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Real-valued-measurable cardinals

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Contents:

Introduction In these notes I seek to describe the theory of real-valued-measurable cardinals, collecting together various results which are scattered around the published literature, and including a good deal of unpublished material.

Almost immediately after Lebesgue introduced his theory of measure and integration, it was established that (subject to the axiom of choice) not every subset of $[0,1]$ is Lebesgue measurable. The question naturally arose, is there any extension of Lebesgue measure to a countably-additive measure defined on $\mathcal{P}[0,1]$? It was early understood that such an extension cannot be translation-invariant. In the seminal paper Ulam 30, S.Ulam showed that a probability space $(X,\mathcal{P}X,\mu)$ in which every subset of X is assigned a probability is either trivial or as complex an object as the set theory of that time could readily envisage. Since then, measure theorists and others have periodically had occasion to wonder whether non-trivial examples do, or can, exist (see, for instance, 6M-6N below), and with less regularity, but with impressive frequency, have turned up some new curiosity concerning them.

The actual definition of 'real-valued-measurable cardinal' involves some technical considerations which I prefer to leave to §1 below; here I will say only that there is a real-valued-measurable cardinal if and only if there is a non-trivial probability space $(X,\mathcal{P}X,\mu)$. Ulam's work already showed, in effect (the terms I use date from later on) that real-valued-measurable cardinals are of two kinds: 'atomlessly-measurable' cardinals, less than or equal to the continuum, associated with extensions of Lebesgue measure to $\mathcal{P}\mathbb{R}$, and 'two-valued-measurable' cardinals, much greater than the continuum, associated with ultrafilters closed under countable intersections.

It was observed by HANF & SCOTT 61 that a two-valued-measurable cardinal has some extraordinary properties from the point of view of mathematical logic. The mixing of combinatorial and metamathematical intuitions and techniques which followed was wonderfully fertile (see, for instance, KEISLER & TARSKI 64), and quickly gave two-valued-measurable cardinals a central place in a rapidly growing theory of 'large cardinals'. I do not propose to describe this theory here; there are accounts in DRAKE 74, KANAMORI & MAGIDOR 78, JECH 78. The relevant measures on a two-valued-measurable cardinal are all purely atomic, and while some of their properties can be described in the language of measure theory, there is little there which is connected with the ordinary concerns of measure theorists or probabilists. I wish rather to look at atomlessly-measurable cardinals, where the deepest concepts of abstract measure theory are both employed and illuminated. Two-valued-measurable cardinals will never be far away, as the work of R.M.Solovay and K.Kunen shows (see SOLOVAY 71 and §2 below); indeed their constructions give general methods for translating ideas about two-valued-measurable cardinals into ideas about atomlessly-measurable cardinals, and many of the results described in this article were suggested in this way.

Recently, the spectacular near-resolution by GITIK & SHELAH 89 of the problem of determining the measure algebra of an atomlessly-measurable cardinal has given new ways of applying Solovay's concept of 'random real forcing', and opens up yet another channel through which ideas developed in other contexts may be applied to the theory of atomlessly-measurable cardinals.

It is fair to say that most of the questions in measure theory depending on the existence of real-valued-measurable cardinals are peripheral. (I do not say this of questions in set theory depending on the existence of, or on supposing the consistency of the existence of, two-valued-measurable cardinals.) However, an atomlessly-measurable cardinal, if one can exist (and there is a problem here; see 1Ee below), necessarily has a structure which makes it as remarkable as anything in measure theory. My aim here is to describe this structure. The investigation will take us into a fascinating blend of measure theory, infinitary combinatorics and metamathematics, drawing on deep ideas from all three.

Out of personal taste and prejudice I will play down the metamathematical aspects, seeking wherever possible to find 'conventional' expressions of the ideas involved. However, deep results from measure theory and set theory are going to be central to my arguments, and many concepts from various branches of mathematics will be called on at some point; so I have written an Appendix to give definitions, statements of theorems I use, and proofs or references. As fundamental references I will take JECH 78 and KUNEN 80 for set theory, ENGELKING 89 for general topology, and my own books FREMLIN 74 and FREMLIN 84 for measure theory and miscellaneous material; I will try to indicate any divergences from these texts in notation.

I have been interested in real-valued-measurable cardinals since 1965, as nearly as I can remember, and cannot trust my memory to give a full list of those from whom I have learnt about them. But I recall that it was W.A.J.Luxemburg who gave a seminar more or less on the subject of 6N, and A.R.D.Mathias who showed me the paper KEISLER & TARSKI 64, the combination being the basis of my first work in this area. My interest was maintained by correspondence with Solovay and conversations with R.G.Haydon in the early seventies. Rather later I learnt of the work of K.Prikry, who introduced me to more of Solovay's ideas. The actual stimulus for writing these notes came from receiving a preprint of GITIK & SHELAH 89 from M.Gitik and from visiting Madison, where Kunen showed me some of his unpublished work. Most recently I have had valuable discussions with Gitik and with S.Shelah. So, pausing for a moment to apologise to those I have missed, I should like to thank all those I have named, for leading me into this garden of delights.

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1. Basic theory

1A The Banach-Ulam problem Can we describe all measure spaces of the form $(X, \mathcal{P}X, \mu)$ in terms enabling us to decide whether μ can be an extension of Lebesgue measure?

There certainly exist measure spaces $(X, \mathcal{P}X, \mu)$, constructed as follows. Let X be any set, $f : X \to [0, \infty[$ any function, and $\mathcal{I}$ any σ-ideal of subsets of X. Write

$$\mu A = \sum_{x\in A} f(x) \text{ if } A \in \mathcal{I},$$
$$= \infty \text{ if } A \in \mathcal{P}X \setminus \mathcal{I}.$$

Then $\mu : \mathcal{P}X \to [0,\infty]$ is countably additive. For the purposes of these notes, I will call a measure space $(X,\mathcal{P}X,\mu)$ **trivial** if it can be obtained by this construction. What I will call the 'Banach-Ulam problem' therefore becomes: is there a non-trivial measure space $(X,\mathcal{P}X,\mu)$?

I include the σ-ideal $\mathcal{I}$ here for the sake of completeness. However, it is clear that a semi-finite trivial measure space (in particular, a trivial probability space) $(X,\mathcal{P}X,\mu)$ must be defined by the function $f(x) = \mu\{x\}$ alone, and that in this case we may take $\mathcal{I} = \mathcal{P}X$, $\mu A = \sum_{x\in A} f(x)$ for every $A \subseteq X$.

1B First reduction Suppose now that there is some non-trivial measure space $(X,\mathcal{P}X,\mu)$. Write

$$f(x) = \mu\{x\} \text{ if } \mu\{x\} < \infty,$$
$$= 0 \text{ if } \mu\{x\} = \infty,$$

and let $\mathcal{I}$ be the σ-ideal of subsets of X generated by $\{A : \mu A < \infty\}$. Let μ' be the measure defined from f and $\mathcal{I}$ by the method of 1A. Then $\mu' A \le \mu A$ for every $A \subseteq X$. By hypothesis, $\mu' \ne \mu$; let $A \subseteq X$ be such that $\mu' A < \mu A$. Surely $A \in \mathcal{I}$; let $\langle A_n\rangle_{n\in\mathbb{N}}$ be an increasing sequence of sets such that $\mu A_n < \infty$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n = A$. Now there must be some $n \in \mathbb{N}$ such that $\mu' A_n < \mu A_n$. Set $\mu'' = \mu\lceil A_n - \mu'\lceil A_n$. Then $(A_n, \mathcal{P}A_n, \mu'')$ is a measure space, with $0 < \mu'' A_n < \infty$ and $\mu''\{x\} = 0$ for every $x \in A_n$. Setting $\hat\mu = (\mu'' A_n)^{-1}\mu''$, we obtain a probability space $(A_n, \mathcal{P}A_n, \hat\mu)$ in which singleton sets are negligible.

Accordingly we can say that there is a positive answer to the Banach-Ulam problem iff there is a probability space $(X,\mathcal{P}X,\mu)$ in which $\mu\{x\} = 0$ for every $x \in X$.

1C Notation For the next step it will be convenient to introduce some phrases which will dominate these notes. Recall that a measure ν is κ**-additive** if $\nu(\bigcup\mathcal{E})$ exists $= \sum_{E\in\mathcal{E}} \nu E$ whenever $\mathcal{E}$ is a disjoint family of ν-measurable sets and $\#(\mathcal{E}) < \kappa$, and that a filter $\mathcal{F}$ is κ**-complete** if $\bigcap\mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{A}) < \kappa$. (See A1B, A2C below.) Now:

(a) A cardinal κ is **real-valued-measurable** if there is a κ-additive probability ν with domain $\mathcal{P}\kappa$ which is zero on singleton sets; in this context I will call such a ν a **witnessing probability** on κ.

(b) A cardinal κ is **two-valued-measurable** (often called just 'measurable') if there is a non-principal κ-complete ultrafilter on κ.

(c) A cardinal κ is **atomlessly-measurable** if there is an atomless κ-additive probability ν with domain $\mathcal{P}\kappa$.

1D Ulam's Theorem (a) Let $(X,\mathcal{P}X,\mu)$ be a probability space, with ideal $\mathcal{N}_\mu$ of negligible sets. Then either it is trivial, and $\mathrm{add}(\mu) = \mathrm{add}(\mathcal{N}_\mu) = \infty$, or $\mathrm{add}(\mu) = \mathrm{add}(\mathcal{N}_\mu)$ is a real-valued-measurable cardinal.

(b) A cardinal is real-valued-measurable iff it is either atomlessly-measurable or two-valued-measurable.

(c) An atomlessly-measurable cardinal is weakly inaccessible and not greater than $\mathfrak{c}$.

(d) A two-valued-measurable cardinal is strongly inaccessible.

(e) There is an extension of Lebesgue measure to a measure defined on every subset of $\mathbb{R}$ iff there is an atomlessly-measurable cardinal.

proof (a) As remarked in A2Cd, $\mathrm{add}(\mu) = \mathrm{add}(\mathcal{N}_\mu)$. If (X,Σ,μ) is trivial, then $\mu X = \sum_{x\in X} \mu\{x\}$ and $\mathrm{add}(\mu) = \infty$. Otherwise, set $H = \{x : \mu\{x\} = 0\}$; then $\mu H > 0$ so $\kappa = \mathrm{add}(\mathcal{N}_\mu) \le \#(X)$. There must be a disjoint family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}_\mu$ such that $E = \bigcup_{\xi<\kappa} E_\xi \notin \mathcal{N}_\mu$. Define $f : E \to \kappa$ by setting $f(x) = \xi$ if $x \in E_\xi$. Write

$$\nu_0 = (\mu\lceil E)f^{-1}$$

(A2Db), so that ν_0 is a κ-additive measure on κ, zero on singletons, and $\nu_0\kappa = \mu E \in]0,\infty[$. Set $\nu = (\mu E)^{-1}\nu_0$; then ν witnesses that κ is real-valued-measurable.

(b)(i) If κ is atomlessly-measurable, of course it is real-valued-measurable. If κ is two-valued-measurable, with witnessing filter $\mathcal{F}$, then we can set

$$\nu A = 1 \ \forall\ A \in \mathcal{F},\ \nu A = 0 \ \forall\ A \in \mathcal{P}\kappa \setminus \mathcal{F},$$

and ν will witness that κ is real-valued-measurable.

(ii) Now suppose that κ is real-valued-measurable, with witnessing probability ν. **($\boldsymbol{\alpha}$)** If $(\kappa, \mathcal{P}\kappa, \nu)$ has an atom $A \subseteq \kappa$, set

$$\mathcal{F} = \{F : F \subseteq \kappa,\ A \setminus F \in \mathcal{N}_\nu\}.$$

Then $\mathcal{F}$ is a κ-complete ultrafilter on κ, so κ is two-valued-measurable. **($\boldsymbol{\beta}$)** Otherwise, $(\kappa, \mathcal{P}\kappa, \nu)$ is atomless, and ν witnesses that κ is atomlessly-measurable.

(c) Let κ be a real-valued-measurable cardinal, with witnessing probability ν.

(i) $\kappa \le \operatorname{add}(\mathcal{N}_\nu) \le \kappa$, because ν is κ-additive and $\kappa = \bigcup \mathcal{N}_\nu$. So $\kappa = \operatorname{add}(\mathcal{N}_\nu)$ is regular (A1Ac).

(ii) ? Now suppose, if possible, that $\kappa = \lambda^+$ for a cardinal $\lambda < \kappa$. For each $\xi < \kappa$, let $f_\xi : \xi \to \lambda$ be an injective function. For $\eta < \kappa$, $\alpha < \lambda$ set

$$A_{\eta\alpha} = \{\xi : \eta < \xi < \kappa,\ f_\xi(\eta) = \alpha\}.$$

For any fixed $\eta < \kappa$,

$$\nu(\bigcup_{\alpha<\lambda} A_{\eta\alpha}) = \nu\{\xi : \eta < \xi < \kappa\} = 1,$$

and ν is λ^+-additive, so there must be a $\beta_\eta < \lambda$ such that $\nu A_{\eta,\beta_\eta} > 0$. Set

$$B_\beta = \{\eta : \eta < \kappa,\ \beta_\eta = \beta\}$$

for $\beta < \lambda$. Then $\nu(\bigcup_{\beta<\lambda} B_\beta) = \nu\kappa = 1$, so there is a $\beta < \lambda$ such that $\nu B_\beta > 0$; in particular, B_β is uncountable. But now observe that if $\eta \in B_\beta$ then $\nu A_{\eta\beta} > 0$, and that $\langle A_{\eta\beta}\rangle_{\eta<\kappa}$ is disjoint, because every f_ξ is injective. So we have

$$1 = \nu\kappa \ge \sum_{\eta<\kappa} \nu A_{\eta\beta} \ge \sum_{\eta\in B_\beta} \nu A_{\eta\beta} = \infty,$$

which is absurd. **X**

(iii) Thus κ is weakly inaccessible. The argument so far assumes only that κ is real-valued-measurable. But if in fact κ is atomlessly-measurable, then $(\kappa, \mathcal{P}\kappa, \nu)$ may be taken to be atomless, in which case there is a function $f : \kappa \to [0,1]$ which is inverse-measure-preserving for ν and Lebesgue measure on $[0,1]$ (A2Kc). Now νf^{-1} is a κ-additive measure on $[0,1]$ which is zero on singletons. Accordingly $\kappa \le \#([0,1]) = \mathfrak{c}$.

(d) Now let κ be a two-valued-measurable cardinal with witnessing ultrafilter $\mathcal{F}$. By (b) and (c-i) above, κ is an uncountable regular cardinal. **?** Suppose, if possible, that $\kappa \le 2^\lambda$ for some cardinal $\lambda < \kappa$. Let $f : \kappa \to \mathcal{P}\lambda$ be any injection. For $\alpha < \lambda$ write

$$A_\alpha = \{\xi : \xi < \kappa,\ \alpha \in f(\xi)\}.$$

Set

$$B = \{\alpha : \alpha < \lambda,\ A_\alpha \in \mathcal{F}\}.$$

Then $\kappa \setminus A_\alpha \in \mathcal{F}$ for $\alpha \in \lambda \setminus B$. Because $\mathcal{F}$ is κ-complete,

$$A = \bigcap_{\alpha\in B} A_\alpha \cap \bigcap_{\alpha\in\lambda\setminus B} (\kappa \setminus A_\alpha) \in \mathcal{F}.$$

Now if $\xi \in A$, $\alpha < \lambda$ then $\alpha \in f(\xi)$ iff $\alpha \in B$; i.e., $f(\xi) = B$ for every $\xi \in A$. But of course A is not a singleton (because $\mathcal{F}$ is non-principal), so f cannot be injective. **X**

Thus κ is strongly inaccessible.

(e) This is now easy. **(i)** If κ is an atomlessly-measurable cardinal, with witnessing probability ν, let $f : \kappa \to [0,1]$ be inverse-measure-preserving for ν and Lebesgue measure on $[0,1]$, as in (c-iii) above. Define $\mu : \mathcal{P}\mathbb{R} \to [0,\infty]$ by writing

$$\mu A = \sum_{n\in\mathbb{Z}} \nu f^{-1}[A+n]\ \ \forall\ \ A \subseteq \mathbb{R};$$

then μ is a countably-additive (in fact, κ-additive) extension of Lebesgue measure to $\mathcal{P}\mathbb{R}$. **(ii)** If μ is a countably-additive extension of Lebesgue measure to $\mathcal{P}\mathbb{R}$, then $\kappa = \operatorname{add}(\mu\lceil[0,1])$ is real-valued-measurable, by (a) above; but as $\kappa \le \mathfrak{c}$, κ cannot be two-valued-measurable, by (d), and (b) tells us that κ is actually atomlessly-measurable.

1E Remarks (a) The theorem above is taken virtually directly from Ulam 30. Evidently (c)-(d) show that the division of real-valued-measurable cardinals into atomlessly-measurable cardinals and two-valued-measurable cardinals is exclusive; this is 'Ulam's Dichotomy'. In §2 below we shall see an extraordinary reunification of these two phenomena at a higher level.

Readers may recognise the family $\langle A_{\eta\alpha}\rangle_{\alpha<\lambda,\eta<\lambda^+}$ of part (c-ii) of the proof above as an 'Ulam matrix'; this concept has many other applications (see Erdös Hajnal Máté & Rado 84) and is one of the principal contributions of the Banach-Ulam problem to mathematics.

(b) It is worth noting explicitly that Ulam's Dichotomy is sharp. If κ is a real-valued-measurable cardinal, with witnessing probability ν, then either $(\kappa, \mathcal{P}\kappa, \nu)$ is atomless and κ is atomlessly-measurable, or it is purely atomic and κ is two-valued-measurable. For in (b-ii-α) of the proof above we saw that if $(\kappa, \mathcal{P}\kappa, \nu)$ is not atomless then κ is two-valued-measurable. While if $(\kappa, \mathcal{P}\kappa, \nu)$ is not purely atomic, there is an $A \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ such that $(A, \mathcal{P}A, \nu\lceil A)$ is atomless; if we set $\nu' B = \nu(B \cap A)/\nu A$ for every $B \subseteq \kappa$, then ν' will witness that κ is atomlessly-measurable. And, as already remarked, no cardinal can be simultaneously atomlessly-measurable and two-valued-measurable.

(c) The phrase 'two-valued-measurable' is used just because there is a natural correspondence between ω_1-complete filters and complete measures taking exactly the values 0 and 1, as described in the formulae of part (b) of the proof above. We shall find that this language enables us to unify certain arguments, as in 1G below. Of course there is not much measure theory to be found in a $\{0,1\}$-valued measure, and the qualities of two-valued-measurable cardinals and atomlessly-measurable cardinals are rather different. At the right metamathematical level, they come together again, as the work of Solovay and Kunen shows; one of the purposes of these notes is to try to describe the combinatorial foundations of this reunification.

(d) Note that Ulam's theorem, while a large step forward, does not give us a working description of all measure spaces $(X, \mathcal{P}X, \mu)$, even if we think we understand real-valued-measurable cardinals. Rather, it gives lower bounds to the possible complexity of a non-trivial $(X, \mathcal{P}X, \mu)$. I will return to this question later (3L-3M, §8).

(e) Because it is relatively consistent with ZFC to suppose that there are no weakly inaccessible cardinals, it is relatively consistent to suppose that every measure space $(X, \mathcal{P}X, \mu)$ is trivial in the sense of 1A. It remains open, in a sense, whether it is relatively consistent with ZFC to suppose that there is a real-valued-measurable cardinal, and therefore a non-trivial measure space $(X, \mathcal{P}X, \mu)$. However, very much stronger assertions have been explored systematically in the last two decades, without so far leading to any contradiction; and at present almost no-one is seriously searching for a proof in ZFC that real-valued-measurable cardinals don't exist. The rest of these notes will tacitly assume that it is consistent to suppose that there is at least one real-valued-measurable cardinal. Those unhappy with such an assumption may however prefer to regard them as preliminary investigations which might eventually lead to a proof by contradiction that there are no real-valued-measurable cardinals. I assure you that such a proof would make you famous enough to justify any effort you put into learning this material.

1F Definition Let κ be a regular uncountable cardinal.

(a) An ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ is **normal** if it is proper, includes κ (or I could say, contains every subset of κ of cardinal less than κ), and whenever $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and $f : S \to \kappa$ is regressive (that is, $f(\xi) < \xi$ for every non-zero $\xi \in S$), then there is a $\xi < \kappa$ such that $f^{-1}[\{\xi\}] \notin \mathcal{I}$.

(b) A filter $\mathcal{F}$ on κ is **normal** if its dual ideal $\{\kappa \setminus F : F \in \mathcal{F}\}$ is normal.

(c) A measure ν on κ is **normal** if the ideal $\mathcal{N}_\nu$ is normal.

Remarks The definition of 'normal' given here is adapted to the needs of the next theorem, but is not quite standard. For the usual definition, and elementary properties of these ideals and filters, see A1E below. It is worth noting immediately that the intersection of any non-empty family of normal ideals or filters is again a normal ideal or filter.

1G Theorem (a) Let κ be an atomlessly-measurable cardinal. Then there is a Maharam homogeneous normal atomless κ-additive probability with domain $\mathcal{P}\kappa$.

(b) Let κ be a two-valued-measurable cardinal. Then there is a normal κ-complete non-principal ultrafilter on κ.

proof In case (a), start from an atomless κ-additive probability ν with domain $\mathcal{P}\kappa$. In case (b), start from a κ-complete non-principal ultrafilter $\mathcal{F}$ on κ, and construct ν from $\mathcal{F}$ by setting $\nu A = 1$ if $A \in \mathcal{F}$, $\nu A = 0$ if $A \in \mathcal{P}\kappa \setminus \mathcal{F}$. Then in both cases ν is a κ-additive probability with domain $\mathcal{P}\kappa$ which is zero on singletons.

Let F be the set of all functions $f : \kappa \to \kappa$ such that $\nu f^{-1}[\zeta] = 0$ for every $\zeta < \kappa$. Then there is an $f_0 \in F$ such that

$$\nu\{\xi : f(\xi) < f_0(\xi)\} = 0 \ \forall \ f \in F.$$

P Note first that if f, $g \in F$ then $f \wedge g \in F$, where $(f \wedge g)(\xi) = \min(f(\xi), g(\xi))$ for every $\xi \in \kappa$. **?** If there is no suitable $f_0 \in F$, then we may define inductively a decreasing family $\langle g_\alpha \rangle_{\alpha<\omega_1}$ in F, as follows. Set $g_0(\xi) = \xi$ for every $\xi < \kappa$; then $g_0 \in F$ because ν is κ-additive and zero on singletons. Given $g_\alpha \in F$, take $g'_\alpha \in F$ such that $\nu E_\alpha > 0$, where $E_\alpha = \{\xi : g'_\alpha(\xi) < g_\alpha(\xi)\}$, and set $g_{\alpha+1} = g_\alpha \wedge g'_\alpha$. Given $\langle g_\beta \rangle_{\beta<\alpha}$, where α is a non-zero countable limit ordinal, set

$$g_\alpha(\xi) = \min_{\beta<\alpha} g_\beta(\xi) \ \forall \ \xi < \kappa;$$

then $g_\alpha \in F$ because

$$\nu g_\alpha^{-1}[\zeta] = \nu(\bigcup_{\beta<\alpha} g_\beta^{-1}[\zeta]) = 0 \ \forall \ \zeta < \kappa.$$

Now consider the family $\langle E_\alpha \rangle_{\alpha<\omega_1}$. By A2Mb, there is a $\xi < \kappa$ such that

$$A = \{\alpha : \xi \in E_\alpha\}$$

is infinite. But if $\langle \alpha(n) \rangle_{n\in\mathbb{N}}$ is any strictly increasing sequence in A,

$$g_{\alpha(n)}(\xi) > g'_{\alpha(n)}(\xi) = g_{\alpha(n)+1}(\xi) \geq g_{\alpha(n+1)}(\xi)$$

for every $n \in \mathbb{N}$, and $\langle g_{\alpha(n)}(\xi) \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

Now set $\nu_0 = \nu f_0^{-1}$. We see that ν_0 is a κ-additive probability defined on $\mathcal{P}\kappa$, because ν is. Because $f_0 \in F$, $\nu_0\{\xi\} \leq \nu f_0^{-1}[\xi + 1] = 0$ for every $\xi < \kappa$, and ν_0 is zero on singletons.

The next step is to show that ν_0 is normal. **P** Take $S \in \mathcal{P}\kappa \setminus \mathcal{N}_{\nu_0}$, and a regressive function $f : S \to \kappa$. Extend f to a function $g : \kappa \to \kappa$ by setting $g(\xi) = \xi$ for $\xi \in \kappa \setminus S$. Consider $f_1 = g \circ f_0$. Then $\nu\{\xi : f_1(\xi) < f_0(\xi)\} \geq \nu f_0^{-1}[S \setminus \{0\}] = \nu_0(S \setminus \{0\}) > 0$, so $f_1 \notin F$ and there is a $\zeta < \kappa$ such that $\nu f_1^{-1}[\zeta] > 0$; because ν is κ-additive, there is a $\xi < \zeta$ such that $0 < \nu f_1^{-1}[\{\xi\}] = \nu_0 f^{-1}[\{\xi\}]$, because $f_1^{-1}[\{\xi\}] \subseteq f_0^{-1}[f^{-1}[\{\xi\}]] \cup f_0^{-1}[\{\xi\}]$. As S and f are arbitrary, ν_0 is normal. **Q**

By A2Hh, there is an $E \subseteq \kappa$ such that $\nu_0 E > 0$ and $\nu_0 \lceil E$ is Maharam homogeneous. Set $\nu_1 A = \nu_0(A \cap E)/\nu_0 E$ for every $A \subseteq \kappa$; then it is easy to check that ν_1 is a Maharam homogeneous normal κ-additive probability on κ.

Now let us re-examine the two cases (a), (b). In case (a), $\kappa \leq \mathfrak{c}$ so by 1D and 1Eb ν_1 must be atomless, and satisfies the requirements of (a). In case (b), ν takes only the values 0 and 1, so ν_0 also takes only these values, and $\mathcal{P}\kappa \setminus \mathcal{N}_{\nu_0}$ is an ultrafilter; an elementary check will confirm that it is κ-complete and normal and non-principal, as demanded by (b).

1H Remarks (a) This theorem is due in the first place to KEISLER & TARSKI 64, who proved it for two-valued-measurable cardinals; the adaptation to atomlessly-measurable cardinals is due independently to Solovay (SOLOVAY 71), R.Jensen and myself. Part of the proof reappears in 2G; an extension of the theorem, due to Solovay, is in 9B.

(b) The original impulse behind this theorem was the question: can the first (weakly) inaccessible cardinal be real-valued-measurable? The answer is spectacularly negative; I give some of the theorems describing how enormously complicated real-valued-measurable cardinals have to be in §4 below.

(c) It will be convenient to use the phrase **normal witnessing probability**, in such contexts as 'Let κ be a real-valued-measurable cardinal, with normal witnessing probability ν', to mean a normal κ-additive probability with domain $\mathcal{P}\kappa$ which is zero on singletons, as in 1Ga.

(d) For a proper discussion of the combinatorial properties of real-valued-measurable cardinals, se §5 below. It is however worth remarking now that if ν is a normal witnessing probability on κ and $f : \kappa \to \kappa$ is any function, then there is a countable set $D \subseteq \kappa$ such that $\nu\{\xi : \xi < \kappa,\ f(\xi) < \xi,\ f(\xi) \notin D\} = 0$. **P** Set $D = \{\zeta : \nu f^{-1}[\{\zeta\}] > 0\}$; then D is countable. Set $S = \{\xi : f(\xi) < \xi,\ f(\xi) \notin D\}$. Then $f\restriction S$ is regressive, and $(f\restriction S)^{-1}[\{\zeta\}] \in \mathcal{N}_\nu$ for every $\zeta < \kappa$, so $S \in \mathcal{N}_\nu$, as claimed. **Q**

(e) Theorem 1Ga speaks of 'Maharam homogeneous' normal witnessing probabilities. These are homogeneous in the sense that they have homogeneous measure algebras. But it should be noted that in other senses they are about as inhomogeneous as can be imagined. If ν is a normal probability on κ, and $f : \kappa \to \kappa$ is any function, set $A = \{\xi : f(\xi) < \xi\}$, $B = \{\xi : f(\xi) = \xi\}$, $C = \{\xi : f(\xi) > \xi\}$. Then there is a countable D such that $\nu(A \setminus f^{-1}[D]) = 0$ ((d) above). On the other hand, looking at $f[C]$, we have a function $g : f[C] \to \kappa$ given by $g(\eta) = \min f^{-1}[\{\eta\}]$ for $\eta \in f[C]$; now g is regressive and injective, so we must have $\nu f[C] = 0$.

Thus our arbitrary f corresponds to a trisection of κ into a negligible piece $A \setminus f^{-1}[D]$, a piece $C \cup f^{-1}[D]$ which is mapped onto a negligible piece, and a piece on which f is the identity. So if we ask, for instance, that $f^{-1}[E]$ should be negligible for every $E \in \mathcal{N}_\nu$, or that $\nu f[E]$ should be equal to νE for every E, we must have $f(\xi) = \xi$ for almost every ξ. Accordingly none of the many automorphisms of $\mathcal{P}\kappa/\mathcal{N}_\nu$, other than the identity, can be represented by functions from κ to itself, and if A, B are disjoint non-negligible sets then $(A, \nu\restriction A)$ and $(B, \nu\restriction B)$ cannot be isomorphic as measure spaces.

(f) It has been recognised since Solovay 71 that many of the properties of real-valued-measurable cardinals depend not on their measures but on the presence of suitably saturated ideals. It would be possible to begin this work with a study of such ideals, later specializing to real-valued-measurable cardinals. However the general theory remains largely dependent on the special case for its inspiration, so I prefer to relegate it to §9 below.

1I Rvm filters and ideals (a) If κ is a real-valued-measurable cardinal, consider

$$\mathcal{W} = \{W : W \subseteq \kappa,\ \nu W = 1 \text{ for every normal witnessing probability } \nu \text{ on } \kappa\}.$$

This is an intersection of normal filters, so is a normal filter on κ; I will call it the **rvm filter** of κ. Similarly, its dual ideal

$$\mathcal{J} = \bigcap\{\mathcal{N}_\nu : \nu \text{ is a normal witnessing probability on } \kappa\}$$

is the **rvm ideal** of κ.

(b) It is perhaps worth noting an elementary fact. If κ is real-valued-measurable and $Z \in \mathcal{P}\kappa \setminus \mathcal{J}$, then there is a Maharam homogeneous normal witnessing probability ν on κ such that $\nu Z = 1$. (See the end of the proof of 1G.) In particular, if κ is two-valued-measurable and $Z \in \mathcal{P}\kappa \setminus \mathcal{J}$ then there is a normal ultrafilter $\mathcal{F}$ on κ containing Z.

***1J** I give here a well-known theorem concerning two-valued-measurable cardinals because it throws light on similar results in §4 below, and it is instructive to contrast the techniques of proof.

Lemma Let κ be a two-valued-measurable cardinal with normal ultrafilter $\mathcal{F}$. Suppose that we have a set $F \in \mathcal{F}$ and for each $\alpha \in F$ an n-place relation C_α on α. Then there is an n-place relation C on κ such that

$$\{\alpha : \alpha \in F,\ C_\alpha = C\restriction\alpha\} \in \mathcal{F}.$$

proof For $\eta_1, \dots, \eta_n < \kappa$ write

$$C(\eta_1, \dots, \eta_n) \iff \{\alpha : C_\alpha(\eta_1 \dots, \eta_n)\} \in \mathcal{F}.$$

Now set

$$\begin{aligned} E(\eta_1, \dots, \eta_n) &= \{\alpha : \alpha \in F,\ C_\alpha(\eta_1, \dots, \eta_n)\} \text{ if } C(\eta_1, \dots, \eta_n), \\ &= \{\alpha : \alpha \in F,\ \neg C_\alpha(\eta_1, \dots, \eta_n)\} \text{ if } \neg C(\eta_1, \dots, \eta_n). \end{aligned}$$

so that $E(\eta_1, \dots, \eta_n) \in \mathcal{F}$. For $\beta < \kappa$ set

$$H_\beta = \bigcap\{E(\eta_1, \dots, \eta_n) : \eta_1, \dots, \eta_n < \beta\} \in \mathcal{F}.$$

Then

$$H = \{\alpha : \alpha \in F \text{ is a limit ordinal},\ \alpha \in H_\beta\ \forall\ \beta < \alpha\} \in \mathcal{F},$$

because $\mathcal{F}$ is normal (cf. A1E(c-iv)) (cf. A1E(c-iv)). But now $C\restriction\alpha = C_\alpha$ for every $\alpha \in H$.

***1K Lemma** Let ϕ be a Π^2_0 formula in the third-order language $\mathcal{L}_3$ of A4K below. Let κ be a two-valued-measurable cardinal with normal ultrafilter $\mathcal{F}$. Suppose that $F_0 \in \mathcal{F}$ and that for each $\alpha \in F_0$ we are given third-order relations $\mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}$ on α. For $i \le l$ let $\mathcal{A}_i$ be the third-order relation on κ defined by writing

$$\begin{aligned}&\mathcal{A}_i(D_1, \dots, D_r, \eta_1, \dots, \eta_s)\\ &\qquad\iff \{\alpha : \eta_j < \alpha \ \forall \ j \le s,\, \mathcal{A}_{\alpha i}(D_1{\restriction}\alpha, \dots, D_r{\restriction}\alpha, \eta_1, \dots, \eta_s)\} \in \mathcal{F}\end{aligned}$$

for any strings $D_1, \dots, D_r$ of relations on κ and $\eta_1, \dots, \eta_s$ of members of κ. If $C_1, \dots, C_k$ are relations on κ and $\xi_1, \dots, \xi_m < \kappa$, then

$$\begin{aligned}&(\kappa; \mathcal{A}_1, \dots, \mathcal{A}_l; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi\\ &\qquad\iff \{\alpha : \alpha < \kappa,\, (\alpha; \mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}; C_1{\restriction}\alpha, \dots, C_k{\restriction}\alpha; \xi_1, \dots, \xi_m) \vDash \phi\} \in \mathcal{F}.\end{aligned}$$

proof Induce on the length of ϕ.

(a) If ϕ is of the form $\mathcal{S}(R_1, \dots, x_n)$ or $R(x_1, \dots, x_n)$ we have just to observe that

$$\mathcal{A}_i(C_{j_1}, \dots, \xi_{j_n}) \iff \{\alpha : \mathcal{A}_{\alpha i}(C_{j_1}{\restriction}\alpha, \dots, \xi_{j_n})\} \in \mathcal{F},$$

$$C_i(\xi_{j_1}, \dots, \xi_{j_n}) \iff \{\alpha : (C_i{\restriction}\alpha)(\xi_{j_1}, \dots, \xi_{j_n})\} \in \mathcal{F}.$$

(b) If ϕ is of one of the forms $\neg\psi$, $\psi \wedge \chi$, $\psi \vee \chi$, $\dots$ the inductive step is easy (using the fact that $\mathcal{F}$ is an ultrafilter for the case $\neg\psi$).

(c) Suppose ϕ is of the form $\forall S\psi$. Set

$$F = \{\alpha : (\alpha; \mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}; C_1{\restriction}\alpha, \dots, C_k{\restriction}\alpha; \xi_1, \dots, \xi_m) \vDash \phi\}.$$

(i) If $F \in \mathcal{F}$ take any relation C on α of the same number of places as the variable S. For each $\alpha \in F$ we have

$$(\alpha; \mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}; C_1{\restriction}\alpha, \dots, C{\restriction}\alpha, \dots, C_k{\restriction}\alpha; \xi_1, \dots, \xi_m) \vDash \psi,$$

where $C{\restriction}\alpha$ is interpolated into the string $C_1{\restriction}\alpha, \dots, C_k{\restriction}\alpha$ in such a way as to assign it to the variable S in ψ. By the inductive hypothesis,

$$(\kappa; \mathcal{A}_1, \dots, \mathcal{A}_l; C_1, \dots, C, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \psi,$$

so, as C is arbitrary,

$$(\kappa; \mathcal{A}_1, \dots, \mathcal{A}_l; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi.$$

(ii) If $F \notin \mathcal{F}$ then for each $\alpha \in \kappa \setminus F$ choose a relation B_α on α, of the same number of places as S, such that

$$(\kappa; \mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}; C_1{\restriction}\alpha, \dots, B_\alpha, \dots, C_k{\restriction}\alpha; \xi_1, \dots, \xi_m) \vDash \neg\psi.$$

Let B be the relation on κ derived from $\langle B_\alpha \rangle_{\alpha \in \kappa \setminus F}$ as in 1J, so that

$$E = \{\alpha : \alpha \in \kappa \setminus F,\, B_\alpha = B{\restriction}\alpha\} \in \mathcal{F}.$$

Then

$$(\alpha; \mathcal{A}_{\alpha 1}, \dots, \mathcal{A}_{\alpha l}; C_1{\restriction}\alpha, \dots, B{\restriction}\alpha, \dots, C_k{\restriction}\alpha; \xi_1, \dots, \xi_m) \nvDash \psi,$$

for every $\alpha \in E$, so

$$(\kappa; \mathcal{A}_1, \dots, \mathcal{A}_l; C_1, \dots, B, \dots, C_k; \xi_1, \dots, \xi_m) \nvDash \psi,$$

by the inductive hypothesis, and

$$(\kappa; \mathcal{A}_1, \dots, \mathcal{A}_l; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \nvDash \forall S\psi.$$

This deals with the inductive step to $\phi = \forall S\psi$.

(d) If ϕ is of the form $\forall x\psi$ the same procedure works. For the case $F \in \mathcal{F}$ we copy (c-i) but with a new ordinal ζ interpolated into the string $\xi_1, \dots, \xi_m$ rather than a new relation B interpolated into the string

$C_1, \ldots, C_k$. For the case $F \notin \mathcal{F}$ we take a witnessing family $\langle \zeta_\alpha \rangle_{\alpha \in \kappa \setminus F}$ in place of $\langle B_\alpha \rangle_{\alpha \in \kappa \setminus F}$, and note that because $\mathcal{F}$ is normal and $\alpha \mapsto \zeta_\alpha$ is regressive, there is a $\zeta < \kappa$ such that $E = \{\alpha : \zeta_\alpha = \zeta\} \in \mathcal{F}$, so that $(\kappa; \mathcal{A}_1, \ldots, \mathcal{A}_l; C_1, \ldots, C_k; \xi_1, \ldots, \zeta, \ldots, \xi_m) \not\models \psi$.

(e) If ϕ is of the form $\exists S \psi$ it is logically equivalent to $\neg \forall S \neg \psi$ so (b)-(c) deal with it. Similarly, (b) and (d) deal with the case $\exists x \psi$.

***1L Theorem** If κ is a two-valued-measurable cardinal, it is Π^2_1-indescribable, and its Π^2_1-filter is included in its rvm filter.

proof Let ϕ be a formula of the language $\mathcal{L}_3$ (A4K) of the form $\forall \mathcal{R}_1 \ldots \forall \mathcal{R}_l \psi$, where ψ is a Π^2_0 formula in $\mathcal{L}_3$ in which the only third-order variables are in the list $\mathcal{R}_1, \ldots, \mathcal{R}_l$, and let $\mathcal{F}$ be a normal ultrafilter on κ. Let $C_1, \ldots, C_k$, $\xi_1, \ldots, \xi_m$ be such that

$$(\kappa; ; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \phi.$$

Set

$$F = \{\alpha : \alpha < \kappa, \ (\alpha; ; C_1, \ldots, \xi_m) \models \phi\}.$$

? If $F \notin \mathcal{F}$ then for $\alpha \in \kappa \setminus F$ let $\mathcal{A}_{\alpha 1}, \ldots, \mathcal{A}_{\alpha l}$ be third-order relations on α such that

$$(\alpha; \mathcal{A}_{\alpha 1}, \ldots, \mathcal{A}_{\alpha l}; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \psi.$$

For each $i \le l$ let $\mathcal{A}_i$ be the third-order relation on κ given by saying that

$$\begin{aligned} &\mathcal{A}_i(D_1, \ldots, D_r, \eta_1, \ldots, \eta_s) \\ &\qquad \iff \{\alpha : \eta_j < \alpha \ \forall \ j \le s, \ \mathcal{A}_{\alpha i}(D_1 \restriction \alpha, \ldots, D_r \restriction \alpha, \eta_1, \ldots, \eta_s)\} \in \mathcal{F} \end{aligned}$$

for all second-order relations $D_1, \ldots, D_r$ on κ and ordinals $\eta_1, \ldots, \eta_s < \kappa$. It follows from Lemma 1K that

$$(\kappa; \mathcal{A}_1, \ldots, \mathcal{A}_r; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \psi)$$

and

$$(\kappa; ; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \models \neg \phi,$$

which is absurd. **X**

Thus $F \in \mathcal{F}$. But sets of the form of F form a base for the Π^2_1-filter of κ, so every set in that filter belongs to $\mathcal{F}$; as $\mathcal{F}$ was arbitrary, the Π^2_1-filter of κ is included in its rvm filter.

Remark This is due to Hanf & Scott 61. See also Drake 74, §9.3, Jech 78, p. 385, Lemma 32.2 and Kanamori & Magidor 78, §I.4.

1M Ergodic theory Going a little deeper into the question considered in 1He above, we have the following, largely due to Zakrzewski 91.

Proposition Let $(X, \mathcal{P}X, \mu)$ be a probability space.

(a) Let G be the group of measure-preserving bijections of X. Then there is a partition $\mathcal{K}$ of X into finite sets such that G is precisely the set of bijections $g : X \to X$ such that $\mu(\bigcup\{K : K \in \mathcal{K}, \ g[K] \ne K\}) = 0$.

(b) Let G^* be the group of bijections $g : X \to X$ such that $\mathcal{N}_\mu = \{g^{-1}[A] : A \in \mathcal{N}_\mu\}$. Then there is a partition $\mathcal{L}$ of X into countable sets such that G^* is precisely the set of bijections $g : X \to X$ such that $\mu(\bigcup\{L : L \in \mathcal{L}, \ g[L] \ne L\}) = 0$.

proof (a)(i) Let us note first that if $H \subseteq G$ is a countable subgroup, $E \subseteq X$ is any set, and $f : E \to X$ is an injection such that $f(x) \in \mathrm{Orb}_H(x) = \{h(x) : h \in H\}$ for every $x \in E$, then $\mu f[E] = \mu E$. **P** Let $\langle h_n \rangle_{n \in \mathbb{N}}$ run over H. For each $n \in \mathbb{N}$ set

$$E_n = \{x : x \in E, \ f(x) = h_n(x), \ f(x) \ne h_i(x) \ \forall \ i < n\}.$$

Then $\mu f[E_n] = \mu h_n[E_n] = \mu E_n$ for each n, so $\mu f[E] = \sum_{n \in \mathbb{N}} \mu f[E_n] = \sum_{n \in \mathbb{N}} \mu E_n = \mu E$. **Q**

(ii) It follows that if $H \subseteq G$ is a countable subgroup then $E = \{x : \mathrm{Orb}_H(x) \text{ is infinite}\}$ is negligible. **P** Of course $h(x) \in E$ for $h \in H$, $x \in E$. Next, E (if it is not empty) has a partition $\langle E_n \rangle_{n \in \mathbb{N}}$ such that for each $x \in E$, $n \in \mathbb{N}$ the intersection $E_n \cap \mathrm{Orb}_H(x)$ consists of a single point. Now for any m, $n \in \mathbb{N}$ we have a bijection $f : E_m \to E_n$ defined by saying that $f(x)$ is the unique member of $\mathrm{Orb}_H(x) \cap E_n$; by (i), $\mu E_n = \mu E_m$. As μE is certainly finite, it must be 0. **Q**

(iii) For any $\epsilon > 0$ there is a finite set $F \subseteq G$ such that $\mu\{x : g(x) \notin \{f(x) : f \in F\}\} \le \epsilon$ for every $g \in G$. **P?** Otherwise we can choose $\langle g_n \rangle_{n\in\mathbb{N}}$ in G inductively so that $\mu E_n \ge \epsilon$ for every $n \in \mathbb{N}$, where $E_n = \{x : g_n(x) \ne g_i(x)\ \forall\ i < n\}$ for each n. Let H be the subgroup of G generated by $\langle g_n \rangle_{n\in\mathbb{N}}$; then H is countable. Set $E = \bigcap_{n\in\mathbb{N}} \bigcup_{i\ge n} E_i$; then $\mu E \ge \epsilon$ and $\mathrm{Orb}_H(x)$ is infinite for every $x \in E$, contradicting (ii) just above. **XQ**

(iv) So there is a countable subgroup H of G such that $\mu\{x : g(x) \notin \mathrm{Orb}_H(x)\} = 0$ for every $g \in G$. By (ii) again, $F = \{x : \mathrm{Orb}_H(x) \text{ is infinite}\}$ is negligible. Set $\mathcal{K} = \{\mathrm{Orb}_H(x) : x \in X \setminus F\} \cup \{\{x\} : x \in F\}$; then $\mathcal{K}$ is a partition of X into finite sets. Let $\langle h_n \rangle_{n\in\mathbb{N}}$ run over H.

(v) Let $g : X \to X$ be a bijection and set $A = \bigcup\{K : K \in \mathcal{K},\ g[K] \ne K\}$.

If $\mu A = 0$, then $g(x) \in \mathrm{Orb}_H(x)$ for every $x \in X \setminus A$. So $\mu g[E] = \mu E$ for every $E \subseteq X \setminus A$, by (i). As $g[A] = A \in \mathcal{N}_\mu$, $\mu g[E] = \mu E$ for every $E \subseteq X$, i.e., $g \in G$.

If $g \in G$ set $D = \{x : g(x) \notin \mathrm{Orb}_H(x)\}$; then $\mu D = 0$, by the choice of H. Set $C = F \cup \bigcup_{h\in H} h[D]$; then $\mu C = 0$. If $K \in \mathcal{K}$ and $g[K] \ne K$ then $K \subseteq C$; thus $A \in \mathcal{N}_\mu$, and $\mathcal{K}$ is the required partition.

(b)(i) If $H \subseteq G^*$ is a subgroup of cardinal at most ω_1, $E \subseteq X$ is any set, and $g : E \to X$ is an injection such that $g(x) \in \mathrm{Orb}_H(x)$ for every $x \in E$, then $E \in \mathcal{N}_\mu$ iff $g[E] \in \mathcal{N}_\mu$; the argument is the same as for (a-i) above, but now using the fact that μ is ω_2-additive (1D).

(ii) Next, if $H \subseteq G^*$ is a subgroup of cardinal at most ω_1, then $\{x : \mathrm{Orb}_H(x) \text{ is uncountable}\}$ is negligible; the argument is the same as for (a-ii) above.

(iii) There is a countable subgroup H of G^* such that

$$\mu\{x : g(x) \notin \mathrm{Orb}_H(x)\} = 0$$

for every $g \in G^*$. **P?** If not, we can choose $\langle g_\xi \rangle_{\xi<\omega_1}$ in G^* inductively so that $\mu E_\xi > 0$ for every ξ, where $E_\xi = \{x : g_\xi(x) \ne g_\eta(x)\ \forall\ \eta < \xi\}$. Let H be the subgroup of G^* generated by $\{g_\xi : \xi < \omega_1\}$, and set $E = \bigcap_{\xi<\omega_1} \bigcup_{\eta\ge\xi} E_\eta$; then (because μ is ω_2-additive) $\mu E \ge \limsup_{\xi\to\omega_1} \mu E_\xi > 0$, while $\mathrm{Orb}_H(x)$ is unocuntable for every $x \in E$, which is impossible. **XQ**

(iv) Now take $\mathcal{L}$ to be the set of orbits of H. Let $g : X \to X$ be a bijection and set $A = \bigcup\{L : L \in \mathcal{L},\ g[L] \ne L\}$.

If $A \in \mathcal{N}_\mu$, then $g(x) \in \mathrm{Orb}_H(x)$ for every $x \in X \setminus A$, while also $g[A] = A$; it follows that $g \in G^*$ as in (a-v) above.

If $g \in G^*$, set $D = \{x : g(x) \notin \mathrm{Orb}_H(x)\} \in \mathcal{N}_\mu$, $C = \bigcup_{h\in H} h[D]$; then $A \subseteq C \in \mathcal{N}_\mu$. Thus $\mathcal{L}$ witnesses the truth of (b).

Remark Compare 9E. In this proposition I have tried to give a succinct but adequate description of the structure involved. Many corollaries can be drawn concerning both G and G^* and the corresponding subgroups of $\mathrm{Aut}(\mathcal{P}X/\mathcal{N}_\mu)$; see Zakrzewski 91.

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2. Solovay's Theorems

As remarked in 1Ee, there cannot be a proof in ZFC that real-valued-measurable cardinals exist. It remains just conceivable that there is a proof that two-valued-measurable cardinals or atomlessly-measurable cardinals do not exist. However, if one of these gives a difficulty, so does the other; this is the main result of Solovay 71, covered in 2A-2D. The method of proof in 2C is important because it provides a technique - to date the only technique known - for proving the relative consistency of further propositions with the existence of an atomlessly-measurable cardinal, as in 2I. I therefore spell out some of the properties of the construction relevant to questions considered later in this paper (2H, 2J, 2K).

This material belongs to the 'metamathematical' part of the subject, and as I said in the Introduction I seek to avoid reliance on such methods; only the second half of §4 and a few paragraphs elsewhere will depend on them in a formal sense. Historically, however, these ideas have been dominant in the development of the subject, and they remain an invaluable guide.

2A Forcing I try to follow KUNEN 80 for the theory and notation of forcing. In particular, a **p.o.set** is a triple $(\mathbb{P}, \leq, \mathbb{1}_{\mathbb{P}})$ such that

$$p \leq p \;\forall\; p \in \mathbb{P},$$

$$\text{if } p \leq q \text{ and } q \leq r \text{ then } p \leq r,$$

$$\mathbb{1}_{\mathbb{P}} \in \mathbb{P} \text{ and } p \leq \mathbb{1}_{\mathbb{P}} \;\forall\; p \in \mathbb{P};$$

and '$p \leq q$' means that p is a stronger condition than q. Two elements p, q of $\mathbb{P}$ are **incompatible** if there is no $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. A set $A \subseteq \mathbb{P}$ is an **antichain** ('down-antichain' in FREMLIN 84) if p and q are incompatible for all distinct p, $q \in A$. A set $D \subseteq \mathbb{P}$ is **dense** ('coinitial' in FREMLIN 84) if for any $p \in \mathbb{P}$ there is a $q \in D$ such that $q \leq p$.

If $\mathbb{P}$ is any p.o.set, it has a natural topology (the 'down-topology' of FREMLIN 84) generated by sets of the form $\{q : q \leq p\}$ as p runs through $\mathbb{P}$. Let $\mathfrak{A}$ be the algebra of regular open sets for this topology (FREMLIN 84, §12). For $p \in \mathbb{P}$ write p^* for the corresponding member of $\mathfrak{A}$, viz.

$$\operatorname{int}\overline{\{q : q \leq p\}} = \{q : \text{ every } r \leq q \text{ is compatible with } p\}.$$

The map $p \mapsto p^* : \mathbb{P} \to \mathfrak{A} \setminus \{\mathbf{0}\}$ is a dense embedding in the sense of KUNEN 80.

2B Random real p.o.sets A **random real p.o.set** is a p.o.set $\mathbb{P}$ such that there is a functional $\mu : \mathbb{P} \to]0,1]$ such that (i) $\mu \mathbb{1}_{\mathbb{P}} = 1$ (ii) if $p \in \mathbb{P}$ and A is a maximal antichain in $\{q : q \leq p\}$, then $\mu p = \sum_{a \in A} \mu a$.

It is fairly easy to see that $\mathbb{P}$ is a random real p.o.set in this sense iff the regular open algebra of $\mathbb{P}$ is a measurable algebra in the sense of A2Fc. Consequently forcing with any random real p.o.set corresponds to forcing with a measurable algebra, which by Maharam's theorem (A2I) is reducible to some assembly of forcings with the standard homogeneous measure algebras based on powers of $\{0, 1\}$ (A2G).

The idea of this definition is to unify the two examples with which we shall be concerned: (i) $\mathbb{P} = \mathfrak{A} \setminus \{\mathbf{0}\}$, where $\mathfrak{A}$ is a non-zero measurable algebra; (ii) $\mathbb{P} = \Sigma \setminus \mathcal{N}_\mu$, where (X, Σ, μ) is a probability space. But it is also worth noting that if $\mathbb{P}$ is a random real p.o.set and $\mathbb{Q}$ is a dense subset of $\mathbb{P}$ containing $\mathbb{1}_{\mathbb{P}}$, then $\mathbb{Q}$ is a random real p.o.set.

2C Theorem If κ is a real-valued-measurable cardinal and $\mathbb{P}$ is a random real p.o.set then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is real-valued-measurable.}$$

proof Let $\mathfrak{A}$ be the regular open algebra of $\mathbb{P}$; fix on a $\bar{\mu}$ such that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, and let ν be a witnessing probability on κ. For each $\mathbb{P}$-name σ for a subset of κ let $\langle a_\xi(\sigma)\rangle_{\xi<\kappa}$ be the family in $\mathfrak{A}$ defined by the formula

$$a_\xi(\sigma) = \sup\{p^* : p \Vdash_{\mathbb{P}} \check{\xi} \in \sigma\}.$$

Define $u_\sigma \in L^\infty(\mathfrak{A})$ by writing

$$\int_a u_\sigma \, d\bar{\mu} = \int \bar{\mu}(a_\xi(\sigma) \cap a) \, \nu(d\xi)$$

for every $a \in \mathfrak{A}$; this is well-defined because the functional

$$a \mapsto \int \bar{\mu}(a_\xi(\sigma) \cap a) \, \nu(d\xi)$$

is additive and dominated by $\bar{\mu}$ (A2Fg). Observe that if σ, σ' are both names for subsets of κ and $p \Vdash_{\mathbb{P}} \sigma = \sigma'$, then $p^* \cap a_\xi(\sigma) = p^* \cap a_\xi(\sigma')$ for every $\xi < \kappa$, so that $u_\sigma \times \chi(p^*) = u_{\sigma'} \times \chi(p^*)$. We therefore have a $\mathbb{P}$-name $\tilde{\nu}$ for a function from $\mathcal{P}\kappa$ to $[0, 1]$ such that, for any rational numbers s, t,

$$p \Vdash_{\mathbb{P}} \check{s} \leq \tilde{\nu}(\sigma) \leq \check{t} \text{ iff } s\chi(p^*) \leq u_\sigma \times \chi(p^*) \leq t\chi(p^*).$$

(The point is that if $p \Vdash_{\mathbb{P}} \sigma = \sigma'$, then $u_\sigma \times \chi(p^*) = u_{\sigma'} \times \chi(p^*)$ so $p \Vdash_{\mathbb{P}} \tilde{\nu}(\sigma) = \tilde{\nu}(\sigma')$.)

Now we have to check the properties of $\tilde{\nu}$.

(i) If σ_1, σ_2 and σ are $\mathbb{P}$-names for subsets of κ and $\Vdash_{\mathbb{P}} \sigma_1 \cap \sigma_2 = \check{\emptyset}$, $\sigma_1 \cup \sigma_2 = \sigma$ then

$$a_\xi(\sigma_1)\cap a_\xi(\sigma_2)=\mathbf{0}\ \forall\ \xi<\kappa,$$

$$a_\xi(\sigma)=a_\xi(\sigma_1)\cup a_\xi(\sigma_2)\ \forall\ \xi<\kappa.$$

So if $a\in\mathfrak{A}$,

$$\bar\mu(a\cap a_\xi(\sigma))=\bar\mu(a\cap a_\xi(\sigma_1))+\bar\mu(a\cap a_\xi(\sigma_2))\ \forall\ \xi<\kappa,$$

$$\int_a u_\sigma\,d\bar\mu=\int_a u_{\sigma_1}\,d\bar\mu+\int_a u_{\sigma_2}\,d\bar\mu.$$

Accordingly

$$u_\sigma=u_{\sigma_1}+u_{\sigma_2}$$

in $L^\infty(\mathfrak{A})$, and

$$\Vdash_{\mathbb{P}}\tilde\nu(\sigma)=\tilde\nu(\sigma_1)+\tilde\nu(\sigma_2).$$

Thus

$$\Vdash_{\mathbb{P}}\tilde\nu\text{ is finitely additive.}$$

(ii) If $\zeta<\kappa$ and σ is a $\mathbb{P}$-name for $\{\zeta\}$ then $a_\xi(\sigma)=\mathbf{0}$ for every $\xi\ne\zeta$ so $\int\bar\mu a_\xi(\sigma)\,\nu(d\xi)=0$ and $u_\sigma=0$ and $\Vdash_{\mathbb{P}}\tilde\nu(\sigma)=\check 0$.

(iii) Similarly, $u_{\check\kappa}=\chi(\mathbf{1})$ so $\Vdash_{\mathbb{P}}\tilde\nu(\check\kappa)=\check 1$.

(iv) If $\lambda<\kappa$ and $\langle\sigma_\alpha\rangle_{\alpha<\lambda}$ is a family of $\mathbb{P}$-names for subsets of κ with

$$\Vdash_{\mathbb{P}}\bigcup_{\alpha<\check\lambda}\sigma_\alpha=\check\kappa,\ \sigma_\alpha\cap\sigma_\beta=\check\emptyset\ \forall\ \alpha\ne\beta,$$

then $a_\xi(\sigma_\alpha)\cap a_\xi(\sigma_\beta)=\mathbf{0}$ whenever $\xi<\kappa$, $\alpha<\beta<\lambda$, and $\sup_{\alpha<\lambda}a_\xi(\sigma_\alpha)=\mathbf{1}$ in $\mathfrak{A}$ for every $\xi<\kappa$. So

$$\bar\mu a=\sum_{\alpha<\lambda}\bar\mu(a_\xi(\sigma_\alpha)\cap a)\ \forall\ \xi<\lambda,\ a\in\mathfrak{A},$$

and (because ν is κ-additive and $\lambda<\kappa$)

$$\bar\mu a=\sum_{\alpha<\lambda}\int\bar\mu(a_\xi(\sigma_\alpha)\cap a)\,\nu(d\xi)\ \forall\ a\in\mathfrak{A}.$$

Accordingly $\sum_{\alpha<\lambda}u_{\sigma_\alpha}=\chi(\mathbf{1})$ in $L^\infty(\mathfrak{A})$. Now if $p\in\mathbb{P}$, $t<1$ there must be a non-zero $a\subseteq p^*$ and $\beta(0),\dots,\beta(n)<\lambda$, $t_0,...,t_n>0$ such that $\sum_{i\le n}t_i\ge t$ and $t_i\chi(a)\le u_{\sigma_{\beta(i)}}$ for each $i\le n$. There is a $p_1\in\mathbb{P}$ such that $p_1^*\subseteq a$, and

$$p_1\Vdash_{\mathbb{P}}\tilde\nu(\sigma_{\beta(i)})\ge\check t_i$$

for every $i\le n$, so that

$$p_1\Vdash_{\mathbb{P}}\sum_{\alpha<\lambda}\tilde\nu(\sigma_\alpha)\ge\check t.$$

As p and t are arbitrary,

$$\Vdash_{\mathbb{P}}\sum_{\alpha<\lambda}\tilde\nu(\sigma_\alpha)\ge\check 1.$$

As $\langle\sigma_\alpha\rangle_{\alpha<\lambda}$ is arbitrary,

$$\Vdash_{\mathbb{P}}\tilde\nu\text{ is }\check\kappa\text{-additive.}$$

Thus $\tilde\nu$ witnesses that

$$\Vdash_{\mathbb{P}}\check\kappa\text{ is real-valued-measurable.}$$

Remark This is due to Solovay and Kunen (Solovay 71, Theorem 7.) For a version of the proof which incorporates the relevant part of the Radon-Nikodým theorem, see Jech 78, p. 423, Lemma 34.6.

2D Theorem If κ is an uncountable cardinal and $\mathcal{I}$ is a proper κ-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons, then

$$L(\mathcal{I})\vDash\text{GCH}+\kappa\text{ is two-valued-measurable.}$$

proof Solovay 71, Theorem 6, or Jech 78, p. 416, Theorem 82a.

2E Corollary The following are equiconsistent:

(a) ‘ZFC + there is a two-valued-measurable cardinal’;
(b) ‘ZFC + there is an atomlessly-measurable cardinal’;
(c) ‘ZFC + there is a real-valued-measurable cardinal’;
(d) ‘ZFC + $\mathfrak{c}$ is atomlessly-measurable’;
(e) ‘ZFC + GCH + there is a two-valued-measurable cardinal’.

proof For (a)⇒(d), use 2C with $\mathbb{P} = \mathfrak{A}_\kappa \setminus \{\mathbf{0}\}$, where $\mathfrak{A}_\kappa$ is the measure algebra of $\{0,1\}^\kappa$ with its usual measure. (d)⇒(b)⇒(c) and (e)⇒(a) are trivial. (c)⇒(e) is covered by 2D.

2F Definition If X and Y are sets and $\mathcal{I}$ is an ideal of subsets of X then I write $\mathrm{Tr}_{\mathcal{I}}(X;Y)$ for

$$\sup\{\#(F) : F \subseteq Y^X,\ \{x : f(x) \neq g(x)\} \in \mathcal{I} \text{ whenever } f,\ g \in F \text{ and } f \neq g\}.$$

2G Lemma Let $(X, \mathcal{P}X, \nu)$ be a probability space and Y any set. Then $\mathrm{Tr}_{\mathcal{N}_\nu}(X;Y)$ is attained, in the sense that there is a set $G \subseteq Y^X$ such that $\#(G) = \mathrm{Tr}_{\mathcal{N}_\nu}(X;Y)$ and $\{x : x \in X,\ g(x) \neq g'(x)\} \in \mathcal{N}_\nu$ for all distinct $g,\ g' \in G$.

proof It is enough to consider the case in which $Y = \lambda$ is a cardinal, and the case of finite λ is elementary, so I suppose from now on that $\lambda \geq \omega$. Set $\theta = \mathrm{Tr}_{\mathcal{N}_\nu}(X;\lambda)$.

(a) If $H \subseteq \lambda^X$ is such that

$$F = \{f : f \in \lambda^X,\ \{x : f(x) \leq h(x)\} \in \mathcal{N}_\nu\ \ \forall\ \ h \in H\} \neq \emptyset,$$

then there is an $f_0 \in F$ such that

$$\{x : f(x) < f_0(x)\} \in \mathcal{N}_\nu \text{ for every } f \in F.$$

P? If not, choose a family $\langle f_\xi \rangle_{\xi<\omega_1}$ in F inductively, as follows. f_0 is to be any member of F. Given f_ξ, there is an $f \in F$ such that $\{x : f(x) < f_\xi(x)\} \notin \mathcal{N}_\nu$; set $f_{\xi+1}(x) = \min(f(x), f_\xi(x))$ for every x; then $f_{\xi+1} \in F$. Given that $f_\eta \in F$ for every $\eta < \xi$, where $\xi < \omega_1$ is a non-zero limit ordinal, set $f_\xi(x) = \min_{\eta<\xi} f_\eta(x)$ for each x; then for any $h \in H$ we shall have

$$\{x : f_\xi(x) \leq h(x)\} = \bigcup_{\eta<\xi}\{x : f_\eta(x) \leq h(x)\} \in \mathcal{N}_\nu,$$

so $f_\xi \in F$ and the induction continues.

Now consider

$$E_\xi = \{x : f_{\xi+1}(x) < f_\xi(x)\} \in \mathcal{P}X \setminus \mathcal{N}_\nu$$

for $\xi < \omega_1$. By A2Mb there is an $x \in X$ such that

$$A = \{\xi : x \in E_\xi\}$$

is infinite. But if $\langle \xi(n) \rangle_{n\in\mathbb{N}}$ is any strictly increasing sequence in A, $\langle f_{\xi(n)}(x) \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(b) We may therefore choose a family $\langle g_\xi \rangle_{\xi<\alpha}$ in λ^X as follows. Given $\langle g_\eta \rangle_{\eta<\xi}$, set

$$F_\xi = \{f : f \in \lambda^X,\ \{x : f(x) \leq g_\eta(x)\} \in \mathcal{N}_\nu\ \ \forall\ \ \eta < \xi\}.$$

If $F_\xi = \emptyset$, set $\alpha = \xi$ and stop. If $F_\xi \neq \emptyset$ choose $g_\xi \in F_\xi$ such that $\{x : f(x) < g_\xi(x)\} \in \mathcal{N}_\nu$ for every $f \in F_\xi$, and continue. Note that for $n < \omega$, $g_n(x) = n$ for ν-almost every $x \in X$ (this is a simple induction on n), so that $\alpha \geq \omega$.

(c) Because $g_\xi \in F_\xi$, $\{x : g_\xi(x) = g_\eta(x)\} \in \mathcal{N}_\nu$ whenever $\eta < \xi < \alpha$, so $\#(\alpha) \leq \theta$. On the other hand, suppose that $F \subseteq \lambda^X$ is such that $\{x : f(x) = f'(x)\} \in \mathcal{N}_\nu$ for all distinct $f,\ f' \in F$. For each $f \in F$, set

$$\zeta'_f = \min\{\xi : \xi \leq \alpha,\ f \notin F_\xi\};$$

this must be defined because $F_\alpha = \emptyset$. Also $F_0 = \lambda^X$ and $F_\xi = \bigcap_{\eta<\xi} F_\eta$ if $\xi \leq \alpha$ is a non-zero limit ordinal, so ζ'_f must be a successor ordinal; let ζ_f be its predecessor. We have $f \in F_{\zeta_f}$ and

$$\{x : f(x) < g_{\zeta_f}(x)\} \in \mathcal{N}_\nu,\ \{x : f(x) \leq g_{\zeta_f}(x)\} \notin \mathcal{N}_\nu,$$

so that

$$E_f = \{x : f(x) = g_{\zeta_f}(x)\} \notin \mathcal{N}_\nu.$$

If f, f' are distinct members of F and $\zeta_f = \zeta_{f'}$, then $E_f \cap E_{f'} \in \mathcal{N}_\nu$. So

$$\{f : f \in F, \zeta_f = \zeta\}$$

must be countable for every $\zeta < \alpha$, and $\#(F) \le \max(\omega, \#(\alpha)) = \#(\alpha)$. As F is arbitrary, $\theta \le \#(\alpha)$.

(d) Accordingly we may take $G = \{g_\xi : \xi < \alpha\}$.

Remark Compare 1G.

2H Proposition Suppose that κ is a two-valued-measurable cardinal and that $\lambda \ge \kappa$. Let $(\mathfrak{A}_\lambda, \bar\mu_\lambda)$ be the measure algebra of $\{0,1\}^\lambda$ with its usual measure, and set $\mathbb{P} = \mathfrak{A}_\lambda \setminus \{\mathbf{0}\}$. Let ν be a $\{0,1\}$-valued κ-additive measure with domain $\mathcal{P}\kappa$, zero on singletons. Write $\mathcal{N}$ for the ideal of Lebesgue negligible subsets of $\mathbb{R}$.

(a) Set $\zeta = \mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda)$. Construct the $\mathbb{P}$-name $\tilde\nu$ for a measure on κ as in 2C. Then

$$\Vdash_{\mathbb{P}} \tilde\nu \text{ is Maharam homogeneous with Maharam type } \check\zeta.$$

(b) Set $\alpha = \lambda^\omega$, the cardinal power. Then

$$\Vdash_{\mathbb{P}} \mathrm{cov}(\mathbb{R}, \mathcal{N}) = \mathfrak{c} = \check\alpha.$$

proof (a)(i) By 2G, there is a family $\langle g_\alpha \rangle_{\alpha<\zeta}$ in λ^κ such that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\} \in \mathcal{N}_\nu$ whenever $\alpha < \beta < \zeta$. Fix a stochastically independent family $\langle e_\eta \rangle_{\eta<\lambda}$ in $\mathfrak{A}_\lambda$ with $\bar\mu_\lambda e_\eta = \frac{1}{2}$ for every η. For each $\alpha < \zeta$ let σ_α be a $\mathbb{P}$-name for a subset of κ such that

$$e_{g_\alpha(\xi)} \Vdash_{\mathbb{P}} \check\xi \in \sigma_\alpha, \quad \mathbf{1} \setminus e_{g_\alpha(\xi)} \Vdash_{\mathbb{P}} \check\xi \notin \sigma_\alpha.$$

Then

$$\Vdash_{\mathbb{P}} \tilde\nu(\textstyle\bigcap_{\alpha\in I} \sigma_\alpha) = 2^{-\#(I)}$$

for every non-empty finite $I \subseteq \zeta$. So

$$\Vdash_{\mathbb{P}} \langle \sigma_\alpha \rangle_{\alpha<\check\zeta} \text{ is stochastically independent}$$

and

$$\Vdash_{\mathbb{P}} \text{for every } A \in \mathcal{P}\check\kappa \setminus \mathcal{N}_{\tilde\nu} \text{ the Maharam type of } \tilde\nu\lceil A \text{ is at least } \check\zeta.$$

(ii) Suppose that $p \in \mathbb{P}$ and that $\langle \sigma_\alpha \rangle_{\alpha<\theta}$ is a family of $\mathbb{P}$-names for subsets of κ such that $p \Vdash_{\mathbb{P}} \tilde\nu(\sigma_\alpha \triangle \sigma_\beta) \ge 3\epsilon > 0$ for all $\alpha < \beta < \theta$. Then, writing $\mathcal{F}$ for the filter $\{A : \nu A = 1\}$,

$$\lim_{\xi\to\mathcal{F}} \bar\mu_\lambda(p \cap (a_\xi(\sigma_\alpha) \triangle a_\xi(\sigma_\beta))) \ge 3\epsilon\bar\mu_\lambda p \ \ \forall \ \alpha < \beta < \theta,$$

defining $a_\xi(\sigma)$ as in 2C, but regarding them as members of $\mathbb{P}$ itself. Take a metrically dense subset D of $\mathfrak{A}_\lambda$ of cardinal λ; take $d_{\alpha\xi} \in D$ with $\bar\mu_\lambda(d_{\alpha\xi} \triangle a_\xi(\sigma_\alpha)) \le \epsilon\bar\mu_\lambda p$ for all $\xi < \kappa$, $\alpha < \theta$; then

$$\lim_{\xi\to\mathcal{F}} \bar\mu_\lambda(p \cap (d_{\alpha\xi} \triangle d_{\beta\xi})) > 0 \ \ \forall \ \alpha < \beta < \theta.$$

Consequently $\langle\langle d_{\alpha\xi} \rangle_{\xi<\kappa}\rangle_{\alpha<\theta}$ witness that $\theta \le \mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; D) = \mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda) = \zeta$. This shows that

$$\Vdash_{\mathbb{P}} \text{the metric density of } \mathcal{P}\check\kappa/\mathcal{N}_{\tilde\nu} \text{ is at most } \check\zeta;$$

but as remarked in A2Hi, this is just

$$\Vdash_{\mathbb{P}} \text{the Maharam type of } \tilde\nu \text{ is at most } \check\zeta.$$

(b)(i) We have

$$\Vdash_{\mathbb{P}} \mathfrak{c} \le \check\alpha$$

because $\#(\mathbb{P}) = \alpha = \#(\mathbb{P})^\omega$ (see A2Hb) and $\mathbb{P}$ is ccc; see Jech 78, Lemma 19.4.

(ii) We need to know the following fact: if $\theta < \alpha$ and $\langle I_\xi \rangle_{\xi<\theta}$ is any family in $[\lambda]^{\le\omega}$, there is a $K \in [\lambda]^\omega$ such that $K \cap I_\xi$ is finite for every $\xi < \theta$. **P** If $\theta < \lambda$ this is trivial. If $\theta \ge \lambda$, let β be the least cardinal such that $\beta^\omega > \theta$; then $\mathfrak{c} < \beta \le \lambda \le \theta$ and $\beta^\omega > \max(\beta, \sup_{\gamma<\beta} \gamma^\omega)$, so $\mathrm{cf}(\beta) = \omega$. Let $\langle \beta_n \rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of infinite cardinals with supremum β. For each $n \in \mathbb{N}$ let $\phi_n : [n \times \beta_n]^{<\omega} \to \beta_{n+1} \setminus \beta_n$ be an injective function, and for $x : \mathbb{N} \to \beta$ set

$$K_x = \{\phi_n(x \cap (n \times \beta_n)) : n \in \mathbb{N}\} \subseteq \beta \subseteq \lambda,$$

identifying x with a subset of $\mathbb{N} \times \beta$. Then each K_x is infinite and $K_x \cap K_y$ is finite whenever $x \neq y$. Consequently, for any $\xi < \theta$,

$$\{x : K_x \cap I_\xi \text{ is infinite}\} = \{x : \exists\ I \in [I_\xi]^\omega,\ I \subseteq K_x\}$$

has cardinal at most $\mathfrak{c}$. Because $\beta^\omega > \max(\mathfrak{c}, \theta)$, there is some x such that $K_x \cap I_\xi$ is finite for every ξ, and this will serve for K. **Q**

(iii) The argument of Theorem 3 of Miller 82 now shows that

$$\Vdash_{\mathbb{P}} \operatorname{cov}(\mathbb{R}, \mathcal{N}) \geq \check{\alpha}.$$

Because $\operatorname{cov}(\mathbb{R}, \mathcal{N}) \leq \mathfrak{c}$, we're done.

Remark The result (b) is well known if $\lambda^\omega = \lambda$; see for instance Kunen 84, 3.14 and 3.19. For $\lambda^\omega > \lambda$ it is less familiar.

Maharam's theorem (A2I) tells us that any random real forcing must correspond to some forcing of the type described in this proposition. Consequently any atomlessly-measurable cardinal constructed by Solovay's method from a $\{0,1\}$-valued measure must have a homogeneous measure algebra.

2I Corollary The following are equiconsistent:

(a) 'ZFC + there is a two-valued-measurable cardinal';

(b) 'ZFC + there is an atomlessly-measurable cardinal κ, with witnessing probability ν, such that the Maharam type of ν is $\mathfrak{c} = 2^\kappa$';

(c) 'ZFC + there is an atomlessly-measurable cardinal κ, with witnessing probability ν, such that the Maharam type of ν is $\kappa^{(+\omega)}$, while $\mathfrak{c} = 2^\kappa = \kappa^{(+\omega+1)}$'.

proof For (b)⇒(a) and (c)⇒(a) we have 2D-2E. For (a)⇒(b), apply 2H, starting from a two-valued-measurable cardinal κ, and using $\lambda = 2^\kappa$. For (a)⇒(c), 2E(a)⇒(e) tells us that we may assume GCH and take a two-valued-measurable cardinal κ with witnessing probability ν. If we now take $\lambda = \kappa^{(+\omega)}$, we shall have $\operatorname{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda) = \lambda$. **P** Take $F \subseteq \lambda^\kappa$ such that $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}_\nu$ for all distinct f, $g \in F$. For each $n \in \mathbb{N}$ set $F_n = \{f : f \in F,\ \nu f^{-1}[\kappa^{(+n)}] > 0\}$. Then $F = \bigcup_{n \in \mathbb{N}} F_n$. If f, g are distinct members of F_n, then $f \cap (\kappa \times \kappa^{(+n)}) \neq g \cap (\kappa \times \kappa^{(+n)})$, and $\#(\mathcal{P}(\kappa \times \kappa^{(+n)})) = \kappa^{(+n+1)} \leq \lambda$. So $\#(F_n) \leq \lambda$ for every n and $\#(F) \leq \lambda$. This shows that $\operatorname{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda) \leq \lambda$; but the reverse inequality is trivial. **Q**

Also $\lambda < \lambda^\omega \leq 2^\lambda = \kappa^{(+\omega+1)}$; so applying 2H we get

$$\Vdash_{\mathbb{P}} \text{the Maharam type of } \tilde{\nu} \text{ is } \check{\kappa}^{(+\omega)},$$

$$\Vdash_{\mathbb{P}} \mathfrak{c} = \check{\kappa}^{(+\omega+1)}.$$

But because $\#(\mathbb{P})^\kappa = \kappa^{(+\omega+1)}$,

$$\Vdash_{\mathbb{P}} 2^{\check{\kappa}} \leq \check{\kappa}^{(+\omega+1)},$$

so we have the (relative) consistency of (c).

2J Proposition Let κ be a real-valued-measurable cardinal with rvm ideal $\mathcal{J}$, and let $\mathbb{P}$ be a random real p.o.set.

(a) If ν is a witnessing probability on κ and $\tilde{\nu}$ the corresponding $\mathbb{P}$-name as in 2C, then

(i) for any $B \subseteq \kappa$,

$$\Vdash_{\mathbb{P}} \tilde{\nu}\check{B} = (\nu B)^\vee.$$

(ii) $\Vdash_{\mathbb{P}} \mathcal{N}_{\tilde{\nu}}$ is the ideal of $\mathcal{P}\check{\kappa}$ generated by $\check{\mathcal{N}}_\nu$.

(iii) If ν is normal, then

$$\Vdash_{\mathbb{P}} \tilde{\nu} \text{ is normal.}$$

(b) If $\dot{\nu}$ is a $\mathbb{P}$-name for a witnessing probability on κ, then there is a witnessing probability ν_1 on κ such that

$$\mathcal{N}_{\nu_1} = \{B : B \subseteq \kappa,\ \Vdash_{\mathbb{P}} \dot{\nu}\check{B} = \check{0}\}.$$

If moreover $\Vdash_{\mathbb{P}} \dot\nu$ is normal, then ν_1 is normal.

(c) $\Vdash_{\mathbb{P}}$ the rvm ideal of $\check\kappa$ is the ideal of $\mathcal{P}\check\kappa$ generated by $\check{\mathcal{J}}$.

proof Take $\mathfrak{A}$ and $\bar\mu$ as in 2C.

(a)(i) If $B \subseteq \kappa$ then $a_\xi(\check B) = \mathbf{1}$ if $\xi \in B$, $\mathbf{0}$ if $\xi \notin B$; so $\int_a \bar\mu(a_\xi(\check B))\nu(d\xi) = \bar\mu a.\nu B$ for every $a \in \mathfrak{A}$ and $\Vdash_{\mathbb{P}} \tilde\nu\check B = (\nu B)^\vee$.

(ii) Let σ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \tilde\nu\sigma = \check 0$. Let $\langle a_\xi(\sigma)\rangle_{\xi<\kappa}$ be the corresponding family in $\mathfrak{A}$. Then $\int \bar\mu(a_\xi(\sigma))\nu(d\xi) = 0$, so $B = \{\xi : \bar\mu(a_\xi(\sigma)) > 0\} \in \mathcal{N}_\nu$. Now $\Vdash_{\mathbb{P}} \sigma \subseteq \check B \in \check{\mathcal{N}}_\nu$.

(iii) Suppose that ν is normal. Let $\langle\sigma_\xi\rangle_{\xi<\kappa}$ be a family of $\mathbb{P}$-names for $\tilde\nu$-negligible subsets of κ and let σ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \sigma = \{\eta : \exists\ \xi < \eta,\ \eta \in \sigma_\xi\}.$$

For each $\xi < \kappa$ we have a $B_\xi \in \mathcal{N}_\nu$ such that $\Vdash_{\mathbb{P}} \sigma_\xi \subseteq \check B_\xi$; set $B = \{\eta : \exists\ \xi < \eta,\ \eta \in B_\xi\}$. Because ν is normal, $B \in \mathcal{N}_\nu$, and $\Vdash_{\mathbb{P}} \tilde\nu\check B = \check 0$. But also if $\eta < \kappa$, $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}} \eta \in \sigma$, then there are $p' \le p$ and $\xi < \eta$ such that $p' \Vdash_{\mathbb{P}} \eta \in \sigma_\xi$, so that $\eta \in B_\xi$ and $\eta \in B$. Thus $\Vdash_{\mathbb{P}} \sigma \subseteq \check B$ and $\Vdash_{\mathbb{P}} \tilde\nu\sigma = \check 0$.

(b) For each $B \subseteq \kappa$ we have a unique $u_B \in L^\infty(\mathfrak{A})$ representing $\dot\nu\check B$ in the sense that for any rational numbers s, t and any $p \in \mathbb{P}$,

$$p \Vdash_{\mathbb{P}} \check s \le \dot\nu\check B \le \check t \iff s\chi(p^*) \le u_B \times \chi(p^*) \le t\chi(p^*).$$

Set

$$\nu_1 B = \int u_B d\bar\mu.$$

Then the same computations as in 2C, taken in reverse, show that ν_1 is a κ-additive probability, zero on singletons; while $\nu_1 B = 0$ iff $u_B = 0$ iff $\Vdash_{\mathbb{P}} \dot\nu\check B = \check 0$.

If $\Vdash_{\mathbb{P}} \dot\nu$ is normal, then take any family $\langle B_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}_{\nu_1}$ and set $B = \{\eta : \exists\ \xi < \eta,\ \eta \in B_\xi\}$. Then

$$\Vdash_{\mathbb{P}} \check B = \{\eta : \exists\ \xi < \eta,\ \eta \in \check B_\xi\};$$

so $\Vdash_{\mathbb{P}} \dot\nu\check B = \check 0$ and $\nu_1 B = 0$. Thus ν_1 is normal.

(c)(i) If $B \in \mathcal{J}$ and $\dot\nu$ is a $\mathbb{P}$-name for a normal witnessing probability on κ, then take ν_1 as in (b); we must have $\nu_1 B = 0$ so $\Vdash_{\mathbb{P}} \dot\nu\check B = \check 0$.

(ii) If σ is a $\mathbb{P}$-name for a member of the rvm filter of κ in $V^{\mathbb{P}}$, let $\langle a_\xi(\sigma)\rangle_{\xi<\kappa}$ be the corresponding family in $\mathfrak{A}$, and set $B = \{\xi : a_\xi(\sigma) \ne \mathbf{0}\}$. Then $\Vdash_{\mathbb{P}} \sigma \subseteq \check B$. If ν is any normal witnessing probability on κ and $\tilde\nu$ the corresponding $\mathbb{P}$-name, then $\Vdash_{\mathbb{P}} \tilde\nu\sigma = \check 0$, so as in (a)(ii) above we must have $\nu B = 0$. As ν is arbitrary, $B \in \mathcal{J}$ and $\Vdash_{\mathbb{P}} \sigma \subseteq \check B \in \check{\mathcal{J}}$.

2K Theorem If κ is a two-valued-measurable cardinal and $\mathcal{F}$ is a normal ultrafilter on κ, then

$$L(\mathcal{F}) \vDash \kappa \text{ is two-valued-measurable and } \mathcal{F} \cap L(\mathcal{F}) \text{ is the rvm filter of } \kappa.$$

proof Kunen 70, §6, or Jech 78, Theorem 76, p. 373.

2L Corollary The following are equiconsistent:

(a) ‘ZFC + there is a two-valued-measurable cardinal’;

(b) ‘ZFC + there is a two-valued-measurable cardinal in which the rvm filter is an ultrafilter’;

(c) ‘ZFC + there is an atomlessly-measurable cardinal in which the rvm ideal is the ideal of negligible sets for a normal witnessing probability’.

proof (a)⇒(b) is covered by 2K. For (b)⇒(c), start with a two-valued-measurable cardinal in which the rvm filter is an ultrafilter, and use 2C with $\mathbb{P} = \mathfrak{A}_\kappa \setminus \{\mathbf{0}\}$, as in 2E. Then the rvm ideal of κ in $V^{\mathbb{P}}$ is generated by $\mathcal{J}$, the rvm ideal of κ in V, by 2Jc. But if $\tilde\nu$ is the $\mathbb{P}$-name for a witnessing probability on κ derived from the $\{0,1\}$-valued probability ν associated with $\mathcal{J}$, then the null ideal of $\tilde\nu$ in $V^{\mathbb{P}}$ is also generated by $\mathcal{J}$, by 2J(a-ii); and therefore coincides with the rvm ideal, as required.

Finally, (c)⇒(a) is covered by 2D-2E above.

3. The Gitik-Shelah theorem

I come now to the most striking development in the theory of real-valued-measurable cardinals since the work of Solovay and Kunen in the late sixties. Recall that Maharam's theorem (A2I) gives us a complete description of all probability algebras, and that consequently almost the first question to ask of any probability space is what its measure algebra is. In particular, we ask this of $(\kappa, \mathcal{P}\kappa, \nu)$ when κ is real-valued-measurable and ν is a κ-additive probability on κ. If κ is two-valued-measurable, $(\kappa, \mathcal{P}\kappa, \nu)$ is purely atomic and there is nothing of interest to say about its measure algebra. But if κ is atomlessly-measurable, there is a great deal more to it. The Gitik-Shelah theorem (3F-3G) tells us that in this case $\mathcal{P}\kappa/\mathcal{N}_\nu$ is at least of the order of complexity achieved if ν is constructed from a κ-complete ultrafilter with Solovay's random reals.

This chapter is devoted to a proof of this theorem. The original paper Gitik & Shelah 89 relied heavily on generic-ultrapower techniques, as does its supplement Gitik & Shelah p91. Here I give what amounts to a translation of their arguments into measure theory. I do not give quite the shortest proof, as many of the intermediate steps seem to be of sufficient interest to be given as separate lemmas in rather greater generality than is needed immediately. A slightly more condensed version may be found in Kamburelis n89, from which some of the ideas below are derived.

3A Theorem Suppose that $(Y, \mathcal{P}Y, \nu)$ is a probability space and that $(X, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon probability space with the topological weight of $(X, \mathfrak{T})$ strictly less than the additivity of ν. Let $f : X \times Y \to \mathbb{R}$ be any bounded function; then

$$\overline{\int}\Big(\int f(x,y)\nu(dy)\Big)\mu(dx) \;\le\; \int\Big(\overline{\int} f(x,y)\mu(dx)\Big)\nu(dy),$$

writing $\overline{\int}$ for the upper integral, as in A2E.

proof Adding a constant function to f if necessary, we need consider only the case of non-negative f. Set $\lambda = \mathrm{w}(X) < \mathrm{add}(\nu)$; let $\langle G_\xi\rangle_{\xi<\lambda}$ enumerate a base for $\mathfrak{T}$. Fix $\epsilon > 0$. For each $y \in Y$, let $h_y : X \to \mathbb{R}$ be a lower semi-continuous function such that $f(x, y) \le h_y(x)$ for every $x \in X$ and

$$\int h_y(x)\mu(dx) \le \epsilon + \overline{\int} f(x,y)\mu(dx)$$

(A2Je). For each $I \subseteq \lambda$, set

$$f_I(x,y) = \sup\{s : \exists\ \xi \in I,\ x \in G_\xi,\ h_y(x') \ge s\ \ \forall\ x' \in G_\xi\}.$$

Then f_I is expressible as $\sup_{\xi\in I, s\in\mathbb{Q}^+} s\chi(G_\xi \times B_{\xi s})$, writing $\mathbb{Q}^+$ for the set of non-negative rational numbers and $\chi(G\times B)$ for the characteristic function of $G\times B \subseteq X\times Y$, and taking $B_{\xi s} = \{y : h_y(x') \ge s\ \ \forall\ x' \in G_\xi\}$. So f_I is $(\mu \times \nu)$-measurable for all countable I, and for such I we shall have

$$\int\int f_I(x,y)\mu(dx)\nu(dy) = \int\int f_I(x,y)\nu(dy)\mu(dx),$$

by Fubini's theorem. Next,

$$\sup_{I\in[\lambda]^{<\omega}} f_I(x,y) = h_y(x)$$

for all $x \in X$, $y \in Y$, because each h_y is lower semi-continuous, so that

$$\sup_{I\in[\lambda]^{<\omega}} \int f_I(x,y)\mu(dx) = \int h_y(x)\mu(dx)$$

for each $y \in Y$ (A2Jf). Because $\lambda < \mathrm{add}(\mu)$, it follows that

$$\sup_{I\in[\lambda]^{<\omega}} \int\int f_I(x,y)\mu(dx)\nu(dy) = \int\int h_y(x)\mu(dx)\nu(dy)$$

(A2Cf). On the other hand, if we write

$$g_I(x) = \int f_I(x,y)\nu(dy)$$

for each $x \in X$, $I \subseteq \lambda$, then (at least for finite I) g_I is also lower semi-continuous, so $g = \sup_{I\in[\lambda]^{<\omega}} g_I$ is lower semi-continuous, and $\int g(x)\mu(dx) = \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)$. Also

$$g(x) = \int h_y(x)\nu(dy) \ge \int f(x,y)\nu(dy)$$

for every $x \in X$, the equality in this formula again depending on A2Cf. So we have

$$\begin{aligned}\overline{\int}\int f(x,y)\nu(dy)\mu(dx) &\le \int g(x)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \int g_I(x)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \int\int f_I(x,y)\nu(dy)\mu(dx)\\ &= \sup_{I\in[\lambda]^{<\omega}} \int\int f_I(x,y)\mu(dx)\nu(dy)\\ &= \int\int h_y(x)\mu(dx)\nu(dy)\\ &\le \epsilon + \int\overline{\int} f(x,y)\mu(dx)\nu(dy).\end{aligned}$$

As ϵ is arbitrary, we have the result.

