_{\xi} = \lambda^+ \} \subseteq \lambda \), \( M = \lambda \setminus L \).

(a) Set \( \delta' = \text{add}(X/G_\delta) \).

(i) Evidently \( \delta' \) is a regular cardinal and \( \delta' \geq \text{add}(X) = \min_{\xi < \lambda} \theta_{\xi} > \lambda \). If \( \delta = \lambda^+ \) then of course \( \delta' \geq \delta \); so suppose that \( \delta > \lambda^+ \). In this case \( L \notin \mathcal{F} \) for any \( \mathcal{F} \in \mathfrak{F}_\delta \) (because if \( L \neq \emptyset \) then \( \text{cf}(\prod_{\xi \in L} \theta_{\xi}) = \lambda^+ \), so \( M \in G_\delta \) and \( \delta' \geq \min_{\xi \in M} \theta_{\xi} > \lambda^+ \).

(ii) If \( \delta' < \delta \) there is a family \( \{ \langle f_\alpha \rangle_{\alpha < \delta} \} \) in \( X \) such that \( f_\alpha \leq g_\delta \) whenever \( \alpha \leq \beta < \delta' \) but there is no \( f \in X \) such that \( f_\alpha \leq g_\delta \) \( f \) for every \( \alpha < \delta' \). Choose \( \langle h_\xi \rangle_{\xi < \lambda^+} \) in \( X \) inductively, as follows. \( h_0 = f_0 \).

Given \( h_\xi \), set
\[ B_{\xi_0} = \{ \xi : \xi \in M, h_\xi(\xi) \approx f_\xi(\xi) \} \]
for each \( \alpha < \delta' \); let \( \alpha_0 < \xi \) such that \( f_{\alpha_0} \leq g_\delta \), so that \( B_{\xi_0} \notin G_\delta \) when \( \alpha_0 \leq \alpha < \delta' \). Choose \( \mathcal{F}_\xi \in \mathfrak{F}_\delta \) such that \( B_{\xi_0} \notin \mathcal{F}_\xi \).

Now, because \( \text{cf}(X/\mathcal{F}_\xi) \geq \delta > \delta' \), there is an \( h_{\xi+1} \in X \) such that \( f_\alpha \leq f_{\xi_0} \) \( h_{\xi+1} \) for every \( \alpha < \delta' \); we may take \( h_{\xi+1} \geq h_\xi \).

For non-zero limit ordinals \( \xi < \lambda^+ \) take \( h_\xi(\xi) = \sup_{\eta < \xi} h_\eta(\xi) \) for every \( \xi < \lambda \).

Set \( \alpha = \sup_{\xi < \lambda, \alpha_0 < \xi} \alpha_0 \). Then \( B_{\xi_0} \in \mathcal{P} \lambda \).

So there must be a \( \xi < \lambda^+ \) such that \( B_{\xi_0} = B_{\xi_0+1} \). But (because \( \alpha \geq \alpha_0 \)) \( B_{\xi_0} \notin \mathcal{F}_\xi \), while (by the choice of \( h_{\xi+1} \)) \( B_{\xi_0+1} \notin \mathcal{F}_\xi \); which is absurd. \( X \)

(b) As in (a-i) above, we must have \( \delta = \min_{\xi < \lambda} \theta_{\xi} > \lambda \), and the case \( \delta = \lambda^+ \) is again trivial, for if \( \delta = \lambda^+ \) we may take \( F \) to be the set of common functions with values less than \( \lambda^+ \). So suppose from now on that \( \delta > \lambda^+ \), so that \( M \in \mathcal{F} \) for every \( \mathcal{F} \in \mathfrak{F}_\delta \). Of course \( \delta \), being the cofinality of a totally ordered set, is regular.

(i) Suppose, if possible, that there is no \( F \) of the required type. We can find families \( \langle f_{\xi_0} \rangle_{\xi < \lambda^+, \alpha < \delta} \)

in \( X \) and \( \langle \mathcal{F}_\xi \rangle_{\xi < \lambda^+} \) in \( \mathfrak{F}_\delta \) such that
\[ (\alpha) \ f_{\xi_0} \leq f_{\xi_0} \text{ whenever } \alpha < \delta, \eta < \xi < \lambda^+; \]
\[ (\beta) \ f_\eta \leq f_{\xi_0} \text{ whenever } \alpha < \delta, \eta < \xi < \lambda^+; \]
\[ (\gamma) \ \{ f_{\xi_0}(\xi_0) : \alpha < \delta \} \text{ is cofinal with } X/\mathcal{F}_\xi \text{ for every } \xi < \lambda^+; \]
\[ (\delta) \text{ if } \xi < \lambda^+, \alpha < \delta \text{ and } \text{cf}(\alpha) = \lambda^+ \text{ then } \] for every \( \xi \in M \):
\[ (\epsilon) \ f_{\xi_0} \leq f_{\xi_0} \text{ whenever } \xi < \lambda^+, \alpha < \delta. \]

Given \( \langle g_{\xi_0} \rangle_{\eta < \xi, \alpha < \delta} \), there must be an \( \mathcal{F}_\xi \in \mathfrak{F}_\delta \) such that \( \{ \pi_{\mathcal{F}_\xi}(g_{\xi_0}) : \eta < \xi, \alpha < \delta \} \) is not cofinal with \( X/\mathcal{F}_\xi \) to take \( \langle g_{\xi_0} \rangle_{\alpha < \delta} \) in \( X \) such that \( \pi_{\mathcal{F}_\xi}(g_{\xi_0}) : \alpha < \delta \) is cofinal with \( X/\mathcal{F}_\xi \).

Choose \( \langle f_{\xi_0} \rangle_{\alpha < \delta} \) inductively so that
\[ f_{\xi_0} \leq f_{\xi_0} \text{ for every } \eta < \xi, \alpha < \delta; \]
\[ \text{if } \alpha < \delta \text{ and } \text{cf}(\alpha) \neq \lambda^+ \text{ then } f_{\xi_0} \leq f_{\xi_0} \text{ for every } \eta < \xi, \]
\[ f_{\xi_0} \leq f_{\xi_0} \text{ for every } \beta < \alpha; \]
\[ \text{if } \alpha < \delta \text{ then } g_{\xi_0} \leq f_{\xi_0+1}; \]
\[ \text{if } \alpha < \delta \text{ and } \text{cf}(\alpha) = \lambda^+ \text{ then } f_{\xi_0}(\xi_0) \leq f_{\xi_0}(\xi_0) \text{ whenever } \eta < \xi, \xi \in L; \]
\[ \text{if } \alpha < \delta \text{ and } \text{cf}(\alpha) = \lambda^+ \text{ then } \]
\[ f_{\xi_0}(\xi_0) \leq \min\{ \sup_{\beta < \delta} f_{\xi_0}(\xi_0) : C \text{ is a closed unbounded set in } \alpha \} \]
for every \( \xi \in M \). (At this point observe that there will be a closed unbounded set \( C \subseteq \alpha \) such that \( f_{\xi_0}(\xi_0) \leq \min_{\beta < \delta} f_{\xi_0}(\xi_0) \) for every \( \xi \in M \); consequently \( f_{\xi_0} \leq f_{\xi_0} \) for every \( \beta < \alpha \), and \( f_{\xi_0} \leq f_{\xi_0} \) for every \( \beta < \alpha \). Also we shall have
\[ f_{\eta_0}(\zeta) \leq \sup_{\beta \in \mathcal{C}} f_{\eta_\beta}(\zeta) \leq f_{\xi_0}(\zeta) \]

for every \( \eta < \xi, \zeta \in M. \)

It is straightforward to check that this procedure works. \( \Box \)

(ii) The next step is to find an increasing family \( \langle h_\eta \rangle_{\eta < \lambda^+} \) in \( X \) and a strictly increasing family \( \langle \gamma(\eta) \rangle_{\eta < \lambda^+} \) in \( \delta \) such that

\[ f_\eta \gamma(\eta)(\zeta) < h_\eta(\zeta) \]

whenever \( \xi < \lambda^+, \eta < \lambda^+, \zeta \in M \) (choosing \( h_\eta \));

\[ h_\eta \leq f_\xi \gamma(\eta+1) \]

whenever \( \xi < \lambda^+, \eta < \lambda^+ \) (choosing \( \gamma(\eta+1) \));

\[ \gamma(\eta) = \sup_{\eta' < \eta} \gamma(\eta') \]

whenever \( \eta < \lambda^+ \) is a limit ordinal (so \( \gamma(0) = 0 \)).

Set \( h(\zeta) = \sup_{\eta < \lambda^+} h_\eta(\zeta) \) for \( \zeta \in M \), \( h(\zeta) = 0 \) for \( \zeta \in L \), \( \alpha = \sup_{\eta < \lambda^+} \gamma(\eta) < \delta \) (because \( \delta = \text{cf}(\delta) > \lambda^+ \)); then \( \text{cf}(\alpha) = \lambda^+ \).

Observe that

\[ f_{\xi_0}(\zeta) \leq \sup_{\eta < \lambda^+} f_{\xi, \gamma(\eta)}(\zeta) \leq h(\zeta) \]

for every \( \xi < \lambda^+, \zeta \in M \), by (i-\( \delta \)). So if we set

\[ A_\xi = \{ \zeta : \zeta \in M, f_{\xi_0}(\zeta) = h(\zeta) \} \quad \forall \xi < \lambda^+ \]

we shall have \( A_\eta \subset A_\xi \) whenever \( \eta \leq \xi < \lambda^+ \), by (i-\( \alpha \)).

(iii) There must therefore be some \( \xi < \lambda^+ \) such that \( A_\xi = A_{\xi+1} \). Let \( C \subset \lambda^+ \) be a closed unbounded set such that

\[ f_{\xi+1, 0}(\zeta) = \sup_{\eta \in C} f_{\xi+1, \gamma(\eta)}(\zeta) \]

for each \( \eta \in C \) write \( \eta' \) for the next member of \( C \) greater than \( \eta \); then

\[ h_\eta \leq f_{\xi+1+1, \gamma(\eta+1)} \leq f_{\xi+1+1, \gamma(\eta')}, \]

\[ f_{\xi_0} \leq f_{\xi+1+1} = f_{\xi+1, 0} = f_{\xi+1, \gamma(0)} \]

so there is a \( \zeta_\eta \in M \)

\[ h_\eta(\zeta_\eta) \leq f_{\xi+1, \gamma(\eta)}(\zeta_\eta), \]

\[ f_{\xi_0}(\zeta_\eta) \leq f_{\xi+1, 0}(\zeta_\eta) < h_0(\zeta_\eta) \leq h(\zeta_\eta) \]

Let \( \zeta \in M \) be such that

\[ B = \{ \eta : \eta \in C, \zeta_\eta = \zeta \} \]

is cofinal with \( C \). Then \( f_{\xi_0}(\zeta) < h(\zeta) \) so \( \zeta \notin A_\xi = A_{\xi+1} \). On the other hand,

\[ f_{\xi+1, 0}(\zeta) = \sup_{\eta \in C} f_{\xi+1, \gamma(\eta)}(\zeta) \geq \sup_{\eta \in B} f_{\xi+1, \gamma(\eta)}(\zeta) \}

\[ \sup_{\eta \in B} h_\eta(\zeta) = h(\zeta) \]

So \( \zeta \in A_{\xi+1} \); which is impossible. \( \Box \)

(c)(i) Set \( \Delta = \{ \delta : \mathcal{G}_\delta \neq \emptyset \}, \mathcal{G} = \bigcup_{\delta \in \Delta} \mathcal{G}_\delta \). Then \( \mathcal{G} \) is a filter on \( \lambda \) so there is an ultrafilter \( \mathcal{H} \) on \( \lambda \) including \( \mathcal{G} \). Now for any \( \delta \in \Delta, \mathcal{H} \supseteq \mathcal{G}_\delta \), so

\[ \text{cf}(X/\mathcal{H}) = \text{add}(X/\mathcal{H}) \geq \text{add}(X/\mathcal{G}_\delta) \geq \delta \]

using (a) above. Consequently \( \delta^* = \text{cf}(X/\mathcal{H}) \) is the greatest element of \( \Delta \).

(ii) For each \( \delta \in \Delta \) choose a set \( F_\delta \in [X]^{<\delta} \) such that \( \pi_x[F_\delta] \) is cofinal with \( X/F \) for every \( F \in \mathcal{F}_\delta \) (using (b) above). Set \( F = \bigcup_{\delta \in \Delta} F_\delta \) and

\[ G = \{ \sup I : I \in [F]^{<\omega} \} \subseteq X \]

Then \( \#(G) \leq \delta^* \). I claim that \( G \) is cofinal with \( X \). \( \textbf{??} \) Suppose, if possible, otherwise; take \( h \in X \) such that \( h \not\leq g \) for every \( g \in G \). Write

\[ A_g = \{ \zeta : h(\zeta) > g(\zeta) \} \]

for each \( g \in G \). Because \( G \) is upwards-directed, \( \{ A_g : g \in G \} \) is a filter base, and there is an ultrafilter \( \mathcal{F} \) on \( \lambda \) containing every \( A_g \). Now there is a \( \delta \in \Delta \) such that \( \mathcal{F} \in \mathcal{G}_\delta \), so that \( \pi_x[F_\delta] \) is cofinal with \( X/F \), and there is an \( f \in F_\delta \) such that \( h \leq_F f \). But in this case \( A = \{ \zeta : h(\zeta) \leq f(\zeta) \} \) and \( A_f = \lambda \setminus A \) both belong to \( \mathcal{F} \). \( \Box \)

(iii) Accordingly \( \text{cf}(X) \leq \#(G) \leq \delta^* \). But also of course \( \text{cf}(X/\mathcal{H}) \leq \text{cf}(X) \), so \( \delta^* \leq \text{cf}(X) \) and they are equal. Now we have \( \mathcal{H} \in \mathcal{G} \), so \( \delta^* = \text{cf}(X) \).
(d) If $\kappa = \lambda^+$ we may take $\theta_\zeta' = \lambda^+$ for every $\zeta$; if $\kappa = \text{cf}(X/F)$ we may take $\theta_\zeta' = \theta_\zeta$; so let us assume that $\lambda^+ < \kappa < \text{cf}(X/F)$. Of course $M \in F$.

(i) For each ordinal $\gamma < \kappa$ choose a relatively closed cofinal set $C_\gamma \subseteq \gamma$ with $\text{otp}(C_\gamma) = \text{cf}(\gamma)$. Choose $(f_\alpha)_{\alpha < \kappa}$ as follows. Given $(f_\beta)_{\beta < \alpha}$, where $\alpha < \kappa$, and $\gamma < \kappa$, define $g_{\alpha \gamma} \in X$ by

$$g_{\alpha \gamma}(\zeta) = \sup \{f_\beta(\zeta) : \beta \in C_\gamma \cap \alpha\} + 1$$

if this is less than $\theta_\zeta$, 

= 0 otherwise.

Now choose $f_{\alpha} \in X$ such that

$$f_\beta \leq F f_\alpha \quad \forall \beta < \alpha, \quad g_{\alpha \gamma} \leq F f_\alpha \quad \forall \gamma < \kappa;$$

this is possible because $\kappa < \text{cf}(X/F)$. Observe that if $\alpha = \beta + 1$ then $C_\alpha = \{\beta\}$ so that $g_{\alpha \alpha} = f_\beta + 1$ and $f_{\alpha} \not\subseteq F f_\beta$. Continue.

(ii) Suppose that for each $\zeta < \lambda$ we are given a set $S_\zeta \subseteq \theta_\zeta$ with $\#(S_\zeta) \leq \lambda$. Then there is an $\alpha < \kappa$ such that

$$\forall \ h \in \prod_{\zeta < \lambda} S_\zeta, \text{ if } f_{\alpha} \leq h \text{ then } f_\beta \leq h \quad \forall \beta < \kappa.$$

If not, then (because $\kappa$ is regular) we can find a family $(h_\xi)_{\zeta < \kappa}$ in $\prod_{\zeta < \lambda} S_\zeta$ and a strictly increasing family $(\phi(\xi))_{\xi < \kappa}$ in $\kappa$ such that

$$f_{\phi(\xi)} \leq h_\zeta \leq F f_{\phi(\zeta + 1)} \quad \forall \xi < \kappa,$$

$$\phi(\xi) = \sup_{\eta < \xi} \phi(\eta) \text{ for limit ordinals } \xi < \kappa.$$

Set

$$C = \{\xi : \xi < \kappa, \phi(\xi) = \xi\},$$

so that $C$ is a closed unbounded set in $\kappa$. Let $\alpha \in C$ be such that $\alpha = \text{sup}(C \cap \alpha)$ and $\text{cf}(\alpha) = \lambda^+$. Then (because $\lambda^+ \geq \omega_1$) $C_\alpha \cap \alpha$ is cofinal with $\alpha$.

For $\beta \in C \cap C_\alpha$, $\zeta < \lambda$ we have $\#(C_\alpha \cap \beta) \leq \text{otp}(C_\alpha \cap \beta) < \text{otp}(C_\alpha) = \lambda^+ < \theta_\zeta$, so

$$\theta_\zeta > \text{sup}_{\xi \in C_\alpha \cap \beta} f_\xi(\zeta) + 1 = g_{\beta \alpha}(\zeta).$$

Now

$$g_{\beta \alpha} \leq F f_\beta = F f_{\phi(\beta)} \leq F h_\beta \leq F f_{\phi(\beta + 1)} \leq F f_{\beta'},$$

where $\beta'$ is the next member of $C \cap C_\alpha$ greater than $\beta$. So there is a $\zeta_\beta < \lambda$ such that

$$g_{\beta \alpha}(\zeta_\beta) \leq h_\beta(\zeta_\beta) \leq f_{\beta'}(\zeta_\beta).$$

Because $\lambda < \text{cf}(\alpha)$ there is a $\zeta < \lambda$ such that

$$B = \{\beta : \beta \in C \cap C_\alpha, \zeta_\beta = \zeta\}$$

is cofinal with $\alpha$. But now observe that if $\beta, \gamma \in B$ and $\beta' < \gamma$ then $\beta' \in C_\alpha \cap \gamma$ so

$$h_\beta(\zeta) \leq f_{\beta'}(\zeta) < g_{\gamma \alpha}(\zeta) \leq h_\gamma(\zeta).$$

It follows that

$$\lambda^+ = \#(B) = \#(\{h_\beta(\zeta) : \beta \in B\}) \leq \#(S_\zeta) \leq \lambda,$$

which is absurd. XQ

(iii) Consequently $E = \{\pi_F(f_\alpha) : \alpha < \kappa\}$ has a least upper bound in $X/F$. P? If not, choose $(h_\xi)_{\xi < \lambda^+}$ inductively, as follows. Because $\kappa < \delta$, there is an $h_0 \in X$ such that $f_\alpha \leq F h_0$ for every $\alpha < \kappa$. Given $h_\xi$ such that $h_\xi^* = \pi_F(h_\xi)$ is an upper bound for $E$, then $h_\xi^*$ cannot be the least upper bound of $E$, so there is an $h_{\xi + 1} \in X$ such that $h_{\xi + 1}^*$ is an upper bound of $E$ strictly less than $h_\xi^*$. For non-zero limit ordinals $\xi < \lambda^+$, set

$$S_{\xi \zeta} = \{h_\eta(\xi) : \eta < \xi\} \subseteq \theta_{\zeta \text{ata}}$$

for each $\zeta < \lambda$. By (ii) above, there is an $\alpha_\zeta < \kappa$ such that

$$h \in \prod_{\xi < \lambda} S_{\xi \zeta} \text{ either } h \leq F f_{\alpha_\xi} \text{ or } f_\alpha \leq F h \quad \forall \alpha < \kappa.$$
Set

\[ h_\xi(\zeta) = \min(\{\eta : \eta \in S_{\xi\zeta}, f_\alpha(\zeta) \leq \eta\}) \cup \{h_0(\zeta)\} \in S_{\xi\zeta} \]

for each \( \zeta < \lambda \). Then \( f_\alpha \leq_h h_\xi \) (because \( f_\alpha(\zeta) \leq h_\xi(\zeta) \) whenever \( f_\alpha(\zeta) \leq h_0(\zeta) \) and \( h_\xi \in \prod_{\zeta < \lambda} S_{\xi\zeta} \), so \( f_\alpha \leq_h h_\xi \) for every \( \alpha < \kappa \) and \( h_\xi^* \) is an upper bound for \( E \). Also, if \( \eta < \zeta \), then \( h_\xi(\zeta) \leq h_\eta(\zeta) \) whenever \( f_\alpha(\zeta) \leq h_\eta(\zeta) \), so \( h_\xi \leq_h h_\eta \). Continue.

Having got the family \( \langle h_\xi \rangle_{\xi < \lambda^+} \), set

\[ S_\xi = \bigcup_{\zeta < \kappa} S_{\xi\zeta} = \{h_\xi(\zeta) : \xi < \kappa\} \subseteq \theta_\xi \]

for each \( \zeta < \lambda \). For each \( \alpha < \kappa, \zeta < \lambda \) set

\[ g_\alpha(\zeta) = \min(\{\eta : f_\alpha(\zeta) \leq \eta \in S_\zeta \cup \{h_0(\zeta)\}\}) \in S_\zeta. \]

Then, by the same arguments as above,

\[ f_\alpha \leq_h f_\alpha \leq_h h_\xi \quad \forall \alpha < \kappa, \xi < \lambda^+. \]

For each \( \alpha < \kappa \) there is a limit ordinal \( \xi < \lambda^+ \) such that \( g_\alpha(\zeta) \in S_{\xi\zeta} \) for each \( \zeta < \lambda \). Because \( \lambda^+ < \kappa \) is order-isomorphic to \( \hat{\theta}_\zeta \), there must be an \( \alpha < \zeta \) such that \( \alpha \geq \alpha_\zeta \). In this case

\[ f_\alpha \leq_h f_\alpha \leq_h g_\alpha \leq_h h_{\xi+1} \leq_h h_{\xi+1}, \]

so there is a \( \zeta < \lambda \) such that

\[ f_\alpha(\zeta) \leq f_\alpha(\zeta) \leq g_\alpha(\zeta) \leq h_{\xi+1}(\zeta) < h_\xi(\zeta). \]

But now observe that

\[ f_\alpha(\zeta) \leq g_\alpha(\zeta) \in S_{\xi\zeta} \]

so \( h_\xi(\zeta) \leq g_\alpha(\zeta) < h_\xi(\zeta) \), which is absurd. \( \textbf{X} \)

(iv) Let \( g \in X \) be such that \( g^* = \sup E \) in \( X/F \) and \( g(\zeta) > 0 \) for every \( \zeta < \lambda \). For each \( \zeta < \lambda \) let \( \hat{\theta}_\zeta = \text{cf}(g(\zeta)) < \theta_\zeta \) and choose a cofinal set \( D_\zeta \subseteq g(\zeta) \) of order type \( \hat{\theta}_\zeta \). For \( \alpha < \kappa, \zeta < \lambda \) set

\[ g_\alpha(\zeta) = \min(\{\eta : f_\alpha(\zeta) \leq \eta \in D_\zeta\}) \]

if \( f_\alpha(\zeta) < g(\zeta) \), \( \min D_\zeta \) otherwise. Then \( g_\alpha \leq_h g_\beta \) whenever \( \alpha \leq \beta < \kappa \). Also if \( h \in Y = \prod_{\xi < \lambda} D_\zeta \) then \( h^* < g^* \) so there is an \( \alpha < \kappa \) such that \( h^* \leq f_\alpha(\zeta) \leq g_\alpha(\zeta) \). Thus \( \{g_\alpha^* : \alpha < \kappa\} \) is cofinal with \( \{h^* : h \in Y\} \).

(v) Because each \( D_\zeta \) is order-isomorphic to \( \hat{\theta}_\zeta \), we can identify \( Y \) with \( X = \prod_{\zeta < \lambda} \hat{\theta}_\zeta \); and see that \( \text{cf}(X/F) \) is either \( 1 \) or \( \kappa \). Of course the former is absurd, because it could be so only if \( \{\zeta : g(\zeta) \) is a successor ordinal\( \} \in F \), and in this case there would have to be an \( \alpha < \kappa \) such that \( g \leq_h f_\alpha \); but we saw in (i) above that \( f_{\alpha+1} \leq_h f_\alpha \).

Accordingly \( \text{cf}(X/F) = \kappa \).

(vi) It may be that some of the \( \hat{\theta}_\zeta \) are less than or equal to \( \lambda \). But taking \( I = \{\zeta : \hat{\theta}_\zeta \leq \lambda\} \), we have \( I \notin F \). \( \textbf{P} \) If \( I \in F \), then for each \( \zeta \in I \) set \( S_\zeta = D_\zeta \) and for \( \zeta \in \lambda \setminus I \) set \( S_\zeta = \{0\} \). By (ii), there is an \( \alpha < \kappa \) such that

\[ \forall h \in \prod_{\zeta < \lambda} S_\zeta \text{ if } f_\alpha \leq_h h \text{ then } f_\beta \leq_h h \forall \beta < \kappa. \]

But as \( f_{\alpha+1} \leq_h g \), and \( I \in F \), there must be an \( h \in \prod_{\zeta < \lambda} S_{\zeta} \) such that \( f_\alpha \leq_h h \), and now \( g \leq_h h \) because \( g^* \) is the least upper bound of \( E \); but \( h(\zeta) < g(\zeta) \) for every \( \zeta \in I \), so this is impossible. \( \textbf{XQ} \)

So \( \{\zeta : \hat{\theta}_\zeta \geq \lambda^+\} \in F \). But this means that if we set \( \theta_\zeta = \max(\lambda^+, \hat{\theta}_\zeta) \) for every \( \zeta < \lambda \), \( X' = \prod_{\zeta < \lambda} \theta_\zeta \)

then \( X'/F \equiv X/F \) so \( \text{cf}(X'/F) = \kappa \), as required.

Version of 10.12.91

**A1J Shelah covering numbers** (a) Let \( \alpha, \beta, \gamma, \delta \) be cardinals. Following SHelah #355 and SHelah #400b, I write

\[ \text{cov}_{\text{Sh}}(\alpha, \beta, \gamma, \delta) \]
for the least cardinal of any family $E \subseteq [\alpha]^{<\gamma}$ such that for every $A \in [\alpha]^{<\gamma}$ there is a $D \in [E]^{<\delta}$ with $A \subseteq \bigcup D$. I diverge insignificantly from the Master in writing $\text{cov}_{Sh}(\alpha, \beta, \gamma, \delta) = \infty$ in the trivial cases in which there is no such family $E$.

(b) For infinite cardinals $\alpha, \gamma$ write $\Theta(\alpha, \gamma)$ for the maximum of $\alpha$ and the supremum of all cofinalities
\[
\text{cf}(\prod_{\xi < \lambda} \theta_{\xi})
\]
for families $(\theta_{\xi})_{\xi < \lambda}$ such that $\lambda < \gamma$ is a cardinal, every $\theta_{\xi}$ is a regular infinite cardinal, and $\lambda < \theta_{\xi} < \alpha$ for every $\xi < \lambda$. (This carries some of the same information as the cardinal $pp_{\alpha}(\alpha)$ of SHELAH #400b.)

**A1K Theorem** For any infinite cardinals $\alpha, \gamma$,
\[
\text{cov}_{Sh}(\alpha, \gamma, \gamma, \omega_1) \leq \Theta(\alpha, \gamma).
\]

**proof (a)** To begin with (down to the end of (f) below) let us suppose that we have $\alpha \geq \gamma = \gamma_0^+ > \text{cf}(\alpha) > \omega$.

Take a family $E \subseteq [\alpha]^{<\gamma}$ such that

(i) $E$ contains all singleton subsets of $\alpha$;

(ii) $E$ contains a cofinal subset of $\alpha$;

(iii) If $E \in \mathcal{E}$ then $\{ \xi : \xi + 1 \in E \} \in \mathcal{E}$;

(iv) if $E \in \mathcal{E}$ then there is an $F \in \mathcal{E}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \leq \text{cf}(\xi) \leq \gamma_0$;

(v) if $E \in \mathcal{E}$ then $\{ \xi : \xi \in E, \text{cf}(\xi) \geq \gamma \} \in \mathcal{E}$;

(vi) if $E \in \mathcal{E}$ and $\text{cf}(\prod_{\eta \in E} \eta) \leq \Theta(\alpha, \gamma)$, then $\{ g : g \in \prod_{\eta \in E} \eta, f[E] \in \mathcal{E} \}$ is cofinal with $\prod_{\eta \in E} \eta$;

(vii) $\#(E) \leq \Theta(\alpha, \gamma)$.

To see that this can be done, observe that whenever $E \in [\alpha]^{<\gamma}$ there is an $F \in [\alpha]^{<\gamma}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \leq \text{cf}(\xi) \leq \gamma_0$; thus condition (iv) can be achieved, like conditions (iii) and (v), by ensuring that $E$ is closed under suitable functions from $[\alpha]^{<\gamma}$ to itself; while condition (vii) requires that for each $E \in \mathcal{E}$ we have an appropriate family of size at most $\Theta(\alpha, \gamma)$ included in $E$.

Write $\mathcal{J}$ for the $\sigma$-ideal of $\mathcal{P}\alpha$ generated by $\mathcal{E}$.

(b) Suppose now that there is some set in $[\alpha]^{<\gamma}$ not covered by any sequence from $\mathcal{E}$, that is, not belonging to $\mathcal{J}$. Then there must be a function $f : \gamma_0 \to \alpha$ such that $f[\gamma_0] \notin \mathcal{J}$. Accordingly $\mathcal{I} = \{ f^{-1}(E) : E \in \mathcal{J} \}$ is a proper $\sigma$-ideal of $\mathcal{P}\gamma_0$. By condition (a-i), $\mathcal{I}$ contains all singletons in $\mathcal{P}\gamma_0$.

Let $H$ be the set of all functions $h : \gamma_0 \to \alpha$ such that $f(\xi) \leq h(\xi)$ for every $\xi < \gamma_0$ and $h[\gamma_0] \in \mathcal{J}$. Because $\mathcal{E}$ contains a cofinal set $C \subseteq \alpha$ (condition (a-ii)), we can find an $h \in H$; just take $h : \gamma_0 \to C$ such that $f(\xi) \leq h(\xi)$ for every $\xi$.

(c) Because $\mathcal{I}$ is a proper $\sigma$-ideal, there cannot be any sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $H$ such that $\{ \xi : h_{n+1}(\xi) \geq h_n(\xi) \} \in \mathcal{I}$ for every $n \in \mathbb{N}$. Consequently there is an $h^* \in H$ such that
\[
\{ \xi : h(\xi) \geq h^*(\xi) \} \notin \mathcal{I} \quad \forall \ h \in H.
\]

We know that $h^*[\gamma_0] \in \mathcal{J}$; let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{E}$ covering $h^*[\gamma_0]$. For $\xi < \gamma_0$ write $\theta_\xi = \text{cf}(h^*(\xi))$, so that each $\theta_\xi$ is 0 or 1 or a regular infinite cardinal less than $\alpha$. Set
\[
I = \{ \xi : \xi + 1 \in \gamma_0, h^*(\xi) = f(\xi) \},
\]
\[
I' = \{ \xi : \xi < \gamma_0, f(\xi) < h^*(\xi), \theta_\xi = 1 \},
\]
\[
I_n = \{ \xi : \xi < \gamma_0, \omega \leq \theta_\xi \leq \gamma, f(\xi) < h^*(\xi), h^*(\xi) \in E_n \} \quad \forall \ n \in \mathbb{N},
\]
\[
J_n = \{ \xi : \xi < \gamma_0, \gamma \leq \theta_\xi \leq \gamma, f(\xi) < h^*(\xi), h^*(\xi) \in E_n \} \quad \forall \ n \in \mathbb{N}.
\]

(d) For each $n \in \mathbb{N}$ set $G_n = \{ \eta : \eta \in E_n, \text{cf}(\eta) \geq \gamma \} \in \mathcal{E}$. Then $\text{cf}(\prod_{\eta \in G_n} \eta) \leq \Theta(\alpha, \gamma)$. For $\eta \in G_n$ set $\theta'_\eta = \text{cf}(\eta)$; then $\theta'_\eta$ is a regular cardinal and $\#(G_n) \leq \gamma_0 < \theta'_\eta < \alpha$ for each $\eta \in G_n$. So
\[
\text{cf}(\prod_{\eta \in G_n} \theta'_\eta) \leq \Theta(\alpha, \gamma)
\]
by the definition of $\Theta(\alpha, \gamma)$. If for each $\eta \in G_n$ we choose a cofinal set $C_\eta \subseteq \eta$ of order type $\theta'_\eta$, then
\[
\text{cf}(\prod_{\eta \in G_n} C_\eta) = \text{cf}(\prod_{\eta \in G_n} \theta'_\eta) \leq \Theta(\alpha, \gamma).
\]
Consequently, by (a-vi),
\[ \{ g : g \in \prod_{\eta \in G_n} \eta, g[G_n] \in \mathcal{E} \} \]
is cofinal with \( \prod_{\eta \in G_n} \eta \).

(e) Define \( h : \gamma_0 \to \alpha \) as follows.

(i) If \( \xi \in I \) set \( h(\xi) = h^*(\xi) \).
(ii) If \( \xi \in I' \) set \( h(\xi) = h^*(\xi) - 1 \).
(iii) For each \( n \in \mathbb{N} \) take \( F_n \in \mathcal{E} \) such that \( \eta = \sup(F_n \cap \eta) \) whenever \( \eta \in E_n \) and \( \omega \leq \text{cf}(\eta) < \gamma \). If \( \xi \in I_n \bigcup_{m < n} I_m \), take \( h(\xi) \in F_n \) such that \( f(\xi) < h(\xi) < h^*(\xi) \).
(iv) For each \( n \in \mathbb{N}, \eta \in G_n \) set
\[ f^*(\eta) = \sup\{ f(\xi) : \xi < \gamma_0, h^*(\xi) = \eta \}. \]
Then \( f^*(\eta) < \eta \), because \( \gamma_0 < \text{cf}(\eta) \). By (d), there is a \( g_n \in \prod_{\eta \in G_n} \eta \) such that \( g_n[G_n] \in \mathcal{E} \) and \( f^*(\eta) < g_n(\eta) \) for every \( \eta \in G_n \). So for \( \xi \in J_n \bigcup_{m < n} J_m \) we may set \( h(\xi) = g_n(h^*(\xi)) \) and see that \( f(\xi) < h(\xi) < h^*(\xi) \), while \( h(\xi) \in g_n[G_n] \).

(f) Now we see that
\[ h[\gamma_0] \subseteq h^*[\gamma_0] \cup \{ \eta : \eta + 1 \in h^*[\gamma_0] \} \cup \bigcup_{n \in \mathbb{N}} F_n \bigcup \bigcup_{n \in \mathbb{N}} \{ g_n[G_n] \in \mathcal{J} \}, \]
while \( f(\xi) \leq h(\xi) \) for every \( \xi < \gamma_0 \), so \( h \in H \). Consequently
\[ I = \{ \xi : h(\xi) \geq h^*(\xi) \} \notin \mathcal{I}. \]
But also
\[ f[I] \subseteq h^*[\gamma_0] \in \mathcal{J}, \]
so \( I \in \mathcal{I} \), which is absurd. \( \square \)

(g) Thus the special case described in (a) is dealt with, and we may return to the general case. I proceed by induction on \( \alpha \) for fixed \( \gamma \geq \omega \).

(i) To start the induction, observe that if \( \alpha < \gamma \) then
\[ \text{cov}_{\text{Sh}}(\alpha, \gamma, \gamma, \omega_1) = 1 \leq \Theta(\alpha, \gamma) \]
for all \( \gamma \).

(ii) For the inductive step to \( \alpha \) when \( \text{cf}(\alpha) \geq \gamma \), observe that in this case \( [\alpha]^* = \bigcup_{\alpha' < \alpha} [\alpha']^* \). For each \( \alpha' < \alpha \), we can find a family \( \mathcal{E}_{\alpha'} \subseteq [\alpha']^* \) such that \( \#(\mathcal{E}_{\alpha'}) = \text{cov}_{\text{Sh}}(\alpha', \gamma, \gamma, \omega_1) \) and every member of \( [\alpha']^* \) can be covered by a sequence from \( \mathcal{E}_{\alpha'} \); now \( \bigcup_{\alpha' < \alpha} \mathcal{E}_{\alpha'} \) witnesses that
\[ \text{cov}_{\text{Sh}}(\alpha, \gamma, \gamma, \omega_1) \leq \max(\alpha, \sup_{\alpha' < \alpha} \text{cov}_{\text{Sh}}(\alpha', \gamma, \gamma, \omega_1)) \]
\[ \leq \max(\alpha, \sup_{\alpha' < \alpha} \Theta(\alpha', \gamma)) \leq \Theta(\alpha, \gamma) \]
by the inductive hypothesis.

(iii) For the inductive step to \( \alpha \) when \( \text{cf}(\alpha) = \omega < \alpha \), take an increasing sequence \( \{ \alpha_n \}_{n \in \mathbb{N}} \) in \( \alpha \) such that \( \sup_{n \in \mathbb{N}} \alpha_n = \alpha \). For each \( n \in \mathbb{N} \) choose a set \( \mathcal{E}_n \subseteq [\alpha_n]^* \) such that every set in \( [\alpha_n]^* \) can be covered by a sequence from \( \mathcal{E}_n \) and \( \#(\mathcal{E}_n) = \text{cov}_{\text{Sh}}(\alpha_n, \gamma, \gamma, \omega_1) \). Set \( \mathcal{E} = \bigcup_{n \in \mathbb{N}} \mathcal{E}_n \subseteq [\alpha]^* \). Then if \( A \in [\alpha]^* \), we can find for each \( n \in \mathbb{N} \) a countable set \( \mathcal{D}_n \subseteq \mathcal{E}_n \) covering \( A \cap \alpha_n \); set \( \mathcal{D} = \bigcup_{n \in \mathbb{N}} \mathcal{D}_n \subseteq [\mathcal{E}]^{\leq \omega} \); then \( A \subseteq \bigcup \mathcal{D} \). As \( A \) is arbitrary, \( \mathcal{E} \) witnesses that
\[ \text{cov}_{\text{Sh}}(\alpha, \gamma, \gamma, \omega_1) \leq \#(\mathcal{E}) \]
\[ \leq \max(\omega, \sup_{n \in \mathbb{N}} \#(\mathcal{E}_n)) \]
\[ = \max(\omega, \sup_{n \in \mathbb{N}} \text{cov}_{\text{Sh}}(\alpha_n, \gamma, \gamma, \omega_1)) \]
\[ \leq \max(\omega, \sup_{n \in \mathbb{N}} \Theta(\alpha_n, \gamma)) \]
by the inductive hypothesis.
\[ \leq \Theta(\alpha, \gamma). \]

(iv) For the inductive step to \( \alpha \) when \( \omega < \text{cf}(\alpha) < \gamma < \alpha \), observe that
\[ [\alpha]_{< \gamma}^\delta = \bigcup_{\delta < \gamma} [\alpha]_{< \delta}^\delta. \]
For each \( \delta < \gamma \) we have a set \( \mathcal{E}_\delta \subseteq [\alpha]_{< \delta}^\delta \) such that \( \#(\mathcal{E}_\delta) \leq \text{cov}_{\text{Sh}}(\alpha, \delta^+, \delta^+, \omega_1) \) and every member of \( [\alpha]_{< \gamma}^\delta \) can be covered by a sequence from \( \mathcal{E}_\delta \). Set \( \mathcal{E} = \bigcup_{\delta < \gamma} \mathcal{E}_\delta \); then every member of \( [\alpha]_{< \gamma}^\gamma \) can be covered by a sequence from \( \mathcal{E} \).

For each \( E \in \mathcal{E} \), choose a family \( \mathcal{H}_E \subseteq [E]_{< \gamma}^\gamma \) such that \( \#(\mathcal{H}_E) = \text{cov}_{\text{Sh}}(\#(E), \gamma, \gamma, \omega_1) \) and every member of \( [E]_{< \gamma}^\gamma \) can be covered by a sequence from \( \mathcal{H}_E \). Set \( \mathcal{H} = \bigcup_{E \in \mathcal{E}} \mathcal{H}_E \). If \( A \in [\alpha]_{< \gamma} \), there is a countable set \( D \subseteq \mathcal{E} \) covering \( A \); now for each \( D \in \mathcal{D} \) there is a countable set \( \mathcal{G}_D \subseteq D \) covering \( A \cap D \); so that \( \mathcal{G} = \bigcup_{D \in \mathcal{D}} \mathcal{G}_D \) is a countable subset of \( \mathcal{H} \) covering \( A \).

Thus
\[
\text{cov}_{\text{Sh}}(\alpha, \gamma, \gamma, \omega_1) \leq \#(\mathcal{H}) \\
\leq \max(\#(\mathcal{E}), \sup_{E \in \mathcal{E}} \#(\mathcal{H}_E)) \\
\leq \max(\alpha, \sup_{\text{cf}(\alpha) < \delta < \gamma} \text{cov}_{\text{Sh}}(\alpha, \delta^+, \delta^+, \omega_1), \sup \text{cov}_{\text{Sh}}(\alpha', \gamma, \gamma, \omega_1)) \\
\leq \max(\alpha, \sup_{\text{cf}(\alpha) < \delta < \gamma} \Theta(\alpha, \delta^+), \sup \Theta(\alpha', \gamma))
\]
by the inductive hypothesis and parts (a)-(f) above
\[
\leq \Theta(\alpha, \gamma).
\]

This completes the proof.

Remark This is taken from Shelah #355, Theorem 5.4, where a stronger result is proved, giving an exact description of many of the numbers \( \text{cov}_{\text{Sh}}(\alpha, \beta, \gamma, \delta) \) in terms of cofinalities of reduced products \( \prod_{\zeta < \lambda} \theta_\zeta / F \).

AII Lemma Let \( \alpha \) and \( \gamma \) be infinite cardinals, with \( \gamma \) regular, and suppose that \( \alpha \geq \Theta(\gamma, \gamma) \). Then
\[ \Theta(\Theta(\alpha, \gamma), \gamma) = \Theta(\alpha, \gamma). \]

Proof Suppose, if possible, otherwise. Note that of course \( \alpha \geq \gamma \).

(a) There must be a family \( \{ \theta_\zeta \}_{\zeta < \lambda} \) of infinite regular cardinals such that \( \lambda < \gamma \), \( \lambda < \theta_\zeta < \Theta(\alpha, \gamma) \) for every \( \zeta < \lambda \), and \( \text{cf}(\prod_{\zeta < \lambda} \theta_\zeta) > \Theta(\alpha, \gamma) \). By AIIc, there is an ultrafilter \( F \) on \( \lambda \) such that \( \text{cf}(\prod_{\zeta < \lambda} \theta_\zeta / F) > \Theta(\alpha, \gamma) \). Set \( L = \{ \zeta : \zeta < \lambda, \theta_\zeta < \alpha \} \); then \( \text{cf}(\prod_{\zeta < L} \theta_\zeta) \leq \Theta(\alpha, \gamma) \), so \( L \not\in F \) and \( M = \lambda \setminus L \in F \). For each \( \zeta \in M \), we have \( \theta_\zeta \leq \Theta(\alpha, \gamma) \), so there must be a family \( \{ \theta_\zeta \eta \}_{\eta < \lambda_\zeta} \) of regular cardinals with \( \lambda_\zeta < \gamma, \lambda_\zeta < \theta_\zeta \eta < \alpha \) for every \( \eta < \lambda_\zeta \) and \( \theta_\zeta \leq \text{cf}(\prod_{\eta < \lambda_\zeta} \theta_\zeta \eta / F_\zeta) \). Again by AIIc, there is an ultrafilter \( F_\zeta \) on \( \lambda_\zeta \) such that \( \theta_\zeta \leq \text{cf}(\prod_{\eta < \lambda_\zeta} \theta_\zeta \eta / F_\zeta) \).

(b) Set
\[ I = \{ (\zeta, \eta) : \zeta \in M, \eta < \lambda_\zeta \}, \]
\[ \mathcal{H} = \{ H : H \subseteq I, \{ \zeta : (\zeta, \eta) \in H, \zeta \notin \mathcal{F}_\zeta \in \mathcal{F} \}, \]
\[ X = \prod_{(\zeta, \eta) \in I} \theta_\eta^{\zeta}. \]
Then \( \mathcal{H} \) is an ultrafilter on \( I \), and \( \text{cf}(X/\mathcal{H}) \geq \text{cf}(\prod_{\zeta \in M} \theta_\zeta / F) \). Let \( F \subseteq X \) be a set of cardinal \( \text{cf}(X/\mathcal{H}) \) such that \( \{ f^*_\zeta : f \in F \} \) is cofinal with \( X/\mathcal{H} \). For each \( f \in F \), \( \zeta \in M \) let \( f_\zeta \in \prod_{\eta < \lambda_\zeta} \theta_\eta^{\zeta} \) be given by \( f_\zeta(\eta) = f(\zeta, \eta) \) for each \( \eta < \lambda_\zeta \), and let \( f^*_\zeta \) be the image of \( f_\zeta \) in \( \prod_{\eta < \lambda_\zeta} \theta_\eta^{\zeta} / F \). For each \( \zeta \in M \) let \( (\zeta, \eta) \notin \mathcal{F}_\zeta \) be a strictly increasing cofinal family in \( \prod_{\eta < \lambda_\zeta} \theta_\eta^{\zeta} / F_\zeta \). Now, for \( f \in F \), take a function \( g_\zeta \in \prod_{\zeta \in M} \theta_\zeta \) such that \( f^*_\zeta \leq g_\zeta_{f(\zeta)} \) for every \( \zeta \in M \).
If \( g \in \prod_{\zeta \in M} \theta_\zeta \), then we can find an \( h \in X \) such that \( h^*_\zeta = u_{\zeta, g(\zeta)} \) for each \( \zeta \in M \). Let \( f \in F \) be such that \( h \leq_f f \). Then \( \{ \zeta : g(\zeta) \leq g_f(\zeta) \} \supseteq \{ \zeta : h^*_\zeta \leq f^*_\zeta \} \in \mathcal{F} \), so \( g \leq_F g_f \). Accordingly \( \{ g_f : f \in F \} \) is cofinal with \( \prod_{\zeta \in M} \theta_\zeta / \mathcal{F} \) and \( \operatorname{cf}(\prod_{\zeta \in M} \theta_\zeta / \mathcal{F}) \leq \#(\mathcal{F}) = \operatorname{cf}(X / \mathcal{H}) \), as claimed. \( \mathbf{Q} \)

(c) Thus \( \operatorname{cf}(X / \mathcal{H}) > \Theta(\alpha, \gamma) \). Set \( J = \{ (\zeta, \eta) : (\zeta, \eta) \in I, \theta_{\zeta, \eta} \geq \gamma \} \). Because \( \gamma \) is regular, \( \#(J) \leq \#(I) < \gamma \), so \( \operatorname{cf}(\prod_{(\zeta, \eta) \in J} \theta_{\zeta, \eta}) \leq \theta(\alpha, \gamma) \), and \( J \notin \mathcal{H} \). It follows that \( K = I \setminus J \in \mathcal{H} \). Set \( M' = \{ (\zeta : \zeta \in M, \{ \eta : (\zeta, \eta) \in K \} \in \mathcal{F}_\zeta \} \in \mathcal{F} \). Then \( \theta_\zeta \leq \Theta(\alpha, \gamma) \leq \alpha \) for \( \zeta \in M' \). So in fact \( \theta_\zeta = \alpha \) for \( \zeta \in M' \) and we have

\[
\alpha \leq \Theta(\alpha, \gamma) < \operatorname{cf}(\prod_{\zeta \in M} \theta_\zeta) = \operatorname{cf}(\prod_{\zeta < \alpha} \alpha),
\]

where \( \delta = \#(M') \), while at the same time \( \alpha \) is regular.

But if \( \alpha \) is regular and \( \delta < \alpha \),

\[
\operatorname{cf}(\prod_{\zeta < \delta} \alpha) = \alpha,
\]

contradicting the last sentence. \( \mathbf{X} \)

This contradiction completes the proof.

A1M Lemma Let \( \alpha \) and \( \gamma \) be infinite cardinals. Set \( \delta = \sup_{\alpha' < \alpha} \Theta(\alpha', \gamma) \).

(a) If \( \operatorname{cf}(\alpha) \geq \gamma \) then \( \Theta(\alpha, \gamma) = \max(\alpha, \delta) \).

(b) If \( \operatorname{cf}(\alpha) < \gamma \) then \( \Theta(\alpha, \gamma) \leq \max(\alpha, \delta^{\operatorname{cf}(\alpha)}) \), where \( \delta^{\operatorname{cf}(\alpha)} \) is the cardinal power.

**Proof** Let \( \{ \theta_\zeta : \zeta < \lambda \} \) be a family of regular cardinals with \( \lambda < \theta_\zeta < \alpha \) for each \( \zeta \) and \( \lambda < \gamma \).

**Case 1** If \( \alpha' = \sup_{\zeta < \lambda} \theta_\zeta < \alpha \), set

\[
I = \{ \zeta : \zeta < \lambda, \theta_\zeta < \alpha' \},
\]

\[
J = \{ \zeta : \zeta < \lambda, \theta_\zeta = \alpha' \}.
\]

Then \( \operatorname{cf}(\prod_{\zeta \in I} \theta_\zeta) \leq \Theta(\alpha', \gamma) \leq \delta, \operatorname{cf}(\prod_{\zeta \in J} \theta_\zeta) \leq \alpha' \leq \delta \) so \( \operatorname{cf}(\prod_{\zeta < \lambda} \theta_\zeta) \leq \delta \). This completes the proof of (a).

**Case 2** If \( \sup_{\zeta < \lambda} \theta_\zeta = \alpha \) let \( (\alpha_\xi)_{\xi < \operatorname{cf}(\alpha)} \) be an increasing family of cardinals with supremum \( \alpha \) and with \( \alpha_\xi = \sup_{\eta < \xi} \alpha_\eta \) for limit ordinals \( \xi < \operatorname{cf}(\alpha) \). Set

\[
P_\xi = \prod_{\zeta < \lambda, \alpha_\xi \leq \theta_\zeta < \alpha_{\xi+1}} \theta_\zeta
\]

for each \( \xi < \operatorname{cf}(\alpha) \). Then \( \operatorname{cf}(P_\xi) \leq \delta \) for each \( \xi < \operatorname{cf}(\alpha) \), so

\[
\operatorname{cf}(\prod_{\zeta < \lambda} \theta_\zeta) = \operatorname{cf}(\prod_{\zeta < \operatorname{cf}(\alpha)} P_\xi) \leq \delta^{\operatorname{cf}(\alpha)}.
\]

This is enough to deal with (b).

A1N The singular cardinals hypothesis (In this paragraph, all powers will be cardinal exponentiation; \( \lambda^\theta \) will be the cardinal of the set of functions from \( \theta \) to \( \lambda \).) Recall that the singular cardinals hypothesis is the statement

whenever \( \lambda \) is an infinite cardinal and \( 2^{\operatorname{cf}(\lambda)} < \lambda \) then \( \lambda^{\operatorname{cf}(\lambda)} = \lambda^+ \)

(JECH 78, §8, p. 61). This is equivalent to

whenever \( \lambda > c \) and \( \operatorname{cf}(\lambda) = \omega \) then \( \lambda^\omega = \lambda^+ \)

(JECH 78, Theorem 23b, p.63); evidently the second statement is implied by

whenever \( \lambda > c \) is a successor cardinal then \( \lambda^\omega = \lambda \),

and by JECH 78, Lemma 8.1, p. 62, the third assertion is implied by the first, so that any of the three may be taken as a statement of the singular cardinals hypothesis.

A1O Miscellaneous definitions (a) If \( X \) is a set, a Jónsson algebra on \( X \) is an algebraic structure with countably many finitary functions and relations such that the only subalgebra of \( X \) of cardinal \( \#(X) \) is \( X \) itself.

(b) If \( \kappa \) is a cardinal, a \( \kappa \)-Aronszajn tree is a tree of height \( \kappa \) in which every level has cardinal strictly less than \( \kappa \) but there are no branches of length \( \kappa \). If \( \kappa \) is weakly \( \Pi^1_1 \)-indescribable, there are no \( \kappa \)-Aronszajn trees (see FREMLIN & KUNEN 87, 2N).
A1P Souslin numbers I say that two elements $p, q$ in a partially ordered set $P$ are **upwards-compatible** if $\{p, q\}$ has an upper bound in $P$, and that a set $A \subseteq P$ is an **up-antichain** if no two elements of $A$ are upwards-compatible.

For any partially ordered set $P$, write

$$S(P) = \min\{\lambda : \text{there is no up-antichain in } P \text{ of cardinal } \lambda\},$$

the **Souslin number** of $P$. Thus $P$ satisfies the ‘$\lambda$-chain condition’ iff $\lambda \geq S(P)$, and is upwards-ccc iff $S(P) \leq \omega_1$. Note that if $Q$ is a cofinal subset of $P$, then $S(Q) = S(P)$.

Similarly, the **Souslin number** of a Boolean algebra $\mathfrak{A}$ is the least cardinal $\lambda$ such that there is no disjoint set of cardinal $\lambda$ in $\mathfrak{A} \setminus \{0\}$; that is, the Souslin number of the partially ordered set $\mathfrak{A} \setminus \{1\}$: it is called sat($\mathfrak{A}$) in Koppelberg 89, 3.8-3.11.

A1Q Lemma Let $R$ be an upwards-ccc partially ordered set and $\langle r_\alpha : \alpha < \omega_1 \rangle$ any family in $R$. Then there is an infinite $M \subseteq \omega_1$ such that $\{r_\alpha : \alpha \in M\}$ is centered, that is, $\{r_\alpha : \alpha \in L\}$ has an upper bound in $R$ for every finite $L \subseteq M$.

**proof** Suppose, if possible, otherwise. For each $\alpha < \omega_1$ let $M_\alpha$ be a maximal subset of $\omega_1$, containing $\alpha$, such that $\{r_\beta : \beta \in M_\alpha\}$ is centered. Because each $M_\alpha$ is finite, there must be an uncountable set $W \subseteq \omega_1$ such that $\alpha \notin M_\beta$ whenever $\alpha, \beta \in W$ and $\beta < \alpha$. Now, for each $\alpha \in W$, let $s_\alpha$ be an upper bound for $\{r_\beta : \beta \in M_\alpha\}$. In this case, for $\alpha, \beta \in W$ and $\beta < \alpha$, $r_\alpha \leq s_\alpha$ but $r_\alpha$ and $s_\beta$ are upwards-incompatible in $R$ (or we should have been able to add $\alpha$ to $M_\beta$). But this means that $\{s_\alpha : \alpha \in W\}$ is an uncountable up-antichain in $R$. $\blacksquare$

A1R Finite-support products If $\{P_\xi : \xi \in I\}$ is any family of partially ordered sets, then its **finite-support product** is the set $P$ of finite functions $p$ such that $\text{dom}(p) \in [I]^{<\omega}$ and $p(\xi) \in P_\xi$ for every $\xi \in \text{dom}(p)$; ordered by saying that $p \leq q$ if $\text{dom}(p) \subseteq \text{dom}(q)$ and $p(\xi) \leq q(\xi)$ for every $\xi \in \text{dom}(p)$.

Now if $\{P_\xi : \xi \in I\}$ is any family of partially ordered sets with finite-support product $P$, and

$$\lambda = \sup\{S(\prod_{\xi \in J} P_\xi) : J \subseteq [I]^{<\omega}\},$$

then $S(P) = \lambda$ if $\lambda$ is regular, and $\lambda^+$ otherwise; see Comfort & Negrepontis 82, Theorem 3.27, where the corresponding theorem is proved for products of topological spaces.

A1S The arrow relation If $\alpha, \beta$ and $\gamma$ are ordinals, write $\alpha \rightarrow (\beta, \gamma)^2$ to mean: if $S \subseteq [\alpha]^2$ is any set, then either there is a $B \subseteq \alpha$ such that $\text{otp}(B) = \beta$ and $[B]^2 \subseteq S$, or there is a $C \subseteq \alpha$ such that $\text{otp}(C) = \gamma$ and $[C]^2 \cap S = \emptyset$. (See Erdős Hajnal Máthé & Rado 84, 8.2.) Then we have, among many other important results,

(i) $\alpha \rightarrow (\alpha, \omega + 1)^2$ whenever $\alpha$ is a regular uncountable cardinal (Erdős Hajnal Máthé & Rado 84, 11.3);

(ii) $\varepsilon \nrightarrow (\omega_1, \omega_1)^2$ (Erdős Hajnal Máthé & Rado 84, 19.7);

(iii) if $\varepsilon = \omega_1$ then $\varepsilon \nrightarrow (\varepsilon, \omega + 1)^2$ (Erdős Hajnal Máthé & Rado 84, 11.5);

(iv) if $\kappa$ is a regular uncountable cardinal then $\kappa \rightarrow (\kappa, \kappa)^2$ iff $\kappa$ is weakly $\Pi^1_2$-indescribable and strongly inaccessible, that is, ‘weakly compact’ (Erdős Hajnal Máthé & Rado 84, 30.3 and 32.1).

A2. Measure theory

My basic texts for this material are Fremlin 74 and Fremlin 89.

A2A General measure spaces In these notes a **measure space** will always be a triple $(X, \Sigma, \mu)$ where $X$ is a set, $\Sigma$ is a $\sigma$-algebra of subsets of $X$, and $\mu : \Sigma \to [0, \infty]$ is a countably-additive functional.

Let $(X, \Sigma, \mu)$ be a measure space.

(a) If $B \subseteq X$, I write

$$\mu^*B = \min\{\mu E : B \subseteq E \in \Sigma\}.$$

(b) If $A$ is any subset of $X$, I write $\mu[A]$ for the measure on $A$ defined by writing
Now $\mu[A]$ is the **subspace measure** on $A$.

(c) An **atom** for $\mu$ is a set $E \in \Sigma$ such that $\mu E > 0$ and if $F \in \Sigma$, $F \subseteq E$ then one of $\mu F$, $\mu (E \setminus F)$ is zero.

(d) I write $\mathcal{N}_\mu$ for the $\sigma$-ideal of $\mathcal{P} X$ consisting of $\mu$-**negligible** sets, that is,

$$\mathcal{N}_\mu = \{E : E \subseteq X, \mu^* E = 0\}.$$  

(e) If $(Y, \mathcal{G})$ is a topological space, a function $f : X \to Y$ is **measurable** if $f^{-1}[H] \in \Sigma$ for every $H \in \mathcal{G}$.

**A2B Taxonomy of measure spaces** (see Fremlin 74). Let $(X, \Sigma, \mu)$ be a measure space.

(a) $(X, \Sigma, \mu)$ is a **probability space** if $\mu X = 1$ (and in this case I call $\mu$ a **probability**); it is **totally finite** if $\mu X$ is finite; it is **$\sigma$-finite** if there is a sequence $(E_n)_{n \in \mathbb{N}}$ of measurable sets of finite measure covering $X$.

(b) $(X, \Sigma, \mu)$ is **semi-finite** if

$$\mu E = \sup \{\mu F : F \subseteq E, F \in \Sigma, \mu F < \infty\}$$

for every $E \in \Sigma$; that is, if whenever $E \in \Sigma$ and $\mu E = \infty$ then there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Any $\sigma$-finite measure space – a fortiori, any probability space – is semi-finite.

(c) $(X, \Sigma, \mu)$ is **locally determined** if it is semi-finite and for every $A \in \mathcal{P} X \setminus \Sigma$ there is an $F \in \Sigma$ such that $\mu F < \infty$ and $F \cap A \notin \Sigma$.

(d) $(X, \Sigma, \mu)$ is **decomposable** (or ‘strictly localizable’) if there is a partition $(X_i)_{i \in I}$ of $X$ such that (i) $\Sigma = \{E : E \subseteq X, E \cap X_i \in \Sigma \forall i \in I\}$, (ii) $\mu E = \sum_{i \in I} \mu (E \cap X_i)$ for every $E \in \Sigma$, (iii) $\mu X_i < \infty$ for every $i \in I$. In this case $(X, \Sigma, \mu)$ is locally determined. (See Fremlin 74, 64G.)

(e) $(X, \Sigma, \mu)$ is **atomless** if there are no atoms for $\mu$.

(f) $(X, \Sigma, \mu)$ is **complete** if $\mathcal{N}_\mu \subseteq \Sigma$. Any measure space of the form $(X, \mathcal{P} X, \mu)$ is surely complete.

(g) $(X, \Sigma, \mu)$ is **purely atomic** if every non-negligible measurable set includes an atom for $\mu$; that is, there is no $E \in \Sigma \setminus \mathcal{N}_\mu$ such that $\mu |E$ is atomless.

**A2C Additivity** Let $(X, \Sigma, \mu)$ be a measure space.

(a) If $\kappa$ is a cardinal, then $\mu$ is $\kappa$-**additive** if $\mu(\bigcup_{\xi < \lambda} E_\xi)$ exists and is equal to $\sum_{\xi < \lambda} \mu E_\xi$ for every disjoint family $(E_\xi)_{\xi < \lambda} \subseteq \Sigma$ indexed by a cardinal $\lambda < \kappa$.

(b) I write $\text{add}(\mu)$, the **additivity** of $\mu$, for the least cardinal $\kappa$, if there is one, such that $\mu$ is not $\kappa$-additive; if $\mu$ is $\kappa$-additive for every cardinal $\kappa$, I write $\text{add}(\mu) = \infty$.

(c) We always have $\text{add}(\mu) \leq \text{add}(\mathcal{N}_\mu)$, defining $\text{add}(\mathcal{N}_\mu)$ as in A1A-B.

(d) If $(X, \Sigma, \mu)$ is complete and locally determined, then $\text{add}(\mu) = \text{add}(\mathcal{N}_\mu)$ (see Fremlin 84, A6O).

(e) If $A \subseteq \Sigma$ is upwards-directed and $\#(A) < \text{add}(\mu)$ then $\mu(\bigcup A) = \sup_{A \in A} \mu A$.

(f) If $(X, \Sigma, \mu)$ is a totally finite measure space and $F$ is a uniformly bounded non-empty upwards-directed family of measurable functions from $X$ to $\mathbb{R}$, of cardinal less than $\text{add}(\mu)$, and if $f_0(x) = \sup_{f \in F} f(x)$ for each $x \in X$, then

$$\int f_0 \, d\mu = \sup_{f \in F} \int f \, d\mu.$$  

(Use (e).)

**A2D Functions between measure spaces** (a) If $(X, \Sigma, \mu)$ and $(Y, \mathcal{T}, \nu)$ are measure spaces, a function $f : X \to Y$ is **inverse-measure-preserving** if $f^{-1}[F] \in \Sigma$ and $\mu f^{-1}[F] = \nu F$ for every $F \in \mathcal{T}$. 
(b) Let \((X, \Sigma, \mu)\) be a measure space, \(Y\) any set and \(f : X \to Y\) any function. I write \(\mu f^{-1}\) for the measure on \(Y\) defined by writing

\[
(\mu f^{-1})(F) = \mu(f^{-1}[F])
\]

for all those \(F \subseteq Y\) for which the right-hand-side is well defined, that is, for which \(f^{-1}[F] \in \Sigma\).

Observe that if \(\nu = \mu f^{-1}\) then

\[
\mathcal{N}_\nu = \{F : F \subseteq Y, f^{-1}[F] \in \mathcal{N}_\mu\},
\]

\(\nu\) is complete if \(\mu\) is, \(\nu\) is a probability iff \(\mu\) is, add\((\nu) \geq \text{add}(\mu)\) and add\((\mathcal{N}_\nu) \geq \text{add}(\mathcal{N}_\mu)\).

**A2E Upper and lower integrals** Let \((X, \Sigma, \mu)\) be any measure space and \(f : X \to \mathbb{R}\) any function. Then its upper integral is

\[
\int f(x) \, \mu(dx) = \inf \{ \int g(x) \, \mu(dx) : g : X \to \mathbb{R}\text{ is integrable}, g(x) \geq f(x) \text{ a.e. } x \in X \},
\]

taking \(\inf \emptyset = \infty\), \(\inf \mathbb{R} = -\infty\), and its lower integral is

\[
\underline{\int} f(x) \, \mu(dx) = -\int (-f(x)) \, \mu(dx).
\]

**A2F Measure algebras** (a) A measure algebra is a pair \((\mathfrak{A}, \mu)\) where \(\mathfrak{A}\) is a Dedekind \(\sigma\)-complete Boolean algebra and \(\mu : \mathfrak{A} \to [0, \infty]\) is a functional such that (i) \(\mu a = 0\) iff \(a = 0\) (ii) \(\mu(\sup_{n \in \mathbb{N}} a_n) = \sum_{n \in \mathbb{N}} \mu a_n\) whenever \(\langle a_n \rangle_{n \in \mathbb{N}}\) is a disjoint sequence in \(\mathfrak{A}\). It is a probability algebra if moreover \(\mu 1 = 1\).

(b) If \((\mathfrak{A}, \mu)\) is a measure algebra with \(\mu 1 < \infty\), it has a natural metric \(\rho\) given by setting

\[
\rho(a, b) = \mu(a \triangle b)
\]

for all \(a, b \in \mathfrak{A}\).

(c) A measurable algebra is a Boolean algebra \(\mathfrak{A}\) for which there is a \(\mu\) making \((\mathfrak{A}, \mu)\) a measure algebra in the sense of (a) above and moreover with \(\mu 1 < \infty\). (See Fremlin 89, 2.1.)

(d) Let \((X, \Sigma, \mu)\) be a measure space. Its measure algebra is the quotient Boolean algebra \(\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{N}_\mu\), endowed with the induced functional \(\hat{\mu}\) defined by saying that \(\hat{\mu} E^* = \mu E\) for every \(E \in \Sigma\).

(e) If \((X, \Sigma, \mu)\) and \((Y, T, \nu)\) are measure spaces and \(f : X \to Y\) is inverse-measure-preserving, then we have a corresponding measure-preserving homomorphism \(\phi : T/\mathcal{N}_\nu \to \Sigma/\mathcal{N}_\mu\) given by writing \(\phi(\mathcal{F}^*) = f^{-1}[\mathcal{F}]^*\) for every \(F \in T\) (Fremlin 89, 2.16).

(f) Let \(\mathfrak{A}\) be any Boolean algebra. Then \(\mathfrak{A}\) is isomorphic to the algebra of open-and-closed sets in its Stone space \(Z\) (Fremlin 74, 41D). I write \(L^\infty(\mathfrak{A})\) for the Banach algebra of continuous real-valued functions on the compact Hausdorff space \(Z\) (Fremlin 74, §43). In this context, given \(a \in \mathfrak{A}\), I write \(\chi(a) \in L^\infty(\mathfrak{A})\) for the characteristic function of the open-and-closed subset of \(Z\) corresponding to \(a\).

(g) Let \((\mathfrak{A}, \mu)\) be a measure algebra. Then we have a unique continuous linear functional \(\int d\mu : L^\infty(\mathfrak{A}) \to \mathbb{R}\) such that \(\int \chi(a) \, d\mu = \mu a\) for every \(a \in \mathfrak{A}\) (Fremlin 74, §52). If \(\nu : \mathfrak{A} \to \mathbb{R}\) is a finitely-additive functional with \(0 \leq \nu a \leq \mu a\) for every \(a \in \mathfrak{A}\), there is a unique \(u \in L^\infty(\mathfrak{A})\) such that \(\int u \, d\mu = \int u \chi(a) \, d\mu = \nu a\) for every \(a \in \mathfrak{A}\); this is a form of the Radon-Nikodým theorem (Fremlin 74, 63J).

**A2G The measure of \(\{0, 1\}^I\)** Let \(I\) be any set.

(a) Write \(\mathcal{C}\) for the family of 'cylinder sets' of the form

\[
C = \{x : x \in \{0, 1\}^I, x|J = z\}
\]

for some finite \(J \subseteq I\), \(z \in \{0, 1\}^J\).

In this case write \(\phi_0(I) = 2^{-\#(J)}\). Define \(\phi : \mathcal{P}(\{0, 1\}^I) \to [0, 1]\) by

\[
\phi(A) = \inf \{\sum_{n \in \mathbb{N}} \phi_0(C_n) : \langle C_n \rangle_{n \in \mathbb{N}} \subseteq \bigcup_{n \in \mathbb{N}} C_n \}
\]

for every \(A \subseteq \{0, 1\}^I\). The usual measure of \(\{0, 1\}^I\) is the measure defined by Carathéodory’s method from the outer measure \(\phi\). For the rest of this paragraph, I will denote it by \(\mu_I\) and its domain by \(\Sigma_I\).

(b) \((\{0, 1\}^N, \mu_0)\) is isomorphic, as measure space, to \((\{0, 1\}, \mu_L)\), where \(\mu_L\) is Lebesgue measure on \([0, 1]\).

(c) Let \(E\) be any set in \(\Sigma_I\). (i) There are sets \(E’, E’’\) belonging to the Baire \(\sigma\)-algebra of \(\{0, 1\}^I\) with \(E’ \subseteq E \subseteq E’’\) and \(\mu_I E’ = \mu_I E = \mu_I E’’\). (ii) \(E\) is expressible in the form \(H \triangle N\) where \(H\) belongs to the
(d) If \((X, \Sigma, \mu)\) is any complete probability space, then any measure-preserving homomorphism \(\psi\) from the measure algebra \(\mathfrak{A}_I\) of \(\{0,1\}^I\) to the measure algebra \(\mathfrak{A}\) of \((X, \Sigma, \mu)\) can be induced by an inverse-measure-preserving function \(f : X \to \{0,1\}^I\), in the manner of \(A2F\); see Fremlin 89, 2.21 and 4.12, and A2Jd below.

(e) If \(J \subseteq I\), \(K = I \setminus J\) then \(\{0,1\}^J\) can be identified with \(\{0,1\}^I \times \{0,1\}^K\). Under this identification we have a form of Fubini’s theorem: If \(f : \{0,1\}^I \to \mathbb{R}\) is \(\mu_I\)-integrable, then \(\int \int f(y,z)\mu_K(dz)\mu_J(dy)\) exists for \(\mu_I\)-almost all \(y\), and \(\int \int f(y,z)\mu_K(dz)\mu_J(dy)\) exists and is equal to \(\int f(x)\mu_I(dx)\). (Schwartz 73, pp. 73-74.)

(f) If \(I\) is infinite, then the measure algebra \(\mathfrak{A}_I\) of \(\{0,1\}^I\) is homogeneous as Boolean algebra, that is, is isomorphic to all its non-zero principal ideals (Fremlin 89, 3.6 and 3.7b, or otherwise).

(g) If \(f : \{0,1\}^I \to \mathbb{R}\) is \(\Sigma_I\)-measurable there are functions \(f', f''\), both measurable with respect to the Baire \(\sigma\)-algebra of \(\{0,1\}^I\), such that \(f'(x) \leq f(x) \leq f''(x)\) for every \(x \in \{0,1\}^I\) and \(\mu_I\{x : f'(x) \neq f''(x)\} = 0\). (Use (c).)

A2H Maharam types Let \(\mathfrak{A}\) be a Boolean algebra.

(a) Write \(\tau(\mathfrak{A})\) for the least cardinal of any subset \(A\) of \(\mathfrak{A}\) which ‘completely generates’ \(\mathfrak{A}\) in the sense that the smallest order-closed (that is, closed-under-suprema or ‘complete’) subalgebra of \(\mathfrak{A}\) including \(A\) is \(\mathfrak{A}\) itself. (See Koppelberg 89, 13.11).

(b) If \(\mathfrak{A}\) is ccc then \(\#(\mathfrak{A})\) is less than or equal to the cardinal power \(\tau(\mathfrak{A})^\omega\). (Fremlin 89, 6.2b.)

(c) If \(a \in \mathfrak{A}\) then \(\tau(\mathfrak{A}[a]) \leq \tau(\mathfrak{A})\), where \(\mathfrak{A}[a]\) is the principal ideal of \(\mathfrak{A}\) generated by \(a\) (Koppelberg 89, 13.12).

(d) If \(\mathfrak{A}\) is a measurable algebra (\(A2F\)) and \(\mathfrak{B} \subseteq \mathfrak{A}\) is an order-closed subalgebra then \(\tau(\mathfrak{B}) \leq \tau(\mathfrak{A})\) (Fremlin 89, 6.3b).

(e) \(\mathfrak{A}\) is \(\tau\)-homogeneous if \(\tau(\mathfrak{A}[a]) = \tau(\mathfrak{A})\) for every \(a \in \mathfrak{A} \setminus \{0\}\).

(f) There is a partition of \(1\) in \(\mathfrak{A}\), \(\langle a_i \rangle_{i \in I}\), say, such that \(\mathfrak{A}[a_i]\) is \(\tau\)-homogeneous for every \(i \in I\).

(g) If \(\mathfrak{A}\) is the measure algebra of a measure space \((X, \Sigma, \mu)\) then \(\tau(\mathfrak{A})\) is the Maharam type of \(\mu\) (or of \((X, \Sigma, \mu)\)). If \(\mathfrak{A}\) is \(\tau\)-homogeneous I say that \(\mu\), or \((X, \Sigma, \mu)\), is Maharam homogeneous.

(h) For any decomposable measure space \((X, \Sigma, \mu)\), there is a partition \(\langle X_i \rangle_{i \in I}\) of \(X\) such that (i) \(\Sigma = \{ E : E \in X, E \cap X_i \in \Sigma \ \forall \ i \in I\}\) (ii) \(\mu E = \sum_{i \in I} \mu(E \cap X_i)\) for every \(E \in \Sigma\) (iii) \(\mu[X_i]\) is Maharam homogeneous and totally finite for every \(i \in I\).

(i) If \(\mathfrak{A}\) is a probability algebra, then either \(\mathfrak{A}\) and \(\tau(\mathfrak{A})\) are both finite, or \(\tau(\mathfrak{A})\) is precisely the topological density of \(\mathfrak{A}\) when \(\mathfrak{A}\) is given the metric \(\rho\) of \(A2FB\); this is because a topologically closed subalgebra of \(\mathfrak{A}\) is order-closed, so a set \(A \subseteq \mathfrak{A}\) completely generates \(\mathfrak{A}\) iff the subalgebra of \(\mathfrak{A}\) generated by \(A\) is topologically dense in \(\mathfrak{A}\) (Fremlin 89, 2.20).

A2I Maharam’s Theorem (a) If \((\mathfrak{A}, \mu)\) is a \(\tau\)-homogeneous probability algebra, with \(\tau(\mathfrak{A}) = \kappa\), then it is isomorphic, as measure algebra, to the measure algebra \(\mathfrak{A}_\kappa\) of \(\{0,1\}^\kappa\) (Fremlin 89, 3.8).

(b) If \((\mathfrak{A}, \mu)\) and \((\mathfrak{B}, \nu)\) are probability algebras, and \(\tau(\mathfrak{A}) \leq \min\{\tau(\mathfrak{B}[b]) : b \in \mathfrak{B} \setminus \{0\}\}\), then there is a measure-preserving Boolean homomorphism from \(\mathfrak{A}\) to \(\mathfrak{B}\) (Fremlin 89, 3.13a, corrected to read ‘if \(\tau(A) \leq \kappa\) and \(\tau(C) < \kappa, \ldots\)’).

(c) If \((\mathfrak{A}, \mu)\) is a probability algebra and \(\tau(\mathfrak{A}) = \kappa\), then it is isomorphic, as measure algebra, to an order-closed subalgebra of \(\mathfrak{A}_\kappa\).