Remark For the case $X = \{0,1\}^\lambda$, this is due to Kunen. Readers uninterested in general quasi-Radon measure spaces should imagine X here and in 3B to be a subset of $\{0,1\}^\lambda$ with the subspace measure.

See also 6A below for a similar result, and 6J for an elementary corollary.

3B Corollary Let κ be a real-valued-measurable cardinal, with witnessing probability ν, and $(X,\mathfrak{T},\Sigma,\mu)$ a quasi-Radon probability space with $\mathrm{w}(X,\mathfrak{T}) < \kappa$.

(a) If $C \subseteq X \times \kappa$ then

$$\overline{\int}\nu C[\{x\}]\mu(dx) \le \int \mu^* C^{-1}[\{\xi\}]\nu(d\xi).$$

(b) If $A \subseteq X$ and $\#(A) \le \kappa$, then there is a $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu^* B = \mu^* A$.

(c) If $\langle C_\xi\rangle_{\xi<\kappa}$ is a family in $\mathcal{P}X \setminus \mathcal{N}_\mu$ such that $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$, then there are distinct ξ, $\eta < \kappa$ such that $\mu^*(C_\xi \cap C_\eta) > 0$.

(d) If we have a family $\langle h_\xi\rangle_{\xi<\kappa}$ of functions such that each $\mathrm{dom}(h_\xi)$ is a non-negligible subset of X and $\#(\bigcup_{\xi<\kappa} h_\xi) < \kappa$ (identifying each h_ξ with its graph), then there are distinct ξ, $\eta < \kappa$ such that

$$\mu^*\{x : x \in \mathrm{dom}(h_\xi) \cap \mathrm{dom}(h_\eta),\ h_\xi(x) = h_\eta(x)\} > 0.$$

proof (a) Apply 3A to $\chi C : X \times \kappa \to \mathbb{R}$.

(b) ? Suppose, if possible, otherwise. Then surely $\#(A) = \kappa$; let $f : \kappa \to A$ be a bijection. Set

$$C = \{(f(\eta),\xi) : \eta \le \xi < \kappa\} \subseteq X \times \kappa.$$

If $x \in A$,

$$\nu C[\{x\}] = \nu\{\xi : f^{-1}(x) \le \xi < \kappa\} = 1,$$

so $\overline{\int}\nu C[\{x\}]\mu(dx) = \mu^* A$. If $\xi < \kappa$,

$$\mu^* C^{-1}[\{\xi\}] = \mu^*\{f(\eta) : \eta < \xi\} < \mu^* A,$$

so $\int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) < \mu^* A$. But this contradicts (a). **X**

(c) Let $\tilde\nu$ be the probability on $\kappa \times \kappa$ defined by writing

$$\tilde\nu A = \int \nu A[\{\xi\}]\,\nu(d\xi)\ \ \forall\ \ A \subseteq \kappa \times \kappa.$$

Then $\tilde\nu$ is κ-additive. Set

$$\begin{aligned}C &= \{(x,(\xi,\eta)) : \xi,\ \eta \text{ are distinct members of } \kappa,\ x \in C_\xi \cap C_\eta\}\\ &\subseteq X \times (\kappa\times\kappa).\end{aligned}$$

Set

$$E = \{x : x \in X,\, \nu\{\xi : x \in C_\xi\} = 0\}.$$

Because $\#(\bigcup_{\xi<\kappa} C_\xi) < \kappa$,

$$\nu\{\xi : E \cap C_\xi \neq \emptyset\} = 0,$$

and there is a $\xi < \kappa$ with $C_\xi \cap E = \emptyset$; thus $\mu^*(X \setminus E) > 0$. Now if $x \in X \setminus E$ then

$$\tilde{\nu}\{(\xi,\eta) : (x,(\xi,\eta)) \in C\} = (\nu\{\xi : x \in C_\xi\})^2 > 0.$$

So we have

$$0 < \overline{\int} \tilde{\nu}\{(\xi,\eta) : (x,(\xi,\eta)) \in C\}\, \mu(dx)$$
$$\leq \int \mu^*\{x : (x,(\xi,\eta)) \in C\}\, \tilde{\nu}(d(\xi,\eta)),$$

by 3A, and there must be distinct ξ, $\eta < \kappa$ such that $\mu^*\{x : (x,(\xi,\eta)) \in C\} > 0$, as required.

(d) Set $Y = \bigcup_{\xi<\kappa} \mathrm{ran}(h_\xi)$. Give $X \times Y$ the measure $\tilde{\mu}$ and topology $\mathfrak{T}'$ defined as follows. The domain of $\tilde{\mu}$ is to be the family $\tilde{\Sigma}$ of subsets H of $X \times Y$ for which there are E, $E' \in \Sigma$ with $\mu(E' \setminus E) = 0$ and $E \times Y \subseteq H \subseteq E' \times Y$; and for such H, $\tilde{\mu}H$ is to be $\mu E = \mu E'$. The topology $\mathfrak{T}'$ is to be just the family $\{G \times Y : G \in \mathfrak{T}\}$. It is easy to check that $(X \times Y, \mathfrak{T}', \tilde{\Sigma}, \tilde{\mu})$ is a quasi-Radon probability space of weight less than κ, and that $\tilde{\mu}^* h_\xi = \mu^*(\mathrm{dom}(h_\xi)) > 0$ for each $\xi < \kappa$. So (c) gives the result.

Remark I have taken the idea of 3Bc from KAMBURELIS N89.

3C Lemma Suppose that λ, ζ, δ are cardinals, with $\delta < \lambda < \mathrm{cf}(\zeta)$, and that S is a stationary subset of ζ. Let $\langle I_\alpha \rangle_{\alpha \in S}$ be a family in $[\lambda]^{\leq\delta}$. Then there is a set $M \subseteq \lambda$ such that $\mathrm{cf}(\#(M)) \leq \delta$ and $\{\alpha : \alpha \in S,\, I_\alpha \subseteq M\}$ is stationary in ζ.

proof For $M \subseteq \lambda$, set $S_M = \{\alpha : \alpha \in S,\, I_\alpha \subseteq M\}$. Let $M \subseteq \lambda$ be a set of minimal cardinality such that S_M is stationary in ζ. Set $\theta = \#(M)$. **?** If $\mathrm{cf}(\theta) > \delta$, enumerate M as $\langle \gamma_\xi \rangle_{\xi<\theta}$. For each $\alpha \in S_M$, set $\beta_\alpha = \sup\{\xi : \gamma_\xi \in I_\alpha\}$; because $\#(I_\alpha) \leq \delta < \mathrm{cf}(\theta)$, $\beta_\alpha < \theta$. Because $\theta \leq \lambda < \mathrm{cf}(\zeta)$, there is a $\beta < \theta$ such that $S' = \{\alpha : \alpha \in S_M,\, \beta_\alpha = \beta\}$ is stationary in ζ. Consider $M' = \{\gamma_\xi : \xi \leq \beta\}$; then $\#(M') < \#(M)$ but $S_{M'} \supseteq S'$ so is stationary in ζ, contrary to the choice of M. **X**

Thus M and $S = S_M$ will serve.

3D Definition Let κ be a cardinal. Write $\mathrm{Tr}(\kappa)$ for

$$\sup\{\#(F) : F \subseteq \kappa^\kappa,\, \#(f \cap g) < \kappa \text{ for all distinct } f,\, g \in F\}.$$

Thus $\mathrm{Tr}(\kappa) = \mathrm{Tr}_{[\kappa]^{<\kappa}}(\kappa;\kappa)$ in the notation of 2F.

3E Lemma (a) For any infinite cardinal κ,

$$\kappa^+ \leq \mathrm{Tr}(\kappa) \leq 2^\kappa.$$

(b) For any infinite cardinal κ,

$$\max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) \geq \min(2^\kappa, \kappa^{(+\omega)}).$$

(c) If κ is such that $2^\delta \leq \kappa$ for every $\delta < \kappa$, then $\mathrm{Tr}(\kappa) = 2^\kappa$.

proof (a) We can build inductively a family $\langle f_\alpha \rangle_{\alpha<\kappa^+}$ in κ^κ, as follows. Given $\langle f_\alpha \rangle_{\alpha<\beta}$, where $\beta < \kappa^+$, let $\theta : \beta \to \kappa$ be any injection. Now choose $f_\beta : \kappa \to \kappa$ so that

$$f_\beta(\xi) \neq f_\alpha(\xi) \text{ whenever } \alpha < \beta \text{ and } \theta(\alpha) \leq \xi.$$

This will mean that if $\alpha < \beta$, then

$$\{\xi : f_\alpha(\xi) = f_\beta(\xi)\} \subseteq \theta(\alpha)$$

has cardinal less than κ. So at the end of the induction, $F = \{f_\alpha : \alpha < \kappa^+\}$ will witness that $\mathrm{Tr}(\kappa) \geq \kappa^+$. On the other hand, $\mathrm{Tr}(\kappa) \leq \#(\kappa^\kappa) = 2^\kappa$.

(b) ? If not, then take $\lambda = \max(\mathrm{Tr}(\kappa), \sup_{\delta<\kappa} 2^\delta) < \min(2^\kappa, \kappa^{(+\omega)})$. For each $\xi < \kappa$ take an injective function $\phi_\xi : \mathcal{P}\xi \to \lambda$. Because $\lambda < 2^\kappa$, we have an injective function $h : \lambda^+ \to \mathcal{P}\kappa$. For $\alpha < \lambda^+$ set

$g_\alpha(\xi) = \phi_\xi(h(\alpha) \cap \xi)$ for every $\xi < \kappa$; then $\langle g_\alpha \rangle_{\alpha<\lambda^+}$ is a family in λ^κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha \neq \beta$.

Apply 3C with $\delta = \kappa$, $\zeta = \lambda^+$, $I_\alpha = g_\alpha[\kappa]$ to see that there is a set $M \subseteq \lambda$ with $\mathrm{cf}(\#(M)) \leq \kappa$ and $S = \{\alpha : \alpha < \lambda^+, g_\alpha[\kappa] \subseteq M\}$ stationary in λ^+. Because $\lambda < \kappa^{(+\omega)}$, we must have $\#(M) \leq \kappa$. If $f : M \to \kappa$ is any injection, $\langle f g_\alpha \rangle_{\alpha \in S}$ will witness that $\mathrm{Tr}(\kappa) \geq \#(S) = \lambda^+$; which is impossible. **X**

(c) For each $\xi < \kappa$, let $\phi_\xi : \mathcal{P}\xi \to \kappa$ be injective. For $A \subseteq \kappa$, define $f_A \in \kappa^\kappa$ by writing

$$f_A(\xi) = \phi_\xi(A \cap \xi) \ \forall \ \xi < \kappa.$$

Then $F = \{f_A : A \subseteq \kappa\}$ witnesses that $\mathrm{Tr}(\kappa) \geq 2^\kappa$.

3F The Gitik-Shelah Theorem Let κ be an atomlessly-measurable cardinal, with witnessing probability ν. Then the Maharam type of $(\kappa, \mathcal{P}\kappa, \nu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$.

proof ? Suppose, if possible, otherwise.

(a) Let $A \subseteq \kappa$ be such that $\nu A > 0$ and $(A, \mathcal{P}A, \nu\lceil A)$ is Maharam homogeneous, of Maharam type λ say; surely λ is not greater than the Maharam type of $(\kappa, \mathcal{P}\kappa, \nu)$, so we have $\lambda < 2^\kappa$ and $\lambda < \kappa^{(+\omega)}$. Also $\lambda \geq \omega$, because ν is atomless.

Set

$$\nu_1 B = (\nu A)^{-1} \nu(A \cap B) \ \forall \ B \subseteq \kappa;$$

then $(\kappa, \mathcal{P}\kappa, \nu_1)$ is Maharam homogeneous, with Maharam type λ.

(b) We shall need some fairly elaborate notation. Let μ_λ be the usual Radon probability on $\{0,1\}^\lambda$ (A2G), Σ_λ its domain, and $\mathfrak{A}_\lambda = \Sigma_\lambda / \mathcal{N}_{\mu_\lambda}$ its measure algebra. For $M \subseteq \lambda$ let $\pi_M : \{0,1\}^\lambda \to \{0,1\}^M$ be the canonical restriction map, and μ_M the usual Radon probability on $\{0,1\}^M$, so that π_M is inverse-measure-preserving for μ_λ and μ_M; write Σ^*_M for the Baire σ-algebra of $\{0,1\}^M$.

Now let us return to the argument. We are supposing that $\mathfrak{A}_\lambda \cong \mathfrak{A} = \mathcal{P}\kappa / \mathcal{N}_{\nu_1}$, the measure algebra of $(\kappa, \mathcal{P}\kappa, \nu_1)$. Let $\phi : \mathfrak{A}_\lambda \to \mathfrak{A}$ be a measure- preserving isomorphism and $f : \kappa \to \{0,1\}^\lambda$ a corresponding inverse-measure -preserving function, so that

$$f^{-1}[E]^\bullet = \phi(E^\bullet) \in \mathfrak{A} \ \forall \ E \in \Sigma_\lambda$$

(A2Jd).

(c) From here on we are going to need a split argument, depending on whether or not $\lambda < \mathrm{Tr}(\kappa)$. Because large parts of the two arguments are the same, I take them together; but readers may prefer to follow through case 1 completely before returning to examine the modifications necessary in case 2.

case 1 Suppose that $\lambda < \mathrm{Tr}(\kappa)$. Set $\zeta = \max(\kappa^+, \lambda^+)$; then $\zeta \leq \mathrm{Tr}(\kappa)$ is a successor cardinal so there is a family $\langle g_\alpha \rangle_{\alpha<\zeta}$ of functions from κ to κ such that $\#(g_\alpha \cap g_\beta) < \kappa$ whenever $\alpha < \beta < \zeta$. Set $S = \zeta \setminus \kappa$, so that S is a stationary set in ζ. We know that $\kappa \leq \mathfrak{c}$; let $h : \kappa \to \{0,1\}^\omega$ be an injection. Set $\delta = \omega$, $\zeta^* = \kappa$, so that we have $g_\alpha : \kappa \to \min(\zeta^*, \alpha)$ for each $\alpha \in S$, and $h : \zeta^* \to \{0,1\}^\delta$.

case 2 Suppose that $\lambda \geq \mathrm{Tr}(\kappa)$. Set $\zeta = \lambda^+ < \kappa^{(+\omega)}$. Then $\mathrm{Tr}(\kappa) < \zeta \leq \min(2^\kappa, \kappa^{(+\omega)})$ so there must be a cardinal $\delta < \kappa$ such that $2^\delta \geq \zeta$, by 3Eb. Set $\zeta^* = \zeta$ and let $h : \zeta^* \to \{0,1\}^\delta$ be an injection. Because $\kappa < \mathrm{Tr}(\kappa) \leq \lambda < \kappa^{(+\omega)}$, $\mathrm{cf}(\lambda) > \omega_1$. By Shelah's lemma A1F-G, we can find a stationary set $S \subseteq \zeta$ and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to $\zeta = \zeta^*$ such that (i) $g_\alpha[\kappa] \subseteq \alpha$ for each $\alpha \in S$ (ii) $\#(g_\alpha \cap g_\beta) < \kappa$ for distinct α, $\beta \in S$ (iii) if θ is a limit ordinal less than κ, and α, $\beta \in S$ are such that $g_\alpha(\theta) = g_\beta(\theta)$, then $g_\alpha\lceil\theta = g_\beta\lceil\theta$.

(d) Now we have a section in which the two arguments run together, if we keep hold of the notations introduced in (c). For each $\alpha \in S$ consider $h g_\alpha : \kappa \to \{0,1\}^\delta$. For $\iota < \delta$ set

$$U_{\alpha\iota} = \{\xi : \xi < \kappa, (h g_\alpha(\xi))(\iota) = 1\},$$

and choose $H_{\alpha\iota} \in \Sigma^*_\lambda$ such that $\phi^{-1}(U^\bullet_{\alpha\iota}) = H^\bullet_{\alpha\iota} \in \mathfrak{A}_\lambda$. Define $\tilde{g}_\alpha : \{0,1\}^\lambda \to \{0,1\}^\delta$ by setting

$$(\tilde{g}_\alpha(x))(\iota) = 1 \text{ if } x \in H_{\alpha\iota}, = 0 \text{ otherwise}.$$

Then

$$\{\xi : \xi < \kappa,\, \tilde{g}_\alpha f(\xi) \ne hg_\alpha(\xi)\} = \bigcup_{\iota<\delta}\{\xi : (\tilde{g}_\alpha f(\xi))(\iota) \ne (hg_\alpha(\xi))(\iota)\}$$
$$= \bigcup_{\iota<\delta} U_{\alpha\iota} \triangle f^{-1}[H_{\alpha\iota}] \in \mathcal{N}_{\nu_1}$$

because $\delta < \kappa = \mathrm{add}(\mathcal{N}_{\nu_1})$. Set $V_\alpha = \{\xi : \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi)\}$, so that $\nu_1 V_\alpha = 1$, for each $\alpha \in S$.

(e) Because every $H_{\alpha\iota}$ is a Baire set, there is for each $\alpha \in S$ a set $I_\alpha \subseteq \lambda$ such that $\#(I_\alpha) \le \delta$ and

$$H_{\alpha\iota} = \pi_{I_\alpha}^{-1}[\pi_{I_\alpha}[H_{\alpha\iota}]] \;\forall\; \iota < \delta.$$

By Lemma 3C there is an $M \subseteq \lambda$ such that

$$S_1 = \{\alpha : \alpha \in S,\, I_\alpha \subseteq M\}$$

is stationary in ζ and $\mathrm{cf}(\#(M)) \le \delta$; because $\lambda < \kappa^{(+\omega)}$ and $\mathrm{cf}(\kappa) > \delta$, $\#(M) < \kappa$. Write $f_M = \pi_M f$, so that $f_M : \kappa \to \{0,1\}^M$ is inverse-measure -preserving for ν_1 and μ_M.

(f) For each $\alpha \in S_1$, there is a $\theta_\alpha < \kappa$ such that $\mu_M^*(f_M[V_\alpha \cap \theta_\alpha]) = 1$. **P** Apply 3Bb to $f_M[V_\alpha] \subseteq \{0,1\}^M$. There must be a set $B \subseteq f_M[V_\alpha]$ such that $\#(B) < \kappa$ and $\mu_M^* B = \mu_M^*(f_M[V_\alpha])$; because κ is regular, there is a $\theta_\alpha < \kappa$ such that $B \subseteq f_M[V_\alpha \cap \theta_\alpha]$. On the other hand, because f_M is inverse-measure -preserving, $\mu_M^*(f_M[V_\alpha]) \ge \nu_1 V_\alpha = 1$. **Q**

Evidently we may take it that every θ_α is a non-zero limit ordinal.

(g) Because $\zeta = \mathrm{cf}(\zeta) > \kappa$, there is a $\theta < \kappa$ such that

$$S_2 = \{\alpha : \alpha \in S_1,\, \theta_\alpha = \theta\}$$

is stationary in ζ. At this point the two cases diverge briefly.

case 1 For $\alpha \in S_2$, $g_\alpha[\theta]$ is bounded in κ; let $\theta' < \kappa$ be such that

$$S_3 = \{\alpha : \alpha \in S_2,\, g_\alpha[\theta] \subseteq \theta'\}$$

is stationary in ζ; write $Y = \theta'$.

case 2 For $\alpha \in S_2$, $g_\alpha(\theta) < \alpha$; by the pressing-down lemma there is a $\theta' < \zeta$ such that

$$S_3 = \{\alpha : \alpha \in S_2,\, g_\alpha(\theta) = \theta'\}$$

is stationary in ζ. Then $g_\alpha \restriction \theta = g_\beta \restriction \theta$ for all α, $\beta \in S_3$; take Y to be the common value of $g_\alpha[\theta]$ for $\alpha \in S_3$.

(h) Thus in both cases we have $Y \subseteq \zeta^*$, $\#(Y) < \kappa$ and $g_\alpha[\theta] \subseteq Y$ for all $\alpha \in S_3$. For each $\alpha \in S_3$, set

$$Q_\alpha = f_M[V_\alpha \cap \theta] = f_M[V_\alpha \cap \theta_\alpha],$$

so that $\mu_M^* Q_\alpha = 1$. Now $I_\alpha \subseteq M$, so we can express $\tilde{g}_\alpha$ as $g_\alpha^* \pi_M$, where $g_\alpha^* : \{0,1\}^M \to \{0,1\}^\delta$ is Baire measurable in each coordinate. If $y \in Q_\alpha$, take $\xi \in V_\alpha \cap \theta$ such that $f_M(\xi) = y$; then

$$g_\alpha^*(y) = g_\alpha^* \pi_M f(\xi) = \tilde{g}_\alpha f(\xi) = hg_\alpha(\xi) \in h[Y].$$

Thus $g_\alpha^* \restriction Q_\alpha \subseteq f_M[\theta] \times h[Y]$ for every $\alpha \in S_3$, and we may apply Corollary 3Bd to $X = \{0,1\}^M$, $\mu = \mu_M$ and the family $\langle g_\alpha^* \restriction Q_\alpha \rangle_{\alpha \in S'}$, where $S' \subseteq S_3$ is a set of cardinal κ, to see that there are distinct α, $\beta \in S_3$ such that $\mu_M^*\{y : y \in Q_\alpha \cap Q_\beta,\, g_\alpha^*(y) = g_\beta^*(y)\} > 0$. Now, however, consider

$$E = \{y : y \in \{0,1\}^M,\, g_\alpha^*(y) = g_\beta^*(y)\}.$$

Then $E = \bigcap_{\iota<\delta} E_\iota$, where

$$E_\iota = \{y : y \in \{0,1\}^M,\, g_\alpha^*(y)(\iota) = g_\beta^*(y)(\iota)\} \in \Sigma_M^*$$

for $\iota < \delta$. Because $\delta < \kappa$,

$$
\begin{aligned}
\nu_1 f_M^{-1}[E] &= \nu_1(\bigcap_{\iota<\delta} f_M^{-1}[E_\iota]) \\
&= \inf_{I\in[\delta]^{<\omega}} \nu_1(\bigcap_{\iota\in I} f_M^{-1}[E_\iota]) \\
&= \inf_{I\in[\delta]^{<\omega}} \mu_M(\bigcap_{\iota\in I} E_\iota) \\
&\geq \mu_M^* E > 0.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
0 &< \nu_1 f_M^{-1}[E] \\
&= \nu_1\{\xi : g_\alpha^*\pi_M f(\xi) = g_\beta^*\pi_M f(\xi)\} \\
&= \nu_1\{\xi : \tilde{g}_\alpha f(\xi) = \tilde{g}_\beta f(\xi)\} \\
&= \nu_1\{\xi : \xi\in V_\alpha\cap V_\beta,\ \tilde{g}_\alpha f(\xi) = \tilde{g}_\beta f(\xi)\} \\
&= \nu_1\{\xi : hg_\alpha(\xi) = hg_\beta(\xi)\} \\
&= \nu_1\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}
\end{aligned}
$$

(because h is injective). But this is absurd, because in (d) above (whether in case 1 or in case 2) we chose g_α, g_β in such a way that $\{\xi : g_\alpha(\xi) = g_\beta(\xi)\}$ would be bounded in κ. **X** This contradiction completes the proof.

3G Theorem Let $(X, \mathcal{P}X, \mu)$ be an atomless probability space. Write $\kappa = \mathrm{add}(\mu)$. Then the Maharam type of $(X, \mathcal{P}X, \mu)$ is at least $\min(\kappa^{(+\omega)}, 2^\kappa)$, and in particular is greater than κ.

proof As in the proof of 1Da above, there is a disjoint family $\langle E_\xi\rangle_{\xi<\kappa}$ in $\mathcal{N}_\mu$ such that $E = \bigcup_{\xi<\kappa} E_\xi \notin \mathcal{N}_\mu$. Let $f : E \to \kappa$ be the corresponding function, and set $\nu = (\mu E)^{-1}((\mu\lceil E)f^{-1})$. Then ν is a witnessing probability on κ. Now observe that if $\mathfrak{A}$, $\mathfrak{B}$ are the measure algebras of $(X, \mathcal{P}X, \mu)$ and $(\kappa, \mathcal{P}\kappa, \nu)$ respectively, the map

$$A^\bullet \mapsto (f^{-1}[A])^\bullet : \mathfrak{B} \to \mathfrak{A}$$

is an injective order-continuous Boolean homomorphism from $\mathfrak{B}$ to the principal ideal of $\mathfrak{A}$ generated by $E^\bullet$. Its range therefore has Maharam type equal to that of $\mathfrak{B}$; it follows easily that the Maharam type of $\mathfrak{A}$ is at least that of $\mathfrak{B}$ (A2Hc-d), and must be at least $\min(2^\kappa, \kappa^{(+\omega)})$, by Theorem 3F.

3H Corollary Let $(X, \mathcal{P}X, \nu)$ be an atomless probability space, and $\kappa = \mathrm{add}(\nu)$. Let $(Z, \mathfrak{T}, \Sigma, \mu)$ be a Radon probability space of Maharam type $\lambda \leq \min(2^\kappa, \kappa^{(+\omega)})$ (e.g., $Z = \{0,1\}^\lambda$). Then there is an inverse-measure -preserving function $f : X \to Z$.

proof By 3G, the Maharam type of $(C, \mathcal{P}C, \nu\lceil C)$ is at least λ whenever $C \subseteq X$ and $\mu C > 0$; that is, $\tau(\mathfrak{A}\lceil c) \geq \lambda$ whenever c is a non-zero element of the measure algebra $\mathfrak{A}$ of $(X, \mathcal{P}X, \nu)$ and $\mathfrak{A}\lceil c$ is the principal ideal of $\mathfrak{A}$ generated by c. Now it follows by A2Ib that there is a measure-preserving homomorphism $\phi : \Sigma/\mathcal{N}_\mu \to \mathfrak{A}$, which by A2Jd corresponds to an inverse-measure -preserving $f : X \to Z$.

3I Corollary If κ is an atomlessly-measurable cardinal, and (Z, μ) is a Radon probability space of Maharam type at most $\min(2^\kappa, \kappa^{(+\omega)})$, then there is an extension of μ to a κ-additive measure defined on $\mathcal{P}Z$.

proof Let ν be a witnessing probability on κ; by 3H, there is an inverse-measure -preserving function $f : X \to Z$; now νf^{-1} extends μ to $\mathcal{P}Z$.

3J Corollary If κ is an atomlessly-measurable cardinal, with witnessing probability ν, and $2^\kappa < \kappa^{(+\omega)}$, then $(\kappa, \mathcal{P}\kappa, \nu)$ is Maharam homogeneous, with Maharam type 2^κ.

proof If $C \in \mathcal{P}\kappa\setminus\mathcal{N}_\nu$, then the Maharam type of $\nu\lceil C$ is at least 2^κ, by 3F-3G; but also it cannot be greater than $\#(\mathcal{P}C) = 2^\kappa$.

3K Remarks The original theorem of Gitik & Shelah 89 was that the Maharam type λ of $(\kappa, \mathcal{P}\kappa, \nu)$ is at least κ^+ for any atomlessly-measurable cardinal κ and witnessing probability ν. Elementary modifications of their arguments showed that it is at least $\min(\mathrm{Tr}(\kappa), \kappa^{(+\omega)})$ (case 1 of 3F above). The extra ideas in case 2 of 3F come from Gitik & Shelah p91.

If there can be measurable cardinals at all, it is consistent to suppose that $\lambda = \kappa^{(+\omega)} < 2^\kappa$ (2Ie above). So there is no obvious sharpening of the result to look for. But it may be that a more delicate combinatorial analysis would give a closer description of λ, and, in particular, determine whether $(\kappa, \mathcal{P}\kappa, \nu)$ is always Maharam homogeneous. See also 7Qb, for a scrap of further information, and P2, for a discussion of the outstanding questions.

For more in the direction of 3I, see 8A.

3L Proposition Let κ be an atomlessly-measurable cardinal.

(a) The following are equivalent: (i) every witnessing probability ν on κ is Maharam homogeneous; (ii) if ν_1 and ν_2 are witnessing probabilities on κ they have the same Maharam type.

(b) The following are equivalent: (i) every normal witnessing probability ν on κ is Maharam homogeneous; (ii) if ν_1 and ν_2 are normal witnessing probabilities on κ they have the same Maharam type.

proof (a)(i)$\Rightarrow$(ii) Suppose that every witnessing probability on κ is Maharam homogeneous, and let ν_1, ν_2 be two such probabilities. Then $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ is another, so is Maharam homogeneous; now A2Y tells us that ν_1, ν_2 and ν all have the same Maharam type.

(ii)$\Rightarrow$(i) ? Suppose, if possible, that (ii) is true and (i) is false. Let ν be a witnessing probability on κ which is not Maharam homogeneous. Then there is an $E \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ such that $\nu\lceil E$ has strictly smaller Maharam type than ν. But now if we set $\nu_1 A = \nu(A \cap E)/\nu E$ for every $A \subseteq \kappa$, we obtain a witnessing probability ν_1 with the same Maharam type as $\nu\lceil E$, and different from that of ν; contradicting (ii).**X**

(b) Argue as for (a); we need note only that if ν_1 and ν_2 are normal so is $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ (because $\mathcal{N}_\nu = \mathcal{N}_{\nu_1} \cap \mathcal{N}_{\nu_2}$), and if ν is normal and ν_1 is constructed as in (ii)$\Rightarrow$(i) above then ν_1 is normal (because $\mathcal{N}_{\nu_1} = \{A : A \cap E \in \mathcal{N}_\nu\}$).

3M Proposition Let κ be an atomlessly-measurable cardinal.

(a) If ν is a Maharam homogeneous witnessing probability on κ with Maharam type λ, then there is a Maharam homogeneous normal witnessing probability ν_1 on κ with Maharam type $\lambda_1 \leq \lambda$.

(b) If ν and ν' are Maharam homogeneous witnessing probabilities on κ with Maharam types λ, λ', then there is a Maharam homogeneous witnessing probability ν'' on κ with Maharam type $\lambda'' \geq \mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda')$.

proof (a) Construct an essentially minimal $f_0 : \kappa \to \kappa$ as in the proof of 1G, and set $\nu_0 = \nu f_0^{-1}$. Then ν_0 is a normal witnessing probability on κ, as observed in 1G; moreover, f_0 is inverse-measure -preserving for ν and ν_0 so induces an embedding of the measure algebra $\mathfrak{A}_0 = \mathcal{P}\kappa/\mathcal{N}_{\nu_0}$ in $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{N}_\nu$. Accordingly $\tau(\mathfrak{A}_0)$, the Maharam type of ν_0, is at most $\tau(\mathfrak{A}) = \lambda$ (A2Hd). I do not know whether ν_0 itself must be Maharam homogeneous, but there is surely an $E \in \mathcal{P}\kappa \setminus \mathcal{N}_{\nu_0}$ such that $\nu_0\lceil E$ is Maharam homogeneous, and now setting $\nu_1 A = \nu_0(A \cap E)/\nu_0 E$ for $A \subseteq \kappa$ we obtain a Maharam homogeneous normal ν_1 of Maharam type less than or equal to λ.

(b) Let ν_1 be the κ-additive probability on $\kappa \times \kappa$ given by

$$\nu_1 C = \int \nu' C[\{\xi\}] \nu(d\xi) \ \forall \ C \subseteq \kappa \times \kappa.$$

Set $\theta = \mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda')$. By 2G there is a family $F \subseteq \lambda'^\kappa$ such that $\#(F) = \theta$ and $\{\xi : f(\xi) = g(\xi)\} \in \mathcal{N}_\nu$ for all distinct f, $g \in F$. Let $\langle E_\xi \rangle_{\xi < \lambda'}$ be a ν'-stochastically independent family of subsets of κ of ν'-measure $\frac{1}{2}$. For each $f \in F$ set

$$C_f = \{(\xi, \eta) : \xi < \kappa,\, \eta \in E_{f(\xi)}\}.$$

Then for any non-empty finite subset I of F, $\nu'(\bigcap_{f \in I} E_{f(\xi)}) = 2^{-\#(I)}$ for ν-almost every ξ, so that

$$\nu_1(\bigcap_{f \in I} C_f) = 2^{-\#(I)}.$$

Thus $\langle C_f \rangle_{f \in F}$ is stochastically independent for ν_1, and the Maharam type of $\nu_1\lceil C$ is at least $\#(F) = \theta$ whenever $\nu_1 C > 0$. Once again, take $\nu_2 C = \nu_1(C \cap D)/\nu_1 D$ for some D for which $\nu_1\lceil D$ is Maharam homogeneous, to obtain a Maharam homogeneous κ-additive probability ν_2 of Maharam type at least θ. Finally, of course, ν_2 can be copied onto a probability ν'' on κ, as asked for.

Remark Evidently the arguments for (b) have extensions to the case in which we have two real-valued-measurable cardinals κ, κ' with witnessing probabilities ν, ν'.

Version of 16.5.91

4. The enormity of real-valued-measurable cardinals.

Under this title I collect together results of the form 'if κ is real-valued-measurable, there are many complex cardinals below it'. Ulam's theorem that a real-valued-measurable cardinal must be weakly inaccessible (1Dc-d) is the first step: if κ is real-valued-measurable, there are κ cardinals below it. But enormously more can be said. To develop these ideas, we need labels for some of the intermediate stages. first 'weakly Mahlo' and 'greatly Mahlo' cardinals (4A), and then 'weakly Π^1_n-indescribable' cardinals (A4C, 4D). Up to the weakly Π^1_1-indescribable cardinals, we can use ordinary infinitary combinatorics (4A-4L); but thereafter we shall need the apparatus of (elementary) model theory from §A4 and forcing from §2. The culminating result is Theorem 4P: if κ is real-valued-measurable, there are many weakly Π^2_0-indescribable cardinals below it. The proof of this theorem includes essentially everything required to prove another remarkable fact: if κ is atomlessly-measurable, and I any structure of cardinal less than κ, then the first- and second-order properties of I are unaffected by random real forcing (Corollary 4Oa). The same arguments provide a general method for proving results of the form 'if κ is real-valued-measurable, there are many $\alpha < \kappa$ such that $\alpha \vDash \phi$' when we have found a proof that $\kappa \vDash \phi$ for every real-valued-measurable cardinal κ (4Ob).

4A Definitions (a) A cardinal κ is a **weakly Mahlo** cardinal if it is weakly inaccessible and the set of weakly inaccessible cardinals below κ is stationary in κ.

(b) If A is any set of ordinals, write $\mathrm{Mh}(A)$ for

$$\{\xi : \xi < \sup A,\ \mathrm{cf}(\xi) > \omega,\ A \cap \xi \text{ is stationary in } \xi\}.$$

Following Baumgartner Taylor & Wagon 77, I will call Mh **Mahlo's operation**.

(This is close to the operation H of Lévy 71, and in a sense dual to the 'Mahlo operation' M of Keisler & Tarski 64.)

(c) A cardinal κ is **greatly Mahlo** if it is uncountable and regular and there is a normal filter $\mathcal{F}$ on κ such that $\mathrm{Mh}(A) \in \mathcal{F}$ for every $A \in \mathcal{F}$.

In this case (because the intersection of any non-empty family of normal filters is a normal filter) there is a minimal normal filter $\mathcal{W}$ on κ which is closed under Mh; I will call $\mathcal{W}$ the **greatly Mahlo** filter of κ.

4B Theorem If κ is greatly Mahlo, it is weakly Mahlo and the set of weakly Mahlo cardinals below κ belongs to the greatly Mahlo filter of κ.

proof This is essentially elementary. Let κ be a greatly Mahlo cardinal and $\mathcal{W}$ its greatly Mahlo filter. For each $\eta < \kappa$, let W_η be the set of ordinals less than κ with cofinality at least η. Then $W_{\eta+1} \supseteq \mathrm{Mh}(W_\eta)$ for every η, so (because $\mathcal{W}$ is κ-complete) $W_\eta \in \mathcal{W}$ for every $\eta < \kappa$. Because $\mathcal{W}$ is normal,

$$W = \{\xi : \omega < \xi < \kappa,\ \xi \in W_\eta\ \ \forall\ \ \eta < \xi\}$$

belongs to $\mathcal{W}$; but W is just the set of regular uncountable cardinals below κ. This forces κ to be a weakly inaccessible cardinal. Next, consider the set W' of weakly inaccessible cardinals below κ; $W' \supseteq W \cap \mathrm{Mh}(W)$, so $W' \in \mathcal{W}$ and must be stationary in κ; thus κ is weakly Mahlo. Finally, $W'' = W' \cap \mathrm{Mh}(W')$ is the set of weakly Mahlo cardinals below κ, and again belongs to $\mathcal{W}$.

Remark The alternation of diagonal intersection and the Mahlo operation Mh can be used to describe an indefinitely complex hierarchy of cardinals; near the bottom, the '$(\alpha+1)$-Mahlo cardinals' are those in which the smaller α-Mahlo cardinals form a stationary set, taking the weakly inaccessible cardinals to be the '0-Mahlo' cardinals. The 'greatly Mahlo' cardinals form a natural staging post well along this progression.

4C Lemma Let κ be a regular uncountable cardinal. Then κ is greatly Mahlo iff whenever $\preccurlyeq$ is a well-ordering of κ and $\langle S_\zeta \rangle_{\zeta<\kappa}$ is a family of subsets of κ such that

$$S_\zeta \supseteq \{\xi : \xi < \kappa,\, \xi \in \mathrm{Mh}(S_\eta) \text{ whenever } \eta < \xi \text{ and } \eta \prec \zeta\}$$

for every $\zeta < \kappa$, then every S_ζ is stationary in κ.

proof (a) If κ is greatly Mahlo, it carries a normal ultrafilter $\mathcal{F}$ which is closed under Mahlo's operation; now an easy induction on the $\preccurlyeq$-position of ζ shows that under the conditions described every S_ζ must belong to $\mathcal{F}$, so is stationary.

(b) Now suppose that κ is not greatly Mahlo. Recall that if NS_κ is the ideal of non-stationary subsets of κ and $\mathfrak{A}$ is the quotient algebra $\mathcal{P}\kappa/\mathrm{NS}_\kappa$, then any subset of $\mathfrak{A}$ of cardinal κ or less has a greatest lower bound in $\mathfrak{A}$. (See 1Fd-e.) Observe also that we have an operation $M : \mathfrak{A} \to \mathfrak{A}$ defined by writing $M(A^\bullet) = (\mathrm{Mh}(A))^\bullet$ for every $A \subseteq \kappa$; the point being that if A, A' are subsets of κ and $A \cap C = A' \cap C$ for a closed unbounded subset C of κ, then $\mathrm{Mh}(A) \cap C' = \mathrm{Mh}(A') \cap C'$, where C' is the set of accumulation points of C in κ.

We may therefore define a family $\langle a_\alpha \rangle_{\alpha < \kappa^+}$ in $\mathfrak{A}$ by setting

$$a_0 = \mathbf{1},$$

$$a_\alpha = \inf_{\beta < \alpha} M(a_\beta) \text{ if } 0 < \alpha < \kappa^+.$$

A simple induction shows that $M(a_\alpha) \subseteq a_\alpha \subseteq a_\beta$ whenever $\beta \le \alpha < \kappa^+$. **?** If no a_α is zero, consider

$$\mathcal{F} = \{A : A \subseteq \kappa,\ \exists\ \alpha < \kappa^+ \text{ such that } a_\alpha \subseteq A^\bullet \in \mathfrak{A}\}.$$

Then it is easy to check that $\mathcal{F}$ is a normal filter on κ which is closed under Mahlo's operation; which is supposed to be impossible. **X**

There must therefore be a $\gamma < \kappa^+$ such that $a_\gamma = \mathbf{0}$; we may suppose that $\gamma \ge \kappa$. Let $h : \gamma + 1 \to \kappa$ be any bijection, and $\preccurlyeq$ the well-ordering on κ corresponding to that of $\gamma + 1$, as transferred to κ by h. If we now define $\langle S_\zeta \rangle_{\zeta < \kappa}$ by the formula

$$S_\zeta = \{\xi : \xi < \kappa,\, \xi \in \ \mathrm{Mh}(S_\eta) \text{ whenever } \eta < \xi \text{ and } \eta \prec \zeta\},$$

we find that $a_\alpha = S^\bullet_{h(\alpha)}$ for every $\alpha \le \gamma$, so that $S^\bullet_{h(\gamma)} = \mathbf{0}$, i.e., $S_{h(\gamma)}$ is stationary. Thus $\preccurlyeq$, $\langle S_\zeta \rangle_{\zeta < \kappa}$ witness that the condition fails.

Remark This is due to Baumgartner Taylor & Wagon 77 (p. 212).

4D Definitions Let κ be a regular infinite cardinal.

(a) Write $\mathrm{Regr}(\kappa)$ for the set of regressive functions from κ to itself. For $F \subseteq \mathrm{Regr}(\kappa)$, write $\mathbf{U}_F$ for the set of uniform ultrafilters $\mathcal{F}$ on κ such that $\lim_{\alpha \to \mathcal{F}} f(\alpha)$ exists for every $f \in F$, that is to say, every $f \in F$ is constant on some member of $\mathcal{F}$.

(b) A set $A \subseteq \kappa$ is **Π^1_1-fully stationary** in κ if for every $F \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$ there is an $\mathcal{F} \in \mathbf{U}_F$ such that $A \in \mathcal{F}$; that is, there is a uniform ultrafilter $\mathcal{F}$ on κ such that $A \in \mathcal{F}$ and $\lim_{\alpha \to \mathcal{F}} f(\alpha)$ exists for every $f \in F$.

(c) From A4H below we see that κ is weakly Π^1_1-indescribable iff κ is Π^1_1-fully stationary in itself, and that in this case its Π^1_1-filter is precisely the set

$$\begin{aligned}\mathcal{W} &= \{W : W \subseteq \kappa,\, W \cap A \ne \emptyset \text{ for every } \Pi^1_1\text{-fully stationary } A \subseteq \kappa\}\\ &= \{W : \kappa \setminus W \text{ is not } \Pi^1_1\text{-fully stationary}\}\\ &= \{W : \exists\ F \in [\mathrm{Regr}(\kappa)]^{\le\kappa},\, \kappa \setminus W \notin \mathcal{F}\ \forall\ \mathcal{F} \in \mathbf{U}_F\}\\ &= \{W : \exists\ F \in [\mathrm{Regr}(\kappa)]^{\le\kappa},\, W \in \bigcap \mathbf{U}_F\}.\end{aligned}$$

Down to 4K below I will use this characterization as if it were the definition of 'weakly Π^1_1-indescribable' and 'Π^1_1-filter'. Of course this is an inefficient procedure in some ways, and many readers will prefer the original proofs of 4F, 4G and 4I as given in Lévy 71 and Baumgartner Taylor & Wagon 77, while 4K is covered by 4P. I take the trouble to spell out the combinatorial arguments because I believe that this alternative route can provide different insights into the relationships between the various concepts here.

4E Lemma Let κ be an infinite cardinal and $\langle C_\xi\rangle_{\xi<\kappa}$ a family of subsets of κ.

(a) If $C_\xi\cap\xi$ is cofinal with ξ for every $\xi<\kappa$, there is an $F\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that $\lim_{\xi\to\mathcal{F}}C_\xi=\{\eta:\{\xi:\eta\in C_\xi\}\in\mathcal{F}\}$ is cofinal with κ for every $\mathcal{F}\in\mathbf{U}_F$.

(b) If $C_\xi\cap\xi$ is relatively closed in ξ for every $\xi<\kappa$, there is an $F\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that $\lim_{\xi\to\mathcal{F}}C_\xi$ is closed in κ for every $\mathcal{F}\in\mathbf{U}_F$.

(c) If $\zeta<\kappa$ and $C_\xi\cap\zeta$ is cofinal with ζ for every $\xi<\kappa$, there is an $F\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that $\zeta\cap\lim_{\xi\to\mathcal{F}}C_\xi$ is cofinal with ζ for every $\mathcal{F}\in\mathbf{U}_F$.

proof (a) For α, $\xi<\kappa$ set

$$f_\alpha(\xi)=\min(C_\xi\cap\xi\setminus\alpha)\text{ if }C_\xi\cap\xi\setminus\alpha\ne\emptyset,$$
$$=0\text{ otherwise.}$$

Set $F=\{f_\alpha:\alpha<\kappa\}\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$. If $\mathcal{F}\in\mathbf{U}_F$, then

$$\gamma_\alpha=\lim_{\xi\to\mathcal{F}}f_\alpha(\xi)$$

is defined and belongs to $C\cup\{0\}$, where $C=\lim_{\xi\to\mathcal{F}}C_\xi$.

Because each $C_\xi\cap\xi$ is cofinal with ξ, we have $\alpha\le f_\alpha(\xi)$ whenever $\xi>\alpha$, so that $\alpha\le\gamma_\alpha$ for every $\alpha<\kappa$, and C must be unbounded in κ, as required.

(b) For α, $\xi<\kappa$ set

$$f_\alpha(\xi)=\sup(\alpha\cap C_\xi)\text{ if }\alpha<\xi,$$
$$=0\text{ otherwise.}$$

Set $F=\{f_\alpha:\alpha<\kappa\}\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$. Let $\mathcal{F}\in\mathbf{U}_F$ and consider $C=\lim_{\xi\to\mathcal{F}}C_\xi$. Suppose that $\gamma\in\kappa\setminus C$ is a non-zero limit ordinal. Set $\delta=\lim_{\xi\to\mathcal{F}}f_\gamma(\xi)<\kappa$. Since $f_\gamma(\xi)\in(C_\xi\cap\gamma)\cup\{0\}$ whenever $\gamma<\xi$, $\delta\in C\cup\{0\}$; but also $\delta\le\gamma$ and $\gamma\notin C$, so $\delta<\gamma$. Now if $\delta<\beta<\gamma$,

$$\{\xi:\beta\notin C_\xi\}\supseteq\{\xi:\xi>\gamma,\,f_\gamma(\xi)=\delta\}\in\mathcal{F},$$

so $\beta\notin C$. Thus C does not meet $]\delta,\gamma]$. Because γ was arbitrary, this shows that C is closed.

(c) Take F, $\mathcal{F}$ and C as in (a). This time we see that $\alpha\le\gamma_\alpha<\zeta$ whenever $\alpha<\zeta$, so that $C\cap\zeta$ must be cofinal with ζ. (If $\zeta=1$ then we have $0\in C_\xi$ for every ξ so that $0\in C$.)

4F Theorem Let κ be a weakly Π^1_1-indescribable cardinal, with Π^1_1-filter $\mathcal{W}$. Then $\mathcal{W}$ is normal and $\mathrm{Mh}(A)\in\mathcal{W}$ for every stationary $A\subseteq\kappa$.

proof For $F\subseteq\mathrm{Regr}(\kappa)$, write $\mathcal{W}_F$ for $\bigcap\mathbf{U}_F$, where $\mathbf{U}_F$ is defined as in 4Da. Then 4Dc shows that $\mathcal{W}=\bigcup\{\mathcal{W}_F:F\in[\mathrm{Regr}(\kappa)]^{\le\kappa}\}$.

(a) Suppose that $\langle W_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{W}$, and that W is its diagonal intersection. Then $W\in\mathcal{W}$. **P** For each $\xi<\kappa$ let $F(\xi)\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$ be such that $W_\xi\in\mathcal{W}_{F(\xi)}$. Define $g\in\mathrm{Regr}(\kappa)$ by setting

$$g(\xi)=\min\{\eta:\xi\notin W_\eta\}\text{ if }\xi\in\kappa\setminus W,$$
$$=0\text{ if }\xi\in W.$$

Set $G=\{g\}\cup\bigcup_{\xi<\kappa}F(\xi)\in[\mathrm{Regr}(\kappa)]^{\le\kappa}$, and consider $\mathcal{W}_G$. If $\mathcal{F}\in\mathbf{U}_G$, set $\gamma=\lim_{\alpha\to\mathcal{F}}g(\alpha)<\kappa$. Then $\mathcal{F}\in\mathbf{U}_{F(\gamma)}$, so $W_\gamma\in\mathcal{F}$ and

$$W\supseteq\{\xi:\xi\in W_\gamma,\,g(\xi)=\gamma\}\in\mathcal{F}.$$

As $\mathcal{F}$ is arbitrary, $W\in\mathcal{W}_G\subseteq\mathcal{W}$. **Q**

(b) Because $\kappa\setminus\zeta\in\mathcal{W}_\emptyset\subseteq\mathcal{W}$ for every $\zeta<\kappa$, it follows that κ is uncountable and regular and that $\mathcal{W}$ is normal.

(c) Now let $A\subseteq\kappa$ be stationary. Then $\mathrm{Mh}(A)\in\mathcal{W}$. **P** Define $\langle C_\xi\rangle_{\xi<\kappa}$ as follows. For $\xi\in\kappa\setminus\mathrm{Mh}(A)$, $\mathrm{cf}(\xi)>\omega$ let $C_\xi\subseteq\xi\setminus A$ be a relatively closed cofinal subset of ξ; for $\xi\in\mathrm{Mh}(A)$, set $C_\xi=\xi$; for $\xi<\kappa$, $\mathrm{cf}(\xi)=\omega$ let C_ξ be a relatively closed cofinal subset of ξ consisting solely of successor ordinals; for $\xi<\kappa$,

$\mathrm{cf}(\xi) = 1$ let C_ξ be the singleton set consisting of the predecessor of ξ; finally, set $C_0 = \emptyset$. By 4Ea-b, there is a set $F \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that $\lim_{\xi\to\mathcal{F}} C_\xi$ is a closed unbounded set in κ for every $\mathcal{F} \in \mathbf{U}_F$. Take any $\mathcal{F} \in \mathbf{U}_F$, and let C be $\lim_{\xi\to\mathcal{F}} C_\xi$, C' the set of non-zero limit ordinals in C. Thus C and C' are closed unbounded sets in κ, and must belong to $\mathcal{W}$, by (b). Also, C' meets A; take $\alpha \in C' \cap A$. Then

$$\mathrm{Mh}(A) \supseteq \{\xi : \xi > \alpha + 1,\, \alpha \in C_\xi\} \in \mathcal{F};$$

as $\mathcal{F}$ is arbitrary, $\mathrm{Mh}(A) \in \mathcal{W}_F \subseteq \mathcal{W}$, as required. **Q**

4G Corollary A weakly Π^1_1-indescribable cardinal is greatly Mahlo, and its Π^1_1-filter includes its greatly Mahlo filter.

proof I remarked in (b) of the proof of 4F that κ must be uncountable and regular; now we need only place 4F and 4Ac together.

4H Lemma Let κ be a weakly Π^1_1-indescribable cardinal. Suppose that for each $\xi < \kappa$ we have a well-ordering $\preccurlyeq_\xi$ of ξ, regarded as a subset of $\xi \times \xi$. Then there is a set $F \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that $\lim_{\xi\to\mathcal{F}} \preccurlyeq_\xi \subseteq \kappa \times \kappa$ is a well-ordering on κ for every $\mathcal{F} \in \mathbf{U}_F$.

proof Let C be the set of infinite cardinals less than κ, so that C is a closed unbounded set in κ (note that from 4G we know that κ is a limit cardinal). For $\zeta < \xi \in C$ let $g_{\xi\zeta} : \zeta \to \xi$ be such that $g_{\xi\zeta}(\eta) < g_{\xi\zeta}(\eta')$ whenever η, $\eta' < \zeta$ and $\eta \prec_\xi \eta'$; such a function exists because the order type of $(\zeta, \preccurlyeq_\xi\restriction \zeta)$ is less than $\#(\zeta)^+$ and therefore less than ξ. For ξ, η, $\zeta < \kappa$ set

$$\begin{aligned} f_{\zeta\eta}(\xi) &= g_{\xi\zeta}(\eta) \text{ if } \eta < \zeta < \xi \in C, \\ &= 0 \text{ otherwise;} \\ f(\xi) &= \sup(C \cap \xi) \text{ if } \xi \in \kappa \setminus C, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Write $F = \{f\} \cup \{f_{\zeta\eta} : \eta, \zeta < \kappa\} \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$.

Now suppose that $\mathcal{F} \in \mathbf{U}_F$. Then $C \in \mathcal{F}$ (because $f^{-1}[\{\gamma\}] \setminus C$ is bounded above in κ for every γ). Set $\preccurlyeq = \lim_{\xi\to\mathcal{F}} \preccurlyeq_\xi$. To see that $\preccurlyeq$ is a total ordering of κ we need note only that for distinct η, η', $\eta'' < \kappa$ the set

$$\begin{aligned} \{\xi :\ & \eta \preccurlyeq_\xi \eta, \\ & \eta \preccurlyeq_\xi \eta' \mathit{\ or\ } \eta' \preccurlyeq_\xi \eta, \\ & \eta \not\preccurlyeq_\xi \eta' \mathit{\ or\ } \eta' \not\preccurlyeq_\xi \eta, \\ & \eta \preccurlyeq_\xi \eta'' \mathit{\ or\ } \eta \not\preccurlyeq_\xi \eta' \mathit{\ or\ } \eta' \not\preccurlyeq_\xi \eta''\} \end{aligned}$$

belongs to $\mathcal{F}$. To see that $\preccurlyeq$ is a well-ordering of κ, take any $\zeta < \kappa$ and consider $h : \zeta \to \kappa$ defined by setting

$$h(\eta) = \lim_{\xi\to\mathcal{F}} f_{\zeta\eta}(\xi)$$

for $\eta < \zeta$; then

$$h(\eta) = \lim_{\xi\to\mathcal{F}} g_{\xi\zeta}(\eta)$$

because $\{\xi : \zeta < \xi \in C\} \in \mathcal{F}$. But now, if η, $\eta' < \zeta$ and $\eta \prec \eta'$, the set

$$\begin{aligned} &\{\xi : \zeta < \xi \in C,\, h(\eta) = g_{\xi\zeta}(\eta),\, h(\eta') = g_{\xi\zeta}(\eta'),\, \eta \preccurlyeq_\xi \eta'\} \\ &\quad = \{\xi : h(\eta) = g_{\xi\zeta}(\eta) < g_{\xi\zeta}(\eta) = h(\eta')\} \end{aligned}$$

belongs to $\mathcal{F}$, so $h(\eta) < h(\eta')$. This means that $\preccurlyeq\restriction \zeta$ is a well-ordering for every $\zeta < \kappa$. Because $\mathrm{cf}(\kappa) > \omega$, it follows that $\preccurlyeq$ is a well-ordering of κ.

4I Theorem If κ is weakly Π^1_1-indescribable, it is greatly Mahlo and the set of greatly Mahlo cardinals below κ belongs to the Π^1_1-filter $\mathcal{W}$ of κ.

proof I have already remarked (4G) that κ is greatly Mahlo; moreover, the set of weakly inaccessible cardinals less than κ belongs to its greatly Mahlo filter (4B) and therefore to $\mathcal{W}$ (4G).

? Suppose, if possible, that the set B of greatly Mahlo cardinals less than κ does not belong to $\mathcal{W}$. Let D be the set of weakly inaccessible cardinals belonging to $\kappa \setminus B$, so that D is Π^1_1-fully stationary in κ.

By Lemma 4C, we may choose for each $\delta \in D$ a well-ordering $\preccurlyeq_\delta$ of δ and a family $\langle S_{\delta\zeta}\rangle_{\zeta<\delta}$ of subsets of δ such that

$$S_{\delta\zeta} \supseteq \{\xi : \xi < \alpha,\, \xi \in \mathrm{Mh}(S_{\delta\eta}) \text{ whenever } \eta < \xi \text{ and } \eta \prec_\delta \zeta\}$$

for every $\zeta < \delta$, but not every $S_{\delta\zeta}$ is stationary in δ. Take a closed unbounded set $C_\delta \subseteq \delta$ and an $h(\delta) < \delta$ such that $S_{\delta,h(\delta)} \cap C_\delta = \emptyset$. Next, for $\zeta < \delta \in D$ and $\xi \in \delta \setminus S_{\delta\zeta}$ choose $g(\delta,\zeta,\xi) < \xi$ such that $g(\delta,\zeta,\xi) \prec_\delta \zeta$ and $\xi \notin \mathrm{Mh}(S_{\delta,g(\delta,\zeta,\xi)})$; if $\mathrm{cf}(\xi) > \omega$, choose a relatively closed unbounded set $C_{\delta\zeta\xi} \subseteq \xi$ such that $C_{\delta\zeta\xi} \cap S_{\delta,g(\delta,\zeta,\xi)} = \emptyset$.

Fill in these definitions by setting

$$C_\alpha = \alpha,\ \preccurlyeq_\alpha = \le\restriction \alpha,\ h(\alpha) = 0,\ g(\alpha,\zeta,\xi) = 0,\ C_{\alpha\zeta\xi} = \xi,\ S_{\alpha\zeta} = \emptyset$$

if α, ζ, $\xi < \kappa$ and we do not have $\zeta < \alpha \in D$, $\xi \in \alpha \setminus S_{\alpha\zeta}$, $\mathrm{cf}(\xi) > \omega$.

By 4E and 4H there is an $F \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$ such that whenever $\mathcal{F} \in \mathbf{U}_F$

$\lim_{\alpha\to\mathcal{F}} C_\alpha$ is a closed unbounded set in κ,
$\lim_{\alpha\to\mathcal{F}} \preccurlyeq_\alpha$ is a well-ordering of κ,
$\lim_{\alpha\to\mathcal{F}} C_{\alpha\zeta\xi}$ is cofinal with ξ,
$\lim_{\alpha\to\mathcal{F}} (C_{\alpha\zeta\xi} \cup \{\xi\})$ is closed in κ

for all ζ, $\xi < \kappa$. We may suppose also that $h \in F$ and that $f_{\zeta\xi} \in F$ for all ζ, $\xi < \kappa$, where $f_{\zeta\xi}(\alpha) = g(\alpha,\zeta,\xi)$ for α, ζ, $\xi < \kappa$.

Because D is supposed to be Π^1_1-fully stationary, there is an $\mathcal{F} \in \mathbf{U}_F$ such that $D \in \mathcal{F}$. Set

$$C = \lim_{\alpha\to\mathcal{F}} C_\alpha,\ \preccurlyeq = \lim_{\alpha\to\mathcal{F}} \preccurlyeq_\alpha,\quad \epsilon = \lim_{\alpha\to\mathcal{F}} h(\alpha),$$

$$C_{\zeta\xi} = \lim_{\alpha\to\mathcal{F}} C_{\alpha\zeta\xi},\quad \gamma(\zeta,\xi) = \lim_{\alpha\to\mathcal{F}} f_{\zeta\xi}(\alpha),$$

$$S_\zeta = \lim_{\alpha\to\mathcal{F}} S_{\alpha\zeta}$$

for ζ, $\xi < \kappa$. Then $\preccurlyeq$ is a well-ordering of κ and $\langle S_\zeta\rangle_{\zeta<\kappa}$ is a family of subsets of κ.

Suppose that $\zeta < \kappa$, that $\xi \in \kappa \setminus S_\zeta$ and that $\mathrm{cf}(\xi) > \omega$. Set $\eta = \gamma(\zeta,\xi)$. Then

$$E = \{\delta : \delta \in D,\ \zeta < \delta,\ \xi \in \delta \setminus S_{\delta\zeta},\ g(\delta,\zeta,\xi) = \eta\}$$

belongs to $\mathcal{F}$. If $\delta \in E$ then $C_{\delta\zeta\xi} \cap S_{\delta\eta} = \emptyset$; it follows that $C_{\zeta\xi} \cap S_\eta = \emptyset$. So $\xi \notin \mathrm{Mh}(S_\eta)$. Also $\eta < \xi$ and $\eta \prec \zeta$ because $g(\delta,\zeta,\xi) < \xi$ and $g(\delta,\zeta,\xi) \prec_\delta \zeta$ for every $\delta \in E$. But this means that

$$S_\zeta \supseteq \{\xi : \xi < \kappa,\ \xi \in \mathrm{Mh}(S_\eta) \text{ whenever } \eta < \xi,\ \eta \prec \zeta\},$$

for every $\zeta < \kappa$.

On the other hand, examine C and S_ϵ. C is a closed unbounded set in κ. Also

$$\{\alpha : h(\alpha) = \epsilon\} \in \mathcal{F},$$

so

$$S_\epsilon = \lim_{\alpha\to\mathcal{F}} S_{\alpha\epsilon} = \lim_{\alpha\to\mathcal{F}} S_{\alpha,h(\alpha)},$$

and

$$C \cap S_\epsilon = \lim_{\alpha\to\mathcal{F}} C_\alpha \cap S_{\alpha,h(\alpha)} = \emptyset.$$

Thus S_ϵ is not stationary. From 4C we conclude that κ cannot be greatly Mahlo. But we know very well that it is. **X**

So $B \in \mathcal{W}$, as claimed.

Remark This theorem is given in Baumgartner Taylor & Wagon 77; the treatment here is taken from Fremlin & Kunen n87.

Version of 10.12.91

4J Lemma Let κ be a real-valued-measurable cardinal, with normal witnessing probability ν. Suppose that for each $\xi < \kappa$ we are given a cofinal relatively closed subset C_ξ of ξ. Then

$$C = \{\alpha : \alpha < \kappa,\, \nu\{\xi : \alpha \in C_\xi\} = 1\}$$

is a closed unbounded set in κ.

proof For α, $\xi < \kappa$ set

$$f_\alpha(\xi) = \min(C_\xi \setminus \alpha) \text{ if } \alpha < \xi,$$
$$= 0 \text{ otherwise.}$$

Then f_α is regressive, so there is a $\zeta_\alpha < \kappa$ such that $\nu f_\alpha^{-1}[\zeta_\alpha] = 1$; that is,

$$\nu\{\xi : C_\xi \cap \zeta_\alpha \setminus \alpha \neq \emptyset\} = 1.$$

Set $C' = \{\beta : \beta < \kappa,\, \zeta_\alpha < \beta\ \forall\ \alpha < \beta\}$. Then C' is a closed unbounded set in κ. But if $\beta \in C'$, then for every $\alpha < \beta$ the set

$$\{\xi : C_\xi \cap \beta \setminus \alpha \neq \emptyset\}$$

has measure 1, so

$$E = \{\xi : \beta < \xi < \kappa,\, C_\xi \cap \beta \setminus \alpha \neq \emptyset\ \forall\ \alpha < \beta\}$$

has measure 1, because ν is κ-additive. But of course $\beta \in C_\xi$ for every $\xi \in E$, because each C_ξ is closed in ξ. Accordingly $\beta \in C$ and we have $C' \subseteq C$.

This means that C is unbounded. But if β is any cluster point of C in κ, the same arguments show that $\beta \in C$, so that C is closed.

4K Theorem Let κ be a real-valued-measurable cardinal, with rvm filter $\mathcal{W}$, rvm ideal $\mathcal{J}$. Then
(a) κ is greatly Mahlo and its greatly Mahlo filter is included in $\mathcal{W}$;
(b) the set of weakly Π^1_1-indescribable cardinals below κ belongs to $\mathcal{W}$;
(c) if $Z \subseteq \kappa$ and $Z \notin \mathcal{J}$, then

$$\{\lambda : \lambda < \kappa \text{ is a cardinal and } Z \cap \lambda \text{ is } \Pi^1_1\text{-fully stationary in } \lambda\} \in \mathcal{W}.$$

proof (a) Let ν be a normal witnessing probability on κ, and set

$$\mathcal{F}_\nu = \{A : A \subseteq \kappa,\, \nu A = 1\};$$

$\mathcal{F}_\nu$ is a normal filter on κ. Let D be the set of ordinals less than κ with uncountable cofinality.

(i) We have $D \in \mathcal{F}_\nu$. **P** For $\xi \in \kappa \setminus (D \cup \{0\})$, let $\langle f_n(\xi)\rangle_{n\in\mathbb{N}}$ run over a cofinal subset of ξ; for $\xi \in D \cup \{0\}$, set $f_n(\xi) = 0$ for every $n \in \mathbb{N}$. Because every f_ξ is regressive and $\mathcal{F}_\nu$ is normal and ω_1-complete, there is a $\zeta < \kappa$ such that $F = \bigcap_{n\in\mathbb{N}} f_n^{-1}[\zeta] \in \mathcal{F}_\nu$; but now $D \supseteq F \setminus (\zeta + 1) \in \mathcal{F}_\nu$. **Q**

(ii) Next, $\mathrm{Mh}(A) \in \mathcal{F}_\nu$ for every stationary $A \subseteq \kappa$. **P** For $\xi \in D \setminus \mathrm{Mh}(A)$, let C_ξ be a relatively closed and unbounded subset of ξ such that $C_\xi \cap A = \emptyset$. For $\xi \in D \cap \mathrm{Mh}(A)$, take $C_\xi = \xi$. 4J tells us that

$$C = \{\alpha : \{\xi : \alpha \in C_\xi\} \in \mathcal{F}_\nu\}$$

is a closed unbounded set in κ. So there is an $\alpha \in A \cap C$. But now $F = \{\xi : \alpha \in C_\xi\} \in \mathcal{F}_\nu$ and $\mathrm{Mh}(A) \supseteq F \cap D \in \mathcal{F}_\nu$. **Q**

(iii) Thus $\mathcal{F}_\nu$ is closed under Mh. Because ν is arbitrary, $\mathcal{W}$ is closed under Mh, and κ is greatly Mahlo, with its greatly Mahlo filter included in $\mathcal{W}$.

(b)-(c) Of course (b) is a consequence of (c) (taking $Z = \kappa$) so I prove (c). I begin with the harder case, in which κ is atomlessly-measurable.

(i) Take a normal witnessing probability ν_1 on κ such that $\nu_1 Z = 1$, and let H be the set of $\xi < \kappa$ such that either ξ is not a weakly Π^1_1-indescribable cardinal or ξ is weakly Π^1_1-indescribable but $Z \cap \xi$ is not Π^1_1-fully stationary in ξ. **?** Suppose, if possible, that $H \notin \mathcal{J}$. Let ν_2 be a normal witnessing probability on κ such that $\nu_2 H = 1$. Let H_0 be the set of regular infinite cardinals belonging to H; then $\nu_2 H_0 = 1$ (using part (a)). For each $\xi \in H_0$, let $\langle f_{\xi\eta}\rangle_{\eta<\xi}$ be a family of regressive functions from ξ to itself such that there is no uniform ultrafilter $\mathcal{F}$ on ξ containing $Z \cap \xi$ for which $\lim_{\alpha\to\mathcal{F}} f_{\xi\eta}(\alpha)$ exists for every $\eta < \xi$. For $\eta < \xi \in \kappa \setminus H_0$, set $f_{\xi\eta}(\alpha) = 0$ for every $\alpha < \xi$.

(ii) Let ν be the probability on $\kappa\times\kappa$ defined by the formula

$$\nu C=\int\nu_2\{\xi:(\alpha,\xi)\in C\}\,\nu_1(d\alpha)\ \ \forall\ C\subseteq\kappa\times\kappa.$$

Let $\mathfrak{A}(\nu)=\mathcal{P}(\kappa\times\kappa)/\mathcal{N}_\nu$ be the corresponding measure algebra. For η, $\beta<\kappa$ write

$$C_{\eta\beta}=\{(\alpha,\xi):\max(\eta,\alpha)<\xi<\kappa,\ f_{\xi\eta}(\alpha)=\beta\}.$$

If $\eta<\kappa$, then $\langle C_{\eta\beta}\rangle_{\beta<\kappa}$ is disjoint, so

$$D_\eta=\{\beta:\beta<\kappa,\ \nu C_{\eta\beta}>0\}$$

is countable. But also

$$\nu(\bigcup_{\beta\in D_\eta}C_{\eta\beta})=1.$$

P For fixed η, α the map $\xi\mapsto f_{\xi\eta}(\alpha):\kappa\to\alpha$ is regressive, therefore ν_2-essentially countably valued (1Hd), therefore ν_2-essentially bounded below α whenever $\mathrm{cf}(\alpha)>\omega$, that is, for ν_1-almost all α, by (a-i) above. Because ν_1 is normal, there is a $\zeta<\kappa$ such that, for ν_1-almost all α, $f_{\xi\eta}(\alpha)<\zeta$ for ν_2-almost all ξ; that is, $\nu(\bigcup_{\zeta\le\beta<\kappa}C_{\eta\beta})=0$. Now the result follows, with $D_\eta\subseteq\zeta$, because ν is κ-additive. **Q**

(iii) Let Σ_0 be the subalgebra of $\mathcal{P}(\kappa\times\kappa)$ generated by

$$\{C_{\eta\beta}:\eta,\ \beta<\kappa\}\cup\{V_\zeta:\zeta<\kappa\},$$

where $V_\zeta=Z\times(\kappa\setminus\zeta)$; note that $\nu V_\zeta=1$ for each $\zeta<\kappa$. Let Σ_1 be the subalgebra of $\mathcal{P}(\kappa\times\kappa)$ generated by

$$\Sigma_0\cup\{\pi_2^{-1}[\pi_2[E]]:E\in\Sigma_0\},$$

writing $\pi_2(\alpha,\xi)=\xi$. Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}(\nu)$ generated by $\{E^\bullet:E\in\Sigma_1\}$. Then $\tau(\mathfrak{B})\le\kappa$. Let $\mathfrak{B}_0$ be

$$\{b:b\in\mathfrak{B},\ \exists\ F\subseteq\kappa\text{ such that }b=(\kappa\times F)^\bullet\}.$$

Then $\mathfrak{B}_0$ is an order-closed subalgebra of $\mathfrak{B}$. Because π_2 is inverse-measure-preserving for ν and ν_2, we have a measure-preserving Boolean homomorphism $\phi_0:\mathfrak{B}_0\to\mathfrak{A}(\nu_2)=\mathcal{P}\kappa/\mathcal{N}_{\nu_2}$ defined by writing

$$\phi_0((\kappa\times F)^\bullet)=F^\bullet$$

whenever $F\subseteq\kappa$ and $(\kappa\times F)^\bullet\in\mathfrak{B}$.

By the Gitik-Shelah theorem (3F), every non-zero principal ideal of $\mathfrak{A}(\nu_2)$ has τ-weight greater than κ. So there is a measure-preserving homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}(\nu_2)$ extending ϕ_0 (A2Ib). For each $\eta<\kappa$ choose a function $h_\eta:\kappa\to D_\eta$ such that

$$\{\xi:h_\eta(\xi)=\beta\}^\bullet=\phi(C^\bullet_{\eta\beta})\ \ \forall\ \beta\in D_\eta;$$

this is possible because $\langle C^\bullet_{\eta\beta}\rangle_{\beta\in D_\eta}$ and $\langle\phi(C^\bullet_{\eta\beta})\rangle_{\beta\in D_\eta}$ are partitions of unity in $\mathfrak{B}$, $\mathfrak{A}(\nu_2)$ respectively. Now if $\xi\in H_0$, there can be no uniform ultrafilter on ξ containing $Z\cap\xi$ and all the sets $\{\alpha:\alpha<\xi,\ f_{\xi\eta}(\alpha)=h_\eta(\xi)\}$ as η runs over ξ. So there must be a $\zeta_\xi<\xi$ and a finite $I_\xi\subseteq\zeta_\xi$ such that

$$\forall\ \alpha\in\xi\cap Z\setminus\zeta_\xi\ \ \exists\ \eta\in I_\xi\text{ such that }f_{\xi\eta}(\alpha)\ne h_\eta(\xi).$$

(iv) Because ν_2 is normal, there are $\zeta<\kappa$, $I\in[\kappa]^{<\omega}$ such that $\nu_2(H_1)>0$, where

$$H_1=\{\xi:\xi\in H_0,\ \zeta_\xi=\zeta,\ I_\xi=I\}.$$

Now $h_\eta(\xi)\in D_\eta$ for every $\xi<\kappa$, $\eta\in I$, and every D_η is countable, so there is a $g:I\to\kappa$ such that $\nu_2(H_2)>0$, where

$$H_2=\{\xi:\xi\in H_1,\ h_\eta(\xi)=g(\eta)\ \ \forall\ \eta\in I\}.$$

But now observe that if $\xi\in H_2$ and $\alpha\in\xi\cap Z\setminus\zeta$ then there is an $\eta\in I$ such that $f_{\xi\eta}(\alpha)\ne g(\eta)$, i.e., $(\alpha,\xi)\notin E^*$, where

$$E^*=\{(\alpha,\xi):\zeta\le\alpha<\xi,\ \alpha\in Z,\ f_{\xi\eta}(\alpha)=g(\eta)\ \ \forall\ \eta\in I\}\in\Sigma_0.$$

Thus $H_2\cap\pi_2[E^*]=\emptyset$. However,

$$
\begin{aligned}
H_2^\bullet &= H_1^\bullet \cap \inf_{\eta\in I}\phi(C^\bullet_{\eta,g(\eta)})\\
&\subseteq \phi(\bigcap_{\eta\in I} C_{\eta,g(\eta)})^\bullet\\
&= \phi(\{(\xi,\alpha) : f_{\xi\eta}(\alpha) = g(\eta)\ \ \forall\ \ \eta\in I\}^\bullet)\\
&= \phi(E^*)^\bullet\\
&\subseteq \phi(\pi_2^{-1}[\pi_2[E^*]]^\bullet)\\
&= \phi_0(\pi_2^{-1}[\pi_2[E^*]]^\bullet)\\
&= \pi_2[E^*]^\bullet
\end{aligned}
$$

because $E^* \in \Sigma_0$ so $\pi_2^{-1}[\pi_2[E^*]] \in \Sigma_1$ and $\pi_2^{-1}[\pi_2[E^*]]^\bullet \in \mathfrak{B}_0$.

Of course this is impossible. **X**

This proves (b)-(c) in the case of atomlessly-measurable κ.

(v) Now let us turn to the case of a two-valued-measurable cardinal κ. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be normal ultrafilters on κ, with $Z \in \mathcal{F}_1$, and ν_1, ν_2 the associated $\{0,1\}$-valued probabilities. Follow the ideas of (b), but noting that there are dramatic simplifications; for instance, ν is also two-valued, and each D_η is a singleton. All the discussion of $\mathfrak{B}$, $\mathfrak{B}_0$ collapses because $\mathfrak{A}(\nu)$ and $\mathfrak{A}(\nu_2)$ are both of the form $\{\mathbf{0},\mathbf{1}\}$. So ϕ_0 and ϕ are both the trivial homomorphism, and there is no call for any measure theory at this point; the functions h_η are all constant. The last string of inequalities also boils down to saying that $\nu_2 H_2 = \nu E^* = 1$ and that this is impossible.

This completes the proof.

4L Remarks (a) This (in the atomlessly-measurable case) is due to Kunen (see 4P, A4L below).

(b) Note that a real-valued-measurable cardinal κ need not itself be weakly Π^1_1-indescribable. For (if there can be real-valued-measurable cardinals at all) $\mathfrak{c}$ can be real-valued-measurable (2Ed). But $\mathfrak{c}$ is never weakly Π^1_1-indescribable (A4Db).

(c) It is worth pausing over 4Ka, as some elementary corollaries will be used repeatedly below. For instance, if ν is a normal witnessing probability on κ, and $\theta < \kappa$, then $\nu\{\xi : \xi < \kappa,\ \mathrm{cf}(\xi) \geq \theta\} = 1$. (This is because the set W of weakly inaccessible cardinals below κ belongs to the greatly Mahlo filter of κ (4B), so $\nu W = 1$ (4Ka) and $\nu(W \setminus \theta) = 1$, while $\mathrm{cf}(\xi) \geq \theta$ for every $\xi \in W \setminus \theta$.) Similarly trivial applications occur frequently.

4M Lemma Let κ be a real-valued-measurable cardinal with normal witnessing probability ν. Let $\mathfrak{A}$ be the measure algebra $\mathcal{P}\kappa/\mathcal{N}_\nu$ and $\mathbb{P}$ the p.o.set $\mathfrak{A} \setminus \{\mathbf{0}\}$; for $a \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$, write $a^\bullet$ for the corresponding element of $\mathbb{P}$.

(a) Suppose that $\langle\xi_\alpha\rangle_{\alpha<\kappa}$ is a family of ordinals such that $\xi_\alpha < \alpha$ for ν-almost every $\alpha < \kappa$. Then we have a $\mathbb{P}$-name $\dot\xi$ for a member of κ defined by saying that

$$a^\bullet \Vdash_{\mathbb{P}} \dot\xi = \check\zeta \text{ iff } \xi_\alpha = \zeta \text{ for } \nu\text{-almost every } \alpha \in a$$

whenever $\zeta < \kappa$, $a \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$.

(b) Suppose that $n \in \mathbb{N}$ and that for ν-almost every $\alpha < \kappa$ we are given an n-place relation C_α on α. Then we have a $\mathbb{P}$-name $\dot C$ for an n-place relation on κ defined by saying that

$$a^\bullet \Vdash_{\mathbb{P}} \dot C(\check\zeta_1, \dots, \check\zeta_n) \text{ iff } C_\alpha(\zeta_1, \dots, \zeta_n) \text{ for } \nu\text{-almost every } \alpha \in a$$

whenever $\zeta_1, \dots, \zeta_n < \kappa$ and $a \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$.

(c) Conversely, given any $\mathbb{P}$-names $\dot\xi$ and $\dot C$ for a member of κ and a relation on κ, we can find corresponding families $\langle\xi_\alpha\rangle_{\alpha<\kappa}$, $\langle C_\alpha\rangle_{\alpha<\kappa}$ for which the formulae of (a) and (b) will be valid.

(d) Now suppose that we have a formula ϕ of the second-order language $\mathcal{L}$ of §A4, and families $\langle C_{\alpha 1}\rangle_{\alpha<\kappa}$, $\dots$, $\langle C_{\alpha k}\rangle_{\alpha<\kappa}$, $\langle\xi_{\alpha 1}\rangle_{\alpha<\kappa}, \dots , \langle\xi_{\alpha m}\rangle_{\alpha<\kappa}$ of relations and ordinals matching the free variables of ϕ; suppose that for each j we have $\xi_{\alpha j} < \alpha$ for ν-almost all α. Let $\dot C_1, \dots , \dot C_k, \dot\xi_1, \dots , \dot\xi_m$ be the corresponding $\mathbb{P}$-names. Then for any $a \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$, $\beta \leq \kappa$,

$$a^\bullet \Vdash_{\mathbb{P}} (\check\beta; \dot C_1, \dots , \dot C_k; \dot\xi_1, \dots , \dot\xi_m) \vDash \phi$$

if and only if

$$(\min(\alpha,\beta);C_{\alpha 1},\ldots,C_{\alpha k};\xi_{\alpha 1},\ldots,\xi_{\alpha m})\vDash\phi \text{ for } \nu\text{-almost every } \alpha<\kappa.$$

proof (a) The point is that $\alpha\mapsto\xi_\alpha$ is essentially regressive and ν is normal. So given $a\in\mathcal{P}\kappa\setminus\mathcal{N}_\nu$ we shall be able to find $\zeta<\kappa$ such that $b=\{\alpha:\alpha\in a,\ \xi_\alpha=\zeta\}\notin\mathcal{N}_\nu$; and now our formula tells us that $b^\bullet\Vdash_{\mathbb{P}}\dot\xi=\check\zeta$. This means that we have a name $\dot\xi$ such that

$$\Vdash_{\mathbb{P}}\exists\ \zeta\in\check\kappa,\ \dot\xi=\zeta,$$

so that $\dot\xi$ is indeed a name for a member of κ.

(b) is elementary.

(c) Because $\mathbb{P}$ is ccc and $\mathcal{N}_\nu$ is a σ-ideal, every maximal antichain in $\mathbb{P}$ corresponds to a partition of κ into non-negligible sets; consequently every $\mathbb{P}$-name $\dot\xi$ for a member of κ corresponds to a family $\langle\xi_\alpha\rangle_{\alpha<\kappa}$ which takes only countably many values, so that $\xi_\alpha<\alpha$ for almost every α.

As in (a)-(b), the corresponding result for relations is really simpler than the result for points. Given the name $\dot C$ for an n-place relation, then for each $\zeta_1,\ldots,\zeta_n<\kappa$ choose $a(\zeta_1,\ldots,\zeta_n)\subseteq\kappa$ such that

$$a(\zeta_1,\ldots,\zeta_n)^\bullet\Vdash_{\mathbb{P}}\dot C(\check\zeta_1,\ldots,\check\zeta_n),$$

$$(\kappa\setminus a(\zeta_1,\ldots,\zeta_n))^\bullet\Vdash_{\mathbb{P}}\neg\dot C(\check\zeta_1,\ldots,\check\zeta_n);$$

and for $\alpha<\kappa$ write

$$C_\alpha(\zeta_1,\ldots,\zeta_n) \text{ for `}\alpha\in a(\zeta_1,\ldots,\zeta_n)\text{'}.$$

It is easy to check that $\langle C_\alpha\rangle_{\alpha<\kappa}$ now represents $\dot C$.

(d) Induce on the length of ϕ.

(i) If ϕ is of the form $R_1(x_1,\ldots,x_n)$ and we are given C_α, $\xi_{\alpha j}$ for $\alpha<\kappa$, $j\le n$ then note first that

$$a^\bullet\Vdash_{\mathbb{P}}\dot\xi_j<\check\beta$$

if and only if

$$\xi_{\alpha j}<\min(\alpha,\beta) \text{ for almost every } \alpha\in a.$$

Next,

$$\begin{aligned}
&a^\bullet\Vdash_{\mathbb{P}}\dot C(\dot\xi_1,\ldots,\dot\xi_n)\\
&\qquad\iff \text{whenever } b\subseteq a,\ \zeta_1,\ldots,\zeta_n<\kappa,\ b^\bullet\Vdash_{\mathbb{P}}\dot\xi_i=\check\zeta_i\ \forall\ i\le n\\
&\qquad\qquad\qquad \text{then } b^\bullet\Vdash_{\mathbb{P}}\dot C(\check\zeta_1,\ldots,\check\zeta_n)\\
&\qquad\iff \text{whenever } b\subseteq a,\ \zeta_1,\ldots,\zeta_n<\kappa,\xi_{\alpha i}=\zeta_i\ \forall\ \alpha\in b,\ i\le n\\
&\qquad\qquad\qquad \text{then } C_\alpha(\zeta_1,\ldots,\zeta_n) \text{ for almost every } \alpha\in b\\
&\qquad\iff C_\alpha(\xi_{\alpha 1},\ldots,\xi_{\alpha n}) \text{ for almost every } \alpha\in a.
\end{aligned}$$

(ii) If ϕ is of the form $\psi\wedge\chi$ or $\psi\vee\chi$ the inductive step to ϕ is elementary. If ϕ is of the form $\neg\psi$ we need the translation

$$\begin{aligned}
&a^\bullet\Vdash_{\mathbb{P}}(\check\beta;\dot C_1,\ldots,\dot\xi_m)\vDash\phi\\
&\qquad\iff b^\bullet\not\Vdash_{\mathbb{P}}(\check\beta;\dot C_1,\ldots,\dot\xi_m)\vDash\psi \text{ for every } b\in\mathcal{P}a\setminus\mathcal{N}_\nu\\
&\qquad\iff \nu\{\alpha:\alpha\in a,\ (\min(\alpha,\beta);C_{\alpha 1},\ldots,\xi_{\alpha m})\vDash\psi\}=0\\
&\qquad\iff (\min(\alpha,\beta);C_{\alpha 1},\ldots,\xi_{\alpha m})\vDash\phi \text{ for almost all } \alpha\in a.
\end{aligned}$$

(iii) If ϕ is of the form $\exists S\psi$ and

$$a^\bullet\Vdash_{\mathbb{P}}(\check\beta;\dot C_1,\ldots,\dot\xi_m)\vDash\phi,$$

then there must be a $\mathbb{P}$-name $\dot{C}$ such that

$$a^\bullet \Vdash_{\mathbb{P}} (\check{\beta}; \dot{C}_1, \dots, \dot{C}, \dots, \dot{\xi}_m) \vDash \psi.$$

Let $\langle C_\alpha \rangle_{\alpha<\kappa}$ be a corresponding family of relations. Then by the inductive hypothesis

$$(\min(\alpha, \beta); C_{\alpha 1}, \dots, C_\alpha, \dots, \xi_{\alpha m}) \vDash \psi$$

for almost every $\alpha \in a$, so that

$$(\min(\alpha, \beta); C_{\alpha 1}, \dots, \xi_{\alpha m}) \vDash \phi$$

for almost every $\alpha \in a$.

(iv) If ϕ is of the form $\exists S\psi$ and $a \in \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ is such that

$$(\min(\alpha, \beta); C_{\alpha 1}, \dots, \xi_{\alpha m}) \vDash \phi$$

for almost every $\alpha \in a$, then we can find relations C_α such that

$$(\min(\alpha, \beta); C_{\alpha 1}, \dots, C_\alpha, \dots, \xi_{\alpha m}) \vDash \psi$$

for almost every $\alpha \in a$. By the inductive hypothesis, taking $\dot{C}$ from $\langle C_\alpha \rangle_{\alpha<\kappa}$ as usual, we have

$$a^\bullet \Vdash_{\mathbb{P}} (\check{\beta}; \dot{C}_1, \dots, \dot{C}, \dots, \dot{\xi}_m) \vDash \psi, \; a^\bullet \Vdash_{\mathbb{P}} (\check{\beta}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi.$$

(v) (iii)-(iv) together deal with the inductive step to $\exists S\psi$. If ϕ is of the form $\forall S\psi$ it is logically equivalent to $\neg\exists S\neg\psi$ so that (ii)-(iv) cover it. Finally, if ϕ is of the form $\exists y\psi$ or $\forall y\psi$ the same ideas suffice.

4N Proposition Let κ be an atomlessly-measurable cardinal, with rvm filter $\mathcal{W}$ and rvm ideal $\mathcal{J}$, and $\lambda > \kappa$ another cardinal. Let $\mathfrak{A}_\lambda$ be the measure algebra of $\{0,1\}^\lambda$ and let $\mathbb{P}_\lambda$ be the p.o.set $\mathfrak{A}_\lambda \setminus \{\mathbf{0}\}$. Let ϕ be a formula of the second-order language $\mathcal{L}$ of §A4 and $C_1, \dots, C_k$ relations on κ, $\xi_1, \dots, \xi_m$ ordinals less than κ. Let $\beta \leq \kappa$. Then the following are equivalent:

(i) $\Vdash_{\mathbb{P}_\lambda} (\check{\beta}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi$,

(ii) $\{\alpha : \alpha < \kappa, (\min(\alpha, \beta); C_1, \dots, \xi_m) \vDash \phi\} \in \mathcal{W}$;

(iii) $\{\alpha : \alpha < \kappa, (\min(\alpha, \beta); C_1, \dots, \xi_m) \vDash \phi\} \notin \mathcal{J}$.

proof Set $A = \{\alpha : (\min(\alpha, \beta); C_1, \dots) \vDash \phi\}$.