A2J Topological measure spaces (a) A quasi-Radon measure space is a quadruple \((X, \mathfrak{T}, \Sigma, \mu)\) such that (i) \((X, \Sigma, \mu)\) is a complete locally determined measure space (ii) \(\mathfrak{T}\) is a topology on \(X\) and \(\mathfrak{T} \subseteq \Sigma\) (iii) if \(\mathcal{G}\) is any upwards-directed family in \(\mathfrak{T}\) then \(\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G\) (iv) for every \(E \in \Sigma\),
\[ \mu E = \sup \{ \mu F : F \subseteq E, F \text{ is closed} \} \] (v) whenever \( E \in \Sigma \) and \( \mu E > 0 \) there is an open set \( G \) such that \( \mu G < \infty \) and \( \mu(E \cap G) > 0 \). For the general theory of quasi-Radon measure spaces, see Fremlin 74, Fremlin N82d and Fremlin 84. In particular, note that any quasi-Radon measure space is decomposable (Fremlin 74, 72B) and that any subspace of a quasi-Radon measure space is a quasi-Radon measure space (Fremlin 84, A7Da).

(b) A Radon measure space is a quadruple \((X, \mathcal{T}, \Sigma, \mu)\) such that (i) \((X, \Sigma, \mu)\) is a complete locally determined measure space (ii) \(\mathcal{T}\) is a Hausdorff topology on \(X\) and \(\mathcal{T} \subseteq \Sigma\) (iii) \(\mu E = \sup \{ F : F \subseteq E, F \text{ is compact} \}\) for every \(E \in \Sigma\) (iv) every point of \(X\) belongs to some open set of finite measure.

(c) Every Radon measure space is a quasi-Radon measure space (Fremlin 74, 73B). The usual measure on \(\{0, 1\}^I\) (A2G above) is always a Radon measure (see the references in Fremlin 89, 1.16). If \((X, \mathcal{T}, \Sigma, \mu)\) is a Radon measure space and \(E \in \Sigma\), then \((E, \mathcal{T}|E, \Sigma \cap \mathcal{P}E, \mu|E)\) is a Radon measure space.

(d) Suppose that \((X, \mathcal{T}, \Sigma, \mu)\) is a Radon measure space, with measure algebra \((\mathfrak{A}, \mu)\), and that \((Y, T, \nu)\) is a complete measure space with measure algebra \((\mathfrak{B}, \phi)\). Suppose that \(\mu X = \nu Y < \infty\) and that \(\phi : \mathfrak{A} \to \mathfrak{B}\) is a measure-preserving homomorphism. Then there is an inverse-measure-preserving function \(f : Y \to X\) such that \(f^{-1}E^* = \phi(E^*)\) in \(\mathfrak{B}\) for every \(E \in \Sigma\). (See Fremlin 89, 4.12-4.13.)

(e) If \((X, \mathcal{T}, \Sigma, \mu)\) is a totally finite quasi-Radon measure space and \(f : X \to \mathbb{R}\) is a bounded function then
\[ \mathcal{f} f \, d\mu = \inf \{ \int h \, d\mu : h : X \to \mathbb{R} \text{ is lower semi-continuous and } f(x) \leq h(x) \ \forall \ x \in X \}. \]
(Use (a-iv). See Schwartz 73, p. 43.)

(f) If \((X, \mathcal{T}, \Sigma, \mu)\) is a totally finite quasi-Radon measure space and \(F\) is a uniformly bounded non-empty upwards-directed family of lower semi-continuous functions from \(X\) to \(\mathbb{R}\), and if \(f_0(x) = \sup_{f \in F} f(x)\) for each \(x \in X\), then
\[ \int f_0 \, d\mu = \sup_{f \in F} \int f \, d\mu. \]
(Use (a-iii). See Schwartz 73, p.42.)

**A2K Corollary** (a) If \((X, \mu)\) is a Maharam homogeneous probability space of Maharam type \(\lambda \geq \omega\), and \((Y, \nu)\) is a Radon probability space of Maharam type not greater than \(\lambda\), then there is an inverse-measure-preserving function \(f : X \to Y\).

(b) If \((X, \mathcal{T}, \Sigma, \mu)\) is a Radon probability space, \(E \subseteq \Sigma\) is a collection of measurable sets, \(\Sigma_0\) is the \(\sigma\)-subalgebra of \(\Sigma\) generated by \(E\), and \(\lambda \geq \max(\omega, \#(E))\) is a cardinal, there is a function \(g : \{0, 1\}^\lambda \to X\) such that \(\mu g^{-1}[E] = \text{ is defined and equal to } \mu E\) for every \(E \in \Sigma\), where \(\mu_\lambda\) is the usual measure of \(\{0, 1\}^\lambda\).

(c) If \((X, \mu)\) is a complete atomless probability space, there is a function \(f : X \to [0, 1]\) which is inverse-measure-preserving for \(\mu\) and Lebesgue measure on \([0, 1]\).

**proof** (a) Let \(\mathfrak{A}\) be the measure algebra of \((X, \mu)\) and \(\mathfrak{B}\) the measure algebra of \((Y, \nu)\). By A2lb there is a measure-preserving homomorphism \(\phi : \mathfrak{B} \to \mathfrak{A}\), which by A2Jd is induced by an inverse-measure-preserving \(f : X \to Y\).

(b) Let \(\kappa\) be the greater of \(\lambda\) and the Maharam type of \((X, \mu)\). Then there is an inverse-measure-preserving \(f : \{0, 1\}^\kappa \to X\), by (a). For each \(E \in \mathcal{E}\) there are Baire sets \(G_E, H_E \subseteq \{0, 1\}^\kappa\) such that \(G_E \subseteq g^{-1}[E] \subseteq H_E\) and \(\mu_E = \mu G_E = \mu H_E\). Now there is a countable set \(I_E \subseteq \kappa\) such that both \(G_E\) and \(H_E\) factor through the canonical map \(\pi_{I_E} : \{0, 1\}^\kappa \to \{0, 1\}^{I_E}\); that is, \(G_E = \pi_{I_E}^{-1}[\pi_{I_E}[G_E]]\), and similarly for \(H_E\). Let \(I \subseteq \kappa\) be a set of cardinal \(\lambda\) including \(I_E\) for every \(E \in \mathcal{E}\). Let \(z\) be any point of \(\{0, 1\}^{I \cup \lambda}\) and set \(h(x) = (x, z)\) for each \(x \in \{0, 1\}^\kappa\), identifying \(\{0, 1\}^\kappa\) with \(\{0, 1\}^\kappa \times \{0, 1\}^\lambda\). If \(E \in \mathcal{E}\) then \(\pi_{I_E}[G_E] \subseteq h^{-1}[f^{-1}[E]] \subseteq \pi_I[H_E]\) so \(\mu_E h^{-1}[f^{-1}[E]] = \mu_E G_E = \mu_E H_E = \mu_E\). So if we set \(g = fh : \{0, 1\}^\kappa \to X\) we shall have \(\mu_E g^{-1}[E] = \mu_E\) for every \(E \in \mathcal{E}\), and therefore for every \(E \in \Sigma_0\). But of course \((\{0, 1\}^\kappa, \mu_\lambda) \cong (\{0, 1\}^\lambda, \mu_\lambda)\), so this proves the result.

(c) Argue as in (a), or as in Fremlin 84, A6lb.

**A2L Hyperstonian spaces** (a) If \((X, \Sigma, \mu)\) is a totally finite measure space, its hyperstonian space is the Radon measure space \((Z, \mathfrak{S}, T, \nu)\), where \((Z, \mathfrak{S})\) is the Stone space of the measure algebra \((\mathfrak{A}, \mu)\) of
\((X, \Sigma, \mu)\), and \(\nu\) is the unique Radon measure on \(Z\) such that \(\nu E^* = \mu E\) for every \(E \in \Sigma\), where \(E^*\) for the open-and-closed subset of \(Z\) corresponding to the image \(E^*\) of \(E\) in \(\mathfrak{A}\). (See Fremlin 89, 2.13-2.14.) In this case, \((\mathfrak{A}, \bar{\mu})\) is isomorphic, as measure algebra, to the measure algebra of \((Z, T, \nu)\) (Fremlin 89, 2.13).

(b) If \((X, \mathfrak{F}, \Sigma, \mu)\) is a compact Radon measure space and \((Z, \mathfrak{G}, T, \nu)\) is its hyperstonian space, then there is a unique continuous function \(f : Z \to X\) such that \(\nu(E^* \Delta f^{-1}[E]) = 0\) for every \(E \in \Sigma\). This \(f\) is of course inverse-measure-preserving; it is the canonical map from \(Z\) to \(X\). (See Fremlin 89, 2.17-2.18.)

(c) If \((X, \Sigma, \mu)\) is any totally finite measure space and \((Z, \mathfrak{G}, T, \nu)\) is hyperstonian space, then for every \(F \in T\), \(\epsilon > 0\) there is a compact open set \(W \subseteq F\) such that \(\nu W \geq \nu F - \epsilon\) (Fremlin 89, 2.14).

**A2M Lemma** Let \((X, \Sigma, \mu)\) be a probability space.

(a) Let \((E_n)_{n \in \mathbb{N}}\) be a family in \(\Sigma\) such that \(\inf_{n \in \mathbb{N}} \mu E_n = \delta > 0\). Then there is an \(x \in X\) such that \(\{n : n \in \mathbb{N}, x \in E_n\}\) is infinite.

(b) Let \((E_\alpha)_{\alpha < \omega_1}\) be a family in \(\Sigma \setminus N_\mu\). Then there is an \(x \in X\) such that \(\{\alpha : \alpha < \omega_1, x \in E_\alpha\}\) is infinite.

**proof** (a) Set \(F_n = \bigcap_{m \geq n} E_m\) for \(n \in \mathbb{N}\); then \((F_n)_{n \in \mathbb{N}}\) is a decreasing sequence and \(\mu F_n \geq \delta > 0\) for every \(n \in \mathbb{N}\). So \(\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset\); take any \(x \in \bigcap_{n \in \mathbb{N}} F_n\); this \(x\) must belong to infinitely many of the \(E_n\).

(b) There is a \(\delta > 0\) such that \(A = \{\alpha : \alpha < \omega_1, \mu E_\alpha > 0\}\) is infinite. Now apply (a) to \((E_\alpha(n))_{n \in \mathbb{N}}\) for any strictly increasing sequence \(\langle \alpha(n) \rangle_{n \in \mathbb{N}}\) in \(A\).

**Remark** For related results see 9A and A2U; also Kanamori & Magidor 78, p. 166, and Fremlin 87, 1E.

**A2N Lemma** Let \((X, \Sigma, \mu)\) be a probability space and \(T\) a subalgebra of \(\Sigma\). Suppose that \((C_n)_{n \in \mathbb{N}}\) is a sequence of members of \(\Sigma\) which are stochastically independent of each other and of \(T\), that is, \(\mu(E \cap \bigcap_{i \in I} C_i) = \mu E \prod_{i \in I} \mu C_i\) for every \(E \in T\), \(I \subseteq \mathbb{N}\). If now \(f\) is a \(\mu\)-integrable real-valued function on \(X\) and \((E_n)_{n \in \mathbb{N}}\) is any sequence in \(T\),

\[
\phi_n(f) = \int_{E_n \cap C_n} f - \mu C_n \cdot \int_{E_n} f \to 0
\]
as \(n \to \infty\).

**proof** Consider first the case \(f = \chi F\) where \(F = E \cap \bigcap_{i \in I} C_i\) for some \(E \in T\), \(I \subseteq \mathbb{N}\); this is trivial. It follows that \(\lim_{n \to \infty} \phi_n(f) = 0\) whenever \(f = \chi F\) for some set \(F\) in the subalgebra \(T_1\) of \(\Sigma\) generated by \(T_0 \cup \{C_n : n \in \mathbb{N}\}\). Also, of course, \(|\phi_n(f) - \phi_n(g)| \leq 2 \int |f - g|\) for all \(n \in \mathbb{N}\), integrable \(f\) and \(g\), so \(\lim_{n \to \infty} \phi_n(f)\) will be 0 for every \(T\)-measurable integrable function \(f\), where \(T\) is the \(\sigma\)-subalgebra of \(\Sigma\) generated by \(T_1\). Finally, for a general \(\mu\)-integrable \(f\), the Radon-Nikodým theorem (Royden 63, chap. 11, §5) tells us that there is a \(T\)-measurable \(g\) such that \(\int_T f = \int_T g\) for every \(F \in T\), so that \(\phi_n(f) = \phi_n(g) \to 0\) as \(n \to \infty\).

**A2O Definition** Let \((X, \mu)\) be a measure space and \(E, F\) two real linear spaces in duality. Say that a function \(x \mapsto u_x : X \to E\) is \(F\)-scalarly measurable if \(x \mapsto (f|u_x) : X \to \mathbb{R}\) is measurable for every \(f \in F\).

**A2P(a)** Let \(X\) be a set and \(\mathcal{A}\) any family of sets. Write

\[
\cov(X, \mathcal{A}) = \min \{\#(B) : B \subseteq \mathcal{A}, X \subseteq \bigcup B\} \text{ if } X \subseteq \bigcup \mathcal{A},
\]

\(= 0\) otherwise.

(b) For a Radon measure space \((X, \mathfrak{F}, \Sigma, \mu)\), we have \(\cov(X, N_\mu) = 0\) iff \(X = \emptyset\) and \(\cov(X, N_\mu) = 1\) iff \(X \neq \emptyset\) and \(\mu X = 0\). In any other case, consider the least cardinal \(\kappa\) for which there is an \(E \in \Sigma\) with \(0 < \mu E < \infty\) and \((E, \Sigma \cap \mathcal{P} E, \mu|E)\) is Maharam homogeneous of Maharam type \(\kappa\); then

\[
\cov(X, N_\mu) = \cov\{0, 1\}^\kappa, N_\mu_{\kappa}\}
\]

where \(\mu_\kappa\) is the usual Radon measure on \(\{0, 1\}^\kappa\). (See Fremlin 89, 6.14c and 6.15c.) If we write
δκ = \text{cov}(\{0, 1\}^\kappa, \mathcal{N}_\mu)

for infinite cardinals κ, then δκ ≤ δλ whenever ω ≤ λ ≤ κ (Fremlin 89, 6.17(d-i), so there is a least κ₀ such that δκ = δκ₀ = δ say for every κ ≥ κ₀, and this δ is precisely the least value of cov(X, N_µ) for any non-trivial Radon measure space (X, µ). Moreover, if δκ ≤ λ ≤ κ then δλ = δκ (Fremlin 89, 6.17(d-ii)); it follows that if δκ ≥ κ then δ ≥ κ and if δκ > κ then δ > κ.

A2Q Lifting Let (X, Σ, µ) be a measure space, with measure algebra Ξ.

(a) A lifting of (X, Σ, µ) is a Boolean homomorphism θ : Ξ → Σ such that (θa)* = a for every a ∈ Ξ.

(b) The Lifting Theorem If (X, Σ, µ) is complete and decomposable it has a lifting (Fremlin 89, 4.4, or Ionescu Tulcea & Ionescu Tulcea 69, chap. IV, Theorem 3).

(c) If (X, Σ, µ) is a complete probability space, Σ₀ ⊆ Σ is a σ-subalgebra and Ξ₀ = {E∗ : E ∈ Σ₀} is the corresponding order-closed subalgebra of Ξ = Σ/\mathcal{N}_µ, then any lifting θ₀ : Ξ₀ → Σ₀ extends to a lifting θ : Ξ → Σ (follow the standard proof of the lifting theorem as given in Fremlin 89 or Ionescu Tulcea & Ionescu Tulcea 69).

(d) If (X, Σ, µ) is a complete probability space and θ : Ξ → Σ is any lifting, then \bigcup_{a ∈ A} θ(a) ∈ Σ for every A ⊆ Ξ. (Ionescu Tulcea & Ionescu Tulcea 69, §V.3.)

A2R Lemma Let (X, Σ, µ) be a probability space, λ a cardinal of uncountable cofinality, (Eξ)ξ<λ a family of measurable non-negligible sets in X, and 1 ≤ l ∈ N. Then there is a set W ⊆ Λ, of cardinal λ, such that \bigcap_{ξ∈\Lambda} E_ξ ≠ Φ for every L ∈ [W]^l.

proof See Argyros & Kalamidas 82, or Comport & Negrepontis 82, Theorem 6.15, or Fremlin 88, Proposition 7, where various stronger results are proved.

Remark For the following results, we need only the case X = \{0, 1\}^I, which is usefully simpler than the general case.

A2S Lemma Let (X, Σ, µ) be a probability space, and Σ₀ a σ-subalgebra of Σ; set µ₀ = µ|Σ₀ and let µ₀⁺ be the outer measure on X defined from µ₀. Let \tau₀ be the Maharam type of µ₀. Suppose that λ is a cardinal with \text{cf}(\lambda) > \text{max}(\omega, \tau₀), and (Eξ)ξ<λ a family in Σ with \text{inf}_{ξ<λ} µE_ξ = γ > 0. Then for any γ’ < γ, 1 ≤ l ∈ N there is a W ∈ [\lambda]^l such that µ₀⁺(\bigcap_{ξ∈\Lambda} E_ξ) ≥ γ’ for every L ∈ [W]^l.

Remark For this paper we need only the case \tau₀ = \omega, λ = \omega₁.

proof (a) Consider first the special case in which X = \{0, 1\}^I for some set I, with µ the usual measure on X, and there is a J ⊆ I of cardinal \tau₀ such that \Sigma₀ = \{\pi_J^{-1}[F] : F ∈ Σ_J\}, writing Σ_J for the domain of the usual measure µ_J on \{0, 1\}^J, and π_J : X → \{0, 1\}^J for the canonical map.

(i) We may regard X as the product \{0, 1\}^I × \{0, 1\}^\backslash J. Set δ = (γ - γ’)/(l + 1) > 0. For ξ < λ, z ∈ \{0, 1\}^\backslash J set

E_ξ = \{y : y ∈ \{0, 1\}^I, (y, z) ∈ E_ξ\}.

Then

\int µ_JE_ξµ_I\backslash J(dz) = µE_ξ ≥ γ

by Fubini’s theorem (A2Ge). Set

G_ξ = \{z : z ∈ \{0, 1\}^\backslash J, µ_JE_ξ exists ≥ γ\};

then µ_I\backslash JG_ξ exists and is greater than 0. Let H_ξ ⊆ X be an open-and-closed set such that µ(E_ξ△H_ξ) ≤ δµ_I\backslash JG_ξ (A2Ge), and set H_ξ = \{y : (y, z) ∈ H_ξ\} for z ∈ \{0, 1\}^\backslash J. Then

\int µ_J(E_ξ△H_ξ)µ_I\backslash J(dz) = µ(E_ξ△H_ξ) ≤ δµ_I\backslash JG_ξ,

so µ_I\backslash JG_ξ > 0, where

G_ξ = \{z : z ∈ G_ξ, µ_J(E_ξ△H_ξ) ≤ δ\}.
Finally, observe that because $H_\varepsilon$ is open-and-closed the set of sections $H_\varepsilon \subseteq \{0,1\}^J$ is finite, and there is an open-and-closed set $F_\xi \subseteq [0,1]^J$ such that $\mu_{\cup \gamma}G_\varepsilon'' > 0$, where

$$G_\varepsilon'' = \{z : z \in G_\varepsilon', H_\varepsilon = F_\xi\}.$$  

(ii) The number of open-and-closed sets in $\{0,1\}^J$ is at most $\max(\omega, \#(J)) < \text{cf}(\lambda)$. So there is an open-and-closed set $F = \{\xi : \xi < \lambda, F_\xi = F\}$ has cardinal $\lambda$. Next, by A2R above, there is a $W \subseteq U$ such that $\#(\xi) = \lambda$ and $\bigcap_{\mu \in W} G_\varepsilon'' \neq \emptyset$ whenever $\xi \in W$. 

Take any $L \in [W]^\varepsilon$. There is a $z \in \bigcap_{\xi \in L} G_\varepsilon''$. For each $\xi \in L$, $\mu_\gamma E_\varepsilon \subseteq \{\xi \in \lambda\}$ so $\mu_\gamma(E_\varepsilon \cap H_\varepsilon) = \delta$ and $H_\varepsilon \subseteq F_\varepsilon = F$. So $\mu_\gamma F \geq \gamma - \delta$ and $\mu_\gamma(\bigcap_{\xi \in L} E_\varepsilon) \geq \mu_\gamma F - \delta \geq \gamma'$. 

Now if $F' \in \Sigma_\gamma$ and $\pi_\gamma^{-1}([F']) \supseteq \bigcap_{\xi \in L} E_\varepsilon$, we must have $F' \supseteq \bigcap_{\xi \in L} E_\varepsilon$, so

$$\mu(\pi_\gamma^{-1}([F'])) = \mu_\gamma F' \geq \gamma'.$$

As $F'$ is arbitrary, $\mu_\gamma'(\bigcap_{\xi \in L} E_\varepsilon) \geq \gamma'$, as required. 

(b) It follows that if $(\mathfrak{A}, \mu)$ is a probability algebra, $\mathfrak{A}_0$ is a order-closed subalgebra of $\mathfrak{A}$, $\lambda$ is a cardinal with $\text{cf}(\lambda) > \max(\omega, \#(\mathfrak{A}_0))$, $(\mathcal{A}_\varepsilon)_{\varepsilon \in \lambda}$ is a family in $\mathfrak{A}$ with $\inf_{\varepsilon \in \lambda} \mu_\varepsilon = \gamma > 0$, $1 \leq l \in \mathbb{N}$ and $\gamma' < \gamma$, then there is a $W \subseteq [\lambda]^\varepsilon$ such that

$$\min\{\#(b) : b \subseteq \mathfrak{A}_0, b \supseteq \inf_{\varepsilon \in L} a_\varepsilon, \gamma' \}$$

whenever $L \subseteq [W]^\varepsilon$. We can embed $\mathfrak{A}$ as a subalgebra of the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ for some set $I$ (A21b). If we take a set $B \subseteq \mathfrak{A}_0$ of cardinal $\tau(\mathfrak{A}_0)$ which completely generates $\mathfrak{A}_0$, then for each $b \in B$ we can find a set $G_b \subseteq X = [0,1]^I$, belonging to the Baire $\sigma$-algebra of $X$, such that $b = G_b^\ast$ in $\mathfrak{A}$; now there is a set $J \subseteq I$, of cardinal at most $\max(\omega, \#(B))$, such that every $G_b$ belongs to $\Sigma_\theta$, if we define $\Sigma_\theta$ from $J$ as in part (a) above. 

Set $\mathfrak{A}_1 = \{G_\varepsilon^\ast : G \in \Sigma_\theta\}$, so that $\mathfrak{A}_1$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$. Let $\Sigma_\gamma$ be the algebra of measurable subsets of $\{0,1\}^J$, and $\mathfrak{A}_\gamma$ the measure algebra of $\{0,1\}^J$. Then the inverse-measure-preserving map $\pi_\gamma : X \to [0,1]^J$ induces an isomorphism $\phi$ between $\mathfrak{A}_1$ and $\mathfrak{A}_\gamma$, taking $\phi(\pi_\gamma^{-1}([G_\varepsilon^\ast])) = G_\varepsilon^\ast$ for every $G \in \Sigma_\gamma$. By the lifting theorem (A2Qb) there is a lifting $\theta : \mathfrak{A}_\gamma \to \Sigma_\theta$, so we have a corresponding Boolean homomorphism $\theta : \mathfrak{A}_1 \to \Sigma_\theta$ given by setting $\theta_1(a) = \pi_\gamma^{-1}[\theta_\gamma(\phi(a))]$ for each $a \in A_1$. As remarked in A2Qc, there is an extension $\theta$ of $\theta_1$ to a lifting from $\mathfrak{A}$ to $\Sigma$. 

Set $E_\varepsilon = \theta(a_\varepsilon)$ for each $\varepsilon < \lambda$. By part (a) above, because $\text{cf}(\lambda) > \max(\omega, \#(\mathfrak{A}_0))$, there is a set $W \subseteq [\lambda]^\varepsilon$ such that $\mu_\gamma(\bigcap_{\varepsilon \in L} E_\varepsilon) \geq \gamma'$ whenever $L \subseteq [W]^\varepsilon$. Now suppose that $L \subseteq [W]^\varepsilon$ and that $b \in \mathfrak{A}_0$, $b \supseteq \inf_{\varepsilon \in L} a_\varepsilon$. Then $\theta(b) \supseteq \bigcap_{\varepsilon \in L} E_\varepsilon$ and $\theta(b) \subseteq \Sigma_\theta$, so

$$\mu_\gamma(\theta(b)) \geq \mu_\gamma(\bigcap_{\varepsilon \in L} E_\varepsilon) \geq \gamma'.$$

Thus we have the set $W$ we need. 

(c) We are now ready for the general case of the lemma. Let $(\mathfrak{A}, \mu)$ be the measure algebra of $(X, \Sigma, \mu)$ and set $\mathfrak{A}_0 = \{G_\varepsilon^\ast : G \in \Sigma_\theta\}$, $a_\varepsilon = E_\varepsilon^\ast$ for each $\varepsilon < \lambda$. By (b), there is a $W \subseteq [\lambda]^\varepsilon$ such that $\#(b) \geq \gamma'$ whenever $L \subseteq [W]^\varepsilon$, $b \subseteq \mathfrak{A}_0$ and $b \supseteq \inf_{\varepsilon \in L} a_\varepsilon$. Now if $L \subseteq [W]^\varepsilon$ and $G \in \Sigma_\theta$ and $G \supseteq \bigcap_{\varepsilon \in L} E_\varepsilon$, $b = G_\varepsilon^\ast$ in $\mathfrak{A}_0$ and $b \supseteq \inf_{\varepsilon \in L} a_\varepsilon$, so that $\mu_\gamma G = \mu G = \mu G_\varepsilon^\ast \geq \gamma'$. As $G$ is arbitrary, $\mu_\gamma(\bigcap_{\varepsilon \in L} E_\varepsilon) \geq \gamma';$ as $L$ is arbitrary, we have the required family $W$. 

A2T Precalculers If $\mathfrak{A}$ is a Boolean algebra and $\lambda$ is a cardinal, then $\lambda$ is a precaliber of $\mathfrak{A}$ if for every family $(a_\varepsilon)_{\varepsilon < \lambda}$ in $\mathfrak{A} \setminus \{0\}$ there is a set $D \in [\lambda]^\varepsilon$ such that $\{a_\varepsilon : \varepsilon < D\}$ is centered, that is, $\inf_{\varepsilon \in D} a_\varepsilon \neq 0$ for any non-empty finite $D \subseteq \varepsilon$. 

A2U Proposition Let $(X, \Sigma, \mu)$ be a complete probability space with measure algebra $(\mathfrak{A}, \mu)$, and $\lambda$ a cardinal of uncountable cofinality which is not a precaliber of $\mathfrak{A}$.

(a) There is a family $(E_\varepsilon)_{\varepsilon < \lambda}$ in $\mathcal{N}_\mu$ such that $\bigcap_{\varepsilon < \lambda} E_\varepsilon \subseteq \Sigma \setminus \mathcal{N}_\mu$.

(b) If $\lambda$ is regular, there is an increasing family $(E_\varepsilon)_{\varepsilon < \lambda}$ in $\mathcal{N}_\mu$ such that $\bigcup_{\varepsilon < \lambda} E_\varepsilon \subseteq \Sigma \setminus \mathcal{N}_\mu$.

proof Let $(a_\varepsilon)_{\varepsilon < \lambda}$ be a family in $\mathfrak{A} \setminus \{0\}$ with no centered subfamily of cardinal $\lambda$. Let $\theta : \mathfrak{A} \to \Sigma$ be a lifting (A2Q). If $D \in [\lambda]^\varepsilon$ then $\{a_\varepsilon : \varepsilon < D\}$ is not centered so there is a finite set $I \subseteq D$ such that $\bigcap_{\varepsilon \in I} \theta(a_\varepsilon) = \theta(\inf_{\varepsilon \in I} a_\varepsilon) = 0$. Consequently $\#(\{\varepsilon : \varepsilon < \theta(a_\varepsilon)\}) < \lambda$ for every $x \in X$. 

(a) Choose inductively countable sets $C_{\alpha} \subseteq \lambda$, for $\alpha < \lambda$, such that
\[
C_{\alpha} \subseteq \lambda \setminus \bigcup_{\beta < \alpha} C_{\beta},
\]
for each $\alpha < \lambda$. Setting $s_{\alpha} = \mu(\{a_{\xi} : \xi \in \lambda \setminus \bigcup_{\beta < \alpha} C_{\beta}\})$ for each $\alpha$, we see that $(s_{\alpha})_{\alpha < \lambda}$ is decreasing; as $\text{cf}(\lambda) > \omega$, there is a $\gamma < \lambda$ such that $s_{\gamma} = s_{\alpha}$ for $\gamma \leq \alpha < \lambda$. Set
\[
H_{\alpha} = \bigcup_{\{a_{\xi} : \xi \in C_{\alpha}\}}
\]
for $\alpha < \lambda$. Then $\bigcap_{\alpha \leq \lambda} H_{\alpha} = \emptyset$, because the $C_{\alpha}$ are disjoint. Also $\mu(H_{\gamma} \setminus H_{\alpha}) = 0$ for every $\alpha$, because $H_{\alpha}^{*} = \{a_{\xi} : \xi \in C_{\alpha}\}$ is finite. So if we set $E_{\xi} = H_{\gamma} \setminus H_{\xi}$, we have a witness for (a).

(b) Now suppose that $\lambda$ is regular. For each $\alpha < \lambda$ set $H_{\alpha}^{'} = \bigcup_{\alpha \leq \xi < \lambda} \theta(a_{\xi})$; then $H_{\alpha}^{'} \in \Sigma$ (A2Qd). The family $(H_{\alpha}^{'})_{\alpha < \lambda}$ is decreasing; because $\text{cf}(\lambda) > \omega$, there is a $\gamma < \lambda$ such that $\mu H_{\gamma}^{'} = \mu H_{\gamma}$ whenever $\gamma \leq \alpha < \lambda$. If $x \in \bigcap_{\alpha \leq \lambda} H_{\alpha}^{'}$, set $D = \{\xi : x \in \theta(a_{\xi})\}$; then $D$ is a cofinal with $\lambda$ (because $\lambda$ is regular) and $\#(D) = \lambda$. Thus $\bigcap_{\alpha \leq \lambda} H_{\alpha} = \emptyset$. But now $(H_{\gamma} \setminus H_{\alpha})_{\alpha < \lambda}$ is an increasing family in $\mathcal{N}_{\mu}$ with union $H_{\gamma} = \Sigma \setminus N_{\mu}$.

A2V More topological measure spaces Let $X$ be a topological space. A Borel measure on $X$ is a measure $\mu$ with domain the algebra $\mathcal{B}$ of Borel subsets of $X$. A Borel measure $\mu$ on $X$ is $\tau$-additive if $\mu(\bigcup G) = \sup_{\xi \in G} \mu G$ for every non-empty upwards-directed family $G$ of open sets in $X$. (If $X$ is regular, a totally finite Borel measure on $X$ is $\tau$-additive iff its completion is a quasi-Radon measure.) $X$ is Borel measure-complete if every totally finite Borel measure on $X$ is $\tau$-additive. A Borel measure $\mu$ on $X$ is inner regular for the compact sets if $\mu E = \sup \{\mu K : K \subseteq E \text{ is compact}\}$ for every $E \in \mathcal{B}$. (If $X$ is Hausdorff and $\mu$ is a totally finite Borel measure on $X$, $\mu$ is inner regular for the compact sets iff its completion is a Radon measure.) $X$ is Radon if it is Hausdorff and every totally finite Borel measure $\mu$ on $X$ is inner regular for the compact sets. Evidently a Radon topological space is Borel measure-complete. See Gardner & Pfeffer 84.

A2W Proposition Let $X$ be a complete metric space.

(a) A semi-finite $\tau$-additive Borel measure on $X$ is inner regular for the compact sets.

(b) $X$ is Radon if it is Borel measure-complete.

proof (a) It will be enough to show that a totally finite $\tau$-additive Borel measure on $X$ is inner regular for the compact sets. Let $\mu$ be such a measure. Let $E \subseteq X$ be a Borel set and $\epsilon > 0$.

For each $n \in \mathbb{N}$ let $G_{n}$ be the family of open sets of diameter less than or equal to $2^{-n};$ then $\bigcup G_{n} = X$. Because $\mu$ is $\tau$-additive there is a finite $\mathcal{H}_{n} \subseteq G_{n}$ such that $\mu(\bigcup \mathcal{H}_{n}) \geq \mu X - 2^{-n} \epsilon$. Set $K_{0} = \bigcap_{n \in \mathbb{N}} \bigcup \mathcal{H}_{n}$; then $K_{0}$ is closed and totally bounded, therefore compact, and $\mu K_{0} \geq \mu X - 2 \epsilon$.

Now $K_{0}$ is compact and metrizable, so the Borel measure $\mu[K_{0}]$ is inner regular for the compact sets (Schwartz 73, p. 117, Proposition 6). Consequently
\[
\mu E \leq \mu(E \cap K_{0}) + 2 \epsilon \leq \sup \{\mu K : K \subseteq E \text{ is compact}\} + 2 \epsilon.
\]
As $E$ and $\epsilon$ are arbitrary, $\mu$ is inner regular, as claimed.

(b) follows at once.

A2X Strong Law of Large Numbers We need this classical result in the following form. If $(X, \Sigma, \mu)$ is a probability space and $(h_{i})_{i \in \mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on $X$, then for almost all $(t_{n})_{i \in \mathbb{N}} \in X^{\mathbb{N}}$ we have
\[
\lim_{n \to \infty} \frac{1}{n+1} \sum_{i \leq n} (h(t_{i}) - \int h_{i}) = 0.
\]
A proof may be found in Shiryaev 84, chap. IV, §3, Theorem 7 (p. 364).

A2Y Lemma If $(X, \Sigma, \mu)$ is a Maharam homogeneous totally finite measure space and $\mu_{1}$ is another measure on $X$ with domain $\Sigma$ such that $\mu_{1} E \leq \mu E$ for every $E \in \Sigma$ and $\mu_{1} X > 0$, then $\mu$ and $\mu_{1}$ have the same Maharam type.

proof Take $E \in \mathcal{N}_{\mu}$, such that $\mu E$ is maximal. Then $\mathcal{N}_{\mu} = \{A : A \subseteq X, A \setminus E \in \mathcal{N}_{\mu}\}$. So the measure algebra $\mathcal{P}X/\mathcal{N}_{\mu}$ is isomorphic (as Boolean algebra) to the principal ideal of $\mathcal{P}X/\mathcal{N}_{\mu}$ generated by $(X \setminus E)^{*}$; which has the same Maharam type as the whole algebra $\mathcal{P}X/\mathcal{N}_{\mu}$.
A3. General topology

We need only a handful of definitions and a couple of standard theorems.

A3A If \( X \) is any topological space, a regular open set in \( X \) is an open subset \( G \) of \( X \) such that \( G = \text{int} \bigcup_{G} \). Now the family \( \mathcal{G} \) of all regular open subsets of \( X \) is a Boolean algebra, taking \( G \cap H = \text{int} \bigcap_{G} \), \( G \cup H = \text{int} \bigcup_{G} \), \( 1 \setminus G = \text{int} (X \setminus G) \) for \( G, H \in \mathcal{G} \); \( \mathcal{G} \) is the regular open algebra of \( X \). (See Engelking 89, 1.36-1.37.)

A3B If \( X \) is any topological space, its Baire \( \sigma \)-algebra is the \( \sigma \)-algebra of subsets of \( X \) generated by the zero sets, that is, by sets of the form \( f^{-1}[[0]] \) where \( f : X \to \mathbb{R} \) is continuous. When \( X = \{0,1\}^{\lambda} \) this is precisely the \( \sigma \)-algebra generated by sets of the form \( \{x : x(\xi) = 1\} \) for \( \xi < \lambda \).

A3C Definitions (a) A Moore space is a regular Hausdorff topological space \( X \) with a sequence \( (G_{n})_{n \in \mathbb{N}} \) of open covers such that for any \( x \in X \), any neighbourhood \( U \) of \( x \) there is an \( n \in \mathbb{N} \) such that \( \bigcup \{G : x \in G \in G_{n}\} \subseteq U \). (Engelking 89, p. 334.)

(b) An indexed family \( \langle F_{i} \rangle_{i \in I} \) of subsets of a topological space \( X \) is discrete if for every \( x \in X \) there is an open set containing \( x \) which meets \( F_{i} \) for at most one \( i \in I \). An indexed family \( \langle x_{i} \rangle_{i \in I} \) of points of \( X \) is discrete if for every \( x \in X \) there is an open set containing \( x \) which contains \( x_{i} \) for at most one \( i \). (Engelking 89, p. 16.)

(c) A topological space \( X \) is collectionwise normal if for every discrete family \( \langle F_{i} \rangle_{i \in I} \) of closed sets in \( X \) there is a disjoint family \( \langle G_{i} \rangle_{i \in I} \) of open sets such that \( F_{i} \subseteq G_{i} \) for every \( i \in I \). (Engelking 89, p. 305.)

A3D Proposition A collectionwise normal Moore space is metrizable.

proof Engelking 89, Theorem 5.4.1.

A3E Definitions Let \( X \) be a topological space.

(a) The density of \( X \), \( d(X) \), is the least cardinal of any dense subset of \( X \). (Engelking 89, p. 25.)

(b) If \( x \in X \) then \( \chi(x, X) \) is the least cardinal of any base of neighbourhoods of \( x \) in \( X \). (Engelking 89, p. 12.)

(c) The weight of \( X \), \( w(X) \), is the least cardinal of any base for the topology of \( X \). (Engelking 89, p. 12.)

A3F Proposition Let \( X \) be a metrizable topological space.

(a) There is a discrete family \( \langle x_{\xi} \rangle_{\xi \in d(X)} \) in \( X \).

(b) There is a base for the topology of \( X \) which is expressible as \( \{G_{ni} : n \in \mathbb{N}, i \in I_{n}\} \) where \( \langle G_{ni} \rangle_{i \in I_{n}} \) is discrete for each \( n \in \mathbb{N} \).

proof (a) is elementary; see Engelking 89, 4.1.15. (b) is Theorem 4.4.3 of Engelking 89.

A4. Indescribable cardinals

For the second half of §4 above, we need some ideas from elementary model theory. The point is that certain results of the form ‘if \( \kappa \) is a real-valued-measurable cardinal, there are many \( \alpha < \kappa \) such that \( \Phi(\alpha) \)' can be effectively approached through an analysis of the logical structure of the assertion ‘\( \Phi(\alpha) \)’. Here I describe a second-order language \( \mathcal{L} \) which provides a suitable classification of most of the sentences we are interested in.

A4A The language \( \mathcal{L} \) Let \( \mathcal{L} \) be the primitive second-order language in which there are countable infinities of first-order variables \( x, y, \ldots \) and second-order relational variables \( R, S, \ldots \); it being understood that each relational variable \( S \) has a determinate number of places, so that an expression ‘\( \forall S \)' must be read
as ‘for every $n$-place relation $S$’ for some $n$, whose value should be evident from the use of $S$ in the rest of the formula. Atomic formulae of $\mathcal{L}$ are of the form $S(y_1, \ldots, y_n)$, where $S$ is an $n$-place relation symbol, and compound formulae are constructed from these with the ordinary logical connectives $\wedge$ (‘and’), $\vee$ (‘or’), $\rightarrow$ (‘implies’) etc., and quantifiers $\forall S, \exists S, \forall x, \exists y$. We can now define a notion of satisfaction

$$(A; C_1, \ldots, C_k; a_1, \ldots, a_m) \models \phi$$

where $A$ is a set, $C_1, \ldots, C_k$ are finitary relations on $A$, $a_1, \ldots, a_m$ are points of $A$, and $\phi$ is a formula of $\mathcal{L}$ with $k$ free relational variables (of place-numbers matching the place-numbers of the $C_i$) and $m$ free first-order variables. See CHANG & KEISLER 73, §1.3, or EBBINGHAUS FLUM & THOMAS 84, §IX.1 for a proper discussion of ‘satisfaction’; intuitively,

$$(A; C_1, \ldots, C_k; a_1, \ldots, a_m) \models \phi$$

means that $\phi(C_1, \ldots, C_k, a_1, \ldots, a_m)$ is ‘true’ when bound variables in $\phi$ are taken to run over the members of $A$ or the relations on $A$.

There is a problem here; of course the assertion

$$(A; C_1, \ldots, C_k; a_1, \ldots, a_m) \models \phi$$

depends on an assumed assignment of the relations $C_i$ and points $a_j$ to the free variables of $\phi$. For our purposes here it will possible to adhere to the convention (an impracticable one for any extended work) that

$$(A; C_1, \ldots, C_k; a_1, \ldots, a_m) \models \phi$$

includes the assertion that the names of the free variables of $\phi$ are $R_i, x_j$, with $1 \leq i \leq k$ and $1 \leq j \leq m$; that each $R_i$ has the same number of places as the corresponding $C_i$; and that in the interpretation of $\phi$ the relations $C_i$ are assigned to the variables $R_i$ and the points $a_j$ are assigned to the variables $x_j$.

It will be convenient on occasion to write the formulae above without checking on the domains of the relations $C_i$ nor on whether every $a_j$ belongs to $A$. In this case it is to be understood that

$$(A; C_1, \ldots, C_k; a_1, \ldots, a_m) \models \phi$$

means

$$a_1, \ldots, a_m \in A \text{ and } (A; C_1|A, \ldots, C_k|A; a_1, \ldots, a_m) \models \phi,$$

where $C_i|A$ is the restriction of the relation $C_i$ to $A$.

**$\Pi^n_1$ formulae** Among the formulae of $\mathcal{L}$ we can distinguish the $\Pi^n_1$ and $\Sigma^n_1$ formulae, as follows. A $\Pi^n_0$ or $\Sigma^n_0$ formula is one in which all quantifiers are of the form $\forall y$ or $\exists y$. A $\Pi^n_{k+1}$ formula is one of the form

$$\forall S_1 \forall S_2 \ldots \forall S_k \phi$$

where $\phi$ is $\Sigma^n_1$: a $\Sigma^{n+1}_1$ formula is one of the form

$$\exists S_1 \ldots \exists S_k \phi$$

where $\phi$ is $\Pi^n_1$. (Here I allow, conventionally, $k = 0$, so that a $\Pi^n_1$ formula is also $\Sigma^{n+1}_1$ and $\Pi^n_{n+1}$.)

**Examples** It may help readers new to these ideas if I give some examples. I concentrate on the properties of cardinals and ordinals because that is the context in which we shall be working.

(i) If we say ‘$(X, \leq)$ is totally ordered’ we are saying just that $(X; =, \leq) \models \phi_1$, where $\phi_1$ is the $\Pi^0_1$ formula

$$\forall u \forall v \forall w \left( (R_2(u, v) \land R_2(v, w)) \rightarrow R_2(u, w) \right) \wedge \left( (R_2(u, v) \land R_2(v, u) \leftrightarrow R_1(u, v)) \right) \wedge \left( R_2(u, v) \lor R_2(v, u) \right).$$

(ii) If we say ‘$(X, \leq)$ is well-ordered’ we are saying just that $(X; =, \leq) \models \phi_2$, where $\phi_2$ is the $\Pi^1_1$ formula
∀S(∃aS(a)) →
∀b(S(b) ∧ ∀c(S(c) → R_2(b, c)))
∧ φ_1.

(iii) If we say ‘(X, ≤) is well-ordered and its order type is a limit cardinal’, we are saying that (X; =, ≤, :) ⊨ φ_3, where φ_3 is the formula

∃G∀F∀x∃y
(χ(G) ∧ G(x, y)
∧ χ(F) → (∃zR_2(z, y) ∧ (∀tR_2(t, x) → ¬F(t, z))))
∧ φ_2,

χ(F) being an abbreviation for the formula

∀u∃v∀w(F(u, v) ∧ F(u, w)) → R_1(v, w).

If we move the quantifier ∀S in φ_2 up with ∀F, we get a Σ^2_2 formula logically equivalent to φ_3.

**A4C Indescribability**

(a) Let n ∈ N. An ordinal α is weakly Π^1_n-indescribable if it is not 0 and whenever C_1, . . . , C_k are relations on α, ξ_1, . . . , ξ_m < α and φ is a Π^1_n formula such that

(α; C_1, . . . , C_k; ξ_1, . . . , ξ_m) ⊨ φ,

then there is an ordinal β < α such that

(β; C_1, . . . , C_k; ξ_1, . . . , ξ_m) ⊨ φ.

I will say that α is weakly Π^2_n-indescribable if it is weakly Π^1_n-indescribable for every n ∈ N.

(b) If α is weakly Π^1_n-indescribable and φ is a Π^1_n formula, C_1, . . . , C_k are relations on α, ξ_1, . . . , ξ_m are ordinals less than α, and

(α; C_1, . . . , C_k; ξ_1, . . . , ξ_m) ⊨ φ,

then we have a non-empty set

A(φ, C_1, . . . , C_k, ξ_1, . . . , ξ_m) = {β : β < α, (β; C_1, . . . , C_k; ξ_1, . . . , ξ_m) ⊨ φ} ⊆ α.

The intersection of two such sets is another (because φ ∧ ψ is logically equivalent to a Π^1_n formula whenever φ and ψ are), so they generate a filter on α; this is the Π^1_n-filter of α. The dual ideal is the Π^1_n-ideal of α.

In the same way, if α is weakly Π^2_n-indescribable then it has a Π^2_n-filter, which is just the union of its Π^1_n-filters, and a dual Π^2_n-ideal.

It will be convenient to say that if α is an ordinal which is not weakly Π^1_n-indescribable, then its ‘Π^1_n-ideal’ is Pα; and similarly that the Π^2_n-ideal of α is Pα if α is not weakly Π^2_n-indescribable.

**Remark** A subset of α belongs to the Π^1_n-filter on α iff it is Π^1_n-enforceable at α in the terminology of Lévy 71. Baumgartner Taylor & Wagon 77 use the phrase ‘ordinal Π^1_n-indescribable’ where I write ‘weakly Π^1_n-indescribable’.

**A4D Proposition**

(a) A weakly Π^1_n-indescribable ordinal θ is an uncountable regular cardinal, and every closed unbounded set in θ belongs to the Π^1_n-filter of θ.

(b) θ is not weakly Π^1_n-indescribable.

**proof**

(a)(i) If α is a successor ordinal then

(α; =; α − 1) ⊨ R_1(x_1, x_1), (β; =; α − 1) ⊭ R_1(x_1, x_1) ∀ β < α.

(ii) If α = ω then

(α; <; 0) ⊨ ∀y_1∃y_2 R_1(y_1, y_2)

but
\[(\beta; \leq 0) \not\forall y_1 \exists y_2 R_1(y_1, y_2)\]

for every \(\beta < \alpha\).

(iii) If \(\omega \leq \text{cf}(\alpha) < \alpha\) take \(\xi < \alpha\) and \(f : \xi \to \alpha\) such that \(f[\xi]\) is cofinal with \(\alpha\); let \(F\) be the corresponding binary relation, so that \(F(\eta, \zeta)\) iff \(f(\eta) = \zeta\); then
\[(\alpha; \leq, F; \xi) \not\forall y(R_1(y, x_1) \to \exists z R_2(y, z)),\]

but
\[(\beta; \leq, F; \xi) \not\forall y(R_1(y, x_1) \to \exists z R_2(y, z)),\]

for every \(\beta < \alpha\).

(iv) Thus \(\theta\) cannot be either a successor, nor \(\omega\), nor of cofinality less than \(\theta\), and must be an uncountable regular cardinal.

Let \(C\) be any closed unbounded set in \(\theta\). Write \(F(\xi, \eta)\) for \(\xi \leq \eta \in C\). Then \((\theta; F; 0) \equiv \forall y \exists z R_1(y, z)\), and if \(\alpha < \theta\) is such that \((\alpha; F; 0) \equiv \forall y \exists z R_1(y, z)\) then \(\alpha \in C\). So \(C\) belongs to the \(\Pi^1_1\)-filter of \(\theta\).

(b) Let \(f : \omega \to P\omega\) be a surjection, and write \(F(\xi, \eta)\) for \(\eta \in f(\xi)'\). Let \(\phi\) be the formula
\[\forall S \forall y \exists z (R_1(z, x_1) \to (S(z) \leftrightarrow R_2(y, z))).\]

Then for \(\alpha \leq \omega\),
\[(\alpha; \leq, F; \omega) \models \phi\]
means just that \(f[\alpha] = P\omega\), which is true iff \(\alpha = \omega\). Because \(\phi\) is a \(\Pi^1_1\) formula, \(\omega\) is not weakly \(\Pi^1_1\)-indescribable.

Remarks (a) In fact, an ordinal \(\alpha\) is weakly \(\Pi^1_1\)-indescribable iff it is a regular uncountable cardinal, and in this case its \(\Pi^1_1\)-filter is just the filter generated by the closed unbounded sets. See Lévy 71, Theorem 6, or Hanf & Scott 61.

(b) Of course the argument above shows that \(2^\lambda\) is not weakly \(\Pi^1_1\)-indescribable for any cardinal \(\lambda\). More generally, no cardinal power \(\lambda^\theta\) can be weakly \(\Pi^1_1\)-indescribable if \(1 < \lambda < \lambda^\theta\).

A4E For the next theorem we need names for some relations on the class \(\text{On}\) of ordinals. (i) Let \(p : \text{On} \times \text{On} \to \text{On}\) be the bijection corresponding to the familiar well-ordering of \(\text{On} \times \text{On}\): \(p(\xi, \eta) < p(\xi', \eta')\) iff either \(\max(\xi, \eta) < \max(\xi', \eta')\) or \(\max(\xi, \eta) = \max(\xi', \eta')\) and \(\xi < \xi'\) or \(\xi = \xi'\) and \(\eta < \eta'\). Let \(P_1\) be the corresponding ternary relation on \(\text{On}\), so that \(P(\xi, \eta, \zeta)\) iff \(p(\xi, \eta) = \zeta\). (ii) Let \(P_2, P_3\) be the ternary relations on \(\text{On}\) corresponding to ordinal addition and multiplication. (iii) Let \(q_1\) and \(q_2\) be the projections of \(p^{-1}\), so that \(\xi = p(q_1(\xi), q_2(\xi))\) for every \(\xi \in \text{On}\). Let \(P_4\) be the ternary relation corresponding to the function \((l, \xi) \mapsto q_2(q_1(\xi)) : \omega \times \text{On} \to \text{On}\).

Theorem For each \(n \geq 1\) there is a \(\Pi^1_n\) formula \(\phi\) such that whenever \(\psi\) is a \(\Pi^1_n\) formula, \(C_1, \ldots, C_k\) are relations on \(\text{On}\), \(\xi_1, \ldots, \xi_m\) are ordinals, then there is a one-place relation \(C\) on \(\text{On}\) such that for every infinite cardinal \(\alpha\)
\[(\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \psi \iff (\alpha; <, P_1, P_2, P_3, P_4, C;) \models \phi\].

proof Lévy 71, Theorem 8.

A4F Theorem Let \(\kappa\) be a weakly \(\Pi^1_\alpha\)-indescribable cardinal, where \(n \geq 1\). Then the \(\Pi^1_n\)-filter on \(\kappa\) is a normal filter closed under Mahlo’s operation.

proof Lévy 71, Theorems 9 and 15.

A4G Theorem Let \(\kappa\) be a weakly \(\Pi^1_{n+1}\)-indescribable cardinal, where \(n \in \mathbb{N}\). Then the set of weakly \(\Pi^1_n\)-indescribable cardinals less than \(\kappa\) belongs to the \(\Pi^1_{n+1}\)-filter on \(\kappa\).

proof Lévy 71, Theorem 16b.

A4H Theorem Let \(\kappa\) be an infinite cardinal and \(A\) a subset of \(\kappa\). Then the following are equivalent:
(a) \(A\) does not belong to the \(\Pi^1_\beta\)-ideal of \(\kappa\) (so that, in particular, \(\kappa\) is weakly \(\Pi^1_\beta\)-indescribable);
(b) A is $\Pi^1_1$-fully stationary in the sense of $\mathsf{4D}$.

**Proof** (a)$\Rightarrow$(b)(i) Suppose, if possible, that (a) is true but (b) is false. Let $\{f_\xi: \xi < \kappa\}$ be a family of regressive functions on $\kappa$ witnessing not-(b); that is, such that there is no uniform ultrafilter $\mathcal{F}$ on $\kappa$, containing $A$, for which $\lim_{\alpha \to \mathcal{F}} f_\xi(\alpha)$ exists for every $\xi$. Because $\kappa$ is regular ($\mathsf{A4Da}$), this means that for any $g \in \kappa^\kappa$ the family

$$\{ A \setminus \zeta : \zeta < \kappa \} \cup \{ f_\xi^{-1}[\{ g(\xi) \}] : \xi < \kappa \}$$

is not included in any ultrafilter on $\kappa$, that is, does not have the finite intersection property. So we have

$$\forall g \in \kappa^\kappa \exists \zeta < \kappa, I \in [\kappa]^{< \omega} \text{ such that } \forall \eta \in A \setminus \zeta \exists \xi \in I \text{ such that } g(\xi) \neq f_\xi(\eta).$$

Evidently (*) can be coded as a $\Pi^1_1$ formula; the details are as follows. Let $h: \kappa \to [\kappa]^{< \omega}$ be any surjection, and write $E(\zeta, \eta)$ for '$\zeta \leq \eta \in A', H(\xi, \gamma, \eta, \delta)$ for '$\xi \in h(\gamma)$ and $f_\xi(\eta) \neq \delta$'. Let $\phi$ be the formula

$$\forall S(\forall u \exists v S(u, v)) \rightarrow \exists z \exists \forall y (R_1(z, y) \rightarrow \exists z \exists d S(x, d, \eta, \delta)).$$

Then (*) says just that

$$\langle \kappa; E, H; \rangle \models \phi.$$ 

Accordingly the set

$$B = \{ \alpha : \alpha < \kappa, (\alpha; E, H;) \models \phi \}$$

belongs to the $\Pi^1_1$-filter $\mathcal{W}$ of $\kappa$.

(ii) Let $C$ be the set of non-zero limit ordinals $\alpha < \kappa$ such that $h[\alpha] = [\alpha]^{< \omega}$. Then $C$ is a closed unbounded set in $\kappa$ so $C \in \mathcal{W}$, by $\mathsf{A4Da}$. Consequently $A \cap B \cap C$ is stationary. But if $\alpha \in B \cap C$ then

$$\forall g \in \alpha^\alpha \exists \zeta < \alpha, I \in [\alpha]^{< \omega} \text{ such that } \forall \eta \in \alpha \setminus \zeta \exists \xi \in I \text{ such that } g(\xi) \neq f_\xi(\eta).$$

It will be helpful to put this into a logically equivalent form. For $0 < \alpha < \kappa$ let $g_\alpha: \kappa \to \alpha$ be given by setting $g_\alpha(\xi) = f_\xi(\alpha)$ for every $\xi < \kappa$. Then for $\alpha \in B \cap C$ we have

$$\forall g \in \alpha^\alpha \exists \zeta < \alpha, I \in [\alpha]^{< \omega} \text{ such that } g_\alpha[I] \neq g_\alpha[I] \forall \eta \in \alpha \setminus \zeta.$$ 

In particular, there are $\zeta_\alpha < \alpha$ and $I_\alpha \in [\alpha]^{< \omega}$ such that $g_\alpha[I_\alpha] \neq g_\alpha[I_\alpha]$ for every $\eta \in \alpha \setminus \alpha \setminus \zeta_\alpha$. Because $\alpha$ is a limit ordinal we may take it that $I_\alpha \cup g_\alpha[I_\alpha] \subseteq \zeta_\alpha$.

Now $A \cap B \cap C$ is stationary, so by the pressing-down lemma there is a $\zeta < \kappa$ such that $A_1 = \{ \alpha : \alpha \in A \cap B \cap C, \zeta = \zeta_\alpha \}$ is stationary; next, $g_\alpha[I_\alpha] \subseteq \zeta \times \alpha$ for every $\alpha \in A_1$, so there is a finite function $e$ such that $A_2 = \{ \alpha : \alpha \in A_1, g_\alpha[I_\alpha] \subseteq \zeta \times \alpha \}$ is stationary. Of course $I_\alpha = \text{dom}(e) = I$ say, for every $\alpha \in A_2$. Now take $\alpha, \beta \in A_2$ such that $\zeta < \beta < \alpha$. Then $\beta \in A \cap B \cap \zeta_\alpha$, so $g_\beta[I_\alpha] \neq g_\beta[I_\alpha]$: but on the other hand $g_\beta[I_\alpha] = e = g_\alpha[I_\alpha]$.

So (a)$\Rightarrow$(b).

(b)$\Rightarrow$(a) Suppose, if possible, that (b) is true but that (a) is false; either $\kappa$ is not weakly $\Pi^1_1$-indescribable or $\kappa \setminus A$ belongs to the $\Pi^1_1$-filter on $\kappa$. In either case there must be a $\Pi^1_1$ formula $\phi$ and relations $C_1, \ldots, C_k$, ordinals $\xi_1, \ldots, \xi_m$ such that

$$\langle \kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m \rangle \models \phi, (\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \not\models \phi \forall \alpha \in A.$$ 

We may suppose that $\phi$ is of the form

$$\forall R_{k+1} \ldots R_{k+r} \exists x_{m+1} \forall x_{m+2} \ldots \exists x_{m+2s-1} \forall x_{m+2s} \forall \psi$$

where $\psi$ has no quantifiers, since $\phi$ is surely logically equivalent to such a formula. Set $A' = \{ \alpha : \alpha \in A, \xi_j < \alpha \forall j \leq m \}$. For $\alpha \in A'$, saying that $(\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \not\models \phi$ means just that there are relations $D_{a_1}, \ldots, D_{a_r}$ on $\alpha$ and functions $f_{a_1}: \alpha \to \alpha, f_{a_2}: \alpha^2 \to \alpha, \ldots, f_{a_s}: \alpha^s \to \alpha$ such that whenever $\eta_1, \ldots, \eta_s < \alpha$ then

$$(\alpha; C_1, \ldots, C_k, D_{a_1}, \ldots, D_{a_r}; \xi_1, \ldots, \xi_m, \eta_1, f_{a_1}(\eta_1), \ldots, \eta_s, f_{a_1}(\eta_1), \ldots, \eta_s) \not\models \psi.$$
Choose such $D_{αi}, f_{αj}$ for $α ∈ A'$. For $η = (η_1, ..., η_j) ∈ κ^j$, where $1 ≤ j ≤ s$, set

\[ g_η(α) = f_{αj}(η_1, ..., η_j) \text{ if } \max(η_1, ..., η_j) < α ∈ A', \]

\[ = 0 \text{ otherwise.} \]

Set $G = \{g_η : η ∈ \bigcup_{1 ≤ j ≤ s} κ^j\} ∈ \mathcal{Regr}(κ)$.

Now (b) tells us that there is a uniform ultrafilter $F$ on $κ$, containing $A$, such that $\lim_{α → F} g(α)$ exists for every $g ∈ G$. Because $F$ is uniform, $A' ∈ F$. Set

\[ h_j(η) = \lim_{α → F} g_η(α) = \lim_{α → F} f_{αj}(η) \]

for $j ≤ s, η ∈ κ^j$. For $i ≤ r$, write $D_i(η)$ for \{\{α : D_{αi}(η) ∈ F\}\}. Now for any formula $χ$ without quantifiers, and any $η_1, ..., η_s < κ$, we have

\[ (κ; C_1, ..., C_k, D_1, ..., D_r; ξ_1, ..., ξ_m, η_1, h_1(η_1), ..., η_s, h_s(η_1, ..., η_s)) \models χ \]

if and only if the set of $α < κ$ for which

\[ (α; C_1, ..., C_k, D_1, ..., D_r; ξ_1, ..., ξ_m, η_1, f_{α1}(η_1), ..., f_{αs}(η_1, ..., η_s)) \models χ \]

belongs to $F$; this is an easy induction on the length of $χ$. In particular, it is valid when $χ$ is $¬ψ$. But this means that

\[ (κ; C_1, ..., C_k, D_1, ..., D_r; ξ_1, ..., ξ_m, η_1, h_1(η_1), ..., η_s, h_s(η_1, ..., η_s)) \models ¬ψ \]

for all $η_1, ..., η_s$, so that $D_1, ..., D_r$ and $h_1, ..., h_s$ witness that

\[ (κ; C_1, ..., C_k; ξ_1, ..., ξ_m) \not\models φ, \]

contrary to hypothesis. X

Thus (b)⇒(a).

Remark This is due to Kunen and J.E.Baumgartner; in the form here it is taken from Fremlin & Kunen n87. The point is that it puts meat onto Theorem A4E by actually exhibiting a universal $Π^1_1$ formula $φ$. In §4 above I show how it can be used to investigate weakly $Π^1_2$-indescribable cardinals without further recourse to the logical characterization. Compare ‘Baumgartner’s principle’ as described in Erdős Hajnal Máthé & Rado 84, 30.6 and 31.3.

It seems that no corresponding combinatorial characterization of weakly $Π^1_2$-indescribable cardinals is known.

A41 Theorem Let $κ$ be an infinite cardinal, $φ$ a formula of $L, C_1, ..., C_k$ relations on $κ, ξ_1, ..., ξ_m$ members of $κ$. For each set $I$ let $A_I$ be the measure algebra of $\{0, 1\}^I$ and $P_I$ the p.o.set $A_I \setminus \{0\}$. Now if $I$ and $J$ are any sets both of cardinal strictly greater than $κ$, and $β ≤ κ$,

\[ \| P_I \models (β; C_1, ..., C_k; ξ_1, ..., ξ_m) \models φ \equiv \| P_J \models (β; C_1, ..., C_k; ξ_1, ..., ξ_m) \models φ. \]

proof I begin with some preliminary remarks. It will be convenient to suppose that $I$ and $J$ are disjoint; this is legitimate because $P_I$ is determined up to isomorphism by the cardinal of $I$. Recall that if $K ⊆ I$ then we may regard $P_K$ as a subset of $P_I$, corresponding to the canonical map from $\{0, 1\}^I$ to $\{0, 1\}^K$ and the induced measure-preserving homomorphism from $A_K$ to $A_I$; and that we may also think of $P_I$ as an iteration $P_K * P_{I \setminus K}$ (Kunen 84, 3.13), if you will allow me to write $P_{I \setminus K}$ rather than $P_{I \setminus K}$ for what is really a $P_I$-name.

Now for the main argument, which proceeds by induction on the length of $φ$.

(a) If $φ$ has no quantifiers the result is trivial, since, for instance,

\[ \| P_I \models C_i(ξ_j, ..., ξ_k) \iff C_i(ξ_j, ..., ξ_k), \]

\[ \| P_I \models ξ_j < β \iff ξ_j < β. \]

(b) Suppose that $φ$ is of the form $∃ Sψ$ and that
where each isomorphism between $P_2$ quantifiers govern first- and second-order variables, and the $\Pi_{110}$ $P$ an isomorphism. If we regard $P$ these; so that we may think of $\dot{\beta}$.

Remark This is due to Kunen. I omit the proof because I shall not rely on it. A sketch of the argument

Because $P_I$ is ccc, the $P_I$-name $\dot{R}$ can be described in terms of not more than $\max(\omega, \kappa) = \kappa$ members of $P_I$; now there is a $K \subseteq I$ of cardinal at most $\kappa$ such that $P_K$, regarded as a subset of $P_I$, contains all of these; so that we may think of $\dot{R}$ as a $P_K$-name.

Let $L \subseteq J$ be a set of the same size as $K$. Then any bijection between $K$ and $L$ gives rise to an isomorphism between $P_K$ and $P_L$. Let $\dot{S}$ be a $P_L$-name for a relation on $\beta$ corresponding to $\dot{R}$ under such an isomorphism. If we regard $P_I$ as the iteration $P_K \ast P_I\setminus K$ then we have

\[
\models_{P_K} (\models_{P_I\setminus K} (\dot{\beta}; \ldots, \dot{R}; \ldots) \models \psi).
\]

So

\[
\models_{P_L} (\models_{P_I\setminus K} (\dot{\beta}; \ldots, \dot{S}; \ldots) \models \psi).
\]

At this point we move to the intermediate model $V^{P_L}$. From this standpoint $\dot{S}$ represents a fixed relation on $\beta$. Also $I \setminus K$ and $J \setminus L$ both have cardinals greater than $\kappa$. So we can use the inductive hypothesis to see that

\[
\models_{P_L} (\models_{P_J\setminus L} (\dot{\beta}; \ldots, \dot{S}; \ldots) \models \psi),
\]

that is,

\[
\models_{P_J} (\dot{\beta}; \ldots, \dot{S}; \ldots) \models \psi.
\]

So $\dot{S}$ witnesses that

\[
\models_{P_J} (\dot{\beta}; \ldots) \models \phi,
\]

and we have

\[
\models_{P_J} (\dot{\beta}; \ldots); \models \phi \Rightarrow \models_{P_J} (\dot{\beta}; \ldots) \models \phi;
\]

of course the reverse implication is equally valid.

(c) If $\phi$ is of the form $\forall R \psi$ we can argue similarly; given

\[
\models_{P_J} (\dot{\beta}; \ldots) \models \phi
\]

and a $P_J$-name $\dot{S}$ for a relation on $\beta$, we express $\dot{S}$ as a $P_L$-name for some $L \subseteq J$ of size at most $\kappa$, copy this into a $P_K$-name $\dot{R}$ for $K \subseteq I$, and use the inductive hypothesis on

\[
\models_{P_K} (\dot{\beta}; \ldots, \dot{R}; \ldots) \models \psi
\]

in $V^{P_J}$ to see that

\[
\models_{P_J} (\dot{\beta}; \ldots, \dot{S}; \ldots) \models \psi,
\]

as required.

(d) If $\phi$ is of the form $\forall x \psi$ or $\exists x \psi$ the same arguments apply, taking $K$ and $L$ to be countable if we wish.

A4J Proposition Let $n \in \mathbb{N}$ and let $\kappa$ be a weakly $\Pi_n^1$-indescribable cardinal such that the cardinal power $\kappa^\omega$ is equal to $\kappa$. Let $\lambda > \kappa$ be any cardinal; let $\mathfrak{A}_{\lambda}$ be the measure algebra of $\{0, 1\}^\lambda$ and $P_{\lambda}$ the p.o.set $\mathfrak{A}_{\lambda} \setminus \{0\}$. Then

\[
\models_{P_{\lambda}} \kappa \text{ is weakly } \Pi_n^1 \text{-indescribable.}
\]

Remark This is due to Kunen. I omit the proof because I shall not rely on it. A sketch of the argument for the corresponding theorem for Cohen reals may be found in KUNEN 71.

A4K $\Pi_3^1$-Indescribability For the next step we need a higher-order language. Let $\mathcal{L}_3$ be the extension of $\mathcal{L}$ in which third-order relational variables $\mathcal{R}$ are added, with corresponding atomic formulae $\mathcal{R}(u_1, \ldots, u_l)$, where each $u_i$ is either a first-order or second-order variable. In $\mathcal{L}_3$, the $\Pi_3^0$ formulae are those in which all quantifiers govern first- and second-order variables, and the $\Pi_3^1$ formulae are those of the form
\[ \forall R_1 \ldots \forall R_k \psi, \]

where \( \psi \) is a \( \Pi_0^2 \) formula. Now a non-zero ordinal \( \kappa \) is weakly \( \Pi_0^2 \)-indescribable if whenever \( \phi \) is a \( \Pi_0^2 \) formula in \( L_3 \) with no free third-order variables and \( C_1, \ldots, C_k \) are relations, \( \xi_1, \ldots, \xi_m \) are members of \( \kappa \) such that

\[ (\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi, \]

then there is an \( \alpha < \kappa \) such that

\[ (\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi. \]

For such \( \kappa \), the sets

\[ \{ \alpha : (\alpha; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi \} \]

where \( \phi, C_1, \ldots \) are such that \( (\kappa; C_1, \ldots) \models \phi \) and \( \phi \) is \( \Pi_0^2 \)-filter on \( \kappa \). Finally, \( \kappa \) is \( \Pi_0^2 \)-indescribable if it is weakly \( \Pi_0^2 \)-indescribable and \( 2^\lambda < \kappa \) for every \( \lambda < \kappa \).

**Theorem** If \( \kappa \) is two-valued-measurable, it is \( \Pi_0^2 \)-indescribable, and its \( \Pi_0^2 \)-filter is included in its rvm filter.

**proof** See Drake 74, §9.3, Jech 78, p. 385, Lemma 32.2 or Kanamori & Magidor 78, §4.4.

Version of 18.9.92

**Problems**

I collect here some of the questions which arise more or less naturally from the work above and seem to be open.

**P1 Construction of atomlessly-measurable cardinals** The most important problem is surely something like this.

(P1) Let \( N \) be a model of ZFC and \( \kappa \in N \) an atomlessly-measurable cardinal in \( N \). Does it follow that there are an inner model \( M \subseteq N, \) containing \( \kappa, \) such that \( \kappa \) is a two-valued-measurable cardinal in \( M, \) and an \( M \)-generic filter \( G \) in a random real p.o.set \( P \in M \) such that \( G \in N \) and \( N \cap P \kappa \subseteq M[G]? \)

Put in this form, it is hard to believe in the possibility of an affirmative answer (though note Theorem 2D). But so long as the question remains open, we have no way of proving consistency results for atomlessly-measurable cardinals except through Solovay’s construction in Theorem 2C above. This construction is not wholly inflexible (see, for instance, 2E, 2I, 2L, 4Lb). But atomlessly-measurable cardinals built in this way share a vast number of special properties. Some are known to be possessed by atomlessly-measurable cardinals in general; many of the results in these notes were suggested by this approach. But others seem inaccessible to present techniques. By and large, positive answers to the other problems here could be taken as (weakly) suggesting a positive answer to P1; while negative answers would often imply a negative answer to P1.

**P2 Measure algebras of atomlessly-measurable cardinals** The Gitik-Shelah theorem (3F) tells us that if \( \kappa \) is an atomlessly-measurable cardinal and \( \nu \) is a witnessing probability on \( \kappa \) then the Maharam type \( \lambda \) of \( \langle \kappa, \mathcal{P}_\kappa, \nu \rangle \) is at least \( \min(2^\kappa, \kappa^{(+\omega)}) \); and of course it cannot be greater than \( 2^\kappa \). So if \( 2^\kappa < \kappa^{(+\omega)} \) then \( \lambda = 2^\kappa \). We get a scrap more information in 7Q: if \( \gamma \) is the least cardinal such that \( 2^\gamma = 2^\delta \) for \( \gamma \leq \delta < \kappa, \) then \( \gamma < \kappa \) and \( \lambda^\gamma = 2^\kappa, \) so that either \( 2^\gamma = 2^\kappa \) or \( 2^\gamma < \lambda. \) Of course this still does not specify \( \lambda \) completely. So we may ask

(P2a) Is there a combinatorial characterization of \( \lambda? \)

An affirmative answer would have a variety of consequences; not least, that any witnessing probability on \( \kappa \) would have the same Maharam type – equivalently, that every witnessing probability on \( \kappa \) would be Maharam homogeneous (3L). So a less ambitious question is
Conceivably it makes a difference if \( \nu \) is normal; so I add

(P2c) If \( \kappa \) is an atomlessly-measurable cardinal and \( \nu \) is a normal witnessing probability on \( \kappa \), must \( (\kappa, \mathcal{P}\kappa, \nu) \) be Maharam homogeneous?

Of course \( \text{Tr}_\nu \) is the standard elementary embedding. We also have a characterization of weak \( \Pi^1_2 \)-indescribable atomlessly-measurable cardinal, must it be weakly \( \Pi^1_2 \)-indescribable?

(P2d) If \( \kappa \) is an atomlessly-measurable cardinal constructed by Solovay's method, then \( \kappa \) is not weakly \( \Pi^1_2 \)-indescribable, or \( \kappa \) is weakly \( \Pi^1_2 \)-indescribable and its \( \Pi^1_2 \)-filter is included in its rvm filter. We also have a characterization of weak \( \Pi^1_2 \)-indescribability in terms of covering numbers (6L). This begs many questions. For instance:

(P3a) If \( \kappa \) is a weakly \( \Pi^1_2 \)-indescribable atomlessly-measurable cardinal, must it be weakly \( \Pi^1_2 \)-indescribable?

(P3b) If \( \kappa \) is a weakly \( \Pi^1_2 \)-indescribable atomlessly-measurable cardinal, must its \( \Pi^1_2 \)-filter be included in its rvm filter?

As a special case of this we have

(P2f) If \( \mathcal{C} \) is atomlessly-measurable, with witnessing probability \( \nu \), does it follow that the Maharam type of \( (\mathcal{C}, \mathcal{P}\mathcal{C}, \nu) \) is \( 2^\kappa \)?

P3 Indescribability In 4R we saw that if \( \kappa \) is an atomlessly-measurable cardinal constructed by Solovay's method, then either \( \kappa \) is not weakly \( \Pi^1_2 \)-indescribable, or \( \kappa \) is weakly \( \Pi^1_2 \)-indescribable and its \( \Pi^1_2 \)-filter is included in its rvm filter. We also have a characterization of weak \( \Pi^1_2 \)-indescribability in terms of covering numbers (6L). This begs many questions. For instance:

(P3e) If \( \kappa \) is an atomlessly-measurable cardinal strictly less than \( \mathcal{C} \), must it be weakly \( \Pi^1_2 \)-indescribable?

P4 The cardinals of analysis A great many cardinals between \( \omega_1 \) and \( \mathcal{C} \) have been named and studied by analysts; see VAUGHAN 90. The existence of an atomlessly-measurable cardinal is known to have a dramatic effect on the patterns formed by these cardinals. For instance, if \( \kappa \) is an atomlessly-measurable cardinal, then

(i) \( \text{non}(\mathcal{R}, \mathcal{N}) = \omega_1 \), where \( \mathcal{N} \) is the ideal of Lebesgue negligible sets (6Ga); it follows that \( \text{cov}(\mathcal{R}, \mathcal{M}) = \text{add}(\mathcal{M}) = \text{add}(\mathcal{N}) = \mathcal{P} = \omega_1 \), where \( \mathcal{M} \) is the ideal of meager subsets of \( \mathcal{R} \) and \( \mathcal{P} = \mathcal{M}_{\sigma\text{-centered}} \), as in FREMLIN 84 and DOUWEN 84; see FREMLIN 85 or BARTOSZYŃSKI & JUDAH p90.

(ii) \( \mathcal{B} \subseteq \kappa \), where \( \mathcal{B} = \text{add}([\mathcal{N}]^\omega / \mathcal{F}_0) \), writing \( \mathcal{F}_0 \) for the Fréchet filter on \( \mathbb{N} \) (5Lb).

(iii) \( \text{cf}(\mathfrak{B}) \neq \kappa \), where \( \mathfrak{B} = \text{cf}([\mathcal{N}]^\omega) \) (7Ka).

(iv) \( \text{cov}(\mathcal{R}, \mathcal{N}) \geq \kappa \) (in fact, \( \text{cov}(\mathcal{X}, \mathcal{N}_\mu) \geq \kappa \) for any non-trivial Radon measure space \( (\mathcal{X}, \mu) \)) (6B). Consequently \( \kappa \leq \text{non}(\mathcal{R}, \mathcal{M}) \leq \text{cf}(\mathcal{M}) \leq \text{cf}(\mathcal{N}) \) (FREMLIN 85 or BARTOSZYŃSKI & JUDAH p90 again).