(i)⇒(ii) Assume (i). Let ν be a Maharam homogeneous normal witnessing probability on κ; let θ be the Maharam type of ν. We know by the Gitik-Shelah theorem (3F) that $\theta > \kappa$. Set $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{N}_\nu$, $\mathbb{P} = \mathfrak{A} \setminus \{\mathbf{0}\}$; then $\mathfrak{A}$ is isomorphic to the measure algebra of $\{0,1\}^\theta$, so Theorem A4I tells us that

$$\Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots) \vDash \phi.$$

Now observe that each $\mathbb{P}$-name $\check{C}_i$, $\check{\xi}_j$ can be identified with the $\mathbb{P}$-names $\dot{C}_i$, $\dot{\xi}_j$ derived from families $\langle C_{\alpha i} \rangle_{\alpha<\kappa}$, $\langle \xi_{\alpha j} \rangle_{\alpha<\kappa}$, setting $C_{\alpha i} = C_i$, $\xi_{\alpha j} = \xi_j$ for every α. So Lemma 4Md tells us that

$$(\min(\alpha, \beta) : C_1, \dots) \vDash \phi$$

for ν-almost every $\alpha < \kappa$, that is, that $\nu A = 1$. As ν is arbitrary, $A \in \mathcal{W}$.

(ii)⇒(iii) is trivial.

(iii)⇒(i) I reverse the argument of (i) ⇒(ii). Let ν be a Maharam homogeneous normal witnessing probability on κ such that $\nu A = 1$; let $\theta > \kappa$ be the Maharam type of ν, and $\mathbb{P} = (\mathcal{P}\kappa/\mathcal{N}_\nu) \setminus \{\mathbf{0}\}$ the corresponding p.o.set. This time 4Md tells us that

$$\Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots) \vDash \phi$$

and A4I that

$$\Vdash_{\mathbb{P}_\lambda} (\check{\beta}; \check{C}_1, \dots) \vDash \phi,$$

as required.

4O Corollary (a) Let κ be an atomlessly-measurable cardinal and I a set of cardinal less than κ. Let ϕ be a formula of the second-order language $\mathcal{L}$, and $C_1, \dots, C_k$ relations on I, $\xi_1, \dots, \xi_m$ members of I. Then for any random real p.o.set $\mathbb{P}$,

$$(I; C_1, \dots, \xi_m) \vDash \phi \iff \Vdash_{\mathbb{P}} (\check{I}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi.$$

(b) Let $C_1, \dots, C_n$ be relations and $\xi_1, \dots, \xi_m$ ordinals, all with definitions which are absolute for random real forcing. Let ϕ be a second-order formula such that

$$\text{ZFC} \vdash \text{'for every atomlessly-measurable cardinal } \kappa > \max_{i \le m} \xi_i,$$
$$(\kappa; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi\text{'}.$$

Then for every atomlessly-measurable cardinal $\kappa > \max_{i \le m} \xi_i$,

$$\{\alpha : \alpha < \kappa,\ (\alpha; C_1, \dots, \xi_m) \vDash \phi\}$$

belongs to the rvm filter of κ.

proof (a) Of course we may take it that $I = \beta$ is an ordinal less than κ. Also, as remarked in 2B, it is enough to consider the case in which $\mathbb{P} = \mathbb{P}_\lambda = \mathfrak{A}_\lambda \setminus \{\mathbf{0}\}$ for some cardinal λ, taking $\mathfrak{A}_\lambda$ to be the measure algebra of $\{0,1\}^\lambda$ as usual.

Set $\theta = \max(\lambda, \kappa^+)$ and let $\mathbb{P}_\theta = \mathfrak{A}_\theta \setminus \{\mathbf{0}\}$ be the corresponding p.o.set. Then

$$(\beta; C_1, \dots) \vDash \phi \iff \Vdash_{\mathbb{P}_\theta} (\check{\beta}; \check{C}_1, \dots) \vDash \phi$$

by 4N. But now note that

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is real-valued-measurable}$$

(2C), so we can repeat the argument in $V^{\mathbb{P}}$ to see that

$$\Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots) \vDash \phi \iff \Vdash_{\mathbb{P}} \big(\Vdash_{\mathbb{P}_\theta} (\check{\beta}; \check{C}_1, \dots) \vDash \phi\big).$$

The iteration $\mathbb{P} * \mathbb{P}_\theta = \mathbb{P}_\lambda * \mathbb{P}_\theta$ is isomorphic to $\mathbb{P}_\theta$ (Kunen 84, 3.13), so we have

$$\Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots) \vDash \phi \iff \Vdash_{\mathbb{P}_\theta} (\check{\beta}; \check{C}_1, \dots) \vDash \phi \iff (\beta; C_1, \dots) \vDash \phi.$$

(b) Let ν be a normal witnessing probability on κ and $\mathbb{P}$ the p.o.set $(\mathcal{P}\kappa/\mathcal{N}_\nu) \setminus \{\mathbf{0}\}$. Then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is real-valued-measurable}$$

by 2C, and also of course

$$\Vdash_{\mathbb{P}} \mathfrak{c} \ge \check{\kappa} > \max_{i \le m} \check{\xi}_i,$$

so, repeating the ZFC proof of $(\kappa; \dots) \vDash \phi$ in $V^{\mathbb{P}}$,

$$\Vdash_{\mathbb{P}} (\check{\kappa}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi.$$

But now, by 4Md,

$$\nu\{\alpha : \alpha < \kappa,\ (\alpha; C_1, \dots, \xi_m) \vDash \phi\} = 1.$$

As ν is arbitrary, we have the result.

Remark Part (a) means that almost any fact about random real forcing is likely to have implications in the presence of an atomlessly-measurable cardinal.

Part (b) is a kind of reflection principle, corresponding to the theorem of Hanf and Scott (A4L) for two-valued-measurable cardinals. The requirements 'ZFC $\vdash \dots$' and 'with definitions absolute for random real forcing' are more restrictive than is absolutely necessary, but we do have to take care, when applying this method, that whatever argument we have used to prove that $(\kappa; \dots) \vDash \phi$ will survive the move to $V^{\mathbb{P}}$.

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4P Theorem Let κ be a real-valued-measurable cardinal with rvm filter $\mathcal{W}$ and rvm ideal $\mathcal{J}$. Then for every $Z \in \mathcal{P}\kappa \setminus \mathcal{J}$,

$$\{\alpha : \alpha < \kappa,\ Z \cap \alpha \text{ belongs to the } \Pi^2_0\text{-ideal of } \alpha\}$$

belongs to $\mathcal{J}$. In particular, setting $Z = \kappa$, the set of weakly Π^2_0-indescribable cardinals less than κ belongs to $\mathcal{W}$.

proof For two-valued-measurable cardinals this (and much more) is essentially covered by A4L; so henceforth I shall assume that κ is atomlessly-measurable. I will as usual write $\mathfrak{A}_I$ for the measure algebra of $\{0,1\}^I$, $\mathbb{P}_I$ for the p.o.set $\mathfrak{A}_I \setminus \{\mathbf{0}\}$, for any set I. Write

$$H = \{\alpha : 0 < \alpha < \kappa,\ Z \cap \alpha \text{ belongs to the } \Pi^2_0\text{-ideal of } \alpha\}.$$

? Suppose, if possible, that $H \notin \mathcal{J}$. For each $\alpha \in H$ we have a formula ϕ_α of the language $\mathcal{L}$ of §A4, integers k_α, $m_\alpha \ge 0$, relations $C_{\alpha 1}, \dots, C_{\alpha k_\alpha}$ on α, and ordinals $\xi_{\alpha_1}, \dots, \xi_{\alpha m_\alpha} < \alpha$ such that

$$(\alpha; C_{\alpha 1}, \dots, \xi_{\alpha m_\alpha}) \vDash \phi_\alpha,$$

$$(\beta; C_{\alpha 1}, \dots, \xi_{\alpha m_\alpha}) \not\vDash \phi_\alpha \text{ for every } \beta < \alpha.$$

Because there are only countably many formulae in $\mathcal{L}$, there must be ϕ, k, m such that $H_1 \notin \mathcal{J}$, where $H_1 = \{\alpha : \alpha \in H,\ \phi_\alpha = \phi,\ k_\alpha = k,\ m_\alpha = m\}$. Let ν_2 be a Maharam homogeneous normal witnessing probability on κ such that $\nu_2 H_1 = 1$. Let $\mathbb{P}$ be the p.o.set $(\mathcal{P}\kappa/\mathcal{N}_{\nu_2}) \setminus \{\mathbf{0}\}$. Let $\dot{C}_1, \dots, \dot{C}_k$, $\dot{\xi}_1, \dots, \dot{\xi}_m$ be the $\mathbb{P}$-names for relations and ordinals corresponding to the families $\langle C_{\alpha i}\rangle_{\alpha<\kappa}$, $\langle \xi_{\alpha j}\rangle_{\alpha<\kappa}$ as in Lemma 4M. Then we have

$$\Vdash_{\mathbb{P}} (\check{\kappa}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi$$

by 4Md. But now observe that the Maharam type λ of ν_2 is greater than κ, by the Gitik-Shelah theorem again (3F). If we identify $\mathbb{P}$ with $\mathbb{P}_\lambda$, we see that there is a set $I \subseteq \lambda$, of cardinal at most κ, such that all the $\mathbb{P}$-names $\dot{C}_1, \dots, \dot{\xi}_m$ can be represented by $\mathbb{P}_I$-names. Let J be a subset of λ, including I, such that $J \setminus I$ and $\lambda \setminus J$ both have cardinal λ. Now we can regard $\mathbb{P} \cong \mathbb{P}_\lambda$ as an iteration $\mathbb{P}_J * \mathbb{P}_{\lambda\setminus J}$ (see Kunen 84, 3.13), so that we have

$$\Vdash_{\mathbb{P}_J} \big(\Vdash_{\mathbb{P}_{\lambda\setminus J}} (\check{\kappa}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi\big).$$

But from the standpoint of $V^{\mathbb{P}_J}$, $\dot{C}_1, \dots, \dot{\xi}_m$ are fixed relations and ordinals. Also we have

$$\Vdash_{\mathbb{P}_J} \check{\kappa} \text{ is real-valued-measurable}$$

(Theorem 2C) and

$$\Vdash_{\mathbb{P}_J} \check{Z} \text{ is not in the rvm ideal of } \check{\kappa}$$

by 2Jc. So we may use Proposition 4N in $V^{\mathbb{P}_J}$ to see that

$$\Vdash_{\mathbb{P}_J} \exists\ \beta \in \check{Z},\ (\beta; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi.$$

Now there is a $\mathbb{P}_J$-name $\dot{\beta}$ for a member of Z such that

$$\Vdash_{\mathbb{P}_J} (\dot{\beta}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi.$$

Let $K \subseteq J$ be a countable set such that the $\mathbb{P}_J$-name $\dot{\beta}$ can be represented by a $\mathbb{P}_K$-name. Regarding $\mathbb{P}_J$ as an iteration $\mathbb{P}_{I\cup K} * \mathbb{P}_{J\setminus(I\cup K)}$ and $\mathbb{P}$ as an iteration $\mathbb{P}_{I\cup K} * \mathbb{P}_{\lambda\setminus(I\cup K)}$, and observing that $J \setminus (I \cup K)$ and $\lambda \setminus (I \cup K)$ both have cardinal $\lambda > \kappa$, we can use A4I in $V^{\mathbb{P}_{I\cup K}}$ to see that

$$\Vdash_{\mathbb{P}} (\dot{\beta}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi.$$

Now there must be $\zeta \in Z$ and $a \in \mathcal{P}\kappa \setminus \mathcal{N}_{\nu_2}$ such that $a^\bullet \Vdash_{\mathbb{P}} \dot{\beta} = \check{\zeta}$, so that

$$a^\bullet \Vdash_{\mathbb{P}} (\check{\zeta}; \dot{C}_1, \dots, \dot{\xi}_m) \vDash \phi.$$

But now 4Md tells us that

$$(\zeta; C_{\alpha 1}, \dots, \xi_{\alpha m}) \vDash \phi$$

for ν_2-almost every $\alpha \in a$. In particular, there is an $\alpha \in a \cap H_1$ such that $\alpha > \zeta$ and $(\zeta; C_{\alpha 1}, \dots) \vDash \phi$; but $\phi = \phi_\alpha$; so this contradicts the choice of $C_{\alpha 1}, \dots$ and ϕ_α. **X**

4Q Proposition Suppose that κ, κ' are two atomlessly-measurable cardinals with $\kappa < \kappa'$. Then κ is weakly Π^2_0-indescribable, and its Π^2_0-filter is included in its rvm filter.

proof Let ν be a Maharam homogeneous normal witnessing probability on κ; let $\mathbb{P} = (\mathcal{P}\kappa/\mathcal{N}_\nu) \setminus \{\mathbf{0}\}$ be the corresponding p.o.set.

Suppose that ϕ is a formula of the language $\mathcal{L}$ of §A4 and that $C_1, \ldots, C_k$ are relations on κ, $\xi_1, \ldots, \xi_m$ are ordinals less than κ such that

$$(\kappa; C_1, \ldots, C_k; \xi_1, \ldots, \xi_m) \vDash \phi.$$

By 4Oa, because there is an atomlessly-measurable cardinal greater than κ,

$$\Vdash_{\mathbb{P}} (\check{\kappa}; \check{C}_1, \ldots) \vDash \phi.$$

So 4Md tells us that

$$(\alpha; C_1, \ldots) \vDash \phi$$

for ν-almost every $\alpha < \kappa$. Because ν is arbitrary,

$$A = \{\alpha : \alpha < \kappa,\ (\alpha; C_1, \ldots) \vDash \phi\}$$

belongs to the rvm filter of κ. As ϕ, $C_1, \ldots, \xi_m$ are arbitrary, we have the result.

4R Proposition Let κ be a real-valued-measurable cardinal and λ any larger cardinal. Let $\mathfrak{A}$ be the measure algebra of $\{0,1\}^\lambda$ and $\mathbb{P}$ the p.o.set $\mathfrak{A}_\lambda \setminus \{\mathbf{0}\}$. Then

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is weakly } \Pi^2_0\text{-indescribable and}$$
$$\text{the rvm filter of } \check{\kappa} \text{ includes the } \Pi^2_0\text{-filter of } \kappa.$$

proof We can use the same ideas as in the last three results. Let ϕ be a formula of the language $\mathcal{L}$, and let $\dot{C}_1, \ldots, \dot{C}_k$, $\dot{\xi}_1, \ldots, \dot{\xi}_m$, σ be $\mathbb{P}_\lambda$-names such that

$$\Vdash_{\mathbb{P}} (\check{\kappa}; \dot{C}_1, \ldots) \vDash \phi,$$

$$\Vdash_{\mathbb{P}} \sigma = \{\alpha : \alpha < \check{\kappa},\ (\alpha; \dot{C}_1, \ldots) \nvDash \phi\}.$$

Then there is a set $I \subseteq \lambda$ such that $\lambda \setminus I$ has cardinal λ and σ and every $\dot{C}_i$, $\dot{\xi}_j$ can be represented by a $\mathbb{P}_I$-name, taking $\mathbb{P}_I$ to be the p.o.set associated with the measure algebra of $\{0,1\}^I$ as usual. Now we have

$$\Vdash_{\mathbb{P}_I} \check{\kappa} \text{ is real-valued-measurable}$$

(2C) and

$$\Vdash_{\mathbb{P}_I} \bigl(\Vdash_{\mathbb{P}_{\lambda\setminus I}} (\check{\kappa}; \dot{C}_1, \ldots) \vDash \phi\bigr),$$

$$\Vdash_{\mathbb{P}_I} \bigl(\Vdash_{\mathbb{P}_{\lambda\setminus I}} \sigma = \{\alpha : \alpha < \check{\kappa},\ (\alpha; \dot{C}_1, \ldots) \nvDash \phi\}\bigr).$$

In $V^{\mathbb{P}_I}$, all of σ, $\dot{C}_1, \ldots$ are definite sets, so we can apply 4N to see that

$$\Vdash_{\mathbb{P}_I} \sigma \text{ belongs to the rvm ideal of } \check{\kappa}.$$

But now 2Jc tells us that

$$\Vdash_{\mathbb{P}_I} \bigl(\Vdash_{\mathbb{P}_{\lambda\setminus I}} \sigma \text{ belongs to the rvm ideal of } \check{\kappa}\bigr),$$

taking the liberty of using the same symbol $\check{\kappa}$ for the new name for κ. Thus

$$\Vdash_{\mathbb{P}} \sigma \text{ belongs to the rvm ideal of } \kappa.$$

As ϕ, $\dot{C}_1, \ldots$ are arbitrary we have the result.

Remark This result shows that if an atomlessly-measurable cardinal κ is constructed by the method of 2C, then

$$\begin{aligned} &\kappa \text{ is weakly } \Pi^1_1\text{-indescribable} \\ &\iff \kappa < \mathfrak{c} \\ &\iff \kappa \text{ is weakly } \Pi^2_0\text{-indescribable} \\ &\qquad \text{and the rvm filter of } \kappa \text{ includes the } \Pi^2_0\text{-filter of } \kappa. \end{aligned}$$

See 6J and P3 below.

4S The next proposition depends on results from §6 below; but it seems natural to place it here, as it again refers to a type of indescribability.

Proposition Let κ be an atomlessly-measurable cardinal with normal witnessing probability ν. Let ϕ be a Σ_1^1 formula in the language $\mathcal{L}$ of §A4, $C_1,\dots,C_k$ relations on κ and $\xi_1,\dots,\xi_m$ ordinals less than κ. Set

$$A=\{\alpha:\alpha<\kappa,\ (\alpha;C_1,\dots,C_k;\xi_1,\dots,\xi_m)\vDash\phi\}.$$

If A is not in the rvm ideal of κ then there is a $B\subseteq\kappa$ such that $\beta\subseteq B$, $\nu B=1$ and

$$(B;C_1,\dots,\xi_m)\vDash\phi.$$

proof (a) By 4N, $\nu A=1$. Of course, we may suppose that $\xi_j<\beta$ for every j. Next, we may take it that ϕ is of the form

$$\exists R_{k+1}\dots\exists R_{k+r}\forall x_{m+1}\exists x_{m+2}\dots\forall x_{m+2s-1}\exists x_{m+2s}\psi$$

where ψ has no quantifiers, since ϕ is surely logically equivalent to such a formula. For every $\alpha\in A$ we have relations $D_{\alpha 1},\dots,D_{\alpha r}$ and functions $h_{\alpha 1}:\alpha\to\alpha$, $h_{\alpha 2}:\alpha^2\to\alpha,\dots,h_{\alpha s}:\alpha^s\to\alpha$ such that

$$(\alpha;C_1,\dots,C_k,D_{\alpha 1},\dots,D_{\alpha r},\xi_1,\dots,\xi_m,\eta_1,h_{\alpha 1}(\eta_1),\dots,\eta_s,h_{\alpha s}(\eta_1,\dots,\eta_s))\vDash\psi$$

for all $\eta_1,\dots,\eta_s<\alpha$. For each $\bar\eta=(\eta_1,\dots,\eta_j)\in\bigcup_{j\le s}\kappa^j$ set

$$M(\bar\eta)=\{\xi:\nu\{\alpha:h_{\alpha j}(\bar\eta)=\xi\}>0\};$$

then $M(\bar\eta)$ is countable. For $I\in[\kappa]^{<\omega}$ let $K(I)$ be the smallest subset of κ such that $I\subseteq K(I)$ and $M(\bar\eta)\subseteq K(I)$ whenever $\bar\eta\in\bigcup_{j\le s}K(I)^j$. Then $K(I)$ is countable.

(b) Let $(Z,\tilde\nu)$ be the hyperstonian space of $(\kappa,\mathcal{P}\kappa,\nu)$, and for $a\subseteq\kappa$ let a^* be the corresponding open-and-closed subset of Z, so that $a\mapsto a^*$ is a Boolean homomorphism and $\tilde\nu a^*=\nu a$ for every $a\subseteq\kappa$. For $j\le s$, $\bar\eta\in\kappa^j$ set

$$Q(\bar\eta)=\{\{\alpha:h_{\alpha j}(\bar\eta)=\xi\}^*:\xi\in M(\bar\eta)\},$$

so that $\tilde\nu(Z\setminus\bigcup Q(\bar\eta))=0$. Set

$$f(I)=\bigcup\{Z\setminus\bigcup Q(\bar\eta):\eta\in\bigcup_{j\le s}K(I)^j\}\in\mathcal{N}_{\tilde\nu}$$

for each $I\in[\kappa]^{<\omega}$.

By Proposition 6E, there is a set $B_0\subseteq\kappa$ such that $\beta\subseteq B_0$, $\nu B_0=1$ and $\bigcup\{f(I):I\in[B_0]^{<\omega}\}\ne Z$. Take $z\in Z\setminus\bigcup\{f(I):I\in[B_0]^{<\omega}\}$ and set $B=\bigcup\{K(I):I\in[B_0]^{<\omega}\}$.

If $j\le s$, we may define $h_j:B^j\to B$ as follows. For each $\bar\eta\in B^j$ there is an $I\in[B_0]^{<\omega}$ such that $\bar\eta\in K(I)^j$ (because the function $I\mapsto K(I)$ is increasing), and now $z\in\bigcup Q(\bar\eta)$, so there is a (unique) ξ such that $z\in\{\alpha:h_{\alpha j}(\bar\eta)=\xi\}^*$. We have $\xi\in M(\bar\eta)\subseteq K(I)\subseteq B$; take this ξ for $h_j(\bar\eta)$.

Next, define relations $D_1,\dots,D_r$ on κ by writing

$$D_i(\eta_1,\dots,\eta_n)\iff z\in\{\alpha:D_{\alpha 1}(\eta_1,\dots,\eta_n)\}^*.$$

Then an easy induction on the length of χ shows that whenever χ is a formula of $\mathcal{L}$ without quantifiers, and $\eta_1,\dots,\eta_s\in B$,

$$(B;C_1,\dots,C_k,D_1,\dots,D_r;\xi_1,\dots,\xi_m,\eta_1,h_1(\eta_1),\dots,h_s(\eta_1,\dots,\eta_s))\vDash\chi$$

at least when $z\in b(\chi,\eta_1,\dots,\eta_s)^*$, where $b(\chi,\eta_1,\dots,\eta_s)$ is the set of those $\alpha<\kappa$ such that

$$(\alpha;C_1,\dots,C_k,D_{\alpha 1},\dots,D_{\alpha r},\eta_1,h_{\alpha 1}(\eta_1),\dots,\eta_s,h_{\alpha s}(\eta_1,\dots,\eta_s))\vDash\chi.$$

In particular, this is true when $\chi=\psi$. But of course $z\in Z=b(\psi,\eta_1,\dots,\eta_s)^*$ for all $\eta_1,\dots,\eta_s<\kappa$, so

$$(B;C_1,\dots,C_k,D_1,\dots,D_r;\xi_1,\dots,\xi_m,\eta_1,h_1(\eta_1),\dots,h_s(\eta_1,\dots,\eta_s))\vDash\psi$$

for all $\eta_1,\dots,\eta_s\in B$, and

$$(B;C_1,\dots,C_k;\xi_1,\dots,\xi_m)\vDash\phi,$$

as required.

4T Corollary Let κ be an atomlessly-measurable cardinal. Let ϕ be a Π^1_1 formula of the language $\mathcal{L}$, and $\xi_1,\dots,\xi_m$ ordinals less than κ. If $(\kappa;<;\xi_1,\dots,\xi_m)\vDash\phi$ then there is an $\alpha<\kappa$ such that $(\alpha;<;\xi_1,\dots,\xi_m)\vDash\phi$.

proof ? Suppose, if possible, otherwise. Applying 4S to (a Σ^1_1 formula logically equivalent to) $\neg\phi$, with β greater than any ξ_j, we obtain a $B\subseteq\kappa$ such that $\beta\subseteq B$, $\#(B)=\kappa$ and

$$(B;<;\xi_1,\dots,\xi_m)\vDash\neg\phi.$$

But $(B;<;\xi_1,\dots,\xi_m)$ is isomorphic to $(\kappa;<;\xi_1,\dots,\xi_m)$, so this is impossible. **X**

Remark 4N-4T are due to Kunen. Most of the ideas are in KUNEN N70.

Version of 18.9.92

5. Combinatorial implications

I turn now to some of the combinatorial consequences of supposing that there is an atomlessly-measurable cardinal. I begin with technical but very useful results on set- valued functions defined on $[\kappa]^{<\omega}$ (5A-5B), with fairly straightforward corollaries (5C-5D), and Prikry's theorem on cardinal powers when $\mathfrak{c}$ is atomlessly -measurable (5E). 5F deals with κ-Aronszajn trees. The rest of the section is devoted to theorems of Kunen concerning ultrafilters on $\mathbb{N}$ (5G), algebras generated by rectangles (5H-5J) with corollaries on families in $\mathcal{P}\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}}$ (5K-5L), $\diamondsuit_{\mathfrak{c}}$ (5N) and a partition relation (5O-5P).

5A Lemma Let $(X,\mathcal{P}X,\mu)$ be a probability space and Y a set of cardinal less than the additivity of μ. Let θ be any cardinal and $f:X\to[Y]^{<\theta}$ a function.

(a) There is an $M\in[Y]^{<\theta}$ such that $\mu\{x:f(x)\subseteq M\}>0$.

(b) If $\mathrm{cf}(\theta)>\omega$ then there is an $M\in[Y]^{<\theta}$ such that $\mu\{x:f(x)\subseteq M\}=1$.

proof (a) If $\theta>\#(Y)$ this is trivial; suppose that $\theta\le\#(Y)<\mathrm{add}(\mu)$. Because $[Y]^{<\theta}=\bigcup_{\alpha<\theta}[Y]^{\le\alpha}$, there is an $\alpha<\theta$ such that $\mu X_0>0$, where $X_0=\{x:\#(f(x))\le\alpha\}$. If $\alpha<\omega$ then $\#([Y]^\alpha)\le\max(\omega,\#(Y))<\mathrm{add}(\mu)$ so there is an $M\in[Y]^\alpha$ such that $\mu\{x:f(x)=M\}>0$. If $\alpha\ge\omega$, then for each $x\in X_0$ let $\langle h_\xi(x)\rangle_{\xi<\alpha}$ run over a set including $f(x)$. For each $\xi<\alpha$,

$$Y_\xi=\{y:\mu h_\xi^{-1}[\{y\}]>0\}$$

is countable, and because $\#(Y)<\mathrm{add}(\mu)$, $\mu h_\xi^{-1}[Y\setminus Y_\xi]=0$. Set $M=\bigcup_{\xi<\alpha}Y_\xi\in[Y]^{\le\alpha}\subseteq[Y]^{<\theta}$. Because $\alpha<\mathrm{add}(\mu)$,

$$\mu\{x:f(x)\subseteq M\}\ge\mu(X_0\setminus\bigcup_{\xi<\theta}h_\xi^{-1}[Y\setminus Y_\xi])>0,$$

as required.

(b) If $\mathrm{cf}(\theta)>\omega$, we can take α such that $\mu X_0=1$, so that $\mu\{x:f(x)\subseteq M\}=1$.

5B Theorem Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. Let $\theta<\kappa$ be a cardinal of uncountable cofinality, and $f:[\kappa]^{<\omega}\to[\kappa]^{<\theta}$ any function. Then there are $C\subseteq\kappa$ and $f^*:[C]^{<\omega}\to[\kappa]^{<\theta}$ such that $\nu C=1$ and $f(I)\cap\eta\subseteq f^*(I\cap\eta)$ whenever $I\in[C]^{<\omega}$ and $\eta<\kappa$.

proof (a) I show by induction on $n\in\mathbb{N}$ that if $g:[\kappa]^{\le n}\to[\kappa]^{<\theta}$ is a function then there are $A\subseteq\kappa$, $g^*:[A]^{<\omega}\to[\kappa]^{<\theta}$ such that $\nu A=1$ and $g(I)\cap\eta\subseteq g^*(I\cap\eta)$ for every $I\in[A]^{<\omega}$, $\eta<\kappa$. **P (i)** If $n=0$ this is trivial; take $A=\kappa$, $g^*(\emptyset)=g(\emptyset)$. **(ii)** For the inductive step to $n+1$, given $g:[\kappa]^{\le n+1}\to[\kappa]^{<\theta}$, then for each $\xi<\kappa$ define $g_\xi:[\kappa]^{\le n}\to[\kappa]^{<\theta}$ by setting $g_\xi(J)=g(J\cup\{\xi\})$ for every $J\in[\kappa]^{\le n}$. Set

$$D=\{\xi:\xi<\kappa,\ \mathrm{cf}(\xi)\ge\theta\};$$

then $\nu D=1$ (4Lc). For $\xi\in D$, $J\in[\kappa]^{\le n}$ set $\zeta_{J\xi}=\sup(\xi\cap g_\xi(J))<\xi$. Then for each $J\in[\kappa]^{\le n}$ the function $\xi\mapsto\zeta_{J\xi}$ is regressive, so there is a $\zeta^*_J<\kappa$ such that $\zeta_{J\xi}<\zeta^*_J$ for almost every $\xi<\kappa$. Now by 5Ab we see that there is an $h(J)\in[\zeta^*_J]^{<\theta}$ such that $\xi\cap g_\xi(J)\subseteq h(J)$ for almost all ξ. By the inductive hypothesis, there are $B\subseteq\kappa$, $h^*:[B]^{\le n}\to[\kappa]^{<\theta}$ such that $\nu B=1$ and $h(J)\cap\eta\subseteq h^*(J\cap\eta)$ for every $J\in[B]^{\le n}$ and $\eta<\kappa$.

Try setting

$$A_J = \{\xi : \xi \cap g_\xi(J) \subseteq h(J)\} \text{ for } J \in [\kappa]^{\le n},$$

$$A = B \cap \{\xi : \xi \in A_J \ \forall \ J \in [\xi]^{\le n}\},$$

$$g^*(I) = g(I) \text{ if } I \in [A]^{n+1},\ g^*(I) = g(I) \cup h^*(I) \text{ if } I \in [A]^{\le n}.$$

Then νA_J is always 1, by the choice of $h(J)$, so $\nu A = 1$, by A1E(c-iv); while $g^*(I) \in [\kappa]^{<\theta}$ for every $I \in [A]^{\le n+1}$. If $\eta < \kappa$ then of course $g(\emptyset) \cap \eta \subseteq g^*(\emptyset \cap \eta)$ for every $\eta < \kappa$. If $I \in [A]^{\le n+1} \setminus \{\emptyset\}$ and $\eta < \kappa$, set $\xi = \max I$, $J = I \setminus \{\xi\}$; then $\xi \in A_J$. If $\eta > \xi$ then $g(I) \cap \eta \subseteq g(I) \subseteq g^*(I) = g^*(I \cap \eta)$. If $\eta \le \xi$ then

$$g(I) \cap \eta = g_\xi(J) \cap \xi \cap \eta \subseteq h(J) \cap \eta \subseteq h^*(J \cap \eta) = h^*(I \cap \eta) \subseteq g^*(I \cap \eta).$$

Thus the induction continues.

(b) Now applying (a) to $f{\restriction}[\kappa]^{\le n}$ we obtain sets $C_n \subseteq \kappa$, functions $f_n^* : [C_n]^{\le n} \to [\kappa]^{<\theta}$ such that $\nu C_n = 1$ and $f(I) \cap \eta \subseteq f_n^*(I \cap \eta)$ whenever $I \in [C_n]^{\le n}$ and $\eta < \kappa$. Set $C = \bigcap_{n \in \mathbb{N}} C_n$ and $f(I) = \bigcup_{i \ge \#(I)} f_i^*(I)$ for each $I \in [C]^{<\omega}$. Then $\nu C = 1$. If $I \in [C]^{<\omega}$ and $\eta < \kappa$, set $n = \#(I)$; then $I \in [C_n]^n$ so $f(I) \cap \eta \subseteq f_n^*(I \cap \eta) \subseteq f^*(I \cap \eta)$, as required.

5C Corollary Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. Let $\theta < \kappa$ be a cardinal of uncountable cofinality.

(a) If Y is a set of cardinal less than κ and $f : [\kappa]^{<\omega} \to [Y]^{<\theta}$ is a function, then there are sets $C \subseteq \kappa$, $M \subseteq Y$ such that $\nu C = 1$, $\#(M) < \theta$ and $f(I) \subseteq M$ for every $I \in [C]^{<\omega}$.

(b) If Y is any set and $g : \kappa \to [Y]^{<\theta}$ any function there are sets $C \subseteq \kappa$, $M \in [Y]^{<\theta}$ such that $\nu C = 1$ and $g(\xi) \cap g(\eta) \subseteq M$ for all distinct ξ, $\eta \in C$.

(c) If $f : [\kappa]^{<\omega} \to \kappa$ is a function such that $f(I) < \min I$ for every non-empty finite $I \subseteq \kappa$ not containing 0, then there are sets C, $M \subseteq \kappa$ such that $\nu C = 1$, M is countable and $f(I) \in M$ for every $I \in [C]^{<\omega}$.

proof (a) We may suppose that Y is actually a subset of κ. In this case, by 5B, we have a set $C_0 \subseteq \kappa$ and a function $f^* : [C_0]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ for all $I \in [C_0]^{<\omega}$ and $\eta < \kappa$. Let $\gamma < \kappa$ be such that $Y \subseteq \gamma$ and set $M = Y \cap f^*(\emptyset) \cap \gamma$, $C = C_0 \setminus \gamma$. Then $\#(M) < \theta$ and $\nu C = 1$ and if $I \in [C]^{<\omega}$ then $f(I) = f(I) \cap \gamma \subseteq Y \cap f^*(I \cap \gamma) = M$.

(b) As above, we may suppose that $\bigcup_{\xi < \kappa} g(\xi) \subseteq \kappa$. Set $f(I) = \bigcup_{\xi \in I} g(\xi) \in [\kappa]^{<\theta}$ for $I \in [\kappa]^{<\omega}$; take $C_0 \subseteq \kappa$, $f^* : [C_0]^{<\omega} \to [\kappa]^{<\theta}$ from 5B, and set $M = Y \cap f^*(\emptyset)$. Write

$$C_1 = \{\xi : \xi < \kappa,\ g(\eta) \subseteq \xi \ \forall \ \eta < \xi\},$$

so that C_1 is a closed unbounded set and $\nu C = 1$, where $C = C_0 \cap C_1$. If ξ, $\eta \in C$ and $\xi < \eta$, then

$$g(\xi) \cap g(\eta) \subseteq \eta \cap g(\eta) = \eta \cap f(\{\eta\}) \subseteq f^*(\{\eta\} \cap \eta) = f^*(\emptyset),$$

so $g(\xi) \cap g(\eta) \subseteq M$, as required.

(c) By 5B we have $f^* : [\kappa]^{<\omega} \to [\kappa]^{\le\omega}$, $C \subseteq \kappa$ such that $\nu C = 1$ and $\{f(I)\} \cap \eta \in f^*(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$. We may suppose that $0 \notin C$. If $I \in [C]^{<\omega}$ and $I \ne \emptyset$ set $\eta = \min I$; then $f(I) \in \{f(I)\} \cap \eta \subseteq f^*(I \cap \eta) = f^*(\emptyset)$. So if we take $M = f^*(\emptyset) \cup \{f(\emptyset)\}$ we have the result.

5D Corollary Let κ be a real-valued-measurable cardinal.

(a) Suppose that Y is a set of cardinal less than κ and θ is a cardinal less than κ of uncountable cofinality. Then

(i) κ is a precaliber of the Boolean algebra $\mathcal{P}Y/[Y]^{<\theta}$;

(ii) $\{\lambda : \lambda < \kappa$ is a precaliber of $\mathcal{P}Y/[Y]^{<\theta}\}$ belongs to the rvm filter of κ;

(iii) the Souslin number of $\mathcal{P}Y/[Y]^{<\theta}$ is less than κ;

(iv) $\mathrm{Tr}_{[Y]^{<\theta}}(Y; Z) < \kappa$ whenever $\#(Z) < \kappa$ (definition: 2F).

(b) If δ is a cardinal less than κ, and $h : [\kappa]^{<\omega} \to [\kappa]^{\le\delta}$ is a function, there is a set $A \subset \kappa$ such that $\#(A) = \kappa$ and $h(I) \subseteq A$ for every $I \in [A]^{<\omega}$.

(c) In particular, there is no Jónsson algebra on κ.

(d) If $f : [\kappa]^{<\omega} \to \omega_1$ is any function there are $C \in [\kappa]^\kappa$, $\zeta < \omega_1$ such that $f(I) \ne \zeta$ for every $I \in [C]^{<\omega}$.

proof (a) Write $\mathfrak{A} = \mathcal{P}Y/[Y]^{<\theta}$.

(i) Let $\langle a_\xi\rangle_{\xi<\kappa}$ be a family in $\mathfrak{A}\setminus\{0\}$. For $\xi<\kappa$ choose $A_\xi\subseteq Y$ such that $A_\xi^\bullet=a_\xi$. For $I\in[\kappa]^{<\omega}$ set $f(I)=\bigcap_{\xi\in I}A_\xi$ if the intersection has cardinal less than θ, $\emptyset$ otherwise. By 5Ca there is a $C\in[\kappa]^\kappa$ such that $M=\bigcup\{f(I):I\in[C]^{<\omega}\}$ has cardinal less than θ. Now there is a $y\in Y\setminus M$ such that $D=\{\xi:\xi\in C,\ y\in A_\xi\}$ has cardinal κ, and we must have $\#(\bigcap_{\xi\in I}A_\xi)\ge\theta$ for every $I\in[D]^{<\omega}$, so that $\langle a_\xi\rangle_{\xi\in D}$ is centered in $\mathfrak{A}$.

(ii) Let W be the set of cardinals less than κ which are not precalibers of $\mathfrak{A}$. For each $\alpha\in W$ choose a family $\langle a_{\alpha\xi}\rangle_{\xi<\alpha}$ in $\mathfrak{A}\setminus\{0\}$ with no centered family of cardinal α, and for $\xi<\alpha\in W$ choose $A_{\alpha\xi}\subseteq Y$ such that $A_{\alpha\xi}^\bullet=a_{\alpha\xi}$. For $I\in[\kappa]^{<\omega}$ set $f(I)=\bigcap_{\xi\in I\cap\alpha}A_{\alpha\xi}$ if $\#(I)\ge2$ and $\max I=\alpha\in W$ and $\#(\bigcap_{\xi\in I\cap\alpha)}A_{\alpha\xi}<\theta$, $\emptyset$ otherwise. **?** If W is not in the rvm ideal of κ, let ν be a normal witnessing probability on κ with $\nu W=1$. By 5Ca there is a $C\subseteq\kappa$ such that $\nu C=1$ and $M=\bigcup\{f(I):I\in[C]^{<\omega}\}$ has cardinal less than θ. Let $\alpha\in C\cap W$ be such that $\#(C\cap\alpha)=\alpha$ and $\mathrm{cf}(\alpha)>\#(Y)$ (using 4Lc). For each $\xi\in C\cap\alpha$ there is a $y\in A_{\alpha\xi}\setminus M$, so there is a $y\in Y$ such that $D=\{\xi:\xi<\alpha,\ y\in A_{\alpha\xi}\}$ has cardinal α. But now $\langle a_{\alpha\xi}\rangle_{\xi\in D}$ is centered, contrary to the choice of $\langle a_{\alpha\xi}\rangle_{\xi<\alpha}$. **X**

Thus W belongs to the rvm ideal of κ, and

$$V=\{\alpha:\alpha<\kappa\text{ is a cardinal}\}\setminus W$$

belongs to the rvm filter of κ. But every member of V is a precaliber of $\mathfrak{A}$.

(iii) If α is any infinite cardinal which is a precaliber of $\mathfrak{A}$, then the Souslin number of $\mathfrak{A}$ is at most α.

(iv) $\mathrm{Tr}_{[Y]^{<\theta}}(Y;Z)\le\mathrm{S}(\mathcal{P}(Y\times Z)/[Y\times Z]^{<\theta})$.

(b)-(c) For $I\in[\kappa]^{<\omega}$ let $g(I)$ be the smallest set such that $I\subseteq g(I)$ and $h(J)\subseteq g(I)$ for every $J\in[g(I)]^{<\omega}$. Then $\#(g(I))\le\delta$ for every I and $g(I)\subseteq g(I')$ whenever $I\subseteq I'\in[\kappa]^{<\omega}$.

Applying 5Ca with $Y=\delta^+=\theta$ and $f(I)=g(I)\cap\theta$, we see that there are sets $C\subseteq\kappa$, $M\subseteq\theta$ such that $\#(C)=\kappa$, $\#(M)\le\delta$ and $g(I)\cap\theta\subseteq M$ for every $I\in[C]^{<\omega}$. Set

$$A=\bigcup\{g(I):I\in[C]^{<\omega}\}\subseteq\kappa.$$

Then $C\subseteq A$, so $\#(A)=\kappa$, and $A\cap\theta\subseteq M$, so $A\ne\kappa$. Finally, $\{g(I):I\in[C]^{<\omega}\}$ is upwards-directed, so if $J\in[A]^{<\omega}$ there is an $I\in[C]^{<\omega}$ such that $J\subseteq g(I)$, in which case $h(J)\subseteq g(I)\subseteq A$.

(d) In fact there are $C\in[\kappa]^\kappa$, $\zeta<\omega_1$ such that $f(I)<\zeta$ for every $I\in[C]^{<\omega}$.

Remarks (a) The ideas here go back to Lemma 14 of Solovay 71 and Lemma 5 of Kunen n70. 5D(a-ii) is a strengthening of Theorem 2d of Prikry 75. 5Dc is due to Shelah. 5Dd corresponds to formula (7) of §53 in Erdös Hajnal Máté & Rado 84, p. 330.

(b) The results above will more often than not be used with successor cardinals θ, so that $[Y]^{<\theta}=[Y]^{\le\delta}$ for some infinite cardinal $\delta<\kappa$. Of course 5B-5C also give information about functions $f:[\kappa]^{<\omega}\to Y$, taking $\theta=\omega_1$ and replacing f by $I\mapsto\{f(I)\}:[\kappa]^{<\omega}\to[Y]^{\le\omega}$.

(c) For possible strengthenings of 5Ab, 5Ca and 5Da see P9 below.

5E Theorem If $\mathfrak{c}$ is atomlessly-measurable, then $2^\lambda\le\mathfrak{c}$ for every cardinal $\lambda<\mathfrak{c}$.

proof The proof is by induction on λ. It starts with the trivial case $\lambda\le\omega$.

(a) For the inductive step to λ, where $\omega<\lambda<\mathfrak{c}$, consider first the case $\mathrm{cf}(\lambda)>\omega$. For each $\xi<\lambda$, let $\theta_\xi:\mathcal{P}\xi\to\mathfrak{c}$ be an injective function. For each $A\subseteq\lambda$, there must be a $\gamma_A<\mathfrak{c}$ such that $\theta_\xi(A\cap\xi)\le\gamma_A$ for every $\xi<\lambda$, because $\mathrm{cf}(\mathfrak{c})=\mathfrak{c}>\lambda$.

? Suppose, if possible, that $2^\lambda>\mathfrak{c}$. Then there must be a $\gamma<\mathfrak{c}$ such that $\mathcal{A}=\{A:A\subseteq\lambda,\ \gamma_A=\gamma\}$ has cardinal greater than $\mathfrak{c}$. Let $h:\mathfrak{c}\to\mathcal{A}$ be injective, and define $f:[\mathfrak{c}]^2\to\lambda$ by setting

$$f(\{\alpha,\beta\})=\min\{\xi:h(\alpha)\cap\xi\ne h(\beta)\cap\xi\}$$

whenever α, $\beta<\mathfrak{c}$ are distinct. By Corollary 5Ca, with $\theta=\omega_1$, $Y=\lambda$, there is a set $C\subseteq\mathfrak{c}$ such that $\#(C)=\mathfrak{c}$ and $M=f[[C]^2]$ is countable. Because $\mathrm{cf}(\lambda)>\omega$, $\zeta=\sup M<\lambda$. But now we see that $h(\alpha)\cap\zeta\ne h(\beta)\cap\zeta$ whenever α, β are distinct members of C. So $\theta_\zeta(h(\alpha)\cap\zeta)\ne\theta_\zeta(h(\beta)\cap\zeta)$ for all distinct α, $\beta\in C$. But $h(\alpha)\in\mathcal{A}$ whenever $\alpha<\mathfrak{c}$, so $\theta_\zeta(h(\alpha)\cap\zeta)\le\gamma$ for every $\alpha\in C$; which is impossible, because $\#(C)>\#(\gamma)$. **X**

So in this case $2^\lambda \le \mathfrak{c}$.

(b) Now if we have $\omega = \mathrm{cf}(\lambda) < \lambda < \mathfrak{c}$, then there is an increasing sequence $\langle \lambda_n \rangle_{n\in\mathbb{N}}$ of cardinals cofinal with λ, so that $2^\lambda \le \#(\prod_{n\in\mathbb{N}} \mathcal{P}\lambda_n) \le \#(\mathfrak{c}^{\mathbb{N}}) = \mathfrak{c}$. So in this case also the induction proceeds.

Remark This is due to PRIKRY 75. See also 7Q below.

5F Proposition If κ is a real-valued-measurable cardinal, then there is no κ-Aronszajn tree.

proof Let T be a tree of height κ in which every level has cardinal strictly less than κ. Then $\#(T) = \kappa$. Let ν be a κ-additive probability with domain $\mathcal{P}T$ which is zero on singletons. For each $t \in T$ write $T^{(t)}$ for the set of elements of T comparable with t, and for $\xi < \kappa$ let T_ξ be the set of elements of T of rank ξ. Set $S = \{t : \nu T^{(t)}\} > 0$. If $s \le t \in S$ then $T^{(s)} \supseteq T^{(t)}$ so $s \in S$; for each $\xi < \kappa$, we have $\#(T_\xi) < \kappa$ and $T = \bigcup_{t\in T_\xi} T^{(t)}$, so $S \cap T_\xi \ne \emptyset$; and $\nu(T^{(s)} \cap T^{(t)}) = 0$ if s, t are distinct members of T_ξ, so $S \cap T_\xi$ is countable. Thus S is a tree of height κ in which every level is countable. Because $\mathrm{cf}(\kappa) > \omega_1$ there must be an $s \in S$ such that $S^{(s)} = S \cap T^{(s)}$ is a branch of length κ. Thus T is not a κ-Aronszajn tree.

Remarks (a) This is due to SILVER 70, Theorem 1.16.

(b) It is also the case that

$$\{\lambda : \lambda < \kappa \text{ is a cardinal and there is a } \lambda\text{-Aronszajn tree}\}$$

belongs to the rvm ideal of κ. This may be deduced from the result above using 4Ob (because it is easy to check that there is a second-order formula ϕ such that, for cardinals λ, $(\lambda; <;) \vDash \phi$ iff there is a λ-Aronszajn tree), or from 4Kb, because if λ is weakly Π^1_1-indescribable there is no λ-Aronszajn tree (FREMLIN & KUNEN N87, 2N; compare KEISLER & TARSKI 64, 4.31).

5G Proposition If there is an atomlessly-measurable cardinal, there is no rapid p-point ultrafilter on $\mathbb{N}$.

proof Let κ be an atomlessly-measurable cardinal, with witnessing probability ν.

? Suppose, if possible, that $\mathcal{F}$ is a rapid p-point ultrafilter on $\mathbb{N}$. Let $\langle C_{kn} \rangle_{k,n\in\mathbb{N}}$ be a stochastically independent double sequence of subsets of κ with $\nu C_{kn} = \frac{1}{2}$ for all k, n. Set

$$\begin{aligned} C_k &= \lim_{n\to\mathcal{F}} C_{kn} \subseteq \kappa, \\ A_k &= \{(\xi, n) : \xi \in C_k \,\&\, \xi \in C_{kn} \ \textit{or} \ \xi \notin C_k \,\&\, \xi \notin C_{kn}\} \\ &\subseteq \kappa \times \mathbb{N} \end{aligned}$$

for each $k \in \mathbb{N}$. Then, for $k \in \mathbb{N}$, $\xi < \kappa$,

$$\begin{aligned} \{n : (\xi, n) \in A_k\} &= \{n : \xi \in C_{kn}\} \in \mathcal{F} \text{ if } \xi \in C_k, \\ &= \{n : \xi \notin C_{kn}\} \in \mathcal{F} \text{ if } \xi \notin C_k. \end{aligned}$$

Because $\mathcal{F}$ is a p-point ultrafilter, there is a set $A \subseteq \kappa \times \mathbb{N}$ such that

$$\{n : (\xi, n) \in A\} \in \mathcal{F},$$

$$\{n : (\xi, n) \in A \setminus A_k\} \text{ is finite}$$

for all $k \in \mathbb{N}$, $\xi < \kappa$.

Now observe that if we write

$$B_{kn} = \{\xi : \xi < \kappa, \, (\xi, n) \in A_i \ \forall \ i < k\},$$

then $\lim_{n\to\infty} \nu(D \cap B_{kn}) = 2^{-k}\nu D$ for every $k \in \mathbb{N}$, $D \subseteq \kappa$. **P** Induce on k. For $k = 0$, this is trivial. For the inductive step to $k + 1$,

$$\begin{aligned} B_{k+1,n} &= \{\xi : \xi \in B_{kn}, \, \xi \in C_k \iff \xi \in C_{kn}\} \\ &= (B_{kn} \cap C_k \cap C_{kn}) \cup ((B_{kn} \setminus C_k) \setminus C_{kn}). \end{aligned}$$

Now if we write $\mathfrak{A}_k$ for the subalgebra of $\mathcal{P}\kappa$ generated by $\{C_{ij} : i < k, \, j \in \mathbb{N}\}$, we see that for any $E \subseteq \kappa$, sequence $\langle D_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_k$ we have

$$\lim_{n\to\infty}\nu(E\cap D_n\cap C_{kn})=\tfrac{1}{2}\lim_{n\to\infty}\nu(E\cap D_n)$$

whenever the right-hand-side is defined, just because the C_{kn} are stochastically independent of each other and the D_n and all have measure $\frac{1}{2}$. (See A2N.) It follows that if we write $\mathfrak{A}'_k$ for the subalgebra of $\mathcal{P}\kappa$ generated by $\mathfrak{A}_k\cup\{C_i : i\le k\}\cup\{D\}$ then

$$\lim_{n\to\infty}\nu(D_n\cap C_{kn})=\tfrac{1}{2}\lim_{n\to\infty}\nu(D_n)$$

for every sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}'_k$ for which the right-hand limit exists. Applying this with $D_n=B_{kn}\cap D\cap C_k$ and with $D_n=B_{kn}\cap D\setminus C_k$ we get, for any $D\subseteq\kappa$,

$$\begin{aligned}\lim_{n\to\infty}\nu(B_{k+1,n}\cap D)&=\lim_{n\to\infty}\nu(B_{kn}\cap D\cap C_k\cap C_{kn})\\&\quad+\lim_{n\to\infty}\nu(B_{kn}\cap D\setminus C_k)\\&\quad-\lim_{n\to\infty}\nu((B_{kn}\cap D\setminus C_k)\cap C_{kn})\\&=\frac{1}{2}\lim_{n\to\infty}\nu(B_{kn}\cap D\cap C_k)+\frac{1}{2}\lim_{n\to\infty}\nu(B_{kn}\cap D\setminus C_k)\\&=\frac{1}{2}\lim_{n\to\infty}\nu(B_{kn}\cap D)=\frac{1}{2}2^{-k}\nu D=2^{-k-1}\nu D,\end{aligned}$$

as required. **Q** In particular, $\lim_{n\to\infty}\nu B_{kn}=2^{-k}$ for every $k\in\mathbb{N}$.

If we now set

$$B_n=\{\xi:(\xi,n)\in A\},$$

the set

$$\{n:n\in\mathbb{N},\ \xi\in B_n\setminus B_{kn}\}$$

is finite for every $\xi<\kappa$, $k\in\mathbb{N}$, and consequently

$$\lim_{n\to\infty}\nu(B_n\setminus B_{kn})=0\ \ \forall\ \ k\in\mathbb{N},$$

so that

$$\lim_{n\to\infty}\nu B_n=0.$$

Because $\mathcal{F}$ is rapid, there is an $F\in\mathcal{F}$ such that $\sum_{n\in F}\nu B_n<\infty$. There is therefore some $\xi<\kappa$ such that $F'=\{n:n\in F,\ \xi\in B_n\}$ is finite. But $F'=\{n:n\in F,\ (\xi,n)\in A\}\in\mathcal{F}$. **X**

Remark This is due to Kunen; compare JECH 78, §38.

5H Proposition Let (X,μ) and (Y,ν) be probability spaces and H a Hilbert space. Suppose that $x\mapsto u_x:X\to H$ and $y\mapsto v_y:Y\to H$ are bounded H-scalarly measurable functions. Then

$$\int\int(u_x|v_y)\mu(dx)\nu(dy)=\int\int(u_x|v_y)\nu(dy)\mu(dx)$$

(and both repeated integrals exist).

proof (a) Let $\mathcal{E}$ be the set of closed separable linear subspaces of H. For each $E\in\mathcal{E}$, let P_E be the orthogonal projection onto E. Observe that (because E is separable) the map $x\mapsto P_E(u_x):X\to E$ will be measurable in the sense of A2Ae for the second-countable norm topology on E, for every $E\in\mathcal{E}$.

(b) There is an $E\in\mathcal{E}$ such that

$$P_E(u_x)=P_F(u_x)\ \ \mu\text{-a.e. }(x)$$

whenever $E\subseteq F\in\mathcal{E}$. **P?** For suppose, if possible, otherwise. Then we can choose inductively an increasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{E}$ such that

$$\mu\{x:P_{E_{\xi+1}}(u_x)\ne P_{E_\xi}(u_x)\}>0\ \ \forall\ \ \xi<\omega_1,$$

$$E_\xi=\overline{\bigcup_{\eta<\xi}E_\eta}\text{ whenever }\xi\text{ is a non-zero countable limit ordinal.}$$

(The set of x for which $P_{E_{\xi+1}}(u_x)\ne P_{E_\xi}(u_x)$ is measurable because E_ξ and $E_{\xi+1}$ are separable, as remarked above.) Now there must be a rational $\delta>0$ such that

$$A = \{\xi : \xi < \omega_1,\, \mu U_\xi \geq \delta\}$$

is infinite, where

$$U_\xi = \{x : \|P_{E_{\xi+1}}(u_x) - P_{E_\xi}(u_x)\| \geq \delta\}$$

for each $\xi < \omega_1$. But in this case there must be an $x \in X$ such that

$$A' = \{\xi : \xi \in A,\, x \in U_\xi\}$$

is infinite (A2Mb). Let ζ be any cluster point of A' in ω_1. Then

$$P_{E_\zeta}(u_x) = \lim_{\xi\uparrow\zeta} P_{E_\xi}(u_x),$$

which is impossible. **XQ**

(c) Applying (b) to both $\langle u_x\rangle_{x\in X}$ and $\langle v_y\rangle_{y\in Y}$, we see that there is an $E \in \mathcal{E}$ such that

$$P_E(u_x) = P_F(u_x)\ \ \mu\text{-a.e.}(x),\ P_E(v_y) = P_F(v_y)\ \ \nu\text{-a.e.}(y)$$

whenever $E \subseteq F \in \mathcal{E}$. Now observe that if $y \in Y$ there is an $F \in \mathcal{E}$ such that $E \subseteq F$ and $v_y \in F$. So

$$\begin{aligned}\int (u_x|v_y)\mu(dx) &= \int (u_x|P_F(v_y))\mu(dx)\\ &= \int (P_F(u_x)|v_y)\mu(dx)\\ &= \int (P_E(u_x)|v_y)\mu(dx)\\ &= \int (P_E(u_x)|P_E(v_y))\mu(dx).\end{aligned}$$

This is true for every y. So

$$\int\int (u_x|v_y)\mu(dx)\nu(dy) = \int\int (P_E(u_x)|P_E(v_y))\mu(dx)\nu(dy).$$

Similarly

$$\int\int (u_x|v_y)\nu(dy)\mu(dx) = \int\int (P_E(u_x)|P_E(v_y))\nu(dy)\mu(dx).$$

But also, because $x \mapsto P_E(u_x)$ and $y \mapsto P_E(v_y)$ are measurable maps to the second-countable space E, and $(u,v) \mapsto (u|v) : E \times E \to \mathbb{R}$ is continuous, $(x,y) \mapsto (P_E(u_x)|P_F(v_y))$ is measurable for the product measure $\mu \times \nu$, and

$$\int\int (P_E(u_x)|P_E(v_y))\mu(dx)\nu(dy) = \int\int (P_E(u_x)|P_E(v_y))\nu(dy)\mu(dx)$$

by Fubini's theorem. Putting these together, we have the result.

5I Corollary Let (X,μ), (Y,ν) and (Z,Σ,σ) be probability spaces. Let $x \mapsto A_x : X \to \Sigma$ and $y \mapsto B_y : Y \to \Sigma$ be functions such that

$$x \mapsto \sigma(A_x \cap C),\ y \mapsto \sigma(B_y \cap C)$$

are measurable for every $C \in \Sigma$. Then

$$\int\int \sigma(A_x \cap B_y)\mu(dx)\nu(dy) = \int\int \sigma(A_x \cap B_y)\nu(dy)\mu(dx).$$

proof Apply 5H with $H = L^2(\sigma)$, $u_x = \chi(A_x)^\bullet$ (the equivalence class in L^2 of the characteristic function of A_x), $v_y = \chi(B_y)^\bullet$.

5J Theorem Let κ be a real-valued-measurable cardinal. For cardinals $\lambda \leq \kappa$ let Σ_λ be the smallest subalgebra of $\mathcal{P}(\lambda \times \lambda)$ containing all sets of the form $E \times F$, where $E,\, F \subseteq \lambda$, and closed under unions of fewer than λ of its members. Set $D_\lambda = \{(\eta,\zeta) : \eta \leq \zeta < \lambda\} \subseteq \lambda\times\lambda$. Then $\{\lambda : \lambda < \kappa$ is a cardinal, $D_\lambda \in \Sigma_\lambda\}$ belongs to the rvm ideal of κ, and $D_\kappa \notin \Sigma_\kappa$.

proof The following notation will be useful: If θ is a cardinal and Z is a set, a **θ-subalgebra** of $\mathcal{P}Z$ will be a subalgebra Σ of $\mathcal{P}Z$ such that $\bigcup\mathcal{A} \in \Sigma$ whenever $\mathcal{A} \subseteq \Sigma$ and $\#(\mathcal{A}) \leq \theta$.

Set

$$A=\{\lambda:\lambda<\kappa \text{ is a cardinal}, D_\lambda\in\Sigma_\lambda\}.$$

(a) ? Suppose, if possible, that A is not in the rvm ideal of κ. Let ν be a normal witnessing probability on κ such that $\nu A=1$. Set

Set

$$A_1=\{\lambda:\lambda\in A,\ \lambda \text{ is uncountable and regular}\};$$

then $\nu A_1=\nu A=1$, by 4Ka.

For cardinals λ, θ let $\Sigma_{\lambda\theta}$ be the smallest θ-subalgebra of $\mathcal{P}(\lambda\times\lambda)$ containing all rectangles $E\times F$ for E, $F\subseteq\lambda$. If $\lambda\in A_1$ it is regular, so $\Sigma_\lambda=\bigcup_{\theta<\lambda}\Sigma_{\lambda\theta}$, and there is an infinite $\theta<\lambda$ such that $D_\lambda\in\Sigma_{\lambda\theta}$. Because ν is normal, there is a $\theta<\kappa$ such that $\nu A_2=1$ where

$$A_2=\{\lambda:\lambda\in A_1,\ D_\lambda\in\Sigma_{\lambda\theta}\}.$$

(b) For each $\lambda\in A_2$ there is a family $\langle E_{\lambda\xi}\rangle_{\xi<\theta}$ of subsets of λ such that D_λ belongs to the smallest θ-subalgebra Σ^*_λ of $\mathcal{P}(\lambda\times\lambda)$ which contains $E_{\lambda\xi}\times E_{\lambda\eta}$ for all ξ, $\eta<\theta$. (For the union of all such subalgebras Σ^*_λ is a θ-subalgebra of $\mathcal{P}(\lambda\times\lambda)$ and must be $\Sigma_{\lambda\theta}$.) Set $X=\{0,1\}^\theta$ and define $f_\lambda:\lambda\to X$ by setting

$$f_\lambda(\eta)(\xi)=1 \text{ if } \eta\in E_{\lambda\xi},\ 0 \text{ otherwise}.$$

Let Σ be the smallest θ-subalgebra of $\mathcal{P}(X\times X)$ containing all the open-and-closed sets. Then every member of Σ^*_λ is of the form

$$\{(\eta,\zeta):(f_\lambda(\eta),f_\lambda(\zeta))\in R\}$$

for some $R\in\Sigma$ (because sets of this form comprise a θ-subalgebra of $\mathcal{P}(\lambda\times\lambda)$ containing all the sets $E_{\lambda\xi}\times E_{\lambda\eta}$); in particular, there is a set $R_\lambda\in\Sigma$ such that

$$D_\lambda=\{(\eta,\zeta):(f_\lambda(\eta),f_\lambda(\zeta))\in R_\lambda\}.$$

(c) Let T be the smallest θ^+-subalgebra of $\mathcal{P}(\kappa\times X\times X)$ containing all sets of the form $B\times U\times V$, where $B\subseteq\kappa$ and U, $V\subseteq X$. Then

$$\{(\lambda,x,y):\lambda\in A_2,\ (x,y)\in R_\lambda\}$$

belongs to T. **P** Define $\mathcal{H}_\gamma$ inductively, for ordinals $\gamma\le\theta^+$, by taking

$\mathcal{H}_0$ to be the family of open-and-closed subsets of $X\times X$,

if $\gamma<\theta^+$ is an even ordinal then

$$\mathcal{H}_{\gamma+1}=\{\textstyle\bigcup_{\xi<\theta}H_\xi:H_\xi\in\mathcal{H}_\gamma\ \ \forall\ \xi<\theta\};$$

if $\gamma<\theta^+$ is an odd ordinal then

$$\mathcal{H}_{\gamma+1}=\{(X\times X)\setminus H:H\in\mathcal{H}_\gamma\};$$

if $\gamma\le\theta^+$ is a non-zero limit ordinal then

$$\mathcal{H}_\gamma=\textstyle\bigcup_{\delta<\gamma}\mathcal{H}_\delta.$$

Every $\mathcal{H}_\gamma$ is closed under finite unions and intersections, and $\mathcal{H}_\gamma\subseteq\mathcal{H}_{\gamma+2}\cap\mathcal{H}_{\gamma+3}$, so $\mathcal{H}_\gamma$ is a subalgebra of $\mathcal{P}(X\times X)$ for limit ordinals γ, and $\mathcal{H}_{\theta^+}=\Sigma$.

Next, an easy induction on γ shows that if $H_\xi\in\mathcal{H}_\gamma$ for every $\xi<\kappa$ then $\{(\xi,x,y):(x,y)\in H_\xi\}\in\mathrm{T}$. The induction starts with the observation that any open-and-closed set $H\subseteq X\times X$ is a finite union of rectangles, so that $B\times H\in\mathrm{T}$ for every $B\subseteq\kappa$. Now $\mathcal{H}_0$ has cardinal θ, so if $H_\xi\in\mathcal{H}_0$ for every $\xi<\kappa$ we get

$$\{(\xi,x,y):(x,y)\in H_\xi\}=\textstyle\bigcup_{H\in\mathcal{H}_0}\{\xi:H_\xi=H\}\times H\in\mathrm{T}.$$

For the inductive step to $\gamma+1$, where γ is even, we have a family $\langle H_\xi\rangle_{\xi<\kappa}$ such that each H_ξ is expressible as $\bigcup_{\eta<\theta}H_{\xi\eta}$ with every $H_{\xi\eta}\in\mathcal{H}_\gamma$; now

$$\{(\xi,x,y):(x,y)\in H_\xi\}=\textstyle\bigcup_{\eta<\theta}\{(\xi,x,y):(x,y)\in H_{\xi\eta}\}\in\mathrm{T},$$

using the inductive hypothesis and the fact that T is a θ-subalgebra. Similarly, the inductive step to $\gamma+1$, where γ is odd, needs only the fact that T is closed under complements. Finally, for the inductive step to a limit $\gamma \leq \theta^+$, we have a family $\langle H_\xi \rangle_{\xi<\kappa} \in \mathcal{H}_\gamma = \bigcup_{\delta<\gamma} \mathcal{H}_\delta$. Set $H_{\delta\xi} = H_\xi$ if $H_\xi \in \mathcal{H}_\delta$, $\emptyset$ otherwise; then

$$\{(\xi, x, y) : (x, y) \in H_\xi\} = \bigcup_{\delta<\gamma} \{(\xi, x, y) : (x, y) \in H_{\delta\xi}\} \in \mathrm{T},$$

using the inductive hypothesis and the fact that T is a θ^+-subalgebra.

Observing that $R_\lambda \in \Sigma = \mathcal{H}_{\theta^+}$ for every $\lambda \in A_2$, we have the result. **Q**

(d) Define two probabilities ϕ, ψ on $\mathcal{P}(\kappa \times X \times X)$ by setting

$$\phi(W) = \int \int \nu\{\lambda : \lambda \in A_2,\ \lambda > \max(\eta, \zeta),\ (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W\} \nu(d\eta)\nu(d\zeta),$$

$$\psi(W) = \int \int \nu\{\lambda : \lambda \in A_2,\ \lambda > \max(\eta, \zeta),\ (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W\} \nu(d\zeta)\nu(d\eta)$$

for all $W \subseteq \kappa \times X \times X$. Then both ϕ and ψ are κ-additive, and $\phi(\kappa \times X \times X) = \psi(\kappa \times X \times X) = 1$. Set

$$\mathcal{S} = \{S : S \subseteq \kappa \times X \times X,\ \phi(S) = \psi(S)\}.$$

Then $\mathcal{S}$ is closed under monotonic and disjoint unions of length less than κ, and also under complements.

(e) The key to the proof is the following fact: If $B \subseteq \kappa$ and U, $V \subseteq X$ then $B \times U \times V \in \mathcal{S}$. **P** Substituting in the formulae above, we get

$$\phi(B \times U \times V) = \int \int \nu(U_\eta \cap V_\zeta)\nu(d\eta)\nu(d\zeta),$$

$$\psi(B \times U \times V) = \int \int \nu(U_\eta \cap V_\zeta)\nu(d\zeta)\nu(d\eta),$$

where

$$U_\eta = \{\lambda : \lambda \in B \cap A_2,\ \eta < \lambda,\ f_\lambda(\eta) \in U\},$$

$$V_\zeta = \{\lambda : \lambda \in A_2,\ \zeta < \lambda,\ f_\lambda(\zeta) \in V\}$$

for η, $\zeta < \kappa$. Now the result is immediate from 5I. **Q**

(f) It follows that $\mathrm{T} \subseteq \mathcal{S}$. **P** Define $\mathcal{S}_\gamma$ inductively, for ordinals $\gamma \leq \theta^{++}$, as follows. Take $\mathcal{S}_0$ to be the subalgebra of $\mathcal{P}(\kappa \times X \times X)$ generated by $\{B \times U \times V : B \subseteq \kappa,\ U \subseteq X,\ V \subseteq X\}$. Then each member of $\mathcal{S}_0$ is a finite disjoint union of cuboids $B \times U \times V$, which by (e) above all belong to $\mathcal{S}$, and $\mathcal{S}_0 \subseteq \mathcal{S}$. Given $\mathcal{S}_\gamma$, for an even ordinal γ, set

$$\mathcal{S}_{\gamma+1} = \{\bigcup_{\xi<\alpha} S_\xi : \alpha \leq \theta^+,\ \langle S_\xi \rangle_{\xi<\alpha} \text{ is an increasing family in } \mathcal{S}_\gamma\};$$

for odd ordinals γ set

$$\mathcal{S}_{\gamma+1} = \{(\kappa \times X \times X) \setminus S : S \in \mathcal{S}_\gamma\};$$

for non-zero limit ordinals γ set

$$\mathcal{S}_\gamma = \bigcup_{\delta<\gamma} \mathcal{S}_\delta.$$

Of course every $\mathcal{S}_\gamma$ is included in $\mathcal{S}$, because $\theta^+ < \kappa$. Next, every $\mathcal{S}_\gamma$ is closed under finite unions and intersections, so $\mathcal{S}_\gamma$ is a subalgebra of $\mathcal{P}(\kappa \times X \times X)$ for every limit ordinal γ; and finally $\mathcal{S}_{\theta^{++}}$ is a subalgebra of $\mathcal{P}(\kappa \times X \times X)$ such that $\bigcup_{\xi<\alpha} S_\xi \in \mathcal{S}_{\theta^{++}}$ whenever $\alpha \leq \theta^+$ and $\langle S_\xi \rangle_{\xi<\alpha}$ is an increasing family in $\mathcal{S}_{\theta^{++}}$. Inducing on α we see that $\bigcup_{\xi<\alpha} S_\xi \in \mathcal{S}_{\theta^{++}}$ whenever $\alpha \leq \theta^+$ and $\langle S_\xi \rangle_{\xi<\alpha}$ is any family in $\mathcal{S}_{\theta^{++}}$; so that $\mathcal{S}_{\theta^{++}}$ is a θ^+-subalgebra of $\mathcal{P}(\kappa \times X \times X)$, and

$$\mathrm{T} \subseteq \mathcal{S}_{\theta^{++}} \subseteq \mathcal{S},$$

as required. **Q**

(g) Putting (c) and (f) together,

$$W^* = \{(\lambda, x, y) : \lambda \in A_2,\ (x, y) \in R_\lambda\}$$

belongs to $\mathcal{S}$, and $\phi(W^*) = \psi(W^*)$. But if we set out to compute these numbers we have

$$\nu\{\lambda : \lambda \in A_2,\, \lambda > \max(\eta,\zeta),\, (\lambda, f_\lambda(\eta), f_\lambda(\zeta)) \in W^*\}$$
$$= \nu\{\lambda : \lambda \in A_2,\, \lambda > \max(\eta,\zeta),\, (f_\lambda(\eta), f_\lambda(\zeta)) \in R_\lambda\}$$
$$= \nu\{\lambda : \lambda \in A_2,\, \lambda > \max(\eta,\zeta),\, (\eta,\zeta) \in D_\lambda\}$$
$$= 1 \text{ if } \eta \le \zeta, = 0 \text{ otherwise.}$$

So $\phi(W^*) = 0$ and $\psi(W^*) = \nu A_2 = 1$; which is impossible. **X**

(h) This contradiction shows that A belongs to the rvm ideal of κ, which is the first part of the theorem. But the second part now follows. **?** For if $D_\kappa \in \Sigma_\kappa$ then (in the notation of (a)) there is a $\theta < \kappa$ such that $D_\kappa \in \Sigma_{\kappa\theta}$; now as $D_\lambda = D_\kappa \cap (\lambda \times \lambda)$ and $\Sigma_{\lambda\theta} = \{E \cap (\lambda \times \lambda) : E \in \Sigma_{\kappa\theta}\}$ for every $\lambda < \kappa$, we get $D_\lambda \in \Sigma_\lambda$ whenever $\theta < \lambda < \kappa$, which cannot be so. **X** Thus $D_\kappa \notin \Sigma_\kappa$ and we are done.

5K Corollary Let κ be a real-valued-measurable cardinal. Then there is a cardinal $\lambda < \kappa$ such that there is no family $\langle a_\eta \rangle_{\eta<\lambda}$ in $\mathcal{P}\mathbb{N}$ for which $a_\eta \setminus a_\zeta$ is finite for $\eta \le \zeta$, infinite for $\eta \ge \zeta$.

proof For if λ is an uncountable cardinal and $\langle a_\eta \rangle_{\eta<\lambda}$ is such a family, set $B_n = \{\eta : n \in a_\eta\} \subseteq \lambda$ for each $n \in \mathbb{N}$. Then, in the notation of 5J,

$$D_\lambda = \bigcup_{n\in\mathbb{N}} \bigcap_{m\ge n} ((\lambda \times B_m) \cup ((\lambda \setminus B_m) \times \lambda)) \in \Sigma_\lambda.$$

So we need only to take an uncountable $\lambda < \kappa$ such that $D_\lambda \notin \Sigma_\lambda$, as provided in abundance by 5J.

5L Remarks (a) 5J-5K are due to Kunen. A version of his original proof (adapted to 5K) is given in FREMLIN & KUNEN 91. Seeking a non-forcing alternative he and I independently (late 1989) devised forms of the argument above. I do not know whether 5H was known earlier.

(b) Note that the cardinal λ of 5K also has the property that there can be no family $\langle f_\xi \rangle_{\xi<\lambda}$ in $\mathbb{N}^\mathbb{N}$ such that $\{n : f_\eta(n) < f_\xi(n)\}$ is finite whenever $\xi \le \eta < \lambda$, infinite when $\xi > \eta$; apply 5K with $a_\xi = \{(n,i) : i \le f_\xi(n)\} \subseteq \mathbb{N} \times \mathbb{N}$ for each $\xi < \lambda$. See FREMLIN & KUNEN 91.

Evidently there are many further results along these lines concerning increasing families in ordered topological spaces of small weight when the ordering has a simple relation to the topology.

(c) In the same way, again taking λ from 5K, there is no $p(\lambda)$-point ultrafilter on $\mathbb{N}$.

Version of 18.9.92

5M Proposition Let κ be a real-valued-measurable cardinal with normal witnessing probability ν, and X any set. Let $\langle f_\xi \rangle_{\xi<\kappa}$ be any family in X^κ such that for every countable $I \subseteq \kappa$ there is a $g \in X^I$ such that $\nu\{\xi : f_\xi \restriction I = g\} > 0$. Then there is an $h \in X^\kappa$ such that $\nu\{\xi : \xi < \kappa,\, f_\xi \restriction \xi = h \restriction \xi\} > 0$.

proof (a) Set

$$G_0 = \bigcup\{X^I : I \in [\kappa]^{\le\omega}\}.$$

For $g \in G_0$, set $E(g) = \{\xi : f_\xi \supseteq g\}$; write

$$G = \{g : g \in G_0,\, \nu E(g) > 0\}.$$

(b) There is a $g \in G$ such that for every $\eta < \kappa$ there is an $x \in X$ such that $E(g) \setminus E(g \cup \{(\eta, x)\}) \in \mathcal{N}_\nu$. **P?** If not, choose $\langle I_\alpha \rangle_{\alpha<\omega_1}$, $\langle H_\alpha \rangle_{\alpha<\omega_1}$ as follows. Start with $I_0 = \emptyset$. Given that I_α is countable, set

$$H_\alpha = \{g : g \in G,\, \mathrm{dom}(g) = I_\alpha\}.$$

Then H_α is non-empty, by the hypothesis of the lemma, and countable, because $E(g) \cap E(g') = \emptyset$ for distinct g, $g' \in H_\alpha$. Now there must be a countable set $I_{\alpha+1} \supseteq I_\alpha$ such that

$$\forall\ g \in H_\alpha\ \exists\ \eta \in I_{\alpha+1} \text{ such that } E(g) \setminus E(g \cup \{(\eta, x)\}) \notin \mathcal{N}_\nu \text{ for every } x \in X.$$

For limit ordinals $\alpha < \omega_1$ set $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$.

For each $\alpha < \omega_1$, take $g_\alpha \in H_\alpha$. By A2Mc there is a $\zeta < \kappa$ such that $D = \{\alpha : \zeta \in E(g_\alpha)\}$ is uncountable. Now for each $\alpha \in D$ there is some $\eta_\alpha \in I_{\alpha+1}$ such that

$$F_\alpha = E(g_\alpha) \setminus E(g_\alpha \cup \{(\eta_\alpha, f_\zeta(\eta_\alpha))\}) \notin \mathcal{N}_\nu.$$

However, if $\beta \in D$ and $\beta > \alpha$, then $g_\beta \restriction I_{\alpha+1} \subseteq g_\beta \restriction I_\beta \subseteq f_\zeta$, so $F_\beta \cap F_\alpha = \emptyset$. Thus $\langle F_\alpha \rangle_{\alpha \in D}$ is an uncountable disjoint family of non-negligible sets, which is impossible. **XQ**

(c) Now for any $\eta < \kappa$ choose $h(\eta) \in X$ such that $F_\eta = E(g) \setminus E(g \cup \{(\eta, h(\eta))\}) \in \mathcal{N}_\nu$. Then we have

$$F = \{\xi : \exists\ \eta < \xi,\ \xi \in F_\eta\} \in \mathcal{N}_\nu,$$

because ν is normal. Set $E = E(g) \setminus F$; then $\nu E > 0$ and $f_\xi \restriction \xi = h \restriction \xi$ for every $\xi \in E$.

5N Theorem Suppose that $\mathfrak{c}$ is atomlessly-measurable. Then $\diamondsuit_{\mathfrak{c}}$ is true, that is, there is a family $\langle A_\xi \rangle_{\xi < \mathfrak{c}}$ such that $\{\xi : A \cap \xi = A_\xi\}$ is stationary in $\mathfrak{c}$ for every $A \subseteq \mathfrak{c}$.

proof (a) Let ν be a normal witnessing probability on $\mathfrak{c}$. Let $\langle p_\xi \rangle_{\xi < \mathfrak{c}}$, $\langle I_\xi \rangle_{\xi < \mathfrak{c}}$ be enumerations of $\bigcup\{\{0,1\}^I : I \in [\mathfrak{c}]^{\le\omega}\}$, $[\mathfrak{c}]^{\le\omega}$ respectively. Define inductively a family $\langle f_\xi \rangle_{\xi < \mathfrak{c}}$ as follows. Given $\langle f_\eta \rangle_{\eta < \xi}$, let F_ξ be the set of those functions $f : \xi \to \{0,1\}$ such that

(i) $f \restriction \eta \ne f_\eta$ for every non-zero limit ordinal $\eta < \xi$,

(ii) for every $\eta < \xi$ there is a $\zeta < \xi$ such that $f \restriction I_\eta = p_\zeta$.

Now if $F_\xi \ne \emptyset$ take $f_\xi \in F_\xi$; otherwise take f_ξ to be any function from ξ to $\{0,1\}$. Set $A_\xi = f_\xi^{-1}[\{1\}]$ for each $\xi < \mathfrak{c}$.

(b) If $A \subseteq \mathfrak{c}$ is any set, then there is a non-zero limit $\xi < \mathfrak{c}$ such that $A \cap \xi = A_\xi$. **P?** Suppose, if possible, otherwise. Set $g(\xi) = 1$ for $\xi \in A$, 0 for $\xi \in \mathfrak{c} \setminus A$. Let C be the set of non-zero limit ordinals $\xi < \mathfrak{c}$ such that for every $\eta < \xi$ there is a $\zeta < \xi$ such that $g \restriction I_\eta \cap \xi = p_\zeta$. Then C is a closed unbounded set in $\mathfrak{c}$, and $g \restriction \xi \in F_\xi$ for every $\xi \in C$, so $f_\xi \in F_\xi$ for every $\xi \in C$.

For any set $J = \{\xi, \eta\} \in [C]^2$, set $h(J) = \min\{\zeta : f_\xi(\zeta) \ne f_\eta(\zeta)\}$; then $h(J) < \min J$ because if $\eta < \xi$ then $f_\xi \restriction \eta \ne f_\eta$. By 5Cc above there are $D \subseteq C$, $M \subseteq \mathfrak{c}$ such that $\nu D = 1$, M is countable and $h(J) \in M$ for all $J \in [D]^2$. Let $\eta < \mathfrak{c}$ be such that $M = I_\eta$; set $B = \{\xi : M \cup \{\eta\} \subseteq \xi \in D\}$, so that $\nu B = 1$. Then for every $\xi \in B$ we shall have a $\zeta(\xi) < \xi$ such that $p_{\zeta(\xi)} = f_\xi \restriction M$. But if ξ, ξ' are distinct members of B we must have $h(\{\xi, \xi'\}) \in M$ so $f_\xi \restriction M \ne f_{\xi'} \restriction M$; thus $\xi \mapsto \zeta(\xi) : B \to \kappa$ is an injective regressive function, which is impossible. **XQ**

(c) Of course the family $\langle A_\xi \rangle_{\xi < \mathfrak{c}}$ constructed in (a) is not necessarily a true $\diamondsuit$-sequence as called for in the statement of the theorem. But if we set $A'_\xi = \{\eta : 2\eta \in A_\xi\}$ for each ξ, we obtain such a sequence (see Devlin 84, Ex. III.3A).

Remark 5M-5N are due to Kunen. Of course $\diamondsuit_\kappa$ is true for every two-valued-measurable cardinal κ, by the same argument. See also 9N.

5O Theorem Let κ be a real-valued-measurable cardinal. Then

(a) $\kappa \to (\kappa, \gamma)^2$ for every ordinal $\gamma < \omega_1$;

(b) the set

$$\{\alpha : \alpha < \kappa,\ \alpha \to (\alpha, \gamma)^2 \text{ for every countable ordinal } \gamma\}$$

belongs to the rvm filter of κ.

Notation See A1S for the definition of '$\alpha \to (\alpha, \beta)^2$'.

proof (a) Let γ be a countable ordinal and $S \subseteq [\kappa]^2$ any set. Suppose that there is no set $B \subseteq \kappa$ of order type κ such that $[B]^2 \cap S = \emptyset$; I seek a set C of order type γ such that $[C]^2 \subseteq S$. For each $\xi < \kappa$ set $S_\xi = \{\eta : \xi < \eta < \kappa,\ \{\xi, \eta\} \in S\}$. Fix a normal witnessing probability ν on κ.

(i) If $E \subseteq \kappa$ then $\nu(E \cap S_\xi) > 0$ for ν-almost every $\xi \in E$. **P** Set

$$E' = \{\xi : \xi \in E,\ \nu(E \cap S_\xi) = 0\},$$

$$A = \{\xi : \xi < \kappa,\ \xi \notin E \cap S_\eta\ \ \forall\ \eta \in \xi \cap E'\};$$

then $\nu A = 1$ (because ν is normal), and also $[A \cap E']^2 \cap S = \emptyset$, so $\nu(A \cap E') = 0$ and $\nu E' = 0$. **Q**

(ii) For each $n \ge 1$ define a κ-additive probability ν_n on κ^n by setting

$$\nu_n V = \int \ldots \int \chi V(\xi_1, \ldots, \xi_n) \nu(d\xi_n) \ldots \nu(d\xi_1)$$

for every $V \subseteq \kappa^n$. Observe that if $V \subseteq \kappa^{n+1}$ then

$$\nu_{n+1}V = \int \nu_n\{t : \xi^\frown t \in V\}\nu(d\xi),$$

writing $\xi^\frown(\xi_1, \dots, \xi_n) = (\xi, \xi_1, \dots, \xi_n)$.

(iii) (The key) If D, $F \subseteq \kappa$ and $n \ge 1$ and

$$\nu_{n+1}(S_\xi^{n+1} \setminus \bigcup_{\eta \in D} S_\eta^{n+1}\}) = 0 \ \forall \ \xi \in F,$$

then for ν-almost every $\xi \in F$

$$\nu_n(S_\xi^n \setminus \bigcup\{S_\eta^n : \eta \in D, \xi \in S_\eta\}) = 0.$$

P? If not, there is a $\delta > 0$ such that $\nu E > 0$, where

$$E = \{\xi : \xi \in F, \nu_n(S_\xi^n \setminus \bigcup\{S_\eta^n : \eta \in D, \xi \in S_\eta\}) \ge \delta\}.$$

Choose a sequence $\langle \xi_i \rangle_{i \in \mathbb{N}}$ as follows. Start with any $\xi_0 \in E$ such that $\nu(E \cap S_{\xi_0}) > 0$; such exists by (i) above. Given $\xi_0, \dots, \xi_k \in F$ such that $\nu(E \cap \bigcap_{i \le k} S_{\xi_i}) > 0$, then

$$\nu_{n+1}(\bigcup_{i \le k} S_{\xi_i}^{n+1} \setminus \bigcup_{\eta \in D} S_\eta^{n+1}) = 0,$$

so for ν-almost all ξ

$$\nu_n(\bigcup\{S_{\xi_i}^n : i \le k, \xi \in S_{\xi_i}\} \setminus \bigcup\{S_\eta^n : \eta \in D, \xi \in S_\eta\}) = 0.$$

Now for ν-almost all $\xi \in \bigcap_{i \le k} S_{\xi_i}$,

$$\nu_n(\bigcup_{i \le k} S_{\xi_i}^n \setminus \bigcup\{S_\eta^n : \eta \in D, \xi \in S_\eta\}) = 0,$$

and for ν-almost all $\xi \in E \cap \bigcap_{i \le k} S_{\xi_i}$ we have

$$\nu_n(S_\xi^n \setminus \bigcup_{i \le k} S_{\xi_i}^n) \ge \delta.$$

We can therefore find a $\xi_{k+1} \in E \cap \bigcap_{i \le k} S_{\xi_i}$ such that

$$\nu(S_{\xi_{k+1}} \cap E \cap \bigcap_{i \le k} S_{\xi_i}) > 0,$$

$$\nu_n(S_{\xi_{k+1}}^n \setminus \bigcup_{i \le k} S_{\xi_i}^n) \ge \delta$$

(using (i) again). Continue.

But we now have a disjoint sequence $\langle S_{\xi_{k+1}}^n \setminus \bigcup_{i \le k} S_{\xi_i}^n \rangle_{i \in \mathbb{N}}$ of subsets of κ^n all of measure at least δ, which is impossible. **XQ**

(iv) If $n \ge 1$ and $V \subseteq \kappa^n$ is ν_n-negligible, then there is a set $A \subseteq \kappa$ such that $\nu A = 1$ and $(\xi_1, \dots, \xi_n) \notin V$ whenever $\xi_1, \dots, \xi_n \in A$ and $\xi_1 < \dots < \xi_n$. **P** Induce on n. For $n = 1$ we may take $A = \kappa \setminus V$. For the inductive step to $n + 1 \ge 2$, set

$$E = \{\xi : \nu_n\{t : \xi^\frown t \in V\} > 0\},$$

so that $\nu E = 0$. For each $\xi \in \kappa \setminus E$, set $V_\xi = \{t : \xi^\frown t \in V\}$; then $\nu_n V_\xi = 0$ so by the inductive hypothesis there is a set $A_\xi \subseteq \kappa$ such that $\nu A_\xi = 1$ and no strictly increasing sequence in A_ξ can belong to V_ξ. Set

$$A = \{\xi : \xi \in \kappa \setminus E, \xi \in A_\eta \ \forall \ \eta \in \xi \setminus E\}.$$

Then $\nu A = 1$ and no strictly increasing family in A can belong to V. **Q**

(v) For each $I \in [\kappa]^{<\omega}$ set

$$R_I = \kappa \cap \bigcap_{\xi \in I} S_\xi,\ R'_I = \{\xi : \xi \in R_I, \nu(R_I \cap S_\xi) = 0\},$$

so that $\nu R'_I = 0$, by (i) once again. Set

$$A = \{\xi : \xi < \kappa, \xi \notin R'_I \ \forall \ I \in [\xi]^{<\omega}\};$$

then $\nu A = 1$ (see A1E(c-iv)). And an easy induction on $\sup I$ shows that $\nu R_I > 0$ whenever $I \in [A]^{<\omega}$ and $R_I \neq \emptyset$.