Once again, there are obvious gaps. The principal one seems to be
(P4a) If $\kappa$ is an atomlessly-measurable cardinal, must $\delta$ be less than $\kappa$?

Concerning covering numbers, we have

(P4b) If there is an atomlessly-measurable cardinal, must $\text{cov}(\mathbb{R}, \mathcal{N})$ be exactly $\theta$?

(see 2Hb). Then concerning the numbers $\text{non}(\ldots)$ we have

(P4c) If $\kappa$ is an atomlessly-measurable cardinal, must $\text{non}(\{0, 1\}^\kappa, \mathcal{N}_{\mu_\kappa})$ be $\omega_1$?

Because we know that $\text{non}(\{0, 1\}^\theta, \mathcal{N}_{\mu_\kappa}) = \omega_1$ for $\theta < \kappa$ (6G), we can apply results from §4; it is not hard to construct a $\Sigma^1_3$ formula $\phi$ such that, for ordinals $\alpha > \omega_1$, $\langle \alpha; \omega, \omega_1 \rangle \models \phi$ iff $\text{non}(\{0, 1\}^\kappa, \mathcal{N}_{\mu_\kappa}) = \omega_1$. Now if $\text{non}(\{0, 1\}^\kappa, \mathcal{N}_{\mu_\kappa}) > \omega_1$, then $\langle \kappa; \omega, \omega_1 \rangle \not\models \neg \phi$ while $\langle \alpha; \omega, \omega_1 \rangle \not\models \neg \phi$ for every $\alpha < \kappa$, and $\kappa$ is not weakly $\Pi^1_2$-indescribable.

An associated question comes from 6F:

(P4d) If $\kappa$ is an atomlessly-measurable cardinal, must there be a set $A \subseteq \mathbb{R}$, of cardinal $\kappa$, such that no uncountable subset of $A$ is Lebesgue negligible?

Again, we get a positive answer if $\kappa$ is weakly $\Pi^1_2$-indescribable, using 6F.

A curious question arises from 3Bb.

(P4e) Let $\kappa$ be an atomlessly-measurable cardinal and $A$ a subset of $\mathbb{R}$. Must there be a set $B \subseteq A$ such that $(\#(B)) < \kappa$ and $\mu_L^B = \mu_L^A$, writing $\mu_L$ for Lebesgue measure?

If $\#(A) \leq \kappa$, yes, by 3Bb; but the general question seems to be open.

A question closely related in form, if not in content, to P4d, is

(P4f) If $\kappa$ is an atomlessly-measurable cardinal, must there be ccc partially ordered sets $P$ and $Q$ such that $S(P \times Q) > \kappa$?

As before, this corresponds to a $\Sigma^1_2$ formula, so that 7D shows that we shall have an affirmative answer if $\kappa$ is weakly $\Pi^1_2$-indescribable. The same route leads from 7F to

(P4g) If $\kappa$ is an atomlessly-measurable cardinal, must there be an $\omega_1$-entangled subset of $\mathbb{R}$ of cardinal $\kappa$?

Returning to named cardinals, let $\alpha$ be the smallest cardinal of any infinite maximal disjoint family in the algebra $\mathcal{P}(\mathbb{N})/\mathcal{N}$; then $\mathfrak{b} \leq \alpha$ (Dowven 84, Theorem 3.1a); but

(P4h) If $\kappa$ is atomlessly-measurable, must $\alpha$ be less than $\kappa$?

P5 Cofinalities The remarkable results from Shelah’s pef theory which give us 7Ka-d leave some natural questions open. The most important has already been listed as P4a. But it seems that even the following bold conjecture might be true:

(P5a) Let $\kappa$ be an atomlessly-measurable cardinal. Let $(P_\zeta)_{\zeta < \lambda}$ be a family of partially ordered sets where $\lambda < \kappa$ and $\text{cf}(P_\zeta) < \kappa$ and $\text{add}(P_\zeta) > \omega$ for every $\zeta < \lambda$. Does it follow that $\text{cf}(\prod_{\zeta < \lambda} P_\zeta) < \kappa$?

If $\kappa = \varpi$, yes (7Kf); if $\text{add}(P_\zeta) > \omega$ for every $\zeta$, yes. A positive answer to P9b below would settle the general question. A natural special case of P5a is

(P5b) Let $\kappa$ be an atomlessly-measurable cardinal. Must $\text{cf}(\omega^{\varpi})$ be less than $\kappa$?

P6 More measure algebras In paragraph P2 above I listed the known facts concerning the Maharam types of witnessing probabilities on atomlessly-measurable cardinals. But if we allow ourselves general probability spaces $(X, \mathcal{P}X, \mu)$ we can expect other phenomena. Indeed, Theorem 8A shows that PMEA implies that there are probability spaces $(X, \mathcal{P}X, \mu)$ of arbitrarily large Maharam type. Now in this context the question arises

(P6) Is it consistent to suppose that there is a cardinal $\lambda_0$ such that for every cardinal $\lambda \geq \lambda_0$ there is a Maharam homogeneous probability space $(X, \mathcal{P}X, \mu)$ of Maharam type $\lambda$?

P7 Qm ideals If a cardinal $\kappa$ is real-valued-measurable, it is of course also quasi-measurable; so we have both an rvm ideal $\mathcal{I}_{\text{rvm}}(\kappa)$ (11) and a qm ideal $\mathcal{J}_{\text{qm}}(\kappa)$ (9C). Evidently $\mathcal{I}_{\text{rvm}}(\kappa) \supseteq \mathcal{J}_{\text{qm}}(\kappa)$, and they are equal if $\kappa$ is two-valued-measurable (9D).

Now suppose that $\kappa$ is two-valued-measurable and that $\mathbb{P}$ is a random real p.o.set. If we have a P-name $\check{I}$ for a normal $\omega_1$-saturated ideal of $\mathcal{P}\kappa$, then $\check{I}_1 = \{ B : B \subseteq \kappa, \check{I} \subseteq \check{I} \}$ is a normal $\omega_1$-saturated ideal
of $\mathcal{P}\kappa$ (because $\mathcal{P}$ is ccc; compare 2Jb), and therefore is the null ideal of some normal probability $\nu_1$ on $\kappa$. Now if $\tilde{\nu}_1$ is the corresponding $\mathcal{P}$-name for a probability on $\kappa$, as in 2C, we have $\|\nu_1\mathcal{N}\subseteq \tilde{\mathcal{I}}$, by 2A(ii). It follows easily that, in $V^\mathcal{P}$, $\mathcal{I}$ is the null ideal for a measure $\tilde{\nu}$ on $\kappa$ (in $V^\mathcal{P}$, take a member $\tilde{E}$ of $\tilde{\mathcal{I}}$ of maximal $\tilde{\nu}_1$-measure, and set $\tilde{\nu}A = \tilde{\nu}_1(A \setminus \tilde{E})$ for every $A$).

So in this case we surely have $\|\nu_1 \mathcal{J}_{rvm}(\kappa) = \mathcal{J}_{qnm}(\kappa)$.

But the question now arises,

$\mathcal{P}7$ If $\kappa$ is a real-valued-measurable cardinal, is it necessarily true that $\mathcal{J}_{rvm}(\kappa) = \mathcal{J}_{qnm}(\kappa)$?

$\mathcal{P}8$ Sequential cardinals A question going back to MAZUR 52 is: for which cardinals $\kappa$, if any, is there a sequentially continuous function $f : \{0, 1\}^\kappa \rightarrow \mathbb{R}$ which is not continuous? Let us call such cardinals ‘sequential’ for the moment. It is easy to see that a real-valued-measurable cardinal is sequential, and it is known that the first sequential cardinal, if there is one, is quasi-measurable. (See PLEBANEK P91 for a survey of known results, with references; also FREMLIN 84, 24D-E.) But the following, raised by KEISLER & TARKSI 64, p. 270, seems still to be open:

$\mathcal{P}8$ Must the first sequential cardinal, if there is one, be real-valued-measurable?

$\mathcal{P}9$ The measure of $\kappa^+$ Let $\kappa$ be a real-valued-measurable cardinal, and $\nu$ a normal witnessing probability on $\kappa$. For any set $I$ we can form the simple (completed) product probability on $\kappa^+$, as described in FREMLIN 84, A6Kb; let us call it $\nu_I$. What can be said about $\nu_I$?

The most important question seems to concern the covering number $\text{cov}(\kappa^+, \mathcal{N}_\nu)$ (see A2P). If $I$ is finite, then there is a $\kappa$-additive probability with domain $\mathcal{P}(\kappa^+)$ extending $\nu_I$ (use the construction of part (a-ii) of the proof of 5O; the same idea appears in 2Mb, 4K and 5J), so that $\text{cov}(\kappa^+, \mathcal{N}_\nu) = \kappa$. But I ask

$\mathcal{P}9a$ If $\kappa$ is a real-valued-measurable cardinal, $\nu$ a normal witnessing probability on $\kappa$, and $\nu_I$ the completed product probability on $\kappa^+$, must $\text{cov}(\kappa^+, \mathcal{N}_\nu) = \kappa$?

A reason for believing that there may be a positive answer to this question is that if $\nu$ is constructed from a normal ultrafilter on a two-valued-measurable cardinal $\kappa$ by the method of 2C, then $\text{cov}(\kappa^+, \mathcal{N}_\nu) = \kappa$ for every non-empty set $I$ of cardinal less than $\kappa$. A reason for taking the problem seriously is that a positive answer would solve the following problem positively:

$\mathcal{P}9b$ If $\kappa$ is a real-valued-measurable cardinal and $\nu$ is a normal witnessing probability on $\kappa$ and $A \subseteq \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ has cardinal less than $\kappa$, does it follow that there is a countable $N \subseteq \kappa$ such that $A \cap N \neq \emptyset$ whenever $A \in A$?

(To answer $\mathcal{P}9b$ from $\mathcal{P}9a$, consider the family $\{((\kappa \setminus A)^\mathcal{P} : A \in A) \}$; if $\text{cov}(\kappa^+, \mathcal{N}_\nu) \geq \kappa$ this cannot cover $\kappa^+$. With a positive answer to $\mathcal{P}9b$ we could deal with $\mathcal{P}5a$, or go on to

$\mathcal{P}9c$ If $\kappa$ is a real-valued-measurable cardinal with normal witnessing probability $\nu$, $Y$ a set of cardinal less than $\kappa$, $\mathcal{I}$ a $\sigma$-algebra of subsets of $Y$ and $f : \kappa \rightarrow \mathcal{I}$ a function, must there be a set $M \in \mathcal{I}$ such that $\nu\{\xi : \xi f(\xi) \subseteq M\} = 1$?

(Set $A_y = \{\xi : y \in f(\xi)\}$ for each $y \in Y$, $A = \bigcup A_y : y \in Y, \nu A_y = 0$. If $N \in [\kappa]^{<\omega}$ is such that $N \cap A_y \neq \emptyset$ whenever $\nu A_y > 0$, set $M = \bigcup_{\xi \in N} f(\xi) \in \mathcal{I}$; then $f(\xi) \subseteq M$ whenever $\xi \in \kappa \setminus A$.)

With a positive answer to $\mathcal{P}9c$ we could now prove generalisations of 5Ca and 5Da with arbitrary $\sigma$-ideals in the place of $[Y]^{<\theta}$. Another class of special cases is suggested by 3Bc.

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Index

References in **bold** type indicate definitions; references in *italics* indicate passing references.

- additivity of a measure 1D, 3A, 3G, 3H, 8A, A2C (A2Cb), A2Db; see also \( \kappa \)-additive measure (A2Ca), \( \tau \)-additive measure (A2V)
- additivity of a partially ordered set 7K, A1Ac, A1Ba, A1H, A1I, P4
- additivity of an ideal A2Cd, A2Db, P4; see also \( \kappa \)-additive ideal (A1Bf)
- antichain 2A, A1P
- Aronszajn tree see \( \kappa \)-Aronszajn tree (A1Ob)
- arrow relation 5O, 5P, A1S
- atom in a measure space A2Ac, A2Bg
- atomless Boolean algebra 9D
- atomless measure space A2Be, A2Kc
- atomlessly-measurable cardinal 1C
- Baire \( \sigma \)-algebra A2G, A3B
- Banach-Ulam problem 1A, 1B
- Baumgartner J.E. A4H
- Borel measure on a topological space A2V
- Borel measure-complete topological space 6M, A2V, A2W
- cardinal exponentiation 5E, 7P, 7Q, 9L, 9O
- ccc Boolean algebra A2Hb
- ccc p.o.set 9F
- ccc partially ordered set 7C, 7D, P4
- chains in \( \mathcal{P}N \) 5K
- club filter A1E, A4D
- cofinal set in a partially ordered set A1Ab
- cofinality of a partially ordered set 7H, 7J, 7K, 7L, 7M, 7N, 9O, A1A (A1Ab), A1Ba, A1H, A1I, P4, P5
- collectionwise normal topological space 8F, 8N, A3Cc, A3D
- compatible elements 2A, A1P
- complete filter see \( \kappa \)-complete filter (A1Bb)
- complete measure space A2Bf, A2Cd, A2Db, A2Gd, A2J, A2Kc, A2Q
- complete metric space 6M, A2W
- completely generating set in a Boolean algebra A2Ha
- consistency of axioms 1E, 2E, 2I, 2L, 8C, 8I, 9G
- covering number see \( \text{cov}(X,A) \) (A2Pa), \( \text{cov}(X,N_\mu) \), \( \text{cov}_{\text{Sh}}(\alpha,\beta,\gamma,\delta) \) (A1J)
- cylinder set A2Ga
- decomposable measure space A2Bd, A2Hb, A2Ja
- decomposable ordinal 8P, A1Dd
- Dedekind \( \sigma \)-complete Boolean algebra 8R
- dense subset of a p.o.set 2A
- density of a topological space 6M, A2Hi, A3Ea
- diagonal intersection of a family of sets 4B, A1E
- diamond see \( \mathcal{D}_\kappa \)
- discrete family of points A3Cb, A3F
- discrete family of sets 8E, A3C (A3Cb), A3F
- down-topology of a p.o.set 2A
- duality between linear spaces A2O
- elementary embedding P2
- entangled set 7E, 7F
- finite-support product of partially-ordered sets A1R
Fleissner W.G. 8M
forcing 2A
Fubini’s theorem see repeated integral
fully stationary see $\Pi^1_1$-fully stationary (4Db)
GCH 2D, 2E
Gitik M. Intro.
Gitik-Shelah theorem 3F, 9H
greatly Mahlo cardinal 4Ac, 4B, 4C, 4G, 4I, 4K, 9J
greatly Mahlo filter 4Ac, 4B, 4G, 4K
Haydon R.G. Intro.
Hilbert space 5H
homogeneous Boolean algebra A2Gf
hyperstonian space A2L (A2La)
ideal of sets A1Ba
image measure A2Db
inaccessible cardinal see weakly inaccessible (A1Da), strongly inaccessible (A1Da)
incompatible elements in a p.o.set 2A
indecomposable ordinal $8P$, A1Dd
indescribable cardinal see $\Pi^m_n$-indescribable, weakly $\Pi^m_n$-indescribable (§A4)
inner model 2D, P1
inner regular measure A2J, A2V, A2W
inverse-measure-preserving function 1J, 3H, A2Da, A2Fe, A2Gd, A2Jd, A2K, A2Lb
Jönsson algebra 5D, 9L, A1Oa
Jensen R. 1H
Juhász I. 8M
Kunen K. Intro., 1E, 2C, 3A, 4L, 4T, 5G, 5L, 5N, 5P, 8C, 8J, 9N, A4H, A4J
Lebesgue H. Intro.
Lebesgue measure Intro., 1A, 1D, A2Gb, A2Ke, P4
lifting of a measure space A2Q (A2Qa)
Lifting Theorem A2Q
locally compact topological space 8N
locally determined measure space A2B (A2Bc), A2Cd, A2J
lower integral 6A, A2E
lower semi-continuous function A2J
Luxemburg W.A.J. Intro.
Maharam homogeneous measure 1G, 1H, 2H, 3J-3M, A2H (A2Hg), A2Ka, A2Y, P2
Maharam type of a measure space 2H, 2I, 3F, 3G, 3H, 3I, 3J, 3K, 3L, 3M, 6D, 6I, 7Q, 8A, A2Hg, A2Ka,
A2Y, P2, P6
Maharam's Theorem A2I
Mahlo cardinal see weakly Mahlo (4Aa), greatly Mahlo (4Ac), $\alpha$-Mahlo (4B)
Mahlo’s operation 4A (4Ab), 4B, 4F, 9J, A4F
Martin’s Axiom 9G
Mathias A.R.D. Intro.
mearger sets in $\mathbb{R}$ P4
measurable algebra 2B, 8R, A2Fc
measurable cardinal see two-valued-measurable (1Cb)
measurable function A2Ae, A2Gg
measure algebra 6C, A2F (A2Fa), A2G, A2I, A2Jd, A2La, A2Q, P2, P6
measure algebra of a measure space A2Fd, A2La, A2U
measure space A2A
measure-preserving homomorphism A2Fe, A2Gd, A2ib, A2Jd
measure-preserving transformation 1J; see also inverse-measure-preserving (A2Da)
metric of a measure algebra A2Fb, A2Hi
metric space 6M, A2W
metrizable space 8G, A3D, A3F
Moore space 8G, A3Ca, A3D

negligible set A2Ad
NMA 8G, 8H, 8I, 8J, 8K, 8M, 8N, 8R, 8S
non-stationary ideal A1E
normal filter 1Fb, 1G, 1H, 4Ac, 4F, 9I, A1E, A4F
normal ideal 1F (1Fa), 9B, 9C, 9K, A1E, P7
normal measure 1Fc, 1G, 8H
normal measure axiom see NMA (8H)
normal topological space 8E, 8F, 8G, 8M, 8N
normal witnessing probability 1H, 1I, 2J, 3L, 6A, P2, P7

Nyikos P.J. 8M

order-closed subalgebra of a Boolean algebra A2H, A2Ic
order-continuous linear functional 6N

partially ordered set A1Aa; see also p.o.set (2A)
partition relations see arrow relation (A1S)
pcf theory A1I

PMEA 8B, 8C-8G, 8J, 8S, 9H
p.o.set 2A; see also partially ordered set (A1Aa)
precaliber of a Boolean algebra 5D, 6C, A2T, A2U

Prikry K. Intro., 6F, 9F
principal ideal in a Boolean algebra A2Gf, A2Hc

probability A2Ba
probability algebra A2Fa
probability space A2Ba
product measure A2G, P9
product measure extension axiom see PMEA (8B)
product of partially ordered sets 7C, 7D, 7H, 7I, 7K-7N, 9O, A1H, A1I, A1R, P5
purely atomic Boolean algebra 9D
purely atomic measure space A2Bg
pushing-down lemma A1E

qm filter 9C, 9J
qm ideal 9C, P7

quasi-Radon measure space 3A, 3B, 6D, A2J (A2Ja), A2V
quasi-measurable cardinal §9 (9C), P7, P8

Radon measure space 3H, 3I, 6A, 6B, 6E, 6I, 6L, 8A, A2J (A2Jb), A2K, A2L, A2Pb, A2V, P4
Radon topological space 6M, A2V, A2W
Radon-Nikodym theorem 2C, A2Fg
random real p.o.set 2B, 2C, 2J, 4O, 8Q, P1, P7
rapid filter 5G, A1Cb
real-valued-measurable cardinal 1C
rectangle algebras 5J
reduced product of partially ordered sets 7I, 7K, 7L, 7M, 9O, A1H , A1I
regressive function 1Fa, A1Db, A1E
regular cardinal 8P, A1Ab
regular open algebra 2A, 2B, 9H, A3A
regular open set A3A
relatively order-closed set 8O, 8P
relatively stationary set $8O$, $8P$
repeated integral $3A$, $5H$, $5I$, $6A$, $6I$, $6J$, $6K$, $A2Ge$
Riesz space $6N$
rvm filter $1I$, $2K$, $2L$, $4K$, $4N$, $4P$, $4Q$, $4R$, $5D$, $6L$, $7K$, $A4L$, $P3$
rvm ideal $1I$, $2J$, $2L$, $4K$, $4N$, $4P$, $4S$, $5F$, $P7$
saturated ideal see $\kappa$-saturated ideal ($A1Bf$)
scalarmy measurable function $5H$, $A2O$
second-order language $A4A$
selective ultrafilter $A1Cc$
semi-finite measure space $A2B$ ($A2Bb$)
sequential cardinal $P8$
sequentially continuous function $P8$
sequentially order-continuous functional $6N$
Shelah S. Intro., $5D$, $A1I$
Shelah covering number see $\text{cov}_{SL}(\alpha, \beta, \gamma, \delta)$ ($A1J$)
simple group $8R$
singular cardinals hypothesis $8K$, $A1N$
Solovay R.M. Intro., $1E$, $1H$, $6F$
Solovay’s construction $2C$, $2H$, $2J$, $4R$, $P1$, $P2$, $P3$, $P7$, $P9$
Souslin number of a partially ordered set $7C$, $7D$, $A1P$, $A1R$, $P4$; of a Boolean algebra $5D$, $A1P$
Souslin’s hypothesis $7D$
estationary set $4Aa$, $4F$, $A1Dc$, $A1F$, $A1G$
strictly localizable measure space $A2Bd$
strong law of large numbers $A2X$
strongly compact cardinal $8C$
strongly inaccessible cardinal $1D$, $A1Da$, $A1S$
subspace of a measure space $A2Ab$, $A2J$
successor cardinal $8P$
supercompact cardinal $8I$
third-order language $A4K$
Todorčević S. $5P$, $7F$
topology base $A3F$
totally finite measure $A2Ba$
totally ordered set $7E$, $A1Ac$
trivial measure space $1A$, $1B$, $1D$
two-valued-measurable cardinal $1Cb$, $1D$, $1E$, $1Gb$, $1Ha$, $1I$, $2D$, $2E$, $2H$, $2I$, $2K$, $2L$, $5N$, $5P$, $9G$, $A4L$, $P1$, $P7$
Ulam S. Intro.
Ulam matrix $1E$
Ulam’s Dichotomy $1E$
Ulam’s Theorem $1D$
uniform filter $A1Bc$
up-antichain in a partially ordered set $A1P$
upper integral $3A$, $6A$, $A2E$, $A2Je$
upwards-ccc partially ordered set $A1Q$
upwards-compatible elements in a partially ordered set $A1P$
usual measure on $\{0, 1\}^I$ $A2G$ ($A2Ga$), $A2I$, $A2Jc$, $A2Pb$
vector lattice $6N$
weakly compact cardinal $A1S$
weakly $\Pi^1_0$-indescribable cardinal $A4C$ ($A4Ca$), $A4D$
weakly $\Pi^1_1$-indescribable cardinal $4Dc$, $4F$, $4G$, $4H$, $4I$, $4K$, $4L$, $4R$, $6L$, $A1Ob$, $A1S$, $A4C$ ($A4Ca$), $A4D$, $A4H$, $P3$
weakly $\Pi^1_\infty$-indescribable cardinal $A_4C, A_4A$
weakly $\Pi^1_\infty$-indescribable cardinal $A_4C$ ($A_4C_a, A_4F, A_4G, A_4J$
weakly $\Pi^1_\infty$-indescribable cardinal $A_4K, A_4L$
weakly Mahlo cardinal $4Aa, 4B, 8P$
weakly inaccessible cardinal $1D, 4Aa, 8P, 9B, A_1Da$
weight of a topological space $3A, 3B, 6J, 6K, A_3Ec$

**$a\ P_4$**
\[a[a A_2Hc\]
add($P$) see additivity of a partially ordered set ($A_1Ac$)
add($\mu$) see additivity of a measure ($A_2Cb$)

**$b\ P_4$**
\[c = 2^\omega \ \ 1D, 4L, 5E, 5N, 9D, 9L, 9N, 9O, A_4D\]
cf($P$) see cofinality of a partially ordered set ($A_1Ab$)
\[\text{cov}(X,A) A_2P (A_2Pa), P_4\]
\[\text{cov}(X,N_\mu) 2H, 6B, 6L, A_2Pb, A_2U, P_4\]
\[\text{cov}_{\text{Sh}}(\alpha,\beta,\gamma,\delta) 7K, 7O, 9O, A_1Ja, A_1K\]
\[d \ 7K, P_4\]
\[\ell_\infty(X) 6N\]
\[L \ A_4A\]
\[L_\lambda \ A_4K\]
\[L(\lambda) 2D, 2K\]
\[L_\infty(\lambda) 2C, A_2FF, A_2Fg\]
\[\text{Mh}(A) \text{ see Mahlo's operation (4Ab)}\]
\[N_\mu 2I, A_2Ad ; \text{ see also cov}(X,N_\mu), \text{non}(X,N_\mu)\]
\[\text{non}(X,A) A_1Be\]
\[\text{non}(X,N_\mu) 6G, P_4\]
\[p \ P_4\]
p-point filter $5G, A_1Ca$
\[p(\kappa)\text{-point filter} 5L, A_1Cd\]
\[\text{Regr}(\kappa) 4Da\]
\[\text{S}(P) \text{ see Souslin number (A1P)}\]
\[\text{sat}(\lambda) \text{ see Souslin number (A1P)}\]
\[\text{Tr}(\kappa) 3D, 3E, 3K\]
\[\text{Tr}_\kappa(X;Y) 2F, 2G, 2H, 3D, 3M, 5D, P_2\]
\[U F 4Da, 4E, 4F\]

**$\alpha\text{-Mahlo cardinal}$**
\[\alpha \rightarrow (\beta,\gamma)^2 5O, 5P, A_1S\]
\[\Theta(\alpha,\gamma) 7K, 9O, A_1Jb, A_1K, A_1L, A_1M\]
\[\kappa\text{-additive ideal} 2D, 9B, 9C, 9E, A_1B (A_1Ba), A_1E\]
\[\kappa\text{-additive measure} 1C, A_2Ca; \text{ see also additivity of a measure (A2Cb)}\]
\[\kappa\text{-additive partially ordered set} A_1Ac\]
\[\kappa\text{-Aronszajn tree} 5F, 9M, A_1Ob\]
\[\kappa\text{-chain condition} A_1P\]
\[\kappa\text{-complete filter} 1C, 1G, A_1Bb, A_1E\]
\[\kappa\text{-measure-bounded partially ordered set} 7G, 7H, 7I, 7J\]
\[\kappa\text{-saturated ideal} 2D, 9A, 9B, 9E, 9I, 9P, A_1Bf\]
\[\mu f^{-1} A_2Db\]
\[ \mu[A A^{2Ab}] \]
\[ \mu^* A^{2Aa} \]
\[ \Pi_1^1 \text{-filter } 4Dc, 4F, 4G, 4I, 6L, A^{4Cb} \]
\[ \Pi_1^1 \text{ formula } 4T, A^{4B} \]
\[ \Pi_1^1 \text{-fully stationary } 4Db, 4Dc, 4K, A^{4H} \]
\[ \Pi_1^1 \text{-ideal } A^{4Cb}, A^{4H} \]
\[ \Pi_0^1 \text{-filter } A^{4Cb}, A^{4F}, A^{4G}, P3 \]
\[ \Pi_0^1 \text{ formula } A^{4B}, A^{4E} \]
\[ \Pi_0^1 \text{-ideal } A^{4Cb} \]
\[ \Pi_0^2 \text{-filter } 4Q, 4R, A^{4Cb} \]
\[ \Pi_0^2 \text{ formula } 4O, 8Q, A^{4K} \]
\[ \Pi_0^2 \text{-ideal } 4P, A^{4Cb} \]
\[ \Pi_0^2 \text{-filter } A^{4K}, A^{4L} \]
\[ \Pi_0^2 \text{-indescribable cardinal } A^{4K}, A^{4L} \]
\[ \sigma \text{-finite measure space } A^{2Ba}, A^{2Bb} \]
\[ \sigma \text{-ideal } 9A, P9 \]
\[ \Sigma_1^1 \text{ formula } 4S, A^{4B} \]
\[ \Sigma_0^1 \text{ formula } A^{4B} \]
\[ \tau \text{-additive measure } A^{2V}, A^{2W} \]
\[ \tau \text{-homogeneous Boolean algebra } A^{2He}, A^{2Hf}, A^{2la} \]
\[ \tau(3) A^{2H} (A^{2Ha}), A^{2I} \]
\[ \chi(a) A^{2Ff} \]
\[ \chi(x, X) 8F, 8M, A^{3Eb} \]
\[ \omega \text{-Tukey function } 7G, 7H, 7I \]
\[ \omega_1 \text{-saturated ideal } 9C, 9H \]
\[ \models A^{4A} \]
\[ \{0, 1\}^7 A^{2G} \]
\[ \diamondsuit 5N, 9N \]
\[ \rightarrow \text{ see } \alpha \rightarrow (\beta, \gamma)^2 (A^{1S}) \]
\[ \lceil \text{ see subspace measure } (A^{2Ab}) \]

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**Supplement to ‘Real-valued-measurable cardinals’**

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This note contains further results on real-valued-measurable cardinals, supplementing my paper ‘Real-valued-measurable cardinals’ (pp. 151-304 in Israel Math. Conference Proc. 6 (1993), ed. H. Judah), with which it should be read. References of the form ‘7C’, ‘A2J’ are directions to paragraphs in that paper, and unexplained notation is defined there. References of the form ‘SA2Ac’ are directions to paragraphs below.

Section numbers follow those of ‘Real-valued measurable cardinals’, except that a new section S6’ (‘Topological implications’) has been added.

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**S5. Combinatorial implications**

**S5A** The arguments of 5E may be refined, as follows.
Theorem  Let $\kappa$ be an atomlessly-measurable cardinal, and $\omega \leq \delta < \lambda < \kappa$. If $2^\delta < \kappa^{\delta+}$, then $2^\lambda = 2^\delta$.

proof: Induce on $\lambda$.

(a) For the inductive step to $\lambda$, where $\text{cf}(\lambda) > \delta$, choose an injection $\theta_\xi : \mathcal{P}(\xi) \rightarrow \xi$ for each $\xi < \lambda$; set $g_\lambda(\xi) = \theta_\xi(A \cap \xi)$ for $A \subseteq \lambda$, $\xi < \lambda$. If $2^\lambda > 2^\delta$, let $N \subseteq 2^\delta$ be a set of minimal cardinal such that

$$A_N = \{A : A \subseteq \lambda, \ g_\lambda^{-1}[N] \text{ is cofinal with } \lambda\}$$

has cardinal greater than $2^\delta$. Then $\text{cf}(\lambda) \leq \text{cf}(\#(N)) \leq \lambda$. If $\gamma < \lambda$ and $N = \bigcup_{\alpha < \gamma} N_\alpha$ then $A_N = \bigcup_{\alpha < \gamma} A_{N_\alpha}$. (ii) If $\gamma > \lambda$ is regular and $\{N_\alpha\}_{\alpha < \gamma}$ is an increasing family with union $N$, then again $A_N = \bigcup_{\alpha < \gamma} A_{N_\alpha}$.

But there is no cardinal $\gamma$ such that $\kappa < \gamma < \kappa^{\delta+}$ and $\delta < \text{cf}(\gamma) \leq \lambda$, so $\#(N) < \kappa$.

Let $h : \kappa \rightarrow A_N$ be injective. For $\alpha < \beta < \kappa$ set $f(\{\alpha, \beta\}) = \min\{\zeta : h(\alpha) \cap \zeta \neq h(\beta) \cap \zeta\}$. By 5Ca, there is a set $C \subseteq \kappa$ such that $\#(C) = \kappa$ and $M = \{f(I) : I \in [C]^2\}$ is countable, therefore bounded above in $\lambda$: set $\zeta = \sup M$, so that $h(\alpha) \cap \zeta \neq h(\beta) \cap \zeta$ for all distinct $\alpha, \beta \in C$. Next, there is a $\xi \in \lambda \setminus C$ such that $C_1 = \{\alpha : \alpha \in C, g_h(\alpha)(\xi) \in N\}$ has cardinal $\kappa$. But now $\alpha \rightarrow g_h(\alpha)(\xi) = h(\alpha \cap \xi)$ is an injective function from $C_1$ to $M$, while $\#(M) < \kappa = \#(C_1)$.

This deals with the inductive step if $\text{cf}(\lambda) > \delta$.

(b) If $\text{cf}(\lambda) \leq \delta$, then $2^\lambda \leq (\sup_{\gamma < \lambda} 2^{\text{cf}(\lambda)}) = 2^\delta$, so the induction proceeds.

S5B Proposition Suppose there is an atomlessly-measurable cardinal $\kappa$. Then there is a family $\{A_\alpha\}_{\alpha < \omega_1}$ of sets such that

(i) $A_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$;

(ii) if $\alpha \leq \beta < \omega_1$, then $A_\alpha \triangle (A_\beta \cap \alpha)$ is finite;

(iii) if $C \subseteq \omega_1$ is uncountable, then

$$\{\alpha : C \cap A_\alpha \text{ is infinite}\}, \quad \{\alpha : (C \cap A_\alpha) \setminus A_\alpha \text{ is infinite}\}$$

are both uncountable.

proof (a) Choose a family $\{e_\alpha\}_{\alpha < \omega_1}$ such that (i) each $e_\alpha$ is an injective function from $\alpha$ to $\mathbb{N}$, (ii) if $\alpha \leq \beta$ then $\{\xi : \xi < \alpha, e_\alpha(\xi) \neq e_\beta(\xi)\}$ is finite (S1A1). Let $\nu$ be an extension of the usual measure $\mu$ of $\mathcal{P}(\mathbb{N})$ (corresponding to the usual measure on $\{0, 1\}^\mathbb{N}$) to $\mathcal{P}(\mathcal{P}(\mathbb{N}))$. For each $\alpha \subseteq \mathbb{N}$ set $A_{e_\alpha} = e_\alpha^{-1}[\alpha] \subseteq \alpha$.

Then whenever $\alpha \leq \beta < \omega_1$,

$$A_{e_\alpha} \triangle (A_\beta \cap A_{e_\alpha}) \subseteq \{\xi : \xi < \alpha, e_\alpha(\xi) \neq e_\beta(\xi)\}$$

is finite. Thus every $\alpha \subseteq \mathbb{N}$ provides a family satisfying conditions (i) and (ii).

(b) Let $S$ be the set of those $\alpha \subseteq \mathbb{N}$ for which there is some uncountable set $C_\alpha \subseteq \omega_1$ such that $\{\alpha : C_\alpha \cap A_{e_\alpha} \text{ is infinite}\}$ is countable. Suppose, if possible, that $\nu S > 0$. Then there is an $\alpha_0 < \omega_1$ such that

$$\{\alpha : C_\alpha \cap A_{e_\alpha} \text{ is infinite}\} \leq \alpha_0$$

for $\nu$-almost every $\alpha \in S$; set

$$T = \{\alpha : \alpha \in S, C_\alpha \cap A_\alpha \text{ is finite for every } \alpha \geq \alpha_0\}.$$

For each $\xi < \omega_1$, set $T_\xi = \{\alpha : \alpha \in T, \ \xi \in C_\alpha\}$; then

$$0 < \nu S = \nu T \leq \sum_{\xi \geq \beta} \nu T_\xi$$

for every $\beta < \omega_1$, so $\{\xi : \nu T_\xi > 0\}$ is uncountable and there is a $\delta > 0$ such that $B$ is uncountable, where $B = \{\xi : \nu T_\xi \geq \delta\}$.

By S6F (I apologize for the forward reference, but you can easily check that there is no circularity) there is an infinite $D \subseteq B$ such that $\mu^*(\bigcap_{\xi \in D} T_\xi) > 0$. Take $\alpha \geq \alpha_0$ such that $D \subseteq \alpha$. If $a \in \bigcap_{\xi \in D} T_\xi$, then $D \subseteq C_a$, while $C_a \cap A_{e_\alpha}$ is finite, so $D \cap A_{e_\alpha}$ is finite. But

$$E = \{a : D \cap A_{e_\alpha} \text{ is finite}\} = \{a : a \cap e_\alpha[D] \text{ is finite}\}$$
is $\mu$-measurable and has zero measure, so $\mu^*(\bigcap_{\xi \in D} T_\xi) = 0$, which is impossible. \textbf{X} Thus $\nu S = 0$.

(c) Now consider the set $S'$ of those $a \subseteq \mathbb{N}$ for which there is some uncountable set $C_a$ such that $\{\alpha : (C_a \cap \alpha) \setminus A_{a\alpha} \text{ is infinite}\} \subseteq \alpha_0$. Suppose, if possible, that $\nu S' > 0$. Then there is an $\alpha_0 < \omega_1$ such that

\[ \{\alpha : C_a \cap \alpha \setminus A_{a\alpha} \text{ is infinite}\} \subseteq \alpha_0 \]

for $\nu$-almost every $a \in S'$; set

\[ T = \{ a : a \in S, C_a \setminus A_{a\alpha} \text{ is finite for every } \alpha \geq \alpha_0 \}. \]

For each $\xi < \omega_1$, set $T_\xi = \{ a : a \in T, \xi \in C_a \}$; then

\[ 0 < \nu S' = \nu T \leq \sum_{\xi \geq \beta} \nu T_\xi \]

for every $\beta < \omega_1$, so $\{\xi : \nu T_\xi > 0\}$ is uncountable and there is a $\delta > 0$ such that $B$ is uncountable, where

\[ B = \{\xi : \nu T_\xi \geq \delta\}. \]

By S6F there is an infinite $D \subseteq B$ such that $\mu^*(\bigcap_{\xi \in D} T_\xi) > 0$. Take $\alpha \geq \alpha_0$ such that $D \subseteq A_{\alpha\alpha}$. If $a \in \bigcap_{\xi \in D} T_\xi$, then $D \subseteq C_a$, while $C_a \cap \alpha \setminus A_{a\alpha}$ is finite, so $D \setminus A_{a\alpha}$ is finite. But

\[ E = \{ a : D \setminus A_{a\alpha} \text{ is finite} \} = \{ a : e_{a[D]} \setminus a \text{ is finite} \} \]

is $\mu$-measurable and has zero measure, so $\mu^*(\bigcap_{\xi \in D} T_\xi) = 0$, which is impossible. \textbf{X} Thus $\nu S' = 0$.

(d) There must therefore be an $a \subseteq \mathbb{N}$ not belonging to either $S$ or $S'$, and the corresponding family $(A_{a\alpha})_{\alpha < \omega}$ has the required property.

\textbf{Remark} This comes from TODORČEVIĆ 93, §7.

S6’. Topological implications

S6’A Proposition Suppose there is an atomlessly-measurable cardinal. Then there is a locally compact locally countable Hausdorff topology $\mathcal{T}$ on $\omega_1$ such that every closed set is either countable or cocountable.

\textbf{proof} (a) Let $\kappa$ be an atomlessly-measurable cardinal, and $\nu_0$ a Maharam homogeneous atomless probability measure with domain $\mathcal{P}_{\kappa}$; let $\lambda$ be the Maharam type of $\nu$. By 3F, $\lambda > \kappa > \omega_1$. Let $g : \kappa \to \{0,1\}^\lambda$ be a function representing an isomorphism from the measure algebra of the usual measure on $X = \{0,1\}^\lambda = X$ and the measure algebra of $\nu_0$ (A2Gd). Let $\nu$ be the image measure $\nu_0 g^{-1}$, so that $\nu$ extends the usual measure on $X$, and setting $E_\xi = \{ x : x \in X, \nu_0(\xi) = 1 \}$ for $\xi < \kappa$, $(E_\xi)_{\xi < \kappa}$ is a stochastic independence family of sets of measure $\frac{1}{2}$ such that $\{ E_\xi : \xi < \lambda \}$ $\tau$-generates the measure algebra of $\nu$. For $\Gamma \subseteq \lambda$ write $\Sigma_\Gamma$ for the $\sigma$-algebra generated by $\{ E_\xi : \xi \in \Gamma \} \cup \{ E : \nu_0E = 0 \}$, so that $\mathcal{P}X = \bigcup \{ \Sigma_\Gamma : \Gamma \subseteq \lambda \}$. Write $\Omega$ for the set of non-zero countable limit ordinals; for $\alpha \in \Omega$, enumerate $\alpha$ as $\langle e_a(n) \rangle_{n \in \mathbb{N}}$. Fix a bijection $\phi : \{0,1\}^\Omega \to [0,1]^\mathbb{N}$ which is an automorphism for the usual measures (Fremlin p007, 341I). For $\alpha \in \Omega$, $n \in \mathbb{N}$ define $\phi_{an} : X \to [0,1]$ by writing $\langle \phi_{an}(x) \rangle_{n \in \mathbb{N}} = \phi(\langle x(\alpha + n) \rangle_{n \in \mathbb{N}})$ for every $x \in X$.

(b) For $x \in X$, $\xi < \omega_1$ choose $B_\xi(\xi)$ as follows. If $\xi < \omega$ then $B_\xi(\xi) = \{ \xi \}$. Suppose that $B_\xi(\xi)$ has been defined for every $\xi < \alpha$, where $\alpha \in \Omega$. Set $C_\alpha(x,n) = B_x(e_a(n)) \setminus \bigcup_{i \leq n} B_x(e_a(i))$ for each $n$. Set

\[ B_\alpha(x+n) = \{x + n\} \cup \{ C_\alpha(x,i) : i \in \mathbb{N}, 2^{-n-1} \leq \phi_{an}(x) \leq 2^{-n} \} \]

for each $n$.

Let $\mathcal{T}_x$ be the topology on $\omega_1$ generated by $\{ B_\alpha(\xi) : \xi < \omega_1 \} \cup \{ \omega_1 \setminus B_\xi(\xi) : \xi < \omega_1 \}$. Then $\mathcal{T}_x$ is Hausdorff and locally countable. \textbf{P} It is Hausdorff because if $\xi < \eta$ then $\xi \in B_\xi(\eta)$ and $\eta \notin B_\xi(\xi)$. (The point is that if $\alpha$ is a limit ordinal and $n \in \mathbb{N}$, then

\[ \alpha + n \in B_\alpha(x+n) \subseteq \{ \alpha + n \} \cup \alpha. \]

It is locally countable because every $B_\alpha(\xi)$ is countable. \textbf{Q}

(c) For $\alpha \in \Omega$, let $\mathcal{T}_{x\alpha}$ be the topology on $\alpha$ generated by $\{ B_\alpha(\xi) : \xi < \alpha \} \cup \{ \alpha \setminus B_\alpha(\xi) : \xi < \alpha \}$. Then an easy induction shows that $C_\alpha(\beta,n) \cap \alpha$ and $B_\alpha(\eta) \cap \alpha$ are open-and-closed in $\mathcal{T}_{x\alpha}$ for every $\beta \in \Omega$, $n \in \mathbb{N}$.
and $\eta \in \omega_1$, so that $\tau_{\omega_1}$ is the subspace topology on $\alpha$ induced by $\tau_x$. Now $B_x(\xi)$ is $\tau_{\omega_1}$-compact whenever $\alpha \in \Omega$, $n \in \mathbb{N}$ and $\xi < \alpha$. P Induce on $\alpha$. For $\alpha = \omega$, every $B_x(\xi)$ is a singleton, so the induction starts. If $\alpha = \beta + \omega$, where $\beta \in \Omega$, and $\xi < \beta$ either $\xi < \beta$ so $B_x(\xi) \subseteq \beta$ is $\tau_{\omega_1}$-compact, therefore $\tau_{\omega_1}$-compact, or $\xi = \alpha + n$ for some $n$. In this case, $B_x(e_0(n))$ is always $\tau_{\omega_1}$-compact and $\tau_{\omega_1}$-open, so $\{C_x(\beta,i)\}_{i \in \mathbb{N}}$ is a partition of $\beta$ into $\tau_{\omega_1}$-compact and $\tau_{\omega_1}$-open sets, and $B_x(\beta + n)$ must be $\tau_{\omega_1}$-compact. If $\alpha$ is a limit member of $\Omega$, then for every $\xi < \alpha$ there is a $\beta \in \Omega$ such that $\xi < \beta < \alpha$, and the inductive hypothesis tells us that $B_x(\xi)$ is $\tau_{\omega_1}$-compact, therefore $\tau_{\omega_1}$-compact. Thus the induction proceeds. Q

It follows that every $B_x(\xi)$ is $\tau_{\omega_1}$-compact, so that $\tau_x$ is locally compact, for every $x \in X$.

(d) The essential fact to note about this construction is that $\{x : \xi \in B_x(\eta)\}$, $\{x : \xi \in C_x(\beta,n)\}$ belong to $\Sigma_\alpha$ whenever $\alpha \in \Omega$, $\xi < \omega_1$, $\eta < \alpha$, $\beta \in \alpha \cap \Omega$ and $n \in \mathbb{N}$.

(e) Suppose, if possible, that for every $x \in X$ there is a set $D_x \subseteq \omega_1$ which is $\tau_x$-closed but neither countable nor cocountable. For $\xi < \omega_1$ set $H_\xi = \{x : \xi \notin D_x\} \subseteq X$. Then there is a countable set $\Gamma \subseteq \omega_1$ such that $H_\xi \subseteq \Sigma_{\Gamma_\xi}$. Note also that, for any $\xi < \omega_1$, $\bigcup_{\eta > \xi} H_\eta = X$, so there is a countable $\Delta_\xi \subseteq \omega_1 \setminus \xi$ such that $\bigcup_{\eta \in \Delta_\xi} H_\eta$ is negligible. Let $A$ be

\[\{\alpha : \alpha \in \Omega, (\Gamma_\alpha \cap \omega_1) \cup \Delta_\xi \subseteq \xi \text{ for every } \xi < \alpha\}\]

then $A$ is a club set in $\omega_1$. Take any $\alpha \in A$. Set $I = \bigcup_{\xi < \alpha} \Gamma_\xi$, so that $\Gamma \cap \omega_1 \subseteq \alpha$.

Let $Y$ be $\{y : y \in X, D_y \cap \alpha \text{ is cofinal with } \alpha\}$, so that $Y \supseteq \bigcap_{\xi < \alpha} \bigcup_{\eta \in \Delta_\xi} H_\eta$ is countable. For every $x \in X, D_x$ does not include $\omega_1 \setminus \alpha$, so there is a least $\zeta_x \geq \alpha$ such that $\zeta_x \notin D_x$; let $\zeta \in \omega_1 \setminus \alpha$ be the least ordinal such that $Q = \{x : \zeta_x = \zeta\}$ is not negligible. Set $Y_1 = \{y : y \in Y, \zeta_y \geq \zeta\}$, so that $Y_1$ is negligible. Express $\zeta$ as $\beta + n$ where $\beta \in \omega_1, n \in \mathbb{N}$. Observe that $H_\xi \subseteq \Sigma_{\Gamma_\xi}$, for every $\xi \in \beta \setminus \alpha$, so $H_\xi \subseteq \Sigma_{\Gamma_\xi}$ for every $\xi < \beta$.

For $x \in X$, set $I_x = \{m : m \in \mathbb{N}, D_x \cap C_x(\beta,m) \neq \emptyset\}$. Observe that

\[\{x : m \in I_x\} = \bigcup_{\xi < \beta} \{x : \xi \in D_x \cap C_x(\beta,m)\} \subseteq \Sigma_{\Gamma_\beta}\]

for every $m \in \mathbb{N}$. If $y \in Y_1, D_y \cap \beta$ is cofinal with $\beta$, whether or not $\beta = \alpha$, because $D_y \cap \alpha$ is cofinal with $\alpha$ and $\beta \setminus \alpha \subseteq D_y$. Since

\[\sup_{C_y(\beta,m)} \leq \sup_{B_y(\xi_\beta(m))} = e_\zeta(m) < \beta\]

for every $m$, $I_y$ is infinite, for every $y \in Y_1$.

On the other hand, setting $F_m = \{x : 2^{-n-1} \leq \phi_m(x) < 2^{-n}\}$ for $m \in \mathbb{N}, \langle F_m\rangle_{m \in \mathbb{N}}$ is a stochastically independent sequence of sets of measure $2^{-n-1}$ all belonging to $\Sigma_{(\beta+\omega)\beta}$, which is independent from $\Sigma_{\Gamma_\beta}$. Set $J_x = \{m : x \in F_m\}$. Then

\[G = \{x : I_x \text{ is infinite, } I_x \cap J_x \text{ is finite}\}\]

is negligible. P Set $h_1(x) = I_x, h_2(x) = J_x, h(x) = (I_x, J_x)$, so that $h_1$ and $h_2$ are functions from $X$ to $\mathcal{P}\mathbb{N}$. Let $\mu_1, \mu_2$ be the Radon measures on $\mathcal{P}\mathbb{N}$ such that $h_1$ and $h_2$ are inverse-measure -preserving. Then $h_1$ is inverse-measure -preserving for $\nu|\Sigma_{\Gamma_\beta}$ and $\mu_1$, and $h_2$ is inverse-measure -preserving for $\nu|\Sigma_{\Gamma_\beta}$. Therefore, $h$ is inverse-measure -preserving for $\nu$ and the product measure $\mu_1 \times \mu_2$ on $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Now the set

\[\{(I, J) : I \in [\mathbb{N}]^\omega, J \subseteq I, I \cap J \text{ is finite}\}\]

is a Borel set in $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ which has $\mu_2$-negligible vertical sections, so is $\mu_1 \times \mu_2$-negligible, and its inverse image $G$ is $\nu$-negligible. Q

There is therefore some $y \in Q \cap Y_1 \setminus G$. But in this case $\zeta \notin D_y$, so $B_y(\zeta) \cap D_y$ is a $\tau_y$-compact set not containing $\zeta$. Since

\[B_y(\zeta) = \{\zeta\} \cup \bigcup_{m \in J_y} C_y(\beta, m),\]

there must be a finite set $K \subseteq J_y$ such that $B_y(\zeta) \cap D_y \subseteq \bigcup_{m \in K} C_y(\beta, m)$. Since $J_y$ is infinite and $y \notin G$, there is an $m \in I_y \cap J_y \setminus K$, and $C_y(\beta, m) \cap D_y$ is not empty; which is impossible. X

This contradiction shows that for at least one $x \in X$, every $\tau_x$-closed set is either countable or cocountable.

**Remarks** This proposition is based on ideas of J.Moore.

Observe that, for a topology $\tau$ as described, $\omega_1$ is hereditarily separable. P? If $A \subseteq \omega_1$ is not separable, of course it is uncountable. For $\xi < \omega_1$, choose $\alpha_\xi, \beta_\xi$ in $A$, and countable open sets $G_\xi \subseteq \omega_1$ inductively,
as follows. Given \((\alpha_n)_{n<\xi}\), let \(\beta_\xi \in A\) be such that \(\beta_\xi \notin \{\alpha_n : n < \xi \} \cup (A \cap \xi)\). Let \(G_\xi\) be a countable open set containing \(\beta_\xi\) and not meeting \(\{\alpha_n : n < \xi \}\). Given \((G_\eta)_{\eta \leq \xi}\), take \(\alpha_\xi \in A \setminus (\xi \cup \bigcup_{\eta \leq \xi} G_\eta)\).

Now consider \(F = \{\alpha_\xi : \xi < \omega_1\}\). Then \(\alpha_\xi \in F\) and \(\beta_\xi \notin F\) for every \(\xi\). Since \(\alpha_\xi, \beta_\xi \geq \xi\) for every \(\xi\), \(F\) is neither countable nor co-countable, which is impossible. \(\square\)

### S6. Measure-theoretic implications

**S6A Proposition** Let \(\kappa\) be an atomlessly-measurable cardinal and \(\lambda < \kappa\) a cardinal. Then there is a cardinal \(\theta < \kappa\) such that whenever \(X\) is a \(T_0\) topological space of cardinal at least \(\theta\) and weight at most \(\lambda\) then there is an atomless \(\tau\)-additive Borel probability on \(X\).

**proof** Suppose, if possible, otherwise.

(a) We are supposing that for each \(\zeta < \kappa\) there is a \(T_0\) space \(X_\zeta\), of cardinal at least \(#(\zeta)\) and of weight at most \(\lambda\), with no atomless \(\tau\)-additive Borel probability. Note that any \(\tau\)-additive Borel probability \(\bar{\mu}\) on any subspace \(D\) of \(X_\zeta\) gives rise to a \(\tau\)-additive Borel probability \(\bar{\mu}_1\) on \(X_\zeta\), writing \(\bar{\mu}_1E = \bar{\mu}(D \cap E)\) for every Borel set \(E \subseteq X_\zeta\), which would be atomless if \(\bar{\mu}\) were atomless; so we may take it that \(#(X_\zeta)\) is precisely \(#(\zeta)\). Let \((\bar{U}_{\zeta})_{\zeta < \lambda}\) run over a base for the topology of \(X_\zeta\). Define \(f_\zeta : X_\zeta \to \{0,1\}^\lambda\) by writing

\[
f_\zeta(x)(\xi) = 1 \text{ if } x \in U_{\zeta,\xi}, \text{ 0 if } x \notin U_{\zeta,\xi}.
\]

Note that \(f_\zeta\) is injective, because \(X_\zeta\) is a \(T_0\) space, so we may enumerate \(Y_\zeta = f_\zeta[X_\zeta]\) as \((y_{\zeta,\xi})_{\xi < \zeta}\).

(b) Let \(\nu\) be a normal witnessing probability on \(\kappa\). Define \(\mu : \mathcal{P}(X \times \kappa) \to [0,1]\) by setting

\[
\mu(E) = \int \nu(\{x : (y_{\zeta,\xi}, \xi) \in E\})\nu(d\zeta)
\]

for \(E \subseteq X \times \kappa\). Then \(\mu\) is a \(\kappa\)-additive probability on \(X \times \kappa\). By SA2B below there is a family \((\mu_\zeta)_{\zeta < \kappa}\) of Radon probabilities on \(X\) such that \(\mu(H \times A) = \int_A \mu_\zeta(H)\nu(d\zeta)\) for every set \(H\) in the Baire \(\sigma\)-algebra of \(X\) and every \(A \subseteq \kappa\).

Let \(\mathcal{V}\) be the algebra of open-and-closed subsets of \(X\).

(c) For any \(C \subseteq X \times \kappa\), \(\mu C \leq \int \mu_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta)\). \(\square\) The set \(W\) of sets expressible as a finite disjoint union of sets \(V \times A\), where \(V \in \mathcal{V}\) and \(A \subseteq \kappa\), is a subalgebra of \(\mathcal{P}(X \times \kappa)\), and for every \(W \in \mathcal{W}\) we have

\[
\mu W = \int \nu(\{y_{\zeta,\xi} : y \in V_{\zeta,\xi} \subseteq \kappa\})\nu(d\zeta).
\]

Now take \(\epsilon > 0\). For each \(\zeta < \kappa\) choose an open set \(G_\zeta \supseteq C^{-1}[\{\zeta\}]\) such that \(\mu_\zeta G_\zeta \leq \mu_\zeta C^{-1}[\{\zeta\}] + \epsilon\). Set

\[
E = \{ (y,\zeta) : \zeta < \kappa, y \in G_\zeta \} = \bigcup_{V \in \mathcal{V}} \{ (y,\zeta) : y \in V \subseteq \kappa \}.
\]

Then \(C \subseteq E\) so \(\mu C \leq \mu E\). Because \(\mu\) is \(\kappa\)-additive and \(#(\mathcal{V}) \leq \max(\lambda, \omega) < \kappa\), there is a finite \(\mathcal{V}_0 \subseteq \mathcal{V}\) such that \(\mu W \geq \mu E - \epsilon\), where

\[
W = \bigcup_{V \in \mathcal{V}_0} \{ (y,\zeta) : y \in V \subseteq \kappa \}.
\]

But \(W \in \mathcal{W}\), so

\[
\mu W = \int \mu_\zeta W^{-1}[\{\zeta\}]\nu(d\zeta) \leq \int \mu_\zeta G_\zeta\nu(d\zeta).
\]

Putting these together,

\[
\mu C \leq \int \mu_\zeta G_\zeta\nu(d\zeta) + \epsilon \leq \int \mu_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta) + 2\epsilon.
\]

As \(\epsilon\) is arbitrary, we have the result. \(\square\)

Taking complements, we see now that

\[
\mu C \geq \int (\mu_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta)
\]

for every \(C \subseteq X \times \kappa\), writing \((\mu_\zeta)_D = \sup \{ \mu_\zeta H : H \subseteq D \} \) is \(\mu_\zeta\)-measurable for every \(D \subseteq X\), \(\zeta < \kappa\).

(d) \(\mu_\zeta Y_\zeta = 1\) for \(\nu\)-almost every \(\zeta\). \(\square\) Set \(C = \{ (y,\zeta) : \zeta < \kappa, y \in Y_\zeta \} \subseteq X \times \kappa\). Then \((y_{\zeta,\xi})_{\zeta < \lambda} \in C\) whenever \(\xi < \zeta < \kappa\), so \(\mu C = 1\). By (c),
Thus $\mu_\zeta Y_\zeta = 1$ for almost all $\zeta$. Q

(b) $\mu_\zeta$ is atomless for almost all $\zeta \in A_0$. Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$.

(c) $\mu_\zeta$ is atomless for almost all $\zeta \in A_0$. Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$.

(d) $\mu_\zeta$ is atomless for almost all $\zeta \in A_0$. Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$.

(e) $\mu_\zeta$ is atomless for almost all $\zeta \in A_0$. Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$.

(f) $\mu_\zeta$ is atomless for almost all $\zeta \in A_0$. Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$.

Set $A_0 = \{ \zeta : \mu_\zeta Y_\zeta = 1 \}$. If $\zeta \in B$, then (because $\mu_\zeta$ is a Radon measure) there is a $y \in X$ such that $\mu_\zeta \{ y \} > 0$; but as $\mu_\zeta Y_\zeta = 1$, $y \in Y_\zeta$ and $(y, \zeta) \in C$. For each $\zeta < \kappa$, $\nu C^{-1}\{ \zeta \}$ is countable, so is expressible as $\{ y_\zeta : \zeta \in I_\zeta \}$, where $I_\zeta \subseteq \zeta$ is countable. Let $I \subseteq \kappa$ be a countable set such that $I_\zeta \subseteq I$ for almost all $\zeta$ (5Ab). Now $\nu \{ \zeta : y_\zeta \in C \} = 0$ for every $\zeta \in \kappa \setminus I$, so $\mu C = 0$. Consequently, by the last remark of (c) above,

$$\int (\mu_\zeta)_* C^{-1}\{ \zeta \} \nu(d\zeta) = 0.$$  

But of course $(\mu_\zeta)_* C^{-1}\{ \zeta \} > 0$ for every $\zeta \in B$, so $\nu B = 0$, as claimed. Q

This contradiction completes the proof.

S6B Corollary Let $\kappa$ be an atomlessly-measurable cardinal and $\lambda < \kappa$ a cardinal. Then there is a cardinal $\theta < \kappa$ such that

(a) whenever $X$ is a regular $T_0$ topological space of cardinal at least $\theta$ and weight at most $\lambda$ then there is an atomless quasi-Radon probability on $X$;

(b) whenever $Z$ is a compact Hausdorff space of weight at most $\lambda$ and $X \subseteq Z$ has cardinal at least $\theta$, then there is an atomless Radon probability $\mu$ on $Z$ such that $\mu^* X = 1$.

proof Take $\theta$ from S6A.

(a) There is an atomless $\tau$-additive Borel probability on $X$, which by SA2A extends to a quasi-Radon probability on $X$.

(b) There is an atomless $\tau$-additive Borel probability $\mu_0$ on $X$; setting $\mu E = \mu_0(X \cap E)$ for Borel sets $E \subseteq Z$, we get a $\tau$-additive Borel probability on $Z$ for which $\mu^* X = 1$; by SA2A the completion $\mu_0^* \mu$ is quasi-Radon, and of course $\mu_0$ is atomless and $\mu_0^* X = 1$. Because $Z$ is compact, $\mu_0$ is inner regular for the compact sets and is therefore a Radon measure.

S6C Corollary Let $\kappa$ be an atomlessly-measurable cardinal. Then there is a $\theta < \kappa$ such that no subset of $\mathbb{R}$ of cardinal $\theta$ or more can be universally negligible.

proof Apply S6Bb with $\lambda = \omega$, $X = [a, b]$, where $a \leq b$ in $\mathbb{R}$.

S6D Proposition If $\kappa$ is an atomlessly-measurable cardinal and $(X, \mu)$ is an atomless Radon probability space of Maharam type at most $\kappa$, then $\text{cf}(\mathcal{N}_\mu) = \text{cf}(\mathcal{N})$, where $\mathcal{N}$ is the Lebesgue null ideal.

proof Use S7B below and 6.14 of Fremlin 89.

S6E Lemma Let $\kappa$ be an atomlessly-measurable cardinal, and $\lambda \leq \kappa$ a cardinal of uncountable cofinality. Let $(X, \mu)$ be a Radon probability space and $(E_\xi)_{\xi < \lambda}$ a family of non-negligible measurable subsets of $X$. Then there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi \}) = \lambda$. 

\[ 1 = \mu C \leq \int \mu_\zeta C^{-1}\{ \zeta \} \nu(d\zeta) = \int \mu_\zeta Y_\zeta \nu(d\zeta). \]
proof By 6C, $\lambda$ is a precaliber of the measure algebra $\mathfrak{A}$ of $(X, \mu)$. For each $\xi < \kappa$, choose a compact non-negligible $F_\xi \subseteq E_\xi$. Then there is a set $D \in |X|^\lambda$ such that $\{F_\xi^* : \xi \in D\}$ is centered. But this means that $\{F_\xi : \xi \in D\}$ has the finite intersection property; because the $F_\xi$ are compact, there is a point in $\bigcap_{\xi \in D} F_\xi$, and this will serve for $x$.

**S6F Lemma** Let $\kappa$ be an atomlessly-measurable cardinal. Let $(X, \Sigma, \mu)$ be a probability space, and $\Sigma_0$ a $\sigma$-subalgebra of $\Sigma$; set $\mu_0 = \mu|\Sigma_0$ and let $\mu_0^+$ be the outer measure on $X$ defined from $\mu_0$. Suppose that the Maharam type $\tau_0$ of $\mu_0$ is less than $\kappa$. Let $\langle E_\xi \xi \in \Lambda \rangle$ be a family in $\Sigma$ with $\inf_{\xi \in \Lambda} \mu_\xi E_\xi = \gamma > 0$, where $\lambda > \max(\omega, \tau_0)$. Then for any $\gamma' < \gamma$ there is an infinite $W \subseteq \lambda$ such that $\mu_0^+\left(\bigcap_{\xi \in W} E_\xi\right) \geq \gamma'$.

**Remark** Compare A2S.

**proof** It is enough to consider the case $\lambda = \max(\omega_1, \tau_0^+)$, so that $\text{cf}(\lambda) > \omega$ and $\lambda \leq \kappa$.

(a) Consider first the special case in which $X = \{0, 1\}$ for some set $I$, with $\mu$ the usual measure on $X$, and there is a $J \subseteq I$ of cardinal $\tau_0$ such that $\Sigma_0 = \{\pi^{-1}_J[F] : F \in \Sigma_J\}$, writing $\Sigma_J$ for the domain of the usual measure $\mu_J$ on $\{0, 1\}^J$, and $\pi_J : X \to \{0, 1\}^J$ for the canonical map.

We may regard $X$ as the product $\{0, 1\}^J \times \{0, 1\}^{I \setminus J}$. For $\xi < \lambda$, $z \in \{0, 1\}^{I \setminus J}$ set

$$E_{\xi, z} = \{y : y \in \{0, 1\}^J, (y, z) \in E_\xi\}.$$ 

Then

$$\int \mu_J E_{\xi, z} \mu_{IJ, J}(dz) = \mu E_\xi \geq \gamma$$

by Fubini’s theorem (A2Ge). Set

$$G_\xi = \{z : z \in \{0, 1\}^{I \setminus J}, \mu_J E_{\xi, z} \text{ exists } \geq \gamma\};$$

then $\mu_{IJ, J} G_\xi$ exists and is greater than 0. By S6E, there is a $z \in \{0, 1\}^{I \setminus J}$ such that $U = \{\xi : z \in E_\xi\}$ has cardinal $\lambda$.

Let $(\mathfrak{A}_0, \mu_0)$ be the measure algebra of $\mu_J$.

Then the topological density of $\mathfrak{A}_0$ is at most $\max(\omega, \tau_0) < \lambda$ (FREMLIN 89, 63b). So there is a $b \in \mathfrak{A}_0$ such that $\langle \xi : \xi \in U, \mu_0(b \Delta E_\xi) \leq \delta\rangle$ is infinite for every $\delta > 0$. Of course $\mu_0 b \geq \inf_{\xi \in U} \mu_0 E_\xi \geq \gamma$. Let $\langle \xi_n \rangle_{n \in \mathbb{N}}$ be a sequence of distinct elements of $U$ such that $\sum_{n \in \mathbb{N}} \mu_0(b \Delta E_{\xi_n}^*) \leq \gamma - \gamma'$.

Then

$$\mu_J(\bigcap_{n \in \mathbb{N}} E_{\xi_n, z}) = \mu_J(\inf_{n \in \mathbb{N}} E_{\xi_n, z}^*) \geq \mu_0 b - \sum_{n \in \mathbb{N}} \mu_0(b \setminus E_{\xi_n, z}^*) \geq \gamma'.$$

Set $W = \{\xi_n : n \in \mathbb{N}\}$. If $F' \in \Sigma_J$ and $\pi_{IJ, J}^{-1}[F'] \supseteq \bigcap_{n \in \mathbb{N}} E_{\xi_n, z}$, we must have $F' \supseteq \bigcap_{n \in \mathbb{N}} E_{\xi_n, z}$, so

$$\mu(\pi_{IJ, J}^{-1}[F']) = \mu F' \geq \gamma'.$$

As $F'$ is arbitrary, $\mu_0^+\left(\bigcap_{\xi \in W} E_\xi\right) \geq \gamma'$, as required.

(b) It follows that if $(\mathfrak{A}, \mu)$ is a probability algebra, $\mathfrak{A}_0$ is an order-closed subalgebra of $\mathfrak{A}$ with $\tau(\mathfrak{A}_0) = \tau_0 < \kappa$, $\lambda$ is a cardinal greater than $\max(\omega, \tau(\mathfrak{A}_0))$, $\langle a_\xi \xi \in \Lambda \rangle$ is a family in $\mathfrak{A}$ with $\inf_{\xi \in \Lambda} \mu_0 a_\xi = \gamma > 0$, and $\gamma' < \gamma$, then there is an infinite $W \subseteq \lambda$ such that

$$\min\{\mu b : b \in \mathfrak{A}_0, b \supseteq \inf_{\xi \in W} a_\xi\} \geq \gamma'.$$

**P** We can embed $\mathfrak{A}$ as a subalgebra of the measure algebra $\mathfrak{A}_I$ of $\{0, 1\}^I$ for some set $I$ (A2Lb). If we take a set $B \subseteq \mathfrak{A}_0$ of cardinal $\tau_0$ which completely generates $\mathfrak{A}_0$, then for each $b \in B$ we can find a set $G_b \subseteq X = \{0, 1\}^I$, belonging to the Baire $\sigma$-algebra of $X$, such that $b = G_b^*$ in $\mathfrak{A}$; now there is a set $J \subseteq I$, of cardinal at most $\max(\omega, \tau_0)$, such that every $G_b$ belongs to $\Sigma_0$, if we define $\Sigma_0$ from $J$ as in part (a) above.

Then $\mathfrak{A}_I = \{G^* : G \in \Sigma_0\}$, so that $\mathfrak{A}_I$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_I$. Let $\Sigma_J$ be the algebra of measurable subsets of $\{0, 1\}^J$, and $\mathfrak{A}_J$ the measure algebra of $\{0, 1\}^J$. Then the inverse-measure-preserving map $\pi_J : X \to \{0, 1\}^J$ induces an isomorphism $\phi$ between $\mathfrak{A}_I$ and $\mathfrak{A}_J$, taking $\phi(\pi_J^{-1}[G^*]) = G^*$ for every $G \in \Sigma_J$. By the lifting theorem (A2Qb) there is a lifting $\theta_J : \mathfrak{A}_I \to \Sigma_J$. So we have a corresponding Boolean homomorphism $\theta_I : \mathfrak{A}_I \to \Sigma_0$ given by setting $\theta_I(a) = \pi_{IJ, J}^{-1}[\theta_J(\phi(a))]$ for each $a \in A_I$. As remarked in A2Qc, there is an extension $\theta$ of $\theta_I$ to a lifting from $\mathfrak{A}$ to $\Sigma$.

Set $E_\xi = \theta(a_\xi)$ for each $\xi < \lambda$. By part (a) above, there is an infinite set $W \subseteq \lambda$ such that $\mu_0^+\left(\bigcap_{\xi \in W} E_\xi\right) \geq \gamma'$. Now suppose that $b \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi \in W} a_\xi$. Then $\theta(b) \supseteq \bigcap_{\xi \in W} E_\xi$ and $\theta(b) \in \Sigma_0$, so
\[ \hat{\mu} b = \mu(\theta(b)) \geq \mu_0(\bigcap_{\xi \in W} E_\xi) \geq \gamma'. \]

Thus we have the set \( W \) we need. \( \square \)

(c) We are now ready for the general case of the lemma. Let \((\mathfrak{A}, \hat{\mu})\) be the measure algebra of \((X, \Sigma, \mu)\) and set \( \mathfrak{A}_0 = \{G^* : G \in \Sigma_0\}, a_\xi = E_\xi^* \) for each \( \xi < \lambda \). By (b), there is an infinite \( W \subseteq \lambda \) such that \( \hat{\mu} b \geq \gamma' \) whenever \( b \in \mathfrak{A}_0 \) and \( b \geq \inf_{\xi \in W} a_\xi \). Now if \( G \in \Sigma_0 \) and \( G \supseteq \bigcap_{\xi \in W} E_\xi \), \( b = G^* \in \mathfrak{A}_0 \) and \( b \geq \inf_{\xi \in W} a_\xi \), so that \( \mu_0 G = \mu G = \mu G^* \geq \gamma' \). As \( G \) is arbitrary, \( \hat{\mu} (\bigcap_{\xi \in W} E_\xi) \geq \gamma' \), and we have a suitable set \( W \).

S6G Proposition Let \( \kappa \) be an atomlessly-measurable cardinal.

(a) Let \( X \) be a Hausdorff space such that \( \chi(X) < \kappa \). Then any Radon probability measure on \( X \) has Maharam type at most \( \alpha \).

(b) Let \( X \) be a Hausdorff space such that \( \chi(x, X) < \kappa \) for every \( x \in X \). Then any Radon probability measure on \( X \) has Maharam type less than \( \kappa \).

\textbf{proof} Suppose, if possible, otherwise. By SA2D, \( X \) has a Maharam homogeneous Radon probability measure \( \mu \) of Maharam type \( \lambda \), where \( \lambda = \max(\omega_1, \chi(X)^+) \) in (a) and \( \lambda = \kappa \) in (b). Let \( \mathfrak{A} \) be the measure algebra of \( \mu \); then \( \mathfrak{A} \) can be expressed as the union of a strictly increasing family \( \{\mathfrak{A}_\xi\}_{\xi < \lambda} \) of closed subalgebras. For each \( x \in X \) choose a base \( \mathcal{U}_x \) of open neighbourhoods of \( x \) with \( \#(\mathcal{U}_x) = \chi(x, X) \). Set

\[ X_\xi = \{x : x \in X, U^* \in \mathfrak{A}_\xi \text{ for every } U \in \mathcal{U}_x\}. \]

Because \( \text{cf}(\lambda) = \lambda > \chi(x, X) \) for every \( x \), \( \bigcup_{\xi < \lambda} X_\xi = X \).

By 6C/S6E, \( \sup_{\xi < \lambda} \mu^* X_\xi = 1 \); because \( \text{cf}(\lambda) > \omega \), there is a \( \xi < \lambda \) such that \( \mu^* X_\xi = 1 \). Now let \( G \) be any open subset of \( X \). Set

\[ \mathcal{H} = \{H : H \subseteq G \text{ is open}, H^* \in \mathfrak{A}_\xi\}, \quad H_0 = \bigcup \mathcal{H}. \]

Then \( H_0 \supseteq G \cap X_\xi \). So \( G \cap H_0 \) must be negligible, and

\[ G^* = H^*_0 = \sup_{H \in \mathcal{H}} H^* \in \mathfrak{A}_\xi. \]

As \( G \) is arbitrary, \( \mathfrak{A} = \mathfrak{A}_\xi \), which is absurd. \( \square \)

Remark This is derived from Plebanek 95.

\textbf{S7'}. Topological implications

\textbf{S7'A Proposition} Suppose there is an atomlessly-measurable cardinal. Then there is a locally compact locally countable Hausdorff topology \( \mathcal{T} \) on \( \omega_1 \) such that every closed set is either countable or cocountable.

\textbf{proof} (a) By 1, there is an \( \omega_2 \)-additive measure \( \nu \) on \( X = [0, 1]^{<\omega_1} \) extending the usual measure on \( X \). We may suppose that \( \nu \) is Maharam homogeneous, with Maharam type \( \lambda \), say, and that we have an independent family \( \{E_\xi\}_{\xi < \lambda} \) of measurable sets generating the measure algebra of \( \nu \), starting with \( E_\xi = \{x : x(\xi) = 1\} \) for \( \xi < \omega_1 \). For \( \Gamma \subseteq \lambda \) write \( \Sigma_\Gamma \) for the \( \sigma \)-algebra generated by \( \{E_\xi : \xi \in \Gamma\} \cup \{E : \nu E = 0\} \), so that \( \mathcal{P}X = \bigcup \{\Sigma_\Gamma : \Gamma \in [\lambda]^{\leq \omega_1}\} \). Write \( \Omega \) for the set of non-zero countable limit ordinals; for \( \alpha \in \Omega \), enumerate \( \alpha \) as \( \langle e_\alpha(n) \rangle_{n \in \mathbb{N}} \). Fix a bijection \( \phi : \{0, 1\}^\mathbb{N} \rightarrow [0, 1]^{\mathbb{N}} \) which is an automorphism for the usual measures (Fremlin p00*, 341I). For \( \alpha \in \Omega \), \( n \in \mathbb{N} \) define \( \phi_{an} : X \rightarrow [0, 1] \) by writing \( \langle \phi_{an}(x) \rangle_{n \in \mathbb{N}} = \phi(x(a+n))_{n \in \mathbb{N}} \) for every \( x \in X \).

(b) For \( x \in X, \xi < \omega_1 \) choose \( B_\xi(\xi) \) as follows. If \( \xi < \omega_1 \) then \( B_\xi(\xi) = \{\xi\} \). Suppose that \( B_\xi(\xi) \) has been defined for every \( \xi < \alpha \), where \( \alpha \in \Omega \). Set \( C_\xi(\alpha, n) = B_\xi(e_\alpha(n)) \cap \bigcup_{i < n} B_\xi(e_\alpha(i)) \) for each \( n \). Set

\[ B_\xi(\alpha + n) = \{\alpha + n\} \cup \bigcup \{C_\xi(\alpha, i) : i \in \mathbb{N}, 2^{-n-1} \leq \phi_{an}(x) < 2^{-n}\} \]

for each \( n \).