(vi) Choose $\langle A_\zeta\rangle_{\zeta<\omega_1}$, $\langle D_\zeta\rangle_{\zeta<\omega_1}$ as follows. $A_0 = A$, as defined in (v) just above. Given A_ζ, $\langle D_\eta\rangle_{\eta<\zeta}$ such that $\nu A_\zeta = 1$ and each D_η is a countable subset of $\alpha_\zeta = \min A_\zeta$, set $D'_\zeta = \bigcup_{\eta<\zeta} D_\eta$ and for $I \in [D'_\zeta]^{<\omega}$ set $F_{\zeta I} = A_\zeta \cap R_I$. Let $D_\zeta \subseteq A_\zeta$ be a countable set such that

$$\nu_n(\bigcup_{\xi\in F_{\zeta I}\cap D_\zeta} S^n_\xi) = \sup\{\nu_n(\bigcup_{\xi\in D} S^n_\xi) : D \subseteq F_{\zeta I} \text{ is countable}\}$$

for every $I \in [D'_\zeta]^{<\omega}$, $n \geq 1$. Then we shall have

$$\nu_n(S^n_\xi \setminus \bigcup_{\eta\in D_\zeta\cap F_{\zeta I}} S^n_\eta) = 0$$

whenever $n \geq 1$, $I \in [D'_\zeta]^{<\omega}$ and $\xi \in F_{\zeta I}$. Consequently, by (iii) above, if $I \in [D'_\zeta]^{<\omega}$ and $n \geq 1$, then

$$\nu_n(S^n_\xi \setminus \bigcup\{S^n_\eta : \eta \in D_\zeta \cap F_{\zeta I},\ \xi \in S_\eta\}) = 0$$

for almost every $\xi \in F_{\zeta I}$. Using (iv) we can now find a set $A_{\zeta+1} \subseteq A_\zeta$ such that

$$\nu A_{\zeta+1} = 1,$$

$$D_\zeta \subseteq \min A_{\zeta+1},$$

$$\nu_n(S^n_\xi \setminus \bigcup\{S^n_\eta : \eta \in D_\zeta \cap F_{\zeta I},\ \xi \in S_\eta\}) = 0$$

whenever $I \in [D'_\zeta]^{<\omega}$, $n \geq 1$, $\xi \in F_{\zeta I} \cap A_{\zeta+1}$,

$$(\xi_1, \dots, \xi_n) \notin S^n_{\xi_0} \setminus \bigcup\{S^n_\eta : \eta \in F_{\zeta I} \cap D_\zeta,\ \xi_0 \in S_\eta\}$$

whenever $I \in [D'_\zeta]^{<\omega}$, $n \geq 1$ and $\xi_0, \dots, \xi_n$ is a strictly increasing family in $A_{\zeta+1}$, with $\xi_0 \in F_{\zeta I}$.

This deals with D_ζ, $A_{\zeta+1}$. For non-zero countable limit ordinals ζ set $A_\zeta = \bigcap_{\eta<\zeta} A_\eta$.

(vii) On completing the induction set $D^* = \bigcup_{\zeta<\omega_1} D_\zeta$; observe that $D_\zeta = D^* \cap \alpha_{\zeta+1} \setminus \alpha_\zeta$ and that $D^* \setminus \alpha_\zeta \subseteq A_\zeta$ for each ζ. Now suppose that $J \in [D^*]^{<\omega}$ is such that $[J]^2 \subseteq S$. Then for any $\zeta < \omega_1$ there is an $\eta \in D_\zeta$ such that $[J \cup \{\eta\}]^2 \subseteq S$. **P** Of course we may suppose that $J \cap D_\zeta = \emptyset$. Moreover, because $J \subseteq A$, $\nu R_J > 0$; so if $J \subseteq D'_{\zeta'}$ then $\nu F_{\zeta' J} > 0$ so there is an $\eta \in F_{\zeta' J}$ with $\nu S_\eta > 0$ and there must be an $\eta' \in D_{\zeta'} \cap F_{\zeta' J}$, in which case $[J \cup \{\eta'\}]^2 \subseteq S$. We may therefore suppose that $J \setminus \alpha_\zeta = J \setminus \alpha_{\zeta+1}$ has at least two members. Set $I = J \cap \alpha_\zeta$ and enumerate $J \setminus I$ in ascending order as $\langle\xi_i\rangle_{i\leq n}$, where $n \geq 1$ and $\xi_i \in A_{\zeta+1}$ for each i. Now $\xi_0 \in F_{\zeta I}$ and

$$(\xi_1, \dots, \xi_n) \notin S^n_{\xi_0} \setminus \bigcup\{S^n_\eta : \eta \in F_{\zeta I} \cap D_\zeta,\ \xi_0 \in S_\eta\}.$$

But as certainly $(\xi_1, \dots, \xi_n) \in S^n_{\xi_0}$, there must be some $\eta \in F_{\zeta I} \cap D_\zeta$ such that $(\xi_0, \dots, \xi_n) \in S^n_\eta$; and this means that $[J \cup \{\eta\}]^2 \subseteq S$, as required. **Q**

(viii) Now, at last, turn to look at γ. Enumerate it as $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ (the case of finite γ is trivial). Choose $\langle\xi_n\rangle_{n\in\mathbb{N}}$ inductively so that $\xi_n \in D_{\zeta_n}$ and $[\{\xi_i : i \leq n\}]^2 \subseteq S$ for each n. Set $C = \{\xi_n : n \in \mathbb{N}\}$; then $[C]^2 \subseteq S$ and the order type of C is γ.

As S and γ are arbitrary, this proves (a).

(b) This now follows. The statement

$$\text{`}\alpha \to (\alpha, \gamma)^2\ \ \forall\ \ \gamma < \omega_1\text{'}$$

can readily be expressed in the form

$$\text{`}(\alpha; <, =; \omega_1) \vDash \phi\text{'},$$

where ϕ is a Π^1_2 formula. So 4Ob gives the result.

5P Remarks This is due to Kunen; I heard of it first from S.Todorčević.

For two-valued-measurable cardinals enormously more can be said. In fact, if κ is two-valued-measurable, then $\kappa \to (\kappa, \kappa)^2$ and $\{\alpha : \alpha < \kappa,\ \alpha \to (\alpha, \alpha)^2\}$ belongs to the rvm filter of κ; this is because, for $\alpha > \omega$, $\alpha \to (\alpha, \alpha)^2$ iff α is a strongly inaccessible weakly Π^1_1-indescribable cardinal (Erdös Hajnal Máté & Rado, 30.3).

I do not know whether (b) above can be strengthened by describing more exactly those $\alpha < \kappa$ for which $\alpha \to (\alpha, \gamma)^2$ for every $\gamma < \omega_1$.

6. Measure-theoretic implications.

In this section I discuss the consequences in measure theory of supposing that there is a real-valued-measurable cardinal. Naturally many of these involve the supposed cardinal and its witnessing measure, and they are most interesting if the cardinal is atomlessly-measurable. A theorem which would be here if it had not already been needed is 3A. I start with a similar result on reversing the order of integration (6A). An elaboration of the same techniques gives some results analogous to those of 5A-5B, but for set-valued functions whose values are small in a different sense (6D-6E). In 6B-6C I look at covering numbers for null ideals, and in 6F-6G I look at small non-negligible sets; the latter analysis leads to a version of Shipman's theorem on changing the order of integration in a multiply-repeated integral (6I) and to a stronger result on repeated integrals of functions with measurable sections (6K). A deeper look at covering numbers gives a description of weakly Π^1_1-indescribable atomlessly-measurable cardinals (6L). I conclude with descriptions of the way in which real-valued-measurable cardinals appear in the theories of metric measure spaces and vector lattices (6M-6N).

6A Theorem Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ; let (X,μ) be a Radon probability space and $f : X\times\kappa\to\mathbb{R}$ a bounded function. Then

$$\underline{\int}\Bigl(\int f(x,\xi)\nu(d\xi)\Bigr)\mu(dx)\le\int\Bigl(\overline{\int} f(x,\xi)\mu(dx)\Bigr)\nu(d\xi).$$

proof ? Suppose, if possible, otherwise. Adding a constant function to f, if necessary, we may suppose that $f(x,\xi)\ge 0$ for all x, ξ.

(a) We are supposing that there is a μ-integrable function $g : X\to\mathbb{R}$ such that $0\le g(x)\le\int f(x,\xi)\nu(d\xi)$ for every $x\in X$ and

$$\int g(x)\,\mu(dx)>\int\overline{\int} f(x,\xi)\mu(dx)\nu(d\xi).$$

Let $F_0,\dots,F_n$ be disjoint non-empty measurable subsets of X such that

$$\sum_{i\le n}t_i\mu F_i>\int\overline{\int} f(x,\xi)\mu(dx)\nu(d\xi),$$

writing $t_i=\inf_{x\in F_i}g(x)$ for each $i\le n$. Then

$$\overline{\int} f(x,\xi)\mu(dx)\ge\sum_{i\le n}\overline{\int}_{F_i}f(x,\xi)\mu(dx)\ \ \forall\ \xi<\kappa,$$

so there must be some $i\le n$ such that

$$t_i\mu F_i>\int\overline{\int}_{F_i}f(x,\xi)\mu(dx)\nu(d\xi).$$

Set $Y=F_i$, $\tilde\mu=(\mu F_i)^{-1}\mu\lceil F_i$, $t=t_i$; then $(Y,\tilde\mu)$ is a Radon probability space and

$$\int\overline{\int} f(y,\xi)\tilde\mu(dy)\nu(d\xi)<t,$$

$$t\le\int f(y,\xi)\nu(d\xi)\text{ for every }y\in Y.$$

(b) For each $\xi<\kappa$ choose a $\tilde\mu$-measurable function $h_\xi : Y\to\mathbb{R}$ such that $f(y,\xi)\le h_\xi(y)$ for every $y\in Y$ and $\int h_\xi(y)\tilde\mu(dy)=\overline{\int} f(y,\xi)\tilde\mu(dy)$. By A2Kb, applied to the family of sets of the form $\{y : h_\xi(y)\ge s\}$ for $\xi<\kappa$ and rational s, there is a function $\phi:\{0,1\}^\kappa\to Y$ such that $\int h_\xi\phi(v)\mu_\kappa(dv)$ exists and is equal to $\int h_\xi(y)\tilde\mu(dy)$ for every $\xi<\kappa$, taking μ_κ to be the usual measure on $\{0,1\}^\kappa$. So setting $f_1(v,\xi)=f(\phi(v),\xi)$ for $v\in\{0,1\}^\kappa$, $\xi<\kappa$ we have

$$\int \overline{\int} f_1(v,\xi)\mu_\kappa(dv)\nu(d\xi) \le \int \int h_\xi \phi(v)\mu_\kappa(dv)\nu(d\xi)$$
$$= \int \int h_\xi(y)\tilde{\mu}(dy)\nu(d\xi)$$
$$= \int \overline{\int} f(y,\xi)\tilde{\mu}(dy)\nu(d\xi) < t,$$

while

$$t \le \int f(\phi(v),\xi)\nu(d\xi) = \int f_1(v,\xi)\nu(d\xi)$$

for every $v \in \{0,1\}^\kappa$.

(c) We may choose for each $\xi < \kappa$ a Baire measurable function $h'_\xi : \{0,1\}^\kappa \to \mathbb{R}$ such that $f_1(v,\xi) \le h'_\xi(v)$ for each $v \in \{0,1\}^\kappa$ and

$$\int h'_\xi(v)\mu_\kappa(dv) = \overline{\int} f_1(v,\xi)\mu_\kappa(dv)$$

(A2Gf). Now there is a countable set $I_\xi \subseteq \kappa$ such that h'_ξ factors through $\{0,1\}^{I_\xi}$, that is, $h'_\xi(v) = h'_\xi(v')$ whenever $v\restriction I_\xi = v'\restriction I_\xi$.

By 5Cb, there are $\Gamma \subseteq \kappa$, $\gamma < \kappa$ such that $\nu\Gamma = 1$ and $I_\xi \cap I_\eta \subseteq \gamma$ whenever ξ, η are distinct members of Γ. Set $\gamma = \min \Gamma$.

(d) Set

$$f'_1(u,\xi) = \int h'_\xi(u^\frown u')\mu_{\kappa\setminus\gamma}(du')$$

for $u \in \{0,1\}^\gamma$, $\xi < \kappa$. Then, applying Fubini's theorem to $\{0,1\}^\kappa \cong \{0,1\}^\gamma \times \{0,1\}^{\kappa\setminus\gamma}$, we have

$$\int f'_1(u,\xi)\mu_\gamma(du) = \int h'_\xi(v)\mu_\kappa(dv)$$

for every ξ, so that

$$\int \int f'_1(u,\xi)\mu_\gamma(du)\nu(d\xi) = \int \overline{\int} f_1(v,\xi)\mu_\kappa(dv)\nu(d\xi) < t,$$

and

$$\overline{\int} \int f'_1(u,\xi)\nu(d\xi)\mu_\gamma(du) < t$$

by Theorem 3A. Accordingly there is a $u \in \{0,1\}^\gamma$ such that

$$\int f'_1(u,\xi)\nu(d\xi) < t.$$

(e) For each $\xi \in \Gamma$ take $u'_\xi \in \{0,1\}^{\kappa\setminus\gamma}$ such that $h'_\xi(u^\frown u'_\xi) \le f'_1(u,\xi)$. Let $w \in \{0,1\}^\kappa$ be such that

$$w\restriction\gamma = u,\ w\restriction I_\xi = (u^\frown u'_\xi)\restriction I_\xi\ \ \forall\ \xi \in \Gamma;$$

such a w exists because if ξ, $\eta \in \Gamma$ and $\xi < \eta$ then $I_\xi \cap I_\eta \subseteq \gamma$. Now

$$f_1(w,\xi) \le h'_\xi(w) = h'_\xi(u^\frown u'_\xi) \le f'_1(u,\xi) \text{ for every } \xi \in \Gamma,$$

so

$$\int f_1(w,\xi)\nu(d\xi) \le \int f'_1(u,\xi)\nu(d\xi) < t,$$

contradicting the last sentence of (b) above. **X**

This completes the proof.

Remark This result was inspired by Lemma 5 of Kunen n70; compare 6E below.

6B Proposition Let κ be an atomlessly-measurable cardinal. If (X,μ) is any Radon measure space with $\mu X > 0$, then $\operatorname{cov}(X,\mathcal{N}_\mu) \ge \kappa$.

proof By A2Pb, it is enough to show that $\operatorname{cov}(\{0,1\}^\kappa, \mathcal{N}_{\mu_\kappa}) \ge \kappa$, where μ_κ is the usual measure on $\{0,1\}^\kappa$. Fix on an atomless κ-additive probability ν with domain $\mathcal{P}\kappa$. By 3H, there is an inverse-measure -preserving function $f : \kappa \to \{0,1\}^\kappa$. Now if $\mathcal{A}$ is any cover of $\{0,1\}^\kappa$ by μ_κ-negligible sets, $\{f^{-1}[A] : A \in \mathcal{A}\}$ is a cover of κ by ν-negligible sets, so must have cardinal at least κ, and $\#(\mathcal{A}) \ge \kappa$, as required.

6C Corollary If κ is an atomlessly-measurable cardinal and $\lambda \le \kappa$ is a cardinal of uncountable cardinality, then λ is a precaliber of every probability algebra.

proof If $\lambda < \kappa$ this is a corollary of 6B and A2Ua. If $\lambda = \kappa$, we can use 6A and A2Ub. For let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon probability space and $\langle E_\xi \rangle_{\xi<\kappa}$ an increasing family in $\mathcal{N}_\mu$ with union $E \in \Sigma$. Set

$$C = \{(x,\xi) : \xi < \kappa,\, x \in E_\xi\} \subseteq X \times \kappa.$$

Then

$$\underline{\int} \nu C[\{x\}]\mu(dx) = \mu E,\ \int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) = 0,$$

so 6A, applied to the characteristic function of C, tells us that $\mu E = 0$; now A2Ub tells us that κ is a precaliber of the measure algebra of X. But as every probability algebra is (isomorphic to) the measure algebra of some Radon probability space (A2La), we have the result.

6D Lemma Let κ be a real-valued-measurable cardinal and ν a normal witnessing probability on κ. If (X, μ) is any quasi-Radon probability space of weight[1] strictly less than κ, and $f : [\kappa]^{<\omega} \to \mathcal{N}_\mu$ is any function, then

$$\bigcap_{V \subseteq \kappa, \nu V = 1} \bigcup_{I \in [V]^{<\omega}} f(I) \in \mathcal{N}_\mu.$$

proof Let $\mathcal{F}$ be the filter $\{A : A \subseteq \kappa,\, \nu A = 1\}$.

(a) I show by induction on $n \in \mathbb{N}$ that if $g : [\kappa]^{\le n} \to \mathcal{N}_\mu$ is any function, then

$$E(g) = \bigcap_{V \in \mathcal{F}} \bigcup_{I \in [V]^{\le n}} f(I) \in \mathcal{N}_\mu.$$

P(i) For $n = 0$ this is trivial; $E(g) = g(\emptyset) \in \mathcal{N}_\mu$. **(ii)** For the inductive step to $n+1$, given $g : [\kappa]^{\le n+1} \to \mathcal{N}_\mu$, then for each $\xi < \kappa$ define $g_\xi : [\kappa]^{\le n} \to \mathcal{N}_\mu$ by setting $g_\xi(I) = g(I \cup \{\xi\})$ for each $I \in [\kappa]^{\le n}$. By the inductive hypothesis, $E(g_\xi) \in \mathcal{N}_\mu$. Set

$$C = \{(x,\xi) : x \in E(g_\xi)\} \subseteq X \times \kappa.$$

Then

$$\int \mu^* C^{-1}[\{\xi\}]\nu(d\xi) = \int \mu^* E(g_\xi)\nu(d\xi) = 0,$$

so by 3A

$$\overline{\int} \nu C[\{x\}]\mu(dx) = 0,$$

and $\mu D = 0$, where $D = \{g(\emptyset)\} \cup \{x : \nu C[\{x\}] > 0\}$.

Take any $x \in X \setminus D$ and set $W = \kappa \setminus C[\{x\}] \in \mathcal{F}$. For each $\xi \in W$, $x \notin E(g_\xi)$, so there is a $V_\xi \in \mathcal{F}$ such that $\nu V_\xi = 1$ and $x \notin g_\xi(I)$ for every $I \in [V_\xi]^{\le n}$. Set

$$V = \{\xi : \xi \in W,\, \xi \in V_\eta\ \ \forall\ \eta \in W \cap \xi\}.$$

Then $V \in \mathcal{F}$. If $I \in [V]^{\le n+1}$, either $I = \emptyset$ and $x \notin g(I)$, or there is a least element ξ of I; in the latter case, $\xi \in W$ so $J = I \setminus \{\xi\} \subseteq V_\xi$ and $x \notin g_\xi(J) = g(I)$. So $x \notin \bigcup\{g(I) : I \in [V]^{\le n+1}\}$. As x is arbitrary, $E(g) \subseteq D \in \mathcal{N}_\mu$ and the induction proceeeds. **Q**

(b) Now consider

$$G = \bigcup_{n \in \mathbb{N}} E(f \restriction [\kappa]^{\le n}) \in \mathcal{N}_\mu.$$

If $x \in X \setminus G$ then for each $n \in \mathbb{N}$ there is a $V_n \in \mathcal{F}$ such that $x \notin \bigcup\{f(I) : I \in [V_n]^{\le n}\}$. Set $V = \bigcap_{n \in \mathbb{N}} V_n \in \mathcal{F}$; then $x \notin \bigcup\{f(I) : I \in [V]^{<\omega}\}$. As x is arbitrary,

[1] In the published version 'Maharam type' was given instead of 'weight'. I do not know if the result is true with this hypothesis.

$$E(f) \subseteq G \in \mathcal{N}_\mu,$$

as required.

Remark This is Lemma 2 of Kunen n70.

6E Proposition Let κ be a real-valued-measurable cardinal with a normal witnessing probability ν. If (X,μ) is a Radon probability space, $f : [\kappa]^{<\omega} \to \mathcal{N}_\mu$ is a function and $\beta < \kappa$, then there is a $V \subseteq \kappa$ such that $\beta \subseteq V$ and $\nu V = 1$ and $\bigcup\{f(I) : I \in [V]^{<\omega}\} \neq X$.

proof (a) Consider first the case $(X,\mu) = (\{0,1\}^\kappa, \mu_\kappa)$, where μ_κ is the usual measure on $\{0,1\}^\kappa$. For any $L \subseteq \kappa$ let μ_L be the usual measure on $\{0,1\}^L$, and $\pi_L : \{0,1\}^\kappa \to \{0,1\}^L$ the canonical map. Write $\mathcal{F} = \{V : V \subseteq \kappa,\, \nu V = 1\}$.

(i) For each $I \in [\kappa]^{<\omega}$, there is a countable set $g(I) \subseteq \kappa$ such that $\mu_{g(I)}(\pi_{g(I)}[f(I)]) = 0$ (see A2Gc); enlarging $f(I)$ if necessary, we may suppose that $f(I) = \pi^{-1}_{g(I)}[\pi_{g(I)}[f(I)]]$. Set $\theta = \max(\omega, \#(\beta))$ and $g^*(I) = \bigcup\{g(I \cup K) : K \in [\beta]^{<\omega}\}$ for each $I \in [\kappa]^{<\omega}$. By 5C there are a set $C \in \mathcal{F}$ and a function $h : [\kappa]^{<\omega} \to [\kappa]^{\le\theta}$ such that $g^*(I) \cap \eta \subseteq h(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$. Set

$$\Gamma = \{\gamma : \beta \le \gamma < \kappa,\, h(I) \subseteq \gamma \ \forall\ I \in [\gamma]^{<\omega}\} \cup \{0\};$$

then Γ is a closed unbounded set in κ, because $\operatorname{cf}(\kappa) > \theta$. Let $\langle \gamma_\eta \rangle_{\eta<\kappa}$ be the increasing enumeration of Γ; note that $\gamma_0 = 0$ and $\gamma_1 \ge \beta$.

(ii) For $\eta < \kappa$, set $M(\eta) = \kappa \setminus \gamma_\eta$ and $L(\eta) = \gamma_{\eta+1} \setminus \gamma_\eta$; then $\mu_{M(\eta)}$ can be identified with the product measure $\mu_{L(\eta)} \times \mu_{M(\eta+1)}$. Choose $u_\eta \in \{0,1\}^{\gamma_\eta}$, $V_\eta \subseteq \kappa$ inductively, as follows. $u_0 = \emptyset$. Given u_η, then for each $I \in [\kappa]^{<\omega}$ set

$$f'_\eta(I) = \{v : v \in \{0,1\}^{L(\eta)},\, \mu_{M(\eta+1)}\{w : u_\eta^\frown v^\frown w \in f(I)\} > 0\},$$

and

$$\begin{aligned} f_\eta(I) &= f'_\eta(I) \text{ if } \mu_{L(\eta)}(f'_\eta(I)) = 0, \\ &= \emptyset \text{ otherwise.} \end{aligned}$$

By 6D, we can find for each $K \in [\gamma_{\eta+1}]^{<\omega}$ a set $E_{\eta K} \subseteq \{0,1\}^{L(\eta)}$ such that $\mu_{L(\eta)} E_{\eta K} = 1$ and for every $v \in E_{\eta K}$ there is a set $V \in \mathcal{F}$ such that $v \notin f_\eta(K \cup J)$ for any $J \in [V]^{<\omega}$. Choose $v_\eta \in \bigcap\{E_{\eta K} : K \in [\gamma_{\eta+1}]^{<\omega}\}$ (using 6B); for $K \in [\gamma_{\eta+1}]^{<\omega}$ choose $V_{\eta K} \in \mathcal{F}$ such that $v_\eta \notin f_\eta(K \cup J)$ for any $J \in [V_{\eta K}]^{<\omega}$. Set $V_\eta = \bigcap\{V_{\eta K} : K \in [\gamma_{\eta+1}]^{<\omega}\} \in \mathcal{F}$ and $u_{\eta+1} = u_\eta^\frown v_\eta \in \{0,1\}^{\gamma_{\eta+1}}$.

At limit ordinals η with $0 < \eta \le \kappa$, set $u_\eta = \bigcup_{\xi<\eta} u_\xi \in \{0,1\}^{\gamma_\eta}$.

(iii) Now consider $u = u_\kappa \in \{0,1\}^\kappa$ and

$$V = \beta \cup \{\xi : \xi \in C,\, \xi \in V_\eta \ \forall\ \eta < \xi\} \in \mathcal{F}.$$

If $I \in [V]^{<\omega}$ then

$$\mu_{M(\eta)}\{w : u_\eta^\frown w \in f(I)\} = 0$$

for every $\eta < \kappa$. **P** Induce on η. For $\eta = 0$ this says just that $\mu_\kappa f(I) = 0$, which was our hypothesis on f. For the inductive step to $\eta + 1$, we have

$$\mu_{M(\eta)}\{w : u_\eta^\frown w \in f(I)\} = 0$$

by the inductive hypothesis, so Fubini's theorem tells us that

$$\mu_{L(\eta)}\{v : \mu_{M(\eta+1)}\{w : u^\frown v^\frown w \in f(I)\} > 0\} = 0,$$

that is, $\mu_{L(\eta)} f'_\eta(I) = 0$, so that $f_\eta(I) = f'_\eta(I)$. Now setting $K = I \cap \gamma_{\eta+1}$, $J = I \setminus \gamma_{\eta+1}$ we see that $J \subseteq V_\eta$ (because of course $\eta < \gamma_{\eta+1}$, while $\beta \subseteq \gamma_{\eta+1}$), therefore $J \subseteq V_{\eta K}$ and $v_\eta \notin f_\eta(K \cup J) = f'_\eta(I)$; but this says just that

$$\mu_{M(\eta+1)}\{w : u_\eta^\frown v_\eta^\frown w \in f(I)\} = 0,$$

that is, that

$$\mu_{M(\eta+1)}\{w : u_{\eta+1}{}^\frown w \in f(I)\} = 0,$$

so that the induction continues.

For the inductive step to a non-zero limit ordinal $\eta \le \kappa$, there is a non-zero $\zeta < \eta$ such that $I \cap \gamma_\eta \subseteq \gamma_\zeta$. Set $J = I \setminus \beta$, $K = I \cap \beta$. Then $J \subseteq C$ so $g(I) = g(J \cup K) \subseteq g^*(J)$ and $g(I) \cap \gamma_\eta \subseteq g^*(J) \cap \gamma_\eta \subseteq h(J \cap \gamma_\eta) = h(J \cap \gamma_\zeta) \subseteq \gamma_\zeta$, by the choice of Γ. But this means that

$$\{w : w \in \{0,1\}^{M(\zeta)},\, u_\zeta{}^\frown w \in f(I)\} = \{0,1\}^{\gamma_\eta \setminus \gamma_\zeta} \times \{w : w \in \{0,1\}^{M(\eta)},\, u_\eta{}^\frown w \in f(I)\}.$$

By the inductive hypothesis,

$$\mu_{M(\zeta)}\{w : u_\zeta{}^\frown w \in f(I)\} = 0,$$

so that

$$\mu_{M(\eta)}\{w : u_\eta{}^\frown w \in f(I)\} = 0$$

and the induction continues. **Q**

(iv) But now, given $I \in [V]^{<\omega}$, there is surely some $\eta < \kappa$ such that $g(I) \subseteq \gamma_\eta$, and in this case

$$\mu_{M(\eta)}\{w : u_\eta{}^\frown w \in f(I)\} = 0$$

implies that $u \notin f(I)$.

Thus we have a point $u \notin \bigcup\{f(I) : I \in [V]^{<\omega}\}$, as required.

(b) For the general case, we have a function $\phi : \{0,1\}^\kappa \to X$ such that $\mu_\kappa \phi^{-1}[f(I)] = 0$ for every $I \in [\kappa]^{<\omega}$, by A2Kb. Now by (a) there are $u \in \{0,1\}^\kappa$, $V \subseteq \kappa$ such that $\beta \subseteq V$, $\nu V = 1$ and $u \notin \phi^{-1}[f(I)]$ for every $I \in [V]^{<\omega}$; in which case $x = \phi(u) \notin f(I)$ for every $I \in [V]^{<\omega}$ and $\bigcup\{f(I) : I \in [V]^{<\omega}\} \ne X$.

Remark 6E is implicit in Lemma 5 of Kunen n70. The clause '$\beta \subseteq V$' is a refinement of a type in which I have generally not indulged; but it is useful here for an application in §4 above.

6F Proposition If κ is an atomlessly-measurable cardinal and λ, θ are infinite cardinals less than κ, then there is a set $A \subseteq \{0,1\}^\theta$ such that $\#(A) = \lambda$ and no uncountable subset of A is negligible for μ_θ, the usual measure on $\{0,1\}^\theta$.

proof Let ν be an atomless κ-additive probability defined on $\mathcal{P}\kappa$. By 3H there is a function $f : \kappa \to (\{0,1\}^\theta)^\lambda$ which is inverse-measure-preserving for ν and the usual measure of $(\{0,1\}^\theta)^\lambda$, identified with $\{0,1\}^{\theta\times\lambda}$. For $\xi < \kappa$, set

$$A_\xi \;=\; \{f(\xi)(\eta) : \eta < \lambda\} \;\subseteq\; \{0,1\}^\theta.$$

? Suppose, if possible, that for every $\xi < \kappa$ there is a set $J_\xi \subseteq \lambda$ such that $\#(J_\xi) = \omega_1$ but $E_\xi = f(\xi)[J_\xi]$ is μ_θ-negligible. For each ξ choose a countable set $I_\xi \subseteq \theta$ such that

$$E'_\xi = \{x : x \in \{0,1\}^\theta,\, \exists\ x' \in E_\xi,\, x{\restriction}I_\xi = x'{\restriction}I_\xi\}$$

is μ_θ-negligible. By 5Ab, there is a countable $I \subseteq \theta$ such that $\nu V = 1$, where $V = \{\xi : I_\xi \subseteq I\}$. For $\xi \in V$ set

$$E^*_\xi = \{x{\restriction}I : x \in E_\xi\} \subseteq \{0,1\}^I,$$

so that $\mu_I E^*_\xi = 0$, where μ_I is the usual measure on $\{0,1\}^I$. Fix a sequence $\langle U_m\rangle_{m\in\mathbb{N}}$ running over the open-and-closed subsets of $\{0,1\}^I$, and for each $\xi \in V$, $n \in \mathbb{N}$ choose an open set $G_{n\xi} \subseteq \{0,1\}^I$ such that $E^*_\xi \subseteq G_{n\xi}$ and $\mu_I(G_{n\xi}) \le 2^{-n}$. For $m, n \in \mathbb{N}$ set

$$D_{nm} = \{\xi : \xi \in V,\, U_m \subseteq G_{n\xi}\}.$$

For each $\alpha < \lambda$, set $f_\alpha(\xi) = f(\xi)(\alpha){\restriction}I \in \{0,1\}^I$ for $\xi < \kappa$; then the functions f_α are all stochastically independent. Consequently, there is for each $\xi < \kappa$ an $\alpha(\xi) \in J_\xi$ such that $f_{\alpha(\xi)}$ is stochastically independent from the countable family $\{D_{nm} : n, m \in \mathbb{N}\} \subseteq \mathcal{P}\kappa$. Because $\lambda < \kappa$ and ν is κ-additive, there is a $\gamma < \lambda$ such that $B = \{\xi : \alpha(\xi) = \gamma\}$ has $\nu B > 0$. Take $n \in \mathbb{N}$ such that $\nu(B) > 2^{-n}$, and examine

$$C = \bigcup_{m\in\mathbb{N}}(D_{nm} \cap f_\gamma^{-1}[U_m]).$$

Because f_γ is independent from all the D_{nm}, and is inverse-measure -preserving for ν and μ_I, $\nu C = (\nu \times \mu_I)(C')$ where

$$C' = \bigcup_{m\in\mathbb{N}}(D_{nm} \times U_m) \subseteq \kappa \times \{0,1\}^I.$$

But, for each $\xi < \kappa$, the vertical section $C'[\{\xi\}]$ is just $\bigcup\{U_m : \xi \in D_{nm}\} = G_{n\xi}$, so

$$(\nu \times \mu_I)(C') = \int \mu_I(G_{n\xi})\nu(d\xi) \le 2^{-n}.$$

Accordingly $\nu C \le 2^{-n} < \nu B$ and there must be a $\xi \in B \cap V \setminus C$. But in this case $f_\gamma(\xi)\restriction I \in E^*_\xi$, because $\gamma = \alpha(\xi) \in J_\xi$, while $f_\gamma(\xi)\restriction I \notin G_{n\xi}$, because there is no m such that $f_\gamma(\xi)\restriction I \in U_m \subseteq G_{n\xi}$; contrary to the choice of $G_{n\xi}$. **X**

So take some $\xi < \kappa$ such that $\mu^*_\theta(f(\xi)[J]) > 0$ for every uncountable $J \subseteq \lambda$. Evidently $f(\xi)$ is countable-to-one, so A_ξ must have cardinal λ (passing over the trivial case of countable λ), and will serve for A.

Remarks (a) The argument above is due to Solovay and Prikry; the form here is lifted from Fremlin p89d.

(b) Using A2Ka this result can easily be converted into a formally more general result about Radon measure spaces of Maharam type less than κ.

(c) I do not know whether, under the hypothesis of this proposition, there is always a set $A \subseteq \{0,1\}^\theta$ with $\#(A) = \kappa$ and no uncountable subset of A negligible for μ_θ; see P4d.

6G Corollary Let κ be an atomlessly-measurable cardinal. Writing $\mathcal{N}_{\mu_\theta}$ for the ideal of negligible subsets of $\{0,1\}^\theta$,

(a) $\mathrm{non}(\{0,1\}^\theta, \mathcal{N}_{\mu_\theta}) = \omega_1$ for $\omega \le \theta < \kappa$,

(b) $\mathrm{non}(\{0,1\}^\theta, \mathcal{N}_{\mu_\theta}) \le \kappa$ for $\kappa \le \theta \le \min(2^\kappa, \kappa^{(+\omega)})$,

(c) $\mathrm{non}(\{0,1\}^\theta, \mathcal{N}_{\mu_\theta}) \le \theta$ for $2^\kappa \le \theta < \kappa^{(+\omega)}$.

proof (a) Immediate from 6F.

(b) If ν is any witnessing probability on κ then we have an inverse-measure -preserving function $f : \kappa \to \{0,1\}^\theta$ (3H); now $f[\kappa]$ witnesses that $\mathrm{non}(\{0,1\}^\theta, \mathcal{N}_{\mu_\theta}) \le \kappa$.

(c) The point is that if $\mathrm{cf}(\theta) > \omega$ then $\mathrm{non}(\mathcal{N}_{\mu_\theta}) \le \max(\theta, \sup_{\alpha<\theta} \mathrm{non}(\mathcal{N}_{\mu_\alpha}))$. **P** Set $\delta = \max(\theta, \sup_{\alpha<\theta} \mathrm{non}(\mathcal{N}_{\mu_\alpha}))$. For each ordinal $\alpha < \theta$ choose $A_\alpha \subseteq \{0,1\}^\alpha$ such that $A_\alpha \notin \mathcal{N}_{\mu_\alpha}$ and $\#(A_\alpha) = \mathrm{non}(\mathcal{N}_{\mu_\alpha})$. Choose $A \subseteq \{0,1\}^\theta$ such that $\#(A) \le \delta$ and $\pi_\alpha[A] \supseteq A_\alpha$ for each α, taking $\pi_\alpha : \{0,1\}^\theta \to \{0,1\}^\alpha$ to be the canonical map. **?** If $A \in \mathrm{non}(\mathcal{N}_{\mu_\theta})$, then there is an H belonging to the Baire σ-algebra of $\{0,1\}^\theta$ such that $A \subseteq H \in \mathcal{N}_{\mu_\theta}$ (A2Gc); now, because $\mathrm{cf}(\theta) > \omega$, there is an $\alpha < \theta$ such that H is expressible as $\pi_\alpha^{-1}[H']$, where $H' \subseteq \{0,1\}^\alpha$. In this case $H' \in \mathcal{N}_{\mu_\alpha}$, so $A_\alpha \not\subseteq H'$ and $A \not\subseteq H$. **X** So A witnesses that $\mathrm{non}(\mathcal{N}_{\mu_\theta}) \le \delta$. **Q**

Now an elementary induction on θ, using (a) when $\theta < \kappa$, shows that $\mathrm{non}(\mathcal{N}_{\mu_\theta}) \le \theta$ whenever $\omega_1 \le \theta < \kappa^{(+\omega)}$.

(See Fremlin 89, 6.17.)

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6H Lemma Let κ be an atomlessly-measurable cardinal, $\theta < \kappa$ a cardinal, and $m \ge 1$ an integer. Set $Z = \{0,1\}^\theta$ with its usual measure, and suppose that for $i < m$, $\mathbf{u} \in Z^{m\setminus\{i\}}$ we are given a negligible set $E(\mathbf{u}) \subseteq Z$. Then there is a $\mathbf{t} = \langle t_i \rangle_{i<m} \in Z^m$ such that $t_i \notin E(\mathbf{t}\restriction m \setminus \{i\})$ for every $i < m$.

proof For each $i < m$ we can find a set $A_i \subseteq Z$ of cardinal ω_{m+1-i} such that no uncountable subset of A_i is negligible (6F). Now choose t_0, t_1 $\ldots$, t_{m-1} in such a way that

$$t_j \in A_j,\ t_j \notin E(\mathbf{u})\ \ \forall\ \ \mathbf{u} \in \prod_{i<j}\{t_i\} \times \prod_{j<i<m} A_i$$

for each $j < m$; this is possible because A_j cannot be covered by ω_{m-j} or fewer negligible sets, while $\#(\prod_{j<i<m} A_i) \le \omega_{m-j}$ for each j. Now $\mathbf{t} = \langle t_i \rangle_{i<m}$ works.

6I Proposition Let κ be an atomlessly-measurable cardinal and $m \ge 1$ an integer. Let $X_0, \ldots , X_{m-1}$ be Radon probability spaces of Maharam type less than κ, and $f : \prod_{i<m} X_i \to \mathbb{R}$ a bounded function. Suppose that $\sigma : m \to m$ is a permutation such that the two repeated integrals

$$I=\int(\dots(\int f(x_0,\dots,x_{m-1})dx_{m-1})\dots)dx_0,$$

$$I'=\int(\dots(\int f(x_0,\dots,x_{m-1})dx_{\sigma(m-1)})\dots)dx_{\sigma(0)}$$

both exist. Then $I=I'$.

Remark The integrals above may all be taken as existing 'in the wide sense', that is, each function being integrated may fail to be defined on a set of measure zero.

proof (a) To begin with, let us suppose that every X_i is $\{0,1\}^\theta$ for some $\theta<\kappa$, with its usual measure. Set $Z=(\{0,1\}^\theta)^{\mathbb{N}}$; then Z also has a natural measure, identifying Z with $\{0,1\}^{\theta\times\mathbb{N}}$.

Define $D_0,\dots,D_m$ as follows. $D_0=\{\emptyset\}=Z^0$. For $0<j\le m$, let D_j be the set of those $(t_0,\dots,t_{j-1})\in Z^j$ such that

$$\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}\int(\dots(\int f(t_{0i},\dots,t_{j-1,i},x_j,\dots,x_{m-1})dx_{m-1})\dots)dx_j$$

exists and is equal to I, taking $t_l=\langle t_{li}\rangle_{i\in\mathbb{N}}$ for each $l<j$. For $j<m$, $\mathbf{u}=\langle u_l\rangle_{l\ne j}\in Z^{m\setminus\{j\}}$ set

$$E(\mathbf{u})=\emptyset \text{ if } (u_0,\dots,u_{j-1})\notin D_j,$$

$$E(\mathbf{u})=\{t:t\in Z,\ (u_0,\dots,u_{j-1},t)\notin D_{j+1}\}$$

if $(u_0,\dots,u_{j-1})\in D_j$. Then $E(\mathbf{u})$ is negligible. **P** We need consider only the case $(u_0,\dots,u_{j-1})\in D_j$. Express u_l as $\langle u_{li}\rangle_{i\in\mathbb{N}}$ for each $l<j$, and for $i\in\mathbb{N}$ define $h_i:\{0,1\}^\theta\to\mathbb{R}$ by setting

$$h_i(x)=\int(\dots(\int f(u_{0i},\dots,u_{j-1,i},x,x_{j+1},\dots,x_{m-1})dx_{m-1})\dots)dx_{j+1}$$

for each $x\in\{0,1\}^\theta$. Then, because $(u_0,\dots,u_{j-1})\in D_j$,

$$\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}\int h_i(x)dx \text{ exists} = I.$$

Also, because f is bounded, the functions h_i are uniformly bounded. By A2X,

$$\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}h_i(t_i)$$

exists and is equal to I for almost all $t=\langle t_i\rangle_{i\in\mathbb{N}}\in Z$; that is, $(u_0,\dots,u_{j-1},t)\in D_{j+1}$ and $t\notin E(\mathbf{u})$ for almost all $t\in Z$. **Q**

Now suppose that $\mathbf{t}=\langle t_j\rangle_{j<m}=\langle\langle t_{ji}\rangle_{i\in\mathbb{N}}\rangle_{j<m}\in Z^m$ and that $t_j\notin E(\mathbf{t}\restriction m\setminus\{j\})$ for each $j<m$. Then

$$(t_0,\dots,t_{j-1})\in D_j$$

for each $j\le m$, so that $\mathbf{t}\in D_m$ and

$$\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}f(t_{0i},\dots,t_{m-1,i})=I.$$

In the same way, we can find for each $\mathbf{u}\in\bigcup_{j<m}Z^{m\setminus\{j\}}$ a negligible set $E'(\mathbf{u})$ such that if $\mathbf{t}\in Z^m$ and $t_j\notin E'(\mathbf{t}\restriction m\setminus\{j\})$ for every $j<m$ then

$$\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}f(t_{0i},\dots,t_{m-1,i})=I'.$$

But by Lemma 6H there is a $\mathbf{t}\in Z^m$ such that $t_j\notin E(\mathbf{t}\restriction m\setminus\{j\})\cup E'(\mathbf{t}\restriction m\setminus\{j\})$ for every $j<m$, and now

$$I=\lim_{n\to\infty}\tfrac{1}{n+1}\sum_{i\le n}f(t_{0i},\dots,t_{m-1,i})=I',$$

as required.

(b) For the general case, there are inverse-measure -preserving functions $g_i:\{0,1\}^\theta\to X_i$, where $\theta<\kappa$ is the maximum of ω and the Maharam types of X_i (A2Ka). Applying (a) to $F:(\{0,1\}^\theta)^m\to\mathbb{R}$, where $F(y_0,\dots,y_{m-1})=f(g_0(y_0),\dots,g_{m-1}(y_{m-1}))$, we obtain the result.

Remark This comes from Theorem 1 of Shipman 90. Compare Zakrzewski p91.

6J The following is an elementary corollary of Theorem 3A.

Proposition Let $(X,\mathfrak{T},\Sigma,\mu)$ be a totally finite quasi-Radon measure space and $(Y,\mathcal{P}Y,\nu)$ a probability space; suppose that $\mathrm{w}(X)<\mathrm{add}(\nu)$. Let $f:X\times Y\to\mathbb{R}$ be a bounded function such that all the sections $x\mapsto f(x,y):X\to\mathbb{R}$ are measurable. Then

$$\int\int f(x,y)\nu(dy)\mu(dx) \text{ exists} = \int\int f(x,y)\mu(dx)\nu(dy).$$

proof If $\mu X = 0$ this is trivial; otherwise, re-scaling μ if necessary, we may suppose that $\mu X = 1$. By 3A,

$$\overline{\int}\int f(x,y)\nu(dy)\mu(dx) \le \int\overline{\int} f(x,y)\mu(dx)\nu(dy) = \int\int f(x,y)\mu(dx)\nu(dy).$$

Similarly

$$\overline{\int}\int(-f(x,y))\nu(dy)\mu(dx) \le \int\int(-f(x,y))\mu(dx)\nu(dy),$$

so that

$$\underline{\int}\int f(x,y)\nu(dy)\mu(dx) \ge \int\int f(x,y)\mu(dx)\nu(dy).$$

Putting these together we have the result.

6K Proposition Let κ be an atomlessly-measurable cardinal and $(X,\mathfrak{T},\Sigma,\ \mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ Radon probability spaces of weights less than κ. Let $f : X\times Y \to \mathbb{R}$ be a function such that all its horizontal and vertical sections

$$x \mapsto f(x,y^*) : X \to \mathbb{R},\ ,\ y \mapsto f(x^*,y) : Y \to \mathbb{R}$$

are measurable. Then

(a) if f is bounded, the repeated integrals

$$\int\int f(x,y)\mu(dx)\nu(dy),\ \int\int f(x,y)\nu(dy)\mu(dx)$$

exist and are equal;

(b) in any case, there is a function $g : X\times Y \to \mathbb{R}$, measurable for the (ordinary) product measure $\mu\times\nu$, such that all the sets $\{x : g(x,y^*) \ne f(x,y^*)\}$, $\{y : g(x^*,y) \ne f(x^*,y)\}$ are negligible.

proof (a) By 3I there is a κ-additive measure $\tilde\nu$ on Y, with domain $\mathcal{P}Y$, extending ν. Now 6J tells us, among other things, that the function

$$x \mapsto \int f(x,y)\nu(dy) = \int f(x,y)\tilde\nu(dy) : X \to \mathbb{R}$$

is μ-measurable. Similarly, $y \mapsto \int f(x,y)\mu(dx)$ is ν-measurable. So returning to 6J we get

$$\begin{aligned}\int\int f(x,y)\mu(dx)\nu(dy) &= \int\int f(x,y)\mu(dx)\tilde\nu(dy)\\ &= \int\int f(x,y)\tilde\nu(dy)\mu(dx) =\\ &\int\int f(x,y)\nu(dy)\mu(dx).\end{aligned}$$

(b) Suppose first that f is bounded. By (a), we can define a measure θ on $X\times Y$ by saying that

$$\theta G = \int \nu G[\{x\}]\mu(dx) = \int \mu G^{-1}[\{y\}]\nu(dy)$$

whenever $G \subseteq X\times Y$ is such that $G[\{x\}] \in \mathrm{T}$ for almost every $x \in X$ and $G^{-1}[\{y\}] \in \Sigma$ for almost every $y \in Y$. This θ extends the ordinary product measure $\mu\times\nu$; writing Ω for the σ-algebra of subsets of $X\times Y$ generated by $\{E\times F : E\in\Sigma,\ F\in\mathrm{T}\}$, the Radon-Nikodým theorem (Royden 63, chap. 11, §5) tells us that there is an Ω-measurable function $h : X\times Y \to \mathbb{R}$ such that $\int_G f(x,y)\theta(dxdy) = \int_G h(x,y)\theta(dxdy)$ for every $G \in \Omega$.

Let $\mathcal{U}$ be a base for the topology $\mathfrak{T}$, with $\#(\mathcal{U}) < \kappa$. For any $U \in \mathcal{U}$ consider

$$V_U = \{y : \int_U f(x,y)\mu(dx) > \int_U h(x,y)\mu(dx)\}.$$

The argument of (a) shows that $y \mapsto \int_U f(x,y)\mu(dx)$ is measurable, so $V_U \in T$, and

$$\int_{V_U}\int_U f(x,y)\mu(dx)\nu(dy) = \int_{U\times V_U} f(x,y)\theta(dxdy)$$
$$= \int_{U\times V_U} h(x,y)\theta(dxdy)$$
$$= \int_{V_U}\int_U h(x,y)\mu(dx)\nu(dy),$$

so $\nu V_U = 0$. Similarly

$$\nu\{y : \int_U f(x,y)\mu(dx) < \int_U h(x,y)\mu(dx)\} = 0.$$

Because $\#(\mathcal{U}) < \kappa$, and no non-negligible measurable set in Y can be covered by fewer than κ negligible sets (6B), we must have

$$\nu^*\{y : \int_U f(x,y)\mu(dx) = \int_U h(x,y)\mu(dx) \ \forall\ U \in \mathcal{U}\} = 1.$$

But because $\mathcal{U}$ is a base for the topology of X, we see that

$$\nu^*\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

But as (again using (a)) the repeated integral $\int\int |f(x,y) - h(x,y)|\mu(dx)\nu(dy)$ exists, it must be 0. Thus

$$\nu\{y : f(x,y) = h(x,y) \text{ for } \mu\text{-almost every } x\} = 1.$$

Similarly,

$$\mu\{x : f(x,y) = h(x,y) \text{ for } \nu\text{-almost every } y\} = 1.$$

But now, changing h on a set of the form $(E \times Y) \cup (X \times F)$ where $\mu E = \nu F = 0$, we can get a function g, still $(\mu\times\nu)$-measurable, such that $\{(x,y) : f(x,y) \neq g(x,y)\}$ has all its horizontal and vertical sections negligible.

This deals with bounded f. But for general f we can look at the truncates $(x,y) \mapsto \max(-n, \min(n, f(x,y))$ for each n to get a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ of functions which will converge at an adequate number of points to provide a suitable g.

Remark (a) above arose in the course of correspondence with P.Zakrzewski. I first learnt of strong Fubini theorems of this type, in the context of random real models, from H.Woodin. See ZAKRZEWSKI P91 for further results along these lines.

I have given these results in a general form, allowing the spaces involved to have relatively large Maharam types; but of course they are chiefly interesting in the case in which each factor is $[0,1]$ with Lebesgue measure.

6L Theorem Let κ be an atomlessly-measurable cardinal with normal witnessing probability ν. Then the following are equivalent:

(i) κ is weakly Π^1_1-indescribable;
(ii) κ is weakly Π^1_1-indescribable and the rvm filter of κ includes the Π^1_1-filter of κ;
(iii) $\mathrm{cov}(\{0,1\}^\kappa, \mathcal{N}_{\mu_\kappa}) > \kappa$, where μ_κ is the usual Radon probability on $\{0,1\}^\kappa$;
(iv) $\mathrm{cov}(X, \mathcal{N}_\mu) > \kappa$ whenever (X,μ) is a Radon measure space and $\mu X > 0$.

proof (i)⇒(iii) Let ν be a normal witnessing probability on κ. Let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a family in $\mathcal{N}_{\mu_\kappa}$. For each $\alpha < \kappa$ let $\langle F_{\alpha n}\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of $\{0,1\}^\kappa \setminus A_\alpha$ such that $\mu_\kappa(\bigcup_{n\in\mathbb{N}} F_{\alpha n}) = 1$. By 3H there is a function $h : \kappa \to \{0,1\}^\kappa$ which is inverse-measure -preserving for ν and μ_κ. Set $H_\alpha = h^{-1}(\bigcup_{n\in\mathbb{N}} F_{\alpha n})$; then $\nu H_\alpha = 1$. Let H be the diagonal intersection of $\langle H_\alpha\rangle_{\alpha<\kappa}$, so that $\nu H = 1$. Let $\langle\gamma_\xi\rangle_{\xi<\kappa}$ be the increasing enumeration of H.

For α, $\xi < \kappa$ set

$$f_\alpha(\xi) = n \text{ if } n < \xi,\ h(\gamma_\xi) \in F_{\alpha n},$$
$$= 0 \text{ otherwise.}$$

Then each $f_\alpha : \kappa \to \kappa$ is regressive, so there is a uniform ultrafilter $\mathcal{F}$ on κ such that $m(\alpha) = \lim_{\xi\to\mathcal{F}} f_\alpha(\xi)$ exists for each $\alpha < \kappa$. Now observe that for any $\alpha < \kappa$ we have $H \setminus H_\alpha \subseteq \alpha + 1$, so that $\{\xi : \gamma_\xi \notin H_\alpha\}$ is bounded above in κ and cannot belong to $\mathcal{F}$. Consequently $\{\xi : h(\gamma_\xi) \in F_{\alpha,m(\alpha)}\} \in \mathcal{F}$. But this implies at once that $\langle F_{\alpha,m(\alpha)}\rangle_{\alpha<\kappa}$ has the finite intersection property; because all the $F_{\alpha n}$ are compact, there is a $y \in \bigcap_{\alpha<\kappa} F_{\alpha,m(\alpha)}$, and now $y \notin \bigcup_{\alpha<\kappa} A_\alpha$.

Because $\langle A_\alpha\rangle_{\alpha<\kappa}$ was arbitrary, $\mathrm{cov}(\{0,1\}^\kappa, \mathcal{N}_{\mu_\kappa}) > \kappa$.

(iii)⇒(iv) This is standard; see A2Pb.

(iv)⇒(ii) Let ν be any normal witnessing probability on κ, and let $(Z, \tilde{\nu})$ be the hyperstonian space of $(\kappa, \mathcal{P}\kappa, \nu)$; for $A \subseteq \kappa$ let A^* be the open-and-closed subset of Z corresponding to the image $A^\bullet$ of A in $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{N}_\nu$ (see A2L).

Now let $\langle f_\alpha\rangle_{\alpha<\kappa}$ be a family of regressive functions on κ and $A \subseteq \kappa$ any set with $\nu A > 0$. Because ν is normal and f_α is regressive, there is for each $\alpha < \kappa$ a countable set $D(\alpha) \subseteq \kappa$ such that $\nu f_\alpha^{-1}[D(\alpha)] = 1$ (1He). For α, $\eta < \kappa$ set $A_{\alpha\eta} = f_\alpha^{-1}[\{\eta\}]$; then $\nu(\bigcup_{\eta\in D(\alpha)} A_{\alpha\eta}) = 1$ so $\tilde{\nu}(\bigcup_{\eta\in D(\alpha)} A^*_{\alpha\eta}) = 1$ and $E_\alpha = Z \setminus \bigcup_{\eta<\kappa} A^*_{\alpha\eta} \in \mathcal{N}_{\tilde{\nu}}$. By hypothesis (iv), $A^* \not\subseteq \bigcup_{\alpha<\kappa} E_\alpha$; take $z \in A^* \setminus \bigcup_{\alpha<\kappa} E_\alpha$. Then for every $\alpha < \kappa$ there must be a $\gamma(\alpha) < \kappa$ such that $z \in A^*_{\alpha,\gamma(\alpha)}$. But this implies that

$$\{A^*\} \cup \{A^*_{\alpha,\gamma(\alpha)} : \alpha < \kappa\}$$

is a centered family of open subsets of Z. It follows that $\{A^\bullet\} \cup \{A^\bullet_{\alpha,\gamma(\alpha)} : \alpha < \kappa\}$ is centered in $\mathfrak{A}$. What this means is that if $I \in [\kappa]^{<\omega}$ then

$$V_I = A \cap \bigcap\nolimits_{\alpha\in I} f_\alpha^{-1}[\{\gamma(\alpha)\}]$$

does not belong to $\mathcal{N}_\nu$, and therefore is unbounded in κ. But now of course we can find a uniform ultrafilter $\mathcal{F}$ on κ containing every V_I, so that $A \in \mathcal{F}$, while $\lim_{\xi\to\mathcal{F}} f_\alpha(\xi) = \gamma(\alpha)$ for every $\alpha < \kappa$.

Look back at where A, $\langle f_\alpha\rangle_{\alpha<\kappa}$ came from. Taking $A = \kappa$ to begin with, we see that κ is indeed weakly Π^1_1-indescribable. But also, letting A vary, we see that any such A must be Π^1_1-fully stationary, that is, its complement cannot belong to the Π^1_1-filter $\mathcal{W}$ of κ; turning this round, we see that $\nu W = 1$ for every $W \in \mathcal{W}$, as demanded by (ii).

(ii)⇒(i) is trivial.

6M I now leave these questions in set theory and logic and turn to two more of the problems in abstract measure theory to which real-valued-measurable cardinals are relevant.

Theorem Let (X, ρ) be a metric space.

(a) X is Borel measure-complete iff there is no real-valued-measurable cardinal less than or equal to $\mathrm{d}(X)$.

(b) If X is complete (as metric space!) then it is Radon iff there is no real-valued-measurable cardinal less than or equal to $\mathrm{d}(X)$.

proof (a)(i) If $\kappa \le \mathrm{d}(X)$ is real-valued-measurable, let ν be a witnessing probability on κ. Let $\langle x_\xi\rangle_{\xi<\kappa}$ be a discrete family in X (see A3Fa). Let μ be the Borel measure on X such that $\mu E = \nu\{\xi : x_\xi \in E\}$ for every Borel set $E \subseteq X$. Let $\mathcal{G}$ be

$$\{G : G \subseteq X \text{is open}, \{\xi : x_\xi \in G\} \text{ is finite}\}.$$

Then $\mathcal{G}$ is an upwards-directed family of open sets in X with union X, so $\mu(\bigcup\mathcal{G}) = \mu X = 1 > 0 = \sup_{G\in\mathcal{G}} \mu G$, and X is not Borel measure-complete.

(ii) If X is not Borel measure-complete, let μ be a totally finite Borel measure on X and $\mathcal{G}$ an upwards-directed family of open subsets of X such that $\mu G^* > \sup_{G\in\mathcal{G}} \mu G$, writing $G^* = \bigcup\mathcal{G}$. Let $\langle\mathcal{U}_n\rangle_{n\in\mathbb{N}}$ be a sequence of discrete families of open sets in X such that $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ is a base for the topology of X (A3Fb). For each Borel set $E \subseteq X$ set

$$\mu_1 E = \mu(E \cap G^*) - \sup\nolimits_{G\in\mathcal{G}} \mu(E \cap G);$$

then μ_1 is a Borel measure on X and $\mu_1(G^*) > 0 = \sup_{G\in\mathcal{G}} \mu_1 G$. For each $n \in \mathbb{N}$ set

$$\mathcal{V}_n = \{U : U \in \mathcal{U}_n,\ \exists\ G \in \mathcal{G},\ U \subseteq G\},\ V_n^* = \bigcup \mathcal{V}_n;$$

then $G^* = \bigcup_{n\in\mathbb{N}} V_n^*$, so there is an $n \in \mathbb{N}$ such that $\mu_1(V_n^*) > 0$. For $\mathcal{E} \subseteq \mathcal{V}_n$, set $\nu\mathcal{E} = \mu_1(\bigcup\mathcal{E})$; then $(\mathcal{V}_n, \mathcal{P}\mathcal{V}_n, \nu)$ is a non- trivial measure space, so its additivity κ is a real-valued-measurable cardinal (1D). But of course $\kappa \le \#(\mathcal{V}_n) \le \mathrm{d}(X)$, because any dense subset of X must meet every non-empty member of $\mathcal{V}_n$, and $\mathcal{V}_n$ is disjoint.

(b) Now (b) follows from (a) by A2Wb.

Remark For an investigation of the exact properties of metric spaces involved in the arguments used here, see GARDNER & PFEFFER 84.

6N Riesz spaces My own introduction to the Banach-Ulam problem was through the following. (For definitions see FREMLIN 74. For an account of the elementary theory of Riesz spaces see also LUXEMBURG & ZAANEN 71.)

Theorem Let E be a Dedekind complete Riesz space (= vector lattice) with a sequentially order-continuous positive linear functional on E which is not order-continuous. Then there are a real-valued-measurable cardinal κ and an order-bounded disjoint set $A \subseteq E^+$, the positive cone of E, of cardinal κ.

proof Suppose that $f : E \to \mathbb{R}$ is a positive linear functional which is sequentially order-continuous but not order-continuous. Let $D \subseteq E^+$ be a non-empty, downwards-directed set with infimum 0 such that $\inf_{x \in D} f(x) > 0$. Let $\langle d_n \rangle_{n \in \mathbb{N}}$ be a decreasing sequence in D such that $\lim_{n\to\infty} f(d_n) = \inf_{x\in D} f(x)$; set $d^* = \inf_{n\in\mathbb{N}} d_n$. Then (because f is sequentially order-continuous) $f(d^*) = \inf_{d\in D} f(d)$, and if $d \in D$ then $f(d \wedge d^*) = \lim_{n\to\infty} f(d \wedge d_n) = f(d^*)$ (using the distributivity of Riesz spaces; FREMLIN 74, 14D). Let C be the set

$$\{x : x \in E,\, x > 0,\, \exists\ d \in D,\, x \leq d^* - (d \wedge d^*)\};$$

then $f(x) = 0$ for every $x \in C$, C is upwards-directed, and $\sup C = d^*$. Now let $A \subseteq C$ be a maximal set such that $x \wedge y = 0$ for all distinct x, $y \in A$, and set $e = \sup A$. Then $d^* = \sup_{n\in\mathbb{N}}(d^* \wedge ne)$. Because f is sequentially order-continuous, there must be some $n \in \mathbb{N}$ such that $f(d^* \wedge ne) > 0$, and $f(e) > 0$. For each $B \subseteq A$ set $e_B = \sup B$, $\nu B = f(e_B)$. Again because f is sequentially order-continuous, $(A, \mathcal{P}A, \nu)$ is a non-trivial totally finite measure space and $\#(A) \geq \operatorname{add}(\nu)$, which is a real-valued-measurable cardinal.

Remark This comes from LUXEMBURG 67. Note that if $(X, \mathcal{P}X, \mu)$ is any non-trivial probability space then we can set $E = \ell^\infty(X)$, the Dedekind complete Riesz space of all bounded real-valued functions on X, and $\int : E \to \mathbb{R}$ will be a sequentially order-continuous positive linear functional on E which is not order-continuous.

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7. Partially ordered sets

I collect here a variety of facts concerning the impact of real-valued-measurable cardinals on partially ordered sets. In 7A-7D I show that if κ is an atomlessly-measurable cardinal there are ccc partially ordered sets P and Q such that $P \times Q$ does not satisfy the κ-chain condition. A similar method shows that there are large 'entangled' subsets of $\mathbb{R}$ (7E-7F). In 7G-7N I discuss the cofinalities of certain partially ordered sets, in particular, of reduced products of families of cardinals. I end the section with an application of these ideas to cardinal exponentiation (7O-7Q).

7A Definition Let a, $b \subseteq \mathbb{N}$. Approximately following TODORČEVIĆ 86, I write

$$\Delta(a,b) = \min(a \triangle b) \text{ if } a \neq b,$$
$$= \infty \text{ if } a = b.$$

We have the following elementary lemma.

7B Lemma Let n, $l \in \mathbb{N}$ and suppose that $m \geq 3(n^2 l)!$. Let $\langle a_{ri} \rangle_{r<m, i<n}$ be any family in $\mathcal{P}\mathbb{N}$, and for r, $s < m$ set $D_{rs} = \{\Delta(a_{ri}, a_{si}) : i < n\} \cap \mathbb{N}$. Then there are $u(0), \dots, u(l), v(0), \dots, v(l) < m$ such that $u(r) \neq v(r)$ for $r \leq l$ and $D_{u(j),v(j)} \cap D_{u(k),v(k)} = \emptyset$ whenever $j < k \leq l$.

proof (a) Given finite sets X and L with $\#(X) \geq 3(\#(L))!$ and any function $\phi : [X]^2 \to L$, there is a $J \in [X]^3$ such that ϕ is constant on $[J]^2$. **P** Induce on $\#(L)$. If $\#(L) \leq 1$ this is trivial. For the inductive step to $\#(L) = n+1$, take any $x \in X$. For each $l \in L$ set $X_l = \{y : y \in X \setminus \{x\},\, \phi(\{x,y\}) = l\}$. Then there must be some $l \in L$ such that $\#(X_l) \geq 3n!$, because $(n+1)(3n!-1)+1 < 3(n+1)! \leq \#(X)$. If there are distinct y, $z \in X_l$ such that $\phi(\{y,z\}) = l$, take $J = \{x,y,z\}$. Otherwise, consider $\phi\restriction[X_l]^2 : [X_l]^2 \to L \setminus \{l\}$. By the inductive hypothesis, there is a $J \in [X_l]^3$ such that ϕ is constant on $[J]^2$, and we're done. **Q**

(b) Now examine the given family $\langle a_{ri}\rangle_{r<m,i<n}$ in $\mathcal{P}\mathbb{N}$. If $L \in [\mathbb{N}]^{<\omega}$ and $m \ge 3(n\#(L))!$, there are distinct r, $s < m$ such that $D_{rs} \cap L = \emptyset$. **P?** Otherwise, we can choose a function $\phi : [m]^2 \to n \times L$ such that $\Delta(a_{ri}, a_{si}) = j$ whenever r, $s < m$ are distinct and $\phi(\{r,s\}) = (i,j)$. By (a), there is a $J \in [m]^3$ such that ϕ is constant on $[J]^2$; suppose that $J = \{r,s,t\}$ and $\phi(\{r,s\}) = \phi(\{r,t\}) = \phi(\{s,t\}) = (i,j)$. Now $\Delta(a_{ri}, a_{si}) = \Delta(a_{ri}, a_{ti}) = \Delta(a_{si}, a_{ti}) = j$. But this means that for each pair from a_{ri}, a_{si}, a_{ti} there is exactly one member of the pair containing j; which is ridiculous. **XQ**

(c) Consequently we can choose $u(0), v(0), \dots, u(l), v(l)$ inductively so that

$$u(r) \ne v(r),\ D_{u(r),v(r)} \cap \bigcup_{s<r} D_{u(s),v(s)} = \emptyset$$

for every $r \le l$; and these will serve.

7C Lemma Let κ be an atomlessly-measurable cardinal and R, S two upwards-ccc partially ordered sets, both of size strictly less than κ. Let $\lambda < \kappa$ be any cardinal. Then there are partially ordered sets P, Q such that

$$\#(P) \le \max(\omega, \lambda),\ \#(Q) \le \max(\omega, \lambda),$$

$$\mathrm{S}(P \times R) \le \omega_1,\ \mathrm{S}(Q \times S) \le \omega_1,\ \mathrm{S}(P \times Q) > \lambda,$$

writing $\mathrm{S}(P \times R)$ for the Souslin number of $P \times R$, as in A1P.

proof (a) Let ν be a κ-additive extension of the usual measure μ on $\{0,1\}^{\mathbb{N}}$ to every subset of $\{0,1\}^{\mathbb{N}}$ (1De, A2Gb). Let $A \subseteq \{0,1\}^{\mathbb{N}}$ be a set of cardinal λ. For each $z \in \{0,1\}^{\mathbb{N}}$ set

$$P_z = \{I : I \in [A]^{<\omega},\ z(\Delta(a,b)) = 1 \text{ for all distinct } a,\ b \in I\},$$

$$Q_z = \{I : I \in [A]^{<\omega},\ z(\Delta(a,b)) = 0 \text{ for all distinct } a,\ b \in I\},$$

ordering both by $\subseteq$. Then $\mathrm{S}(P_z \times Q_z) > \lambda$ because $\{(\{a\},\{a\}) : a \in A\}$ is an up-antichain in $P_z \times Q_z$. Also $\#(P_z) \le \max(\omega, \lambda)$ and $\#(Q_z) \le \max(\omega, \lambda)$.

(b) I shall therefore be done if I can find a $z \in \{0,1\}^{\mathbb{N}}$ such that $P_z \times R$ and $Q_z \times S$ are both upwards-ccc. In fact I show that $P_z \times R$ is upwards-ccc for ν-almost every z; as the same argument will work for $Q_z \times S$, that will be more than enough.

Set

$$H_0 = \{z : z \in \{0,1\}^{\mathbb{N}},\ \mathrm{S}(P_z \times R) > \omega_1\}.$$

For each $z \in H_0$ there is a family $\langle (I_\alpha(z), r_\alpha(z))\rangle_{\alpha<\omega_1}$ enumerating an uncountable up-antichain in $P_z \times R$. There is an $n_z \in \mathbb{N}$ such that $B_z = \{\alpha : \#(I_\alpha(z)) = n_z\}$ is uncountable. Let $\langle (J_\alpha(z), s_\alpha(z))\rangle_{\alpha<\omega_1}$ be a re-enumeration of the up-antichain $\langle (I_\alpha(z), r_\alpha(z))\rangle_{\alpha\in B_z}$ in $P_z \times R$.

(c) ? Suppose now that $\nu H_0 > 0$. In this case there is an $n \in \mathbb{N}$ such that $\nu H_1 > 0$, where

$$H_1 = \{z : z \in H_0,\ n_z = n\}.$$

Because A and R both have cardinal less than κ, we can find for each $\alpha < \omega_1$ a finite set $J^*_\alpha \subseteq A$ and an $s^*_\alpha \in R$ such that $\nu E_\alpha > 0$, where

$$E_\alpha = \{z : z \in H_1,\ J_\alpha(z) = J^*_\alpha,\ s_\alpha(z) = s^*_\alpha\}.$$

Enumerate J^*_α as $\langle a_{\alpha i}\rangle_{i<n}$ for each $i < n$. For α, $\beta < \omega_1$ set $D_{\alpha\beta} = \{\Delta(a_{\alpha i}, a_{\beta i}) : i < n\} \cap \mathbb{N}$; note that $\#(D_{\alpha\beta}) \le n$.

(d) Let $\gamma > 0$ be such that

$$T = \{\alpha : \alpha < \omega_1,\ \nu E_\alpha \ge \gamma\}$$

is uncountable. For each $\alpha \in T$ set

$$m_\alpha = \sup\{\Delta(a_{\alpha i}, a_{\alpha j}) : i < j < n\} + 1.$$

Let m be such that

$$U = \{\alpha : \alpha \in T,\, m_\alpha = m\}$$

is uncountable. Because there are only finitely many possibilities for the family $\langle a_{\alpha i} \cap m\rangle_{i<n}$ (identifying m with the set of its predecessors), there is a family $\langle b_i\rangle_{i<n}$ in $\mathcal{P}m$ such that

$$V = \{\alpha : \alpha \in U,\, a_{\alpha i} \cap m = b_i \ \forall \ i < n\}$$

is uncountable.

(e) If α, β are distinct members of V, $z \in E_\alpha \cap E_\beta$, and s^*_α, s^*_β are upwards-compatible in R, there is a $k \in D_{\alpha\beta}$ such that $z(k) = 0$. **P** $(J^*_\alpha, s^*_\alpha) = (J_\alpha(z), s_\alpha(z))$ and (J^*_β, s^*_β) are upwards-incompatible in $P_z \times R$; we are supposing that s^*_α and s^*_β are upwards-compatible in R, so $J^*_\alpha \cup J^*_\beta \notin P_z$. In this case, there must be a, $b \in J^*_\alpha \cup J^*_\beta$ such that $z(\Delta(a, b)) = 0$. But J^*_α and J^*_β do belong to P_z, so neither can contain both a and b, and we may take it that $a \in J^*_\alpha$ and $b \in J^*_\beta$. Thus there must be i, $j < n$ such that $z(\Delta(a_{\alpha i}, a_{\beta j})) = 0$.

? If $i \neq j$, then

$$\Delta(a_{\alpha i}, a_{\alpha j}) = \Delta(b_i, b_j) = \Delta(a_{\alpha i}, a_{\beta j}),$$

so

$$z(\Delta(a_{\alpha i}, a_{\beta j})) = z(\Delta(a_{\alpha i}, a_{\alpha j})) = 1. \ \mathbf{X}$$

Thus $i = j$ and $z(\Delta(a_{\alpha i}, a_{\beta i})) = 0$, where $\Delta(a_{\alpha i}, a_{\beta i}) \in D_{\alpha\beta}$. **Q**

(f) Let k be so large that $(1-2^{-n})^{k+1} \leq \frac{1}{3}\gamma$; set $l = 3(n^2k)!$. By A2S, there is an uncountable set $W \subseteq V$ such that $\mu^*(\bigcap_{\alpha\in L} E_\alpha) \geq \frac{2}{3}\gamma$ whenever $L \in [W]^l$. By A1Q, there is an $L \in [W]^l$ such that $\{s^*_\alpha : \alpha \in L\}$ is bounded above. By 7B, there are $\alpha(0), \dots, \alpha(k), \beta(0), \dots, \beta(k)$ in L such that $\alpha(j) \neq \beta(j)$ for $j \leq k$ and $\langle D_{\alpha(j),\beta(j)}\rangle_{j\leq k}$ is disjoint. Now however observe that if $z \in \bigcap_{\alpha\in L} E_\alpha$ and $i \leq k$ then $s^*_{\alpha(i)}$ and $s^*_{\beta(i)}$ are upwards-compatible in R, so there is a $j \in D_{\alpha(i),\beta(i)}$ such that $z(j) = 0$, by (e) above. Thus $\bigcap_{\alpha\in L} E_\alpha \subseteq F$ where

$$F = \{z : \forall \ i \leq k \ \exists \ j \in D_{\alpha(i),\beta(i)} \text{ such that } z(j) = 0\}.$$

But, because $\langle D_{\alpha(i),\beta(i)}\rangle_{i\leq k}$ is disjoint and $\#(D_{\alpha(i),\beta(i)}) \leq n$ for each i,

$$\begin{aligned} \mu F &= \prod_{i\leq k} \mu\{z : \exists \ j \in D_{\alpha(i),\beta(i)},\, z(j) = 0\} \\ &\leq \prod_{i\leq k} (1 - 2^{-n}) = (1 - 2^{-n})^{k+1} \leq \gamma/3. \end{aligned}$$

So $\frac{2}{3}\gamma \leq \mu^*(\bigcap_{\alpha\in L} E_\alpha) \leq \frac{1}{3}\gamma$. **X**

This contradiction shows that $\mathrm{S}(P_z \times R) \leq \omega_1$ for ν-almost all z, as required.

7D Theorem Let κ be an atomlessly-measurable cardinal. Then there are ccc partially ordered sets P, Q such that $\mathrm{S}(P \times Q) = \kappa$.

proof Construct $\langle P_\xi\rangle_{\xi<\kappa}$, $\langle Q_\xi\rangle_{\xi<\kappa}$, $\langle P^*_\xi\rangle_{\xi\leq\kappa}$, $\langle Q^*_\xi\rangle_{\xi\leq\kappa}$ as follows. Given $\langle P_\eta\rangle_{\eta<\xi}$ and $\langle Q_\eta\rangle_{\eta<\xi}$, then P^*_ξ and Q^*_ξ are to be the finite-support products of these families. Given that P^*_ξ and Q^*_ξ are ccc and that their cardinals are at most $\max(\omega, \#(\xi))$, for $\xi < \kappa$, then P_ξ and Q_ξ are to be partially ordered sets of size at most $\max(\omega, \#(\xi))$ such that $\mathrm{S}(P_\xi \times P^*_\xi) \leq \omega_1$, $\mathrm{S}(Q_\xi \times Q^*_\xi) \leq \omega_1$, and $\mathrm{S}(P_\xi \times Q_\xi) > \#(\xi)$; Lemma 7C tells us that such can be found. The induction continues because at successor stages $P_\xi \times P^*_\xi$ and $Q_\xi \times Q^*_\xi$ can be embedded as cofinal subsets of $P^*_{\xi+1}$ and $Q^*_{\xi+1}$ respectively, so that $P^*_{\xi+1}$ and $Q^*_{\xi+1}$ will be ccc if P^*_ξ and Q^*_ξ are, while for limit ordinals $\xi > 0$, $P^*_\xi = \bigcup_{\eta<\xi} P^*_\eta$ will be ccc because all the preceding P^*_η are, by the remarks in A1R.

On completing the induction, set $P = P^*_\kappa$, $Q = Q^*_\kappa$. Observe that the finite-support product R of $\langle P_\xi \times Q_\xi\rangle_{\xi<\kappa}$ can be embedded as a cofinal subset of $P \times Q$, so that $\mathrm{S}(P \times Q) = \mathrm{S}(R)$. Also $\sup\{\mathrm{S}(\prod_{\xi\in J} P_\xi \times Q_\xi) : J \in [\kappa]^{<\omega}\} = \kappa$, because $\mathrm{S}(P_\xi \times Q_\xi) > \#(\xi)$ for every $\xi < \kappa$, while $\#(\prod_{\xi\in J} P_\xi \times Q_\xi) < \kappa$ for every $J \in [\kappa]^{<\omega}$. Because κ is regular, A1R tells us that $\mathrm{S}(R) = \kappa$, so that $\mathrm{S}(P \times Q) = \kappa$, as claimed.

Remark Note that (if it is consistent to suppose that there is a two-valued-measurable cardinal) it is consistent to suppose that there is an atomlessly-measurable cardinal and that Souslin's hypothesis is true; see Laver 87.

7E Definition Let $(X,\le)$ be a totally ordered set. Let $\le_1$ be the original order $\le$ and $\le_{-1}$ the reverse total ordering $\ge$. We say that a set $A\subseteq X$ is ω_1**-entangled** if for any $n\in\mathbb{N}$, $\epsilon\in\{-1,1\}^n$ and every family $\langle x_{\xi i}\rangle_{\xi<\omega_1,i<n}$ of distinct elements of A there are distinct ξ, $\eta<\omega_1$ such that $x_{\xi i}\le_{\epsilon(i)}x_{\eta i}$ for every $i<n$.

7F Proposition Let κ be an atomlessly-measurable cardinal. Then for any $\lambda<\kappa$ there is an ω_1-entangled subset of $\mathbb{R}$ of cardinal λ.

proof (a) Let X be $\{-1,1\}^{\mathbb{N}}$ with its lexicographic total ordering, so that $x<y$ iff $x(\Delta(x,y))=-1$, $y(\Delta(x,y))=1$, writing $\Delta(x,y)=\min\{m:x(m)\ne y(m)\}$ (cf. 7A). The map $x\mapsto\sum_{m\in\mathbb{N}}3^{-m}x(m):X\to\mathbb{R}$ is strictly increasing, so that any ω_1-entangled subset of X has an ω_1-entangled direct image in $\mathbb{R}$, and it will be enough to find an ω_1-entangled subset of X of cardinal λ.

(b) Set $S=\bigcup_{n\in\mathbb{N}}\{-1,1\}^n$, $Z=\{-1,1\}^S$; let μ be the usual measure on Z (as in A2G), and ν a κ-additive extension of μ to $\mathcal{P}Z$ (see 1De, A2Gb). For $z\in Z$ define $h_z:X\to X$ by writing

$$h_z(x)(m)=x(m)\times z(x\restriction m)\ \ \forall\ m\in\mathbb{N},\,x\in X;$$

then h_z is a bijection. Observe that if x, $y\in X$ and $\Delta(x,y)=m$ then $h_z(x)<_\epsilon h_z(y)$ iff $z(x\restriction m)=y(m)\times\epsilon$.

(c) Let $B\subseteq X$ be any set of cardinal λ, and for $z\in Z$ write $A_z=h_z[B]\in[X]^\lambda$. I aim to show that A_z is ω_1-entangled for some, in fact for ν-almost every, $z\in Z$.

Let E be $\{z:z\in Z,\,A_z\text{ is not }\omega_1\text{-entangled}\}$. For each $z\in E$ we can find $n_z\in\mathbb{N}$, $\epsilon_z\in\{-1,1\}^{n_z}$ and a family $\langle x_{z\xi i}\rangle_{\xi<\omega_1,i<n_z}$ of distinct elements of B such that

$$\forall\ \xi<\eta<\omega_1\ \exists\ i<n,\,h_z(x_{z\xi i})\not\le_{\epsilon_z(i)}h_z(x_{z\eta i}).$$

Because all the $x_{z\xi i}$ are distinct, there will now be a $k_z\in\mathbb{N}$ such that

$$C_z=\{\xi:x_{z\xi i}\restriction k_z\ne x_{z\xi j}\restriction k_z\ \ \forall\ i<j<n_z\}$$

is uncountable; next, there will be a family $\langle u_{zi}\rangle_{i<n_z}$ of distinct elements of $\{-1,1\}^{k_z}$ such that

$$C'_z=\{\xi:\xi\in C_z,\,x_{z\xi i}\restriction k_z=u_{zi}\ \ \forall\ i<n_z\}$$

is uncountable. Let $\langle y_{z\xi i}\rangle_{\xi<\omega_1,i<n_z}$ be a re-enumeration of $\langle x_{z\xi i}\rangle_{\xi\in C'_z,i<n_z}$, so that

$$\langle y_{z\xi i}\rangle_{\xi<\omega_1,i<n_z}\text{ is a family of distinct elements of }B,$$

$$y_{z\xi i}\restriction k_z=u_{zi}\text{ for }i<n_z,\,\xi<\omega_1,$$

$$\forall\ \xi<\eta<\omega_1\ \exists\ i<n_z,\,h_z(y_{z\xi i})\not\le_{\epsilon_z(i)}h_z(y_{z\eta i}).$$

(d) ? Suppose, if possible, that $\nu E>0$. Then there are n, $k\in\mathbb{N}$, $\epsilon\in\{-1,1\}^n$ and a family $\langle u_i\rangle_{i<n}$ in $\{-1,1\}^k$ such that $\nu F>0$, where

$$F=\{z:z\in E,\,n_z=n,\,k_z=k,\,\epsilon_z=\epsilon,\,u_{zi}=u_i\ \ \forall\ i<n\}.$$

Note that the u_i must all be different. Next, because $\#(B^n)<\kappa$, we can find for each $\xi<\omega_1$ a family $\langle y_{\xi i}\rangle_{i<n}$ in B such that $\nu F_\xi>0$, where

$$F_\xi=\{z:z\in F,\,y_{z\xi i}=y_{\xi i}\ \ \forall\ i<n\}.$$

Thus

$$y_{\xi i}\restriction k=u_i\text{ for }i<n,\,\xi<\omega_1,$$

and if $z\in F_\xi\cap F_\eta$, where $\xi<\eta<\omega_1$, there must be an $i<n$ such that

$$h_z(y_{\xi i})\not\le_{\epsilon(i)}h_z(y_{\eta i}).$$

(e) Let $\gamma>0$ be such that $U=\{\xi:\nu F_\xi\ge\gamma\}$ is uncountable. Let $l\ge1$ be such that $(1-2^{-n})^l\le\frac{1}{3}\gamma$. By A2S, there is an uncountable $V\subseteq U$ such that

$$\mu^*(\textstyle\bigcap_{\xi\in L}F_\xi)\ge\frac{2}{3}\gamma\text{ for every }L\in[V]^{l+1}.$$

Let $\langle y_i\rangle_{i<n}$ be a cluster point of $\langle\langle y_{\xi i}\rangle_{i<n}\rangle_{\xi\in V}$ in the sense that

$$\forall\ k\in\mathbb{N},\,\zeta<\omega_1\ \exists\ \xi\in V\setminus\zeta\text{ such that }y_{\xi i}\restriction k=y_i\restriction k\ \ \forall\ i<n.$$

Note that because $F_\xi \cap F_\eta \neq \emptyset$ for ξ, $\eta \in V$, all the $y_{\xi i}$, for $\xi \in V$ and $i < n$, are distinct; so that there is a cofinite $W \subseteq V$ such that $y_{\xi i} \neq y_i$ for $\xi \in W$, $i < n$.

(f) Choose strictly increasing sequences $\langle r_j \rangle_{j\in\mathbb{N}}$ in $\mathbb{N}$, $\langle \xi_j \rangle_{j\in\mathbb{N}}$ in W such that $r_0 = k$ and

$$y_{\xi_j i}\restriction r_j = y_i\restriction r_j \ \ \forall\ \ i < n,$$

$$y_{\xi_j i}\restriction r_{j+1} \neq y_i\restriction r_{j+1} \ \ \forall\ \ i < n$$

for every $j \in \mathbb{N}$. For $j \in \mathbb{N}$, $i < n$ set

$$m_{ji} = \Delta(y_{\xi_j i}, y_i),$$

so that $r_j \leq m_{ji} < r_{j+1}$ and also

$$m_{ji} = \Delta(y_{\xi_{j+1} i}, y_{\xi_j i}).$$

Set

$$s_{ji} = y_i\restriction m_{ji} = y_{\xi_j i}\restriction m_{ji} = y_{\xi_{j+1} i}\restriction m_{ji} \in S.$$

Observe that all the s_{ji} are distinct; this is because $r_j \leq m_{ji} < r_{j+1}$ for all i, j, so that $s_{ji} \neq s_{j'i'}$ if $j \neq j'$, while $r_0 = k$, so that $s_{ji}\restriction k = u_i$ for all i, j and $s_{ji} \neq s_{j'i'}$ if $i \neq i'$.

(g) If $z \in Z$, $j \in \mathbb{N}$, $i < n$ then

$$h_z(y_{\xi_j i}) \leq_{\epsilon(i)} h_z(y_{\xi_{j+1} i}) \iff z(s_{ji}) = y_i(m_{ji}) \times \epsilon(i).$$

So if we set $G = \bigcap_{j\leq l} F_{\xi_j}$, then for any $z \in G$ we have $y_{\xi_j i} = y_{z\xi_j i}$ for every $j \leq l$, $i < n$, so for each $j < l$ there must be an $i < n$ such that

$$h_z(y_{\xi_j i}) \not\leq_{\epsilon(i)} h_z(y_{\xi_{j+1} i}),$$

that is,

$$z(s_{ji}) \neq y_i(m_{ji}) \times \epsilon(i).$$

Accordingly $G \subseteq H$, where

$$H = \{z : \forall\ \ j < l\ \ \exists\ \ i < n \text{ such that } z(s_{ji}) \neq y_i(m_{ji}) \times \epsilon(i)\}.$$

Because all the s_{ji} are distinct, $\mu H = (1 - 2^{-n})^l \leq \frac{1}{3}\gamma$, so $\mu^* G \leq \frac{1}{3}\gamma$. But by the choice of V, $\mu^* G \geq \frac{2}{3}\gamma$. **X**

This contradiction shows that $\nu E = 0$. So we can take a $z \in Z \setminus E$ to yield the required entangled set $A_z \subseteq X$.

Remark 7F is due to S.Todorčević; it corresponds to Theorem 2 of TODORČEVIĆ 85, from which it may be deduced using Corollary 4Oa.

Version of 18.9.92

7G κ-measure-bounded partially ordered sets (a) Let P be a partially ordered set and κ a real-valued-measurable cardinal. I will say that P is **κ-measure-bounded** (upwards) if for every κ-additive probability μ on P with domain $\mathcal{P}P$ there is a $p \in P$ such that $\mu\{p' : p' \leq p\} > 0$.

(b) Let P and Q be partially ordered sets. I say that a function $f : P \to Q$ is an **ω-Tukey** function if for every $q \in Q$ there is a countable set $D \subseteq P$ such that whenever $p \in P$ and $f(p) \leq q$ there is a $d \in D$ such that $p \leq d$.

(For a systematic discussion of this concept see FREMLIN P90.)

7H Elementary facts Let κ be a real-valued-measurable cardinal.

(a) If P and Q are partially ordered sets, $f : P \to Q$ is an ω-Tukey function, and Q is κ-measure-bounded, then P is κ-measure-bounded. **P** Let μ be a κ-additive probability with domain $\mathcal{P}P$. Let $\nu = \mu f^{-1} : \mathcal{P}Q \to [0, 1]$. Then ν is κ-additive, so there is a $q \in Q$ such that $\nu\{q' : q' \leq q\} > 0$. Let $D \subseteq P$ be a countable set such that whenever $f(p) \leq q$ there is a $d \in D$ such that $p \leq d$. Then

$$0 < \nu\{q' : q' \leq q\} = \mu\{p : f(p) \leq q\} \leq \textstyle\sum_{d\in D} \mu\{p : p \leq d\},$$

so there is a $d \in D$ such that $\mu\{p : p \le d\} > 0$. As μ is arbitrary, P is κ-measure-bounded. **Q**

(b) If P is a partially ordered set and $Q \subseteq P$ is cofinal with P, then P is κ-measure-bounded iff Q is. **P** The identity function from Q to P is ω-Tukey, and any function f from P to Q such that $f(p) \le p$ for every p is also ω-Tukey. **Q**

(c) If P is any partially ordered set such that $\mathrm{cf}(P) < \kappa$ then P is κ-measure-bounded.