Let \( \mathcal{T}_x \) be the topology on \( \omega_1 \) generated by \( \{B_\xi(\xi) : \xi < \omega_1\} \cup \{\omega_1 \setminus B_\xi(\xi) : \xi < \omega_1\} \). Then \( \mathcal{T}_x \) is Hausdorff, locally countable and locally compact. \( \square \) It is Hausdorff because if \( \xi < \eta \) then \( \xi \in B_\xi(\xi) \) and \( \eta \notin B_\xi(\xi) \). (The point is that if \( \alpha \) is a limit ordinal and \( n \in \mathbb{N} \), then

\[ \alpha + n \in B_\xi(\alpha + n) \subseteq \{\alpha + n\} \cup \alpha. \]
It is locally countable because every $B_\alpha(\xi)$ is countable. Finally, an induction on $\xi$ shows that every $B_\alpha(\xi)$ is compact, because if, for $\alpha \in \Omega$, we take $\mathcal{T}_\alpha$ to be the topology on $\alpha$ generated by $\{B_\alpha(\xi) : \xi < \alpha\} \cup \{\alpha \setminus B_\alpha(\xi) : \xi < \alpha\}$, then every $C_\alpha(x, \eta) \cap \alpha$ and every $B_\alpha(\eta) \cap \alpha$, for $\beta \in \Omega \setminus \alpha$, $\eta \in \omega_1 \setminus \alpha$ are open-and-closed for $\mathcal{T}_\alpha$. Now, given $\alpha \in \Omega$, the inductive hypothesis tells us that $C_\alpha(\alpha, i)$ is always compact for $\mathcal{T}_\alpha$, so that $B_\alpha(\alpha + n)$ is compact for $\mathcal{T}_{x, \alpha + n}$ and therefore for $\mathcal{T}_x$. So $\mathcal{T}_x$ is locally compact. 

(c) The essential fact to note about this construction is that $\{x : \xi \in B_\alpha(\eta)\}$, $\{x : \xi \in C_\beta(\alpha, n)\}$ belong to $\Sigma_\alpha$ whenever $\alpha \in \Omega$, $\xi < \omega_1$, $\eta < \alpha$, $\beta \in \alpha \cap \Omega$ and $n \in \mathbb{N}$.

Suppose, if possible, that for every $x \in X$ there is a set $D_x \subseteq \omega_1$ which is $\mathcal{T}_x$-closed but neither countable nor cocountable. For $\xi < \omega_1$ set $H_\xi = \{x : \xi \in D_x\} \subseteq X$. Then there is a countable set $\Gamma_\xi \subseteq \lambda$ such that $H_\xi \in \Sigma_{\Gamma_\xi}$. Note also that, for any $\xi < \omega_1$, $\bigcup_{\eta > \xi} H_\eta = X$, so there is a countable $\Delta_\xi \subseteq \omega_1$ such that $\bigcup_{\eta \in \Delta_\xi, \eta > \xi} H_\eta$ is conegligible. Let $A$ be $\{\alpha : \alpha \in \Omega, (\Gamma_\xi \cap \omega_1) \cup \Delta_\xi \subseteq \alpha \text{ for every } \xi < \alpha\}$; then $A$ is a club set in $\omega_1$. Let $\alpha$ be the supremum of a strictly increasing set in $\Lambda$. Set $\Gamma = \bigcup_{\xi < \alpha} \Gamma_\xi$, so that $\Gamma \cap \omega_1 \subseteq \alpha$.

Let $Y$ be $\{y : D_y \cap \alpha$ is cofinal with $\alpha\}$, so that $Y$ is conegligible. For every $x \in X$, $D_x$ does not include $\omega_1 \setminus \alpha$, so there is a least $\xi \geq \alpha$ such that $\xi \in \omega_1 \setminus \alpha$ at least ordinal such that $Y \alpha = \{y : y \in Y, \xi \subseteq \xi\}$ is not negligible. Set $Y_1 = \{y : y \in Y, \xi \subseteq \xi\}$, so that $Y_1$ is conegligible. Express $\xi$ as $\beta + n$ where $\beta \in \Omega$, $n \in \mathbb{N}$. Observe that $H_\xi$ is conegligible, so belongs to $\Sigma_\Gamma$, for every $\xi \in \beta \setminus \alpha$, so $H_\xi \in \Sigma_\Gamma$ for every $\xi < \beta$.

For $x \in X$, set $I_x = \{m : m \in \mathbb{N}, D_x \cap C_\beta(\beta, m) \neq \emptyset\}$. Observe that $\{x \in I_x : \xi \in D_x \cap C_\beta(\beta, m)\} \in \Sigma_{\Gamma_\beta}$ for every $m \in \mathbb{N}$. If $y \in Y_1$, $D_y \cap \beta$ is cofinal with $\beta$, whether or not $\beta = \alpha$, because $D_y \cap \alpha$ is cofinal with $\alpha$ and $\beta \setminus \alpha \subseteq D_y$. Since

$$\sup_{(\beta, m)} C_\beta(\beta, m) = B_y(e_{\beta}(m)) = e_{\beta}(m) < \beta$$

for every $m$, $I_y$ is infinite, for every $y \in Y_1$.

On the other hand, setting $F_m = \{x : 2^{-m-1} \leq \phi_{m}(x) < 2^{-n}\}$ for $m \in \mathbb{N}$, $(F_m)_{m \in \mathbb{N}}$ is a stochastically independent sequence of sets of measure $2^{-n-1}$ all belonging to $\Sigma_{(\beta^*+\omega)^\beta}$, which is independent from $\Sigma_{\Gamma, \beta}$. Set $J_y = \{m : y \in F_m\}$. Then

$$G = \{y : I_y \text{ is infinite, } I_y \cap J_y \text{ is finite}\}$$

is negligible. (Set $h_1(y) = I_y$, $h_2(y) = J_y$, $h(y) = (I_y, J_y)$, so that $h_1$ and $h_2$ are functions from $X$ to $\mathbb{P}_n$.) Let $\mu_1$, $\mu_2$ be the Radon measures on $\mathbb{P}_n$ such that $h_1$ and $h_2$ are inverse-measure-preserving, and $h$ is inverse-measure-preserving for the product measure $\mu_1 \times \mu_2$ on $\mathbb{P}_n \times \mathbb{P}_n$. Now the set

$$(\{I, J\} : I \in [\mathbb{N}]^0, J \subseteq I, I \cap J \text{ is finite}\}$$

is a Borel set in $\mathbb{P}_n \times \mathbb{P}_n$ which has $\mu_2$-negligible vertical sections, so is $\mu_1 \times \mu_2$-negligible, and its inverse image $G$ is $\mu_2$-negligible.)

There is therefore some $y \in Y_2 \setminus G$. But in this case, $\xi \notin D_y$, so $B_y(\xi) \cap D_y$ is a $\mathcal{T}_\xi$-compact set not containing $\xi$. Since

$$B_y(\xi) = \{\xi\} \cup \bigcup_{m \in I_y} C_\beta(\beta, m),$$

there must be a finite set $K \subseteq J_y$ such that $B_y(\xi) \cap D_y \subseteq \bigcup_{m \in K} C_\beta(\beta, m)$. Since $I_y$ is infinite and $y \notin G$, there is an $m \in I_y \cap J_y \setminus K$, and $C_\beta(\beta, m) \cap D_y$ is not empty; which is impossible. 

This contradiction shows that for at least one $x \in X$, every $\mathcal{T}_x$-closed set is either countable or cocountable.

**Remarks** This proposition is based on ideas of J.Moore.

Observe that, for a topology $\mathcal{T}$ as described, $\omega_1$ is hereditarily separable. P? If $A \subseteq \omega_1$ is not separable, of course it is uncountable. For $\xi < \omega_1$, choose $\alpha_\xi$, $\beta_\xi$ in $A$, and countable open sets $G_{\xi} \subseteq \omega_1 \setminus \alpha_\xi$ inductively, as follows. Given $(\alpha_n)_{n \leq \xi}$, let $\beta_n \in \alpha_n$ such that $\beta_n \notin \{\alpha_n : \eta < \xi\} \cup (A \setminus \xi)$. Let $G_{\xi}$ be a countable open set containing $\beta_\xi$ and not meeting $\{\alpha_n : \eta < \xi\}$. Given $(G_{\eta})_{\eta \leq \xi}$, take $\alpha_\xi \in A \setminus (\xi \cup \bigcup_{\eta \leq \xi} G_{\eta})$. 

Now consider $F = \{ \alpha_\xi : \xi < \omega_1 \}$. Then $\alpha_\xi \in F$ and $\beta_\xi \notin F$ for every $\xi$. Since $\alpha_\xi, \beta_\xi \geq \xi$ for every $\xi$, $F$ is neither countable nor cocountable, which is impossible. XQ

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S7. Partially ordered sets

S7A Cofinalities III: Lemma Let $\kappa$ be a real-valued-measurable cardinal and $\langle \alpha_\xi \rangle_{\xi \in I}$ a countable family of ordinals less than $\kappa$ and of cofinality at least $\omega_2$. Then there is a set $F \subseteq \prod_{\xi \in I} \alpha_\xi$ such that (i) $F$ is cofinal with $\prod_{\xi \in I} \alpha_\xi$ (ii) if $\langle f_\xi \rangle_{\xi < \omega_1}$ is an increasing family in $F$ then $\sup_{\xi < \omega_1} f_\xi \in F$ (iii) $\#(F) < \kappa$.

proof We have $\text{cf}(\prod_{\xi \in I} \alpha_\xi) = \text{cf}(\prod_{\xi \in I} \text{cf}(\alpha_\xi)) < \kappa$, by 7Kb. So we may find a cofinal set $F_0 \subseteq \prod_{\xi \in I} \alpha_\xi$ of cardinal less than $\kappa$. Now for $0 < \zeta < \omega_2$ define $F_\zeta$ by saying that

$$F_{\zeta+1} = \{ \sup_{\xi < \omega_1} f_\xi : \langle f_\xi \rangle_{\xi < \omega_1} \text{ is an increasing family in } F_\zeta \}.$$ 

Then $\#(F_\zeta) < \kappa$ for every $\zeta$. P Induce on $\zeta$. For the inductive step to $\zeta + 1$, ? suppose, if possible, that $\#(F_\zeta) < \kappa$ but $\#(F_{\zeta+1}) \geq \kappa$. For each $h \in F_{\zeta+1}$ choose an increasing family $\langle f_{h,\zeta} \rangle_{\xi < \omega_1}$ in $F_\zeta$ with supremum $h$. The set $\{ h[I] \}$ of values of $h$ is a countable subset of $h = \bigcup_{\xi \in I} \alpha_\xi$. By 5A, taking $X = F_{\zeta+1}$ and $\mu$ a non-trivial $\kappa$-additive measure on $X$, there is a set $H \subseteq F_{\zeta+1}$ of cardinal $\kappa$, such that $M = \bigcup_{i \in H} h[I]$ is countable. Now, for each $h \in H$, there is a $\gamma(h) < \omega_2$ such that whenever $i \in I$ and $\beta \in M$ then $h(i) > \beta$ iff $f_{h,\gamma(h)}(i) > \beta$. If $g, h \in H$ and $i \in I$ and $g(i) < h(i)$, then $f_{h,\gamma(h)}(i) > g(i) < f_{h,\gamma(h)}(i)$, because $g(i) \in M$. Thus $h \mapsto f_{h,\gamma(h)} : H \rightarrow F_\zeta$ is injective; but $\#(F_\zeta) < \kappa = \#(H)$. X

Thus $\#(F_{\zeta+1}) < \kappa$ if $\#(F_\zeta) < \kappa$. At limit ordinals $\zeta$ the induction proceeds without difficulty because $\text{cf}(\kappa) > \zeta$. Q

So $\#(F_{\omega_1}) < \kappa$ and we may take $F = F_{\omega_1}$.

S7B Theorem Let $\kappa$ be a real-valued-measurable cardinal.

(a) For any cardinal $\theta$, $\text{cf}(\kappa^{<\theta}) \leq \kappa$.

(b) For any cardinal $\lambda < \kappa$, any $\theta$, $\text{cf}(\lambda^{<\theta}) < \kappa$.

proof (a)(i) Consider first the case $\theta = \omega_1$. Write $G_1$ for the set of ordinals less than $\kappa$ of cofinality less than or equal to $\omega_1$; for $\delta \in G_1$ let $\psi_\delta : \text{cf}(\delta) \rightarrow \delta$ enumerate a cofinal subset of $\delta$. Next, write $G_2$ for $\kappa \setminus G_1$, and for every countable set $A \subseteq G_2$ let $F(A) \subseteq \prod_{\alpha \in A} \alpha$ be a cofinal set, of cardinal less than $\kappa$, closed under suprema of increasing families of length $\omega_1$; such exists by S7A above.

(ii) It is worth observing at this point that if $\langle A_\xi \rangle_{\xi < \omega_1}$ is any family of countable subsets of $G_2$, $D = \bigcup_{\xi < \omega_1} A_\xi$, and $g \in \prod_{\alpha \in D} \alpha$, then there is an $f \in \prod_{\alpha \in D} \alpha$ such that $f \geq g$ and $f| A_\xi \in F(A_\xi)$ for every $\xi < \omega_1$. P Let $\langle \phi(\xi) \rangle_{\xi < \omega_1}$ run over $\omega_1$ taking every uncountably often. Choose an increasing family $\langle f_\xi \rangle_{\xi < \omega_1}$ in $\prod_{\alpha \in D} \alpha$ in such a way that $f_0 = g$ and $f_{\xi+1} A_{\phi(\xi)} \in F(A_{\phi(\xi)})$ for every $\xi$; this is possible because $F(A)$ is cofinal with $\bigcap_{\alpha \in A} \alpha$ for every $A$. Set $f = \sup_{\xi < \omega_1} f_\xi$; this works because every $F(A)$ is closed under suprema of increasing families of length $\omega_1$. Q

(iii) We can now find a family $A$ of countable subsets of $\kappa$ such that

- $\langle \alpha \rangle \in A$ for every $\alpha < \kappa$;
- (\beta) whenever $A, A' \in A$, $\zeta < \omega_1$ then $A \cup A', A \cap G_2, \{ \psi_\alpha(\xi) : \alpha \in A \cap G_1, \xi < \text{min}(\zeta, \text{cf}(\alpha)) \}$ all belong to $A$;
- (\gamma) whenever $A \in A, A \subseteq \bigcap_{\alpha \in G_2} \text{ for every } f \in F(A)$;
- (\delta) $\#(A) \leq \kappa$.

(iv) ? Suppose, if possible, that $\text{cf}(\kappa^{<\omega_1}) > \kappa$. Because $\kappa^{<\omega_1} = \bigcup_{\lambda < \kappa} \lambda^{<\omega_1}$, there is a cardinal $\lambda < \kappa$ such that $\text{cf}(\lambda^{<\omega_1}) > \kappa$. We can therefore choose inductively a family $\langle a_\xi \rangle_{\xi < \kappa}$ of countable subsets of $\lambda$ such that

$$a_\xi \not\subseteq \bigcup_{\eta \in A \cap G_1} a_\eta$$ 

whenever $\xi < \kappa, A \in A$. By 5A, there is a set $W \subseteq \kappa$, of cardinal $\kappa$, such that $\bigcup_{\xi \in W} a_\xi$ is countable. Let $\delta < \kappa$ be such that $W \cap \delta$ is cofinal with $\delta$ and of order type $\omega_1$. 
(v) I choose a family \( \langle k_\zeta \rangle_{\zeta < \omega_1} \in A \) as follows. Start by setting \( A_0 = \psi_3(\zeta) \) for every \( \zeta < \omega_1 \); then \( A_\zeta \in A \) by (iii)(\( \alpha \beta \)). Given \( \langle A_\zeta \rangle_{\zeta < \omega_1} \), set \( A'_\zeta = A_\zeta \cap G_2 \) for each \( \zeta < \omega_1 \). For \( \alpha \in D_k = \bigcup_{\zeta < \omega_1} A_\zeta \), set \( g_\alpha(k) \alpha = \sup(\alpha \cap W \cap \delta) < \alpha \); choose \( f_k \in \prod_{\alpha \in D_k} \alpha \) such that \( g_\alpha(k) \leq f_\alpha k \) and \( f_\alpha k A'_\zeta \in F(A'_\zeta) \) for every \( \zeta \); this is possible by (ii) above. Set

\[
A_{k+1} = A_k \cup f[k A'_k] \cup \{ \psi_4(\zeta) : \alpha \in A_\zeta \cap G_1, \zeta < \min(\zeta, cf(\alpha)) \} \in A
\]

for each \( \zeta < \omega_1 \), and continue. An easy induction on \( k \) shows that \( \langle A_\zeta \rangle_{\zeta < \omega_1} \) is increasing for every \( k \); also, every \( A_\zeta \) is a subset of \( \delta \).

(vi) Set \( V_k = \bigcup_{\zeta < \omega_1} A_\zeta, b_k = \bigcup_{\{ \alpha : \xi \in W \cap V_\zeta \}} \{ \alpha : \xi \in W \cap A_\zeta(\beta(k)) \} \). Now \( \bigcup_{\zeta < \omega_1} A_\zeta(\beta(k)) \) is a countable subset of \( \delta \), so there is a member \( \gamma \) of \( W \cap \delta \) greater than its supremum. We have

\[
a_\gamma \not\in \bigcup\{ a_\eta : \eta \in A_\zeta(\beta(k)) \}
\]

for every \( k \), so \( a_\gamma \not\in b_k \) and \( \gamma \notin V_k \), for each \( k \).

Set \( V = \bigcup_{k \in \mathbb{N}} V_k \), \( \gamma_0 = \min(W \setminus \delta \setminus V) \). Because \( V_0 = \psi_3(\omega_1) \) is cofinal with \( \delta \), \( V \setminus \gamma_0 \neq \emptyset \); let \( \gamma_1 \) be its least member. Then \( \gamma_1 > \gamma_0 \). Suppose \( \gamma_1 \in A_\zeta \). Observe that if \( \alpha \in V \cap G_1 \) then \( V \cap \alpha \) is cofinal with \( \alpha \); but \( V \cap \gamma_1 \leq \gamma_0 \), so \( \gamma_1 \notin G_1 \) and \( \gamma_1 \in D_k \). But now \( f_k(\gamma_1) \in A_{k+1} \subseteq V \) and \( \gamma_0 \leq f_k(\gamma_1) \leq f_k(\gamma_1) < \gamma_1 \), so \( \gamma_1 \neq \min(V \setminus \gamma_0) \).

(vii) This contradiction shows that \( cf([k]^{\leq \omega}) \leq k \). Now consider \( cf([k]^{\leq \delta}) \), where \( \delta < \kappa \) is an infinite cardinal. Then

\[
cov_{SH}(\kappa, \delta^+, \delta^+, \omega_1) = \sup_{\lambda < \kappa} \cov_{SH}(\lambda, \delta^+, \delta^+, \omega_1) \leq \kappa
\]

by 7Kd. (See A1 for the definition of \( cov_{SH} \).) So there is a family \( B \subseteq [k]^{\leq \delta} \), of cardinal at most \( \kappa \), such that every member of \( [k]^{\leq \delta} \) is covered by a sequence in \( B \). But now there is a family \( C \) of countable subsets of \( B \) which is cofinal with \( [B]^{\leq \omega} \) and of cardinal at most \( \kappa \); setting \( D = \{ \cup C : C \in C \} \), we have \( D \) cofinal with \( [k]^{\leq \delta} \) and of cardinal at most \( \kappa \).

Finally, of course, \( [k]^{< \theta} = \bigcup_{\alpha < \theta} [k]^{\leq \delta}, \) so \( cf([k]^{< \theta}) \leq \sup_{\theta < \theta} cf([k]^{\leq \delta}) \leq \kappa \) whenever \( \theta \leq \kappa \). For \( \theta > \kappa \) we have \( cf([k]^{< \theta}) = 1 \), so \( cf([k]^{< \theta}) \leq \kappa \) for every \( \theta \).

(b) If \( A \) is cofinal with \( [k]^{< \theta} \) then \( \{ A \cap \lambda : \lambda \in A \} \) is cofinal with \( [\lambda]^{< \theta}, \) so \( cf([\lambda]^{< \theta}) \leq cf([k]^{< \theta}) \leq cf([k]^{< \theta}) \), by (a). But by 7ld and 7Jb, \( cf([\lambda]^{< \theta}) \neq \kappa \). Thus \( cf([\lambda]^{< \theta}) < \kappa \), as claimed.

Remark This result is taken from Shelah #430.

S7C Corollary Let \( \kappa \) be a real-valued-measurable cardinal. Let \( \langle P_\zeta \rangle_{\zeta < \lambda} \) be a family of partially ordered sets such that \( \lambda < add(P_\zeta) \leq cf(P_\zeta) < \kappa \) for every \( \zeta < \kappa \). Then \( cf(\prod_{\zeta < \lambda} P_\zeta) < \kappa \).

Proof For each \( \zeta < \lambda \) let \( Q_\zeta \) be a cofinal subset of \( P_\zeta \) of cardinal less than \( \kappa \). Set \( P = \prod_{\zeta < \lambda} P_\zeta, Z = \bigcup_{\zeta < \lambda} Q_\zeta \); then \( \#(Z) < \kappa \) so \( cf([Z]^{\leq \delta}) < \kappa \), by 57Bb. Let \( A \) be a cofinal subset of \( [Z]^{\leq \kappa} \) with \( \#(A) < \kappa \). For each \( \alpha \in A \) choose \( f_\alpha \in P \) such that \( f_\alpha(\zeta) \) is an upper bound for \( A \cap P_\zeta \) for every \( \zeta \); this is possible because \( add(P_\zeta) > \#(A) \).

If \( g \in P \), then there is an \( h \in \prod_{\zeta < \lambda} Q_\zeta \) such that \( g \leq h \). Now \( h[\zeta] \in [Z]^{\leq \kappa} \) so there is an \( \alpha \in A \) such that \( h[\zeta] \subseteq A \). In this case \( h \leq f_\alpha \).

Accordingly \( P \) is cofinal with \( P \) and \( cf(P) \leq \#(F) < \kappa \), as required.

S7D The techniques of 7C can be adapted to give a further combinatorial result.

Theorem Let \( \kappa \) be an atomlessly-measurable cardinal, and \( \lambda < \kappa \) an infinite cardinal. Then there is a function \( f : [\lambda^+]^2 \to \mathbb{N} \) such that whenever \( I \) is a disjoint family of finite subsets of \( \lambda^+ \) with \( \#(I) = \lambda^+ \), and \( k \in \mathbb{N} \), there are distinct \( I, J \in I \) such that \( f(I, J) = k \) for every \( \xi \in I, \eta \in J \).

Proof (a) For \( \xi < \lambda^+ \) let \( e_\xi : \xi \to \lambda \) be an injective function. Give \( \mathbb{N} \) the Radon measure \( \mu_0 \) defined by saying that \( \mu_0(n) = 2^{-n-1} \) for every \( n \in \mathbb{N} \), and give \( \mathbb{N}^\lambda \) the product measure \( \mu \). Then \( \mu \) is an atomless Radon probability measure with Maharam type \( \lambda \), so there is a \( \kappa \)-additive measure \( \nu \), defined on every \( \kappa \)-subset of \( \mathbb{N}^\lambda \), extending \( \mu \) (31).

For \( w \in \mathbb{N}^\lambda \), define \( f_w : [\lambda^+]^2 \to \mathbb{N} \) by setting

\[
f_w(I, J) = w(e_\eta(\xi))
\]
whenever $\xi < \eta < \lambda^+$. 

(b) Suppose, if possible, that no $f_w$ witnesses the truth of the theorem. Then for every $w \in \mathbb{N}^\lambda$ there are a $k_w \in \mathbb{N}$ and a disjoint $I_w \subseteq [\lambda^+]^{<\omega}$ such that $\#(I_w) = \lambda^+$ and whenever $I, J$ are distinct members of $I_w$ there are $\xi \in I$, $\eta \in J$ such that $f_w(\langle \xi, \eta \rangle) \neq k_w$. Take $k \in \mathbb{N}$ such that $\nu R > 0$, where 

$$R = \{w : w \in \mathbb{N}^\lambda, k_w = k\}.$$ 

For every $\alpha < \lambda^+$, $w \in \mathbb{N}^\lambda$ there is an $I \in I_w$ such that $I \cap \alpha = \emptyset$; because $\#([\lambda^+ \setminus \alpha]^{<\omega}) < \kappa$, there is an $I \subseteq \lambda^+ \setminus \alpha$ such that $\nu \{w : w \in R, I \in I_w\} > 0$. We may therefore choose $(I_\alpha)_{\alpha < \lambda^+}$ inductively so that $\xi < \eta$ whenever $\xi \in I_\alpha$, $\eta \in I_\beta$ and $\alpha < \beta < \lambda^+$, 

$$\nu S_\alpha > 0 \text{ for every } \alpha < \lambda,$$

where $S_\alpha = \{w : w \in R, I_\alpha \in I_w\}$. 

Take $\delta > 0$, $n \in \mathbb{N}$ such that $\#(A) = \lambda^+$, where 

$$A = \{\alpha : \alpha < \lambda^+, (\#(I_\alpha) = n, \nu S_\alpha \geq \delta\}.$$ 

Take $l \in \mathbb{N}$ such that $(1 - 2^{-(k+1)n^2})^l \leq \frac{1}{3}\delta$. 

(b) By A2S, there is a set $B \subseteq A$, of cardinal $\lambda^+$, such that $\mu^*(\bigcap_{\alpha \in L} S_\alpha) \geq \frac{2}{3}\delta$ whenever $L \subseteq B$ and $\#(L) \leq l + 1$. Take $\beta \in B$ such that $B \cap \beta$ is infinite. Now for each $\eta \in I_\beta$, $e_\eta : \eta \to \lambda$ is injective. Accordingly we can find a set $L \subseteq B \cap \beta$ such that $\#(L) = l$ and 

$$e_\eta[I_\beta] \cap e_{\eta'}[I_\alpha] = \emptyset$$

whenever $\alpha$, $\alpha'$ are distinct members of $L$ and $\eta$, $\eta' \in I_\beta$. For $\alpha \in L$ set 

$$K_\alpha = \{e_\eta(\xi) : \xi \in I_\alpha, \eta \in I_\beta\};$$

then $(K_\alpha)_{\alpha \in L}$ is disjoint and $\#(K_\alpha) \leq n^2$ for every $\alpha$. 

Consider 

$$E = \{w : w \in \mathbb{N}^\lambda, \forall \alpha \in L \exists \xi \in K_\alpha \text{ such that } w(\xi) \neq k\}.$$ 

Then 

$$\mu E = \prod_{\alpha \in L}(1 - 2^{-(k+1)\#(K_\alpha)}) \leq \frac{1}{3}\delta.$$ 

On the other hand, if $w \in S_\lambda \cap \bigcap_{\alpha \in L} S_\alpha$, $I_\alpha \in I_w$ and $I_\beta \in I_w$ for every $\alpha \in L$. So for every $\alpha \in L$ there are $\xi \in I_\alpha$, $\eta \in I_\beta$ such that $f_w(\langle \xi, \eta \rangle) \neq k_w$, that is, $w(e_\eta(\xi)) \neq k$; and we have a $\zeta \in K_\alpha$ such that $w(\zeta) \neq k$. Thus 

$$\bigcap_{\alpha \in L \cup \beta} S_\alpha \subseteq E,$$

and (by the choice of $B$) we get 

$$\frac{2}{3}\delta \leq \mu^*(\bigcap_{\alpha \in L \cup \beta} S_\alpha) \leq \mu E \leq \frac{1}{3}\delta,$$

which is absurd. \textbf{X}

(d) Thus there is a $w \in \mathbb{N}^\lambda$ for which $f_w$ is a suitable function.

**S7E Corollary** Let $\kappa$ be an atomlessly-measurable cardinal, and $\lambda < \kappa$ an infinite cardinal. Then there is a function $g : [\lambda^+]^2 \to \mathbb{N}$ such that whenever $I$ is a disjoint family of finite subsets of $\lambda^+$ with $\#(I) = \lambda^+$, and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function, there are distinct $I, J \in I$ such that $g(\langle \xi, \eta \rangle) = h(\#(I \cap \xi), \#(J \cap \eta))$ whenever $\xi \in I$, $\eta \in J$ and $\xi < \eta$.

**proof** Take $f : [\lambda^+]^2 \to \mathbb{N}$ from S7D. Enumerate as $\langle h_k \rangle_{k \in \mathbb{N}}$ the set of all maps from finite products $\{0,1\}^m \times \{0,1\}^m$ to $\mathbb{N}$; take the domain of $h_k$ to be $\{0,1\}^m \times \{0,1\}^m$. Choose a family $\langle w_\xi \rangle_{\xi < \lambda^+}$ of distinct elements of $\{0,1\}^\mathbb{N}$. Now, given $\xi < \eta < \lambda^+$, set $k = f(\langle \xi, \eta \rangle)$ and try $g(\langle \xi, \eta \rangle) = h_k(w_\xi | \eta, w_\eta | \eta)$. 

Suppose that $I$ is a disjoint family of finite subsets of $\lambda^+$ with cardinal $\lambda^+$, and that $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is any function. Let $m, n$ be such that
\[
\mathcal{J} = \{ I : I \in \mathcal{I}, \#(I) = n, w_\xi |m \neq w_\eta |m \text{ for all distinct } \xi, \eta \in I \}
\]
has cardinal \( \lambda^+ \), and \((v_i)_{i < \kappa} \text{ such that} \)
\[
K = \{ I : I \in \mathcal{J}, w_\xi |m = v_i \text{ whenever } \xi \in I, \#(\xi \cap I) = i < n \}
\]
has cardinal \( \lambda^+ \). Take \( k \in \mathbb{N} \) such that \( m_k = m \) and \( h_k(v_i, v_j) = h(i, j) \) for all \( i, j < n \). (Note that \( v_i \neq v_j \) if \( i \neq j \), because \( w_\xi |m \neq w_\eta |m \) if \( \xi, \eta \) are distinct members of any \( I \in \mathcal{J} \).) Then there are distinct \( I, J \in \mathcal{K} \) such that \( f(\{ \xi, \eta \}) = k \) for \( \xi \in I, \eta \in J \). So if \( \xi \in I, \eta \in J, \xi < \eta \) we have
\[
g(\{ \xi, \eta \}) = h_k(w_\xi |m, w_\eta |m) = h_k(v_i, v_j) = h(i, j),
\]
where \( i = \#(I \cap \xi), j = \#(J \cap \eta) \). And this is what we need.

**S7F Corollary** If there is an atomlessly-measurable cardinal, there is a ccc Hausdorff abelian topological group whose square is not ccc.

**proof (a)** Let \( g : [\omega_1]^2 \to \mathbb{N} \) be such that whenever \( \mathcal{I} \) is an uncountable disjoint family of finite subsets of \( \omega_1 \), and \( h : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) is a function, there are distinct \( I, J \in \mathcal{I} \) such that \( g(\{ \xi, \eta \}) = h(\#(I \cap \xi), \#(J \cap \eta)) \) whenever \( \xi \in I, \eta \in J \) and \( \xi < \eta \). Set \( X = \omega_1 \times \{0, 1\} \), and let \( G \) be the family of finite subsets of \( X \) with the group operation \( a + b = a \Delta b \). For \( \beta < \omega_1 \) let \( H_\beta \) be the set of those \( a \in G \) such that
\[
(\beta, 0), (\beta, 1) \notin a,
\]
if \( \alpha < \beta \) and \( g(\{ \alpha, \beta \}) = i \leq 1 \) then \((\alpha, i) \notin a \),
\[
\text{if } \alpha < \beta \text{ and } g(\{ \alpha, \beta \}) \geq 2 \text{ then } (\alpha, 0) \in a \text{ iff } (\alpha, 1) \in a.
\]
Note that \( H_\beta \) is a subgroup of \( G \). Give \( G \) the topology generated by the subgroups \( H_\beta \) and their cosets for \( \beta < \omega_1 \). This is a topological-group topology, and because \( \bigcap_{\beta < \omega_1} H_\beta = \{0\} \) it is Hausdorff.

(b) It may make the idea behind this definition clearer if I show first that \( G \times G \) is not ccc. **P** Consider
\[
W_\xi = (\{ (\xi, 0) \} + H_\xi) \times (\{ (\xi, 1) \} + H_\xi)
\]
for \( \xi < \omega_1 \).

If \( \xi < \eta < \omega_1 \) and \( a \in \{(\xi, 0) + H_\xi \} \cap \{(\eta, 0) + H_\eta \} \), then \( a + \{(\xi, 0) \} \in H_\xi \), so \( (\xi, 0) \in a \) and \( (\xi, 1) \notin a \). But now \( (\xi, 0) \notin a \) and \( (\xi, 1) \notin a \); since \( a + \{(\eta, 0) \} \in H_\eta \), we must have \( g(\{\xi, \eta\}) = 1 \).

If \( \xi < \eta < \omega_1 \) and \( b \in \{(\xi, 1) + H_\xi \} \cap \{(\eta, 1) + H_\eta \} \), then \( b + \{(\xi, 1) \} \in H_\xi \), so \( (\xi, 0) \notin b \) and \( (\xi, 1) \in b \). But now \( (\xi, 0) \notin b \) and \( (\xi, 1) \notin b \); since \( b + \{(\eta, 1) \} \in H_\eta \), we must have \( g(\{\xi, \eta\}) = 0 \).

Since these cannot happen together, \( W_\xi \cap W_\eta = \emptyset \) whenever \( \xi < \eta < \omega_1 \), and we have an uncountable disjoint family of open subsets of \( G \times G \). **Q**

(c) \( G \) is ccc. **P** Let \( \{ U_\xi \}_{\xi \in A} \) be a family of non-empty open subsets of \( G \). For each \( \xi < \omega_1 \) take \( a_\xi \in U_\xi \), \( I_\xi \in [\omega_1]^{<\omega_1} \) such that \( a_\xi + \bigcap_{\xi \in I_\xi} H_\beta \subseteq \bigcup_{\xi \in I_\xi} U_\xi \). Let \( A \subseteq \omega_1 \) be an uncountable set such that \( \langle a_\xi \rangle_{\xi \in A}, \langle I_\xi \rangle_{\xi \in A} \) are \( \Delta \)-systems with roots \( a, I \) respectively; set \( a'_\xi = a_\xi \setminus a \) for \( \xi \in A \). Set
\[
J_\xi = (I_\xi \setminus I) \cup \{ (\alpha, 0) : \alpha : (a_\xi, 0) \in a_\xi' \} \cup \{ (\alpha, 1) : (a_\xi, 1) \in a_\xi' \},
\]
\[
m_\xi = \#(J_\xi),
\]
and enumerate \( J_\xi \) in ascending order as \( \langle \alpha_\xi i \rangle_{i < m_\xi} \). Define \( h_\xi : m_\xi \to \{0, 1, 2\} \) by setting
\[
h_\xi(0) = 0 \text{ if } (\alpha_\xi, 0) \notin a_\xi' \text{, } (\alpha_\xi, 1) \in a_\xi',
\]
\[
= 1 \text{ if } (\alpha_\xi, 0) \in a_\xi' \text{, } (\alpha_\xi, 1) \notin a_\xi',
\]
\[
= 2 \text{ otherwise.}
\]
Let \( m, h : m \to \{0, 1, 2\} \) be such that
\[
B = \{ \xi : \xi \in A, m_\xi = m, h_\xi = h \}
\]
is uncountable. Because any \( \alpha < \omega_1 \) can belong to at most three of the \( J_\xi \), there is an uncountable \( C \subseteq B \) such that (i) \( \alpha < \beta \) whenever \( \alpha \in J \) and \( \beta \in J_\xi \) for some \( \xi \in C \) (ii) \( \alpha < \beta \) whenever \( \xi, \eta \in C, \xi < \eta, \)
Let \( \alpha, \beta \in J_\xi \) and \( \beta \in J_\eta \). Now, by the choice of \( g \), there must be \( \xi, \eta \in C \) such that \( \xi < \eta \) and \( g(\{\alpha_\xi, \beta\}) = h(i) \) whenever \( i < m, \beta \in J_\eta \).

What this means is that for \( \alpha \in J_\xi, \beta \in J_\eta \),

\[
g(\{\alpha, \beta\}) = \begin{cases} 0 & \text{if } (\alpha, 0) \notin a'_\xi, \, (\alpha, 1) \in a'_\xi, \\ 1 & \text{if } (\alpha, 0) \in a'_\xi, \, (\alpha, 1) \notin a'_\xi, \\ 2 & \text{otherwise}. \end{cases}
\]

Set

\[
b = a_\xi \cup a_\eta = a_\xi + a'_\eta = a_\eta + a'_\xi.
\]

I seek to show that \( b \in U_\xi \cap U_\eta \); it will be enough if I can show that \( a'_\eta \in H_\beta \) for every \( \beta \in I_\xi \) and \( a'_\xi \in H_\beta \) for every \( \beta \in I_\eta \).

(i) Suppose \( \beta \in I \). In this case \( \beta < \alpha \) whenever either \( (\alpha, 0) \) or \( (\alpha, 1) \) belongs to \( a'_\eta \). Consequently \( a'_\xi, \, a'_\eta \) belong to \( H_\beta \).

(ii) Suppose \( \beta \in I_\xi \setminus I \). In this case \( \beta < \alpha \) whenever either \( (\alpha, 0) \) or \( (\alpha, 1) \) belongs to \( a'_\eta \). So \( a'_\eta \in H_\beta \).

(iii) Suppose \( \beta \in I_\eta \setminus I \). In this case \( \alpha < \beta \) whenever either \( (\alpha, 0) \) or \( (\alpha, 1) \) belongs to \( a'_\eta \). In particular, neither \( (\beta, 0) \) nor \( (\beta, 1) \) belongs to \( a'_\xi \). If \( \alpha < \beta \) and \( g(\{\alpha, \beta\}) = i \leq 1 \), then either \( \alpha \notin J_\xi \), in which case \( (\alpha, i) \) certainly does not belong to \( a'_\xi \), or \( \alpha \in J_\xi \), in which case again \( (\alpha, i) \notin a'_\xi \). While if \( \alpha < \beta \) and \( g(\{\alpha, \beta\}) \geq 2 \), then either \( \alpha \notin J_\xi \) and neither \( (\alpha, 0) \) nor \( (\alpha, 1) \) belongs to \( a'_\xi \), or \( \alpha \in J_\xi \) and \( (\alpha, 0) \in a'_\xi \) iff \( (\alpha, 1) \in a'_\xi \). Thus \( a'_\xi \in H_\beta \).

Putting these together, we see that all the requirements for \( b \in U_\xi \cap U_\eta \) are met, so that \( U_\xi \cap U_\eta \neq \emptyset \). As \( (U_\xi)_{\xi \prec \omega_1} \) is arbitrary, \( G \) is ccc. \( \Box \)

**Remark** S7D-S7F are taken from §2 of Todorčević 93.

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**S9. Quasi-measurable cardinals**

**S9A Theorem** Let \( \kappa \) be a quasi-measurable cardinal.

(a) \( \text{cf}(\kappa)^{\leq \theta} \leq \kappa \) for every cardinal \( \theta \).

(b) If \( \lambda < \kappa \) then \( \text{cf}(\lambda)^{< \theta} < \kappa \) for every cardinal \( \theta \).

(c) If \( (P_\xi)_{\xi < \lambda} \) is a family of partially ordered sets with \( \lambda < \text{add}(P_\xi) \leq \text{cf}(P_\xi) < \kappa \) for every \( \xi < \lambda \), then \( \text{cf}(\prod_{\xi < \kappa} P_\xi) < \kappa \).

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**Appendix: Useful Facts**

**SA2. Measure theory**

**SA2A \( \tau \)-additive Borel measures** Let \( X \) be a regular topological space and \( \mu \) a totally finite \( \tau \)-additive Borel measure on \( X \). Then the completion \( \tilde{\mu} \) of \( \mu \) is a quasi-Radon measure on \( X \). If \( H \subseteq X \) is open, then (because \( X \) is regular) \( H = \bigcup G \), where \( G = \{G : G \subseteq X \text{ is open}, \overline{G} \subseteq H\} \). Consequently

\[
\mu H = \sup_{G \in G} \mu G = \sup\{\mu L : L \subseteq H \text{ is closed}\}.
\]

Now let \( F \) be the family of closed subsets in \( X \) and write \( \phi F = \mu F \) for \( F \in F \). Then (i) \( \phi \emptyset = 0 \) (ii) if \( E, F \) are closed subsets of \( X \) and \( E \subseteq F \),

\[
\phi F - \phi E = \mu F - \mu E = \mu((X \setminus E) \cap F) = \sup\{\mu(L \cap F) : L \subseteq X \setminus E \text{ is closed}\} = \sup\{\phi L : L \subseteq F \setminus E \text{ is closed}\}.
\]
(iii) whenever \( K \subseteq \mathcal{F} \) is non-empty and downwards-directed and \( \bigcap K = \emptyset \), then \( \inf_{K \in \mathcal{K}} \phi K = \mu X - \sup_{K \in \mathcal{K}} \mu (X \setminus K) = 0 \) because \( \{ X \setminus K : K \in \mathcal{K} \} \) is an upwards-directed family of open sets with union \( X \). By Fremlin 74, Theorem 72A, there is a complete measure \( \mu_1 \) on \( X \), extending \( \phi \), which is inner regular for the closed sets. Now for any Borel set \( E \subseteq X \),

\[
\mu_1 E = \sup \{ \mu_1 F : F \subseteq E \text{ is closed} \} = \sup \{ \phi F : F \subseteq E \text{ is closed} \} = \sup \{ \mu F : F \subseteq E \text{ is closed} \} \leq \mu E;
\]
similarly, \( \mu_1 (X \setminus E) \leq \mu (X \setminus E) \); but \( \mu_1 X = \phi X = \mu X \), so in fact \( \mu_1 E = \mu E \), and \( \mu_1 \) extends \( \mu \). It follows at once that \( \mu_1 \) is actually the completion of \( \mu \). Now \( \mu_1 = \hat{\mu} \) is \( \tau \)-additive and inner regular for the closed sets, therefore quasi-Radon. \( \Box \) (Compare Gardner & Pfeffer 84.,)

**SA2B Disintegrations** Let \( (X, \mathcal{F}) \) be a compact Hausdorff space and \((Y, T, \nu)\) a complete probability space. Let \( \mu \) be a probability on \( X \times Y \) such that (i) \( \mu (E \times F) \) is defined whenever \( E \subseteq \mathcal{F} \), the Baire \( \sigma \)-algebra of \( X \), and \( F \in T \) (ii) \( \mu (X \times F) = \nu F \) for every \( F \subseteq T \). Then there is a family \( \{ \nu_y \}_{y \in Y} \) of Radon probabilities on \( X \) such that \( \int_F \nu_y E \nu(dy) \) exists and is equal to \( \rho (E \times F) \) whenever \( E \subseteq \mathcal{F} \) and \( F \subseteq T \).

**proof** Write \( L^\infty \) for the space of bounded \( T \)-measurable functions from \( Y \) to \( \mathbb{R} \). Because \((Y, T, \nu)\) is complete, there is a ‘linear lifting’ \( \rho : L^\infty \to L^\infty \) such that

\[
\rho (f + g) = \rho (f) + \rho (g), \quad \rho (\alpha f) = \alpha \rho (f) \quad \text{for all } f, g \in L^\infty \text{ and } \alpha \in \mathbb{R};
\]
if \( f = 0 \) \( \nu \)-a.e., then \( \rho (f) = 0 \);

\[
\rho (f) = f \quad \text{\( \nu \)-a.e. for every } f \in L^\infty;
\]
if \( f \geq 0 \) then \( \rho (f) \geq 0 \);

if \( f \) is constant then \( \rho (f) = f \).

(Ionescu Tulcea & Ionescu Tulcea 69, p. 46, Theorem 3, or Fremlin 89, ). Write \( C(X) \) for the set of continuous functions \( g : X \to \mathbb{R} \). For each \( g \in C(X) \), we have a functional \( \nu_g : T \to [0, 1] \) given by the formula \( \nu_g F = \int_{X \times F} g(x) \mu (dx dy) \) for every \( F \subseteq T \); now \( \nu_g \) is additive and dominated by \( \nu \), so by the Radon-Nikodým theorem (Fremlin 74, 63J) there is a bounded \( T \)-measurable function \( f_g : Y \to \mathbb{R} \) such that \( \nu_g F = \int_F f_g \nu \, d\nu \) for every \( F \subseteq T \). Set \( f'_g = \rho (f_g) \). If \( g, h \in C(X) \), then \( \nu_{g+h} = \nu_g + \nu_h \) so \( f_{g+h} = f_g + f_h \) \( \nu \)-a.e. and \( f'_{g+h} = f'_g + f'_h \). Similarly, \( f''_g = \alpha f_g \) for every \( g \in C(X) \), \( \alpha \in \mathbb{R} \). Also, if \( g \geq 0 \) in \( C(X) \), then \( \nu_g \geq 0 \) so \( f_g \geq 0 \) a.e. and \( f''_g \geq 0 \).

For each \( g \in Y \), define \( \phi_g : C(X) \to \mathbb{R} \) by setting \( \phi_g (g) = f'_g (y) \) for every \( g \in C(X) \). Then \( \phi_g \) is a positive linear functional. Also, writing \( 1 \) for the function with constant value 1, \( \nu_1 = \nu \) so \( f_1 = 1 \) a.e. and \( f'_1 = 1 \) everywhere; thus \( \phi_1 (1) = 1 \) for every \( y \). By the Riesz representation theorem (Fremlin 74, 73D) there is for each \( y \in Y \) a Radon probability \( \mu_\gamma \) on \( X \) such that \( \int g d\mu_\gamma = \phi_g (g) \) for every \( g \in C(X) \). Consequently, if \( g \in C(X) \) and \( F \subseteq T \),

\[
\int_F (\int g d\mu_\gamma) \nu (dy) = \int_F f'_g (y) \nu (dy) = \int_F f_g \nu = \nu_g F = \int_{X \times F} g(x) \mu (dx dy).
\]

Consider the set \( H \) of functions \( h : X \to [0, 1] \) such that

\[
\int_F \left( \int h(x) \mu_y (dx) \right) \nu (dy) = \int_{X \times F} h(x) \mu (dx dy) \quad \forall \ F \subseteq T.
\]

\( H \) contains all continuous functions from \( X \) to \([0, 1]\) and \( \lim_{n \to \infty} h_n \in H \) whenever \( \langle h_n \rangle_{n \in \mathbb{N}} \) is a pointwise convergent sequence in \( H \). Consequently \( H \) contains the characteristic function of any set \( E \subseteq \mathcal{F} \); that is,

\[
\int_F \mu_y E \mu (dy) = \mu (E \times F)
\]

for every \( E \subseteq \mathcal{F} \), \( F \subseteq T \), as required. (Compare Ionescu Tulcea & Ionescu Tulcea 69, chap. IX.)

**SA2C Universally negligible sets** If \( X \) is a Polish space (that is, a separable metrizable space in which the topology can be defined by a complete metric) a set \( A \subseteq X \) is **universally negligible** if it is negligible for every Radon measure on \( X \).
Appendix: Useful Facts

SA1 Combinatorics

SA1A Lemma There is a family \( (e_{\alpha})_{\alpha<\omega_1} \) such that

(i) for each \( \alpha \), \( e_{\alpha} \) is an injective function from \( \alpha \) to \( \mathbb{N} \);

(ii) if \( \alpha \leq \beta < \omega_1 \), \( \{ \xi : \xi < \alpha, e_{\alpha}(\xi) \neq e_{\beta}(\xi) \} \) is finite.

proof Construct the \( e_{\alpha} \) inductively. The inductive hypothesis must include a third condition

(iii) \( N \setminus e_{\alpha}[\alpha] \) is infinite for every \( \alpha \).

The induction starts trivially with \( e_0 = \emptyset \). For the inductive step to \( \beta + 1 \), take any \( n \in N \setminus e_{\beta}[\beta] \), and set

\[ e_{\beta+1}(\xi) = e_{\beta}(\xi) \text{ if } \xi < \beta , \quad e_{\beta+1}(\beta) = n. \]

For the inductive step to a non-zero limit ordinal \( \beta < \omega_1 \), take a strictly increasing sequence \( \{ \beta_n \}_{n \in \mathbb{N}} \) with supremum \( \beta \), starting with \( \beta_0 = 0 \). Observe that, for any \( n \),

\[ A_n = N \setminus \bigcup_{i \leq n} e_{\beta_i}[\beta_i] \supseteq (N \setminus e_{\beta_n}[\beta_n]) \setminus \{ e_{\beta_i}(\xi) : i \leq n, \xi < \beta_i , \ e_{\beta_i}(\xi) \neq e_{\beta_n}(\xi) \} \]

is infinite. So we can choose a sequence \( \{ m_n \}_{n \in \mathbb{N}} \) of distinct integers such that \( m_n \in A_n \) for every \( n \). Set \( B = \{ m_n : n \in \mathbb{N} \} \); then \( B \) is infinite and \( B \cap e_{\beta_n}[\beta_n] \) is finite for every \( n \).

Set \( C_{mn} = \{ \xi : \xi < \beta_m , \ e_{\beta_m}(\xi) \neq e_{\beta_n}(\xi) \} \) for \( m \leq n \in \mathbb{N} \). These sets are all finite, so

\[ C_n = e_{\beta_n+1}^{-1}[B \cup \bigcup_{m \leq n} e_{\beta_m}[C_{m,n+1}]] \]

is always finite. If \( \xi \in D_n = \beta_{n+1} \setminus (\beta_n \cup C_n) \), then \( e_{\beta_{n+1}}(\xi) \notin \bigcup_{m \leq n} e_{\beta_m}[\beta_m] \). If \( m \leq n \) and \( \eta < \beta_m \), then either \( \eta \notin C_{m,n+1} \), so that \( e_{\beta_m}(\eta) = e_{\beta_{n+1}}(\eta) \neq e_{\beta_{n+1}}(\xi) \) or \( \eta \in C_{m,n+1} \) so that \( e_{\beta_{n+1}}(\xi) \neq e_{\beta_m}(\eta) \) because \( \xi \notin e_{\beta_n}^{-1}[e_{\beta_m}[C_{m,n+1}]] \).

Set \( C = \bigcup_{n \in \mathbb{N}} C_n \setminus \beta_n = \beta \setminus \bigcup_{n \in \mathbb{N}} D_n \), and let \( f : C \to B \) be an injective function such that \( B \setminus f[C] \) is infinite. Now define \( e_{\beta} : \beta \to N \) by setting

\[ e_{\beta}(\alpha) = e_{\beta_{n+1}}(\alpha) \text{ if } \alpha \in D_n, \]

\[ = f(\alpha) \text{ if } \alpha \in C. \]

I check the three requirements of the inductive hypothesis in turn.

(i) Because \( f \) and all the \( e_{\beta_m} \) are injective and

\[ e_{\beta_{n+1}}[D_n] \cap e_{\beta_m}[\beta_m] = e_{\beta_{n+1}}[D_n] \cap B = \emptyset \]

whenever \( m \leq n \), \( e_{\beta} \) is injective.

(ii) For \( n \in \mathbb{N} \), set \( E_n = \{ \xi : \xi < \beta_n , \ e_{\beta_n}(\xi) \neq e_{\beta}(\xi) \} \). Then

\[ E_{n+1} \subseteq \bigcup_{m \leq n} (E_m \cup C_{m,n+1}) \cup C_n \]

for each \( n \), so every \( E_n \) is finite. If \( \alpha < \beta \), then there is some \( n \) such that \( \alpha \leq \beta_n \), so that

\[ \{ \xi : \xi < \alpha , \ e_{\alpha}(\xi) \neq e_{\beta}(\xi) \} \subseteq E_n \cup \{ \xi : \xi < \alpha , \ e_{\alpha}(\xi) \neq e_{\beta_n}(\xi) \} \]

is finite.

(iii) Because \( e_{\beta_{n+1}}^{-1}[B] \subseteq C_n \), \( e_{\beta_{n+1}}[D_n] \cap B = \emptyset \) for every \( n \), so \( e_{\beta}[\beta] \) does not meet \( B \setminus f[C] \) is \( N \setminus e_{\beta}[\beta] \) is infinite.

Thus the induction continues.

Remark This is due to .

Version of 22.5.00

SA2. Measure theory

SA2A \( \tau \)-additive Borel measures Let \( X \) be a regular topological space and \( \mu \) a totally finite \( \tau \)-additive Borel measure on \( X \). Then the completion \( \hat{\mu} \) of \( \mu \) is a quasi-Radon measure on \( X \). \[ \text{FREMLIN p00*}, \text{415Cb}. \]

Q
**SA2B Disintegrations** Let \((X, \Sigma)\) be a compact Hausdorff space and \((Y, T, \nu)\) a complete probability space. Let \(\mu\) be a probability on \(X \times Y\) such that (i) \(\mu(E \times F)\) is defined whenever \(E \in \Sigma\), the Baire \(\sigma\)-algebra of \(X\), and \(F \in T\) (ii) \(\mu(X \times F) = \nu F\) for every \(F \in T\). Then there is a family \(\{\mu_y\}_{y \in Y}\) of Radon probabilities on \(X\) such that \(\int_F \mu_y E \nu(dy) = \nu F\) whenever \(E \in \Sigma\) and \(F \in T\).

**proof** Write \(\mathcal{L}^\infty\) for the space of bounded \(T\)-measurable functions from \(Y\) to \(\mathbb{R}\). Because \((Y, T, \nu)\) is complete, there is a ‘linear lifting’ \(\rho: \mathcal{L}^\infty \rightarrow \mathcal{L}^\infty\) such that
\[
\rho(f + g) = \rho(f) + \rho(g), \quad \rho(\alpha f) = \alpha \rho(f) \quad \text{for all } f, g \in \mathcal{L}^\infty \text{ and } \alpha \in \mathbb{R};
\]
if \(f = 0\) \(\nu\)-a.e. then \(\rho(f) = 0\);
\[
\rho(f) = f \text{ } \nu\text{-a.e. for every } f \in \mathcal{L}^\infty;
\]
if \(f \geq 0\) then \(\rho(f) \geq 0\);
\[
\text{if } f \text{ is constant then } \rho(f) = f.
\]
(Ionescu Tulcea & Ionescu Tulcea 69, p. 46, Theorem 3, or Fremlin p00*, §341). Write \(C(X)\) for the set of continuous functions \(g: X \rightarrow \mathbb{R}\). For each \(g \in C(X)\), we have a functional \(\nu_g : T \rightarrow [0, 1]\) given by the formula \(\nu_g g = \int_{X \times F} g(x) \mu(dx\,dy)\) for every \(F \in T\); now \(\nu_g\) is additive and dominated by \(\nu\), so by the Radon-Nikodym theorem (Fremlin 74, 63J) there is a bounded \(T\)-measurable function \(f_g : Y \rightarrow \mathbb{R}\) such that \(\nu_g g = \int_F f_g \, d\nu\) for every \(F \in T\). Set \(f'_g = \rho(f_g)\). If \(g, h \in C(X)\), then \(\nu_{g+h} = \nu_g + \nu_h\) so \(f'_{g+h} = f'_g + f'_h\). Similarly, \(f'_{\alpha g} = \alpha f_g\) for every \(g \in C(X)\), \(\alpha \in \mathbb{R}\). Also, if \(g \geq 0\) in \(C(X)\), then \(\nu_g \geq 0\) so \(f'_g \geq 0\) and \(f'_g \geq 0\).

For each \(y \in Y\), define \(\phi_y : C(X) \rightarrow \mathbb{R}\) by setting \(\phi_y(g) = f'_g(y)\) for every \(g \in C(X)\). Then \(\phi_y\) is a positive linear functional. Also, writing \(1\) for the function with constant value 1, \(\nu_1 = \nu \) so \(f'_1 = 1 \text{ } \nu\text{-a.e. and } f'_1 = 1 \text{ } \mu\text{-a.e. everywhere; thus } \phi_y(1) = 1\) for every \(y\). By the Riesz representation theorem (Fremlin 74, 73D) there is for each \(y \in Y\) a Radon probability \(\mu_y\) on \(X\) such that \(\int g \, d\mu_y = \phi_y(g)\) for every \(g \in C(X)\). Consequently, if \(g \in C(X)\) and \(F \in T\),
\[
\int_F \int g \, d\mu_y \, d\nu(dy) = \int_F f'_g \, d\nu = \nu g d\nu = \int_{X \times F} g(x) \mu(dx\,dy).
\]
Consider the set \(H\) of functions \(h: X \rightarrow [0, 1]\) such that
\[
\int_F \left( \int h(x) \mu_y(dx) \right) \nu(dy) = \int_{X \times F} h(x) \mu(dx\,dy) \quad \forall \ F \in T.
\]
\(H\) contains all continuous functions from \(X\) to \([0, 1]\) and \(\lim_{n \to \infty} h_n \in H\) whenever \((h_n)_{n \in \mathbb{N}}\) is a pointwise convergent sequence in \(H\). Consequently \(H\) contains the characteristic function of any set \(E \in \Sigma\); that is,
\[
\int_F \mu_y \, E \mu(dy) = \mu(E \times F)
\]
for every \(E \in \Sigma\), \(F \in T\), as required. (Compare Ionescu Tulcea & Ionescu Tulcea 69, chap. IX, and Fremlin p00*, §452.)

**SA2C Universally negligible sets** If \(X\) is a Polish space (that is, a separable metrizable space in which the topology can be defined by a complete metric) a set \(A \subseteq X\) is universally negligible if it is negligible for every Radon measure on \(X\).

**SA2D Theorem** Let \(X\) be a Hausdorff space and suppose that there is a Radon probability measure \(\mu\) on \(X\) with Maharam type \(\kappa \geq \omega\). Then whenever \(\omega \leq \lambda < \kappa\) there is a Maharam homogeneous Radon probability measure \(\mu_1\) on \(X\) with Maharam type \(\lambda\).

**proof**
antichain 2A, A1P
Aronszajn tree see $\kappa$-Aronszajn tree (A1Ob)
arrow relation 5O, 5P, A1S
atom in a measure space A2Ac, A2Bg
atomless Boolean algebra 9D
atomless measure space A2Be, A2Kc
atomlessly-measurable cardinal 1C
Baire $\sigma$-algebra A2G, A3B
Banach-Ulam problem 1A, 1B
Baumgartner J.E. A4H
Borel measure on a topological space A2V
Borel measure-complete topological space 6M, A2V, A2W
cardinal exponentiation 5E, 7P, 7Q, 9L, 9O
ccc Boolean algebra A2Hb
ccc p.o.set 9F
ccc partially ordered set 7C, 7D, P4
chains in $\mathcal{P}N$ 5K
club filter A1E, A4D
cofinal set in a partially ordered set A1Ab
cofinality of a partially ordered set 7H, 7J, 7K, 7L, 7M, 7N, 9O, A1A (A1Ab ), S6D, S7B, S7C, S9A (A1Ab), A1Ba, A1H, A1I, P4, P5
collectionwise normal topological space 8F, 8N, A3Cc, A3D
compatible elements 2A, A1P
complete filter see $\kappa$-complete filter (A1Bb)
complete measure space A2Bf, A2Cd, A2Db, A2Gd, A2J, A2Kc, A2Q
complete metric space 6M, A2W
completely generating set in a Boolean algebra A2Ha
consistency of axioms 1E, 2E, 2I, 2L, 8C, 8I, 9G
covering number see cov($X, \mathcal{A}$) (A2Pa), cov($X, \mathcal{N}_\mu$), cov$_{Sh}(\alpha, \beta, \gamma, \delta)$ (A1J)
cylinder set A2Ga
decomposable measure space A2Bd, A2Hh, A2Ja
decomposable ordinal 8P, A1Dd
Dedekind $\sigma$-complete Boolean algebra 8R
dense subset of a p.o.set 2A
density of a topological space 6M, A2Hi, A3Ea
diagonal intersection of a family of sets 4B, A1E
diamond see $\Diamond_c$
discrete family of points A3Cb, A3F
discrete family of sets 8E, A3C (A3Cb), A3F
disintegration of a measure SA2B
down-topology of a p.o.set 2A
duality between linear spaces A2O
elementary embedding P2
entangled set 7E, 7F
finite-support product of partially-ordered sets A1R
Fleissner W.G. 8M
forcing 2A
Fubini’s theorem see repeated integral
fully stationary see $\Pi^1_2$-fully stationary (4Db)
GCH 2D, 2E
Gitik M. Intro.
Gitik-Shelah theorem 3F, 9H
greatly Mahlo cardinal 4Ac, 4B, 4C, 4G, 4I, 4K, 9J
greatly Mahlo filter 4Ac, 4B, 4G, 4K

Haydon R.G. Intro. hereditarily separable topological space S7'A
Hilbert space 5H
homogeneous Boolean algebra A2Gf
hyperstonian space A2L (A2La)

ideal of sets A1Ba
image measure A2Db
inaccessible cardinal see weakly inaccessible (A1Da), strongly inaccessible (A1Da)
incompatible elements in a p.o.set 2A
indecomposable ordinal 8P, A1Dd
indescribable cardinal see \(\Pi^m_n\)-indescribable, weakly \(\Pi^m_n\)-indescribable (§A4)
in inner model 2D, P1
inner regular measure A2J, A2V, A2W
inverse-measure-preserving function 1J, 3H, A2Da, A2Fe, A2Gd, A2Jd, A2K, A2Lb

Jónsson algebra 5D, 9L, A1Oa
Jensen R. 1H
Juhász I. 8M

Kunen K. Intro., 1E, 2C, 3A, 4L, 4T, 5G, 5L, 5N, 5P, 8C, 8J, 9N, A4H, A4J

Lebesgue H. Intro.
Lebesgue measure Intro., 1A, 1D, A2Gb, A2Kc, P4
lifting of a measure space A2Q (A2Qa)
Lifting Theorem A2Q locally compact topological space 8N , S7'A locally countable topological space S7'A
locally determined measure space A2B (A2Bc), A2Cd, A2J
lower integral 6A, A2E
lower semi-continuous function 2J
Luxemburg W.A.J. Intro.

Maharam homogeneous measure 1G, 1H, 2H, 3J-3M, A2H (A2Hg), A2Ka, A2Y, P2
Maharam type of a measure space 2H, 2I, 3F, 3G, 3H, 3I, 3J, 3K, 3L, 3M, 6D, 6I, 7Q, 8A, A2Hg, A2Ka, A2Y, P2, P6
Maharam's Theorem A2I
Mahlo cardinal see weakly Mahlo (4Aa), greatly Mahlo (4Ac), \(\alpha\)-Mahlo (4B)
Mahlo's operation 4A (4Ab), 4B, 4F, 9J, A4F
Martin's Axiom 9G
Mathias A.R.D. Intro.
meager sets in \(\mathbb{R}\) P4
measurable algebra 2B, 8R, A2Fc
measurable cardinal see two-valued-measurable (1Cb)
measurable function A2Ae, A2Gg
measure algebra 6C, A2F (A2Fa), A2G, A2I, A2Jd, A2La, A2Q, P2, P6
measure algebra of a measure space A2Fd, A2La, A2U
measure space A2A
measure-preserving homomorphism A2Fe, A2Gd, A2Jb, A2Jd
measure-preserving transformation 1J; see also inverse-measure-preserving (A2Da)
metric of a measure algebra A2Fb, A2Hi
metric space 6M, A2W
metrizable space 8G, A3D, A3F
Moore space 8G, A3Ca, A3D

negligible set A2Ad
NMA $8G, 8H, 8I, 8J, 8K, 8M, 8N, 8R, 8S$
non-stationary ideal A1E
normal filter 1Fb, 1G, 1I, 4Ac, 4F, 9I, A1E, A4F
normal ideal 1F (1Fa), 9B, 9C, 9K, A1E, P7
normal measure 1Fc, 1G, 8H
normal measure axiom see NMA (8H)
normal topological space 8E, 8F, 8G, 8M, 8N
normal witnessing probability 1H, 1I, 2J, 3L, 6A, P2, P7
Nyikos P.J. 8M
order-closed subalgebra of a Boolean algebra A2H, A2Ic
order-continuous linear functional 6N
partially ordered set $A1Aa$; see also p.o.set (2A)
partition relations see arrow relation (A1S)
pcf theory A11
PMEA $8B, 8C-8G, 8J, 8S, 9H$
p.o.set 2A; see also partially ordered set (A1Aa)
precaliber of a Boolean algebra 5D, 6C, A2T, A2U
Prikry K. Intro., 6F, 9F
principal ideal in a Boolean algebra A2Gf, A2Hc
probability $A2Ba$
probability algebra $A2Fa$
probability space $A2Ba$
product measure A2G, P9
product measure extension axiom see PMEA (8B)
product of partially ordered sets 7C, 7D, 7I, 7L-7N, 9O, $A1H$, A1I, A1R, P5, S7C, S9A
purely atomic Boolean algebra 9D
purely atomic measure space $A2Bc$
pushing-down lemma A1E
qm filter 9C, 9J
qm ideal 9C, P7
quasi-Radon measure space 3A, 3B, 6D, A2J (A2Ja), A2V, S6B, SA2A
quasi-measurable cardinal §9 (9C), P7, P8
Radon measure space 3H, 3I, 6A, 6B, 6I, 6L, 8A, A2J (A2Jb), A2K, A2L, A2Pb, A2V, P4, S6B, SA2B
Radon topological space 6M, $A2V$, A2W
Radon-Nikodým theorem 2C, A2Fg
random real p.o.set 2B, 2C, 2J, 4O, 8Q, P1, P7
rapid filter 5G, $A1Cb$
real-valued-measurable cardinal 1C
rectangle algebras 5J
reduced product of partially ordered sets 7I, 7K, 7L, 7M, 9O, $A1H$, A1I, A1Db, A1E
regressive function 1Fa, $A1Db$, A1E
regular cardinal $8P, A1Ab$
regular open algebra 2A, 2B, 9H, $A3A$
regular open set $A3A$
relatively order-closed set 8O, 8P
relatively stationary set 8O, 8P
repeated integral 3A, 5H, 5I, 6A, 6I, 6J, 6K, A2Ge
Riesz space 6N
rvm filter 1I, 2K, 2L, 4K, 4N, 4P, 4Q, 4R, 5D, 6L, 7K, A4L, P3
rvm ideal 1I, 2J, 2L, 4K, 4N, 4P, 4S, 5F, P7
saturated ideal see $\kappa$-saturated ideal (A1Bf)
scalarly measurable function $5H$, $A2O$
second-order language $A4A$
selective ultrafilter $A1Cc$
semi-finite measure space $A2B$ ($A2Bb$)
sequential cardinal $P8$
sequentially continuous function $P8$
sequentially order-continuous functional $6N$
Shelah S. Intro., $5D$, $A1I$
Shelah covering number see $cov_{Sh}(\alpha, \beta, \gamma, \delta)$ ($A1J$
simple group $8R$
singular cardinals hypothesis $8K$, $A1N$
Solovay R.M. Intro., $1E$, $1H$, $6F$
Solovay’s construction $2C$, $2H$, $2I$, $4R$, $P1$, $P2$, $P3$, $P7$, $P9$
Souslin number of a partially ordered set $7C$, $7D$, $A1P$, $A1R$, $P4$; of a Boolean algebra $5D$, $A1P$
Souslin’s hypothesis $7D$
stationary set $4Aa$, $4F$, $A1Dc$, $A1F$, $A1G$
strictly localizable measure space $A2Bd$
strong law of large numbers $A2X$
strongly compact cardinal $8C$
strongly inaccessible cardinal $1D$, $A1Da$, $A1S$
subspace of a measure space $A2Ab$, $A2J$
successor cardinal $8P$
supercompact cardinal $8I$
third-order language $A4K$
Todorčević S. $5P$, $7F$
topological group $S7E$
topology base $A3F$
totally finite measure $A2Ba$
totally ordered set $7E$, $A1Ac$
trivial measure space $1A$, $1B$, $1D$
two-valued-measurable cardinal $1Cb$, $1D$, $1E$, $1Gb$, $1Ha$, $1I$, $2D$, $2E$, $2H$, $2I$, $2K$, $2L$, $5N$, $5P$, $9G$, $A4L$, $P1$, $P7$
Ulam S. Intro.
Ulam matrix $1E$
Ulam’s Dichotomy $1E$
Ulam’s Theorem $1D$
uniform filter $A1Bc$
universally negligible set $S6C$, $SA2C$
up-antichain in a partially ordered set $A1P$
upper integral $3A$, $6A$, $A2E$, $A2Je$
upwards-ccc partially ordered set $A1Q$
upwards-compatible elements in a partially ordered set $A1P$
usual measure on $\{0, 1\}^I$ $A2G$ ($A2Ga$), $A2I$, $A2Jc$, $A2Pb$
vector lattice $6N$
weakly compact cardinal $A1S$
weakly $\Pi^1_0$-indescribable cardinal $A4C$ ($A4Ca$), $A4D$
weakly $\Pi^1_1$-indescribable cardinal $4Dc$, $4F$, $4G$, $4H$, $4I$, $4K$, $4L$, $4R$, $6L$, $A1Ob$, $A1S$, $A4C$ ($A4Ca$), $A4D$, $A4H$, $P3$
weakly $\Pi^1_1$-indescribable cardinal $A4Ca$, $P4$
weakly $\Pi^1_0$-indescribable cardinal $A4C$ ($A4Ca$), $A4F$, $A4G$, $A4J$
weakly $\Pi^2_1$-indescribable cardinal $4P$, $4Q$, $4R$, $A4C$ ($A4Ca$), $P3$
weakly $\Pi^2_2$-indescribable cardinal $A4K$, $A4L$
weakly Mahlo cardinal 4Aa, 4B, 8P
weakly inaccessible cardinal 1D, 4Aa, 8P, 9B, A1Da
weight of a topological space 3A, 3B, 6J, 6K, A3Ec
witnessing ideal 9C
witnessing probability 1C, P2
Woodin H. 6K
Zakrzewski P. 1J, 6K

a P4
\[ \mathfrak{a} \mathbf{[a] A2Hc} \]
add(\(P\)) see additivity of a partially ordered set (A1Ac)
add(\(\mu\)) see additivity of a measure (A2Cb)

b P4
c = 2^c 1D, 4L, 5E, 5N, 9D, 9L, 9N, 9O, A4D
cf(\(P\)) see cofinality of a partially ordered set (A1Ab)
cov(X, A) A2P (A2Pa), P4
cov(X, \(\mathcal{N}^\mu\)) 2H, 6B, 6L, A2Ph, A2U, P4
cov_{Sh}(\alpha, \beta, \gamma, \delta) 7K, 7O, 9O, A1Ja, A1K

\[ \ell^\infty(X) 6N \]
\[ \mathcal{L} \mathbf{A4A} \]
\[ \mathcal{L}_3 \mathbf{A4K} \]
\[ L(I) 2D, 2K \]
\[ L^\infty(\mathfrak{a}) 2C, \mathbf{A2FF, A2Fg} \]
Mh(A) see Mahlo’s operation (4Ab)
\[ \mathcal{N}^\mu_{\mathfrak{a}} 2I, \mathbf{A2Ad}, S6D; \text{see also} \ cov(X, \mathcal{N}^\mu), \text{non}(X, \mathcal{N}^\mu) \]
non(X, A) A1Be
non(X, \(\mathcal{N}^\mu\)) 6G, P4

d P4
\[ p \mathbf{P4} \]
\[ p(\kappa) \text{-point filter} 5G, \mathbf{A1Ca} \]
\[ p(\kappa) \text{-point filter} 5L, \mathbf{A1Cd} \]
Regr(\(\kappa\)) 4Da
S(\(\mathfrak{R}\)) see Souslin number (A1P)
sat(\(\mathfrak{R}\)) see Souslin number (A1P)
Tr(\(\kappa\)) 3D, 3E, 3K
Tr_{F}(X; Y) 2F, 2G, 2H, 3D, 3M, 5D, P2
\[ U_F \mathbf{4Da, 4E}, 4H \]
\[ \alpha \text{-Mahlo cardinal} 4B \]
\[ \alpha \rightarrow (\beta, \gamma)^2 5Q, 5P, \mathbf{A1S} \]
\[ \Theta(\alpha, \gamma) 7K, 9O, \mathbf{A1Jb, A1K, A1L, A1M} \]
\[ \kappa \text{-additive ideal} 2D, 9B, 9C, 9E, A1B (A1Ba), A1E \]
\[ \kappa \text{-additive measure} 1C, \mathbf{A2Ca}; \text{see also} \ additivity \ of \ a \ measure \ (A2Cb) \]
\[ \kappa \text{-additive partially ordered set} \mathbf{A1Ac} \]
\[ \kappa \text{-Aronszajn tree} 5F, 9M, \mathbf{A1Ob} \]
\[ \kappa \text{-chain condition} A1P \]
\[ \kappa \text{-complete filter} 1C, 1G, \mathbf{A1Bb, A1E} \]
\[ \kappa \text{-measure-bounded partially ordered set} 7G, 7H, 7I, 7J \]
\[ \kappa \text{-saturated ideal} 2D, 9A, 9B, 9E, 9I, 9P, \mathbf{A1Bf} \]
\[ \mu^{f^{-1}} \mathbf{A2Db} \]
\[ \mu[A \mathbf{A2Ab} \]
\[ \mu^+ \mathbf{A2Aa} \]
\[ \Pi_1^1 \text{-filter} 4Dc, 4F, 4G, 4I, 6L, \mathbf{A4Cb} \]
\[ \Pi_\delta \text{formula} 4T, \mathbf{A4B} \]
$\Pi_1^1$-fully stationary $4Db$, $4Dc$, $4K$, $A4H$

$\Pi_1^1$-ideal $A4Cb$, $A4H$

$\Pi_2^1$-filter $A4Cb$, $A4G$, $P3$

$\Pi_2^1$ formula $A4B$, $A4E$

$\Pi_3^1$-ideal $A4Cb$

$\Pi_2^2$-filter $4Q$, $4R$, $A4Cb$

$\Pi_2^2$ formula $4O$, $8Q$, $A4K$

$\Pi_2^2$-ideal $4P$, $A4Cb$

$\Omega_1^1$-filter $A4K$, $A4L$

$\Pi_2^2$-indescribable cardinal $A4K$, $A4L$

$\sigma$-finite measure space $A2Ba$, $A2Bb$

$\sigma$-ideal $9A$, $P9$

$\Sigma_1^n$ formula $4S$, $A4B$

$\Sigma_2^n$ formula $A4B$

$\tau$-additive measure $A2V$, $A2W$, $S6A$, $SA2A$

$\tau$-homogeneous Boolean algebra $A2He$, $A2Hf$, $A2J$

$\tau(3)$ $A2H$ ($A2Ha$), $A2I$

$\chi(a)$ $A2Ff$

$\chi(x, X)$ $8F$, $8M$, $A3Eb$

$\omega$-Tukey function $7G$, $7H$, $7I$

$\omega_1$-saturated ideal $9C$, $9H$

$\models A4A$

$\{0, 1\}^\omega$ $A2G$

$\Diamond_4$ $5N$, $9N$

$\rightarrow$ see $\alpha \rightarrow (\beta, \gamma)^2$ ($A1S$)

$\upharpoonright$ see subspace measure ($A2Ab$)

References

The list here comprises only works referred to in this supplement, but shows where they have been used in the main text as well as the supplement.


Fremlin D.H. [p00*] *Measure Theory*, in preparation. *Drafts may be obtained by anonymous ftp from ftp.essex.ac.uk/pub/measuretheory.*