(d) Suppose that P is a κ-measure-bounded partially ordered set and that μ is a κ-additive probability on P with domain $\mathcal{P}P$. (i) If P is upwards-directed then $\sup_{p\in P} \mu\{p' : p' \le p\} = 1$. (ii) If $\mathrm{add}(P) > \omega$ (P is 'countably closed') then there is a $p \in P$ such that $\mu\{p' : p' \le p\} = 1$. **P** (i) For $A \subseteq P$ set

$$\nu A = \mu A - \sup_{p\in P} \mu\{p' : p' \in A,\, p' \le p\}.$$

Then ν is a κ-additive measure on P and $\nu\{p' : p' \le p\} = 0$ for every $p \in P$. Because P is κ-measure-bounded, νP must be 0, that is, $\sup_{p\in P} \mu\{p' : p' \le p\} = 1$. (ii) If $\mathrm{add}(P) > \omega$, take a sequence $\langle p_n\rangle_{n\in\mathbb{N}}$ in P such that $\lim_{n\to\infty} \mu\{p' : p' \le p_n\} = 1$; there is a $p \in P$ such that $p_n \le p$ for every n, and now $\mu\{p' : p' \le p\} = 1$. **Q**

(e) (i) If $\langle P_n\rangle_{n\in\mathbb{N}}$ is a sequence of upwards-directed κ-measure-bounded partially ordered sets then $P = \prod_{n\in\mathbb{N}} P_n$ is κ-measure-bounded. (ii) If $\langle P_\zeta\rangle_{\zeta<\lambda}$ is a family of κ-measure-bounded partially ordered sets such that $\lambda < \kappa$ and $\mathrm{add}(P_\zeta) > \omega$ for every $\zeta < \kappa$, then $P = \prod_{\zeta<\lambda} P_\zeta$ is κ-measure-bounded. **P** (i) Let μ be a κ-additive probability with domain $\mathcal{P}P$. For each $n \in \mathbb{N}$ set $\mu_n = \mu\pi_n^{-1} : \mathcal{P}P_n \to [0,1]$, where $\pi_n : P \to P_n$ is the canonical map. By (d-i) there is a $p_n \in P_n$ such that $\mu_n\{q : q \in P_n,\, q \le p_n\} \ge 1 - 2^{-n-2}$. Now if we set $p = \langle p_n\rangle_{n\in\mathbb{N}} \in P$, we see that $\mu\{p' : p' \le p\} \ge \frac{1}{2}$. (ii) Argue as in (i), but taking $p_\zeta \in P_\zeta$ such that $\mu_\zeta\{q : q \le p_\zeta\} = 1$, so that $\mu\{p' : p' \le p\} = 1$, because $\lambda < \kappa$. **Q**

(f) If P is a κ-measure-bounded partially ordered set, Q is another partially ordered set, and $f : P \to Q$ is an order-preserving surjection, then Q is κ-measure-bounded. **P** Take any $g : Q \to P$ such that $f(g(q)) = q$ for every $q \in Q$; then g is ω-Tukey. **Q**

7I Examples (a) $\mathbb{N}^{\mathbb{N}}$ is κ-measure-bounded for any real-valued-measurable cardinal κ. (Use 7H(e-i).)

(b) $\omega_1^{\omega_1}$ is κ-measure-bounded for any real-valued-measurable cardinal κ. (Use 7H(e-ii).)

(c) $\mathbb{N}^{\mathbb{N}}/\mathcal{F}$ is κ-measure-bounded for any real-valued-measurable cardinal κ, any filter $\mathcal{F}$ on $\mathbb{N}$. (Use 7Hf.)

(d) If λ and θ are cardinals then $[\lambda]^{<\theta}$ is κ-measure-bounded for any real-valued-measurable cardinal $\kappa > \lambda$. **P** If μ is a κ-additive probability with domain $\mathcal{P}[\lambda]^{<\theta}$ we may apply 5Aa with $X = [\lambda]^{<\theta}$, $Y = \lambda$ and f the identity map to find an $M \in X$ such that $\mu(\mathcal{P}M) > 0$. **Q**

(e) Let $\mathcal{E}$ be the family of closed Lebesgue negligible subsets of $[0,1]$, ordered by $\subseteq$. Then $\mathcal{E}$ is not κ-measure-bounded for any atomlessly-measurable cardinal κ. **P** If κ is atomlessly-measurable, there is a κ-additive extension μ of Lebesgue measure to $\mathcal{P}[0,1]$. Define $f : [0,1] \to \mathcal{E}$ by setting $f(a) = \{a\}$ for every $a \in [0,1]$, and set $\nu = \mu f^{-1} : \mathcal{P}\mathcal{E} \to [0,1]$. Then $\nu\{E : E \subseteq F\} = \mu F = 0$ for every $F \in \mathcal{E}$, so ν witnesses that $\mathcal{E}$ is not κ-measure-bounded. **Q**

Remark I include (e) because (as explained in FREMLIN P90) many of the natural partially ordered sets P of analysis allow ω-Tukey functions from $\mathcal{E}$ to P, so cannot be κ-measure-bounded.

7J Proposition Suppose that κ is a real-valued-measurable cardinal and that P is a κ-measure-bounded partially ordered set.

(a) If $\langle Q_\xi\rangle_{\xi<\kappa}$ is an increasing family of subsets of P and $Q = \bigcup_{\xi<\kappa} Q_\xi$ is cofinal with P, then there is a $\xi < \kappa$ such that Q_ξ is cofinal with P.

(b) $\mathrm{cf}(\mathrm{cf}(P)) \ne \kappa$.

proof (a) ? If no Q_ξ is cofinal with P, then we may choose a function $f : \kappa \to P$ such that $f(\xi) \not\le q$ whenever $\xi < \kappa$, $q \in Q_\xi$. Let ν be a witnessing probability on the real-valued-measurable cardinal κ, and set $\mu = \nu f^{-1} : \mathcal{P}P \to [0,1]$. Then there should be a $p \in P$ such that $\mu\{p' : p' \le p\} > 0$. But now there are a $q \in Q$ such that $p \le q$ and a $\xi < \kappa$ such that $q \in Q_\xi$, in which case

$$0 < \mu\{p' : p' \le p\} \le \nu\{\eta : f(\eta) \le q\} \le \nu\{\eta : \eta < \xi\},$$

which is impossible. **X**

(b) Set $\lambda = \mathrm{cf}(P)$. **?** If $\mathrm{cf}(\lambda) = \kappa$, take a cofinal set $Q \subseteq P$ of cardinal λ. Then there is an increasing family $\langle Q_\xi \rangle_{\xi<\kappa}$ of subsets of Q, all of cardinal strictly less than λ, therefore not cofinal with P, but with union Q, contradicting (a). **X**

7K Corollaries Let κ be a real-valued-measurable cardinal.

(a) If $\mathcal{F}$ is any filter on $\mathbb{N}$, then $\mathrm{cf}(\mathrm{cf}(\mathbb{N}^{\mathbb{N}}/\mathcal{F})) \neq \kappa$; in particular, $\mathrm{cf}(\mathfrak{d}) \neq \kappa$.

(b) If λ is a cardinal and $\langle \theta_\zeta \rangle_{\zeta<\lambda}$ is a family of regular cardinals all greater than λ and less than κ, then $\mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta) < \kappa$.

(c) If α and γ are cardinals less than κ then $\Theta(\alpha, \gamma)$ (see A1Jb) is less than κ.

(d) If α, β, γ and δ are cardinals, with $\delta \geq \omega_1$, $\gamma \leq \beta$, $\alpha < \kappa$, then $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta)$ (see A1Ja) is less than κ.

(e) $\Theta(\kappa, \kappa) = \kappa$ and

$$\{\alpha : \alpha < \kappa \text{ is a cardinal, } \Theta(\alpha, \alpha) = \alpha\}$$

belongs to the rvm filter of κ.

(f) If $\kappa = \mathfrak{c}$, $\lambda < \kappa$ and $\langle P_\zeta \rangle_{\zeta<\lambda}$ is a family of partially ordered sets such that $\omega < \mathrm{add}(P_\zeta)$, $\mathrm{cf}(P_\zeta) < \kappa$ for every $\zeta < \lambda$, then $\mathrm{cf}(\prod_{\zeta<\lambda} P_\zeta) < \kappa$.

proof (a) Use 7Ic and 7Jb.

(b) **?** If $\kappa \leq \delta = \mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta)$, then by A1Ic there is an ultrafilter $\mathcal{F}$ on λ such that $\delta = \mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta/\mathcal{F})$, and by A1Id there is a family $\langle \theta'_\zeta \rangle_{\zeta<\lambda}$ such that $\theta'_\zeta \leq \theta_\zeta$ for every $\zeta < \lambda$ and $\kappa = \mathrm{cf}(\prod_{\zeta<\lambda} \theta'_\zeta/\mathcal{F})$. But by 7H(d-ii) and 7Hf, $\prod_{\zeta<\lambda} \theta'_\zeta/\mathcal{F}$ is κ-measure-bounded and its cofinality cannot be κ, by 7Jb. **X**

(c) **?** Suppose, if possible, otherwise. Then for each $\xi < \kappa$ there must be a family $\langle \theta_{\xi\zeta} \rangle_{\zeta<\lambda_\xi}$ of regular cardinals less than α such that $\lambda_\xi < \theta_{\xi\zeta}$ for every $\zeta < \lambda_\xi$ and $\mathrm{cf}(\prod_{\zeta<\lambda_\xi} \theta_{\xi\zeta}) \geq \xi$. Because $\alpha < \kappa$ there is a cardinal $\lambda < \kappa$ such that

$$A = \{\xi : \xi < \kappa,\ \lambda_\xi = \lambda\}$$

is unbounded in κ. Now by 5Ab there is a set $M \subseteq \alpha$, of cardinal at most λ, such that

$$B = \{\xi : \xi \in A,\ \theta_{\xi\zeta} \in M\ \ \forall\ \zeta < \lambda\}$$

is unbounded in κ. Let $\langle \theta_\zeta \rangle_{\zeta<\lambda'}$ be any enumeration of M. By (b), there is a cofinal set $F \subseteq \prod_{\zeta<\lambda'} \theta_\zeta$ with $\#(F) < \kappa$. Let $\xi \in B$ be such that $\xi > \#(F)$. For each $f \in F$ define $g_f \in \prod_{\zeta<\lambda} \theta_{\xi\zeta}$ by setting

$$g_f(\zeta) = f(\zeta') \text{ whenever } \theta_{\xi\zeta} = \theta_{\zeta'}.$$

Then $\{g_f : f \in F\}$ is cofinal with $\prod_{\zeta<\lambda} \theta_{\xi\zeta}$, because if $h \in \prod_{\zeta<\lambda} \theta_{\xi\zeta}$ there is an $f \in F$ such that

$$f(\zeta') \geq \sup\{h(\zeta) : \zeta < \lambda,\ \theta_{\xi\zeta} = \theta_{\zeta'}\}$$

for every $\zeta' < \lambda'$, and in this case $h \leq g_f$. So

$$\#(F) < \xi \leq \mathrm{cf}(\textstyle\prod_{\zeta<\lambda} \theta_{\xi\zeta}) \leq \#(F),$$

which is absurd. **X**

(d) This is trivial if any of the cardinals α, β, γ is finite; let us take it that they are all infinite. In this case $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta) \leq \mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1)$, and $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) \leq \Theta(\alpha, \gamma) < \kappa$, by A1K and (c) above.

(e) From (d) we see that

$$C = \{\alpha : \alpha < \kappa \text{ is a cardinal, } \Theta(\beta, \beta) \leq \alpha\ \ \forall\ \beta < \alpha\}$$

is a closed unbounded set in κ, so D belongs to the rvm filter of κ, where D is the set of regular cardinals in C (using a fraction of 4K). But if $\alpha \in D$, and $\langle \theta_\zeta \rangle_{\zeta<\lambda}$ is a family of regular cardinals with $\lambda < \theta_\zeta < \alpha$ for every $\zeta < \lambda$, set $\beta = \sup_{\zeta\in I} \theta_\zeta^+$; then

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda} \theta_\zeta) \leq \Theta(\beta, \beta) \leq \alpha.$$

Accordingly $\Theta(\alpha, \alpha) = \alpha$ for every $\alpha \in D$. In the same way $\Theta(\kappa, \kappa) = \kappa$.

(f) By 5E, we have $\mathrm{cf}(\prod_{\zeta<\lambda} P_\zeta) \le \mathfrak{c}$; by 7H(d-ii) and 7Jb, $\mathrm{cf}(\prod_{\zeta<\lambda} P_\zeta) \ne \kappa$.

Remarks Part (f) comes from Prikry 75, Theorem 2c. Parts (b)-(d) come from Gitik & Shelah p91.

7L Cofinalities II: Proposition Suppose that κ is an atomlessly-measurable cardinal. Let X be a set of cardinal at most $\min(2^\kappa, \kappa^{(+\omega)})$ and $\mathcal{F}$ a filter on X which has a base of cardinal less than κ and contains no countable set. Then $\mathrm{cf}(\mathbb{N}^X/\mathcal{F}) \ge \kappa$.

proof Let ν be a witnessing probability on κ. By the Gitik-Shelah theorem (3F-3H), there is a stochastically independent family $\langle E_{xn}\rangle_{x\in X,n\in\mathbb{N}}$ of subsets of κ of measure $\frac{1}{2}$. Define $\phi : \kappa \to \mathbb{N}^X$ by

$$\phi(\xi)(x) = \min\{n : \xi \in E_{xn}\} \text{ if } \xi \in \bigcup_{n\in\mathbb{N}} E_{xn},$$
$$= 0 \text{ otherwise.}$$

Now let $\mathcal{A} \subseteq \mathcal{F}$ be a base of cardinal $< \kappa$. For $f \in \mathbb{N}^X$, $A \in \mathcal{A}$ set

$$W_{fA} = \{\xi : \xi < \kappa,\ \phi(\xi)(x) \le f(x)\ \forall\ x \in A\}.$$

Then

$$\nu W_{fA} = \textstyle\prod_{x\in A}(1 - 2^{-f(x)-1}) = 0$$

because A is uncountable. If $F \subseteq \mathbb{N}^X$ is any set of cardinal less than κ, then

$$\{\xi : \exists\ f \in F,\ \phi(\xi) \le_{\mathcal{F}} f\} = \textstyle\bigcup_{f\in F, A\in\mathcal{A}} W_{fA} \in \mathcal{N}_\nu,$$

so there is a $\xi < \kappa$ such that $\phi(\xi) \not\le_{\mathcal{F}} f$ for every $f \in F$, and $\{f^\bullet : f \in F\}$ is not cofinal with $\mathbb{N}^X/\mathcal{F}$. As F is arbitrary, $\mathrm{cf}(\mathbb{N}^X/\mathcal{F}) \ge \kappa$.

7M Corollary Let κ be an atomlessly-measurable cardinal.
(a) $\mathrm{cf}(\mathbb{N}^\lambda) \ge \kappa$ for every $\lambda > \omega$. (For $\mathrm{cf}(\mathbb{N}^\lambda) \ge \mathrm{cf}(\mathbb{N}^{\omega_1})$.)
(b) If $\lambda < \kappa$ is a regular uncountable cardinal, and $\mathcal{F} = \{F : F \subseteq \lambda,\ \#(\lambda \setminus F) < \lambda\}$ then $\mathrm{cf}(\mathbb{N}^\lambda/\mathcal{F}) \ge \kappa$.
(c) Suppose that $\mathfrak{c}$ is atomlessly-measurable. Let X be a set of cardinal less than $\mathfrak{c}$ and $\mathcal{F}$ a filter on X with a base of cardinal less than $\mathfrak{c}$, not containing any countable set. Then $\mathrm{cf}(\mathbb{N}^X/\mathcal{F}) = \mathfrak{c}$. (Use 5E to see that $\#(\mathbb{N}^X) = \mathfrak{c}$.)

Remark 7L-7M come from Jech & Prikry 84.

7N Remark The results 7Kb-e are interesting; the theory in which they are embedded, partially described in A1H-A1J, is astonishing. But it may well be that the last word has not been said. From 7Kb we see, for instance, that if κ is real-valued-measurable then $\mathrm{cf}(\prod_{n\in\mathbb{N}} \omega_n) < \kappa$. But in fact $\mathrm{cf}(\prod_{n\in\mathbb{N}} \omega_n) < \omega_{\omega_4}$ (Burke & Magidor 90, Theorem 6.1). We can hope that further inequalities of this type will swallow up 7Kb-e completely.

7O Lemma Let κ be an atomlessly-measurable cardinal, and γ, δ cardinals such that $\omega \le \gamma < \delta < \kappa$, $2^\beta = 2^\gamma$ for $\gamma \le \beta < \delta$, but $2^\delta > 2^\gamma$. Then δ is regular and $2^\delta = \mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) = \mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) = \mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2)$.

proof δ is regular because 2^δ is at most the cardinal power $(\max_{\beta<\delta} 2^\beta)^{\mathrm{cf}(\delta)}$. Of course

$$\mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta) \le \mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \omega_1) \le \mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, 2) \le \#([2^\gamma]^{\le\delta}) \le 2^\delta.$$

For the reverse inequality, let $\mathcal{E} \subseteq [2^\gamma]^{<\kappa}$ be a set of cardinal $\mathrm{cov}_{\mathrm{Sh}}(2^\gamma, \kappa, \delta^+, \delta)$ such that every member of $[2^\gamma]^{\le\delta}$ is covered by fewer than δ members of $\mathcal{E}$. For each ordinal $\xi < \delta$ let $\phi_\xi : \mathcal{P}\xi \to 2^\gamma$ be an injective function. For $A \subseteq \delta$ define $f_A : \delta \to 2^\gamma$ by

$$f_A(\xi) = \phi_\xi(A \cap \xi)\ \forall\ \xi < \delta.$$

Choose $E_A \in \mathcal{E}$ such that $f_A^{-1}[E_A]$ is cofinal with δ; such must exist because δ is regular and $f_A[\delta]$ can be covered by fewer than δ members of $\mathcal{E}$.

? If $2^\delta > \#(\mathcal{E})$ then there must be an $E \in \mathcal{E}$ and an $\mathcal{A} \subseteq \mathcal{P}\delta$ such that $\#(\mathcal{A}) = \kappa$ and $E_A = E$ for every $A \in \mathcal{A}$. For each pair A, B of distinct members of $\mathcal{A}$ set $\xi_{AB} = \min(A \triangle B) < \delta$. By 5B, there is a set $\mathcal{B} \subseteq \mathcal{A}$, of cardinal κ, such that $M = \{\xi_{AB} : A,\ B \in \mathcal{B},\ A \ne B\}$ is countable. Set $\zeta = \sup M < \delta$. Next, for each $A \in \mathcal{B}$, take $\eta_A > \zeta$ such that $f_A(\eta_A) \in E$. Let $\eta < \delta$ be such that $\mathcal{C} = \{A : A \in \mathcal{B}, \eta_A = \eta\}$ has cardinal κ. Then we have a map

$$A \mapsto f_A(\eta) = \phi_\eta(A \cap \eta) : \mathcal{C} \to E$$

which is injective, because if A, B are distinct members of $\mathcal{C}$ then $\xi_{AB} \le \zeta < \eta$, so $A \cap \eta \ne B \cap \eta$. So $\#(E) \ge \kappa$; but $E \in \mathcal{E} \subseteq [2^\gamma]^{<\kappa}$. **X**

7P Theorem If κ is an atomlessly-measurable cardinal,

$$\{2^\gamma : \omega \le \gamma < \kappa\}$$

is finite.

proof ? Suppose, if possible, otherwise.

(a) Define a sequence $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ of cardinals by setting

$$\gamma_0 = \omega,\ \gamma_{n+1} = \min\{\gamma : 2^\gamma > 2^{\gamma_n}\}\ \ \forall\ n \in \mathbb{N}.$$

Then we are supposing that $\gamma_n < \kappa$ for every n, so by Lemma 7O every γ_n is regular and

$$2^{\gamma_{n+1}} = \mathrm{cov}_{\mathrm{Sh}}(2^{\gamma_n}, \kappa, \gamma_{n+1}^+, \omega_1)\ \ \forall\ n \in \mathbb{N},$$

and by Theorem A1K

$$2^{\gamma_{n+1}} \le \Theta(2^{\gamma_n}, \gamma_{n+1}^+);$$

also, of course, $\Theta(2^{\gamma_n}, \gamma_{n+1}^+) \le 2^{\gamma_{n+1}}$, for every n. (For the definition of $\Theta(\alpha, \beta)$, see A1J-A1K.) Thus $2^{\gamma_{n+1}} = \Theta(2^{\gamma_n}, \gamma_{n+1}^+)$ for every n.

(b) Now $\Theta(2^{\gamma_n}, \gamma) = \Theta(\mathfrak{c}, \gamma)$ whenever $n \in \mathbb{N}$ and γ is a regular cardinal with $\gamma_n < \gamma < \kappa$. **P** Induce on n. For $n = 0$ we have $\mathfrak{c} = 2^{\gamma_0}$. For the inductive step to $n+1$, if γ is regular and $\gamma_{n+1} < \gamma < \kappa$, then $\mathfrak{c} \ge \kappa \ge \Theta(\gamma, \gamma)$ (7Kb-c), so

$$\begin{aligned}\Theta(2^{\gamma_{n+1}}, \gamma) &= \Theta(\Theta(2^{\gamma_n}, \gamma_{n+1}^+), \gamma)\\ &\le \Theta(\Theta(2^{\gamma_n}, \gamma), \gamma)\end{aligned}$$

because $\gamma \ge \gamma_{n+1}^+$

$$= \Theta(\Theta(\mathfrak{c}, \gamma), \gamma)$$

by the inductive hypothesis

$$= \Theta(\mathfrak{c}, \gamma)$$

by Lemma A1L

$$\le \Theta(2^{\gamma_{n+1}}, \gamma)$$

because $2^{\gamma_{n+1}} \ge \mathfrak{c}$. **Q** In particular, $2^{\gamma_n} = \Theta(\mathfrak{c}, \gamma_n^+)$ for every $n \ge 1$ (as well as for $n = 0$).

(c) For each $n \in \mathbb{N}$, let α_n be the least cardinal such that $\Theta(\alpha_n, \gamma_n^+) > \mathfrak{c}$. Then $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ is non-increasing and $\alpha_1 \le \mathfrak{c}$, so there are $n \ge 1$, $\alpha \le \mathfrak{c}$ such that $\alpha_m = \alpha$ for every $m \ge n$. Now for $m \ge n$ we have

$$\begin{aligned}\mathfrak{c} < \Theta(\alpha, \gamma_m^+) &\le \max(\alpha, (\sup_{\alpha' < \alpha} \Theta(\alpha', \gamma_m^+))^{\mathrm{cf}(\alpha)})\\ &\le \max(\alpha, \mathfrak{c}^{\mathrm{cf}(\alpha)}) = 2^{\mathrm{cf}(\alpha)},\end{aligned}$$

using A1M. Also we still have $\alpha \ge \kappa > \Theta(\gamma_m^+, \gamma_m^+)$ because $\Theta(\alpha', \gamma_m^+) < \kappa$ for every $\alpha' < \kappa$. So

$$\Theta(\alpha, \gamma_m^+) = \Theta(\Theta(\alpha, \gamma_m^+), \gamma_m^+) \ge \Theta(\mathfrak{c}, \gamma_m^+) = 2^{\gamma_m}$$

for every $m \ge n$; consequently

$$2^{\gamma_m} < 2^{\gamma_{m+1}} \le 2^{\mathrm{cf}(\alpha)}$$

and $\mathrm{cf}(\alpha) > \gamma_m$ for every m. But this means that

$$\Theta(\alpha,\gamma_m^+)=\sup_{\alpha'<\alpha}\Theta(\alpha',\gamma_m^+)\le\mathfrak{c}$$

for each m, which is absurd. **X**

This contradiction proves the theorem.

Remark This comes from GITIK & SHELAH P91.

7Q Corollary Let κ be an atomlessly-measurable cardinal.

(a) There is a least cardinal $\gamma<\kappa$ such that $2^\gamma=2^\delta$ for $\gamma\le\delta<\kappa$.

(b) If ν is a witnessing probability on κ and λ is the Maharam type of $(\kappa,\mathcal{P}\kappa,\nu)$ then the cardinal power λ^γ is 2^κ.

(c) If $\mathfrak{c}<\kappa^{(+\omega_1)}$ then $\gamma=\omega$, that is, $2^\delta=\mathfrak{c}$ for $\omega\le\delta<\kappa$.

proof (a) Immediate from 7P.

(b) Let $\mathfrak{A}$ be the measure algebra $\mathcal{P}\kappa/\mathcal{N}_\nu$. For $\xi<\kappa$ let $\phi_\xi:\mathcal{P}\xi\to\mathcal{P}\gamma$ be an injective function. For $\eta<\gamma$, $A\subseteq\kappa$ set

$$d_{A\eta}=\{\xi:\eta\in\phi_\xi(A\cap\xi)\}^\bullet\in\mathfrak{A}.$$

If A, $B\subseteq\kappa$ are distinct then there is a $\zeta<\kappa$ such that $\phi_\xi(A\cap\xi)\ne\phi_\xi(B\cap\xi)$ for $\zeta\le\xi<\kappa$, so that (because $\gamma<\kappa$) there is an $\eta<\gamma$ such that $d_{A\eta}\ne d_{B\eta}$. Thus $2^\kappa\le\#(\mathfrak{A})^\gamma\le(\lambda^\omega)^\gamma=\lambda^\gamma$, using A2Hb. On the other hand, $\lambda\le 2^\kappa$ and $\gamma<\kappa$, so $\lambda^\gamma\le 2^\kappa$.

(c)(i) We need the following elementary fact: if α, β, δ are infinite cardinals, with δ regular and $\operatorname{cf}(\alpha)\ne\delta$, then

$$\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\delta^+,\delta)\le\max(\alpha,\sup_{\alpha'<\alpha}\operatorname{cov}_{\mathrm{Sh}}(\alpha',\beta,\delta^+,\delta)).$$

P Set $\theta=\max(\alpha,\sup_{\alpha'<\alpha}\operatorname{cov}_{\mathrm{Sh}}(\alpha',\beta,\delta^+,\delta))$. For each $\xi<\alpha$ choose $\mathcal{E}_\xi\subseteq[\xi]^{<\beta}$ such that $\#(\mathcal{E}_\xi)=\operatorname{cov}_{\mathrm{Sh}}(\#(\xi),\beta,\delta^+,\delta)$ and every member of $[\xi]^{\le\delta}$ can be covered by fewer than δ members of $\mathcal{E}_\xi$. Set $\mathcal{E}=\bigcup_{\xi<\alpha}\mathcal{E}_\xi$; then $\#(\mathcal{E})\le\theta$. Take any $A\in[\alpha]^{\le\delta}$. If $\operatorname{cf}(\alpha)>\delta$ then $A\subseteq\xi$ for some $\xi<\alpha$ so A is covered by fewer than δ members of $\mathcal{E}_\xi\subseteq\mathcal{E}$. If $\operatorname{cf}(\alpha)<\delta$, take a cofinal set $C\subseteq\alpha$ of cardinal less than δ; then for each $\xi\in C$ there is an $\mathcal{A}_\xi\in[\mathcal{E}_\xi]^{<\delta}$ covering $A\cap\xi$, so $\mathcal{A}=\bigcup_{\xi\in C}\mathcal{A}_\xi\in[\mathcal{E}]^{<\delta}$ (because δ is regular), and $A\subseteq\bigcup\mathcal{A}$. As A is arbitrary, $\mathcal{E}$ witnesses that $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\beta,\delta^+,\delta)\le\theta$, as required. **Q**

(ii) ? Now suppose that $\gamma>\omega$, that is, that $2^\gamma>\mathfrak{c}$. Let γ_1 be the least cardinal such that $2^{\gamma_1}>\mathfrak{c}$. Then an easy induction on α, using (i) just above, shows that $\operatorname{cov}_{\mathrm{Sh}}(\alpha,\kappa,\gamma_1^+,\gamma_1)\le\alpha$ whenever $1\le\alpha<\kappa^{(+\gamma_1)}$. In particular, $\operatorname{cov}_{\mathrm{Sh}}(\mathfrak{c},\kappa,\gamma_1^+,\gamma_1)\le\mathfrak{c}$. But in 7O we saw that $\operatorname{cov}_{\mathrm{Sh}}(\mathfrak{c},\kappa,\gamma_1^+,\gamma_1)=2^{\gamma_1}$. **X**

Remark The case $\kappa=\mathfrak{c}$ of (b) is due to PRIKRY 75.

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8. PMEA and NMA

In this section I give a brief description of two axioms, both much stronger than the assertion '$\mathfrak{c}$ is real-valued-measurable' but nevertheless apparently consistent, with some of their consequences in set theory and general topology.

8A Theorem If one of the following statements is true, so are the others:

(i) for every cardinal λ, there is a probability space $(X,\mathcal{P}X,\mu)$ of Maharam type at least λ, and with $\operatorname{add}(\mu)=\mathfrak{c}$;

(ii) for every cardinal λ, there is a $\mathfrak{c}$-additive probability with domain $\mathcal{P}(\{0,1\}^\lambda)$ which extends the usual measure μ_λ of $\{0,1\}^\lambda$;

(iii) for every Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$, there is a $\mathfrak{c}$-additive probability with domain $\mathcal{P}X$ which extends μ.

proof (a)(i)⇒(ii) Assume (i). Let λ be any cardinal; of course (ii) is surely true for finite λ, so we may take it that $\lambda\ge\omega$. Let $(X,\mathcal{P}X,\mu)$ be a probability space of Maharam type greater than λ and with $\operatorname{add}(\mu)=\mathfrak{c}$. Let $E\in\mathcal{P}X\setminus\mathcal{N}_\mu$ be such that $(E,\mathcal{P}E,\mu\lceil E)$ is Maharam homogeneous with Maharam type at least λ (A2Hh). Setting $\mu'A=\mu A/\mu E$ for $A\subseteq E$, $(E,\mathcal{P}E,\mu')$ is a Maharam homogeneous probability space of

Maharam type at least λ, and $\mathrm{add}(\mu') \geq \mathfrak{c}$. By A2Ka, there is an inverse-measure -preserving function $f : E \to \{0,1\}^\lambda$. Now $\nu = \mu' f^{-1}$ (A2Db) is a $\mathfrak{c}$-additive extension of μ_λ to $\mathcal{P}(\{0,1\}^\lambda)$.

(b)(ii)⇒(iii) Assume (ii). Let $(X, \mathfrak{T}, \Sigma, \mu)$ be any Radon measure space.

case 1 Suppose that $\mu X = 1$ and that μ has Maharam type λ. Then there is an inverse-measure -preserving function $f : \{0,1\}^\lambda \to X$ (A2Ka). If ν is a $\mathfrak{c}$-additive extension of μ_λ to $\mathcal{P}(\{0,1\}^\lambda)$, then νf^{-1} is a $\mathfrak{c}$-additive extension of μ to $\mathcal{P}X$.

case 2 In general, there is a partition $\langle X_i \rangle_{i \in I}$ of X into measurable sets of finite measure such that $\mu E = \sum_{i \in I} \mu(E \cap X_i)$ for any $E \in \Sigma$ (A2Ja). By case 1, applied to a suitable normalization of $\mu\lceil X_i$ (the case $\mu X_i = 0$ being trivial), there is an extension of $\mu\lceil X_i$ to a $\mathfrak{c}$-additive measure ν_i with domain $\mathcal{P}X_i$, for each $i \in I$. Now setting $\nu A = \sum_{i \in I} \nu_i(A \cap X_i)$ for each $A \subseteq X$, we get a $\mathfrak{c}$-additive extension of μ to $\mathcal{P}X$.

(c)(iii)⇒(ii) and **(ii)⇒(i)** are trivial.

8B Definition PMEA (the '**product measure extension axiom**') is the assertion that the statements (i)-(iii) of 8A are true.

8C Theorem [Kunen] If 'ZFC + there is a strongly compact cardinal' is consistent, so is 'ZFC + PMEA'.

proof See Fleissner 84, Theorem 3.4.

8D Proposition Assume PMEA. Then $\mathfrak{c}$ is atomlessly-measurable.

proof Use 8A(ii), with $\lambda = \omega$, and 1D.

8E Lemma Assume PMEA. Let X be a normal topological space and $\langle F_i \rangle_{i \in I}$ a discrete family of subsets of X. Suppose that for each $x \in X$ we are given a downwards-directed family $\mathcal{U}_x$ of sets such that $\#(\mathcal{U}_x) < \mathfrak{c}$ and whenever $G \subseteq X$ is open, $i \in I$, and $x \in F_i \subseteq G$ there is a $U \in \mathcal{U}_x$ such that $U \subseteq G$. Then there is a family $\langle U_x \rangle_{x \in X}$ such that $U_x \in \mathcal{U}_x$ for every $x \in X$ and whenever i, $j \in I$ are distinct, $x \in F_i$ and $y \in F_j$ then $U_x \cap U_y = \emptyset$.

proof For each $z \in \{0,1\}^I$ let G_z, H_z be disjoint open sets such that $F_i \subseteq G_z$ whenever $z(i) = 1$ and $F_i \subseteq H_z$ whenever $z(i) = 0$. Let ν be a $\mathfrak{c}$-additive probability on $\{0,1\}^I$ extending the usual measure on $\{0,1\}^I$. Write $F = \bigcup_{i \in I} F_i$.

For $x \in F$, $U \in \mathcal{U}_x$ write

$$A(x, U) = \{z : z \in \{0,1\}^I, \text{ either } z(i) = 1 \text{ and } U \subseteq G_z \text{ or } z(i) = 0 \text{ and } U \subseteq H_z\},$$

where i is that member of I such that $x \in F_i$. For each $x \in F$, $\{A(x, U) : U \in \mathcal{U}_x\}$ is an upwards-directed family in $\mathcal{P}(\{0,1\}^I)$, of cardinal less than $\mathfrak{c}$, covering $\{0,1\}^I$. Consequently there is a $U_x \in \mathcal{U}_x$ such that $\nu A(x, U_x) > \frac{3}{4}$ (A2Ce).

If now $x \in F_i$, $y \in F_j$ where $i \neq j$, then

$$\nu\{z : z \in A(x, U_x) \cap A(y, U_y),\ z(i) \neq z(j)\} > 0.$$

Take any $z \in A(x, U_x) \cap A(y, U_y)$ such that $z(i) \neq z(j)$. Then one of U_x, U_y is included in G_z and the other in H_z, so $U_x \cap U_y = \emptyset$, as required.

8F Theorem Assume PMEA. Let X be a normal topological space in which $\chi(x, X) < \mathfrak{c}$ for every $x \in X$. Then X is collectionwise normal.

proof Let $\langle F_i \rangle_{i \in I}$ be a discrete family of subsets of X. For each $x \in X$ let $\mathcal{U}_x$ be a base of neighbourhoods of x with $\#(\mathcal{U}_x) = \chi(x, X) < \mathfrak{c}$. By 8E there is a family $\langle U_x \rangle_{x \in X}$ such that $U_x \in \mathcal{U}_x$ for each $x \in X$ and $U_x \cap U_y = \emptyset$ whenever x, $y \in X$ belong to different F_i. Set $G_i = \bigcup\{U_x : x \in F_i\}$ for each $i \in I$; then $\langle G_i \rangle_{i \in I}$ is a disjoint family of open sets and $F_i \subseteq G_i$ for every i. As $\langle F_i \rangle_{i \in I}$ is arbitrary, X is collectionwise normal.

8G Corollary Assume PMEA. Then every normal Moore space is metrizable.

proof Moore spaces are first-countable; by 8F, assuming PMEA, normal Moore spaces are collectionwise normal. By A3D, a collectionwise normal Moore space is metrizable. So we have the result.

Remarks (a) 8G is due to Nyikos 80; 8F, as stated, is due to Junnila 83.

(b) Of course we can do something with the ideas of 8E-8F without the full strength of PMEA. If we have an extension of the usual measure on $\{0,1\}^I$ to a κ-additive measure with domain $\mathcal{P}(\{0,1\}^I)$, then we can make 8E work if $\#(\mathcal{U}_x) < \kappa$ for every $x \in X$. (Compare KULESZA LEVY & NYIKOS 91, §4.) Thus if κ is atomlessly-measurable, we shall have every normal Moore space of weight at most $\min(2^\kappa, \kappa^{(+\omega)})$ metrizable, using the Gitik-Shelah theorem (3F, 3I).

(c) Under NMA, stronger results my be obtained by putting 8M below together with 8E-8G.

8H Definition NMA (the '**normal measure axiom**') is the assertion

For every set I there is a $\mathfrak{c}$-additive probability ν on $S = [I]^{<\mathfrak{c}}$, with domain $\mathcal{P}S$, such that

(i) $\nu\{s : i \in s \in S\} = 1$ for every $i \in I$;

(ii) if $A \subseteq S$ and $\nu A > 0$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then there is an $i \in I$ such that $\nu\{s : s \in A,\ f(s) = i\} > 0$.

Remarks (a) If we read the above statement as 'NMA(I)', with the set I as a parameter, then we see that the truth of NMA(I) depends only on $\#(I)$, so that if seeking to prove NMA we need consider only NMA(λ) for cardinals λ. More importantly, we see that NMA(I) implies NMA(J) for every $J \subseteq I$; for if ν is a probability on $[I]^{<\mathfrak{c}}$ as above, and we define $f : [I]^{<\mathfrak{c}} \to [J]^{<\mathfrak{c}}$ by setting $f(s) = s \cap J$ for every $s \in [I]^{<\mathfrak{c}}$, then νf^{-1} will witness NMA(J). Consequently NMA will be true iff NMA(λ) is true for arbitrarily large cardinals λ; e.g. for all regular $\lambda \geq \mathfrak{c}$.

(b) Observe that the condition (ii) of the statement above can be replaced with

(ii)$'$ if $A \subseteq S$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then there is a countable $D \subseteq I$ such that $f(s) \in D$ for ν-almost all $s \in A$,

or with

(ii)$''$ if $A \subseteq S$ and $f : A \to I$ is such that $f(s) \in s$ for every $s \in A$, then for every $\delta > 0$ there is a finite $D \subseteq I$ such that $\nu(A \setminus f^{-1}[D]) \leq \delta$.

8I Theorem If 'ZFC + there is a supercompact cardinal' is consistent, so is 'ZFC + NMA'.

proof PRIKRY 75, FLEISSNER 89.

8J Proposition NMA implies PMEA.

proof Assume NMA. Let λ be any cardinal. Let θ be a regular cardinal greater than the cardinal power λ^ω. Let ν be a probability on $[\theta]^{<\mathfrak{c}}$ as in 8H. For $\xi < \theta$ define $f_\xi : [\theta]^{<\mathfrak{c}} \to \mathfrak{c}$ by setting

$$f_\xi(s) = \mathrm{otp}(s \cap \xi)\ \ \forall\ \ s \in [\theta]^{<\mathfrak{c}}.$$

Then if $\xi < \eta < \theta$ we have $f_\xi(s) < f_\eta(s)$ whenever $\xi \in s$, that is, for ν-almost all s.

Let $g : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be any injection. For $\xi < \theta$, $n \in \mathbb{N}$ let $a_{\xi n}$ be the equivalence class of $\{s : n \in g(f_\xi(s))\}$ in the measure algebra $\mathfrak{A}$ of $([\theta]^{<\mathfrak{c}}, \mathcal{P}([\theta]^{<\mathfrak{c}}), \nu)$. Then for any $\xi < \eta < \theta$ there must be an $n \in \mathbb{N}$ such that $a_{\xi n} \neq a_{\eta n}$, so $\#(\mathfrak{A})^\omega \geq \theta > \lambda^\omega$. Now $\#(\mathfrak{A}) \leq \tau(\mathfrak{A})^\omega$ (A2Hb), so $\tau(\mathfrak{A}) > \lambda$.

Thus we have (i) of 8A, and PMEA is true.

Remark This result is due to Kunen; the argument above is taken from FLEISSNER 89.

8K Theorem Assume NMA. Then the singular cardinals hypothesis is true.

proof (a) As noted in A1N, it will be more than enough if I can show that $\#([\lambda]^{<\mathfrak{c}}) \leq \lambda$ for every regular cardinal $\lambda \geq \mathfrak{c}$. Let ν be a measure on $S = [\lambda]^{<\mathfrak{c}}$ as described in 8H. For each $\xi < \lambda$ choose a set $c_\xi \subseteq \xi$, cofinal with ξ, of cardinality $\mathrm{cf}(\xi)$.

(b) If $\zeta < \lambda$ there is a $\zeta' < \lambda$ such that

$$\nu\{s : s \in S,\ c_{\sup s} \cap \zeta' \setminus \zeta \neq \emptyset\} = 1.$$

P Set $E = \{s : \zeta + 1 \in s \in S\}$; then $\nu E = 1$. For each $s \in E$, $\zeta < \sup s$ so there is an $f_0(s) \in c_{\sup s}$ such that $\zeta \leq f_0(s)$; now $f_0(s) < \sup s$ so there is an $f(s) \in s$ such that $f_0(s) < f(s)$. As remarked in 8H, there is now a countable $D \subseteq \lambda$ such that $\nu\{s : s \in E,\ f(s) \in D\} = 1$; as λ is regular, there is a $\zeta' < \lambda$ such that $D \subseteq \zeta'$. Then $f_0(s) \in c_{\sup s} \cap \zeta' \setminus \zeta$ whenever $s \in E$ and $f(s) \in D$, so ζ' serves. **Q**

(c) So (again because λ is regular) there is a family $\langle \zeta_\xi \rangle_{\xi < \lambda}$ such that

$$\nu\{s : c_{\sup s} \cap \zeta_{\xi+1} \setminus \zeta_\xi \neq \emptyset\} = 1 \ \forall \ \xi < \lambda,$$

$$\zeta_\xi = \sup_{\eta<\xi} \zeta_\eta \text{ for limit ordinals } \xi < \lambda.$$

Set $a_\xi = \{\eta : c_\xi \cap \zeta_{\eta+1} \setminus \zeta_\eta \neq \emptyset\}$ for each $\xi < \lambda$; then $\#(a_\xi) \le \#(c_\xi) = \mathrm{cf}(\xi)$ for each ξ. Now

$$\nu\{s : \eta \in a_{\sup s}\} = \nu\{s : c_{\sup s} \cap \zeta_{\eta+1} \setminus \zeta_\eta \neq \emptyset\} = 1$$

for each $\eta < \lambda$.

(d) Set

$$A = \{a : \exists \ \xi < \lambda, \ \mathrm{cf}(\xi) < \mathfrak{c}, \ a \subseteq a_\xi\}.$$

Then $\#(A) \le \max(\lambda, \sup_{\delta<\mathfrak{c}} 2^\delta) = \lambda$, by 5E. Also, for any $a \in S$,

$$\nu\{s : a \subseteq a_{\sup s}\} = 1;$$

but if $s \in S$ then $\mathrm{cf}(\sup s) < \mathfrak{c}$, so we have $a \in A$. Thus $A = S$ and $\#(S) \le \lambda$, as required.

Remark This is Theorem 3(a-b) of Prikry 75. Compare the corresponding theorem concerning strongly compact cardinals (Drake 74, Theorem 3.6 and Corollary 3.8, or Solovay 74, Theorem 1).

8L Theorem Suppose that I is a set and that ν is a measure on $S = [I]^{<\mathfrak{c}}$ as in 8H. Then

(a) If $\langle A_i \rangle_{i\in I}$ is any family of subsets of S with $\nu A_i = 1$ for each $i \in I$, then

$$\nu\{s : s \in S, \ s \in A_i \ \forall \ i \in s\} = 1.$$

(b) If θ is a cardinal, $\omega < \mathrm{cf}(\theta) \le \theta < \mathfrak{c}$, and $f : S \to [I]^{<\theta}$ is any function, then there is an $M \in [I]^{<\theta}$ such that

$$\nu\{s : s \in S, \ f(s) \cap s \subseteq M\} = 1.$$

(c) If $\theta < \mathfrak{c}$ is a cardinal and $C \subseteq S$ is cofinal with S then

$$\nu\{s : s \in S, \ \exists \ D \subseteq C, \ \mathrm{add}(D) > \theta, \ \bigcup D = s\} = 1.$$

(Here $\mathrm{add}(D)$ is the additivity of the partially ordered set $(D, \subseteq)$, as in A1Ac.)

(d) If $f : [I]^{<\omega} \to S$ is any function then

$$\nu\{s : s \in S, \ f(J) \subseteq s \ \forall \ J \in [s]^{<\omega}\} = 1.$$

proof (a) Set

$$A = \{s : s \in A_i \ \forall \ i \in s\}.$$

For each $s \in S \setminus A$ choose $f(s) \in s$ such that $s \notin A_{f(s)}$. **?** If $\nu A < 1$ then $\nu(S \setminus A) > 0$ so there is an $i \in I$ such that $\nu\{s : s \in S \setminus A, \ f(s) = i\} > 0$. But now $\nu A_i \le 1 - \nu\{s : f(s) = i\} < 1$. **X** (Compare A1Ea.)

(b) As $\omega < \mathrm{cf}(\theta) < \mathfrak{c}$, there is an infinite cardinal $\delta < \theta$ such that $\nu A = 1$, where $A = \{s : s \in S, \ \#(f(s)) \le \delta\}$. For $s \in A$ let $\langle f_\xi(s) \rangle_{\xi<\delta}$ run over $f(s) \cup \{0\}$. Set

$$M = \{i : i \in I, \ \exists \ \xi < \delta, \ \nu\{s : f_\xi(s) = i\} > 0\};$$

then $\#(M) \le \max(\omega, \delta) < \theta$. Set

$$B = \{s : s \in A, \ f(s) \cap s \not\subseteq M\}.$$

For each $s \in B$ choose $g(s) \in f(s) \cap s \setminus M$. **?** If $\nu B > 0$ there is an $i \in I$ such that $\nu\{s : s \in B, \ g(s) = i\} > 0$; but now there is an $\eta < \delta$ such that $\nu B_1 > 0$, where

$$B_1 = \{s : s \in B, \ f_\eta(s) = i\},$$

so that $i \in M$; which is absurd. **X**

Thus $\nu B = 0$ and $\nu\{s : f(s) \cap s \subseteq M\} = 1$. (Compare 5A.)

(c) We need consider only the case $\theta > \omega$. If $\#(I) < \mathfrak{c}$ the result is trivial (since $I \in S$ and $\nu\{I\} = 1$), so let us take it that $\#(I) \ge \mathfrak{c}$; for convenience, let us suppose that $\mathfrak{c} \subseteq I$. For $s \in S$ write $s^* = s \cup \{\#(s)\}$; as $\#(s) \in \mathfrak{c} \subseteq I$, $s^* \in S$. For each $i \in I$ choose $c_i \in C$ such that $i \in c_i$, and set

$$A_i = \{s : c_i^* \subseteq s \in S\}.$$

Then $\nu A_i = 1$, because $\#(c_i^*) < \mathfrak{c} = \mathrm{add}(\nu)$. Set

$$A = \{s : s \in S,\, s \in A_i \ \forall \ i \in s\};$$

then $\nu A = 1$, by (a) above. For $s \in A$ set

$$D_s = \{c : c \in C,\, c^* \subseteq s\}.$$

Then $s = \bigcup D_s$ because $i \in c_i \in D_s$ for every $i \in s$.

Set

$$B = \{s : s \in A,\, \mathrm{add}(D_s) \le \theta\}.$$

For each $s \in B$ choose a family $D'_s \subseteq D_s$ with $\#(D'_s) \le \theta$ and with no upper bound in D_s; set $f(s) = \{\#(d) : d \in D'_s\} \subseteq s \cap \mathfrak{c}$. By (b), there is an $M \subseteq \mathfrak{c}$ such that $\#(M) < \theta^+$ and $\nu B = \nu B_1$, where

$$B_1 = \{s : s \in B,\, f(s) \subseteq M\}.$$

As $\theta < \mathfrak{c} = \mathrm{cf}(\mathfrak{c})$, $\delta = \sup(M \cup \{\theta\}) < \mathfrak{c}$. Now, for $s \in B_1$, $d_s = \bigcup D'_s$ has cardinal at most δ. So by (b) again there is an $N \subseteq I$ such that $\#(N) < \delta^+$ and $\nu B_2 = \nu B_1$ where

$$B_2 = \{s : s \in B_1,\, d_s \cap s \subseteq N\}.$$

Let $c \in C$ be such that $N \subseteq c$. Then $E = \{s : c^* \subseteq s\}$ has measure 1. But if $s \in B_2$ then D'_s has no upper bound in D_s, while $\bigcup D'_s \subseteq N \subseteq c$, so $c \notin D_s$ and $c^* \not\subseteq s$ and $s \notin E$. Thus $E \cap B_2 = \emptyset$ and $\nu B = \nu B_2 = 0$.

Accordingly $\nu(A \setminus B) = 1$. But if $s \in A \setminus B$ then $D_s \subseteq C$ has $\mathrm{add}(D_s) > \theta$ and $\bigcup D_s = s$, so we are done.

(d) Apply (c) with $\theta = \omega$,

$$C = \{s : f(J) \subseteq s \ \forall \ J \in [s]^{<\omega}\};$$

then

$$C \supseteq \{s : \exists \ D \subseteq C, \bigcup D = s,\, \mathrm{add}(D) \ge \omega\},$$

so $\nu C = 1$, as required.

Remark These results may be found, implicitly or explicitly, in FLEISSNER 89.

8M Proposition Assume NMA. Let X be a Hausdorff space such that $\chi(x, X) < \mathfrak{c}$ for every $x \in X$ and every closed set of cardinal at most $\mathfrak{c}$ is normal. Then X is normal.

proof Let E, F be disjoint closed sets in X. For each $x \in X$ let $\mathcal{U}_x$ be a base of neighbourhoods of x of cardinal less than $\mathfrak{c}$. Let ν be a measure on $S = [X]^{<\mathfrak{c}}$ as in 8H.

For each $s \in S$, its closure $\bar{s}$ has cardinal at most $\mathfrak{c}$, because if x, y are distinct points of $\bar{s}$ then $\{U \cap s : U \in \mathcal{U}_x\}$ and $\{U \cap s : U \in \mathcal{U}_y\}$ are distinct members of $[\mathcal{P}s]^{<\mathfrak{c}}$, and $\#([\mathcal{P}s]^{<\mathfrak{c}}) \le \mathfrak{c}$, by 5E. So there are disjoint relatively open sets G_s, $H_s \subseteq \bar{s}$ such that $E \cap \bar{s} \subseteq G_s$, $F \cap \bar{s} \subseteq H_s$. For $U \subseteq X$ set

$$A(U) = \{s : U \cap \bar{s} \subseteq G_s\},\ B(U) = \{s : U \cap \bar{s} \subseteq H_s\}.$$

Then for each $x \in E$, $y \in F$ there are $U_x \in \mathcal{U}_x$, $V_y \in \mathcal{U}_y$ such that $\nu A(U_x) > \frac{1}{2}$, $\nu B(V_y) > \frac{1}{2}$. But in this case $U_x \cap V_y = \emptyset$. **P?** Otherwise, take $z \in U_x \cap V_y$; there is an s such that $U_x \cap \bar{s} \subseteq G_s$, $V_y \cap \bar{s} \subseteq H_s$, $z \in s$; but in this case $z \in G_s \cap H_s$. **XQ**

So $G = \bigcup\{U_x : x \in E\}$ and $H = \bigcup\{V_y : y \in F\}$ are open sets separating E from F.

Remark This is due to W.G.Fleissner and I.Juhász (see JUHÁSZ 89). I am grateful to P.J.Nyikos for the reference.

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8N Theorem Assume NMA. Then any locally compact normal space is collectionwise normal.

proof (a) Let $(X, \mathfrak{T})$ be a locally compact normal Hausdorff space; without loss of generality we may suppose that $X \cap \mathfrak{T} = \emptyset$. Let $\langle F_i \rangle_{i \in I}$ be a discrete family of subsets of X. Let ν be a measure on $S = [X \cup \mathfrak{T}]^{<\mathfrak{c}}$ as in 8H.

For $s \in S$ set

$$B_x(s) = X \cap \bigcap\{\overline{U} : x \in U \in s \cap \mathfrak{T}\} \subseteq X$$

for each $x \in X$,

$$Y_i(s) = \bigcup\{B_x(s) : x \in F_i \cap s\}$$

for each $i \in I$.

(b) Set

$$A_0 = \{s : s \in S,\ \overline{Y_i(s)} \cap \overline{\bigcup_{j \neq i} Y_j(s)} = \emptyset \ \forall \ i \in I\};$$

then $\nu A_0 = 1$. **P** For each $x \in X$ choose G_x, $H_x \in \mathfrak{T}$ in such a way that if $x \in F_i$ then

$$F_i \subseteq G_x \subseteq \overline{G}_x \subseteq H_x \subseteq \overline{H}_x \subseteq X \setminus \bigcup_{j \neq i} F_j;$$

this is possible because $\overline{F}_i \cap \overline{\bigcup_{j \neq i} F_j} = \emptyset$ and X is normal. Set

$$A'_0 = \{s : s \in S,\ \{G_x, X \setminus \overline{H}_x\} \subseteq s \ \forall \ x \in s \cap X\};$$

by 8Ld, $\nu A'_0 = 1$. But if $s \in A'_0$ and $Y_i(s) \neq \emptyset$, take $x \in s \cap F_i$; then $Y_i(s) \subseteq \overline{G}_x$ and $\bigcup_{j \neq i} Y_j(s) \subseteq X \setminus H_x$, so $\overline{Y_i(s)} \cap \overline{\bigcup_{j \neq i} Y_j(s)} = \emptyset$. Thus $A'_0 \subseteq A_0$ and $\nu A_0 = 1$. **Q**

(c) Set

$$A_1 = \{s : s \in S,\ \langle Y_i(s)\rangle_{i \in I} \text{ is discrete}\}.$$

Then A_1 is cofinal with S. **P** Take any $s \in S$. Because $\nu A_0 > 0$, there is an $s' \in A_0$ such that $s' \supseteq s$. Set

$$\mathcal{H} = \{H : H \in \mathfrak{T},\ \#(\{i : H \cap Y_i(s') \neq \emptyset\}) \leq 1\}.$$

Then $\bigcup_{i \in I} \overline{Y_i(s')} \subseteq \bigcup \mathcal{H}$, because $s' \in A_0$, so $F = \bigcup_{i \in I} \overline{F_i \cap s'} \subseteq \bigcup \mathcal{H}$; but F is closed, because $\langle F_i\rangle_{i \in I}$ is locally finite. Take $G \in \mathfrak{T}$ such that $F \subseteq G \subseteq \overline{G} \subseteq \bigcup \mathcal{H}$, and set $s'' = s' \cup \{G\} \in S$. Then $s'' \cap F_i = s' \cap F_i \subseteq G$ for each i (this is where I use the requirement '$X \cap \mathfrak{T} = \emptyset$'), so $Y_i(s'') \subseteq Y_i(s') \cap \overline{G}$ for every i. Now we see that H meets at most one $Y_i(s'')$ for every $H \in \mathcal{H}$ and also for $H = X \setminus \overline{G}$, so that $s'' \in A_1$. **Q**

(d) Set

$$A_2 = \{s : s \in S,\ \langle Y_i(s)\rangle_{i \in I} \text{ is locally finite}\};$$

then $\nu A_2 = 1$. **P** Set

$$A'_2 = \{s : \exists \ D \subseteq A_1,\ \mathrm{add}(D) > \omega,\ \bigcup D = s\}.$$

By 8Lc, $\nu A'_2 = 1$. **?** If $A'_2 \not\subseteq A_2$, take $s \in A'_2 \setminus A_2$. Let $x \in X$ be such that $\{i : U \cap Y_i(s) \neq \emptyset\}$ is infinite for every neighbourhood U of x. Let K be a compact neighbourhood of x, and $J \subseteq I$ a countably infinite set such that $K \cap Y_i(s) \neq \emptyset$ for every $i \in J$. For each $i \in J$ there is an $x(i) \in F_i \cap s$ such that $K \cap B_{x(i)}(s) \neq \emptyset$. Now there is a $D \subseteq A_1$ such that $\mathrm{add}(D) > \omega$ and $s = \bigcup D$; so there is a $d \in D$ such that $x(i) \in d$ for every $i \in J$. In this case $B_{x(i)}(d) \supseteq B_{x(i)}(s)$ for each i, so $K \cap Y_i(d) \neq \emptyset$ for every $i \in J$. However $\langle Y_i(d)\rangle_{i \in I}$ is supposed to be discrete, and K is supposed to be compact, so this is impossible. **X**

Thus $A'_2 \subseteq A_2$ and $\nu A_2 = 1$, as claimed. **Q**

(e) Consequently $\nu A_1 = 1$, as $A_1 = A_0 \cap A_2$.

(f) Let A_3 be the set of those $s \in S$ for which there is a family $\langle W_x\rangle_{x \in X}$ in $s \cap \mathfrak{T}$ such that $x \in W_x$ for every $x \in s \cap X$ and if i, j are distinct members of I, $x \in F_i$ and $y \in F_j$ then $W_x \cap W_y = \emptyset$. Then $\nu A_3 = 1$. **P** For $x \in X$ take a compact neighbourhood K_x of x. Set

$$A'_3 = \{s : s \in S,\ \emptyset \in s,\ U \cap V \in s \ \forall \ U, V \in s \cap \mathfrak{T},\ \mathrm{int}(K_x) \in s \ \forall \ x \in s \cap X\}.$$

Then $\nu A'_3 = 1$ by 8Ld.

Take $s \in A'_3 \cap A_1$. Note that as $\langle Y_i(s)\rangle_{i \in I}$ is disjoint, $s \cap Y_i(s) \cap \bigcup_{j \in I} F_j = s \cap F_i$ for every $i \in I$. For each $x \in s \cap \bigcup_{i \in I} F_i$, set $\mathcal{U}_x = \{U : x \in U \in s \cap \mathfrak{T}\}$; for $x \in X \setminus (s \cap \bigcup_{i \in I} F_i)$, set $\mathcal{U}_x = \{\emptyset\}$. Then $\mathcal{U}_x$ is downwards-directed and $\#(\mathcal{U}_x) < \mathfrak{c}$ for every x. If $i \in I$ and $x \in Y_i(s)$ and G is an open neighbourhood of $Y_i(s)$, then either $x \notin s \cap F_i$ and $\emptyset \in \mathcal{U}_x$, $\emptyset \subseteq G$, or $x \in s \cap F_i$; in the latter case, $\bigcap\{\overline{U} : U \in \mathcal{U}_x\} = B_x(s) \subseteq G$; but as $\mathcal{U}_x$ is downwards-directed and contains the relatively compact set $\mathrm{int}(K_x)$, there must be a member of $\mathcal{U}_x$ included in G.

We may therefore apply 8E to X, $\langle Y_i(s)\rangle_{i\in I}$ and $\langle\mathcal{U}_x\rangle_{x\in X}$ to see that there is a family $\langle W_x\rangle_{x\in X}$ such that $W_x\in\mathcal{U}_x$ for every $x\in X$ and if $x\in Y_i(s)$, $y\in Y_j(s)$ with $i\neq j$ then $W_x\cap W_y=\emptyset$. Evidently $\langle W_x\rangle_{x\in X}$ witnesses that $s\in A_3$.

Thus $A_3'\cap A_1\subseteq A_3$ and $\nu A_3=1$. **Q**

(g) For each $s\in A_3$ take a family $\langle W_x(s)\rangle_{x\in X}$ in $s\cap\mathfrak{T}$ as in (f). Now, given any $x\in X$,

$$\nu\{s : x\in s\in A_3\}=1,$$

so there is a finite set $\mathcal{V}_x$ of open neighbourhoods of x such that

$$\nu\{s : x\in s\in A_3,\, W_x(s)\in\mathcal{V}_x\}>\tfrac{1}{2}$$

(see (ii)$''$ of Remark (b) in 8H). Set $V_x=\bigcap\mathcal{V}_x$; then V_x is an open neighbourhood of x for each $x\in X$.

If i, $j\in I$ are distinct and $x\in F_i$, $y\in F_j$ then there is an $s\in A_3$ such that

$$x\in s,\ y\in s,\ W_x(s)\in\mathcal{V}_x,\ W_y(s)\in\mathcal{V}_y.$$

Now $V_x\cap V_y\subseteq W_x(s)\cap W_y(s)=\emptyset$. So if we set

$$H_i=\bigcup\{V_x : x\in F_i\}\ \ \forall\ i\in I,$$

$\langle H_i\rangle_{i\in I}$ is a disjoint family of open sets and $F_i\subseteq H_i$ for each $i\in I$.

(h) As $\langle F_i\rangle_{i\in I}$ is arbitrary, X is collectionwise normal, as claimed.

Remark This comes from FLEISSNER 89. Some of the ideas were originally developed in BALOGH 91.

8O Definition Let s be a set of ordinals.

(a) A set $C\subseteq s$ is **relatively order-closed** if it is closed in the intrinsic order topology of s, that is, if

$$\min\{\xi : \xi\in s,\, \eta\leq\xi\ \ \forall\ \eta\in C'\}\in C$$

whenever $C'\subseteq C$ is non-empty and bounded above in s.

(b) A set $A\subseteq s$ is **relatively stationary** in s if it meets every cofinal set $C\subseteq s$ which is relatively order-closed in s.

Observe that these definitions correspond to the ordinary notions of 'closed' and 'stationary' set in the ordinal $\operatorname{otp}(s)$.

8P Theorem Let α be an ordinal and ν a measure on $S=[\alpha]^{<\mathfrak{c}}$ as in 8H.

(a) If $\beta<\alpha$ is an ordinal, then $\operatorname{otp}(s\cap\beta)<\operatorname{otp}(s)$ for ν-almost every $s\in S$.

(b) If α is a decomposable ordinal then $\operatorname{otp}(s)$ is a decomposable ordinal for ν-almost every $s\in S$.

(c) If α is an indecomposable ordinal then $\operatorname{otp}(s)$ is an indecomposable ordinal for ν-almost every $s\in S$.

(d) $\#(s)=\operatorname{otp}(s\cap\#(\alpha))$ for ν-almost every $s\in S$. Consequently (i) if α is a cardinal then $\operatorname{otp}(s)=\#(s)$ for ν-almost every $s\in S$ (ii) if α is not a cardinal then $\operatorname{otp}(s)$ is not a cardinal, for ν-almost every $s\in S$.

(e) If $\alpha=\lambda^+$ is a successor cardinal then $\#(s)=(\#(s\cap\lambda))^+$ for ν-almost every $s\in S$.

(f) $\operatorname{cf}(s)=\operatorname{cf}(s\cap\operatorname{cf}(\alpha))$ for ν-almost every $s\in S$.

(g) If α is a regular infinite cardinal then $\#(s)$ is a regular infinite cardinal for ν-almost every $s\in S$.

(h) If α is a weakly inaccessible cardinal then $\#(s)$ is a weakly inaccessible cardinal for ν-almost every $s\in S$.

(i) If $C\subseteq\alpha$ is closed then $s\cap C$ is relatively order-closed in s for ν-almost every $s\in S$.

(j) If $C\subseteq\alpha$ is cofinal with α then $s\cap C$ is cofinal with s for ν-almost every $s\in S$.

(k) If $A\subseteq\alpha$ is stationary in α then $s\cap A$ is relatively stationary in s for ν-almost every $s\in S$.

(l) If α is a weakly Mahlo cardinal then $\#(s)$ is a weakly Mahlo cardinal for ν-almost every $s\in S$.

(m) If α is a non-weakly-Mahlo cardinal then $\#(s)$ is a non- weakly-Mahlo cardinal for ν-almost every $s\in S$.

proof (a) $\beta\in s$ for almost all s.

(b) Take β, $\gamma<\alpha$ such that $\alpha=\beta+\gamma$. Let $h:\alpha\setminus\beta\to\gamma$ be an order-isomorphism. By 8Ld, with $f(J)=h[J]\cup h^{-1}[J]$ for each $J\in[\alpha]^{<\omega}$, $h[s\setminus\beta]=s\cap\gamma$ for almost every $s\in S$. So $\operatorname{otp}(s\setminus\beta)=\operatorname{otp}(s\cap\gamma)<\operatorname{otp}(s)$ for almost every s, using (a), and $\operatorname{otp}(s)$ is decomposable for almost every s.

(c) Set $A = \{s : s \in S, \text{otp}(s) \text{ is decomposable}\}$. For $s \in A$ take $g(s), h(s) \in s$ and an order-isomorphism $f_s : s \setminus g(s) \to s \cap h(s)$. **?** If $\nu A > 0$, there are $\beta, \gamma < \alpha$ such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, g(s) = \beta, h(s) = \gamma\}$. So $\text{otp}(s \setminus \beta) = \text{otp}(s \cap \gamma)$ for every $s \in A_1$. Let $\phi : \gamma \to (\beta + \gamma) \setminus \beta$ be an order-isomorphism. Then there is an $s \in A_1$ such that $\phi[s \cap \gamma] = s \cap (\beta + \gamma) \setminus \beta$ (using 8Ld again) and also $\beta + \gamma \in s$, because $\beta + \gamma < \alpha$. But now

$$\text{otp}(s \setminus \beta) > \text{otp}(s \cap (\beta + \gamma) \setminus \beta) = \text{otp}(s \cap \gamma),$$

which is impossible. **X**

(d) If $\alpha < \omega_1$ this is trivial; suppose that α is uncountable.

Set $A = \{s : \#(s) < \text{otp}(s \cap \#(\alpha))\}$. For $s \in A$, take a surjective function $f_s : s \cap g(s) \to s$, where $g(s) \in s \cap \#(\alpha)$. **?** If $\nu A > 0$, let $\zeta < \#(\alpha)$ be such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, g(s) = \zeta\}$. By 8Lb, with $f(s) = f_s[s \cap \zeta]$, $\theta = \max(\omega, \#(\zeta))^+$ there is a set $M \subseteq \alpha$ with $\#(M) \le \max(\omega, \#(\zeta))$ and $f_s[s \cap \zeta] \subseteq M$ for almost every s. But also $f_s[s \cap \zeta] = s \not\subseteq M$ for almost all $s \in A_1$. **X** Thus $\#(s) \ge \text{otp}(s \cap \#(\alpha))$ for almost all s.

Let $g : \alpha \to \#(\alpha)$ be any bijection. Then $g[s] = s \cap \#(\alpha)$ for ν-almost every $s \in S$ (using 8Ld with $f(J) = g[J] \cup g^{-1}[J]$ for $J \in [\alpha]^{<\omega}$). So $\#(s) = \#(s \cap \#(\alpha)) \le \text{otp}(s \cap \#(\alpha))$ for almost all s.

(e) For $\lambda < \omega$ this is trivial; take $\lambda \ge \omega$. By (a) and (d), $\#(s) = \text{otp}(s) > \text{otp}(s \cap \lambda) \ge \#(s \cap \lambda)$ for almost all s. Set $A = \{s : \#(s) > \#(s \cap \lambda)^+\}$. For $s \in A$ take $f(s) \in s$ such that $\#(s \cap f(s)) \ge \#(s \cap \lambda)^+$. **?** If $\nu A > 0$, let $\zeta < \alpha$ be such that $\zeta \ge \lambda$ and $\nu A_1 > 0$, where $A_1 = \{s : s \in A, f(s) \le \zeta\}$. Let $h : \zeta \to \lambda$ be any bijection. Then $s \cap \zeta = h^{-1}[s \cap \lambda]$ for almost all $s \in A_1$. But of course if $s \in A_1$ then $\#(s \cap \zeta) > \#(s \cap \lambda)$. **X** So $\nu A = 0$ and $\#(s) \le \#(s \cap \lambda)^+$ for almost all s.

(f) If α is a successor ordinal, or 0, this is trivial; let us take it that $\text{cf}(\alpha) \ge \omega$. Let C be a closed cofinal subset of α of order type $\text{cf}(\alpha)$, and let $h : \text{cf}(\alpha) \to C$ be the increasing enumeration of C. Then (using 8Ld again)

$$B = \{s : \forall\ \xi < \text{cf}(\alpha), \xi \in s \iff s \cap h(\xi + 1) \setminus h(\xi) \ne \emptyset \iff \xi + 1 \in s\}$$

has $\nu B = 1$. And $\text{cf}(s \cap \alpha) = \text{cf}(s \cap \text{cf}(\alpha))$ for every $s \in B$.

(g) For $\alpha = \omega$ this is trivial; take $\alpha > \omega$. Set $A = \{s : \text{cf}(s) < \#(s)\}$. For $s \in A$ choose $f(s) \in s$ such that $\text{cf}(s) = \text{otp}(s \cap f(s))$. Let $\zeta < \alpha$ be such that $f(s) \le \zeta$ for almost all $s \in A$; say $A_1 = \{s : s \in A, f(s) \le \zeta\}$. For $s \in A_1$ let $g_s : s \cap \zeta \to s$ be a function with range cofinal in s. By 8Lb, there is a $M \subseteq \alpha$, with $\#(M) < \alpha$, such that $g_s[s \cap \zeta] \subseteq M$ for almost all $s \in A_1$. However $s \cap M$ is not cofinal in s, for almost all s, because $\sup M < \alpha$. Thus $\nu A_1 = 0$ and $\nu A = 0$. Finally, $\text{cf}(s) = \text{cf}(\#(s))$ for almost all s, by (d-i) above; so $\text{cf}(\#(s)) = \#(s)$ and $\#(s)$ is regular, for almost all s.

(h) By (g), $\#(s)$ is a regular infinite cardinal for almost all s. Set

$$A = \{s : \#(s) \text{ is a successor cardinal}\}.$$

For $s \in A$ choose $f(s) \in s$ such that $\#(s) = \#(s \cap f(s))^+$. **?** If $\nu A > 0$, there is a $\zeta < \alpha$ such that $\nu A_1 > 0$, where $A_1 = \{s : s \in A, f(s) = \zeta\}$. Now consider $\delta = \#(\zeta)^+$. Recall that, as remarked in 8H, we have a measure $\nu_1 = \nu\phi^{-1}$ on $[\delta]^{<\mathfrak{c}}$, where $\phi(s) = s \cap \delta$ for each $s \in S$, with the same properties as ν. So we may apply (d) to δ, ν_1 to see that $\#(s \cap \zeta) < \#(s \cap \delta)$ for almost all s. Also, by (d) as written, $\#(s \cap \delta) < \#(s)$ for almost all s. So we have $\#(s \cap \zeta)^+ < \#(s)$ for almost all s, and $\nu A_1 = 0$; which is absurd. **X**

(i) For each $\zeta < \alpha$ set $h(\zeta) = \sup(C \cap \zeta) \in C \cup \{0\}$. Set $A = \{s : h(\zeta) \in s\ \forall\ \zeta \in s\}$; then $\nu A = 1$. If $s \in A$ and $C' \subseteq C \cap s$ is non-empty and bounded above in s, set $\zeta = \min\{\xi : \xi \in s, \eta \le \xi\ \forall\ \eta \in C'\}$. Then $h(\zeta) \in s$; but of course $h(\zeta)$ is also an upper bound for C', so $h(\zeta) = \zeta$ and $\zeta \in C$. Thus $s \cap C$ is relatively order-closed in s for every $s \in A$.

(j) Let $h : \alpha \to \alpha$ be such that $\xi \le h(\xi) \in C$ for every $\xi < \alpha$. Then $h[s] \subseteq s$ for almost all s; and $C \cap s$ is cofinal with s whenever $h[s] \subseteq s$.

(k)(i) If $\text{cf}(\alpha) \le \omega$ then there is a $\zeta < \alpha$ such that $\alpha \setminus \zeta \subseteq A$; now $s \cap A$ is stationary in s whenever $\zeta \in s$. **(ii)** If $\text{cf}(\alpha) > \omega$, set $D = \{s : s \cap A \text{ is not relatively stationary in } s\}$. For $s \in D$ let C_s be a cofinal relatively order-closed subset of s disjoint from A; for $s \in S \setminus D$, set $C_s = s$. Set

$$C = \{\xi : \nu\{s : \xi \in C_s\} = 1\}.$$

Then C is closed and unbounded in α. **P** (α) If $\zeta < \alpha$, then $C_s \not\subseteq \zeta$ for almost all s, so there is a countable set $K \subseteq \alpha \setminus \zeta$ such that $C_s \cap K \neq \emptyset$ for almost all s. Because $\mathrm{cf}(\alpha) > \omega$, we can find an increasing sequence $\langle \zeta_n \rangle_{n\in\mathbb{N}}$, starting with $\zeta_0 = \zeta$, such that $C_s \cap \zeta_{n+1} \setminus \zeta_n \neq \emptyset$ for almost all s, for every $n \in \mathbb{N}$. Take $\zeta^* = \sup_{n\in\mathbb{N}} \zeta_n$; then $\zeta^* \in C_s$ for almost all s, that is, $\zeta^* \in C$, and $\zeta^* \geq \zeta$. This shows that C is unbounded in α. (β) If $\zeta \in \alpha \setminus (C \cup \{0\})$, then $\nu E > 0$, where $E = \{s : \zeta \notin C_s\}$. For each $s \in E$, there is a $\beta \in s \cap \zeta$ such that $C_s \cap \zeta \subseteq \beta$. So there is a $\beta < \alpha$ such that $\nu E_1 > 0$, where $E_1 = \{s : C_s \cap \zeta \subseteq \beta\}$. Now $C \cap \zeta \subseteq \beta$. This shows that C is closed in α. **Q**

Accordingly there is a $\zeta \in A \cap C$. But now $\nu D \leq \nu\{s : \zeta \notin C_s\} = 0$.

(l) Let A be the set of weakly inaccessible cardinals less than α; then A is stationary in α. For each $\lambda \in A$, $\mathrm{otp}(s \cap \lambda)$ is a weakly inaccessible cardinal for almost all s, applying (d) and (h) in $[\lambda]^{<\mathfrak{c}}$. So $\nu D = 1$, where

$$D = \{s : \mathrm{otp}(s \cap \lambda) \text{ is weakly inaccessible for every } \lambda \in s \cap A\}.$$

On the other hand, we also have $s \cap A$ relatively stationary in s for almost all s, by (i). Copying these facts into $\mathrm{otp}(s)$ for each s, we see that $\#(s)$ is a weakly Mahlo cardinal for almost all s.

(m) Let C be a closed unbounded set in α consisting of ordinals which are not weakly inaccessible cardinals. For each $\gamma \in C$, $\mathrm{otp}(s \cap \gamma)$ is not a weakly inaccessible cardinal, for almost all s, by (e) and (f). So $\nu D = 1$, where

$$D = \{s : \mathrm{otp}(s \cap \gamma) \text{ is not a weakly inaccessible cardinal for every } \gamma \in s \cap C\}.$$

But also $C \cap s$ is relatively order-closed and cofinal with s for almost all s, by (i) and (j), so $\mathrm{otp}(s)$ is not a weakly Mahlo cardinal, for almost all s. On the other hand, $\#(s) = \mathrm{otp}(s)$ for almost all s, so $\#(s)$ is non-weakly-Mahlo for almost all s.

8Q I have given 'elementary' proofs of the results in 8P. They are of course just reflection properties, and all can be reached by means of the following.

Theorem (a) Let I be a set and ν a measure on $S = [I]^{<\mathfrak{c}}$ as in 8H. Let $\mathbb{P}$ be the random real p.o.set $\mathcal{P}S \setminus \mathcal{N}_\nu$. Let ϕ be a formula of the second- order language $\mathcal{L}$ of §A4; let $C_1, \dots, C_k$ be relations on I and $\xi_1, \dots, \xi_m$ members of I. Then the following are equivalent:

(i) $\Vdash_{\mathbb{P}} (\check{I}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi$;

(ii) $\{s : s \in S,\ (s; C_1, \dots, \xi_m) \nvDash \phi\} \in \mathcal{N}_\nu$.

(b) Suppose that ϕ is a second-order formula and that α is an ordinal such that $\Vdash_{\mathbb{P}} (\check{\alpha}; <) \vDash \phi$ for every random real p.o.set $\mathbb{P}$. Then for any measure ν on $S = [\alpha]^{<\mathfrak{c}}$ as in 8H, we shall have $(\mathrm{otp}(s); <) \vDash \phi$ for ν-almost every $s \in S$.

proof (a) This is a matter of re-writing the proof of 4M. For instance, the step corresponding to 4Ma becomes: Suppose that $\langle \xi_s \rangle_{s\in S}$ is a family in I such that $\xi_s \in s$ for almost all $s \in S$. Then we have a $\mathbb{P}$-name $\dot{\xi}$ for a member of I given by

$$p \Vdash_{\mathbb{P}} \dot{\xi} = \check{\zeta} \text{ iff } \xi_s = \zeta \text{ for almost every } s \in p.$$

(b) This now follows at once, because $(\mathrm{otp}(s); <)$ is isomorphic to $(s; <)$.

8R I have concentrated here on properties of well-ordered sets. But 8H-8I allow much more general contexts. For instance, if I is a simple group, then s will be a simple subgroup of I for almost every $s \in [I]^{<\mathfrak{c}}$. As a further example depending on the presence of a measure ν (rather than just on the ideal $\mathcal{N}_\nu$) I give the following.

Proposition Assume NMA. Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra such that whenever $\mathfrak{B} \subseteq \mathfrak{A}$ is a subalgebra of cardinal less than $\mathfrak{c}$ there is a functional $\mu : \mathfrak{B} \to [0, 1]$ such that

(i) $\mu b = \sum_{n\in\mathbb{N}} \mu b_n$ whenever $\langle b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{B}$ and $b \in \mathfrak{B}$ is the supremum of $\{b_n : n \in \mathbb{N}\}$ in $\mathfrak{A}$;

(ii) $\mu b > 0$ for every $b \in \mathfrak{B} \setminus \{\mathbf{0}\}$.

Then $\mathfrak{A}$ is a measurable algebra.

proof Let ν be a measure on $[\mathfrak{A}]^{<\mathfrak{c}}$ as in 8H. For each $s \in [\mathfrak{A}]^{<\mathfrak{c}}$ let $\mathfrak{B}_s$ be the subalgebra of $\mathfrak{A}$ generated by s, so that $\#(\mathfrak{B}_s) \le \max(\omega, \#(s)) < \mathfrak{c}$, and let $\mu_s : \mathfrak{B}_s \to [0,1]$ be a functional satisfying (i)-(ii) of the hypothesis above. For $a \in \mathfrak{A}$ set

$$\bar{\mu}a = \int \mu_s(a)\nu(ds);$$

note that $a \in s \subseteq \mathfrak{B}_s$ for almost all s, so that the integral is well-defined. Then we have

(i) whenever $\langle b_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$ with supremum $b \in \mathfrak{A}$, then $\nu E = 1$, where

$$E = \{s : s \in [\mathfrak{A}]^{<\mathfrak{c}},\, b \in s,\, b_n \in s \;\forall\; n \in \mathbb{N}\},$$

and $\mu_s b = \sum_{n\in\mathbb{N}} \mu_s b_n$ for every $s \in E$, so by B.Levi's theorem

$$\bar{\mu}b = \textstyle\sum_{n\in\mathbb{N}} \bar{\mu}b_n.$$

(ii) If $a \in \mathfrak{A} \setminus \{\mathbf{0}\}$ then $\mu_s a > 0$ for almost all s so $\bar{\mu}a > 0$.

Now $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and $\mathfrak{A}$ is a measurable algebra.

8S Remark For further topological consequences of PMEA and NMA see Fleissner Hansell & Junnila 82 (with Fleissner 79 and Fremlin Hansell & Junnila 83), Junnila 83, Burke 84, Fleissner 84b, Tall 84, Fleissner 89, Kulesza Levy & Nyikos 91.

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9. Quasi-measurable cardinals

Many of the ideas above can be generalized to contexts outside ordinary measure theory. Here I discuss some of these extensions, concentrating on those which do not involve new concepts. With a few exceptions, the proofs are straightforward adaptations of arguments above, and I therefore omit many of the details.

9A Lemma Let X be a set, λ a cardinal of uncountable cofinality and $\mathcal{I}$ a λ-saturated σ-ideal of $\mathcal{P}X$.

(a) If $\mathcal{E} \subseteq \mathcal{P}X$ there is a set $\mathcal{E}' \subseteq \mathcal{E}$ such that $\#(\mathcal{E}') < \lambda$ and $E \setminus \bigcup \mathcal{E}' \in \mathcal{I}$ for every $E \in \mathcal{E}$.

(b) If $\langle E_\xi \rangle_{\xi<\lambda}$ is any family in $\mathcal{P}X \setminus \mathcal{I}$ then there is an $x \in X$ such that $\{\xi : \xi < \lambda,\, x \in E_\xi\}$ is infinite.

proof (a) ? Suppose, if possible, otherwise. Then we can choose inductively a family $\langle F_\xi \rangle_{\xi<\lambda}$ in $\mathcal{E}$ such that $G_\xi = F_\xi \setminus \bigcup_{\eta<\xi} F_\eta \notin \mathcal{I}$ for every $\xi < \lambda$. But now $\langle G_\xi \rangle_{\xi<\lambda}$ is a disjoint family in $\mathcal{P}\lambda \setminus \mathcal{I}$, which is impossible, because $\mathcal{I}$ is supposed to be λ-saturated. **X**

(b) Applying (a) to families of the form

$$\mathcal{E} = \{E_\eta : \beta(n) \le \eta < \lambda\},$$

we can build inductively a strictly increasing sequence $\langle \beta(n) \rangle_{n\in\mathbb{N}}$ in λ such that

$$E_\xi \setminus \textstyle\bigcup_{\beta(n)\le\eta<\beta(n+1)} E_\eta \in \mathcal{I} \;\forall\; n \in \mathbb{N},\, \xi \in \lambda \setminus \beta(n).$$

Set $\beta^* = \sup_{n\in\mathbb{N}} \beta(n) < \lambda$, and consider

$$G_n = E_{\beta^*} \setminus \textstyle\bigcup_{\beta(n)\le\eta<\beta(n+1)} E_\eta \in \mathcal{I}$$

for each $n \in \mathbb{N}$. Because $\mathcal{I}$ is a σ-ideal and $E_{\beta^*} \notin \mathcal{I}$, there is an $x \in E_{\beta^*} \setminus \bigcup_{n\in\mathbb{N}} G_n$, and now $\{\eta : x \in E_\eta\}$ meets $\beta(n+1) \setminus \beta(n)$ for every $n \in \mathbb{N}$, so is infinite.

Remark For a slightly different expression of the same idea, see Fremlin 87, Lemma 1E.

9B Theorem Suppose that κ and λ are cardinals, with $\omega < \mathrm{cf}(\lambda) \le \lambda \le \kappa$, and that there is a proper κ-additive λ-saturated ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ which contains all singletons. Then

(a) κ is weakly inaccessible;

(b) $\mathcal{P}\kappa$ has a λ-saturated normal ideal.

proof The argument amounts to extracting the essential ideas from 1D and 1G above.

(a) $\kappa = \mathrm{add}(\mathcal{I})$ is regular. **?** If $\kappa = \theta^+$, choose an injective function $f_\xi : \xi \to \theta$ for each $\xi < \kappa$. Set

$$A_{\eta\alpha} = \{\xi : \eta < \xi < \kappa,\, f_\xi(\eta) = \alpha\}$$

for $\eta < \kappa$, $\alpha < \theta$. Because $\mathcal{I}$ is κ-additive, there is for each $\eta < \kappa$ a $\beta_\eta < \theta$ such that $A_{\eta,\beta_\eta} \notin \mathcal{I}$. Set

$$B_\beta = \{\eta : \eta < \kappa,\ \beta_\eta = \beta\}$$

for $\beta < \theta$. Again because $\mathcal{I}$ is κ-additive, there is a $\beta < \theta$ such that $B_\beta \notin \mathcal{I}$. Now $\#(B_\beta) = \kappa \geq \lambda$ and $\langle A_{\eta\beta}\rangle_{\eta\in B_\beta}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{I}$, which is impossible, because $\mathcal{I}$ is λ-saturated. **X**

This shows that κ is weakly inaccessible.

(b) Let F be the family of functions $f : \kappa \to \kappa$ such that $f^{-1}[\zeta] \in \mathcal{I}$ for every $\zeta < \kappa$. Then there is an $f_0 \in F$ such that $\{\xi : \xi < \kappa,\ f(\xi) < f_0(\xi)\} \in \mathcal{I}$ for every $f \in F$. **P?** If not, we can find a decreasing family $\langle g_\alpha\rangle_{\alpha<\lambda}$ in F such that

$$E_\alpha = \{\xi : g_{\alpha+1}(\xi) < g_\alpha(\xi)\} \notin \mathcal{I}\ \forall\ \alpha < \lambda,$$

just as in 1G; we use the κ-additivity of $\mathcal{I}$ to be sure that g_α, defined as $\inf_{\beta<\alpha} g_\beta$, belongs to F for every non-zero limit ordinal $\alpha < \lambda$. By Lemma 9Bb there is a $\xi < \kappa$ such that $A = \{\alpha : \alpha < \lambda,\ \xi \in E_\alpha\}$ is infinite. But now we can find a strictly increasing sequence $\langle\alpha(n)\rangle_{n\in\mathbb{N}}$ in A, and $\langle g_{\alpha(n)}(\xi)\rangle_{n\in\mathbb{N}}$ will be a strictly decreasing sequence of ordinals, which is impossible. **XQ**

Set $\mathcal{J} = \{A : A \subseteq \kappa,\ f_0^{-1}[A] \in \mathcal{I}\}$. Then $\mathcal{J}$ is a proper λ-saturated κ-additive ideal of $\mathcal{P}\kappa$ because $\mathcal{I}$ is. $\mathcal{J}$ contains all singletons because $f_0 \in F$. Finally, let $f : \kappa \to \kappa$ be any function. Because $\mathcal{J}$ is κ-saturated and κ is regular, 9Aa tells us that there is an $\alpha < \kappa$ such that $f^{-1}[\zeta \setminus \alpha] \in \mathcal{J}$ for every $\zeta < \kappa$. Set $E = \{\xi : \alpha \leq f(\xi) < \xi < \kappa\}$. Define $f_1 : \kappa \to \kappa$ by setting

$$\begin{aligned} f_1(\xi) &= f(f_0(\xi)) \text{ if } f_0(\xi) \in E, \\ &= f_0(\xi) \text{ otherwise.} \end{aligned}$$

Then $f_1 \in F$ so

$$f_0^{-1}[E] = \{\xi : f_1(\xi) < f_0(\xi)\} \in \mathcal{I},$$

and $E \in \mathcal{J}$. Because f is arbitrary, $\mathcal{J}$ is normal (A1Ed).

Thus $\mathcal{J}$ is an ideal of the required kind.

Remarks The results of 9A-9B above are due to Solovay 71.

9C Definition I will call a cardinal κ **quasi-measurable** if it is uncountable and there is a proper ω_1-saturated κ-additive ideal of $\mathcal{P}\kappa$ containing singletons; such an ideal being a **witnessing ideal**.

Observe that every real-valued-measurable cardinal is quasi-measurable, and that a quasi-measurable cardinal κ carries a normal ω_1-saturated ideal.

In the spirit of 1I and 4Ac, we can speak of the **qm ideal** of a quasi-measurable cardinal κ, being the intersection of all its normal witnessing ideals, and the dual filter, the **qm filter** of κ.

For results involving quasi-measurable cardinals, see Fremlin 75c, Jech 78, Fremlin Hansell & Junnila 83, Fremlin & Jasiński 86, Fremlin 87, Kamburelis p89 and Główczyński 91.

9D Theorem If κ is a quasi-measurable cardinal, with witnessing ideal $\mathcal{I} \lhd \mathcal{P}\kappa$, then either $\kappa \leq \mathfrak{c}$ and $\mathcal{P}\kappa/\mathcal{I}$ is atomless, or κ is two-valued-measurable and $\mathcal{P}\kappa/\mathcal{I}$ is purely atomic.

proof (a) If $\mathcal{P}\kappa/\mathcal{I}$ has an atom a, take $A \subseteq \kappa$ such that $A^\bullet = a$, and see that

$$\{F : F \subseteq \kappa,\ A \setminus F \in \mathcal{I}\}$$

is a κ-complete ultrafilter on κ, so that κ is two-valued-measurable.

(b) If $\mathcal{P}\kappa/\mathcal{I}$ is not purely atomic, take $A \in \mathcal{P}\kappa \setminus \mathcal{I}$ such that no atom of $\mathcal{P}\kappa/\mathcal{I}$ is included in $A^\bullet$. Choose $\langle\mathcal{A}_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $\mathcal{A}_0 = \{A\}$. Given that $\mathcal{A}_\xi$ is a disjoint family in $\mathcal{P}A \setminus \mathcal{I}$, then for each $B \in \mathcal{A}_\xi$ choose disjoint B', $B'' \subseteq B$ such that $B' \cup B'' = B$ and neither belongs to $\mathcal{I}$; this is possible because $B^\bullet$ is not an atom in $\mathcal{P}\kappa/\mathcal{I}$. Given $\langle\mathcal{A}_\eta\rangle_{\eta<\xi}$, where $\xi < \omega_1$ is a non-zero limit ordinal, let $\mathcal{A}_\xi$ be a maximal disjoint family in $\mathcal{P}A \setminus \mathcal{I}$ such that for every $B \in \mathcal{A}_\xi$, $\eta < \xi$ there is a $C \in \mathcal{A}_\eta$ such that $B \subseteq C$. Continue.

? If $\bigcap_{\xi<\omega_1} \bigcup \mathcal{A}_\xi \neq \emptyset$, take $x \in \bigcap_{\xi<\omega_1} \bigcup \mathcal{A}_\xi$ and for each $\xi < \omega_1$ take $A_\xi \in \mathcal{A}_\xi$ such that $x \in A_\xi$. Let $B_\xi \in \mathcal{A}_{\xi+1}$ be whichever of A'_ξ, A''_ξ does not contain x. Then $\langle B_\xi\rangle_{\xi<\omega_1}$ is a disjoint family in $\mathcal{P}\kappa \setminus \mathcal{I}$; which is impossible. **X**

So $\bigcap_{\xi<\omega_1} \bigcup \mathcal{A}_\xi = \emptyset$, and there is a first $\xi < \omega_1$ such that $D = A \setminus \bigcup \mathcal{A}_\xi \notin \mathcal{I}$. Of course ξ cannot be a successor, because $\bigcup \mathcal{A}_{\eta+1} = \bigcup \mathcal{A}_\eta$ for every η. Now $E = D \cap \bigcap_{\eta<\xi} \bigcup \mathcal{A}_\eta \notin \mathcal{I}$. On E consider the equivalence relation $\sim$ given by

$$x \sim y \iff \forall\ \eta < \xi\ \exists\ B \in \mathcal{A}_\eta \text{ containing both } x \text{ and } y.$$

Because every $\mathcal{A}_\eta$ is countable, and $\xi < \omega_1$, there are at most $\mathfrak{c}$ equivalence classes for $\sim$. Also, every equivalence class must belong to $\mathcal{I}$, since otherwise it would have been a candidate for membership of $\mathcal{A}_\xi$. So E is covered by at most $\mathfrak{c}$ members of $\mathcal{I}$, and $\kappa = \mathrm{add}(\mathcal{I}) \le \mathfrak{c}$.

(c) Because any two-valued-measurable cardinal is greater than $\mathfrak{c}$, this completes the proof.

9E Proposition Let X be a set and $\mathcal{I}$ an ω_1-saturated σ-ideal of $\mathcal{P}X$. Let G^* be the set of bijections $g : X \to X$ such that $\mathcal{I} = \{A : A \subseteq X,\ g^{-1}[A] \in \mathcal{I}\}$. Then there is a partition $\mathcal{L}$ of X into countable sets such that G^* is precisely the set of bijections $g : X \to X$ such that $\bigcup\{L : L \in \mathcal{L},\ g[L] \ne L\} \in \mathcal{I}$.

proof As 1Mb.

Remark See ZAKRZEWSKI 91.

9F Theorem If κ is a quasi-measurable cardinal and $\mathbb{P}$ is a ccc p.o.set, then $\Vdash_{\mathbb{P}} \check\kappa$ is quasi-measurable.

proof Let $\mathcal{I}$ be a proper κ-additive ω_1-saturated ideal of $\mathcal{P}\kappa$ containing singletons. In $V^{\mathbb{P}}$, let $\mathcal{J}$ be the ideal of $\mathcal{P}\kappa$ generated by $\mathcal{I}$, that is, if σ is a $\mathbb{P}$-name for a subset of κ,

$$\Vdash_{\mathbb{P}} (\sigma \in \mathcal{J} \iff \exists\ A \in \check{\mathcal{I}},\ \sigma \subseteq A);$$

or, $p \Vdash_{\mathbb{P}} \sigma \in \mathcal{J}$ iff for every $p' \le p$ there are $p'' \le p'$ and $A \in \mathcal{I}$ such that $p'' \Vdash_{\mathbb{P}} \sigma \subseteq \check{A}$. But as $\mathbb{P}$ is ccc and $\mathcal{I}$ is a σ-ideal,

$$p \Vdash_{\mathbb{P}} \sigma \in \mathcal{J} \text{ iff } \exists\ A \in \mathcal{I} \text{ such that } p \Vdash_{\mathbb{P}} \sigma \subseteq \check{A}.$$

Clearly,

$$\Vdash_{\mathbb{P}} \sigma \subseteq \tau \in \mathcal{J} \Rightarrow \sigma \in \mathcal{J}.$$

Because singleton subsets of κ all belong to $\mathcal{I}$, we have

$$\Vdash_{\mathbb{P}} \{\xi\} \in \mathcal{J}\ \forall\ \xi < \check\kappa.$$

Now suppose that $\lambda < \kappa$ and that $\langle\sigma_\alpha\rangle_{\alpha<\lambda}$ is a family of $\mathbb{P}$-names for subsets of κ. If

$$\Vdash_{\mathbb{P}} \sigma_\alpha \in \mathcal{J}\ \forall\ \alpha < \lambda,$$

then for each $\alpha < \lambda$ we can find an $A_\alpha \in \mathcal{I}$ such that

$$\Vdash_{\mathbb{P}} \sigma_\alpha \subseteq \check{A}_\alpha.$$

But now $A = \bigcup_{\alpha<\lambda} A_\alpha \in \mathcal{I}$, because $\mathcal{I}$ is κ-additive, and

$$\Vdash_{\mathbb{P}} \textstyle\bigcup_{\alpha<\check\kappa} \sigma_\alpha \subseteq \check{A},$$

so that $\Vdash_{\mathbb{P}} \bigcup_{\alpha<\check\kappa} \sigma_\alpha \in \mathcal{J}$. Thus $\Vdash_{\mathbb{P}} \mathcal{J}$ is $\check\kappa$-additive.

Finally, suppose that $\langle\sigma_\alpha\rangle_{\alpha<\omega_1}$ is a family of $\mathbb{P}$-names for subsets of κ, and that

$$\Vdash_{\mathbb{P}} \sigma_\alpha \cap \sigma_\beta = \emptyset\ \forall\ \alpha \ne \beta.$$

For each $\xi < \kappa$ we may write

$$a_{\alpha\xi} = \sup\{p^* : p \Vdash_{\mathbb{P}} \check\xi \in \sigma_\alpha\} \in \mathfrak{A},$$

where $\mathfrak{A}$ is the regular open algebra of $\mathbb{P}$. In this case we shall have $a_{\alpha\xi} \cap a_{\beta\xi} = \mathbf{0}$ whenever $\alpha \ne \beta$ and $\xi < \kappa$. Because $\mathbb{P}$ is ccc, there is for each $\xi < \kappa$ a $\gamma_\xi < \omega_1$ such that $a_{\alpha\xi} = \mathbf{0}$ whenever $\gamma_\xi \le \alpha < \omega_1$. Because $\mathcal{I}$ is ω_1-saturated, there is a $\gamma < \omega_1$ such that $\{\xi : \gamma_\xi = \gamma'\} \in \mathcal{I}$ for every $\gamma' \ge \gamma$; because $\mathcal{I}$ is ω_2-additive,

$$A = \{\xi : \gamma_\xi \ge \gamma\} \in \mathcal{I}.$$

Now $a_{\alpha\xi} = \mathbf{0}$ for all $\xi \in \kappa \setminus A$, $\alpha \ge \gamma$. So

$$\Vdash_{\mathbb{P}} \sigma_\alpha \subseteq \check{A} \in \check{\mathcal{I}}$$

for every $\alpha \ge \gamma$. Because $\langle\sigma_\alpha\rangle_{\alpha<\omega_1}$ is arbitrary,

$$\Vdash_{\mathbb{P}} \mathcal{J} \text{ is } \check\omega_1\text{-saturated},$$

as required.

Remark This is due to Prikry (see Solovay 71, Theorem 8).

9G Proposition The following are equiconsistent:
(a) 'ZFC + there is a two-valued-measurable cardinal';
(b) 'ZFC + there is a quasi-measurable cardinal';
(c) 'ZFC + Martin's Axiom + there is a quasi-measurable cardinal $\kappa < \mathfrak{c}$';
(d) 'ZFC + Martin's Axiom + $\mathfrak{c}$ c is quasi-measurable'.

proof (a)⇒(c) Start with a two-valued-measurable cardinal κ. Let $\mathbb{P}$ be a ccc p.o.set such that

$$\Vdash_{\mathbb{P}} \mathfrak{c} = \check{\kappa}^+.$$

Let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for a ccc p.o.set of cardinal κ^+ such that

$$\Vdash_{\mathbb{P}} (\Vdash_{\dot{\mathbb{Q}}} \mathrm{MA})$$

(Kunen 80, §VIII.6). Then $\mathbb{P} * \dot{\mathbb{Q}}$ is ccc so $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \check{\kappa}$ is quasi-measurable.

(a)⇒(d) As above, reading κ for κ^+ at each opportunity.

(c)⇒(b) and **(d)⇒(b)** are trivial, and **(b)⇒(a)** is covered by 2D.

9H Algebras $\mathcal{P}X/\mathcal{I}$ The Gitik-Shelah theorem (3F) may be regarded as an attack on the problem: which measurable algebras $\mathfrak{A}$ are isomorphic to quotient algebras of the form $\mathcal{P}X/\mathcal{I}$ where X is a set and $\mathcal{I}$ is a σ-ideal of $\mathcal{P}X$? (For if $(\mathcal{P}X/\mathcal{I}, \bar{\mu})$ is a probability algebra, then it will be the measure algebra of $(X, \mathcal{P}X, \mu)$, where $\mu A = \bar{\mu}A^\bullet$ for every $A \subseteq X$.) The Gitik-Shelah theorem tells us that if $\mathfrak{A}$ is an atomless non-zero measurable algebra isomorphic to $\mathcal{P}X/\mathcal{I}$, then $\tau(\mathfrak{A}) \geq \min(2^\kappa, \kappa^{(+\omega)})$ for some atomlessly -measurable cardinal κ. The product measure extension axiom (8A-8B) is the assertion that there are such algebras of arbitrarily large size.

Suppose now we extend the question, and seek to describe the class $\mathfrak{Quot}_\sigma$ of Boolean algebras isomorphic to algebras of the form $\mathcal{P}X/\mathcal{I}$ where $\mathcal{I}$ is a σ-ideal of $\mathcal{P}X$. If $\mathcal{P}X/\mathcal{I}$ is non-zero, ccc and atomless then $\mathcal{I}$ is ω_1-saturated and $\mathrm{add}(\mathcal{I})$ is quasi-measurable. In Gitik & Shelah 89 and Gitik & Shelah p91 a variety of special types of algebra are considered; for instance, writing $\mathcal{G}_\lambda$ for the regular open algebra of $\{0,1\}^\lambda$, then if $\mathcal{G}_\lambda \in \mathfrak{Quot}_\sigma$, there is a quasi-measurable cardinal less than λ. On the other hand, Główczyński 91 points out that (if there can be measurable cardinals) $\mathfrak{Quot}_\sigma$ can contain countably-generated ccc algebras. Such algebras are always quotients of the algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$ (Fremlin 84, 12F). But the ideals involved in Główczyński's construction are necessarily very irregular, and we may reasonably ask if any naturally arising ideal $\mathcal{I}$ of $\mathcal{B}$ can have $\mathcal{B}/\mathcal{I}$ in $\mathfrak{Quot}_\sigma$. Similarly, we may ask whether the regular open algebra of a 'simple' partially ordered set $(P, \leq)$ (e.g., one in which P is an analytic set in a Polish space, and $\leq$ is an analytic, or Borel, subset of $P \times P$) can belong to $\mathfrak{Quot}_\sigma$.

I ought perhaps to mention in passing that *any* Dedekind complete Boolean algebra is isomorphic to a quotient $\mathcal{P}X/\mathcal{I}$ for some set X and some ideal $\mathcal{I}$; the problems here depend on the insistence on taking σ-ideals.

9I Theorem Let κ be a regular uncountable cardinal, $\mathcal{F}$ a normal filter on κ, $\mathcal{I}$ the dual ideal; suppose that $\mathcal{I}$ is κ^+-saturated. Let $\mathbb{P}$ be *either* the p.o.set $(\mathcal{P}\kappa/\mathcal{I}) \setminus \{\mathbf{0}\}$ *or* the p.o.set $\mathcal{P}\kappa \setminus \mathcal{I}$. Let ϕ be a formula of the second-order language $\mathcal{L}$ of §A4 and $C_1, \dots, C_k$ relations on κ, $\xi_1, \dots, \xi_m$ ordinals less than κ. Let $\beta \leq \kappa$. Then the following are equivalent:

(i) $\Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi$

(ii) $\{\alpha : \alpha < \kappa, (\min(\alpha, \beta); C_1, \dots, \xi_m) \vDash \phi\} \in \mathcal{F}$.

In particular, if $\beta < \kappa$ then

$$(\beta; C_1, \dots, \xi_m) \vDash \phi \iff \Vdash_{\mathbb{P}} (\check{\beta}; \check{C}_1, \dots, \check{\xi}_m) \vDash \phi.$$

proof As in 4M-4N.

9J Theorem Let κ be a quasi-measurable cardinal with qm filter $\mathcal{W}$. Then
(i) κ is greatly Mahlo;
(ii) $\mathrm{Mh}(A) \in \mathcal{W}$ for every stationary $A \subseteq \kappa$ – in particular, $\mathcal{W}$ is closed under the operation Mh;
(iii) the set of greatly Mahlo cardinals below κ belongs to $\mathcal{W}$.

proof The arguments of 4J-4Ka cover (i) and (ii). Now let A be the set of greatly Mahlo cardinals below κ. If $\mathcal{I}$ is any normal witnessing ideal for κ, with dual filter $\mathcal{F}$, then $\mathbb{P} = \mathcal{P}\kappa \setminus \mathcal{I}$ is ccc; so by 9F

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is qm},$$

and

$$\Vdash_{\mathbb{P}} \check{\kappa} \text{ is greatly Mahlo}.$$

Next, it is evident from 4C that there is a second-order formula ϕ such that for any ordinal α

$$(\alpha; <;) \vDash \phi \iff \alpha \text{ is a greatly Mahlo cardinal.}$$

So 9I tells us that $A \in \mathcal{F}$. As $\mathcal{I}$, $\mathcal{F}$ are arbitrary, $A \in \mathcal{W}$.

9K Theorem Let κ be a quasi-measurable cardinal, with normal witnessing ideal $\mathcal{I}$ and dual filter $\mathcal{F}$; let $\theta < \kappa$ be a cardinal of uncountable cofinality. Then for any function $f : [\kappa]^{<\omega} \to [\kappa]^{<\theta}$ there are $C \in \mathcal{F}$, $f^* : [C]^{<\omega} \to [\kappa]^{<\theta}$ such that $f(I) \cap \eta \subseteq f^*(I \cap \eta)$ whenever $I \in [C]^{<\omega}$ and $\eta < \kappa$.

proof As in 5A-5B.

9L Corollaries (a) If κ is quasi-measurable then there is no Jónsson algebra on κ.
(b) If $\mathfrak{c}$ is quasi-measurable then $2^\lambda = \mathfrak{c}$ for $\omega \le \lambda < \mathfrak{c}$.

proof As 5D-5E.

9M Proposition If κ is a quasi-measurable cardinal with qm ideal $\mathcal{J}$, then there is no κ-Aronszajn tree, and moreover $A = \{\theta : \theta < \kappa$, there is a θ-Aronszajn tree$\}$ belongs to $\mathcal{J}$.

proof The arguments of 5F show that there is no κ-Aronszajn tree. But it is easy to find a second-order formula ϕ such that, for any cardinal θ, $(\theta; <;) \vDash \phi$ iff there is a θ-Aronszajn tree. So the arguments of 9J show that $A \in \mathcal{J}$.

9N Proposition If $\mathfrak{c}$ is quasi-measurable, then $\diamondsuit_{\mathfrak{c}}$ is true.

proof As 5N.

Remark This is due to Kunen.

9O Theorem Let κ be a quasi-measurable cardinal.
(a) If $\mathcal{F}$ is any filter on $\mathbb{N}$ then $\mathrm{cf}(\mathrm{cf}(\mathbb{N}^{\mathbb{N}}/\mathcal{F})) \ne \kappa$.
(b) $\Theta(\alpha, \gamma) < \kappa$ for all cardinals α, $\gamma < \kappa$.
(c) $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta) < \kappa$ whenever $\alpha < \kappa$, $\gamma \le \beta$ and $\delta \ge \omega_1$.
(d) If $\kappa \le \mathfrak{c}$ then $\{2^\gamma : \omega \le \gamma < \kappa\}$ is finite.

proof Use the ideas of 7G-7K and 7O-7Q, but replacing 'κ-measure-bounded' with the property

if $\mathcal{I} \lhd \mathcal{P}P$ is an ω_1-saturated κ-additive ideal then there is a $p \in P$ such that $\{p' : p' \le p\} \notin \mathcal{I}$.

9P Remarks 9Od is given in Gitik & Shelah p91.
Many of the results in 9J-9O have generalizations to cardinals κ carrying non-trivial κ-additive ideals which are λ-saturated for some $\lambda < \kappa$.

Version of 18.9.92

Appendix: Useful Facts

In this appendix I seek to support the main text by giving definitions and theorems which may not be universally familiar, with some proofs.

A1. Combinatorics

I begin with material in (infinitary) combinatorics and set theory.

A1A Partially ordered sets (a) Recall that a **partially ordered set** is a set P together with a relation $\le$ such that, for p, q, $r \in P$,

$$p \leq q \;\&\; q \leq r \Rightarrow p \leq r,$$

$$p \leq q \;\&\; q \leq p \iff p = q.$$

(b) For a partially ordered set P, say that a subset Q of P is **cofinal** with P if for every $p \in P$ there is a $q \in Q$ such that $p \leq q$. Now write

$$\mathrm{cf}(P) = \min\{\#(Q) : Q \subseteq P \text{ is cofinal with } P\},$$

the **cofinality** of P.

If P is totally ordered then P has a well-ordered cofinal subset of order type $\mathrm{cf}(P)$, which in this case is either 0 (if $P = \emptyset$) or 1 (if P has a greatest element) or an infinite cardinal which is **regular**, that is, equal to its own cofinality. (But the cofinalities of general partially ordered sets need not be regular.)

(c) For a partially ordered set P with no greatest element, write

$$\mathrm{add}(P) = \min\{\#(A) : A \subseteq P \text{ has no upper bound in } P\},$$

the **additivity** of P. Then $\mathrm{add}(P)$ is either 0 (if $P = \emptyset$) or 2 (if P is not upwards-directed) or a regular infinite cardinal. In the last case, there is a family $\langle p_\xi \rangle_{\xi < \mathrm{add}(P)}$ in P such that $p_\xi \leq p_\eta$ whenever $\xi \leq \eta < \mathrm{add}(P)$ and $\{p_\xi : \xi < \mathrm{add}(P)\}$ has no upper bound in P.

If P has a greatest element I will write $\mathrm{add}(P) = \infty$.

If κ is a cardinal less than or equal to $\mathrm{add}(P)$ (allowing $\kappa < \infty = \mathrm{add}(P)$) we say that P is **κ-additive**; that is, P is κ-additive iff every subset of P of cardinal less than κ has an upper bound in P.

If $\mathrm{add}(P) \neq \infty$ then $\mathrm{add}(P) \leq \mathrm{cf}(P)$. If P is totally ordered and has no greatest element then $\mathrm{add}(P) = \mathrm{cf}(P)$.

A1B Filters and ideals (a) If X is a set and $\mathcal{I}$ is an ideal of subsets of X, we may think of $\mathcal{I}$ as partially ordered by $\subseteq$, and discuss its additivity (and cofinality) as in A1A. I allow $\mathcal{P}X$ as an (improper) ideal of itself.

(b) If X is a set, κ is a cardinal and $\mathcal{F}$ is a filter on X, then $\mathcal{F}$ is **κ-complete** if $\bigcap \mathcal{A} \in \mathcal{F}$ whenever $\mathcal{A} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{A}) < \kappa$; that is, iff the dual ideal $\{X \setminus F : F \in \mathcal{F}\}$ is κ-additive.

(c) Let X be a set and $\mathcal{F}$ a filter on X. Then $\mathcal{F}$ is **uniform** if $X \setminus A \in \mathcal{F}$ whenever $A \subseteq X$ and $\#(A) < \#(X)$.

(d) Let X and Y be sets, $\mathcal{F}$ a filter on X, and $f : X \to Y$ a function. I write $f[[\mathcal{F}]]$ for the filter $\{G : G \subseteq Y,\ f^{-1}[G] \in \mathcal{F}\}$, that is, the filter on Y generated by $\{f[F] : F \in \mathcal{F}\}$.

(e) If X is a set and $\mathcal{A}$ is any family of sets, I write

$$\mathrm{non}(X, \mathcal{A}) = \min\{\#(Y) : Y \subseteq X,\ Y \not\subseteq A \ \forall \ A \in \mathcal{A}\},$$

allowing $\mathrm{non}(X, \mathcal{A}) = \infty$ if $X \subseteq A \in \mathcal{A}$.

(f) If X is a set, $\mathcal{I}$ is an ideal of subsets of X, and κ is a cardinal, then $\mathcal{I}$ is **κ-saturated** if there is no family $\langle A_\xi \rangle_{\xi < \kappa}$ in $\mathcal{P}X \setminus \mathcal{I}$ such that $A_\xi \cap A_\eta \in \mathcal{I}$ whenever $\xi < \eta < \kappa$; that is, if there is no disjoint family of size κ in $(\mathcal{P}X/\mathcal{I}) \setminus \{\mathbf{0}\}$. If $\mathcal{I}$ is κ-additive, then it is κ-saturated iff there is no disjoint family of size κ in $\mathcal{P}X \setminus \mathcal{I}$.

A1C Filters on $\mathbb{N}$ (a) A **p-point filter** on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that for every sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{F}$ there is an $F \in \mathcal{F}$ such that $F \setminus F_n$ is finite for every $n \in \mathbb{N}$.

(b) A **rapid** filter on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that for every sequence $\langle t_n \rangle_{n \in \mathbb{N}}$ of non-negative real numbers converging to 0 there is an $F \in \mathcal{F}$ such that $\sum_{n \in F} t_n < \infty$.

(c) A **selective** ultrafilter on $\mathbb{N}$ is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that whenever $\langle A_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathcal{P}\mathbb{N} \setminus \mathcal{F}$ there is an $F \in \mathcal{F}$ such that $\#(F \cap A_n) \leq 1$ for every $n \in \mathbb{N}$.

(d) If λ is a cardinal, a **$p(\lambda)$-point** filter on $\mathbb{N}$ is a uniform filter $\mathcal{F}$ on $\mathbb{N}$ such that whenever $\mathcal{A} \subseteq \mathcal{F}$ and $\#(\mathcal{A}) < \lambda$ there is an $F \in \mathcal{F}$ such that $F \setminus A$ is finite for every $A \in \mathcal{A}$. (Thus a p-point filter is a $p(\omega_1)$-point filter.)

A1D Cardinals and ordinals (a) Let κ be a cardinal. **(i)** κ is **weakly inaccessible** if it is uncountable and regular and $\lambda^+ < \kappa$ for every cardinal $\lambda < \kappa$. **(ii)** κ is **strongly inaccessible** if it is uncountable and regular and $2^\lambda < \kappa$ for every cardinal $\lambda < \kappa$.

(b) Let S be any class of ordinals and f an ordinal-valued function defined on S; then f is **regressive** if $f(\xi) < \xi$ for every $\xi \in S \setminus \{0\}$.

(c) If ξ is any ordinal and $A \subseteq \xi$, then A is **stationary** in ξ if $A \cap C \neq \emptyset$ for every set $C \subseteq \xi$ which is cofinal with ξ and closed (for the order-topology of ξ).

(d) An ordinal α is **decomposable** if there are smaller ordinals β, γ such that α is equal to the ordinal sum $\beta + \gamma$; that is, if there is an ordinal $\beta < \alpha$ such that $\mathrm{otp}(\alpha \setminus \beta) < \alpha$. Otherwise α is **indecomposable**.

A1E Lemma Let κ be a regular uncountable cardinal, $\mathcal{F}$ a uniform filter on κ, $\mathcal{I}$ the dual ideal $\{\kappa \setminus F : F \in \mathcal{F}\}$.

(a) $\mathcal{F}$ and $\mathcal{I}$ are normal (see 1F) iff for every family $\langle F_\xi \rangle_{\xi<\kappa}$ in $\mathcal{F}$ its **diagonal intersection**

$$\{\xi : \xi < \kappa,\, \xi \in F_\eta \ \ \forall\ \eta < \xi\}$$

belongs to $\mathcal{F}$.

(b) The 'club filter' on κ, generated by the closed unbounded sets in κ, is normal; the 'non-stationary ideal', consisting of the non-stationary subsets of κ, is normal.

(c) If $\mathcal{F}$ and $\mathcal{I}$ are normal, then

(i) $\mathcal{F}$ is κ-complete, $\mathcal{I}$ is κ-additive;

(ii) every closed unbounded subset of κ belongs to $\mathcal{F}$;

(iii) if H is a subset of the quotient algebra $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$ and $\#(H) \le \kappa$, then $\sup H$ and $\inf H$ are defined in $\mathfrak{A}$;

(iv) if $\langle F_I \rangle_{I\in[\kappa]^{<\omega}}$ is any family in $\mathcal{F}$, then $\{\xi : \xi < \kappa,\, \xi \in F_I \ \ \forall\ I \in [\xi]^{<\omega}\}$ belongs to $\mathcal{F}$.

(d) The following are equivalent:

(i) $\mathcal{I}$ is normal and κ-saturated in $\mathcal{P}\kappa$;

(ii) $\mathcal{I}$ is κ-additive and for every function $f : \kappa \to \kappa$ there is a $\zeta < \kappa$ such that $\{\xi : \zeta \le f(\xi) < \xi\} \in \mathcal{I}$;

(iii) $\mathcal{F}$ is κ-complete and for every regressive $f : \kappa \to \kappa$ there is a $\zeta < \kappa$ such that $f^{-1}[\zeta] \in \mathcal{F}$.

proof (a)(i) Suppose that $\mathcal{F}$ is normal, and that $\langle F_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$ with diagonal intersection F. Set $S = \kappa \setminus F$, and for $\xi \in S$ take $f(\xi) < \xi$ such that $\xi \notin F_{f(\xi)}$. Then f is regressive, but $f^{-1}[\{\xi\}] \subseteq \kappa \setminus F_\xi \in \mathcal{I}$ for every $\xi < \kappa$; so $S \in \mathcal{I}$ and $F \in \mathcal{F}$.

(ii) Now suppose that $\mathcal{F}$ is closed under diagonal intersections, and that we are given $S \in \mathcal{P}\kappa \setminus \mathcal{I}$ and a regressive function $f : S \to \kappa$. **?** If $A_\xi = f^{-1}[\{\xi\}] \in \mathcal{I}$ for every $\xi < \kappa$, set $F_\xi = \kappa \setminus A_\xi \in \mathcal{F}$ for each ξ, and take F to be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$, so that $F \in \mathcal{F}$. Then there must be a $\xi \in F \cap S \setminus \{0\}$, so that $f(\xi) < \xi$ and $\xi \in A_{f(\xi)}$ and $\xi \notin F_{f(\xi)}$ and $\xi \notin F$, which is absurd. **X**

(b) It is easy to check that if $\langle C_\xi \rangle_{\xi<\kappa}$ is any family of closed unbounded sets in κ, then its diagonal intersection is closed and also (because κ is regular) unbounded. So the club filter is normal and its dual ideal, the non-stationary ideal, is normal.

(Of course the definition of 'normal' ideal in 1F above corresponds to the 'pushing-down lemma'; see Kunen 80, II.6.15, or Jech 78, Theorem 22.)

(c)(i) If $\lambda < \kappa$ and $\langle F_\xi \rangle_{\xi<\lambda}$ is any family in $\mathcal{F}$, set $F_\xi = \kappa$ for $\xi \in \kappa \setminus \lambda$, and consider the diagonal intersection F of $\langle F_\xi \rangle_{\xi<\kappa}$. Then F and $F \setminus \lambda$ belong to $\mathcal{F}$; but $F \setminus \lambda \subseteq \bigcap_{\xi<\lambda} F_\xi$. Thus $\mathcal{F}$ is κ-complete and $\mathcal{I}$ is κ-additive.

(ii) Now take any closed unbounded set $C \subseteq \kappa$. For each $\xi < \kappa$ set $\gamma_\xi = \min(C \setminus (\xi+1))$, $F_\xi = \kappa \setminus \gamma_\xi \in \mathcal{F}$; let F be diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$. Then $F \in \mathcal{F}$ and $F \setminus \gamma_0 \subseteq C$, so $C \in \mathcal{F}$.

(iii) If $H = \emptyset$ then $\inf H = \mathbf{1}$. Otherwise, let $\langle a_\xi \rangle_{\xi<\kappa}$ run over H. For each $\xi < \kappa$ choose $A_\xi \subseteq \kappa$ such that $A_\xi^\bullet = a_\xi$ in $\mathfrak{A}$. Let A be the diagonal intersection of $\langle A_\xi \rangle_{\xi<\kappa}$, and consider $a = A^\bullet$. Because $A \setminus A_\xi \subseteq \xi + 1 \in \mathcal{I}$, $a \subseteq a_\xi$ for each $\xi < \kappa$, and a is a lower bound for H. Now let b be any other lower bound for H in $\mathfrak{A}$, and take $B \subseteq \kappa$ such that $B^\bullet = b$. For each $\xi < \kappa$, we have $b \subseteq a_\xi$, that is, $B \setminus A_\xi \in \mathcal{I}$; set $F_\xi = \kappa \setminus (B \setminus A_\xi) \in \mathcal{F}$. Let F be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$, so that $F \in \mathcal{F}$. **?** If there is a $\xi \in (B \setminus A) \cap F$, then there must be an $\eta < \xi$ such that $\xi \notin A_\eta$; but now $\xi \in B \setminus A_\eta$ so $\xi \notin F_\eta$ and $\xi \notin F$. **X** Thus $B \setminus A \subseteq \kappa \setminus F \in \mathcal{I}$, and $b \subseteq a$ in $\mathfrak{A}$. As b is arbitrary, $a = \inf H$ in $\mathfrak{A}$.

Thus H has an infimum in $\mathfrak{A}$. But applying the argument above to $\{\mathbf{1}\setminus a : a\in H\}$ we see that H also has a supremum.

(iv) Set $E_\xi=\bigcap_{I\in[\xi]^{<\omega}}F_I$ for each $\xi<\kappa$; by (i), $E_\xi\in\mathcal{F}$. Let C be the closed unbounded set of limit ordinals less than κ; set

$$F=\{\xi:\xi\in C,\,\xi\in E_\eta\ \forall\ \eta<\xi\}.$$

Then $\{\xi:\xi\in F_I\ \forall\ I\in[\xi]^{<\omega}\}\supseteq F\in\mathcal{F}$.

(d)(i)⇒(ii) Of course $\mathcal{I}$ is κ-additive, by (c-i). Now let $f:\kappa\to\kappa$ be any function. Because $\mathcal{I}$ is κ-saturated, the set $A=\{\alpha:f^{-1}[\{\alpha\}]\notin\mathcal{I}\}$ has cardinal less than κ; because κ is regular, A is bounded above in κ — say $A\subseteq\zeta<\kappa$. For $\alpha\in A$ set $F_\alpha=\kappa$; for $\alpha\in\kappa\setminus A$ set $F_\alpha=\kappa\setminus f^{-1}[\{\alpha\}]$. Then

$$F=\{\xi:\xi\in F_\alpha\ \forall\ \alpha<\xi\}$$

belongs to $\mathcal{F}$. If $\zeta\le f(\xi)<\xi<\kappa$ then $f(\xi)\notin A$, so $\xi\notin F_{f(\xi)}$ and $\xi\notin F$; thus $\{\xi:\zeta\le f(\xi)<\xi\}\subseteq\kappa\setminus F\in\mathcal{I}$.

(ii)⇒(iii) is elementary.

(iii)⇒(i) Of course $\mathcal{I}$ is κ-additive. **?** If it is not κ-saturated, choose a disjoint family $\langle A_\xi\rangle_{\xi<\kappa}$ in $\mathcal{P}\kappa\setminus\mathcal{I}$; set $f(\eta)=\xi$ for $\eta\in A_\xi\setminus(\xi+1)$, $f(\eta)=0$ for $\eta\in\kappa\setminus\bigcup_{\xi<\kappa}(A_\xi\setminus(\xi+1))$. Then f is regressive, and $f^{-1}[\zeta]$ does not meet $A_\zeta\setminus(\zeta+1)$, so does not belong to $\mathcal{F}$, for any $\zeta<\kappa$. **X**

Now suppose that $\langle F_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{F}$. Let F be its diagonal intersection, and define $f:\kappa\to\kappa$ by setting $f(\xi)=0$ if $\xi\in F$, $f(\xi)=\min\{\eta:\xi\notin F_\eta\}$ if $\xi\in\kappa\setminus F$. Let $\zeta<\kappa$ be such that $f^{-1}[\zeta]\in\mathcal{F}$. Then $F\supseteq f^{-1}[\zeta]\cap\bigcap_{\eta<\zeta}F_\eta\in\mathcal{F}$. This shows that $\mathcal{F}$ is normal.

Remark For another version of this material, see Baumgartner Taylor & Wagon 82.

A1F Almost-square-sequences I give here one of Shelah's lemmas (Shelah #351, Lemma 4.4; Burke & Magidor 90, 7.7) in a form appropriate to Theorem 3F.

Lemma Let κ, λ be infinite cardinals, with κ regular and $\lambda>\kappa$, $\mathrm{cf}(\lambda)>\omega_1$. Then we can find a stationary set $S\subseteq\lambda^+$ and a family $\langle C_\alpha\rangle_{\alpha\in S}$ of sets such that (i) for each $\alpha\in S$, C_α is a closed unbounded set in α of order type κ (ii) if $\alpha,\beta\in S$ and γ is a limit point of both C_α and C_β then $C_\alpha\cap\gamma=C_\beta\cap\gamma$.

proof For each $\gamma<\lambda^+$ fix an injection $f_\gamma:\gamma\to\lambda$. Let S_0 be the set of ordinals $\alpha<\lambda^+$ of cofinality κ; then S_0 is stationary in λ^+. For each $\alpha\in S_0$ choose an increasing family $\langle N_{\alpha\delta}\rangle_{\delta<\lambda}$ of subsets of λ^+ such that

(α) $N_{\alpha0}$ is a cofinal subset of α of cardinal κ;

(β) if $\delta<\lambda$ then

$$N_{\alpha,\delta+1}=\bigcup\{f_\gamma[N_{\alpha\delta}]\cup f_\gamma^{-1}[\delta]:\gamma\in N_{\alpha\delta}\}\cup\overline{N}_{\alpha\delta}\cup\delta$$

(taking the closure $\overline{N}_{\alpha\delta}$ in the order topology of λ^+);

(γ) if $\delta<\lambda$ is a non-zero limit ordinal then $N_{\alpha\delta}=\bigcup_{\delta'<\delta}N_{\alpha\delta'}$.

Then $\#(N_{\alpha\delta})\le\max(\kappa,\#(\delta))<\lambda$ for each $\delta<\lambda$. Now observe that $\{\delta:\delta<\lambda,\,N_{\alpha\delta}\cap\lambda=\delta\}$ is a closed unbounded set in λ, and in particular contains an ordinal of cofinality ω_1, for every $\alpha\in S_0$. Let $\delta<\lambda$ be such that $\mathrm{cf}(\delta)=\omega_1$ and

$$S_1=\{\alpha:\alpha\in S_0,\,N_{\alpha\delta}\cap\lambda=\delta\}$$

is stationary in λ^+. For $\alpha\in S_1$, set $C^*_\alpha=\alpha\cap\overline{N}_{\alpha\delta}$; then C^*_α is a closed unbounded set in α and $\#(C^*_\alpha)<\lambda$ so $\mathrm{otp}(C^*_\alpha)<\lambda$. Let $\zeta<\lambda$ be such that

$$S=\{\alpha:\alpha\in S_1,\,\mathrm{otp}(C^*_\alpha)=\zeta\}$$

is stationary in λ^+. Observe that as $\mathrm{cf}(C^*_\alpha)=\mathrm{cf}(\alpha)=\kappa$ for each $\alpha\in S$, $\mathrm{cf}(\zeta)=\kappa$.

Take any closed unbounded set $C\subseteq\zeta$ of order type κ and for each $\alpha\in S$ let C_α be the image of C in C^*_α under the order-isomorphism between ζ and C^*_α. Then C_α will be a closed unbounded subset of α of order type κ.

I claim that if $\alpha,\beta\in S$ and γ is a common limit point of C_α, C_β then $C_\alpha\cap\gamma=C_\beta\cap\gamma$.

P case 1 Suppose $\kappa=\omega$. In this case the only limit point of C_α will be α itself, and similarly for β, so that in this case we have $\alpha=\beta$ and there is nothing more to do.

case 2 Suppose $\mathrm{cf}(\gamma)=\omega<\kappa$. Then γ is a limit point of $C_\alpha\subseteq\overline{N}_{\alpha\delta}$, so there is an increasing sequence in $N_{\alpha\delta}$ with supremum γ; as $N_{\alpha\delta}=\bigcup_{\delta'<\delta}N_{\alpha\delta'}$ and $\mathrm{cf}(\delta)=\omega_1$, this sequence lies entirely within $N_{\alpha\delta'}$ for some $\delta' < \delta$, and $\gamma \in \overline{N}_{\alpha\delta'} \subseteq N_{\alpha,\delta'+1}$. Now, for $\delta' + 1 \le \xi < \delta$, $N_{\alpha,\xi+1} \supseteq f_\gamma^{-1}[\xi] \cup f_\gamma[N_{\alpha\xi}]$; consequently $N_{\alpha\delta} \cap \gamma = f_\gamma^{-1}[N_{\alpha\delta} \cap \lambda] = f_\gamma^{-1}[\delta]$. Similarly, $N_{\beta\delta} \cap \gamma = f_\gamma^{-1}[\delta]$. Now

$$C_\alpha^* \cap \gamma = \overline{N}_{\alpha\delta} \cap \gamma = \overline{f_\gamma^{-1}[\delta]} \cap \gamma = C_\beta^* \cap \gamma.$$

Thus $C_\alpha \cap \gamma$ and $C_\beta \cap \gamma$ must be equal.

case 3 Suppose that $\mathrm{cf}(\gamma) > \omega$, $\kappa > \omega$. Because $\gamma = \sup(C_\alpha \cap \gamma) = \sup(C_\beta \cap \gamma)$,

$$D = \{\gamma' : \gamma' \text{ is a limit point of both } C_\alpha \text{ and } C_\beta,\ \gamma' < \gamma,\ \mathrm{cf}(\gamma') = \omega\}$$

is cofinal with γ, and $C_\alpha \cap \gamma = \bigcup_{\gamma' \in D} C_\alpha \cap \gamma' = C_\beta \cap \gamma$, using case 2. **Q**

Thus S, $\langle C_\alpha \rangle_{\alpha \in S}$ have the required properties.

A1G Corollary Let κ, λ be infinite cardinals with κ regular and $\lambda > \kappa$, $\mathrm{cf}(\lambda) > \omega_1$. Then we can find a stationary subset S of λ^+ and a family $\langle g_\alpha \rangle_{\alpha \in S}$ of functions from κ to λ^+ such that, for all distinct α, $\beta \in S$, (i) $g_\alpha[\kappa] \subseteq \alpha$ for each $\alpha \in S$ (ii) $\#(g_\alpha \cap g_\beta) < \kappa$ for each α (iii) if $\theta < \kappa$ is a limit ordinal and α, $\beta \in S$ and $g_\alpha(\theta) = g_\beta(\theta)$ then $g_\alpha \restriction \theta = g_\beta \restriction \theta$.

proof Take $\langle C_\alpha \rangle_{\alpha \in S}$ from A1F above and let g_α be the increasing enumeration of C_α.

A1H Products of partially ordered sets (a) Let $\langle P_i \rangle_{i \in I}$ be a family of partially ordered sets. Then $X = \prod_{i \in I} P_i$ is a partially ordered set, if we say that $x \le y$ iff $x(i) \le y(i)$ for every $i \in I$.

(b) Now suppose that $\mathcal{F}$ is a filter on I. Then we have an equivalence relation $\equiv_{\mathcal{F}}$ on X, given by saying that $f \equiv_{\mathcal{F}} g$ if $\{i : f(i) = g(i)\} \in \mathcal{F}$. I write $X/\mathcal{F}$ for the set of equivalence classes under this relation, the 'reduced product' of $\langle P_i \rangle_{i \in I}$ modulo $\mathcal{F}$. Now $X/\mathcal{F}$ is again a partially ordered set, writing

$$f^\bullet \le g^\bullet \iff f \le_{\mathcal{F}} g \iff \{i : f(i) \le g(i)\} \in \mathcal{F}.$$

Observe that if every P_i is totally ordered and $\mathcal{F}$ is an ultrafilter, then $X/\mathcal{F}$ is totally ordered.

(c) For any filter $\mathcal{F}$ on I we have

$$\mathrm{add}(X) \le \sup_{F \in \mathcal{F}} \mathrm{add}(\textstyle\prod_{i \in F} P_i) = \sup_{F \in \mathcal{F}} \min_{i \in F} \mathrm{add}(P_i) \le \mathrm{add}(X/\mathcal{F}),$$

$$\mathrm{cf}(X/\mathcal{F}) \le \min_{F \in \mathcal{F}} \mathrm{cf}(\textstyle\prod_{i \in F} P_i) \le \mathrm{cf}(X).$$

Version of 13.11.91

A1I Scraps of pcf In §7 I need to call on certain results from Shelah's pcf theory. An admirable exposition of some of this extraordinary development may be found in Burke & Magidor 90, from which most of the ideas here are drawn; but for the reader's convenience I extract and reproduce the material I wish to use.

Theorem [Shelah] Let $\lambda > 0$ be a cardinal and $\langle \theta_\zeta \rangle_{\zeta < \lambda}$ a family of regular infinite cardinals, all greater than λ. Set $X = \prod_{\zeta < \lambda} \theta_\zeta$, ordered as in A1H. For any filter $\mathcal{F}$ on λ, let $\pi_{\mathcal{F}} : X \to X/\mathcal{F}$ be the canonical map. For any cardinal δ set

$$\mathfrak{F}_\delta = \{\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \lambda,\ \mathrm{cf}(X/\mathcal{F}) = \delta\},$$

$$\mathfrak{F}_\delta^* = \textstyle\bigcup_{\delta' \ge \delta} \mathfrak{F}_{\delta'};$$

if $\mathfrak{F}_\delta^* \ne \emptyset$, let $\mathcal{G}_\delta$ be the filter $\bigcap \mathfrak{F}_\delta^*$. Now

(a) if $\mathfrak{F}_\delta^* \ne \emptyset$, then $\mathrm{add}(X/\mathcal{G}_\delta) \ge \delta$ (Burke & Magidor 90, 1.1);

(b) if $\mathfrak{F}_\delta \ne \emptyset$ then there is a set $F \subseteq X$ such that $\#(F) \le \delta$ and $\pi_{\mathcal{F}}[F]$ is cofinal with $X/\mathcal{F}$ for every $F \in \mathfrak{F}_\delta$ (Burke & Magidor 90, 7.3);

(c) $\mathfrak{F}_{\mathrm{cf}(X)} \ne \emptyset$ (Burke & Magidor 90, 7.10);

(d) if $\mathcal{F}$ is an ultrafilter on λ and κ is a regular cardinal with $\lambda < \kappa \le \mathrm{cf}(X/\mathcal{F})$ then there is a family $\langle \theta'_\zeta \rangle_{\zeta < \lambda}$ of regular cardinals such that $\lambda < \theta'_\zeta \le \theta_\zeta$ for every $\zeta < \lambda$ and $\mathrm{cf}(X'/\mathcal{F}) = \kappa$, where $X' = \prod_{\zeta < \lambda} \theta'_\zeta$ (Burke & Magidor 90, 2.1).

proof The case of finite λ is trivial throughout, as then

$$\mathrm{cf}(X)=\max_{\zeta<\lambda}\theta_\zeta,$$

$$\mathfrak{F}_\delta=\{\mathcal{F}:\exists\ \zeta,\,\theta_\zeta=\delta,\,\{\zeta\}\in\mathcal{F}\},$$

$$\mathfrak{F}^*_\delta=\{\mathcal{F}:\{\zeta:\theta_\zeta\ge\delta\}\in\mathcal{F}\},$$

$$\mathcal{G}_\delta=\{G:\{\zeta:\theta_\zeta\ge\delta\}\subseteq G\subseteq\lambda\}.$$

So henceforth let us take it that λ is infinite.

For any filter $\mathcal{F}$ on λ, write $f\le_{\mathcal{F}}g$ if f, $g\in X$ and $\{\zeta:f(\zeta)\le g(\zeta)\}\in\mathcal{F}$, that is, if $\pi_{\mathcal{F}}f\le\pi_{\mathcal{F}}g$ in $X/\mathcal{F}$. Write $L=\{\zeta:\theta_\zeta=\lambda^+\}\subseteq\lambda$, $M=\lambda\setminus L$.

(a) Set $\delta'=\mathrm{add}(X/\mathcal{G}_\delta)$.

(i) Evidently δ' is a regular cardinal and $\delta'\ge\mathrm{add}(X)=\min_{\zeta<\lambda}\theta_\zeta>\lambda$. If $\delta=\lambda^+$ then of course $\delta'\ge\delta$; so suppose that $\delta>\lambda^+$. In this case $L\notin\mathcal{F}$ for any $\mathcal{F}\in\mathfrak{F}^*_\delta$ (because if $L\ne\emptyset$ then $\mathrm{cf}(\prod_{\zeta\in L}\theta_\zeta)=\lambda^+$), so $M\in\mathcal{G}_\delta$ and $\delta'\ge\min_{\zeta\in M}\theta_\zeta>\lambda^+$.

(ii) ? If $\delta'<\delta$ there is a family $\langle f_\alpha\rangle_{\alpha<\delta'}$ in X such that $f_\alpha\le_{\mathcal{G}_\delta}f_\beta$ whenever $\alpha\le\beta<\delta'$ but there is no $f\in X$ such that $f_\alpha\le_{\mathcal{G}_\delta}f$ for every $\alpha<\delta'$. Choose $\langle h_\xi\rangle_{\xi<\lambda^+}$ in X inductively, as follows. $h_0=f_0$. Given h_ξ, set

$$B_{\xi\alpha}=\{\zeta:\zeta\in M,\,h_\xi(\zeta)\ge f_\alpha(\zeta)\}$$

for each $\alpha<\delta'$; let $\alpha_\xi<\delta'$ be such that $f_{\alpha_\xi}\not\le_{\mathcal{G}_\delta}h_\xi$, so that $B_{\xi\alpha}\notin\mathcal{G}_\delta$ when $\alpha_\xi\le\alpha<\delta'$. Choose $\mathcal{F}_\xi\in\mathfrak{F}^*_\delta$ such that $B_{\xi,\alpha_\xi}\notin\mathcal{F}_\xi$. Now, because $\mathrm{cf}(X/\mathcal{F}_\xi)\ge\delta>\delta'$, there is an $h_{\xi+1}\in X$ such that $f_\alpha\le_{\mathcal{F}_\xi}h_{\xi+1}$ for every $\alpha<\delta'$; we may take $h_{\xi+1}\ge h_\xi$.

For non-zero limit ordinals $\xi<\lambda^+$ take $h_\xi(\zeta)=\sup_{\eta<\xi}h_\eta(\zeta)$ for every $\zeta<\lambda$.

Set $\alpha=\sup_{\xi<\lambda^+}\alpha_\xi<\delta'$. Then $\langle B_{\xi\alpha}\rangle_{\xi<\lambda^+}$ is an increasing family in $\mathcal{P}\lambda$. So there must be a $\xi<\lambda^+$ such that $B_{\xi\alpha}=B_{\xi+1,\alpha}$. But (because $\alpha\ge\alpha_\xi$) $B_{\xi\alpha}\notin\mathcal{F}_\xi$, while (by the choice of $h_{\xi+1}$) $B_{\xi+1,\alpha}\in\mathcal{F}_\xi$; which is absurd. **X**

(b) As in (a-i) above, we must have $\delta\ge\min_{\zeta<\lambda}\theta_\zeta>\lambda$, and the case $\delta=\lambda^+$ is again trivial, for if $\delta=\lambda^+$ we may take F to be the set of constant functions with values less than λ^+. So suppose from now on that $\delta>\lambda^+$, so that $M\in\mathcal{F}$ for every $\mathcal{F}\in\mathfrak{F}_\delta$. Of course δ, being the cofinality of a totally ordered set, is regular.

(i) ? Suppose, if possible, that there is no F of the required type. We can find families $\langle f_{\xi\alpha}\rangle_{\xi<\lambda^+,\alpha<\delta}$ in X and $\langle\mathcal{F}_\xi\rangle_{\xi<\lambda^+}$ in $\mathfrak{F}_\delta$ such that

(α) $f_{\eta\alpha}\le f_{\xi\alpha}$ whenever $\alpha<\delta$, $\eta\le\xi<\lambda^+$;

(β) $f_{\eta\alpha}\le_{\mathcal{F}_\xi}f_{\xi,0}$ whenever $\alpha<\delta$, $\eta<\xi<\lambda^+$;

(γ) $\{\pi_{\mathcal{F}_\xi}(f_{\xi\alpha}):\alpha<\delta\}$ is cofinal with $X/\mathcal{F}_\xi$ for every $\xi<\lambda^+$;

(δ) if $\xi<\lambda^+$, $\alpha<\delta$ and $\mathrm{cf}(\alpha)=\lambda^+$ then

$$f_{\xi\alpha}(\zeta)=\min\{\sup_{\beta\in C}f_{\xi\beta}(\zeta):C\text{ is a closed unbounded set in }\alpha\}$$

for every $\zeta\in M$;

(ϵ) $f_{\xi\alpha}\le_{\mathcal{F}_\xi}f_{\xi\beta}$ whenever $\xi<\lambda^+$, $\alpha\le\beta<\delta$.

P Given $\langle f_{\eta\alpha}\rangle_{\eta<\xi,\alpha<\delta}$, there must be an $\mathcal{F}_\xi\in\mathfrak{F}_\delta$ such that $\{\pi_{\mathcal{F}_\xi}(f_{\eta\alpha}):\eta<\xi,\alpha<\delta\}$ is not cofinal with $X/\mathcal{F}_\xi$; take $\langle g_{\xi\alpha}\rangle_{\alpha<\delta}$ in X such that $\{\pi_{\mathcal{F}_\xi}(g_{\xi\alpha}):\alpha<\delta\}$ is cofinal with $X/\mathcal{F}_\xi$. Choose $\langle f_{\xi\alpha}\rangle_{\alpha<\delta}$ inductively so that

$f_{\eta\alpha}\le_{\mathcal{F}_\xi}f_{\xi0}$ for every $\eta<\xi$, $\alpha<\delta$;

if $\alpha<\delta$ and $\mathrm{cf}(\alpha)\ne\lambda^+$ then $f_{\eta\alpha}\le f_{\xi\alpha}$ for every $\eta<\xi$,

$f_{\xi\beta}\le_{\mathcal{F}_\xi}f_{\xi\alpha}$ for every $\beta<\alpha$;

if $\alpha<\delta$ then $g_{\xi\alpha}\le f_{\xi,\alpha+1}$;

if $\alpha<\delta$ and $\mathrm{cf}(\alpha)=\lambda^+$ then $f_{\eta\alpha}(\zeta)\le f_{\xi\alpha}(\zeta)$ whenever $\eta<\xi$, $\zeta\in L$;

if $\alpha<\delta$ and $\mathrm{cf}(\alpha)=\lambda^+$ then

$$f_{\xi\alpha}(\zeta)=\min\{\sup_{\beta\in C}f_{\xi\beta}(\zeta):C\text{ is a closed unbounded set in }\alpha\}$$

for every $\zeta\in M$. (At this point observe that there will be a closed unbounded set $C\subseteq\alpha$ such that $f_{\xi\alpha}(\zeta)=\sup_{\beta\in C}f_{\xi\beta}(\zeta)$ for every $\zeta\in M$; consequently $f_{\xi\beta}\le_{\mathcal{F}_\xi}f_{\xi\alpha}$ for every $\beta\in C$, and $f_{\xi\beta}\le_{\mathcal{F}_\xi}f_{\xi\alpha}$ for every $\beta<\alpha$. Also we shall have

$$f_{\eta\alpha}(\zeta) \le \sup_{\beta\in C} f_{\eta\beta}(\zeta) \le f_{\xi\alpha}(\zeta)$$

for every $\eta < \xi$, $\zeta \in M$.)

It is straightforward to check that this procedure works. **Q**

(ii) The next step is to find an increasing family $\langle h_\eta\rangle_{\eta<\lambda^+}$ in X and a strictly increasing family $\langle\gamma(\eta)\rangle_{\eta<\lambda^+}$ in δ such that

$f_{\xi,\gamma(\eta)}(\zeta) < h_\eta(\zeta)$ whenever $\xi < \lambda^+$, $\eta < \lambda^+$, $\zeta \in M$ (choosing h_η);

$h_\eta \le_{\mathcal{F}_\xi} f_{\xi,\gamma(\eta+1)}$ whenever $\xi < \lambda^+$, $\eta < \lambda^+$ (choosing $\gamma(\eta+1)$);

$\gamma(\eta) = \sup_{\eta'<\eta}\gamma(\eta')$ whenever $\eta < \lambda^+$ is a limit ordinal (so $\gamma(0) = 0$).

Set $h(\zeta) = \sup_{\eta<\lambda^+} h_\eta(\zeta)$ for $\zeta \in M$, $h(\zeta) = 0$ for $\zeta \in L$, $\alpha = \sup_{\eta<\lambda^+}\gamma(\eta) < \delta$ (because $\delta = \mathrm{cf}(\delta) > \lambda^+$); then $\mathrm{cf}(\alpha) = \lambda^+$. Observe that

$$f_{\xi\alpha}(\zeta) \le \sup_{\eta<\lambda^+} f_{\xi,\gamma(\eta)}(\zeta) \le h(\zeta)$$

for every $\xi < \lambda^+$, $\zeta \in M$, by (i-δ). So if we set

$$A_\xi = \{\zeta : \zeta \in M,\ f_{\xi\alpha}(\zeta) = h(\zeta)\}\ \forall\ \xi < \lambda^+,$$

we shall have $A_\eta \subseteq A_\xi$ whenever $\eta \le \xi < \lambda^+$, by (i-α).

(iii) There must therefore be some $\xi < \lambda^+$ such that $A_\xi = A_{\xi+1}$. Let $C \subseteq \lambda^+$ be a closed unbounded set such that

$$f_{\xi+1,\alpha}(\zeta) = \sup_{\eta\in C} f_{\xi+1,\gamma(\eta)}(\zeta)\ \forall\ \zeta \in M.$$

For each $\eta \in C$ write η' for the next member of C greater than η; then

$$h_\eta \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta+1)} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,\gamma(\eta')},$$

$$f_{\xi\alpha} \le_{\mathcal{F}_{\xi+1}} f_{\xi+1,0} = f_{\xi+1,\gamma(0)}$$

so there is a $\zeta_\eta \in M$ such that

$$h_\eta(\zeta_\eta) \le f_{\xi+1,\gamma(\eta')}(\zeta_\eta),\ f_{\xi\alpha}(\zeta_\eta) \le f_{\xi+1,\gamma(0)}(\zeta_\eta) < h_0(\zeta_\eta) \le h(\zeta_\eta).$$

Let $\zeta \in M$ be such that

$$B = \{\eta : \eta \in C,\ \zeta_\eta = \zeta\}$$

is cofinal with C. Then $f_{\xi\alpha}(\zeta) < h(\zeta)$ so $\zeta \notin A_\xi = A_{\xi+1}$. On the other hand,

$$f_{\xi+1,\alpha}(\zeta) = \sup_{\eta\in C} f_{\xi+1,\gamma(\eta)}(\zeta) \ge \sup_{\eta\in B} f_{\xi+1,\gamma(\eta')}(\zeta) \ge \sup_{\eta\in B} h_\eta(\zeta) = h(\zeta).$$

So $\zeta \in A_{\xi+1}$; which is impossible. **X**

(c)(i) Set $\Delta = \{\delta : \mathfrak{F}_\delta \ne \emptyset\}$, $\mathcal{G} = \bigcup_{\delta\in\Delta}\mathcal{G}_\delta$. Then $\mathcal{G}$ is a filter on λ so there is an ultrafilter $\mathcal{H}$ on λ including $\mathcal{G}$. Now for any $\delta \in \Delta$, $\mathcal{H} \supseteq \mathcal{G}_\delta$, so

$$\mathrm{cf}(X/\mathcal{H}) = \mathrm{add}(X/\mathcal{H}) \ge \mathrm{add}(X/\mathcal{G}_\delta) \ge \delta,$$

using (a) above. Consequently $\delta^* = \mathrm{cf}(X/\mathcal{H})$ is the greatest element of Δ.

(ii) For each $\delta \in \Delta$ choose a set $F_\delta \in [X]^{\le\delta}$ such that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $X/\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}_\delta$ (using (b) above). Set $F = \bigcup_{\delta\in\Delta} F_\delta$ and

$$G = \{\sup I : I \in [F]^{<\omega}\} \subseteq X.$$

Then $\#(G) \le \delta^*$. I claim that G is cofinal with X. **P?** Suppose, if possible, otherwise; take $h \in X$ such that $h \not\le g$ for every $g \in G$. Write

$$A_g = \{\zeta : h(\zeta) > g(\zeta)\}$$

for each $g \in G$. Because G is upwards-directed, $\{A_g : g \in G\}$ is a filter base, and there is an ultrafilter $\mathcal{F}$ on λ containing every A_g. Now there is a $\delta \in \Delta$ such that $\mathcal{F} \in \mathfrak{F}_\delta$, so that $\pi_{\mathcal{F}}[F_\delta]$ is cofinal with $X/\mathcal{F}$, and there is an $f \in F_\delta$ such that $h \le_{\mathcal{F}} f$. But in this case $A = \{\zeta : h(\zeta) \le f(\zeta)\}$ and $A_f = \lambda \setminus A$ both belong to $\mathcal{F}$. **XQ**

(iii) Accordingly $\mathrm{cf}(X) \le \#(G) \le \delta^*$. But also of course $\mathrm{cf}(X/\mathcal{H}) \le \mathrm{cf}(X)$, so $\delta^* \le \mathrm{cf}(X)$ and they are equal. Now we have $\mathcal{H} \in \mathfrak{F}_{\delta^*} = \mathfrak{F}_{\mathrm{cf}(X)}$.

(d) If $\kappa = \lambda^+$ we may take $\theta'_\zeta = \lambda^+$ for every ζ; if $\kappa = \mathrm{cf}(X/\mathcal{F})$ we may take $\theta'_\zeta = \theta_\zeta$; so let us assume that $\lambda^+ < \kappa < \mathrm{cf}(X/\mathcal{F})$. Of course $M \in \mathcal{F}$.

(i) For each ordinal $\gamma < \kappa$ choose a relatively closed cofinal set $C_\gamma \subseteq \gamma$ with $\mathrm{otp}(C_\gamma) = \mathrm{cf}(\gamma)$. Choose $\langle f_\alpha \rangle_{\alpha<\kappa}$ as follows. Given $\langle f_\beta \rangle_{\beta<\alpha}$, where $\alpha < \kappa$, and $\gamma < \kappa$, define $g_{\alpha\gamma} \in X$ by

$$\begin{aligned} g_{\alpha\gamma}(\zeta) &= \sup\{f_\beta(\zeta) : \beta \in C_\gamma \cap \alpha\} + 1 \text{ if this is less than } \theta_\zeta, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Now choose $f_\alpha \in X$ such that

$$f_\beta \leq_{\mathcal{F}} f_\alpha \ \forall \ \beta < \alpha, \, g_{\alpha\gamma} \leq_{\mathcal{F}} f_\alpha \ \forall \ \gamma < \kappa;$$

this is possible because $\kappa < \mathrm{cf}(X/\mathcal{F})$. Observe that if $\alpha = \beta + 1$ then $C_\alpha = \{\beta\}$ so that $g_{\alpha\alpha} = f_\beta + 1$ and $f_\alpha \not\leq_{\mathcal{F}} f_\beta$. Continue.

(ii) Suppose that for each $\zeta < \lambda$ we are given a set $S_\zeta \subseteq \theta_\zeta$ with $\#(S_\zeta) \leq \lambda$. Then there is an $\alpha < \kappa$ such that

$$\forall \ h \in \prod_{\zeta<\lambda} S_\zeta, \text{ if } f_\alpha \leq_{\mathcal{F}} h \text{ then } f_\beta \leq_{\mathcal{F}} h \ \forall \ \beta < \kappa.$$

P? If not, then (because κ is regular) we can find a family $\langle h_\xi \rangle_{\xi<\kappa}$ in $\prod_{\zeta<\lambda} S_\zeta$ and a strictly increasing family $\langle \phi(\xi) \rangle_{\xi<\kappa}$ in κ such that

$$f_{\phi(\xi)} \leq_{\mathcal{F}} h_\xi \leq_{\mathcal{F}} f_{\phi(\xi+1)} \text{ for all } \xi < \kappa,$$

$$\phi(\xi) = \sup_{\eta<\xi} \phi(\eta) \text{ for limit ordinals } \xi < \kappa.$$

Set

$$C = \{\xi : \xi < \kappa, \, \phi(\xi) = \xi\},$$

so that C is a closed unbounded set in κ. Let $\alpha \in C$ be such that $\alpha = \sup(C \cap \alpha)$ and $\mathrm{cf}(\alpha) = \lambda^+$. Then (because $\lambda^+ \geq \omega_1$) $C_\alpha \cap C$ is cofinal with α.

For $\beta \in C \cap C_\alpha$, $\zeta < \lambda$ we have $\#(C_\alpha \cap \beta) \leq \mathrm{otp}(C_\alpha \cap \beta) < \mathrm{otp}(C_\alpha) = \lambda^+ \leq \theta_\zeta$, so

$$\theta_\zeta > \sup_{\xi \in C_\alpha \cap \beta} f_\xi(\zeta) + 1 = g_{\beta\alpha}(\zeta).$$

Now

$$g_{\beta\alpha} \leq_{\mathcal{F}} f_\beta = f_{\phi(\beta)} \leq_{\mathcal{F}} h_\beta \leq_{\mathcal{F}} f_{\phi(\beta+1)} \leq_{\mathcal{F}} f_{\beta'},$$

where β' is the next member of $C \cap C_\alpha$ greater than β. So there is a $\zeta_\beta < \lambda$ such that

$$g_{\beta\alpha}(\zeta_\beta) \leq h_\beta(\zeta_\beta) \leq f_{\beta'}(\zeta_\beta).$$

Because $\lambda < \mathrm{cf}(\alpha)$ there is a $\zeta < \lambda$ such that

$$B = \{\beta : \beta \in C \cap C_\alpha, \, \zeta_\beta = \zeta\}$$

is cofinal with α. But now observe that if $\beta, \gamma \in B$ and $\beta' < \gamma$ then $\beta' \in C_\alpha \cap \gamma$ so

$$h_\beta(\zeta) \leq f_{\beta'}(\zeta) < g_{\gamma\alpha}(\zeta) \leq h_\gamma(\zeta).$$

It follows that

$$\lambda^+ = \#(B) = \#(\{h_\beta(\zeta) : \beta \in B\}) \leq \#(S_\zeta) \leq \lambda,$$

which is absurd. **XQ**

(iii) Consequently $E = \{\pi_{\mathcal{F}}(f_\alpha) : \alpha < \kappa\}$ has a least upper bound in $X/\mathcal{F}$. **P?** If not, choose $\langle h_\xi \rangle_{\xi<\lambda^+}$ inductively, as follows. Because $\kappa < \delta$, there is an $h_0 \in X$ such that $f_\alpha \leq_{\mathcal{F}} h_0$ for every $\alpha < \kappa$. Given h_ξ such that $h^\bullet_\xi = \pi_{\mathcal{F}}(h_\xi)$ is an upper bound for E, then $h^\bullet_\xi$ cannot be the least upper bound of E, so there is an $h_{\xi+1} \in X$ such that $h^\bullet_{\xi+1}$ is an upper bound of E strictly less than $h^\bullet_\xi$. For non-zero limit ordinals $\xi < \lambda^+$, set

$$S_{\xi\zeta} = \{h_\eta(\zeta) : \eta < \xi\} \subseteq \theta_{\ zeta}$$

for each $\zeta < \lambda$. By (ii) above, there is an $\alpha_\xi < \kappa$ such that

$$\forall \ h \in \prod_{\zeta<\lambda} S_{\xi\zeta} \text{ either } h \leq_{\mathcal{F}} f_{\alpha_\xi} \text{ or } f_\alpha \leq_{\mathcal{F}} h \ \forall \ \alpha < \kappa.$$

Set

$$h_\xi(\zeta)=\min(\{\eta:\eta\in S_{\xi\zeta},\, f_{\alpha_\xi}(\zeta)\le\eta\}\cup\{h_0(\zeta)\})\in S_{\xi\zeta}$$

for each $\zeta<\lambda$. Then $f_{\alpha_\xi}\le_{\mathcal{F}} h_\xi$ (because $f_{\alpha_\xi}(\zeta)\le h_\xi(\zeta)$ whenever $f_{\alpha_\xi}(\zeta)\le h_0(\zeta)$) and $h_\xi\in\prod_{\zeta<\lambda}S_{\xi\zeta}$, so $f_\alpha\le_{\mathcal{F}} h_\xi$ for every $\alpha<\kappa$ and $h^\bullet_\xi$ is an upper bound for E. Also, if $\eta<\xi$, then $h_\xi(\zeta)\le h_\eta(\zeta)$ whenever $f_{\alpha_\xi}(\zeta)\le h_\eta(\zeta)$, so $h_\xi\le_{\mathcal{F}} h_\eta$. Continue.

Having got the family $\langle h_\xi\rangle_{\xi<\lambda^+}$, set

$$S_\zeta=\bigcup_{\xi<\kappa}S_{\xi\zeta}=\{h_\xi(\zeta):\xi<\kappa\}\subseteq\theta_\zeta$$

for each $\zeta<\lambda$. For each $\alpha<\kappa$, $\zeta<\lambda$ set

$$g_\alpha(\zeta)=\min(\{\eta:f_\alpha(\zeta)\le\eta\in S_\zeta\}\cup\{h_0(\zeta)\})\in S_\zeta.$$

Then, by the same arguments as above,

$$f_\alpha\le_{\mathcal{F}} g_\alpha\le_{\mathcal{F}} h_\xi\ \ \forall\ \alpha<\kappa,\,\xi<\lambda^+.$$

For each $\alpha<\kappa$ there is a limit ordinal $\xi<\lambda^+$ such that $g_\alpha(\zeta)\in S_{\xi\zeta}$ for every $\zeta<\lambda$. Because $\lambda^+<\kappa$ there is a limit ordinal $\xi<\lambda^+$ such that

$$A=\{\alpha:g_\alpha(\zeta)\in S_{\xi\zeta}\ \ \forall\ \zeta<\lambda\}$$

is cofinal with κ. In particular, there is an $\alpha\in A$ such that $\alpha\ge\alpha_\xi$. In this case

$$f_{\alpha_\xi}\le_{\mathcal{F}} f_\alpha\le_{\mathcal{F}} g_\alpha\le_{\mathcal{F}} h_{\xi+1}\le_{\mathcal{F}} h_\xi\not\le_{\mathcal{F}} h_{\xi+1},$$

so there is a $\zeta<\lambda$ such that

$$f_{\alpha_\xi}(\zeta)\le f_\alpha(\zeta)\le g_\alpha(\zeta)\le h_{\xi+1}(\zeta)<h_\xi(\zeta).$$

But now observe that

$$f_{\alpha_\xi}(\zeta)\le g_\alpha(\zeta)\in S_{\xi\zeta}$$

so $h_\xi(\zeta)\le g_\alpha(\zeta)<h_\xi(\zeta)$, which is absurd. **X**

(iv) Let $g\in X$ be such that $g^\bullet=\sup E$ in $X/\mathcal{F}$ and $g(\zeta)>0$ for every $\zeta<\lambda$. For each $\zeta<\lambda$ set $\hat\theta_\zeta=\mathrm{cf}(g(\zeta))<\theta_\zeta$ and choose a cofinal set $D_\zeta\subseteq g(\zeta)$ of order type $\hat\theta_\zeta$. For $\alpha<\kappa$, $\zeta<\lambda$ set

$$g_\alpha(\zeta)=\min\{\eta:f_\alpha(\zeta)\le\eta\in D_\zeta\}$$

if $f_\alpha(\zeta)<g(\zeta)$, $\min D_\zeta$ otherwise. Then $g_\alpha\le_{\mathcal{F}} g_\beta$ whenever $\alpha\le\beta<\kappa$. Also if $h\in Y=\prod_{\zeta<\lambda}D_\zeta$ then $h^\bullet<g^\bullet$ so there is an $\alpha<\kappa$ such that $h^\bullet\le f^\bullet_\alpha\le g^\bullet_\alpha$. Thus $\{g^\bullet_\alpha:\alpha<\kappa\}$ is cofinal with $\{h^\bullet:h\in Y\}$.

(v) Because each D_ζ is order-isomorphic to $\hat\theta_\zeta$, we can identify Y with $\hat X=\prod_{\zeta<\lambda}\hat\theta_\zeta$, and see that $\mathrm{cf}(\hat X/\mathcal{F})$ is either 1 or κ. But of course the former is absurd, because it could be so only if $\{\zeta: g(\zeta)$ is a successor ordinal$\}\in\mathcal{F}$, and in this case there would have to be an $\alpha<\kappa$ such that $g\le_{\mathcal{F}} f_\alpha$; but we saw in (i) above that $f_{\alpha+1}\not\le_{\mathcal{F}} f_\alpha$.

Accordingly $\mathrm{cf}(\hat X/\mathcal{F})=\kappa$.

(vi) It may be that some of the $\hat\theta_\zeta$ are less than or equal to λ. But taking $I=\{\zeta:\hat\theta_\zeta\le\lambda\}$, we have $I\notin\mathcal{F}$. **P?** If $I\in\mathcal{F}$, then for each $\zeta\in I$ set $S_\zeta=D_\zeta$ and for $\zeta\in\lambda\setminus I$ set $S_\zeta=\{0\}$. By (ii), there is an $\alpha<\kappa$ such that

$$\forall\ h\in\prod_{\zeta<\lambda}S_\zeta,\text{ if } f_\alpha\le_{\mathcal{F}} h\text{ then } f_\beta\le_{\mathcal{F}} h\ \ \forall\ \beta<\kappa.$$

But as $f_{\alpha+1}\le_{\mathcal{F}} g$, and $I\in\mathcal{F}$, there must be an $h\in\prod_{\zeta<\lambda}S_\zeta$ such that $f_\alpha\le_{\mathcal{F}} h$, and now $g\le_{\mathcal{F}} h$ because $g^\bullet$ is the least upper bound of E; but $h(\zeta)<g(\zeta)$ for every $\zeta\in I$, so this is impossible. **XQ**

So $\{\zeta:\hat\theta_\zeta\ge\lambda^+\}\in\mathcal{F}$. But this means that if we set $\theta'_\zeta=\max(\lambda^+,\hat\theta_\zeta)$ for every $\zeta<\lambda$, $X'=\prod_{\zeta<\lambda}\theta'_\zeta$ then $X'/\mathcal{F}\cong\hat X/\mathcal{F}$ so $\mathrm{cf}(X'/\mathcal{F})=\kappa$, as required.

Version of 10.12.91

A1J Shelah covering numbers (a) Let α, β, γ, δ be cardinals. Following Shelah #355 and Shelah #400b, I write

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha,\beta,\gamma,\delta)$$

for the least cardinal of any family $\mathcal{E} \subseteq [\alpha]^{<\beta}$ such that for every $A \in [\alpha]^{<\gamma}$ there is a $\mathcal{D} \in [\mathcal{E}]^{<\delta}$ with $A \subseteq \bigcup \mathcal{D}$. I diverge insignificantly from the Master in writing $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta) = \infty$ in the trivial cases in which there is no such family $\mathcal{E}$.

(b) For infinite cardinals α, γ write $\Theta(\alpha, \gamma)$ for the maximum of α and the supremum of all cofinalities

$$\mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta)$$

for families $\langle \theta_\zeta \rangle_{\zeta<\lambda}$ such that $\lambda < \gamma$ is a cardinal, every θ_ζ is a regular infinite cardinal, and $\lambda < \theta_\zeta < \alpha$ for every $\zeta < \lambda$. (This carries some of the same information as the cardinal $\mathrm{pp}_\kappa(\alpha)$ of SHELAH #400B.)

A1K Theorem For any infinite cardinals α, γ,

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) \leq \Theta(\alpha, \gamma).$$

proof (a) To begin with (down to the end of (f) below) let us suppose that we have $\alpha \geq \gamma = \gamma_0^+ > \mathrm{cf}(\alpha) > \omega$.

Take a family $\mathcal{E} \subseteq [\alpha]^{\leq \gamma_0}$ such that

(i) $\mathcal{E}$ contains all singleton subsets of α;

(ii) $\mathcal{E}$ contains a cofinal subset of α;

(iii) If $E \in \mathcal{E}$ then $\{\xi : \xi + 1 \in E\} \in \mathcal{E}$;

(iv) if $E \in \mathcal{E}$ then there is an $F \in \mathcal{E}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \leq \mathrm{cf}(\xi) \leq \gamma_0$;

(v) if $E \in \mathcal{E}$ then $\{\xi : \xi \in E,\ \mathrm{cf}(\xi) \geq \gamma\} \in \mathcal{E}$;

(vi) if $E \in \mathcal{E}$ and $\mathrm{cf}(\prod_{\eta \in E} \eta) \leq \Theta(\alpha, \gamma)$, then $\{g : g \in \prod_{\eta \in E} \eta,\ g[E] \in \mathcal{E}\}$ is cofinal with $\prod_{\eta \in E} \eta$;

(vii) $\#(\mathcal{E}) \leq \Theta(\alpha, \gamma)$.

To see that this can be done, observe that whenever $E \in [\alpha]^{\leq \gamma_0}$ there is an $F \in [\alpha]^{\leq \gamma_0}$ such that $\sup(F \cap \xi) = \xi$ whenever $\xi \in E$ and $\omega \leq \mathrm{cf}(\xi) \leq \gamma_0$; thus condition (iv) can be achieved, like conditions (iii) and (v), by ensuring that $\mathcal{E}$ is closed under suitable functions from $[\alpha]^{\leq \gamma_0}$ to itself; while condition (vi) requires that for each $E \in \mathcal{E}$ we have an appropriate family of size at most $\Theta(\alpha, \gamma)$ included in $\mathcal{E}$.

Write $\mathcal{J}$ for the σ-ideal of $\mathcal{P}\alpha$ generated by $\mathcal{E}$.

(b) ? Suppose now that there is some set in $[\alpha]^{<\gamma}$ not covered by any sequence from $\mathcal{E}$, that is, not belonging to $\mathcal{J}$. Then there must be a function $f : \gamma_0 \to \alpha$ such that $f[\gamma_0] \notin \mathcal{J}$. Accordingly $\mathcal{I} = \{f^{-1}[E] : E \in \mathcal{J}\}$ is a proper σ-ideal of $\mathcal{P}\gamma_0$. By condition (a-i), $\mathcal{I}$ contains all singletons in $\mathcal{P}\gamma_0$.

Let H be the set of all functions $h : \gamma_0 \to \alpha$ such that $f(\xi) \leq h(\xi)$ for every $\xi < \gamma_0$ and $h[\gamma_0] \in \mathcal{J}$. Because $\mathcal{E}$ contains a cofinal set $C \subseteq \alpha$ (condition (a-ii)), we can find an $h \in H$; just take $h : \gamma_0 \to C$ such that $f(\xi) \leq h(\xi)$ for every ξ.

(c) Because $\mathcal{I}$ is a proper σ-ideal, there cannot be any sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in H such that $\{\xi : h_{n+1}(\xi) \geq h_n(\xi)\} \in \mathcal{I}$ for every $n \in \mathbb{N}$. Consequently there is an $h^* \in H$ such that

$$\{\xi : h(\xi) \geq h^*(\xi)\} \notin \mathcal{I} \ \ \forall \ h \in H.$$

We know that $h^*[\gamma_0] \in \mathcal{J}$; let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{E}$ covering $h^*[\gamma_0]$. For $\xi < \gamma_0$ write $\theta_\xi = \mathrm{cf}(h^*(\xi))$, so that each θ_ξ is 0 or 1 or a regular infinite cardinal less than α. Set

$$I = \{\xi : \xi < \gamma_0,\ h^*(\xi) = f(\xi)\},$$

$$I' = \{\xi : \xi < \gamma_0,\ f(\xi) < h^*(\xi),\ \theta_\xi = 1\}\ ,$$

$$I_n = \{\xi : \xi < \gamma_0,\ \omega \leq \theta_\xi \leq \gamma,\ f(\xi) < h^*(\xi),\ h^*(\xi) \in E_n\} \ \ \forall \ n \in \mathbb{N},$$

$$J_n = \{\xi : \xi < \gamma_0,\ \gamma \leq \theta_\xi,\ f(\xi) < h^*(\xi),\ h^*(\xi) \in E_n\} \ \ \forall \ n \in \mathbb{N}.$$

(d) For each $n \in \mathbb{N}$ set $G_n = \{\eta : \eta \in E_n,\ \mathrm{cf}(\eta) \geq \gamma\} \in \mathcal{E}$. Then $\mathrm{cf}(\prod_{\eta \in G_n} \eta) \leq \Theta(\alpha, \gamma)$. **P** For $\eta \in G_n$ set $\theta'_\eta = \mathrm{cf}(\eta)$; then θ'_η is a regular cardinal and $\#(G_n) \leq \gamma_0 < \theta'_\eta < \alpha$ for each $\eta \in G_n$. So

$$\mathrm{cf}(\prod_{\eta \in G_n} \theta'_\eta) \leq \Theta(\alpha, \gamma)$$

by the definition of $\Theta(\alpha, \gamma)$. If for each $\eta \in G_n$ we choose a cofinal set $C_\eta \subseteq \eta$ of order type θ'_η, then

$$\mathrm{cf}(\prod_{\eta \in G_n} \eta) = \mathrm{cf}(\prod_{\eta \in G_n} C_\eta) = \mathrm{cf}(\prod_{\eta \in G_n} \theta'_\eta) \leq \Theta(\alpha, \gamma). \ \mathbf{Q}$$

Consequently, by (a-vi),

$$\{g : g \in \prod_{\eta\in G_n} \eta,\, g[G_n] \in \mathcal{E}\}$$

is cofinal with $\prod_{\eta\in G_n} \eta$.

(e) Define $h : \gamma_0 \to \alpha$ as follows.

(i) If $\xi \in I$ set $h(\xi) = h^*(\xi)$.

(ii) If $\xi \in I'$ set $h(\xi) = h^*(\xi) - 1$.

(iii) For each $n \in \mathbb{N}$ take $F_n \in \mathcal{E}$ such that $\eta = \sup(F_n \cap \eta)$ whenever $\eta \in E_n$ and $\omega \le \mathrm{cf}(\eta) < \gamma$. If $\xi \in I_n \setminus \bigcup_{m<n} I_m$, take $h(\xi) \in F_n$ such that $f(\xi) < h(\xi) < h^*(\xi)$.

(iv) For each $n \in \mathbb{N}$, $\eta \in G_n$ set

$$f^*(\eta) = \sup\{f(\xi) : \xi < \gamma_0,\, h^*(\xi) = \eta\}.$$

Then $f^*(\eta) < \eta$, because $\gamma_0 < \mathrm{cf}(\eta)$. By (d), there is a $g_n \in \prod_{\eta\in G_n} \eta$ such that $g_n[G_n] \in \mathcal{E}$ and $f^*(\eta) < g_n(\eta)$ for every $\eta \in G_n$. So for $\xi \in J_n \setminus \bigcup_{m<n} J_m$ we may set $h(\xi) = g_n(h^*(\xi))$ and see that $f(\xi) < h(\xi) < h^*(\xi)$, while $h(\xi) \in g_n[G_n]$.

(f) Now we see that

$$h[\gamma_0] \subseteq h^*[\gamma_0] \cup \{\eta : \eta + 1 \in h^*[\gamma_0]\} \cup \bigcup_{n\in\mathbb{N}} F_n \cup \bigcup_{n\in\mathbb{N}} g_n[G_n] \in \mathcal{J},$$

while $f(\xi) \le h(\xi)$ for every $\xi < \gamma_0$, so $h \in H$. Consequently

$$I = \{\xi : h(\xi) \ge h^*(\xi)\} \notin \mathcal{I}.$$

But also

$$f[I] \subseteq h^*[\gamma_0] \in \mathcal{J},$$

so $I \in \mathcal{I}$, which is absurd. **X**

(g) Thus the special case described in (a) is dealt with, and we may return to the general case. I proceed by induction on α for fixed $\gamma \ge \omega$.

(i) To start the induction, observe that if $\alpha < \gamma$ then

$$\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) = 1 \le \Theta(\alpha, \gamma)$$

for all γ.

(ii) For the inductive step to α when $\mathrm{cf}(\alpha) \ge \gamma$, observe that in this case $[\alpha]^{<\gamma} = \bigcup_{\alpha'<\alpha} [\alpha']^{<\gamma}$. For each $\alpha' < \alpha$, we can find a family $\mathcal{E}_{\alpha'} \subseteq [\alpha']^{<\gamma}$ such that $\#(\mathcal{E}_{\alpha'}) = \mathrm{cov}_{\mathrm{Sh}}(\alpha', \gamma, \gamma, \omega_1)$ and every member of $[\alpha']^{<\gamma}$ can be covered by a sequence from $\mathcal{E}_{\alpha'}$; now $\bigcup_{\alpha'<\alpha} \mathcal{E}_{\alpha'}$ witnesses that

$$\begin{aligned}\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) &\le \max(\alpha, \sup_{\alpha'<\alpha} \mathrm{cov}_{\mathrm{Sh}}(\alpha', \gamma, \gamma, \omega_1)) \\ &\le \max(\alpha, \sup_{\alpha'<\alpha} \Theta(\alpha', \gamma)) \le \Theta(\alpha, \gamma)\end{aligned}$$

by the inductive hypothesis.

(iii) For the inductive step to α when $\mathrm{cf}(\alpha) = \omega < \alpha$, take an increasing sequence $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ in α such that $\sup_{n\in\mathbb{N}} \alpha_n = \alpha$. For each $n \in \mathbb{N}$ choose a set $\mathcal{E}_n \subseteq [\alpha_n]^{<\gamma}$ such that every set in $[\alpha_n]^{<\gamma}$ can be covered by a sequence from $\mathcal{E}_n$ and $\#(\mathcal{E}_n) = \mathrm{cov}_{\mathrm{Sh}}(\alpha_n, \gamma, \gamma, \omega_1)$. Set $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n \subseteq [\alpha]^{<\gamma}$. Then if $A \in [\alpha]^{<\gamma}$, we can find for each $n \in \mathbb{N}$ a countable set $\mathcal{D}_n \subseteq \mathcal{E}_n$ covering $A \cap \alpha_n$; set $\mathcal{D} = \bigcup_{n\in\mathbb{N}} \mathcal{D}_n \in [\mathcal{E}]^{\le\omega}$; then $A \subseteq \bigcup \mathcal{D}$. As A is arbitrary, $\mathcal{E}$ witnesses that

$$\begin{aligned}\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) &\le \#(\mathcal{E}) \\ &\le \max(\omega, \sup_{n\in\mathbb{N}} \#(\mathcal{E}_n)) \\ &= \max(\omega, \sup_{n\in\mathbb{N}} \mathrm{cov}_{\mathrm{Sh}}(\alpha_n, \gamma, \gamma, \omega_1)) \\ &\le \max(\omega, \sup_{n\in\mathbb{N}} \Theta(\alpha_n, \gamma))\end{aligned}$$

by the inductive hypothesis

$$\leq \Theta(\alpha, \gamma).$$

(iv) For the inductive step to α when $\omega < \mathrm{cf}(\alpha) < \gamma < \alpha$, observe that

$$[\alpha]^{<\gamma} = \bigcup_{\delta<\gamma} [\alpha]^{\leq\delta}.$$

For each $\delta < \gamma$ we have a set $\mathcal{E}_\delta \subseteq [\alpha]^{\leq\delta}$ such that $\#(\mathcal{E}_\delta) \leq \mathrm{cov}_{\mathrm{Sh}}(\alpha, \delta^+, \delta^+, \omega_1)$ and every member of $[\alpha]^{\leq\delta}$ can be covered by a sequence from $\mathcal{E}_\delta$. Set $\mathcal{E} = \bigcup_{\mathrm{cf}(\alpha)\leq\delta<\gamma} \mathcal{E}_\delta$; then every member of $[\alpha]^{<\gamma}$ can be covered by a sequence from $\mathcal{E}$.

For each $E \in \mathcal{E}$, choose a family $\mathcal{H}_E \subseteq [E]^{<\gamma}$ such that $\#(\mathcal{H}_E)= \mathrm{cov}_{\mathrm{Sh}}(\#(E), \gamma, \gamma, \omega_1)$ and every member of $[E]^{<\gamma}$ can be covered by a sequence from $\mathcal{H}_E$. Set $\mathcal{H} = \bigcup_{E\in\mathcal{E}} \mathcal{H}_E$. If $A \in [\alpha]^{<\gamma}$, there is a countable set $\mathcal{D} \subseteq \mathcal{E}$ covering A; now for each $D \in \mathcal{D}$ there is a countable set $\mathcal{G}_D \subseteq \mathcal{H}_D$ covering $A \cap D$; so that $\mathcal{G} = \bigcup_{D\in\mathcal{D}} \mathcal{G}_D$ is a countable subset of $\mathcal{H}$ covering A.

Thus

$$\begin{aligned}
\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma, \gamma, \omega_1) &\leq \#(\mathcal{H}) \\
&\leq \max(\#(\mathcal{E}), \sup_{E\in\mathcal{E}} \#(\mathcal{H}_E)) \\
&\leq \max(\alpha, \sup_{\mathrm{cf}(\alpha)\leq\delta<\gamma} \mathrm{cov}_{\mathrm{Sh}}(\alpha, \delta^+, \delta^+, \omega_1), \sup_{\alpha'<\alpha} \mathrm{cov}_{\mathrm{Sh}}(\alpha', \gamma, \gamma, \omega_1)) \\
&\leq \max(\alpha, \sup_{\mathrm{cf}(\alpha)\leq\delta<\gamma} \Theta(\alpha, \delta^+), \sup_{\alpha'<\alpha} \Theta(\alpha', \gamma))
\end{aligned}$$

by the inductive hypothesis and parts (a)-(f) above

$$\leq \Theta(\alpha, \gamma).$$

This completes the proof.

Remark This is taken from Shelah #355, Theorem 5.4, where a stronger result is proved, giving an exact description of many of the numbers $\mathrm{cov}_{\mathrm{Sh}}(\alpha, \beta, \gamma, \delta)$ in terms of cofinalities of reduced products $\prod_{\zeta<\lambda} \theta_\zeta/\mathcal{F}$.

A1L Lemma Let α and γ be infinite cardinals, with γ regular, and suppose that $\alpha \geq \Theta(\gamma, \gamma)$. Then $\Theta(\Theta(\alpha, \gamma), \gamma) = \Theta(\alpha, \gamma)$.

proof ? Suppose, if possible, otherwise. Note that of course $\alpha \geq \gamma$.

(a) There must be a family $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ of infinite regular cardinals such that $\lambda < \gamma$, $\lambda < \theta_\zeta < \Theta(\alpha, \gamma)$ for every $\zeta < \lambda$, and $\mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta) > \Theta(\alpha, \gamma)$. By A1Ic, there is an ultrafilter $\mathcal{F}$ on λ such that $\mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta/\mathcal{F}) > \Theta(\alpha, \gamma)$. Set $L = \{\zeta : \zeta < \lambda, \theta_\zeta < \alpha\}$; then $\mathrm{cf}(\prod_{\zeta\in L} \theta_\zeta) \leq \Theta(\alpha, \gamma)$, so $L \notin \mathcal{F}$ and $M = \lambda \setminus L \in \mathcal{F}$. For each $\zeta \in M$, we have $\theta_\zeta < \Theta(\alpha, \gamma)$, so there must be a family $\langle\theta_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular cardinals with $\lambda_\zeta < \gamma$, $\lambda_\zeta < \theta_{\zeta\eta} < \alpha$ for every $\eta < \lambda_\zeta$ and $\theta_\zeta \leq \mathrm{cf}(\prod_{\eta<\lambda_\zeta} \theta_{\zeta\eta})$. Again by A1Ic, there is an ultrafilter $\mathcal{F}_\zeta$ on λ_ζ such that $\theta_\zeta \leq \mathrm{cf}(\prod_{\eta<\lambda_\zeta} \theta_{\zeta\eta}/\mathcal{F}_\zeta)$. Because $\lambda_\zeta < \gamma \leq \alpha \leq \theta_\zeta$, A1Id tells us that there is a family $\langle\theta'_{\zeta\eta}\rangle_{\eta<\lambda_\zeta}$ of regular cardinals such that $\lambda_\zeta < \theta'_{\zeta\eta} \leq \theta_{\zeta\eta}$ for every η and $\theta_\zeta = \mathrm{cf}(\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}/\mathcal{F}_\zeta)$.

(b) Set

$$I = \{(\zeta, \eta) : \zeta \in M, \eta < \lambda_\zeta\},$$

$$\mathcal{H} = \{H : H \subseteq I, \{\zeta : \{\eta : (\zeta, \eta) \in H\} \in \mathcal{F}_\zeta\} \in \mathcal{F}\},$$

$$X = \prod_{(\zeta,\eta)\in I} \theta'_{\zeta\eta}.$$

Then $\mathcal{H}$ is an ultrafilter on I, and $\mathrm{cf}(X/\mathcal{H}) \geq \mathrm{cf}(\prod_{\zeta\in M} \theta_\zeta/\mathcal{F})$. **P** Let $F \subseteq X$ be a set of cardinal $\mathrm{cf}(X/\mathcal{H})$ such that $\{f^\bullet : f \in F\}$ is cofinal with $X/\mathcal{H}$. For each $f \in X$, $\zeta \in M$ let $f_\zeta \in \prod_{\eta<\lambda_\zeta} \theta'_\zeta$ be given by $f_\zeta(\eta) = f(\zeta, \eta)$ for each $\eta < \lambda_\zeta$, and let $f^\bullet_\zeta$ be the image of f_ζ in $\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}/\mathcal{F}_\zeta$. For each $\zeta \in M$ let $\langle u_{\zeta\xi}\rangle_{\xi<\theta_\zeta}$ be a strictly increasing cofinal family in $\prod_{\eta<\lambda_\zeta} \theta'_{\zeta\eta}/\mathcal{F}_\zeta$. Now, for $f \in F$, take a function $g_f \in \prod_{\zeta\in M} \theta_\zeta$ such that $f^\bullet_\zeta \leq u_{\zeta,g_f(\zeta)}$ for every $\zeta \in M$.

If $g \in \prod_{\zeta\in M} \theta_\zeta$, then we can find an $h \in X$ such that $h^\bullet_\zeta = u_{\zeta,g(\zeta)}$ for each $\zeta \in M$. Let $f \in F$ be such that $h \leq_{\mathcal{H}} f$. Then $\{\zeta : g(\zeta) \leq g_f(\zeta)\} \supseteq \{\zeta : h^\bullet_\zeta \leq f^\bullet_\zeta\} \in \mathcal{F}$, so $g \leq_{\mathcal{F}} g_f$. Accordingly $\{g_f : f \in F\}$ is cofinal with $\prod_{\zeta\in M} \theta_\zeta/\mathcal{F}$ and $\mathrm{cf}(\prod_{\zeta\in M} \theta_\zeta/\mathcal{F}) \leq \#(F) = \mathrm{cf}(X/\mathcal{H})$, as claimed. **Q**

(c) Thus $\mathrm{cf}(X/\mathcal{H}) > \Theta(\alpha, \gamma)$. Set $J = \{(\zeta, \eta) : (\zeta, \eta) \in I,\ \theta'_{\zeta\eta} \geq \gamma\}$. Because γ is regular, $\#(J) \leq \#(I) < \gamma$, so $\mathrm{cf}(\prod_{(\zeta,\eta)\in J} \theta'_{\zeta\eta}) \leq \Theta(\alpha, \gamma)$, and $J \notin \mathcal{H}$. It follows that $K = I \setminus J \in \mathcal{H}$. Set $M' = \{\zeta : \zeta \in M, \{\eta : (\zeta, \eta) \in K\} \in \mathcal{F}_\zeta\} \in \mathcal{F}$. Then $\theta_\zeta \leq \Theta(\gamma, \gamma) \leq \alpha$ for $\zeta \in M'$. So in fact $\theta_\zeta = \alpha$ for $\zeta \in M'$ and we have

$$\alpha \leq \Theta(\alpha, \gamma) < \mathrm{cf}(\textstyle\prod_{\zeta\in M'} \theta_\zeta) = \mathrm{cf}(\prod_{\zeta<\delta} \alpha),$$

where $\delta = \#(M')$, while at the same time α is regular.

But if α is regular and $\delta < \alpha$,

$$\mathrm{cf}(\textstyle\prod_{\zeta<\delta} \alpha) = \alpha,$$

contradicting the last sentence. **X**

This contradiction completes the proof.

A1M Lemma Let α and γ be infinite cardinals. Set $\delta = \sup_{\alpha'<\alpha} \Theta(\alpha', \gamma)$.

(a) If $\mathrm{cf}(\alpha) \geq \gamma$ then $\Theta(\alpha, \gamma) = \max(\alpha, \delta)$.

(b) If $\mathrm{cf}(\alpha) < \gamma$ then $\Theta(\alpha, \gamma) \leq \max(\alpha, \delta^{\mathrm{cf}(\alpha)})$, where $\delta^{\mathrm{cf}(\alpha)}$ is the cardinal power.

proof Let $\langle\theta_\zeta\rangle_{\zeta<\lambda}$ be a family of regular cardinals with $\lambda < \theta_\zeta < \alpha$ for each ζ and $\lambda < \gamma$.

case 1 If $\alpha' = \sup_{\zeta<\lambda} \theta_\zeta < \alpha$, set

$$I = \{\zeta : \zeta < \lambda,\ \theta_\zeta < \alpha'\},$$

$$J = \{\zeta : \zeta < \lambda,\ \theta_\zeta = \alpha'\}.$$

Then $\mathrm{cf}(\prod_{\zeta\in I} \theta_\zeta) \leq \Theta(\alpha', \gamma) \leq \delta$, $\mathrm{cf}(\prod_{\zeta\in J} \theta_\zeta) \leq \alpha' \leq \delta$ so $\mathrm{cf}(\prod_{\zeta<\lambda} \theta_\zeta) \leq \delta$. This completes the proof of (a).

case 2 If $\sup_{\zeta<\lambda} \theta_\zeta = \alpha$ let $\langle\alpha_\xi\rangle_{\xi<\mathrm{cf}(\alpha)}$ be an increasing family of cardinals with supremum α and with $\alpha_\xi = \sup_{\eta<\xi} \alpha_\eta$ for limit ordinals $\xi < \mathrm{cf}(\alpha)$. Set

$$P_\xi = \textstyle\prod_{\zeta<\lambda, \alpha_\xi\leq\theta_\zeta<\alpha_{\xi+1}} \theta_\zeta$$

for each $\xi < \mathrm{cf}(\alpha)$. Then $\mathrm{cf}(P_\xi) \leq \delta$ for each $\xi < \mathrm{cf}(\alpha)$, so

$$\mathrm{cf}(\textstyle\prod_{\zeta<\lambda} \theta_\zeta) = \mathrm{cf}(\prod_{\xi<\mathrm{cf}(\alpha)} P_\xi) \leq \delta^{\mathrm{cf}(\alpha)}.$$

This is enough to deal with (b).

A1N The singular cardinals hypothesis (In this paragraph, all powers will be cardinal exponentiation; λ^θ will be the cardinal of the set of functions from θ to λ.) Recall that the **singular cardinals hypothesis** is the statement

whenever λ is an infinite cardinal and $2^{\mathrm{cf}(\lambda)} < \lambda$ then $\lambda^{\mathrm{cf}(\lambda)} = \lambda^+$

(Jech 78, §8, p. 61). This is equivalent to

whenever $\lambda > \mathfrak{c}$ and $\mathrm{cf}(\lambda) = \omega$ then $\lambda^\omega = \lambda^+$

(Jech 78, Theorem 23b, p.63); evidently the second statement is implied by

whenever $\lambda > \mathfrak{c}$ is a successor cardinal then $\lambda^\omega = \lambda$,

and by Jech 78, Lemma 8.1, p. 62, the third assertion is implied by the first, so that any of the three may be taken as a statement of the singular cardinals hypothesis.

A1O Miscellaneous definitions (a) If X is a set, a **Jónsson algebra** on X is an algebraic structure with countably many finitary functions and relations such that the only subalgebra of X of cardinal $\#(X)$ is X itself.

(b) If κ is a cardinal, a κ**-Aronszajn** tree is a tree of height κ in which every level has cardinal strictly less than κ but there are no branches of length κ. If κ is weakly Π^1_1-indescribable, there are no κ-Aronszajn trees (see Fremlin & Kunen n87, 2N).

A1P Souslin numbers I say that two elements p, q in a partially ordered set P are **upwards-compatible** if $\{p,q\}$ has an upper bound in P, and that a set $A \subseteq P$ is an **up-antichain** if no two elements of A are upwards-compatible.

For any partially ordered set P, write

$$\mathrm{S}(P) = \min\{\lambda : \text{ there is no up-antichain in } P \text{ of cardinal } \lambda\},$$

the **Souslin number** of P. Thus P satisfies the 'λ-chain condition' iff $\lambda \ge \mathrm{S}(P)$, and is upwards-ccc iff $\mathrm{S}(P) \le \omega_1$. Note that if Q is a cofinal subset of P, then $\mathrm{S}(Q) = \mathrm{S}(P)$.

Similarly, the **Souslin number** of a Boolean algebra $\mathfrak{A}$ is the least cardinal λ such that there is no disjoint set of cardinal λ in $\mathfrak{A} \setminus \{\mathbf{0}\}$; that is, the Souslin number of the partially ordered set $\mathfrak{A} \setminus \{\mathbf{1}\}$: it is called $\mathrm{sat}(\mathfrak{A})$ in Koppelberg 89, 3.8-3.11.

A1Q Lemma Let R be an upwards-ccc partially ordered set and $\langle r_\alpha\rangle_{\alpha<\omega_1}$ any family in R. Then there is an infinite $M \subseteq \omega_1$ such that $\{r_\alpha : \alpha \in M\}$ is centered, that is, $\{r_\alpha : \alpha \in L\}$ has an upper bound in R for every finite $L \subseteq M$.

proof ? Suppose, if possible, otherwise. For each $\alpha < \omega_1$ let M_α be a maximal subset of ω_1, containing α, such that $\{r_\beta : \beta \in M_\alpha\}$ is centered. Because each M_α is finite, there must be an uncountable set $W \subseteq \omega_1$ such that $\alpha \notin M_\beta$ whenever α, $\beta \in W$ and $\beta < \alpha$. Now, for each $\alpha \in W$, let s_α be an upper bound for $\{r_\beta : \beta \in M_\alpha\}$. In this case, for α, $\beta \in W$ and $\beta < \alpha$, $r_\alpha \le s_\alpha$ but r_α and s_β are upwards-incompatible in R (or we should have been able to add α to M_β). But this means that $\{s_\alpha : \alpha \in W\}$ is an uncountable up-antichain in R. **X**

A1R Finite-support products If $\langle P_\xi\rangle_{\xi\in I}$ is any family of partially ordered sets, then its **finite-support product** is the set P of finite functions p such that $\mathrm{dom}(p) \in [I]^{<\omega}$ and $p(\xi) \in P_\xi$ for every $\xi \in \mathrm{dom}(p)$; ordered by saying that $p \le q$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $p(\xi) \le q(\xi)$ for every $\xi \in \mathrm{dom}(p)$.

Now if $\langle P_\xi\rangle_{\xi\in I}$ is any family of partially ordered sets with finite-support product P, and

$$\lambda = \sup\{\mathrm{S}(\textstyle\prod_{\xi\in J} P_\xi) : J \in [I]^{<\omega}\},$$

then $\mathrm{S}(P) = \lambda$ if λ is regular, and λ^+ otherwise; see Comfort & Negrepontis 82, Theorem 3.27, where the corresponding theorem is proved for products of topological spaces.

A1S The arrow relation If α, β and γ are ordinals, write $\alpha \to (\beta,\gamma)^2$ to mean: if $S \subseteq [\alpha]^2$ is any set, then either there is a $B \subseteq \alpha$ such that $\mathrm{otp}(B) = \beta$ and $[B]^2 \subseteq S$, or there is a $C \subseteq \alpha$ such that $\mathrm{otp}(C) = \gamma$ and $[C]^2 \cap S = \emptyset$. (See Erdös Hajnal Máté & Rado 84, 8.2.) Then we have, among many other important results,

(i) $\alpha \to (\alpha, \omega+1)^2$ whenever α is a regular uncountable cardinal (Erdös Hajnal Máté & Rado 84, 11.3);

(ii) $\mathfrak{c} \not\to (\omega_1,\omega_1)^2$ (Erdös Hajnal Máté & Rado 84, 19.7);

(iii) if $\mathfrak{c} = \omega_1$ then $\mathfrak{c} \not\to (\mathfrak{c}, \omega+2)^2$ (Erdös Hajnal Máté & Rado 84, 11.5);

(iv) if κ is a regular uncountable cardinal then $\kappa \to (\kappa,\kappa)^2$ iff κ is weakly Π^1_1-indescribable and strongly inaccessible, that is, 'weakly compact' (Erdös Hajnal Máté & Rado 84, 30.3 and 32.1).

Version of 10.12.91

A2. Measure theory

My basic texts for this material are Fremlin 74 and Fremlin 89.

A2A General measure spaces In these notes a **measure space** will always be a triple (X,Σ,μ) where X is a set, Σ is a σ- algebra of subsets of X, and $\mu : \Sigma \to [0,\infty]$ is a countably- additive functional.

Let (X,Σ,μ) be a measure space.

(a) If $B \subseteq X$, I write

$$\mu^* B = \min\{\mu E : B \subseteq E \in \Sigma\}.$$

(b) If A is any subset of X, I write $\mu\lceil A$ for the measure on A defined by writing

$$\mathrm{dom}(\mu\lceil A) = \{A \cap E : E \in \Sigma\},$$

$$(\mu\lceil A)(B) = \mu^*(B) \ \forall \ B \in \mathrm{dom}(\mu\lceil A).$$

Now $\mu\lceil A$ is the **subspace measure** on A.

(c) An **atom** for μ is a set $E \in \Sigma$ such that $\mu E > 0$ and if $F \in \Sigma$, $F \subseteq E$ then one of μF, $\mu(E\setminus F)$ is zero.

(d) I write $\mathcal{N}_\mu$ for the σ-ideal of $\mathcal{P}X$ consisting of **μ-negligible** sets, that is,

$$\mathcal{N}_\mu = \{E : E \subseteq X,\ \mu^* E = 0\}.$$

(e) If $(Y, \mathfrak{S})$ is a topological space, a function $f : X \to Y$ is **measurable** if $f^{-1}[H] \in \Sigma$ for every $H \in \mathfrak{S}$.

A2B Taxonomy of measure spaces (see Fremlin 74). Let (X, Σ, μ) be a measure space.

(a) (X, Σ, μ) is a **probability space** if $\mu X = 1$ (and in this case I call μ a **probability**); it is **totally finite** if μX is finite; it is **σ-finite** if there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure covering X.

(b) (X, Σ, μ) is **semi-finite** if

$$\mu E = \sup\{\mu F : F \subseteq E,\ F \in \Sigma,\ \mu F < \infty\}$$

for every $E \in \Sigma$; that is, if whenever $E \in \Sigma$ and $\mu E = \infty$ then there is an $F \in \Sigma$ such that $F \subseteq E$ and $0 < \mu F < \infty$. Any σ-finite measure space – a fortiori, any probability space – is semi-finite.

(c) (X, Σ, μ) is **locally determined** if it is semi-finite and for every $A \in \mathcal{P}X \setminus \Sigma$ there is an $F \in \Sigma$ such that $\mu F < \infty$ and $F \cap A \notin \Sigma$.

(d) (X, Σ, μ) is **decomposable** (or 'strictly localizable') if there is a partition $\langle X_i\rangle_{i\in I}$ of X such that (i) $\Sigma = \{E : E \subseteq X,\ E \cap X_i \in \Sigma \ \forall \ i \in I\}$ (ii) $\mu E = \sum_{i\in I} \mu(E \cap X_i)$ for every $E \in \Sigma$ (iii) $\mu X_i < \infty$ for every $i \in I$. In this case (X, Σ, μ) is locally determined. (See Fremlin 74, 64G.)

(e) (X, Σ, μ) is **atomless** if there are no atoms for μ.

(f) (X, Σ, μ) is **complete** if $\mathcal{N}_\mu \subseteq \Sigma$. Any measure space of the form $(X, \mathcal{P}X, \mu)$ is surely complete.

(g) (X, Σ, μ) is **purely atomic** if every non-negligible measurable set includes an atom for μ; that is, there is no $E \in \Sigma \setminus \mathcal{N}_\mu$ such that $\mu\lceil E$ is atomless.

A2C Additivity Let (X, Σ, μ) be a measure space.

(a) If κ is a cardinal, then μ is **κ-additive** if $\mu(\bigcup_{\xi<\lambda} E_\xi)$ exists and is equal to $\sum_{\xi<\lambda} \mu E_\xi$ for every disjoint family $\langle E_\xi\rangle_{\xi<\lambda}$ in Σ indexed by a cardinal $\lambda < \kappa$.

(b) I write **add(μ)**, the **additivity** of μ, for the least cardinal κ, if there is one, such that μ is not κ-additive; if μ is κ-additive for every cardinal κ, I write $\mathrm{add}(\mu) = \infty$.

(c) We always have $\mathrm{add}(\mu) \le \mathrm{add}(\mathcal{N}_\mu)$, defining $\mathrm{add}(\mathcal{N}_\mu)$ as in A1A-B.

(d) If (X, Σ, μ) is complete and locally determined, then $\mathrm{add}(\mu) = \mathrm{add}(\mathcal{N}_\mu)$ (see Fremlin 84, A6O).

(e) If $\mathcal{A} \subseteq \Sigma$ is upwards-directed and $\#(\mathcal{A}) < \mathrm{add}(\mu)$ then $\mu(\bigcup \mathcal{A}) = \sup_{A\in\mathcal{A}} \mu A$.

(f) If (X, Σ, μ) is a totally finite measure space and F is a uniformly bounded non-empty upwards-directed family of measurable functions from X to $\mathbb{R}$, of cardinal less than $\mathrm{add}(\mu)$, and if $f_0(x) = \sup_{f\in F} f(x)$ for each $x \in X$, then

$$\int f_0\, d\mu = \sup_{f\in F} \int f\, d\mu.$$

(Use (e).)

A2D Functions between measure spaces (a) If (X, Σ, μ) and (Y, T, ν) are measure spaces, a function $f : X \to Y$ is **inverse-measure -preserving** if $f^{-1}[F] \in \Sigma$ and $\mu f^{-1}[F] = \nu F$ for every $F \in \mathrm{T}$.

(b) Let (X, Σ, μ) be a measure space, Y any set and $f : X \to Y$ any function. I write μf^{-1} for the measure on Y defined by writing

$$(\mu f^{-1})(F) = \mu(f^{-1}[F])$$

for all those $F \subseteq Y$ for which the right-hand-side is well defined, that is, for which $f^{-1}[F] \in \Sigma$.

Observe that if $\nu = \mu f^{-1}$ then

$$\mathcal{N}_\nu = \{F : F \subseteq Y,\ f^{-1}[F] \in \mathcal{N}_\mu\},$$

ν is complete if μ is, ν is a probability iff μ is, $\operatorname{add}(\nu) \geq \operatorname{add}(\mu)$ and $\operatorname{add}(\mathcal{N}_\nu) \geq \operatorname{add}(\mathcal{N}_\mu)$.

A2E Upper and lower integrals Let (X, Σ, μ) be any measure space and $f : X \to \mathbb{R}$ any function. Then its **upper integral** is

$$\overline{\int} f(x)\,\mu(dx) = \inf\{\textstyle\int g(x)\,\mu(dx) : g : X \to \mathbb{R} \text{ is integrable, } g(x) \geq f(x) \text{ a.e. } x \in X\},$$

taking $\inf \emptyset = \infty$, $\inf \mathbb{R} = -\infty$, and its **lower integral** is

$$\underline{\int} f(x)\,\mu(dx) = -\overline{\int}(-f(x))\,\mu(dx).$$

A2F Measure algebras (a) A **measure algebra** is a pair $(\mathfrak{A}, \bar\mu)$ where $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra and $\bar\mu : \mathfrak{A} \to [0, \infty]$ is a functional such that (i) $\bar\mu a = 0$ iff $a = \mathbf{0}$ (ii) $\bar\mu(\sup_{n\in\mathbb{N}} a_n) = \sum_{n\in\mathbb{N}} \bar\mu a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$. It is a **probability algebra** if moreover $\bar\mu\mathbf{1} = 1$.

(b) If $(\mathfrak{A}, \bar\mu)$ is a measure algebra with $\bar\mu\mathbf{1} < \infty$, it has a natural metric ρ given by setting

$$\rho(a, b) = \bar\mu(a \triangle b) \text{ for all } a,\ b \in \mathfrak{A}.$$

(c) A **measurable algebra** is a Boolean algebra $\mathfrak{A}$ for which there is a $\bar\mu$ making $(\mathfrak{A}, \bar\mu)$ a measure algebra in the sense of (a) above and moreover with $\bar\mu\mathbf{1} < \infty$. (See Fremlin 89, 2.1.)

(d) Let (X, Σ, μ) be a measure space. Its **measure algebra** is the quotient Boolean algebra $\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{N}_\mu$, endowed with the induced functional $\bar\mu$ defined by saying that $\bar\mu E^\bullet = \mu E$ for every $E \in \Sigma$.

(e) If (X, Σ, μ) and (Y, T, ν) are measure spaces and $f : X \to Y$ is inverse-measure -preserving, then we have a corresponding measure-preserving homomorphism $\phi : \mathrm{T}/\mathcal{N}_\nu \to \Sigma/\mathcal{N}_\mu$ given by writing $\phi(F^\bullet) = f^{-1}[F]^\bullet$ for every $F \in \mathrm{T}$ (Fremlin 89, 2.16).

(f) Let $\mathfrak{A}$ be any Boolean algebra. Then $\mathfrak{A}$ is isomorphic to the algebra of open-and-closed sets in its Stone space Z (Fremlin 74, 41D). I write $L^\infty(\mathfrak{A})$ for the Banach algebra of continuous real-valued functions on the compact Hausdorff space Z (Fremlin 74, §43). In this context, given $a \in \mathfrak{A}$, I write $\chi(a) \in L^\infty(\mathfrak{A})$ for the characteristic function of the open-and-closed subset of Z corresponding to a.

(g) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Then we have a unique continuous linear functional $\int d\bar\mu : L^\infty(\mathfrak{A}) \to \mathbb{R}$ such that $\int \chi(a) d\bar\mu = \bar\mu a$ for every $a \in \mathfrak{A}$ (Fremlin 74, §52). If $\nu : \mathfrak{A} \to \mathbb{R}$ is a finitely-additive functional with $0 \leq \nu a \leq \bar\mu a$ for every $a \in \mathfrak{A}$, there is a unique $u \in L^\infty(\mathfrak{A})$ such that $\int_a u d\bar\mu = \int u \times \chi(a) d\bar\mu = \nu a$ for every $a \in \mathfrak{A}$; this is a form of the Radon-Nikodým theorem (Fremlin 74, 63J).

A2G The measure of $\{0,1\}^I$ Let I be any set.

(a) Write $\mathcal{C}$ for the family of 'cylinder sets' of the form

$$C = \{x : x \in \{0,1\}^I,\ x \restriction J = z\}$$

for some finite $J \subseteq I$, $z \in \{0,1\}^J$. In this case write $\phi_0(C) = 2^{-\#(J)}$. Define $\phi : \mathcal{P}(\{0,1\}^I) \to [0,1]$ by

$$\phi(A) = \inf\{\textstyle\sum_{n\in\mathbb{N}} \phi_0(C_n) : \langle C_n\rangle_{n\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}},\ A \subseteq \bigcup_{n\in\mathbb{N}} C_n\}$$

for every $A \subseteq \{0,1\}^I$. The **usual measure** of $\{0,1\}^I$ is the measure defined by Carathéodory's method from the outer measure ϕ. For the rest of this paragraph, I will denote it by μ_I and its domain by Σ_I.

(b) $(\{0,1\}^{\mathbb{N}}, \mu_{\mathbb{N}})$ is isomorphic, as measure space, to $([0,1], \mu_L)$, where μ_L is Lebesgue measure on $[0,1]$.

(c) Let E be any set in Σ_I. **(i)** There are sets E', E'' belonging to the Baire σ-algebra of $\{0,1\}^I$ with $E' \subseteq E \subseteq E''$ and $\mu_I E' = \mu_I E = \mu_I E''$. **(ii)** E is expressible in the form $H \triangle N$ where H belongs to the

Baire σ-algebra of $\{0,1\}^I$ and $N \in \mathcal{N}_{\mu_I}$. **(iii)** For every $\epsilon > 0$ there is an open-and-closed set $H \subseteq \{0,1\}^I$ such that $\mu_I(E \triangle H) \le \epsilon$.

(d) If (X, Σ, μ) is any complete probability space, then any measure-preserving homomorphism ψ from the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ to the measure algebra $\mathfrak{A}$ of (X, Σ, μ) can be induced by an inverse-measure -preserving function $f : X \to \{0,1\}^I$, in the manner of A2Fe; see FREMLIN 89, 2.21 and 4.12, and A2Jd below.

(e) If $J \subseteq I$, $K = I \setminus J$ then $\{0,1\}^I$ can be identified with $\{0,1\}^J \times \{0,1\}^K$. Under this identification we have a form of Fubini's theorem: if $f : \{0,1\}^I \to \mathbb{R}$ is μ_I-integrable, then $\int f(y,z)\mu_K(dz)$ exists for μ_J-almost all y, and $\int\int f(y,z)\mu_K(dz)\mu_J(dy)$ exists and is equal to $\int f(x)\mu_I(dx)$. (SCHWARTZ 73, pp. 73-74.)

(f) If I is infinite, then the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ is homogeneous as Boolean algebra, that is, is isomorphic to all its non-zero principal ideals (FREMLIN 89, 3.6 and 3.7b, or otherwise).

(g) If $f : \{0,1\}^I \to \mathbb{R}$ is Σ_I-measurable there are functions f', f'', both measurable with respect to the Baire σ-algebra of $\{0,1\}^I$, such that $f'(x) \le f(x) \le f''(x)$ for every $x \in \{0,1\}^I$ and $\mu_I\{x : f'(x) \ne f''(x)\} = 0$. (Use (c).)

A2H Maharam types Let $\mathfrak{A}$ be a Boolean algebra.

(a) Write $\tau(\mathfrak{A})$ for the least cardinal of any subset A of $\mathfrak{A}$ which 'completely generates' $\mathfrak{A}$ in the sense that the smallest order-closed (that is, closed-under-suprema or 'complete') subalgebra of $\mathfrak{A}$ including A is $\mathfrak{A}$ itself. (See KOPPELBERG 89, 13.11).

(b) If $\mathfrak{A}$ is ccc then $\#(\mathfrak{A})$ is less than or equal to the cardinal power $\tau(\mathfrak{A})^\omega$. (FREMLIN 89, 6.2b.)

(c) If $a \in \mathfrak{A}$ then $\tau(\mathfrak{A}\lceil a) \le \tau(\mathfrak{A})$, where $\mathfrak{A}\lceil a$ is the principal ideal of $\mathfrak{A}$ generated by a (KOPPELBERG 89, 13.12).

(d) If $\mathfrak{A}$ is a measurable algebra (A2Fc) and $\mathfrak{B} \subseteq \mathfrak{A}$ is an order-closed subalgebra then $\tau(\mathfrak{B}) \le \tau(\mathfrak{A})$ (FREMLIN 89, 6.3b).

(e) $\mathfrak{A}$ is **τ-homogeneous** if $\tau(\mathfrak{A}\lceil a) = \tau(\mathfrak{A})$ for every $a \in \mathfrak{A} \setminus \{\mathbf{0}\}$.

(f) There is a partition of $\mathbf{1}$ in $\mathfrak{A}$, $\langle a_i \rangle_{i \in I}$ say, such that $\mathfrak{A}\lceil a_i$ is τ-homogeneous for every $i \in I$.

(g) If $\mathfrak{A}$ is the measure algebra of a measure space (X, Σ, μ) then $\tau(\mathfrak{A})$ is the **Maharam type** of μ (or of (X, Σ, μ)). If $\mathfrak{A}$ is τ-homogeneous I say that μ, or (X, Σ, μ), is **Maharam homogeneous**.

(h) For any decomposable measure space (X, Σ, μ), there is a partition $\langle X_i \rangle_{i \in I}$ of X such that (i) $\Sigma = \{E : E \in X,\ E \cap X_i \in \Sigma\ \forall\ i \in I\}$ (ii) $\mu E = \sum_{i \in I} \mu(E \cap X_i)$ for every $E \in \Sigma$ (iii) $\mu\lceil X_i$ is Maharam homogeneous and totally finite for every $i \in I$.

(i) If $(\mathfrak{A}, \bar\mu)$ is a probability algebra, then either $\mathfrak{A}$ and $\tau(\mathfrak{A})$ are both finite, or $\tau(\mathfrak{A})$ is precisely the topological density of $\mathfrak{A}$ when $\mathfrak{A}$ is given the metric ρ of A2Fb; this is because a topologically closed subalgebra of $\mathfrak{A}$ is order-closed, so a set $A \subseteq \mathfrak{A}$ completely generates A iff the subalgebra of $\mathfrak{A}$ generated by A is topologically dense in $\mathfrak{A}$ (FREMLIN 89, 2.20).

A2I Maharam's Theorem (a) If $(\mathfrak{A}, \bar\mu)$ is a τ-homogeneous probability algebra, with $\tau(\mathfrak{A}) = \kappa$, then it is isomorphic, as measure algebra, to the measure algebra $\mathfrak{A}_\kappa$ of $\{0,1\}^\kappa$ (FREMLIN 89, 3.8).

(b) If $(\mathfrak{A}, \bar\mu)$ and $(\mathfrak{B}, \bar\nu)$ are probability algebras, and $\tau(\mathfrak{A}) \le \min\{\tau(\mathfrak{B}\lceil b) : b \in \mathfrak{B} \setminus \{\mathbf{0}\}\}$, then there is a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ (FREMLIN 89, 3.13a, corrected to read 'if $\tau(A) \le \kappa$ and $\tau(C) < \kappa, \dots$').

(c) If $(\mathfrak{A}, \bar\mu)$ is a probability algebra and $\tau(\mathfrak{A}) = \kappa$, then it is isomorphic, as measure algebra, to an order-closed subalgebra of $\mathfrak{A}_\kappa$.

A2J Topological measure spaces (a) A **quasi-Radon measure space** is a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is a complete locally determined measure space (ii) $\mathfrak{T}$ is a topology on X and $\mathfrak{T} \subseteq \Sigma$ (iii) if $\mathcal{G}$ is any upwards-directed family in $\mathfrak{T}$ then $\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G$ (iv) for every $E \in \Sigma$,

$\mu E = \sup\{\mu F : F \subseteq E,\ F$ is closed$\}$ (v) whenever $E \in \Sigma$ and $\mu E > 0$ there is an open set G such that $\mu G < \infty$ and $\mu(E \cap G) > 0$. For the general theory of quasi-Radon measure spaces, see FREMLIN 74, FREMLIN N82D and FREMLIN 84. In particular, note that any quasi-Radon measure space is decomposable (FREMLIN 74, 72B) and that any subspace of a quasi-Radon measure space is a quasi-Radon measure space (FREMLIN 84, A7Da).

(b) A **Radon measure space** is a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is a complete locally determined measure space (ii) $\mathfrak{T}$ is a Hausdorff topology on X and $\mathfrak{T} \subseteq \Sigma$ (iii) $\mu E = \sup\{F : F \subseteq E,\ F$ is compact$\}$ for every $E \in \Sigma$ (iv) every point of X belongs to some open set of finite measure.

(c) Every Radon measure space is a quasi-Radon measure space (FREMLIN 74, 73B). The usual measure on $\{0,1\}^I$ (A2G above) is always a Radon measure (see the references in FREMLIN 89, 1.16). If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space and $E \in \Sigma$, then $(E, \mathfrak{T}\lceil E, \Sigma \cap \mathcal{P}E, \mu\lceil E)$ is a Radon measure space.

(d) Suppose that $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, and that (Y, T, ν) is a complete measure space with measure algebra $(\mathfrak{B}, \bar{\nu})$. Suppose that $\mu X = \nu Y < \infty$ and that $\phi : \mathfrak{A} \to \mathfrak{B}$ is a measure-preserving homomorphism. Then there is an inverse-measure -preserving function $f : Y \to X$ such that $f^{-1}[E]^\bullet = \phi(E^\bullet)$ in $\mathfrak{B}$ for every $E \in \Sigma$. (See FREMLIN 89, 4.12-4.13.)

(e) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a totally finite quasi-Radon measure space and $f : X \to \mathbb{R}$ is a bounded function then

$$\overline{\int} f\,d\mu = \inf\{\textstyle\int h\,d\mu : h : X \to \mathbb{R} \text{ is lower semi-continuous and } f(x) \leq h(x)\ \ \forall\ \ x \in X\}.$$

(Use (a-iv). See SCHWARTZ 73, p. 43.)

(f) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a totally finite quasi-Radon measure space and F is a uniformly bounded non-empty upwards-directed family of lower semi-continuous functions from X to $\mathbb{R}$, and if $f_0(x) = \sup_{f \in F} f(x)$ for each $x \in X$, then

$$\int f_0\,d\mu = \sup\nolimits_{f \in F} \int f\,d\mu.$$

(Use (a-iii). See SCHWARTZ 73, p.42.)

A2K Corollary (a) If (X, μ) is a Maharam homogeneous probability space of Maharam type $\lambda \geq \omega$, and (Y, ν) is a Radon probability space of Maharam type not greater than λ, then there is an inverse-measure -preserving function $f : X \to Y$.

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon probability space, $\mathcal{E} \subseteq \Sigma$ is a collection of measurable sets, Σ_0 is the σ-subalgebra of Σ generated by $\mathcal{E}$, and $\lambda \geq \max(\omega, \#(\mathcal{E}))$ is a cardinal, there is a function $g : \{0,1\}^\lambda \to X$ such that $\mu_\lambda g^{-1}[E]$ is defined and equal to μE for every $E \in \mathcal{E}$, where μ_λ is the usual measure of $\{0,1\}^\lambda$.

(c) If (X, μ) is a complete atomless probability space, there is a function $f : X \to [0,1]$ which is inverse-measure -preserving for μ and Lebesgue measure on $[0,1]$.

proof (a) Let $\mathfrak{A}$ be the measure algebra of (X, μ) and $\mathfrak{B}$ the measure algebra of (Y, ν). By A2Ib there is a measure-preserving homomorphism $\phi : \mathfrak{B} \to \mathfrak{A}$, which by A2Jd is induced by an inverse-measure -preserving $f : X \to Y$.

(b) Let κ be the greater of λ and the Maharam type of (X, μ). Then there is an inverse-measure -preserving $f : \{0,1\}^\kappa \to X$, by (a). For each $E \in \mathcal{E}$ there are Baire sets G_E, $H_E \subseteq \{0,1\}^\kappa$ such that $G_E \subseteq g^{-1}[E] \subseteq H_E$ and $\mu_\kappa H_E = \mu E = \mu_\kappa G_E$. Now there is a countable set $I_E \subseteq \kappa$ such that both G_E and H_E factor through the canonical map $\pi_{I_E} : \{0,1\}^\kappa \to \{0,1\}^{I_E}$; that is, $G_E = \pi_{I_E}^{-1}[\pi_{I_E}[G_E]]$, and similarly for H_E. Let $I \subseteq \kappa$ be a set of cardinal λ including I_E for every $E \in \mathcal{E}$. Let z be any point of $\{0,1\}^{\kappa \setminus I}$ and set $h(x) = (x, z)$ for each $x \in \{0,1\}^I$, identifying $\{0,1\}^\kappa$ with $\{0,1\}^I \times \{0,1\}^{\kappa \setminus I}$. If $E \in \mathcal{E}$ then $\pi_I[G_E] \subseteq h^{-1}[f^{-1}[E]] \subseteq \pi_I[H_E]$ so $\mu_I h^{-1}[f^{-1}[E]] = \mu_\kappa G_E = \mu_\kappa H_E = \mu E$. So if we set $g = fh : \{0,1\}^I \to X$ we shall have $\mu_I g^{-1}[E] = \mu E$ for every $E \in \mathcal{E}$, and therefore for every $E \in \Sigma_0$. But of course $(\{0,1\}^I, \mu_I) \cong (\{0,1\}^\lambda, \mu_\lambda)$, so this proves the result.

(c) Argue as in (a), or as in FREMLIN 84, A6Ib.

A2L Hyperstonian spaces (a) If (X, Σ, μ) is a totally finite measure space, its **hyperstonian space** is the Radon measure space $(Z, \mathfrak{S}, \mathrm{T}, \nu)$, where $(Z, \mathfrak{S})$ is the Stone space of the measure algebra $(\mathfrak{A}, \bar{\mu})$ of

(X,Σ,μ), and ν is the unique Radon measure on Z such that $\nu E^* = \mu E$ for every $E\in\Sigma$, where I write E^* for the open-and-closed subset of Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. (See FREMLIN 89, 2.13-2.14.) In this case, $(\mathfrak{A},\bar\mu)$ is isomorphic, as measure algebra, to the measure algebra of (Z,T,ν) (FREMLIN 89, 2.13).

(b) If $(X,\mathfrak{T},\Sigma,\mu)$ is a compact Radon measure space and $(Z,\mathfrak{S},\mathrm{T},\nu)$ is its hyperstonian space, then there is a unique continuous function $f:Z\to X$ such that $\nu(E^*\triangle f^{-1}[E])=0$ for every $E\in\Sigma$. This f is of course inverse-measure -preserving; it is the **canonical map** from Z to X. (See FREMLIN 89, 2.17-2.18.)

(c) If (X,Σ,μ) is any totally finite measure space and $(Z,\mathfrak{S},\mathrm{T},\nu)$ is hyperstonian space, then for every $F\in\mathrm{T}$, $\epsilon>0$ there is a compact open set $W\subseteq F$ such that $\nu W\ge\nu F-\epsilon$ (FREMLIN 89, 2.14).

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A2M Lemma Let (X,Σ,μ) be a probability space.

(a) Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a family in Σ such that $\inf_{n\in\mathbb{N}}\mu E_n=\delta>0$. Then there is an $x\in X$ such that $\{n:n\in\mathbb{N},\ x\in E_n\}$ is infinite.

(b) Let $\langle E_\alpha\rangle_{\alpha<\omega_1}$ be a family in $\Sigma\setminus\mathcal{N}_\mu$. Then there is an $x\in X$ such that $\{\alpha:\alpha<\omega_1,\ x\in E_\alpha\}$ is infinite.

proof (a) Set $F_n=\bigcup_{m\ge n}E_m$ for $n\in\mathbb{N}$; then $\langle F_n\rangle_{n\in\mathbb{N}}$ is a decreasing sequence and $\mu F_n\ge\delta>0$ for every $n\in\mathbb{N}$. So $\bigcap_{n\in\mathbb{N}}F_n\ne\emptyset$; take any $x\in\bigcap_{n\in\mathbb{N}}F_n$; this x must belong to infinitely many of the E_n.

(b) There is a $\delta>0$ such that $A=\{\alpha:\alpha<\omega_1,\ \mu E_\alpha>0\}$ is infinite. Now apply (a) to $\langle E_{\alpha(n)}\rangle_{n\in\mathbb{N}}$ for any strictly increasing sequence $\langle\alpha(n)\rangle_{n\in\mathbb{N}}$ in A.

Remark For related results see 9A and A2U; also KANAMORI & MAGIDOR 78, p. 166, and FREMLIN 87, 1E.

A2N Lemma Let (X,Σ,μ) be a probability space and T_0 a subalgebra of Σ. Suppose that $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence of members of Σ which are stochastically independent of each other and of T_0, that is, $\mu(E\cap\bigcap_{i\in I}C_i)=\mu E\prod_{i\in I}\mu C_i$ for every $E\in\mathrm{T}_0$, $I\subseteq\mathbb{N}$. If now f is a μ-integrable real-valued function on X and $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence in T_0,

$$\phi_n(f)=\int_{E_n\cap C_n}f-\mu C_n.\int_{E_n}f\to 0$$

as $n\to\infty$.

proof Consider first the case $f=\chi F$ where $F=E\cap\bigcap_{i\in I}C_i$ for some $E\in\mathrm{T}_0$, $I\in[\mathbb{N}]^{<\omega}$; this is trivial. It follows that $\lim_{n\to\infty}\phi_n(f)=0$ whenever $f=\chi F$ for some set F in the subalgebra T_1 of Σ generated by $\mathrm{T}_0\cup\{C_n:n\in\mathbb{N}\}$. Also, of course, $|\phi_n(f)-\phi_n(g)|\le 2\int|f-g|$ for all $n\in\mathbb{N}$, integrable f and g, so $\lim_{n\to\infty}\phi_n(f)$ will be 0 for every T-measurable integrable function f, where T is the σ-subalgebra of Σ generated by T_1. Finally, for a general μ-integrable f, the Radon-Nikodým theorem (ROYDEN 63, chap. 11, §5) tells us that there is a T-measurable g such that $\int_F f=\int_F g$ for every $F\in\mathrm{T}$, so that $\phi_n(f)=\phi_n(g)\to 0$ as $n\to\infty$.

A2O Definition Let (X,μ) be a measure space and E, F two real linear spaces in duality. Say that a function $x\mapsto u_x:X\to E$ is **F-scalarly measurable** if $x\mapsto(f|u_x):X\to\mathbb{R}$ is measurable for every $f\in F$.

A2P(a) Let X be a set and $\mathcal{A}$ any family of sets. Write

$$\begin{aligned}\mathrm{cov}(X,\mathcal{A})&=\min\{\#(\mathcal{B}):\mathcal{B}\subseteq\mathcal{A},\ X\subseteq\bigcup\mathcal{B}\}\text{ if }X\subseteq\bigcup\mathcal{A},\\&=\infty\text{ otherwise.}\end{aligned}$$

(b) For a Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$, we have $\mathrm{cov}(X,\mathcal{N}_\mu)=0$ iff $X=\emptyset$ and $\mathrm{cov}(X,\mathcal{N}_\mu)=1$ iff $X\ne\emptyset$ and $\mu X=0$. In any other case, consider the least cardinal κ for which there is an $E\in\Sigma$ with $0<\mu E<\infty$ and $(E,\Sigma\cap\mathcal{P}E,\mu\lceil E)$ is Maharam homogeneous of Maharam type κ; then

$$\mathrm{cov}(X,\mathcal{N}_\mu)=\mathrm{cov}(\{0,1\}^\kappa,\mathcal{N}_{\mu_\kappa}),$$

where μ_κ is the usual Radon measure on $\{0,1\}^\kappa$. (See FREMLIN 89, 6.14c and 6.15c.) If we write

$$\delta_\kappa = \mathrm{cov}(\{0,1\}^\kappa, \mathcal{N}_{\mu_\kappa})$$

for infinite cardinals κ, then $\delta_\kappa \le \delta_\lambda$ whenever $\omega \le \lambda \le \kappa$ (FREMLIN 89, 6.17(d-i), so there is a least κ_0 such that $\delta_\kappa = \delta_{\kappa_0} = \delta$ say for every $\kappa \ge \kappa_0$, and this δ is precisely the least value of $\mathrm{cov}(X, \mathcal{N}_\mu)$ for any non-trivial Radon measure space (X, μ). Moreover, if $\delta_\kappa \le \lambda \le \kappa$ then $\delta_\lambda = \delta_\kappa$ (FREMLIN 89, 6.17(d-ii)); it follows that if $\delta_\kappa \ge \kappa$ then $\delta \ge \kappa$ and if $\delta_\kappa > \kappa$ then $\delta > \kappa$.

A2Q Liftings Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$.

(a) A **lifting** of (X, Σ, μ) is a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$.

(b) The Lifting Theorem If (X, Σ, μ) is complete and decomposable it has a lifting (FREMLIN 89, 4.4, or IONESCU TULCEA & IONESCU TULCEA 69, chap. IV, Theorem 3).

(c) If (X, Σ, μ) is a complete probability space, $\Sigma_0 \subseteq \Sigma$ is a σ-subalgebra and $\mathfrak{A}_0 = \{E^\bullet : E \in \Sigma_0\}$ is the corresponding order-closed subalgebra of $\mathfrak{A} = \Sigma/\mathcal{N}_\mu$, then any lifting $\theta_0 : \mathfrak{A}_0 \to \Sigma_0$ extends to a lifting $\theta : \mathfrak{A} \to \Sigma$ (follow the standard proof of the lifting theorem as given in FREMLIN 89 or IONESCU TULCEA & IONESCU TULCEA 69).

(d) If (X, Σ, μ) is a complete probability space and $\theta : \mathfrak{A} \to \Sigma$ is any lifting, then $\bigcup_{a \in A} \theta(a) \in \Sigma$ for every $A \subseteq \mathfrak{A}$. (IONESCU TULCEA & IONESCU TULCEA 69, §V.3.)

A2R Lemma Let (X, Σ, μ) be a probability space, λ a cardinal of uncountable cofinality, $\langle E_\xi \rangle_{\xi < \lambda}$ a family of measurable non-negligible sets in X, and $1 \le l \in \mathbb{N}$. Then there is a set $W \subseteq \lambda$, of cardinal λ, such that $\bigcap_{\xi \in L} E_\xi \ne \emptyset$ for every $L \in [W]^l$.

proof See ARGYROS & KALAMIDAS 82, or COMFORT & NEGREPONTIS 82, Theorem 6.15, or FREMLIN 88, Proposition 7, where various stronger results are proved.

Remark For the following results, we need only the case $X = \{0,1\}^I$, which is usefully simpler than the general case.

A2S Lemma Let (X, Σ, μ) be a probability space, and Σ_0 a σ-subalgebra of Σ; set $\mu_0 = \mu \restriction \Sigma_0$ and let μ_0^* be the outer measure on X defined from μ_0. Let τ_0 be the Maharam type of μ_0. Suppose that λ is a cardinal with $\mathrm{cf}(\lambda) > \max(\omega, \tau_0)$, and $\langle E_\xi \rangle_{\xi < \lambda}$ a family in Σ with $\inf_{\xi < \lambda} \mu E_\xi = \gamma > 0$. Then for any $\gamma' < \gamma$, $1 \le l \in \mathbb{N}$ there is a $W \in [\lambda]^\lambda$ such that $\mu_0^*(\bigcap_{\xi \in L} E_\xi) \ge \gamma'$ for every $L \in [W]^l$.

Remark For this paper we need only the case $\tau_0 = \omega$, $\lambda = \omega_1$.

proof (a) Consider first the special case in which $X = \{0,1\}^I$ for some set I, with μ the usual measure on X, and there is a $J \subseteq I$ of cardinal τ_0 such that $\Sigma_0 = \{\pi_J^{-1}[F] : F \in \Sigma_J\}$, writing Σ_J for the domain of the usual measure μ_J on $\{0,1\}^J$, and $\pi_J : X \to \{0,1\}^J$ for the canonical map.

(i) We may regard X as the product $\{0,1\}^J \times \{0,1\}^{I \setminus J}$. Set $\delta = (\gamma - \gamma')/(l+1) > 0$. For $\xi < \lambda$, $z \in \{0,1\}^{I \setminus J}$ set

$$E_{\xi z} = \{y : y \in \{0,1\}^J,\ (y, z) \in E_\xi\}.$$

Then

$$\int \mu_J E_{\xi z} \mu_{I \setminus J}(dz) = \mu E_\xi \ge \gamma$$

by Fubini's theorem (A2Ge). Set

$$G_\xi = \{z : z \in \{0,1\}^{I \setminus J},\ \mu_J E_{\xi z} \text{ exists } \ge \gamma\};$$

then $\mu_{I \setminus J} G_\xi$ exists and is greater than 0. Let $H_\xi \subseteq X$ be an open-and-closed set such that $\mu(E_\xi \triangle H_\xi) \le \delta \mu_{I \setminus J} G_\xi$ (A2Gc), and set $H_{\xi z} = \{y : (y, z) \in H_\xi\}$ for $z \in \{0,1\}^{I \setminus J}$. Then

$$\int \mu_J(E_{\xi z} \triangle H_{\xi z}) \mu_{I \setminus J}(dz) = \mu(E_\xi \triangle H_\xi) \le \delta \mu_{I \setminus J} G_\xi,$$

so $\mu_{I \setminus J} G'_\xi > 0$, where

$$G'_\xi = \{z : z \in G_\xi,\ \mu_J(E_{\xi z} \triangle H_{\xi z}) \le \delta\}.$$

Finally, observe that because H_ξ is open-and-closed the set of sections $H_{\xi z}$ is finite, and there is an open-and-closed set $F_\xi \subseteq \{0,1\}^J$ such that $\mu_{I\setminus J} G''_\xi > 0$, where

$$G''_\xi = \{z : z \in G'_\xi,\ H_{\xi z} = F_\xi\}.$$

(ii) The number of open-and-closed sets in $\{0,1\}^J$ is at most $\max(\omega, \#(J)) < \mathrm{cf}(\lambda)$. So there is an open-and-closed set $F \subseteq \{0,1\}^J$ such that $U = \{\xi : \xi < \lambda,\ F_\xi = F\}$ has cardinal λ. Next, by A2R above, there is a $W \subseteq U$ such that $\#(W) = \lambda$ and $\bigcap_{\xi\in L} G''_\xi \neq \emptyset$ whenever $L \in [W]^l$.

Take any $L \in [W]^l$. There is a $z \in \bigcap_{\xi\in L} G''_\xi$. For each $\xi \in L$, $\mu_J E_{\xi z}$ exists $\geq \gamma$, $\mu_J(E_{\xi z} \triangle H_{\xi z}) \leq \delta$ and $H_{\xi z} = F_\xi = F$. So $\mu_J F \geq \gamma - \delta$ and $\mu_J(\bigcap_{\xi\in L} E_{\xi z}) \geq \mu_J F - l\delta \geq \gamma'$.

Now if $F' \in \Sigma_J$ and $\pi_J^{-1}[F'] \supseteq \bigcap_{\xi\in L} E_\xi$, we must have $F' \supseteq \bigcap_{\xi\in L} E_{\xi z}$, so

$$\mu(\pi_J^{-1}[F']) = \mu_J F' \geq \gamma'.$$

As F' is arbitrary, $\mu_0^*(\bigcap_{\xi\in L} E_\xi) \geq \gamma$, as required.

(b) It follows that if $(\mathfrak{A}, \bar\mu)$ is a probability algebra, $\mathfrak{A}_0$ is a order-closed subalgebra of $\mathfrak{A}$, λ is a cardinal with $\mathrm{cf}(\lambda) > \max(\omega, \tau(\mathfrak{A}_0))$, $\langle a_\xi\rangle_{\xi<\lambda}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\lambda} \bar\mu a_\xi = \gamma > 0$, $1 \leq l \in \mathbb{N}$ and $\gamma' < \gamma$, then there is a $W \in [\lambda]^\lambda$ such that

$$\min\{\bar\mu b : b \in \mathfrak{A}_0,\ b \supseteq \inf_{\xi\in L} a_\xi\} \geq \gamma'$$

whenever $L \in [W]^l$. **P** We can embed $\mathfrak{A}$ as a subalgebra of the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ for some set I (A2Ib). If we take a set $B \subseteq \mathfrak{A}_0$ of cardinal $\tau(\mathfrak{A}_0)$ which completely generates $\mathfrak{A}_0$, then for each $b \in B$ we can find a set $G_b \subseteq X = \{0,1\}^I$, belonging to the Baire σ-algebra of X, such that $b = G_b^\bullet$ in $\mathfrak{A}$; now there is a set $J \subseteq I$, of cardinal at most $\max(\omega, \#(B))$, such that every G_b belongs to Σ_0, if we define Σ_0 from J as in part (a) above.

Set $\mathfrak{A}_1 = \{G^\bullet : G \in \Sigma_0\}$, so that $\mathfrak{A}_1$ is a order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_0 \subseteq \mathfrak{A}_1$. Let Σ_J be the algebra of measurable subsets of $\{0,1\}^J$, and $\mathfrak{A}_J$ the measure algebra of $\{0,1\}^J$. Then the inverse-measure-preserving map $\pi_J : X \to \{0,1\}^J$ induces an isomorphism ϕ between $\mathfrak{A}_1$ and $\mathfrak{A}_J$, taking $\phi(\pi_J^{-1}[G]^\bullet) = G^\bullet$ for every $G \in \Sigma_J$. By the lifting theorem (A2Qb) there is a lifting $\theta_J : \mathfrak{A}_J \to \Sigma_J$. So we have a corresponding Boolean homomorphism $\theta_1 : \mathfrak{A}_1 \to \Sigma_0$ given by setting $\theta_1(a) = \pi_J^{-1}[\theta_J(\phi(a))]$ for each $a \in A_1$. As remarked in A2Qc, there is an extension θ of θ_1 to a lifting from $\mathfrak{A}$ to Σ.

Set $E_\xi = \theta(a_\xi)$ for each $\xi < \lambda$. By part (a) above, because $\mathrm{cf}(\lambda) > \max(\omega, \#(J))$, there is a set $W \in [\lambda]^\lambda$ such that $\mu_0^*(\bigcap_{\xi\in L} E_\xi) \geq \gamma'$ whenever $L \in [W]^l$. Now suppose that $L \in [W]^l$ and that $b \in \mathfrak{A}_0$, $b \supseteq \inf_{\xi\in L} a_\xi$. Then $\theta(b) \supseteq \bigcap_{\xi\in L} E_\xi$ and $\theta(b) \in \Sigma_0$, so

$$\bar\mu b = \mu(\theta(b)) \geq \mu_0^*(\textstyle\bigcap_{\xi\in L} E_\xi) \geq \gamma'.$$

Thus we have the set W we need. **Q**

(c) We are now ready for the general case of the lemma. Let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of (X, Σ, μ) and set $\mathfrak{A}_0 = \{G^\bullet : G \in \Sigma_0\}$, $a_\xi = E_\xi^\bullet$ for each $\xi < \lambda$. By (b), there is a $W \in [\lambda]^\lambda$ such that $\bar\mu b \geq \gamma'$ whenever $L \in [W]^l$, $b \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in L} a_\xi$. Now if $L \in [W]^l$ and $G \in \Sigma_0$ and $G \supseteq \bigcap_{\xi\in L} E_\xi$, $b = G^\bullet \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in L} a_\xi$, so that $\mu_0 G = \mu G = \bar\mu G^\bullet \geq \gamma'$. As G is arbitrary, $\mu_0^*(\bigcap_{\xi\in L} E_\xi) \geq \gamma'$; as L is arbitrary, we have the required family W.

A2T Precalibers If $\mathfrak{A}$ is a Boolean algebra and λ is a cardinal, then λ is a **precaliber** of $\mathfrak{A}$ if for every family $\langle a_\xi\rangle_{\xi<\lambda}$ in $\mathfrak{A} \setminus \{\mathbf{0}\}$ there is a set $D \in [\lambda]^\lambda$ such that $\{a_\xi : \xi \in D\}$ is **centered**, that is, $\inf_{\xi\in I} a_\xi \neq \mathbf{0}$ for any non-empty finite $I \subseteq D$.

A2U Proposition Let (X, Σ, μ) be a complete probability space with measure algebra $(\mathfrak{A}, \bar\mu)$, and λ a cardinal of uncountable cofinality which is not a precaliber of $\mathfrak{A}$.

(a) There is a family $\langle E_\xi\rangle_{\xi<\lambda}$ in $\mathcal{N}_\nu$ such that $\bigcup_{\xi<\lambda} E_\xi \in \Sigma \setminus \mathcal{N}_\mu$.

(b) If λ is regular, there is an increasing family $\langle E_\xi\rangle_{\xi<\lambda}$ in $\mathcal{N}_\mu$ such that $\bigcup_{\xi<\lambda} E_\xi \in \Sigma \setminus \mathcal{N}_\mu$.

proof Let $\langle a_\xi\rangle_{\xi<\lambda}$ be a family in $\mathfrak{A} \setminus \{\mathbf{0}\}$ with no centered subfamily of cardinal λ. Let $\theta : \mathfrak{A} \to \Sigma$ be a lifting (A2Q). If $D \in [\lambda]^\lambda$ then $\{a_\xi : \xi \in D\}$ is not centered so there is a finite set $I \subseteq D$ such that $\bigcap_{\xi\in I} \theta(a_\xi) = \theta(\inf_{\xi\in I} a_\xi) = \emptyset$. Consequently $\#(\{\xi : x \in \theta(a_\xi)\}) < \lambda$ for every $x \in X$.

(a) Choose inductively countable sets $C_\alpha \subseteq \lambda$, for $\alpha < \lambda$, such that

$$C_\alpha \subseteq \lambda \setminus \bigcup_{\beta<\alpha} C_\beta,$$

$$\sup\{a_\xi : \xi \in C_\alpha\} = \sup\{a_\xi : \xi \in \lambda \setminus \bigcup_{\beta<\alpha} C_\beta\}$$

for each $\alpha < \lambda$. Setting $s_\alpha = \bar{\mu}(\sup\{a_\xi : \xi \in C_\alpha\})$ for each α, we see that $\langle s_\alpha\rangle_{\alpha<\lambda}$ is decreasing; as $\mathrm{cf}(\lambda) > \omega$, there is a $\gamma < \lambda$ such that $s_\gamma = s_\alpha$ for $\gamma \le \alpha < \lambda$. Set

$$H_\alpha = \bigcup\{\theta(a_\xi) : \xi \in C_\alpha\}$$

for $\alpha < \lambda$. Then $\bigcap_{\alpha<\lambda} H_\alpha = \emptyset$, because the C_α are disjoint. Also $\mu(H_\gamma \setminus H_\alpha) = 0$ for every α, because $H_\alpha^\bullet = \sup\{a_\xi : \xi \in C_\alpha\}$ in $\mathfrak{A}$. So if we set $E_\xi = H_\gamma \setminus H_\xi$, we have a witness for (a).

(b) Now suppose that λ is regular. For each $\alpha < \lambda$ set $H'_\alpha = \bigcup_{\alpha\le\xi<\lambda} \theta(a_\xi)$; then $H'_\alpha \in \Sigma$ (A2Qd). The family $\langle H'_\alpha\rangle_{\alpha<\lambda}$ is decreasing; because $\mathrm{cf}(\lambda) > \omega$, there is a $\gamma < \lambda$ such that $\mu H'_\alpha = \mu H_\gamma$ whenever $\gamma \le \alpha < \lambda$. **?** If $x \in \bigcap_{\alpha<\lambda} H'_\alpha$, set $D = \{\xi : x \in \theta(a_\xi)\}$; then D is cofinal with λ so (because λ is regular) $\#(D) = \lambda$. **X** Thus $\bigcap_{\alpha<\lambda} H_\alpha = \emptyset$. But now $\langle H_\gamma \setminus H_\alpha\rangle_{\alpha<\lambda}$ is an increasing family in $\mathcal{N}_\mu$ with union $H_\gamma \in \Sigma \setminus \mathcal{N}_\mu$.

A2V More topological measure spaces Let X be a topological space. A **Borel measure** on X is a measure μ with domain the algebra $\mathcal{B}$ of Borel subsets of X. A Borel measure μ on X is **τ-additive** if $\mu(\bigcup \mathcal{G}) = \sup_{G\in\mathcal{G}} \mu G$ for every non-empty upwards-directed family $\mathcal{G}$ of open sets in X. (If X is regular, a totally finite Borel measure on X is τ-additive iff its completion is a quasi-Radon measure.) X is **Borel measure-complete** if every totally finite Borel measure on X is τ-additive. A Borel measure μ on X is **inner regular for the compact sets** if $\mu E = \sup\{\mu K : K \subseteq E \text{ is compact}\}$ for every $E \in \mathcal{B}$. (If X is Hausdorff and μ is a totally finite Borel measure on X, μ is inner regular for the compact sets iff its completion is a Radon measure.) X is **Radon** if it is Hausdorff and every totally finite Borel measure μ on X is inner regular for the compact sets. Evidently a Radon topological space is Borel measure-complete. See Gardner & Pfeffer 84.

A2W Proposition Let X be a complete metric space.

(a) A semi-finite τ-additive Borel measure on X is inner regular for the compact sets.

(b) X is Radon iff it is Borel measure-complete.

proof (a) It will be enough to show that a totally finite τ-additive Borel measure on X is inner regular for the compact sets. Let μ be such a measure. Let $E \subseteq X$ be a Borel set and $\epsilon > 0$.

For each $n \in \mathbb{N}$ let $\mathcal{G}_n$ be the family of open sets of diameter less than or equal to 2^{-n}; then $\bigcup \mathcal{G}_n = X$. Because μ is τ-additive there is a finite $\mathcal{H}_n \subseteq \mathcal{G}_n$ such that $\mu(\bigcup \mathcal{H}_n) \ge \mu X - 2^{-n}\epsilon$. Set $K_0 = \bigcap_{n\in\mathbb{N}} \overline{\bigcup \mathcal{H}_n}$; then K_0 is closed and totally bounded, therefore compact, and $\mu K_0 \ge \mu X - 2\epsilon$.

Now K_0 is compact and metrizable, so the Borel measure $\mu\lceil K_0$ is inner regular for the compact sets (Schwartz 73, p. 117, Proposition 6). Consequently

$$\mu E \le \mu(E \cap K_0) + 2\epsilon \le \sup\{\mu K : K \subseteq E \text{ is compact}\} + 2\epsilon.$$

As E and ϵ are arbitrary, μ is inner regular, as claimed.

(b) follows at once.

A2X Strong Law of Large Numbers We need this classical result in the following form. If (X, Σ, μ) is a probability space and $\langle h_i\rangle_{i\in\mathbb{N}}$ is a uniformly bounded sequence of measurable real-valued functions on X, then for almost all $\langle t_i\rangle_{i\in\mathbb{N}} \in X^{\mathbb{N}}$ we have

$$\lim_{n\to\infty} \tfrac{1}{n+1} \textstyle\sum_{i\le n}(h(t_i) - \int h_i) = 0.$$

A proof may be found in Shiryaev 84, chap. IV, §3, Theorem 2 (p. 364).

A2Y Lemma If (X, Σ, μ) is a Maharam homogeneous totally finite measure space and μ_1 is another measure on X with domain Σ such that $\mu_1 E \le \mu E$ for every $E \in \Sigma$ and $\mu_1 X > 0$, then μ and μ_1 have the same Maharam type.

proof Take $E \in \mathcal{N}_{\mu_1}$ such that μE is maximal. Then $\mathcal{N}_{\mu_1} = \{A : A \subseteq X,\ A \setminus E \in \mathcal{N}_\mu\}$. So the measure algebra $\mathcal{P}X/\mathcal{N}_{\mu_1}$ is isomorphic (as Boolean algebra) to the principal ideal of $\mathcal{P}X/\mathcal{N}_\mu$ generated by $(X \setminus E)^\bullet$; which has the same Maharam type as the whole algebra $\mathcal{P}X/\mathcal{N}_\mu$.

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A3. General topology

We need only a handful of definitions and a couple of standard theorems.

A3A If X is any topological space, a **regular open** set in X is an open subset G of X such that $G = \operatorname{int}\overline{G}$. Now the family $\mathcal{G}$ of all regular open subsets of X is a Boolean algebra, taking $G \wedge H = G \cap H$, $G \vee H = \operatorname{int}\overline{G \cup H}$, $\mathbf{1} \setminus G = \operatorname{int}(X \setminus G)$ for $G, H \in \mathcal{G}$; $\mathcal{G}$ is the **regular open algebra** of X. (See Koppelberg 89, 1.36-1.37.)

A3B If X is any topological space, its **Baire σ-algebra** is the σ-algebra of subsets of X generated by the zero sets, that is, by sets of the form $f^{-1}[\{0\}]$ where $f : X \to \mathbb{R}$ is continuous. When $X = \{0,1\}^\lambda$ this is precisely the σ-algebra generated by sets of the form $\{x : x(\xi) = 1\}$ for $\xi < \lambda$.

A3C Definitions (a) A **Moore space** is a regular Hausdorff topological space X with a sequence $\langle \mathcal{G}_n \rangle_{n \in \mathbb{N}}$ of open covers such that for any $x \in X$, any neighbourhood U of x there is an $n \in \mathbb{N}$ such that $\bigcup\{G : x \in G \in \mathcal{G}_n\} \subseteq U$. (Engleking 89, p. 334.)

(b) An indexed family $\langle F_i \rangle_{i \in I}$ of subsets of a topological space X is **discrete** if for every $x \in X$ there is an open set containing x which meets F_i for at most one $i \in I$. An indexed family $\langle x_i \rangle_{i \in I}$ of points of X is **discrete** if for every $x \in X$ there is an open set containing x which contains x_i for at most one i. (Engelking 89, p. 16.)

(c) A topological space X is **collectionwise normal** if for every discrete family $\langle F_i \rangle_{i \in I}$ of closed sets in X there is a disjoint family $\langle G_i \rangle_{i \in I}$ of open sets such that $F_i \subseteq G_i$ for every $i \in I$. (Engelking 89, p. 305.)

A3D Proposition A collectionwise normal Moore space is metrizable.

proof Engelking 89, Theorem 5.4.1.

A3E Definitions Let X be a topological space.

(a) The **density** of X, $\mathrm{d}(X)$, is the least cardinal of any dense subset of X. (Engelking 89, p. 25.)

(b) If $x \in X$ then $\chi(x, X)$ is the least cardinal of any base of neighbourhoods of x in X. (Engelking 89, p. 12.)

(c) The **weight** of X, $\mathrm{w}(X)$, is the least cardinal of any base for the topology of X. (Engleking 89, p. 12.)

A3F Proposition Let X be a metrizable topological space.
(a) There is a discrete family $\langle x_\xi \rangle_{\xi < \mathrm{d}(X)}$ in X.
(b) There is a base for the topology of X which is expressible as $\{G_{ni} : n \in \mathbb{N}, i \in I_n\}$ where $\langle G_{ni} \rangle_{i \in I_n}$ is discrete for each $n \in \mathbb{N}$.

proof (a) is elementary; see Engelking 89, 4.1.15. (b) is Theorem 4.4.3 of Engelking 89.

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A4. Indescribable cardinals

For the second half of §4 above, we need some ideas from elementary model theory. The point is that certain results of the form 'if κ is a real-valued-measurable cardinal, there are many $\alpha < \kappa$ such that $\Phi(\alpha)$' can be effectively approached through an analysis of the logical structure of the assertion '$\Phi(\alpha)$'. Here I describe a second-order language $\mathcal{L}$ which provides a suitable classification of most of the sentences we are interested in.

A4A The language $\mathcal{L}$ Let $\mathcal{L}$ be the primitive second-order language in which there are countable infinities of first-order variables x, $y, \ldots$ and second-order relational variables R, $S, \ldots$; it being understood that each relational variable S has a determinate number of places, so that an expression '$\forall S$' must be read as 'for every n-place relation S' for some n, whose value should be evident from the use of S in the rest of the formula. Atomic formulae of $\mathcal{L}$ are of the form $S(y_1, \dots, y_n)$, where S is an n-place relation symbol, and compound formulae are constructed from these with the ordinary logical connectives $\wedge$ ('and'), $\vee$ ('or'), $\neg$ ('not'), $\rightarrow$ ('implies') etc., and quantifiers $\forall S$, $\exists S$, $\forall y$, $\exists y$. We can now define a notion of satisfaction

$$(A; C_1, \dots, C_k; a_1, \dots, a_m) \vDash \phi$$

where A is a set, $C_1, \dots, C_k$ are finitary relations on A, $a_1, \dots, a_m$ are points of A, and ϕ is a formula of $\mathcal{L}$ with k free relational variables (of place-numbers matching the place-numbers of the C_i) and m free first-order variables. See Chang & Keisler 73, §1.3, or Ebbinghaus Flum & Thomas 84, §IX.1 for a proper discussion of 'satisfaction'; intuitively,

$$(A; C_1, \dots, C_k; a_1, \dots, a_m) \vDash \phi$$

means that $\phi(C_1, \dots, C_k, a_1, \dots, a_m)$ is 'true' when bound variables in ϕ are taken to run over the members of A or the relations on A.

There is a problem here; of course the assertion

$$(A; C_1, \dots, C_k; a_1, \dots, a_m) \vDash \phi$$

depends on an assumed assignment of the relations C_i and points a_j to the free variables of ϕ. For our purposes here it will possible to adhere to the convention (an impracticable one for any extended work) that

$$(A; C_1, \dots, C_k; a_1, \dots, a_m) \vDash \phi$$

includes the assertion that the names of the free variables of ϕ are R_i, x_j, with $1 \le i \le k$ and $1 \le j \le m$; that each R_i has the same number of places as the corresponding C_i; and that in the interpretation of ϕ the relations C_i are assigned to the variables R_i and the points a_j are assigned to the variables x_j.

It will be convenient on occasion to write the formulae above without checking on the domains of the relations C_i nor on whether every a_j belongs to A. In this case it is to be understood that

$$(A; C_1, \dots, C_k; a_1, \dots, a_m) \vDash \phi$$

means

$$a_1, \dots, a_m \in A \text{ and } (A; C_1 \restriction A, \dots, C_k \restriction A; a_1, \dots, a_m) \vDash \phi,$$

where $C_i \restriction A$ is the restriction of the relation C_i to A.

A4B Π^1_n formulae Among the formulae of $\mathcal{L}$ we can distinguish the Π^1_n and Σ^1_n formulae, as follows. A Π^1_0 or Σ^1_0 formula is one in which all quantifiers are of the form $\forall y$ or $\exists y$. A Π^1_{n+1} formula is one of the form

$$\forall S_1 \forall S_2 \dots \forall S_k \phi$$

where ϕ is Σ^1_n; a Σ^1_{n+1} formula is one of the form

$$\exists S_1 \dots \exists S_k \phi$$

where ϕ is Π^1_n. (Here I allow, conventionally, $k = 0$, so that a Π^1_n formula is also Σ^1_{n+1} and Π^1_{n+1}.)

Examples It may help readers new to these ideas if I give some examples. I concentrate on the properties of cardinals and ordinals because that is the context in which we shall be working.

(i) If we say '$(X, \le)$ is totally ordered' we are saying just that $(X; =, \le;) \vDash \phi_1$, where ϕ_1 is the Π^1_0 formula

$$\begin{aligned}
&\forall u \forall v \forall w \\
&\qquad \big((R_2(u, v) \wedge R_2(v, w)) \rightarrow R_2(u, w)\big) \\
&\qquad \wedge \big((R_2(u, v) \wedge R_2(v, u) \leftrightarrow R_1(u, v)\big) \\
&\qquad \wedge \big(R_2(u, v) \vee R_2(v, u)\big).
\end{aligned}$$

(ii) If we say '$(X, \le)$ is well-ordered' we are saying just that $(X; =, \le;) \vDash \phi_2$, where ϕ_2 is the Π^1_1 formula

$$\begin{aligned}&\forall S(\exists a S(a)) \rightarrow\\&\quad \exists b\big(S(b) \wedge \forall c(S(c) \rightarrow R_2(b,c))\big)\\&\wedge \phi_1.\end{aligned}$$

(iii) If we say '$(X, \leq)$ is well-ordered and its order type is a limit cardinal', we are saying that $(X; =, \leq ;) \vDash \phi_3$, where ϕ_3 is the formula

$$\begin{aligned}&\exists G \forall F \forall x \exists y\\&\quad\big(\chi(G) \wedge G(x,y)\\&\qquad \wedge \chi(F) \rightarrow (\exists z R_2(z,y) \wedge (\forall t R_2(t,x) \rightarrow \neg F(t,z)))\big)\\&\wedge \phi_2,\end{aligned}$$

$\chi(F)$ being an abbreviation for the formula

$$\forall u \forall v \forall w (F(u,v) \wedge F(u,w)) \rightarrow R_1(v,w).$$

If we move the quantifier $\forall S$ in ϕ_2 up with $\forall F$, we get a Σ^1_2 formula logically equivalent to ϕ_3.

A4C Indescribability (a) Let $n \in \mathbb{N}$. An ordinal α is **weakly Π^1_n-indescribable** if it is not 0 and whenever $C_1, \dots, C_k$ are relations on α, $\xi_1, \dots, \xi_m < \alpha$ and ϕ is a Π^1_n formula such that

$$(\alpha; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi,$$

then there is an ordinal $\beta < \alpha$ such that

$$(\beta; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi.$$

I will say that α is **weakly Π^2_0-indescribable** if it is weakly Π^1_n-indescribable for every $n \in \mathbb{N}$.

(b) If α is weakly Π^1_n-indescribable and ϕ is a Π^1_n formula, $C_1, \dots, C_k$ are relations on α, $\xi_1, \dots, \xi_m$ are ordinals less than α, and

$$(\alpha; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi,$$

then we have a non-empty set

$$A(\phi, C_1, \dots, C_k, \xi_1, \dots, \xi_m) = \{\beta : \beta < \alpha,\ (\beta; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi\} \subseteq \alpha.$$

The intersection of two such sets is another (because $\phi \wedge \psi$ is logically equivalent to a Π^1_n formula whenever ϕ and ψ are), so they generate a filter on α; this is the **Π^1_n-filter** of α. The dual ideal is the **Π^1_n-ideal** of α.

In the same way, if α is weakly Π^2_0-indescribable then it has a **Π^2_0-filter**, which is just the union of its Π^1_n-filters, and a dual **Π^2_0-ideal**.

It will be convenient to say that if α is an ordinal which is not weakly Π^1_n-indescribable, then its 'Π^1_n-ideal' is $\mathcal{P}\alpha$; and similarly that the Π^2_0-ideal of α is $\mathcal{P}\alpha$ if α is not weakly Π^2_0-indescribable.

Remark A subset of α belongs to the Π^1_n-filter on α iff it is Π^1_n-enforceable at α in the terminology of Lévy 71. Baumgartner Taylor & Wagon 77 use the phrase 'ordinal Π^1_1-indescribable' where I write 'weakly Π^1_1-indescribable'.

A4D Proposition (a) A weakly Π^1_0-indescribable ordinal θ is an uncountable regular cardinal, and every closed unbounded set in θ belongs to the Π^1_0-filter of θ.

(b) $\mathfrak{c}$ is not weakly Π^1_1-indescribable.

proof (a)(i) If α is a successor ordinal then

$$(\alpha; =; \alpha - 1) \vDash R_1(x_1, x_1),\ (\beta; =; \alpha - 1) \nvDash R_1(x_1, x_1)\ \ \forall\ \beta < \alpha.$$

(ii) If $\alpha = \omega$ then

$$(\alpha; <; 0) \vDash \forall y_1 \exists y_2\, R_1(y_1, y_2)$$

but

$$(\beta; <; 0) \not\models \forall y_1 \exists y_2\, R_1(y_1, y_2)$$

for every $\beta < \alpha$.

(iii) If $\omega \le \mathrm{cf}(\alpha) < \alpha$ take $\xi < \alpha$ and $f : \xi \to \alpha$ such that $f[\xi]$ is cofinal with α; let F be the corresponding binary relation, so that $F(\eta, \zeta)$ iff $f(\eta) = \zeta$; then

$$(\alpha; <, F; \xi) \models \forall y(R_1(y, x_1) \to \exists z\, R_2(y, z)),$$

but

$$(\beta; <, F; \xi) \not\models \forall y(R_1(y, x_1) \to \exists z\, R_2(y, z)),$$

for every $\beta < \alpha$.

(iv) Thus θ cannot be either a successor, nor ω, nor of cofinality less than θ, and must be an uncountable regular cardinal.

Let C be any closed unbounded set in θ. Write $F(\xi, \eta)$ for '$\xi \le \eta \in C$'. Then $(\theta; F; 0) \models \forall y \exists z\, R_1(y, z)$, and if $\alpha < \theta$ is such that $(\alpha; F; 0) \models \forall y \exists z\, R_1(y, z)$ then $\alpha \in C$. So C belongs to the Π^1_0-filter of θ.

(b) Let $f : \mathfrak{c} \to \mathcal{P}\omega$ be a surjection, and write $F(\xi, \eta)$ for '$\eta \in f(\xi)$'. Let ϕ be the formula

$$\forall S \exists y \forall z (R_1(z, x_1) \to (S(z) \leftrightarrow R_2(y, z))).$$

Then for $\alpha \le \mathfrak{c}$,

$$(\alpha; <, F; \omega) \models \phi$$

means just that $f[\alpha] = \mathcal{P}\omega$, which is true iff $\alpha = \mathfrak{c}$. Because ϕ is a Π^1_1 formula, $\mathfrak{c}$ is not weakly Π^1_1-indescribable.

Remarks (a) In fact, an ordinal α is weakly Π^1_0-indescribable iff it is a regular uncountable cardinal, and in this case its Π^1_0-filter is just the filter generated by the closed unbounded sets. See Lévy 71, Theorem 6, or Hanf & Scott 61.

(b) Of course the argument above shows that 2^λ is not weakly Π^1_1-indescribable for any cardinal λ. More generally, no cardinal power λ^θ can be weakly Π^1_1-indescribable if $1 < \lambda < \lambda^\theta$.

A4E For the next theorem we need names for some relations on the class On of ordinals. (i) Let $p : \mathrm{On} \times \mathrm{On} \to \mathrm{On}$ be the bijection corresponding to the familiar well-ordering of $\mathrm{On} \times \mathrm{On}$: $p(\xi, \eta) < p(\xi', \eta')$ iff *either* $\max(\xi, \eta) < \max(\xi', \eta')$ *or* $\max(\xi, \eta) = \max(\xi', \eta')$ and $\xi < \xi'$ *or* $\xi = \xi'$ and $\eta < \eta'$. Let P_1 be the corresponding ternary relation on On, so that $P(\xi, \eta, \zeta)$ iff $p(\xi, \eta) = \zeta$. (ii) Let P_2, P_3 be the ternary relations on On corresponding to ordinal addition and multiplication. (iii) Let q_1 and q_2 be the projections of p^{-1}, so that $\xi = p(q_1(\xi), q_2(\xi))$ for every $\xi \in \mathrm{On}$. Let P_4 be the ternary relation corresponding to the function $(l, \xi) \mapsto q_2(q_1^l(\xi)) : \omega \times \mathrm{On} \to \mathrm{On}$.

Theorem For each $n \ge 1$ there is a Π^1_n formula ϕ such that whenever ψ is a Π^1_n formula, $C_1, \dots, C_k$ are relations on On, $\xi_1, \dots, \xi_m$ are ordinals, then there is a one-place relation C on On such that for every infinite cardinal α

$$(\alpha; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \models \psi \iff (\alpha; <, P_1, P_2, P_3, P_4, C;) \models \phi.$$

proof Lévy 71, Theorem 8.

A4F Theorem Let κ be a weakly Π^1_n-indescribable cardinal, where $n \ge 1$. Then the Π^1_n-filter on κ is a normal filter closed under Mahlo's operation.

proof Lévy 71, Theorems 9 and 15.

A4G Theorem Let κ be a weakly Π^1_{n+1}-indescribable cardinal, where $n \in \mathbb{N}$. Then the set of weakly Π^1_n-indescribable cardinals less than κ belongs to the Π^1_{n+1}-filter on κ.

proof Lévy 71, Theorem 16b.

A4H Theorem Let κ be an infinite cardinal and A a subset of κ. Then the following are equivalent:

(a) A does not belong to the Π^1_1-ideal of κ (so that, in particular, κ is weakly Π^1_1-indescribable);

(b) A is Π^1_1-fully stationary in the sense of 4D.

proof (a)⇒(b)(i) ? Suppose, if possible, that (a) is true but (b) is false. Let $\langle f_\xi\rangle_{\xi<\kappa}$ be a family of regressive functions on κ witnessing not-(b); that is, such that there is no uniform ultrafilter $\mathcal{F}$ on κ, containing A, for which $\lim_{\alpha\to\mathcal{F}} f_\xi(\alpha)$ exists for every ξ. Because κ is regular (A4Da), this means that for any $g\in\kappa^\kappa$ the family

$$\{A\setminus\zeta:\zeta<\kappa\}\cup\{f_\xi^{-1}[\{g(\xi)\}]:\xi<\kappa\}$$

is not included in any ultrafilter on κ, that is, does not have the finite intersection property. So we have

$$\forall\ g\in\kappa^\kappa\ \exists\ \zeta<\kappa,\ I\in[\kappa]^{<\omega}\text{ such that }\forall\ \eta\in A\setminus\zeta\ \exists\ \xi\in I\text{ such that }g(\xi)\ne f_\xi(\eta).\qquad(*)$$

Evidently (*) can be coded as a Π^1_1 formula; the details are as follows. Let $h:\kappa\to[\kappa]^{<\omega}$ be any surjection, and write $E(\zeta,\eta)$ for '$\zeta\le\eta\in A$', $H(\xi,\gamma,\eta,\delta)$ for '$\xi\in h(\gamma)$ and $f_\xi(\eta)\ne\delta$'. Let ϕ be the formula

$$\begin{aligned}&\forall S(\forall u\exists vS(u,v))\to\\&\quad\exists z\exists c\forall y\big(R_1(z,y)\to\\&\qquad\exists x\exists dS(x,d)\wedge R_2(x,c,y,d)\big).\end{aligned}$$

Then (*) says just that

$$(\kappa;E,H;)\vDash\phi.$$

Accordingly the set

$$B=\{\alpha:\alpha<\kappa,\ (\alpha;E,H;)\vDash\phi\}$$

belongs to the Π^1_1-filter $\mathcal{W}$ of κ.

(ii) Let C be the set of non-zero limit ordinals $\alpha<\kappa$ such that $h[\alpha]=[\alpha]^{<\omega}$. Then C is a closed unbounded set in κ so $C\in\mathcal{W}$, by A4Da. Consequently $A\cap B\cap C$ is stationary. But if $\alpha\in B\cap C$ then

$$\begin{aligned}&\forall\ g\in\alpha^\alpha\ \exists\ \zeta<\alpha,I\in[\alpha]^{<\omega}\text{ such that}\\&\qquad\qquad\forall\ \eta\in\alpha\cap A\setminus\zeta\ \exists\ \xi\in I\text{ such that }g(\xi)\ne f_\xi(\eta).\end{aligned}$$

It will be helpful to put this into a logically equivalent form. For $0<\alpha<\kappa$ let $g_\alpha:\kappa\to\alpha$ be given by setting $g_\alpha(\xi)=f_\xi(\alpha)$ for every $\xi<\kappa$. Then for $\alpha\in B\cap C$ we have

$$\forall\ g\in\alpha^\alpha\ \exists\ \zeta<\alpha,\ I\in[\alpha]^{<\omega}\text{ such that }g_\eta\restriction I\ne g\restriction I\ \forall\ \eta\in\alpha\cap A\setminus\zeta.$$

In particular, there are $\zeta_\alpha<\alpha$ and $I_\alpha\in[\alpha]^{<\omega}$ such that $g_\eta\restriction I_\alpha\ne g_\alpha\restriction I_\alpha$ for every $\eta\in\alpha\cap A\setminus\zeta_\alpha$. Because α is a limit ordinal we may take it that $I_\alpha\cup g_\alpha[I_\alpha]\subseteq\zeta_\alpha$.

Now $A\cap B\cap C$ is stationary, so by the pressing-down lemma there is a $\zeta<\kappa$ such that $A_1=\{\alpha:\alpha\in A\cap B\cap C,\ \zeta_\alpha=\zeta\}$ is stationary; next, $g_\alpha\restriction I_\alpha\subseteq\zeta\times\zeta$ for every $\alpha\in A_1$, so there is a finite function e such that $A_2=\{\alpha:\alpha\in A_1,\ g_\alpha\restriction I_\alpha=e\}$ is stationary. Of course $I_\alpha=\mathrm{dom}(e)=I$ say, for every $\alpha\in A_2$. Now take $\alpha,\ \beta\in A_2$ such that $\zeta<\beta<\alpha$. Then $\beta\in\alpha\cap A\setminus\zeta_\alpha$, so $g_\beta\restriction I_\alpha\ne g_\alpha\restriction I_\alpha$; but on the other hand $g_\beta\restriction I_\alpha=e=g_\alpha\restriction I_\alpha$. **X**

So (a)⇒(b).

(b)⇒(a) ? Suppose, if possible, that (b) is true but that (a) is false; either κ is not weakly Π^1_1-indescribable or $\kappa\setminus A$ belongs to the Π^1_1-filter on κ. In either case there must be a Π^1_1 formula ϕ and relations $C_1,\dots,C_k$, ordinals $\xi_1,\dots,\xi_m$ such that

$$(\kappa;C_1,\dots,C_k;\xi_1,\dots,\xi_m)\vDash\phi,\ (\alpha;C_1,\dots,C_k;\xi_1,\dots,\xi_m)\nvDash\phi\ \forall\ \alpha\in A.$$

We may suppose that ϕ is of the form

$$\forall R_{k+1}\dots R_{k+r}\exists x_{m+1}\forall x_{m+2}\dots\exists x_{m+2s-1}\forall x_{m+2s}\psi$$

where ψ has no quantifiers, since ϕ is surely logically equivalent to such a formula. Set $A'=\{\alpha:\alpha\in A,\ \xi_j<\alpha\ \forall\ j\le m\}$. For $\alpha\in A'$, saying that $(\alpha;C_1,\dots,C_k;\xi_1,\dots,\xi_m)\nvDash\phi$ means just that there are relations $D_{\alpha1},\dots,D_{\alpha r}$ on α and functions $f_{\alpha1}:\alpha\to\alpha$, $f_{\alpha2}:\alpha^2\to\alpha,\dots,f_{\alpha s}:\alpha^s\to\alpha$ such that whenever $\eta_1,\dots,\eta_s<\alpha$ then

$$(\alpha;C_1,\dots,C_k,D_{\alpha1},\dots,D_{\alpha r};\xi_1,\dots,\xi_m,\eta_1,f_{\alpha1}(\eta_1),\dots,\eta_s,f_{\alpha s}(\eta_1,\dots,\eta_s))\vDash\neg\psi.$$

Choose such $D_{\alpha i}$, $f_{\alpha j}$ for $\alpha \in A'$. For $\bar\eta = (\eta_1, \dots, \eta_j) \in \kappa^j$, where $1 \le j \le s$, set

$$\begin{aligned} g_{\bar\eta}(\alpha) &= f_{\alpha j}(\eta_1, \dots, \eta_j) \text{ if } \max(\eta_1, \dots, \eta_j) < \alpha \in A', \\ &= 0 \text{ otherwise.} \end{aligned}$$

Set $G = \{g_{\bar\eta} : \bar\eta \in \bigcup_{1 \le j \le s} \kappa^j\} \in [\mathrm{Regr}(\kappa)]^{\le\kappa}$.

Now (b) tells us that there is a uniform ultrafilter $\mathcal{F}$ on κ, containing A, such that $\lim_{\alpha\to\mathcal{F}} g(\alpha)$ exists for every $g \in G$. Because $\mathcal{F}$ is uniform, $A' \in \mathcal{F}$. Set

$$h_j(\bar\eta) = \lim_{\alpha\to\mathcal{F}} g_{\bar\eta}(\alpha) = \lim_{\alpha\to\mathcal{F}} f_\alpha(\bar\eta)$$

for $j \le s$, $\bar\eta \in \kappa^j$. For $i \le r$, write $D_i(\bar\eta)$ for '$\{\alpha : D_{\alpha i}(\bar\eta) \in \mathcal{F}\}$'. Now for any formula χ without quantifiers, and any $\eta_1, \dots, \eta_s < \kappa$, we have

$$(\kappa; C_1, \dots, C_k, D_1, \dots, D_r; \xi_1, \dots, \xi_m, \eta_1, h_1(\eta_1), \dots, \eta_s, h_s(\eta_1, \dots, \eta_s)) \vDash \chi$$

if and only if the set of $\alpha < \kappa$ for which

$$(\alpha; C_1, \dots, C_k, D_{\alpha 1}, \dots, D_{\alpha r}; \xi_1, \dots, \xi_m, \eta_1, f_{\alpha 1}(\eta_1), \dots, f_{\alpha s}(\eta_1, \dots, \eta_s)) \vDash \chi$$

belongs to $\mathcal{F}$; this is an easy induction on the length of χ. In particular, it is valid when χ is $\neg\psi$. But this means that

$$(\kappa; C_1, \dots, C_k, D_1, \dots, D_r; \xi_1, \dots, \xi_m, \eta_1, h_1(\eta_1), \dots, \eta_s, h_s(\eta_1, \dots, \eta_s)) \vDash \neg\psi$$

for all $\eta_1, \dots, \eta_s$, so that $D_1, \dots, D_r$ and $h_1, \dots, h_s$ witness that

$$(\kappa; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \nvDash \phi,$$

contrary to hypothesis. **X**

Thus (b)⇒(a).

Remark This is due to Kunen and J.E.Baumgartner; in the form here it is taken from Fremlin & Kunen n87. The point is that it puts meat onto Theorem A4E by actually exhibiting a universal Π^1_1 formula ϕ. In §4 above I show how it can be used to investigate weakly Π^1_1-indescribable cardinals without further recourse to the logical characterization. Compare 'Baumgartner's principle' as described in Erdös Hajnal Máté & Rado 84, 30.6 and 31.3.

It seems that no corresponding combinatorial characterization of weakly Π^1_2-indescribable cardinals is known.

A4I Theorem Let κ be an infinite cardinal, ϕ a formula of $\mathcal{L}$, $C_1, \dots, C_k$ relations on κ, $\xi_1, \dots, \xi_m$ members of κ. For each set I let $\mathfrak{A}_I$ be the measure algebra of $\{0,1\}^I$ and $\mathbb{P}_I$ the p.o.set $\mathfrak{A}_I \setminus \{\mathbf{0}\}$. Now if I and J are any sets both of cardinal strictly greater than κ, and $\beta \le \kappa$,

$$\begin{aligned} &\Vdash_{\mathbb{P}_I} (\check\beta; \check C_1, \dots, \check C_k; \check\xi_1, \dots, \check\xi_m) \vDash \phi \\ &\qquad\iff \Vdash_{\mathbb{P}_J} (\check\beta; \check C_1, \dots, \check C_k; \check\xi_1, \dots, \check\xi_m) \vDash \phi. \end{aligned}$$

proof I begin with some preliminary remarks. It will be convenient to suppose that I and J are disjoint; this is legitimate because $\mathbb{P}_I$ is determined up to isomorphism by the cardinal of I. Recall that if $K \subseteq I$ then we may regard $\mathbb{P}_K$ as a subset of $\mathbb{P}_I$, corresponding to the canonical map from $\{0,1\}^I$ to $\{0,1\}^K$ and the induced measure-preserving homomorphism from $\mathfrak{A}_K$ to $\mathfrak{A}_I$; and that we may also think of $\mathbb{P}_I$ as an iteration $\mathbb{P}_K * \mathbb{P}_{I\setminus K}$ (Kunen 84, 3.13), if you will allow me to write $\mathbb{P}_{I\setminus K}$ rather than $\dot{\mathbb{P}}_{\check I\setminus\check K}$ for what is really a $\mathbb{P}_I$-name.

Now for the main argument, which proceeds by induction on the length of ϕ.

(a) If ϕ has no quantifiers the result is trivial, since, for instance,

$$\Vdash_{\mathbb{P}_I} \check C_i(\check\xi_{j_1}, \dots, \check\xi_{j_n}) \iff C_i(\xi_{j_1}, \dots, \xi_{j_n}),$$

$$\Vdash_{\mathbb{P}_I} \check\xi_j < \check\beta \iff \xi_j < \beta.$$

(b) Suppose that ϕ is of the form $\exists S\psi$ and that

$$\Vdash_{\mathbb{P}_I} (\check{\beta};\dots) \vDash \phi.$$

Then there must be a $\mathbb{P}_I$-name $\dot{R}$ for a relation on β such that

$$\Vdash_{\mathbb{P}_I} (\check{\beta};\dots,\dot{R},\dots) \vDash \psi.$$

Because $\mathbb{P}_I$ is ccc, the P_I-name $\dot{R}$ can be described in terms of not more than $\max(\omega,\kappa)=\kappa$ members of $\mathbb{P}_I$; now there is a $K \subseteq I$ of cardinal at most κ such that $\mathbb{P}_K$, regarded as a subset of $\mathbb{P}_I$, contains all of these; so that we may think of $\dot{R}$ as a $\mathbb{P}_K$-name.

Let $L \subseteq J$ be a set of the same size as K. Then any bijection between K and L gives rise to an isomorphism between $\mathbb{P}_K$ and $\mathbb{P}_L$. Let $\dot{S}$ be a $\mathbb{P}_L$-name for a relation on β corresponding to $\dot{R}$ under such an isomorphism. If we regard $\mathbb{P}_I$ as the iteration $\mathbb{P}_K * \mathbb{P}_{I\setminus K}$ then we have

$$\Vdash_{\mathbb{P}_K} \bigl(\Vdash_{\mathbb{P}_{I\setminus K}} (\check{\beta};\dots,\dot{R},\dots) \vDash \psi\bigr).$$

So

$$\Vdash_{\mathbb{P}_L} \bigl(\Vdash_{\mathbb{P}_{I\setminus K}} (\check{\beta};\dots,\dot{S},\dots) \vDash \psi\bigr).$$

At this point we move to the intermediate model $V^{\mathbb{P}_L}$. From this standpoint $\dot{S}$ represents a fixed relation on β. Also $I \setminus K$ and $J \setminus L$ both have cardinals greater than κ. So we can use the inductive hypothesis to see that

$$\Vdash_{\mathbb{P}_L} \bigl(\Vdash_{\mathbb{P}_{J\setminus L}} (\check{\beta};\dots,\dot{S},\dots) \vDash \psi\bigr),$$

that is,

$$\Vdash_{\mathbb{P}_J} (\check{\beta};\dots,\dot{S},\dots) \vDash \psi.$$

So $\dot{S}$ witnesses that

$$\Vdash_{\mathbb{P}_J} (\check{\beta};\dots) \vDash \phi,$$

and we have

$$\Vdash_{\mathbb{P}_I} (\check{\beta};\dots) \vDash \phi \;\Rightarrow\; \Vdash_{\mathbb{P}_J} (\check{\beta};\dots) \vDash \phi;$$

of course the reverse implication is equally valid.

(c) If ϕ is of the form $\forall R\psi$ we can argue similarly; given

$$\Vdash_{\mathbb{P}_I} (\check{\beta};\dots) \vDash \phi$$

and a $\mathbb{P}_J$-name $\dot{S}$ for a relation on β, we express $\dot{S}$ as a $\mathbb{P}_L$-name for some $L \subseteq J$ of size at most κ, copy this into a $\mathbb{P}_K$-name $\dot{R}$ for $K \subseteq I$, and use the inductive hypothesis on

$$\Vdash_{\mathbb{P}_{I\setminus K}} (\check{\beta};\dots,\dot{R},\dots) \vDash \psi$$

in $V^{\mathbb{P}_J}$ to see that

$$\Vdash_{\mathbb{P}_J} (\check{\beta};\dots,\dot{S},\dots) \vDash \psi,$$

as required.

(d) If ϕ is of the form $\forall x\psi$ or $\exists x\psi$ the same arguments apply, taking K and L to be countable if we wish.

A4J Proposition Let $n \in \mathbb{N}$ and let κ be a weakly Π^1_n-indescribable cardinal such that the cardinal power κ^ω is equal to κ. Let $\lambda > \kappa$ be any cardinal; let $\mathfrak{A}_\lambda$ be the measure algebra of $\{0,1\}^\lambda$ and $\mathbb{P}_\lambda$ the p.o.set $\mathfrak{A}_\lambda \setminus \{\mathbf{0}\}$. Then

$$\Vdash_{\mathbb{P}_\lambda} \check{\kappa} \text{ is weakly } \Pi^1_n\text{-indescribable.}$$

Remark This is due to Kunen. I omit the proof because I shall not rely on it. A sketch of the argument for the corresponding theorem for Cohen reals may be found in Kunen 71.

A4K Π^2_1-indescribability For the next step we need a higher-order language. Let $\mathcal{L}_3$ be the extension of $\mathcal{L}$ in which third-order relational variables $\mathcal{R}$ are added, with corresponding atomic formulae $\mathcal{R}(u_1,\dots,u_l)$, where each u_i is either a first-order or second-order variable. In $\mathcal{L}_3$, the Π^2_0 formulae are those in which all quantifiers govern first- and second-order variables, and the Π^2_1 formulae are those of the form

$$\forall \mathcal{R}_1 \dots \forall \mathcal{R}_k \psi,$$

where ψ is a Π^2_0 formula. Now a non-zero ordinal κ is **weakly Π^2_1-indescribable** if whenever ϕ is a Π^2_1 formula in $\mathcal{L}_3$ *with no free third-order variables* and $C_1, \dots, C_k$ are relations, $\xi_1, \dots, \xi_m$ are members of κ such that

$$(\kappa; ; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi,$$

then there is an $\alpha < \kappa$ such that

$$(\alpha; ; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi.$$

For such κ, the sets

$$\{\alpha : (\alpha; ; C_1, \dots, C_k; \xi_1, \dots, \xi_m) \vDash \phi\}$$

where ϕ, $C_1, \dots$ are such that $(\kappa; ; C_1, \dots) \vDash \phi$ and ϕ is Π^2_1, generate the **Π^2_1-filter** on κ. Finally, κ is **Π^2_1-indescribable** if it is weakly Π^2_1-indescribable and $2^\lambda < \kappa$ for every $\lambda < \kappa$.

A4L Now I can state one of the basic results of the theory of large cardinals, due originally to Hanf & Scott 61. I omit the proof, as it has appeared more than once in hard covers, but I recommend comparing it with the arguments given in these notes for the corresponding results for atomlessly-measurable cardinals (4P *et seq.*).

Theorem If κ is two-valued-measurable, it is Π^2_1- indescribable, and its Π^2_1-filter is included in its rvm filter.

proof See Drake 74, §9.3, Jech 78, p. 385, Lemma 32.2 or Kanamori & Magidor 78, §I.4.

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Problems

I collect here some of the questions which arise more or less naturally from the work above and seem to be open.

P1 Construction of atomlessly-measurable cardinals The most important problem is surely something like this.

> (P1) Let N be a model of ZFC and $\kappa \in N$ an atomlessly-measurable cardinal in N. Does it follow that there are an inner model $M \subseteq N$, containing κ, such that κ is a two-valued-measurable cardinal in M, and an M-generic filter G in a random real p.o.set $\mathbb{P} \in M$ such that $G \in N$ and $N \cap \mathcal{P}\kappa \subseteq M[G]$?

Put in this form, it is hard to believe in the possibility of an affirmative answer (though note Theorem 2D). But so long as the question remains open, we have no way of proving consistency results for atomlessly -measurable cardinals except through Solovay's construction in Theorem 2C above. This construction is not wholly inflexible (see, for instance, 2E, 2I, 2L, 4Lb). But atomlessly-measurable cardinals built in this way share a vast number of special properties. Some are known to be possessed by atomlessly-measurable cardinals in general; many of the results in these notes were suggested by this approach. But others seem inaccessible to present techniques. By and large, positive answers to the other problems here could be taken as (weakly) suggesting a positive answer to P1; while negative answers would often imply a negative answer to P1.

P2 Measure algebras of atomlessly-measurable cardinals The Gitik-Shelah theorem (3F) tells us that if κ is an atomlessly-measurable cardinal and ν is a witnessing probability on κ then the Maharam type λ of $(\kappa, \mathcal{P}\kappa, \nu)$ is at least $\min(2^\kappa, \kappa^{(+\omega)})$; and of course it cannot be greater than 2^κ. So if $2^\kappa < \kappa^{(+\omega)}$ then $\lambda = 2^\kappa$. We get a scrap more information in 7Q; if γ is the least cardinal such that $2^\gamma = 2^\delta$ for $\gamma \leq \delta < \kappa$, then $\gamma < \kappa$ and $\lambda^\gamma = 2^\kappa$, so that either $2^\gamma = 2^\kappa$ or $2^\gamma < \lambda$. Of course this still does not specify λ completely. So we may ask

> (P2a) Is there a combinatorial characterization of λ?

An affirmative answer would have a variety of consequences; not least, that any witnessing probability on κ would have the same Maharam type – equivalently, that every witnessing probability on κ would be Maharam homogeneous (3L). So a less ambitious question is

(P2b) If κ is an atomlessly-measurable cardinal and ν is a witnessing probability on κ, must $(\kappa, \mathcal{P}\kappa, \nu)$ be Maharam homogeneous?

Conceivably it makes a difference if ν is normal; so I add

(P2c) If κ is an atomlessly-measurable cardinal and ν is a normal witnessing probability on κ, must $(\kappa, \mathcal{P}\kappa, \nu)$ be Maharam homogeneous?

It may be that a negative answer can be achieved using Solovay's construction on an appropriately complex two-valued-measurable cardinal. For in 2H we saw that if κ is a two-valued-measurable cardinal and $\mathbb{P}$ is a random real p.o.set, then the possible Maharam types of $(\kappa, \mathcal{P}\kappa, \tilde{\nu})$ in $V^{\mathbb{P}}$ are determined by the cardinals $\mathrm{Tr}_{\mathcal{I}}(\kappa; \lambda)$. So I ask

(P2d) If κ is a two-valued-measurable cardinal with two κ-additive ultrafilters $\mathcal{F}_1$, $\mathcal{F}_2$ and corresponding maximal ideals $\mathcal{I}_1$, $\mathcal{I}_2$, and if $\lambda \geq \kappa$ is a cardinal, are $\mathrm{Tr}_{\mathcal{I}_1}(\kappa; \lambda)$ and $\mathrm{Tr}_{\mathcal{I}_2}(\kappa; \lambda)$ necessarily equal?

Of course $\mathrm{Tr}_{\mathcal{I}_1}(\kappa; \kappa) = 2^\kappa = \mathrm{Tr}_{\mathcal{I}_2}(\kappa; \kappa)$. Note also that in the langauage of JECH 78, §28, $\mathrm{Tr}_{\mathcal{I}_i}(\kappa; \lambda) = \#(j_{\mathcal{F}_i}(\lambda))$, where $j_{\mathcal{F}_i} : V \to V^\kappa/\mathcal{F}_i$ is the standard elementary embedding.

The construction of 3Mb is relevant to P2a-P2b above. For take $\gamma < \kappa$ such that $2^\gamma = 2^\delta$ for $\gamma \leq \delta < \kappa$ (7P). Then $\mathrm{Tr}_{[\kappa]^{<\kappa}}(\kappa; 2^\gamma) = 2^\kappa$. Assume that $2^\gamma < 2^\kappa$. In this case, if ν is any Maharam homogeneous normal witnessing probability on κ, with Maharam type λ, then (as remarked above) $2^\gamma < \lambda$, so $\mathrm{Tr}_{\mathcal{N}_\nu}(\kappa; \lambda) = 2^\kappa$, and there is a witnessing probability ν_1 on κ of Maharam type 2^κ. We may therefore ask

(P2e) If κ is an atomlessly-measurable cardinal with witnessing probability ν, and $2^\delta < 2^\kappa$ for every $\delta < \kappa$, does it follow that the Maharam type of $(\kappa, \mathcal{P}\kappa, \nu)$ is 2^κ?

As a special case of this we have

(P2f) If $\mathfrak{c}$ is atomlessly-measurable, with witnessing probability ν, does it follow that the Maharam type of $(\mathfrak{c}, \mathcal{P}\mathfrak{c}, \nu)$ is $2^\mathfrak{c}$?

P3 Indescribability In 4R we saw that if κ is an atomlessly-measurable cardinal constructed by Solovay's method, then *either* κ is not weakly Π^1_1-indescribable, *or* κ is weakly Π^2_0-indescribable and its Π^2_0-filter is included in its rvm filter. We also have a characterization of weak Π^1_1-indescribability in terms of covering numbers (6L). This begs many questions. For instance:

(P3a) If κ is a weakly Π^1_1-indescribable atomlessly-measurable cardinal, must it be weakly Π^2_0-indescribable?

(P3b) If κ is a weakly Π^1_n-indescribable atomlessly-measurable cardinal, must its Π^1_n-filter be included in its rvm filter?

(For $n = 0$ and $n = 1$, yes; see 6L.)

(P3c) If κ is an atomlessly-measurable cardinal strictly less than $\mathfrak{c}$, must it be weakly Π^1_1-indescribable?

P4 The cardinals of analysis A great many cardinals between ω_1 and $\mathfrak{c}$ have been named and studied by analysts; see VAUGHAN 90. The existence of an atomlessly-measurable cardinal is known to have a dramatic effect on the patterns formed by these cardinals. For instance, if κ is an atomlessly-measurable cardinal, then

(i) $\mathrm{non}(\mathbb{R}, \mathcal{N}) = \omega_1$, where $\mathcal{N}$ is the ideal of Lebesgue negligible sets (6Ga); it follows that $\mathrm{cov}(\mathbb{R}, \mathcal{M}) = \mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{N}) = \mathfrak{p} = \omega_1$, where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$ and $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centered}}$, as in FREMLIN 84 and DOUWEN 84; see FREMLIN 85 or BARTOSZYŃSKI & JUDAH P90.

(ii) $\mathfrak{b} < \kappa$, where $\mathfrak{b} = \mathrm{add}(\mathbb{N}^\mathbb{N}/\mathcal{F}_0)$, writing $\mathcal{F}_0$ for the Fréchet filter on $\mathbb{N}$ (5Lb).

(iii) $\mathrm{cf}(\mathfrak{d}) \neq \kappa$, where $\mathfrak{d} = \mathrm{cf}(\mathbb{N}^\mathbb{N})$ (7Ka).

(iv) $\mathrm{cov}(\mathbb{R}, \mathcal{N}) \geq \kappa$ (in fact, $\mathrm{cov}(X, \mathcal{N}_\mu) \geq \kappa$ for any non-trivial Radon measure space (X, μ)) (6B). Consequently $\kappa \leq \mathrm{non}(\mathbb{R}, \mathcal{M}) \leq \mathrm{cf}(\mathcal{M}) \leq \mathrm{cf}(\mathcal{N})$ (FREMLIN 85 or BARTOSZYŃSKI & JUDAH P90 again).

Once again, there are obvious gaps. The principal one seems to be

(P4a) If κ is an atomlessly-measurable cardinal, must $\mathfrak{d}$ be less than κ?

Concerning covering numbers, we have

(P4b) If there is an atomlessly-measurable cardinal, must $\operatorname{cov}(\mathbb{R},\mathcal{N})$ be exactly $\mathfrak{c}$?

(see 2Hb). Then concerning the numbers non(.,.) we have

(P4c) If κ is an atomlessly-measurable cardinal, must $\operatorname{non}(\{0,1\}^\kappa,\mathcal{N}_{\mu_\kappa})$ be ω_1?

Because we know that $\operatorname{non}(\{0,1\}^\theta,\mathcal{N}_{\mu_\theta}) = \omega_1$ for $\theta < \kappa$ (6G), we can apply results from §4; it is not hard to construct a Σ^1_2 formula ϕ such that, for ordinals $\alpha > \omega_1$, $(\alpha;;\omega,\omega_1) \vDash \phi$ iff $\operatorname{non}(\{0,1\}^\alpha,\mathcal{N}_{\mu_\alpha}) = \omega_1$. Now if $\operatorname{non}(\{0,1\}^\kappa,\mathcal{N}_{\mu_\kappa}) > \omega_1$, then $(\kappa;;\omega,\omega_1) \vDash \neg\phi$ while $(\alpha;;\omega,\omega_1) \nvDash \neg\phi$ for every $\alpha < \kappa$, and κ is not weakly Π^1_2-indescribable.

An associated question comes from 6F:

(P4d) If κ is an atomlessly-measurable cardinal, must there be a set $A \subseteq \mathbb{R}$, of cardinal κ, such that no uncountable subset of A is Lebesgue negligible?

Again, we get a positive answer if κ is weakly Π^1_2-indescribable, using 6F.

A curious question arises from 3Bb.

(P4e) Let κ be an atomlessly-measurable cardinal and A a subset of $\mathbb{R}$. Must there be a set $B \subseteq A$ such that $\#(B) < \kappa$ and $\mu_L^* B = \mu_L^* A$, writing μ_L for Lebesgue measure?

If $\#(A) \le \kappa$, yes, by 3Bb; but the general question seems to be open.

A question closely related in form, if not in content, to P4d, is

(P4f) If κ is an atomlessly-measurable cardinal, must there be ccc partially ordered sets P and Q such that $\mathrm{S}(P \times Q) > \kappa$?

As before, this corresponds to a Σ^1_2 formula, so that 7D shows that we shall have an affirmative answer if κ is weakly Π^1_2-indescribable. The same route leads from 7F to

(P4g) If κ is an atomlessly-measurable cardinal, must there be an ω_1-entangled subset of $\mathbb{R}$ of cardinal κ?

Returning to named cardinals, let $\mathfrak{a}$ be the smallest cardinal of any infinite maximal disjoint family in the algebra $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$; then $\mathfrak{b} \le \mathfrak{a}$ (Douwen 84, Theorem 3.1a); but

(P4h) If κ is atomlessly-measurable, must $\mathfrak{a}$ be less than κ?

P5 Cofinalities The remarkable results from Shelah's pcf theory which give us 7Ka-d leave some natural questions open. The most important has already been listed as P4a. But it seems that even the following bold conjecture might be true:

(P5a) Let κ be an atomlessly-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets where $\lambda < \kappa$ and $\operatorname{cf}(P_\zeta) < \kappa$ and $\operatorname{add}(P_\zeta) > \omega$ for every $\zeta < \lambda$. Does it follow that $\operatorname{cf}(\prod_{\zeta<\lambda} P_\zeta) < \kappa$?

If $\kappa = \mathfrak{c}$, yes (7Kf); if $\operatorname{add}(P_\zeta) > \lambda$ for every ζ, yes. A positive answer to P9b below would settle the general question. A natural special case of P5a is

(P5b) Let κ be an atomlessly-measurable cardinal. Must $\operatorname{cf}(\omega_1^{\omega_1})$ be less than κ?

P6 More measure algebras In paragraph P2 above I listed the known facts concerning the Maharam types of witnessing probabilities on atomlessly-measurable cardinals. But if we allow ourselves general probability spaces $(X,\mathcal{P}X,\mu)$ we can expect other phenomena. Indeed, Theorem 8A shows that PMEA implies that there are probability spaces $(X,\mathcal{P}X,\mu)$ of arbitrarily large Maharam type. Now in this context the question arises

(P6) Is it consistent to suppose that there is a cardinal λ_0 such that for every cardinal $\lambda \ge \lambda_0$ there is a Maharam homogeneous probability space $(X,\mathcal{P}X,\mu)$ of Maharam type λ?

P7 Qm ideals If a cardinal κ is real-valued-measurable, it is of course also quasi-measurable; so we have both an rvm ideal $\mathcal{J}_{\mathrm{rvm}}(\kappa)$ (1I) and a qm ideal $\mathcal{J}_{\mathrm{qm}}(\kappa)$ (9C). Evidently $\mathcal{J}_{\mathrm{rvm}}(\kappa) \supseteq \mathcal{J}_{\mathrm{qm}}(\kappa)$, and they are equal if κ is two-valued-measurable (9D).

Now suppose that κ is two-valued-measurable and that $\mathbb{P}$ is a random real p.o.set. If we have a $\mathbb{P}$-name $\dot{\mathcal{I}}$ for a normal ω_1-saturated ideal of $\mathcal{P}\kappa$, then $\mathcal{I}_1 = \{B : B \subseteq \kappa, \Vdash_{\mathbb{P}} \check{B} \in \dot{\mathcal{I}}\}$ is a normal ω_1-saturated ideal of $\mathcal{P}\kappa$ (because $\mathbb{P}$ is ccc; compare 2Jb), and therefore is the null ideal of some normal probability ν_1 on κ. Now if $\tilde{\nu}_1$ is the corresponding $\mathbb{P}$-name for a probability on κ, as in 2C, we have $\Vdash_{\mathbb{P}} \mathcal{N}_{\tilde{\nu}_1} \subseteq \dot{\mathcal{I}}$, by 2J(a-ii). It follows easily that, in $V^{\mathbb{P}}$, $\dot{\mathcal{I}}$ is the null ideal for a measure $\dot{\nu}$ on κ (in $V^{\mathbb{P}}$, take a member $\dot{E}$ of $\dot{\mathcal{I}}$ of maximal $\tilde{\nu}_1$-measure, and set $\dot{\nu}\dot{A} = \tilde{\nu}_1(\dot{A} \setminus \dot{E})$ for every $\dot{A}$).

So in this case we surely have

$$\Vdash_{\mathbb{P}} \mathcal{J}_{\mathrm{rvm}}(\check{\kappa}) = \mathcal{J}_{\mathrm{qm}}(\check{\kappa}).$$

But the question now arises,

(P7) If κ is a real-valued-measurable cardinal, is it necessarily true that $\mathcal{J}_{\mathrm{rvm}}(\kappa) = \mathcal{J}_{\mathrm{qm}}(\kappa)$?

P8 Sequential cardinals A question going back to Mazur 52 is: for which cardinals κ, if any, is there a sequentially continuous function $f : \{0,1\}^\kappa \to \mathbb{R}$ which is not continuous? Let us call such cardinals 'sequential' for the moment. It is easy to see that a real-valued-measurable cardinal is sequential, and it is known that the first sequential cardinal, if there is one, is quasi-measurable. (See Plebanek p91 for a survey of known results, with references; also Fremlin 84, 24D-E.) But the following, raised by Keisler & Tarski 64, p. 270, seems still to be open:

(P8) Must the first sequential cardinal, if there is one, be real-valued-measurable?

P9 The measure of κ^I Let κ be a real-valued-measurable cardinal, and ν a normal witnessing probability on κ. For any set I we can form the simple (completed) product probability on κ^I, as described in Fremlin 84, A6Kb; let us call it ν_I. What can be said about ν_I?

The most important question seems to concern the covering number $\mathrm{cov}(\kappa^I, \mathcal{N}_{\nu_I})$ (see A2P). If I is finite, then there is a κ-additive probability with domain $\mathcal{P}(\kappa^I)$ extending ν_I (use the construction of part (a-ii) of the proof of 5O; the same idea appears in 2Mb, 4K and 5J), so that $\mathrm{cov}(\kappa^I, \mathcal{N}_{\nu_I}) \geq \kappa$. But I ask

(P9a) If κ is a real-valued-measurable cardinal, ν a normal witnessing probability on κ, and $\nu_{\mathbb{N}}$ the completed product probability on $\kappa^{\mathbb{N}}$, must $\mathrm{cov}(\kappa^{\mathbb{N}}, \mathcal{N}_{\nu_{\mathbb{N}}})$ be κ?

A reason for believing that there may be a positive answer to this question is that if ν is constructed from a normal ultrafilter on a two-valued-measurable cardinal κ by the method of 2C, then $\mathrm{cov}(\kappa^I, \mathcal{N}_{\nu_I}) = \kappa$ for every non-empty set I of cardinal less than κ. A reason for taking the problem seriously is that a positive answer would solve the following problem positively:

(P9b) If κ is a real-valued-measurable cardinal and ν is a normal witnessing probability on κ and $\mathcal{A} \subseteq \mathcal{P}\kappa \setminus \mathcal{N}_\nu$ has cardinal less than κ, does it follow that there is a countable $N \subseteq \kappa$ such that $A \cap N \neq \emptyset$ whenever $A \in \mathcal{A}$?

(To answer P9b from P9a, consider the family $\{(\kappa \setminus A)^{\mathbb{N}} : A \in \mathcal{A}\}$; if $\mathrm{cov}(\kappa^{\mathbb{N}}, \mathcal{N}_{\nu_{\mathbb{N}}}) \geq \kappa$ this cannot cover $\kappa^{\mathbb{N}}$.) With a positive answer to P9b we could deal with P5a, or go on to

(P9c) If κ is a real-valued-measurable cardinal with normal witnessing probability ν, Y a set of cardinal less than κ, $\mathcal{I}$ a σ-algebra of subsets of Y and $f : \kappa \to \mathcal{I}$ a function, must there be a set $M \in \mathcal{I}$ such that $\nu\{\xi : f(\xi) \subseteq M\} = 1$?

(Set $A_y = \{\xi : y \in f(\xi)\}$ for each $y \in Y$, $A = \bigcup\{A_y : y \in Y, \nu A_y = 0\}$. If $N \in [\kappa]^{\leq\omega}$ is such that $N \cap A_y \neq \emptyset$ whenever $\nu A_y > 0$, set $M = \bigcup_{\xi \in N} f(\xi) \in \mathcal{I}$; then $f(\xi) \subseteq M$ whenever $\xi \in \kappa \setminus A$.)

With a positive answer to P9c we could now prove generalisations of 5Ca and 5Da with arbitrary σ-ideals in the place of $[Y]^{<\theta}$. Another class of special cases is suggested by 3Bc.

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References in **bold** type indicate definitions; references in *italics* indicate passing references.

Version of 22.5.00

Supplement to 'Real-valued-measurable cardinals'

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This note contains further results on real-valued-measurable cardinals, supplementing my paper 'Real-valued-measurable cardinals' (pp. 151-304 in Israel Math. Conference Proc. 6 (1993), ed. H.Judah), with which it should be read. References of the form '7C', 'A2J' are directions to paragraphs in that paper, and unexplained notation is defined there. References of the form 'SA2Ac' are directions to paragraphs below.

Section numbers follow those of 'Real-valued measurable cardinals', except that a new section S6′ ('Topological implications') has been added.

Version of 27.10.94

S5. Combinatorial implications

S5A The arguments of 5E may be refined, as follows.

Theorem Let κ be an atomlessly-measurable cardinal, and $\omega \le \delta \le \lambda < \kappa$. If $2^\delta < \kappa^{\delta^+}$, then $2^\lambda = 2^\delta$.

proof Induce on λ.

(a) For the inductive step to λ, where $\mathrm{cf}(\lambda) > \delta$, choose an injection $\theta_\xi : \mathcal{P}\xi \to \mathfrak{c}$ for each $\xi < \lambda$; set $g_A(\xi) = \theta_\xi(A \cap \xi)$ for $A \subseteq \lambda$, $\xi < \lambda$. **?** If $2^\lambda > 2^\delta$, let $N \subseteq 2^\delta$ be a set of minimal cardinal such that

$$\mathcal{A}_N = \{A : A \subseteq \lambda,\, g_A^{-1}[N] \text{ is cofinal with } \lambda\}$$

has cardinal greater than 2^δ. Then $\mathrm{cf}(\lambda) \le \mathrm{cf}(\#(N)) \le \lambda$. **P** **(i)** If $\gamma < \lambda$ and $N = \bigcup_{\alpha<\gamma} N_\alpha$ then $\mathcal{A}_N = \bigcup_{\alpha<\gamma} \mathcal{A}_{N_\alpha}$. **(ii)** If $\gamma > \lambda$ is regular and $\langle N_\alpha \rangle_{\alpha<\gamma}$ is an increasing family with union N, then again $\mathcal{A}_N = \bigcup_{\alpha<\gamma} \mathcal{A}_{N_\alpha}$. **Q**

But there is no cardinal γ such that $\kappa \le \gamma < \kappa^{+\delta^+}$ and $\delta < \mathrm{cf}(\gamma) \le \lambda$, so $\#(N) < \kappa$.

Let $h : \kappa \to \mathcal{A}_N$ be injective. For $\alpha < \beta < \kappa$ set $f(\{\alpha, \beta\}) = \min\{\zeta : h(\alpha) \cap \zeta \ne h(\beta) \cap \zeta\}$. By 5Ca, there is a set $C \subseteq \kappa$ such that $\#(C) = \kappa$ and $M = \{f(I) : I \in [C]^2\}$ is countable, therefore bounded above in λ; set $\zeta = \sup M$, so that $h(\alpha) \cap \zeta \ne h(\beta) \cap \zeta$ for all distinct α, $\beta \in C$. Next, there is a $\xi \in \lambda \setminus \zeta$ such that $C_1 = \{\alpha : \alpha \in C,\, g_{h(\alpha)}(\xi) \in N\}$ has cardinal κ. But now $\alpha \mapsto g_{h(\alpha)}(\xi) = \theta_\xi(h(\alpha \cap \xi))$ is an injective function from C_1 to M, while $\#(M) < \kappa = \#(C_1)$. **X**

This deals with the inductive step if $\mathrm{cf}(\lambda) > \delta$.

(b) If $\mathrm{cf}(\lambda) \le \delta$, then $2^\lambda \le (\sup_{\gamma<\lambda} 2^\gamma)^{\mathrm{cf}(\lambda)} = 2^\delta$, so the induction proceeds.

S5B Proposition Suppose there is an atomlessly-measurable cardinal κ. Then there is a family $\langle A_\alpha \rangle_{\alpha<\omega_1}$ of sets such that

(i) $A_\alpha \subseteq \alpha$ for every $\alpha < \omega_1$;
(ii) if $\alpha \le \beta < \omega_1$, then $A_\alpha \triangle (A_\beta \cap \alpha)$ is finite;
(iii) if $C \subseteq \omega_1$ is uncountable, then

$$\{\alpha : C \cap A_\alpha \text{ is infinite}\}, \quad \{\alpha : (C \cap \alpha) \setminus A_\alpha \text{ is infinite}\}$$

are both uncountable.

proof (a) Choose a family $\langle e_\alpha \rangle_{\alpha<\omega_1}$ such that (i) each e_α is an injective function from α to $\mathbb{N}$ (ii) if $\alpha \le \beta$ then $\{\xi : \xi < \alpha,\, e_\alpha(\xi) \ne e_\beta(\xi)\}$ is finite (SA1A). Let ν be an extension of the usual measure μ of $\mathcal{P}\mathbb{N}$ (corresponding to the usual measure on $\{0,1\}^{\mathbb{N}}$) to $\mathcal{P}(\mathcal{P}\mathbb{N})$. For each $a \subseteq \mathbb{N}$ set

$$A_{a\alpha} = e_\alpha^{-1}[a] \subseteq \alpha.$$

Then whenever $\alpha \le \beta < \omega_1$,

$$A_{a\alpha} \triangle (\alpha \cap A_{a\beta}) \subseteq \{\xi : \xi < \alpha,\, e_\alpha(\xi) \ne e_\beta(\xi)\}$$

is finite. Thus every $a \subseteq \mathbb{N}$ provides a family satisfying conditions (i) and (ii).

(b) Let S be the set of those $a \subseteq \mathbb{N}$ for which there is some uncountable set $C_a \subseteq \omega_1$ such that $\{\alpha : C_a \cap A_{a\alpha}$ is infinite$\}$ is countable. **?** Suppose, if possible, that $\nu S > 0$. Then there is an $\alpha_0 < \omega_1$ such that

$$\{\alpha : C_a \cap A_{a\alpha} \text{ is infinite}\} \subseteq \alpha_0$$

for ν-almost every $a \in S$; set

$$T = \{a : a \in S,\, C_a \cap A_\alpha \text{ is finite for every } \alpha \ge \alpha_0\}.$$

For each $\xi < \omega_1$, set $T_\xi = \{a : a \in T,\, \xi \in C_a\}$; then

$$0 < \nu S = \nu T \le \textstyle\sum_{\xi \ge \beta} \nu T_\xi$$

for every $\beta < \omega_1$, so $\{\xi : \nu T_\xi > 0\}$ is uncountable and there is a $\delta > 0$ such that B is uncountable, where

$$B = \{\xi : \nu T_\xi \ge \delta\}.$$

By S6F (I apologize for the forward reference, but you can easily check that there is no circularity) there is an infinite $D \subseteq B$ such that $\mu^*(\bigcap_{\xi\in D} T_\xi) > 0$. Take $\alpha \ge \alpha_0$ such that $D \subseteq \alpha$. If $a \in \bigcap_{\xi\in D} T_\xi$, then $D \subseteq C_a$, while $C_a \cap A_{a\alpha}$ is finite, so $D \cap A_{a\alpha}$ is finite. But

$$E = \{a : D \cap A_{a\alpha} \text{ is finite}\} = \{a : a \cap e_\alpha[D] \text{ is finite}\}$$

is μ-measurable and has zero measure, so $\mu^*(\bigcap_{\xi\in D} T_\xi)=0$, which is impossible. **X** Thus $\nu S=0$.

(c) Now consider the set S' of those $a\subseteq\mathbb{N}$ for which there is some uncountable set C_a such that $\{\alpha:(C_a\cap\alpha)\setminus A_{a\alpha}$ is infinite$\}$ is countable. **?** Suppose, if possible, that $\nu S'>0$. Then there is an $\alpha_0<\omega_1$ such that

$$\{\alpha: C_a\cap\alpha\setminus A_{a\alpha}\text{ is infinite}\}\subseteq\alpha_0$$

for ν-almost every $a\in S'$; set

$$T=\{a:a\in S,\,C_a\cap\alpha\setminus A_\alpha\text{ is finite for every }\alpha\ge\alpha_0\}.$$

For each $\xi<\omega_1$, set $T_\xi=\{a:a\in T,\,\xi\in C_a\}$; then

$$0<\nu S'=\nu T\le\textstyle\sum_{\xi\ge\beta}\nu T_\xi$$

for every $\beta<\omega_1$, so $\{\xi:\nu T_\xi>0\}$ is uncountable and there is a $\delta>0$ such that B is uncountable, where

$$B=\{\xi:\nu T_\xi\ge\delta\}.$$

By S6F there is an infinite $D\subseteq B$ such that $\mu^*(\bigcap_{\xi\in D}T_\xi)>0$. Take $\alpha\ge\alpha_0$ such that $D\subseteq\alpha$. If $a\in\bigcap_{\xi\in D}T_\xi$, then $D\subseteq C_a$, while $C_a\cap\alpha\setminus A_{a\alpha}$ is finite, so $D\setminus A_{a\alpha}$ is finite. But

$$E=\{a:D\setminus A_{a\alpha}\text{ is finite}\}=\{a:e_\alpha[D]\setminus a\text{ is finite}\}$$

is μ-measurable and has zero measure, so $\mu^*(\bigcap_{\xi\in D}T_\xi)=0$, which is impossible. **X** Thus $\nu S'=0$.

(d) There must therefore be an $a\subseteq\mathbb{N}$ not belonging to either S or S', and the corresponding family $\langle A_{a\alpha}\rangle_{\alpha<\omega_1}$ has the required property.

Remark This comes from TODORČEVIĆ 93, §7.

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S6′. Topological implications

S6′A Proposition Suppose there is an atomlessly-measurable cardinal. Then there is a locally compact locally countable Hausdorff topology $\mathfrak{T}$ on ω_1 such that every closed set is either countable or cocountable.

proof (a) Let κ be an atomlessly-measurable cardinal, and ν_0 a Maharam homogeneous atomless probability measure with domain $\mathcal{P}\kappa$; let λ be the Maharam type of ν. By 3F, $\lambda>\kappa>\omega_1$. Let $g:\kappa\to\{0,1\}^\lambda$ be a function representing an isomorphism from the measure algebra of the usual measure on $X=\{0,1\}^\lambda=X$ and the measure algebra of ν_0 (A2Gd). Let ν be the image measure ν_0g^{-1}, so that ν extends the usual measure on X, and setting $E_\xi=\{x:x\in X,\,x(\xi)=1\}$ for $\xi<\lambda$, $\langle E_\xi\rangle_{\xi<\lambda}$ is a stochastically independent family of sets of measure $\frac12$ such that $\{E_\xi^\bullet:\xi<\lambda\}$ τ-generates the measure algebra of ν. For $\Gamma\subseteq\lambda$ write Σ_Γ for the σ-algebra generated by $\{E_\xi:\xi\in\Gamma\}\cup\{E:\nu E=0\}$, so that $\mathcal{P}X=\bigcup\{\Sigma_\Gamma:\Gamma\in[\lambda]^{\le\omega}\}$. Write Ω for the set of non-zero countable limit ordinals; for $\alpha\in\Omega$, enumerate α as $\langle e_\alpha(n)\rangle_{n\in\mathbb{N}}$. Fix a bijection $\phi:\{0,1\}^{\mathbb{N}}\to[0,1]^{\mathbb{N}}$ which is an automorphism for the usual measures (FREMLIN P00*, 344I). For $\alpha\in\Omega$, $n\in\mathbb{N}$ define $\phi_{\alpha n}:X\to[0,1]$ by writing $\langle\phi_{\alpha n}(x)\rangle_{n\in\mathbb{N}}=\phi(\langle x(\alpha+n)\rangle_{n\in\mathbb{N}})$ for every $x\in X$.

(b) For $x\in X$, $\xi<\omega_1$ choose $B_x(\xi)$ as follows. If $\xi<\omega$ then $B_x(\xi)=\{\xi\}$. Suppose that $B_x(\xi)$ has been defined for every $\xi<\alpha$, where $\alpha\in\Omega$. Set $C_x(\alpha,n)=B_x(e_\alpha(n))\setminus\bigcup_{i<n}B_x(e_\alpha(i))$ for each n. Set

$$B_x(\alpha+n)=\{\alpha+n\}\cup\bigcup\{C_x(\alpha,i):i\in\mathbb{N},\,2^{-n-1}\le\phi_{\alpha i}(x)<2^{-n}\}$$

for each n.

Let $\mathfrak{T}_x$ be the topology on ω_1 generated by $\{B_x(\xi):\xi<\omega_1\}\cup\{\omega_1\setminus B_x(\xi):\xi<\omega_1\}$. Then $\mathfrak{T}_x$ is Hausdorff and locally countable. **P** It is Hausdorff because if $\xi<\eta$ then $\xi\in B_x(\xi)$ and $\eta\notin B_x(\xi)$. (The point is that if α is a limit ordinal and $n\in\mathbb{N}$, then

$$\alpha+n\in B_x(\alpha+n)\subseteq\{\alpha+n\}\cup\alpha.)$$

It is locally countable because every $B_x(\xi)$ is countable. **Q**

(c) For $\alpha\in\Omega$, let $\mathfrak{T}_{x\alpha}$ be the topology on α generated by $\{B_x(\xi):\xi<\alpha\}\cup\{\alpha\setminus B_x(\xi):\xi<\alpha\}$. Then an easy induction shows that $C_x(\beta,n)\cap\alpha$ and $B_x(\eta)\cap\alpha$ are open-and-closed in $\mathfrak{T}_{x\alpha}$ for every $\beta\in\Omega$, $n\in\mathbb{N}$

and $\eta \in \omega_1$, so that $\mathfrak{T}_{x\alpha}$ is the subspace topology on α induced by $\mathfrak{T}_x$. Now $B_x(\xi)$ is $\mathfrak{T}_{x\alpha}$-compact whenever $\alpha \in \Omega$, $n \in \mathbb{N}$ and $\xi < \alpha$. **P** Induce on α. For $\alpha = \omega$, every $B_x(\xi)$ is a singleton, so the induction starts. If $\alpha = \beta + \omega$, where $\beta \in \Omega$, and $\xi < \alpha$, either $\xi < \beta$ so $B_x(\xi) \subseteq \beta$ is $\mathfrak{T}_{x\beta}$-compact, therefore $\mathfrak{T}_{x\alpha}$-compact, or $\xi = \alpha + n$ for some n. In this case, $B_x(e_\beta(n))$ is always $\mathfrak{T}_{x\beta}$-compact and $\mathfrak{T}_{x\beta}$-open, so $\langle C_x(\beta, i)\rangle_{i\in\mathbb{N}}$ is a partition of β into $\mathfrak{T}_{x\beta}$-compact and $\mathfrak{T}_{x\beta}$-open sets, and $B_x(\beta + n)$ must be $\mathfrak{T}_{x,\beta+\omega}$-compact. If α is a limit member of Ω, then for every $\xi < \alpha$ there is a $\beta \in \Omega$ such that $\xi < \beta < \alpha$, and the inductive hypothesis tells us that $B_x(\xi)$ is $\mathfrak{T}_{x\beta}$-compact, therefore $\mathfrak{T}_{x\alpha}$-compact. Thus the induction proceeds. **Q**

It follows that every $B_x(\xi)$ is $\mathfrak{T}_x$-compact, so that $\mathfrak{T}_x$ is locally compact, for every $x \in X$.

(d) The essential fact to note about this construction is that $\{x : \xi \in B_x(\eta)\}$, $\{x : \xi \in C_{\beta,n}\}$ belong to Σ_α whenever $\alpha \in \Omega$, $\xi < \omega_1$, $\eta < \alpha$, $\beta \in \alpha \cap \Omega$ and $n \in \mathbb{N}$.

(e) ? Suppose, if possible, that for every $x \in X$ there is a set $D_x \subseteq \omega_1$ which is $\mathfrak{T}_x$-closed but neither countable nor cocountable. For $\xi < \omega_1$ set $H_\xi = \{x : \xi \in D_x\} \subseteq X$. Then there is a countable set $\Gamma_\xi \subseteq \lambda$ such that $H_\xi \in \Sigma_{\Gamma_\xi}$. Note also that, for any $\xi < \omega_1$, $\bigcup_{\eta>\xi} H_\eta = X$, so there is a countable $\Delta_\xi \subseteq \omega_1 \setminus \xi$ such that $\bigcup_{\eta\in\Delta_\xi} H_\eta$ is conegligible. Let A be

$$\{\alpha : \alpha \in \Omega,\ (\Gamma_\xi \cap \omega_1) \cup \Delta_\xi \subseteq \alpha \text{ for every } \xi < \alpha\};$$

then A is a club set in ω_1. Take any $\alpha \in A$. Set $\Gamma = \bigcup_{\xi<\alpha} \Gamma_\xi$, so that $\Gamma \cap \omega_1 \subseteq \alpha$.

Let Y be $\{y : y \in X,\ D_y \cap \alpha \text{ is cofinal with } \alpha\}$, so that $Y \supseteq \bigcap_{\xi<\alpha} \bigcup_{\eta\in\Delta_\xi} H_\eta$ is conegligible. For every $x \in X$, D_x does not include $\omega_1 \setminus \alpha$, so there is a least $\zeta_x \geq \alpha$ such that $\zeta_x \notin D_x$; let $\zeta \in \omega_1 \setminus \alpha$ be the least ordinal such that $Q = \{x : \zeta_x = \zeta\}$ is not negligible. Set $Y_1 = \{y : y \in Y,\ \zeta_y \geq \zeta\}$, so that Y_1 is conegligible. Express ζ as $\beta + n$ where $\beta \in \Omega$, $n \in \mathbb{N}$. Observe that H_ξ is conegligible, so belongs to Σ_Γ, for every $\xi \in \beta \setminus \alpha$, so $H_\xi \in \Sigma_\Gamma$ for every $\xi < \beta$.

For $x \in X$, set $I_x = \{m : m \in \mathbb{N},\ D_x \cap C_x(\beta, m) \neq \emptyset\}$. Observe that

$$\{x : m \in I_x\} = \bigcup_{\xi<\beta} \{x : \xi \in D_x \cap C_x(\beta, m)\} \in \Sigma_{\Gamma\cup\beta}$$

for every $m \in \mathbb{N}$. If $y \in Y_1$, $D_y \cap \beta$ is cofinal with β, whether or not $\beta = \alpha$, because $D_y \cap \alpha$ is cofinal with α and $\beta \setminus \alpha \subseteq D_y$. Since

$$\sup C_y(\beta, m) \leq \sup B_y(e_\beta(m)) = e_\beta(m) < \beta$$

for every m, I_y is infinite, for every $y \in Y_1$.

On the other hand, setting $F_m = \{x : 2^{-n-1} \leq \phi_{\beta m}(x) < 2^{-n}\}$ for $m \in \mathbb{N}$, $\langle F_m\rangle_{m\in\mathbb{N}}$ is a stochastically independent sequence of sets of measure 2^{-n-1} all belonging to $\Sigma_{(\beta+\omega)\setminus\beta}$, which is independent from $\Sigma_{\Gamma\cup\beta}$. Set $J_x = \{m : x \in F_m\}$. Then

$$G = \{x : I_x \text{ is infinite, } I_x \cap J_x \text{ is finite}\}$$

is negligible. **P** Set $h_1(x) = I_x$, $h_2(x) = J_x$, $h(x) = (I_x, J_x)$, so that h_1 and h_2 are functions from X to $\mathcal{P}\mathbb{N}$. Let μ_1, μ_2 be the Radon measures on $\mathcal{P}\mathbb{N}$ such that h_1 and h_2 are inverse-measure -preserving. Then h_1 is inverse-measure -preserving for $\nu{\restriction}\Sigma_{\Gamma\cup\beta}$ and μ_1, and h_2 is inverse-measure -preserving for $\nu{\restriction}\Sigma_{(\beta\cup\omega)\setminus\beta}$ and μ_2, so h is inverse-measure -preserving for ν and the product measure $\mu_1 \times \mu_2$ on $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Now the set

$$\{(I, J) : I \in [\mathbb{N}]^\omega,\ J \subseteq \mathbb{N},\ I \cap J \text{ is finite}\}$$

is a Borel set in $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ which has μ_2-negligible vertical sections, so is $\mu_1 \times \mu_2$-negligible, and its inverse image G is ν-negligible. **Q**

There is therefore some $y \in Q \cap Y_1 \setminus G$. But in this case $\zeta \notin D_y$, so $B_y(\zeta) \cap D_y$ is a $\mathfrak{T}_y$-compact set not containing ζ. Since

$$B_y(\zeta) = \{\zeta\} \cup \bigcup_{m\in J_y} C_y(\beta, m),$$

there must be a finite set $K \subseteq J_y$ such that $B_y(\zeta) \cap D_y \subseteq \bigcup_{m\in K} C_y(\beta, m)$. Since I_y is infinite and $y \notin G$, there is an $m \in I_y \cap J_y \setminus K$, and $C_y(\beta, m) \cap D_y$ is not empty; which is impossible. **X**

This contradiction shows that for at least one $x \in X$, every $\mathfrak{T}_x$-closed set is either countable or cocountable.

Remarks This proposition is based on ideas of J.Moore.

Observe that, for a topology $\mathfrak{T}$ as described, ω_1 is hereditarily separable. **P?** If $A \subseteq \omega_1$ is not separable, of course it is uncountable. For $\xi < \omega_1$, choose α_ξ, β_ξ in A, and countable open sets $G_\xi \subseteq \omega_1$ inductively, as follows. Given $\langle\alpha_\eta\rangle_{\eta<\xi}$, let $\beta_\xi \in A$ be such that $\beta_\xi \notin \overline{\{\alpha_\eta : \eta < \xi\} \cup (A \cap \xi)}$. Let G_ξ be a countable open set containing β_ξ and not meeting $\overline{\{\alpha_\eta : \eta < \xi\}}$. Given $\langle G_\eta\rangle_{\eta\le\xi}$, take $\alpha_\xi \in A \setminus (\xi \cup \bigcup_{\eta\le\xi} G_\eta)$.

Now consider $F = \overline{\{\alpha_\xi : \xi < \omega_1\}}$. Then $\alpha_\xi \in F$ and $\beta_\xi \notin F$ for every ξ. Since α_ξ, $\beta_\xi \ge \xi$ for every ξ, F is neither countable nor cocountable, which is impossible. **XQ**

Version of 21.5.00

S6. Measure-theoretic implications

S6A Proposition Let κ be an atomlessly-measurable cardinal and $\lambda < \kappa$ a cardinal. Then there is a cardinal $\theta < \kappa$ such that whenever X is a T_0 topological space of cardinal at least θ and weight at most λ then there is an atomless τ-additive Borel probability on X.

proof ? Suppose, if possible, otherwise.

(a) We are supposing that for each $\zeta < \kappa$ there is a T_0 space X_ζ, of cardinal at least $\#(\zeta)$ and of weight at most λ, with no atomless τ-additive Borel probability. Note that any τ-additive Borel probability $\tilde\mu$ on any subspace D of X_ζ gives rise to a τ-additive Borel probability $\tilde\mu_1$ on X_ζ, writing $\tilde\mu_1 E = \tilde\mu(D \cap E)$ for every Borel set $E \subseteq X_\zeta$, which would be atomless if $\tilde\mu$ were atomless; so we may take it that $\#(X_\zeta)$ is precisely $\#(\zeta)$. Let $\langle U_{\zeta\xi}\rangle_{\xi<\lambda}$ run over a base for the topology of X_ζ. Define $f_\zeta : X_\zeta \to X = \{0,1\}^\lambda$ by writing

$$f_\zeta(x)(\xi) = 1 \text{ if } x \in U_{\zeta\xi},\ 0 \text{ if } x \in X_\zeta \setminus U_{\zeta\xi}.$$

Note that f_ζ is injective, because X_ζ is a T_0 space, so we may enumerate $Y_\zeta = f_\zeta[X_\zeta]$ as $\langle y_{\zeta\xi}\rangle_{\xi<\zeta}$.

(b) Let ν be a normal witnessing probability on κ. Define $\mu : \mathcal{P}(X \times \kappa) \to [0,1]$ by setting

$$\mu E = \int \nu\{\zeta : (y_{\zeta\xi}, \zeta) \in E\}\nu(d\xi)$$

for $E \subseteq X \times \kappa$. Then μ is a κ-additive probability on $X \times \kappa$. By SA2B below there is a family $\langle\mu_\zeta\rangle_{\zeta<\kappa}$ of Radon probabilities on X such that $\mu(H \times A) = \int_A \mu_\zeta H\nu(d\zeta)$ for every set H in the Baire σ-algebra of X and every $A \subseteq \kappa$.

Let $\mathcal{V}$ be the algebra of open-and-closed subsets of X.

(c) For any $C \subseteq X \times \kappa$, $\mu C \le \int \mu_\zeta^* C^{-1}[\{\zeta\}]\nu(d\zeta)$. **P** The set $\mathcal{W}$ of sets expressible as a finite disjoint union of sets $V \times A$, where $V \in \mathcal{V}$ and $A \subseteq \kappa$, is a subalgebra of $\mathcal{P}(X \times \kappa)$, and for every $W \in \mathcal{W}$ we have

$$\mu W = \int \mu_\zeta W^{-1}[\{\zeta\}]\nu(d\zeta).$$

Now take $\epsilon > 0$. For each $\zeta < \kappa$ choose an open set $G_\zeta \supseteq C^{-1}[\{\zeta\}]$ such that $\mu_\zeta G_\zeta \le \mu_\zeta^* C^{-1}[\{\zeta\}] + \epsilon$. Set

$$E = \{(y,\zeta) : \zeta < \kappa,\ y \in G_\zeta\} = \bigcup_{V\in\mathcal{V}}\{(y,\zeta) : y \in V \subseteq G_\zeta\}.$$

Then $C \subseteq E$ so $\mu C \le \mu E$. Because μ is κ-additive and $\#(\mathcal{V}) \le \max(\lambda,\omega) < \kappa$, there is a finite $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $\mu W \ge \mu E - \epsilon$, where

$$W = \bigcup_{V\in\mathcal{V}_0}\{(y,\zeta) : y \in V \subseteq G_\zeta\}.$$

But $W \in \mathcal{W}$, so

$$\mu W = \int \mu_\zeta W^{-1}[\{\zeta\}]\nu(d\zeta) \le \int \mu_\zeta G_\zeta \nu(d\zeta).$$

Putting these together,

$$\mu C \le \int \mu_\zeta G_\zeta \nu(d\zeta) + \epsilon \le \int \mu_\zeta^* C^{-1}[\{\zeta\}]\nu(d\zeta) + 2\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

Taking complements, we see now that

$$\mu C \ge \int (\mu_\zeta)_* C^{-1}[\{\zeta\}]\nu(d\zeta)$$

for every $C \subseteq X \times \kappa$, writing $(\mu_\zeta)_* D = \sup\{\mu_\zeta H : H \subseteq D \text{ is } \mu_\zeta\text{-measurable}\}$ for every $D \subseteq X$, $\zeta < \kappa$.

(d) $\mu_\zeta^* Y_\zeta = 1$ for ν-almost every ζ. **P** Set $C = \{(y,\zeta) : \zeta < \kappa,\ y \in Y_\zeta\} \subseteq X \times \kappa$. Then $(y_{\zeta\xi}, \zeta) \in C$ whenever $\xi < \zeta < \kappa$, so $\mu C = 1$. By (c),

$$1 = \mu C \le \int \mu^*_\zeta C^{-1}[\{\zeta\}]\nu(d\zeta) = \int \mu^*_\zeta Y_\zeta \nu(d\zeta).$$

Thus $\mu^*_\zeta Y_\zeta = 1$ for almost all ζ. **Q**

Set $A_0 = \{\zeta : \mu^*_\zeta Y_\zeta = 1\}$.

(e) μ_ζ is atomless for almost all $\zeta \in A_0$. **P** Set

$$B = \{\zeta : \zeta \in A_0,\ \mu_\zeta \text{ has an atom}\},$$

$$C = \{(y, \zeta) : \zeta < \kappa,\ y \in Y_\zeta,\ \mu_\zeta\{y\} > 0\}.$$

If $\zeta \in B$, then (because μ_ζ is a Radon measure) there is a $y \in X$ such that $\mu_\zeta\{y\} > 0$; but as $\mu^*_\zeta Y_\zeta = 1$, $y \in Y_\zeta$ and $(y, \zeta) \in C$. For each $\zeta < \kappa$, $C^{-1}[\{\zeta\}]$ is countable, so is expressible as $\{y_{\zeta\xi} : \xi \in I_\zeta\}$, where $I_\zeta \subseteq \zeta$ is countable. Let $I \subseteq \kappa$ be a countable set such that $I_\zeta \subseteq I$ for almost all ζ (5Ab). Now $\nu\{\zeta : y_{\zeta\xi} \in C\} = 0$ for every $\xi \in \kappa \setminus I$, so $\mu C = 0$. Consequently, by the last remark of (c) above,

$$\int (\mu_\zeta)_* C^{-1}[\{\zeta\}]\nu(d\zeta) = 0.$$

But of course $(\mu_\zeta)_* C^{-1}[\{\zeta\}] > 0$ for every $\zeta \in B$, so $\nu B = 0$, as claimed. **Q**

(f) There must therefore be some $\zeta < \kappa$ such that μ_ζ is atomless and $\mu^*_\zeta Y_\zeta = 1$.

Write μ'_ζ for the quasi-Radon subspace probability $\mu_\zeta \lceil Y_\zeta$ (A2Ja). $f_\zeta^{-1} : Y_\zeta \to X_\zeta$ is continuous, so we have a τ-additive Borel probability $\tilde\mu$ on X_ζ defined by saying that $\tilde\mu E = \mu'_\zeta(f_\zeta[E])$ for every Borel set $E \subseteq X_\zeta$. Now take $\epsilon > 0$. Because μ_ζ is atomless, every point of X is contained in an open set of μ_ζ-measure at most ϵ. Because X is compact, we have a finite cover of X by basic cylinder sets of the form $U_w = \{y : y \restriction I = w\}$, where $I \subseteq \lambda$ is finite and $w \in \{0, 1\}^I$, all of μ_ζ-measure at most ϵ. Now $f_\zeta^{-1}[U_w]$ is a Borel set in X_ζ for every w (examine the original definition of f_ζ in (a) above), and $\tilde\mu f_\zeta^{-1}[U_w] = \mu'_\zeta(Y_\zeta \cap U_w) = \mu_\zeta U_w$, so X_ζ has a partition into finitely many Borel sets of $\tilde\mu$-measure at most ϵ. As ϵ is arbitrary, $\tilde\mu$ is atomless.

But this contradicts the choice of X_ζ. **X**

This contradiction completes the proof.

S6B Corollary Let κ be an atomlessly-measurable cardinal and $\lambda < \kappa$ a cardinal. Then there is a cardinal $\theta < \kappa$ such that

(a) whenever X is a regular T_0 topological space of cardinal at least θ and weight at most λ then there is an atomless quasi-Radon probability on X;

(b) whenever Z is a compact Hausdorff space of weight at most λ and $X \subseteq Z$ has cardinal at least θ, then there is an atomless Radon probability μ on Z such that $\mu^* X = 1$.

proof Take θ from S6A.

(a) There is an atomless τ-additive Borel probability on X, which by SA2A extends to a quasi-Radon probability on X.

(b) There is an atomless τ-additive Borel probability μ_0 on X; setting $\mu E = \mu_0(X \cap E)$ for Borel sets $E \subseteq Z$, we get a τ-additive Borel probability on Z for which $\mu^* X = 1$; by SA2A the completion $\hat\mu$ of μ is quasi-Radon, and of course $\hat\mu$ is atomless and $\hat\mu^* X = 1$. Because Z is compact, $\hat\mu$ is inner regular for the compact sets and is therefore a Radon measure.

S6C Corollary Let κ be an atomlessly-measurable cardinal. Then there is a $\theta < \kappa$ such that no subset of $\mathbb{R}$ of cardinal θ or more can be universally negligible.

proof Apply S6Bb with $\lambda = \omega$, $X = [a, b]$, where $a \le b$ in $\mathbb{R}$.

S6D Proposition If κ is an atomlessly-measurable cardinal and (X, μ) is an atomless Radon probability space of Maharam type at most κ, then $\mathrm{cf}(\mathcal{N}_\mu) = \mathrm{cf}(\mathcal{N})$, where $\mathcal{N}$ is the Lebesgue null ideal.

proof Use S7B below and 6.14 of Fremlin 89.

S6E Lemma Let κ be an atomlessly-measurable cardinal, and $\lambda \le \kappa$ a cardinal of uncountable cofinality. Let (X, μ) be a Radon probability space and $\langle E_\xi \rangle_{\xi < \lambda}$ a family of non-negligible measurable subsets of X. Then there is an $x \in X$ such that $\#(\{\xi : x \in E_\xi\}) = \lambda$.

proof By 6C, λ is a precaliber of the measure algebra $\mathfrak{A}$ of (X,μ). For each $\xi<\kappa$, choose a compact non-negligible $F_\xi\subseteq E_\xi$. Then there is a set $D\in[\lambda]^\lambda$ such that $\{F_\xi^\bullet:\xi\in D\}$ is centered. But this means that $\{F_\xi:\xi\in D\}$ has the finite intersection property; because the F_ξ are compact, there is a point in $\bigcap_{\xi\in D}F_\xi$, and this will serve for x.

S6F Lemma Let κ be an atomlessly-measurable cardinal. Let (X,Σ,μ) be a probability space, and Σ_0 a σ-subalgebra of Σ; set $\mu_0=\mu\restriction\Sigma_0$ and let μ_0^* be the outer measure on X defined from μ_0. Suppose that the Maharam type τ_0 of μ_0 is less than κ. Let $\langle E_\xi\rangle_{\xi<\lambda}$ be a family in Σ with $\inf_{\xi<\lambda}\mu E_\xi=\gamma>0$, where $\lambda>\max(\omega,\tau_0)$. Then for any $\gamma'<\gamma$ there is an infinite $W\subseteq\lambda$ such that $\mu_0^*(\bigcap_{\xi\in W}E_\xi)\ge\gamma'$.

Remark Compare A2S.

proof It is enough to consider the case $\lambda=\max(\omega_1,\tau_0^+)$, so that $\mathrm{cf}(\lambda)>\omega$ and $\lambda\le\kappa$.

(a) Consider first the special case in which $X=\{0,1\}^I$ for some set I, with μ the usual measure on X, and there is a $J\subseteq I$ of cardinal τ_0 such that $\Sigma_0=\{\pi_J^{-1}[F]:F\in\Sigma_J\}$, writing Σ_J for the domain of the usual measure μ_J on $\{0,1\}^J$, and $\pi_J:X\to\{0,1\}^J$ for the canonical map.

We may regard X as the product $\{0,1\}^J\times\{0,1\}^{I\setminus J}$. For $\xi<\lambda$, $z\in\{0,1\}^{I\setminus J}$ set

$$E_{\xi z}=\{y:y\in\{0,1\}^J,\ (y,z)\in E_\xi\}.$$

Then

$$\int\mu_J E_{\xi z}\mu_{I\setminus J}(dz)=\mu E_\xi\ge\gamma$$

by Fubini's theorem (A2Ge). Set

$$G_\xi=\{z:z\in\{0,1\}^{I\setminus J},\ \mu_J E_{\xi z}\text{ exists }\ge\gamma\};$$

then $\mu_{I\setminus J}G_\xi$ exists and is greater than 0. By S6E, there is a $z\in\{0,1\}^{I\setminus J}$ such that $U=\{\xi:z\in G_\xi\}$ has cardinal λ.

Let $(\mathfrak{A}_0,\bar\mu_J)$ be the measure algebra of μ_J. Then the topological density of $\mathfrak{A}_0$ is at most $\max(\omega,\tau_0)<\lambda$ (Fremlin 89, 6.3b). So there is a $b\in\mathfrak{A}_0$ such that $\{\xi:\xi\in U,\ \bar\mu_J(b\triangle E_{\xi z}^\bullet)\le\delta\}$ is infinite for every $\delta>0$. Of course $\bar\mu_J b\ge\inf_{\xi\in U}\bar\mu_J E_{\xi z}^\bullet\ge\gamma$. Let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a sequence of distinct elements of U such that $\sum_{n\in\mathbb{N}}\bar\mu_J(b\triangle E_{\xi_n z}^\bullet)\le\gamma-\gamma'$. Then

$$\mu_J(\textstyle\bigcap_{n\in\mathbb{N}}E_{\xi_n z})=\bar\mu_J(\inf_{n\in\mathbb{N}}E_{\xi_n z}^\bullet)\ge\bar\mu_J b-\sum_{n\in\mathbb{N}}\bar\mu_J(b\setminus E_{\xi_n z}^\bullet)\ge\gamma'.$$

Set $W=\{\xi_n:n\in\mathbb{N}\}$. If $F'\in\Sigma_J$ and $\pi_J^{-1}[F']\supseteq\bigcap_{\xi\in W}E_\xi$, we must have $F'\supseteq\bigcap_{n\in\mathbb{N}}E_{\xi_n z}$, so

$$\mu(\pi_J^{-1}[F'])=\mu_J F'\ge\gamma'.$$

As F' is arbitrary, $\mu_0^*(\bigcap_{\xi\in W}E_\xi)\ge\gamma'$, as required.

(b) It follows that if $(\mathfrak{A},\bar\mu)$ is a probability algebra, $\mathfrak{A}_0$ is an order-closed subalgebra of $\mathfrak{A}$ with $\tau(\mathfrak{A}_0)=\tau_0<\kappa$, λ is a cardinal greater than $\max(\omega,\tau(\mathfrak{A}_0))$, $\langle a_\xi\rangle_{\xi<\lambda}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\lambda}\bar\mu a_\xi=\gamma>0$, and $\gamma'<\gamma$, then there is an infinite $W\subseteq\lambda$ such that

$$\min\{\bar\mu b:b\in\mathfrak{A}_0,\ b\supseteq\inf_{\xi\in W}a_\xi\}\ge\gamma'.$$

$\mathbf{P}$ We can embed $\mathfrak{A}$ as a subalgebra of the measure algebra $\mathfrak{A}_I$ of $\{0,1\}^I$ for some set I (A2Ib). If we take a set $B\subseteq\mathfrak{A}_0$ of cardinal τ_0 which completely generates $\mathfrak{A}_0$, then for each $b\in B$ we can find a set $G_b\subseteq X=\{0,1\}^I$, belonging to the Baire σ-algebra of X, such that $b=G_b^\bullet$ in $\mathfrak{A}$; now there is a set $J\subseteq I$, of cardinal at most $\max(\omega,\tau_0)$, such that every G_b belongs to Σ_0, if we define Σ_0 from J as in part (a) above.

Set $\mathfrak{A}_1=\{G^\bullet:G\in\Sigma_0\}$, so that $\mathfrak{A}_1$ is an order-closed subalgebra of $\mathfrak{A}$ and $\mathfrak{A}_0\subseteq\mathfrak{A}_1$. Let Σ_J be the algebra of measurable subsets of $\{0,1\}^J$, and $\mathfrak{A}_J$ the measure algebra of $\{0,1\}^J$. Then the inverse-measure-preserving map $\pi_J:X\to\{0,1\}^J$ induces an isomorphism ϕ between $\mathfrak{A}_1$ and $\mathfrak{A}_J$, taking $\phi(\pi_J^{-1}[G]^\bullet)=G^\bullet$ for every $G\in\Sigma_J$. By the lifting theorem (A2Qb) there is a lifting $\theta_J:\mathfrak{A}_J\to\Sigma_J$. So we have a corresponding Boolean homomorphism $\theta_1:\mathfrak{A}_1\to\Sigma_0$ given by setting $\theta_1(a)=\pi_J^{-1}[\theta_J(\phi(a))]$ for each $a\in A_1$. As remarked in A2Qc, there is an extension θ of θ_1 to a lifting from $\mathfrak{A}$ to Σ.

Set $E_\xi=\theta(a_\xi)$ for each $\xi<\lambda$. By part (a) above, there is an infinite set $W\subseteq\lambda$ such that $\mu_0^*(\bigcap_{\xi\in W}E_\xi)\ge\gamma'$. Now suppose that $b\in\mathfrak{A}_0$ and $b\supseteq\inf_{\xi\in W}a_\xi$. Then $\theta(b)\supseteq\bigcap_{\xi\in L}E_\xi$ and $\theta(b)\in\Sigma_0$, so

$$\bar{\mu}b = \mu(\theta(b)) \ge \mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'.$$

Thus we have the set W we need. **Q**

(c) We are now ready for the general case of the lemma. Let $(\mathfrak{A},\bar{\mu})$ be the measure algebra of (X,Σ,μ) and set $\mathfrak{A}_0 = \{G^\bullet : G \in \Sigma_0\}$, $a_\xi = E_\xi^\bullet$ for each $\xi < \lambda$. By (b), there is an infinite $W \subseteq \lambda$ such that $\bar{\mu}b \ge \gamma'$ whenever $b \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in W} a_\xi$. Now if $G \in \Sigma_0$ and $G \supseteq \bigcap_{\xi\in W} E_\xi$, $b = G^\bullet \in \mathfrak{A}_0$ and $b \supseteq \inf_{\xi\in W} a_\xi$, so that $\mu_0 G = \mu G = \bar{\mu}G^\bullet \ge \gamma'$. As G is arbitrary, $\mu_0^*(\bigcap_{\xi\in W} E_\xi) \ge \gamma'$, and we have a suitable set W.

S6G Proposition Let κ be an atomlessly-measurable cardinal.

(a) Let X be a Hausdorff space such that $\chi(X) < \kappa$. Then any Radon probability measure on X has Maharam type at most $\max(\omega,\chi(X))$.

(b) Let X be a Hausdorff space such that $\chi(x,X) < \kappa$ for every $x \in X$. Then any Radon probability measure on X has Maharam type less than κ.

proof ? Suppose, if possible, otherwise. By SA2D, X has a Maharam homogeneous Radon probability measure μ of Maharam type λ, where $\lambda = \max(\omega_1,\chi(X)^+)$ in (a) and $\lambda = \kappa$ in (b). Let $\mathfrak{A}$ be the measure algebra of μ; then $\mathfrak{A}$ can be expressed as the union of a strictly increasing family $\langle\mathfrak{A}_\xi\rangle_{\xi<\lambda}$ of closed subalgebras. For each $x \in X$ choose a base $\mathcal{U}_x$ of open neighbourhoods of x with $\#(\mathcal{U}_x) = \chi(x,X)$. Set

$$X_\xi = \{x : x \in X,\ U^\bullet \in \mathfrak{A}_\xi \text{ for every } U \in \mathcal{U}_x\}.$$

Because $\mathrm{cf}(\lambda) = \lambda > \chi(x,X)$ for every x, $\bigcup_{\xi<\lambda} X_\xi = X$.

By 6C/S6E, $\sup_{\xi<\lambda}\mu^* X_\xi = 1$; because $\mathrm{cf}(\lambda) > \omega$, there is a $\xi < \lambda$ such that $\mu^* X_\xi = 1$. Now let G be any open subset of X. Set

$$\mathcal{H} = \{H : H \subseteq G \text{ is open, } H^\bullet \in \mathfrak{A}_\xi\}, \quad H_0 = \bigcup\mathcal{H}.$$

Then $H_0 \supseteq G \cap X_\xi$. So $G \setminus H_0$ must be negligible, and

$$G^\bullet = H_0^\bullet = \sup_{H\in\mathcal{H}} H^\bullet \in \mathfrak{A}_\xi.$$

As G is arbitrary, $\mathfrak{A} = \mathfrak{A}_\xi$, which is absurd. **X**

Remark This is derived from Plebanek 95.

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S7′. Topological implications

S7′A Proposition Suppose there is an atomlessly-measurable cardinal. Then there is a locally compact locally countable Hausdorff topology $\mathfrak{T}$ on ω_1 such that every closed set is either countable or cocountable.

proof (a) By , there is an ω_2-additive measure ν on $X = \{0,1\}^{\omega_1}$ extending the usual measure on X. We may suppose that ν is Maharam homogeneous, with Maharam type λ say, and that we have an independent family $\langle E_\xi\rangle_{\xi<\lambda}$ of measurable sets generating the measure algebra of ν, starting with $E_\xi = \{x : x(\xi) = 1\}$ for $\xi < \omega_1$ (). For $\Gamma \subseteq \lambda$ write Σ_Γ for the σ-algebra generated by $\{E_\xi : \xi \in \Gamma\} \cup \{E : \nu E = 0\}$, so that $\mathcal{P}X = \bigcup\{\Sigma_\Gamma : \Gamma \in [\lambda]^{\le\omega}\}$. Write Ω for the set of non-zero countable limit ordinals; for $\alpha \in \Omega$, enumerate α as $\langle e_\alpha(n)\rangle_{n\in\mathbb{N}}$. Fix a bijection $\phi : \{0,1\}^{\mathbb{N}} \to [0,1]^{\mathbb{N}}$ which is an automorphism for the usual measures (Fremlin p00*, 344I). For $\alpha \in \Omega$, $n \in \mathbb{N}$ define $\phi_{\alpha n} : X \to [0,1]$ by writing $\langle\phi_{\alpha n}(x)\rangle_{n\in\mathbb{N}} = \phi(\langle x(\alpha+n)\rangle_{n\in\mathbb{N}})$ for every $x \in X$.

(b) For $x \in X$, $\xi < \omega_1$ choose $B_x(\xi)$ as follows. If $\xi < \omega$ then $B_x(\xi) = \{\xi\}$. Suppose that $B_x(\xi)$ has been defined for every $\xi < \alpha$, where $\alpha \in \Omega$. Set $C_x(\alpha,n) = B_x(e_\alpha(n)) \setminus \bigcup_{i<n} B_x(e_\alpha(i))$ for each n. Set

$$B_x(\alpha+n) = \{\alpha+n\} \cup \bigcup\{C_x(\alpha,i) : i \in \mathbb{N},\ 2^{-n-1} \le \phi_{\alpha i}(x) < 2^{-n}\}$$

for each n.

Let $\mathfrak{T}_x$ be the topology on ω_1 generated by $\{B_x(\xi) : \xi < \omega_1\} \cup \{\omega_1 \setminus B_x(\xi) : \xi < \omega_1\}$. Then $\mathfrak{T}_x$ is Hausdorff, locally countable and locally compact. **P** It is Hausdorff because if $\xi < \eta$ then $\xi \in B_x(\xi)$ and $\eta \notin B_x(\xi)$. (The point is that if α is a limit ordinal and $n \in \mathbb{N}$, then

$$\alpha+n \in B_x(\alpha+n) \subseteq \{\alpha+n\} \cup \alpha.)$$

It is locally countable because every $B_x(\xi)$ is countable. Finally, an induction on ξ shows that every $B_x(\xi)$ is compact, because if, for $\alpha \in \Omega$, we take $\mathfrak{T}_{x\alpha}$ to be the topology on α generated by $\{B_x(\xi) : \xi < \alpha\} \cup \{\alpha \setminus B_x(\xi) : \xi < \alpha\}$, then every $C_x(\beta, n) \cap \alpha$ and every $B_x(\eta) \cap \alpha$, for $\beta \in \Omega \setminus \alpha$, $\eta \in \omega_1 \setminus \alpha$ are open-and-closed for $\mathfrak{T}_\alpha$. Now, given $\alpha \in \Omega$, the inductive hypothesis tells us that $C_x(\alpha, i)$ is always compact for $\mathfrak{T}_{x\alpha}$, so that $B_x(\alpha + n)$ is compact for $\mathfrak{T}_{x,\alpha+\omega}$ and therefore for $\mathfrak{T}_x$. So $\mathfrak{T}_x$ is locally compact. **Q**

(c) The essential fact to note about this construction is that $\{x : \xi \in B_x(\eta)\}$, $\{x : \xi \in C_{\beta,n}\}$ belong to Σ_α whenever $\alpha \in \Omega$, $\xi < \omega_1$, $\eta < \alpha$, $\beta \in \alpha \cap \Omega$ and $n \in \mathbb{N}$.

? Suppose, if possible, that for every $x \in X$ there is a set $D_x \subseteq \omega_1$ which is $\mathfrak{T}_x$-closed but neither countable nor cocountable. For $\xi < \omega_1$ set $H_\xi = \{x : \xi \in D_x\} \subseteq X$. Then there is a countable set $\Gamma_\xi \subseteq \lambda$ such that $H_\xi \in \Sigma_{\Gamma_\xi}$. Note also that, for any $\xi < \omega_1$, $\bigcup_{\eta > \xi} H_\eta = X$, so there is a countable $\Delta_\xi \subseteq \omega_1$ such that $\bigcup_{\eta \in \Delta_\xi, \eta > \xi} H_\eta$ is conegligible. Let A be

$$\{\alpha : \alpha \in \Omega,\ (\Gamma_\xi \cap \omega_1) \cup \Delta_\xi \subseteq \alpha \text{ for every } \xi < \alpha\};$$

then A is a club set in ω_1. Let α be the supremum of a strictly increasing set in A. Set $\Gamma = \bigcup_{\xi<\alpha} \Gamma_\xi$, so that $\Gamma \cap \omega_1 \subseteq \alpha$.

Let Y be $\{y : D_y \cap \alpha$ is cofinal with $\alpha\}$, so that Y is conegligible. For every $x \in X$, D_x does not include $\omega_1 \setminus \alpha$, so there is a least $\zeta_x \ge \alpha$ such that $\zeta_x \notin D_x$; let $\zeta \in \omega_1 \setminus \alpha$ be the least ordinal such that $Y_2 = \{y : y \in Y,\ \zeta_y = \zeta\}$ is not negligible. Set $Y_1 = \{y : y \in Y,\ \zeta_y \ge \zeta\}$, so that Y_1 is conegligible. Express ζ as $\beta + n$ where $\beta \in \Omega$, $n \in \mathbb{N}$. Observe that H_ξ is conegligible, so belongs to Σ_Γ, for every $\xi \in \beta \setminus \alpha$, so $H_\xi \in \Sigma_\Gamma$ for every $\xi < \beta$.

For $x \in X$, set $I_x = \{m : m \in \mathbb{N},\ D_x \cap C_y(\beta, m) \ne \emptyset\}$. Observe that

$$\{x : m \in I_x\} = \bigcup_{\xi<\beta} \{x : \xi \in D_x \cap C_x(\beta, m)\} \in \Sigma_{\Gamma \cup \beta}$$

for every $m \in \mathbb{N}$. If $y \in Y_1$, $D_y \cap \beta$ is cofinal with β, whether or not $\beta = \alpha$, because $D_y \cap \alpha$ is cofinal with α and $\beta \setminus \alpha \subseteq D_y$. Since

$$\sup C_y(\beta, m) \le B_y(e_\beta(m)) = e_\beta(m) < \beta$$

for every m, I_y is infinite, for every $y \in Y_1$.

On the other hand, setting $F_m = \{x : 2^{-n-1} \le \phi_{\beta m}(x) < 2^{-n}\}$ for $m \in \mathbb{N}$, $\langle F_m \rangle_{m\in\mathbb{N}}$ is a stochastically independent sequence of sets of measure 2^{-n-1} all belonging to $\Sigma_{(\beta+\omega)\setminus\beta}$, which is independent from $\Sigma_{\Gamma\cup\beta}$. Set $J_y = \{m : y \in F_m\}$. Then

$$G = \{y : I_y \text{ is infinite, } I_y \cap J_y \text{ is finite}\}$$

is negligible. (Set $h_1(y) = I_y$, $h_2(y) = J_y$, $h(y) + (I_y, J_y)$, so that h_1 and h_2 are functions from X to $\mathcal{P}\mathbb{N}$. Let μ_1, μ_2 be the Radon measures on $\mathcal{P}\mathbb{N}$ such that h_1 and h_2 are inverse-measure -preserving, and h is inverse-measure -preserving for the product measure $\mu_1 \times \mu_2$ on $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Now the set

$$\{(I, J) : I \in [\mathbb{N}]^\omega,\ J \subseteq \mathbb{N},\ I \cap J \text{ is finite}\}$$

is a Borel set in $\mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$ which has μ_2-negligible vertical sections, so is $\mu_1 \times \mu_2$-negligible, and its inverse image G is ν-negligible.)

There is therefore some $y \in Y_2 \setminus G$. But in this case, $\zeta \notin D_y$, so $B_y(\zeta) \cap D_y$ is a $\mathfrak{T}_y$-compact set not containing ζ. Since

$$B_y(\zeta) = \{\zeta\} \cup \bigcup_{m \in J_y} C_y(\beta, m),$$

there must be a finite set $K \subseteq J_y$ such that $B_y(\zeta) \cap D_y \subseteq \bigcup_{m\in K} C_y(\beta, m)$. Since I_y is infinite and $y \notin G$, there is an $m \in I_y \cap J_y \setminus K$, and $C_y(\beta, m) \cap D_y$ is not empty; which is impossible. **X**

This contradiction shows that for at least one $x \in X$, every $\mathfrak{T}_x$-closed set is either countable or cocountable.

Remarks This proposition is based on ideas of J.Moore.

Observe that, for a topology $\mathfrak{T}$ as described, ω_1 is hereditarily separable. **P?** If $A \subseteq \omega_1$ is not separable, of course it is uncountable. For $\xi < \omega_1$, choose α_ξ, β_ξ in A, and countable open sets $G_\xi \subseteq \omega_1$ inductively, as follows. Given $\langle \alpha_\eta \rangle_{\eta<\xi}$, let $\beta_\xi \in A$ be such that $\beta_\xi \notin \overline{\{\alpha_\eta : \eta < \xi\} \cup (A \cap \xi)}$. Let G_ξ be a countable open set containing β_ξ and not meeting $\{\alpha_\eta : \eta < \xi\}$. Given $\langle G_\eta \rangle_{\eta\le\xi}$, take $\alpha_\xi \in A \setminus (\xi \cup \bigcup_{\eta\le\xi} G_\eta)$.

Now consider $F = \overline{\{\alpha_\xi : \xi < \omega_1\}}$. Then $\alpha_\xi \in F$ and $\beta_\xi \notin F$ for every ξ. Since α_ξ, $\beta_\xi \ge \xi$ for every ξ, F is neither countable nor cocountable, which is impossible. **XQ**

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S7. Partially ordered sets

S7A Cofinalities III: Lemma Let κ be a real-valued-measurable cardinal and $\langle\alpha_i\rangle_{i\in I}$ a countable family of ordinals less than κ and of cofinality at least ω_2. Then there is a set $F \subseteq \prod_{i\in I}\alpha_i$ such that (i) F is cofinal with $\prod_{i\in I}\alpha_i$ (ii) if $\langle f_\xi\rangle_{\xi<\omega_1}$ is an increasing family in F then $\sup_{\xi<\omega_1} f_\xi \in F$ (iii) $\#(F) < \kappa$.

proof We have $\mathrm{cf}(\prod_{i\in I}\alpha_i) = \mathrm{cf}(\prod_{i\in I}\mathrm{cf}(\alpha_i)) < \kappa$, by 7Kb. So we may find a cofinal set $F_0 \subseteq \prod_{i\in I}\alpha_i$ of cardinal less than κ. Now for $0 < \zeta \le \omega_2$ define F_ζ by saying that

$$F_{\zeta+1} = \{\sup_{\xi<\omega_1} f_\xi : \langle f_\xi\rangle_{\xi<\omega_1} \text{ is an increasing family in } F_\zeta\},$$

$$F_\zeta = \bigcup_{\eta<\zeta} F_\eta \text{ for non-zero limit ordinals } \zeta \le \omega_2.$$

Then $\#(F_\zeta) < \kappa$ for every ζ. **P** Induce on ζ. For the inductive step to $\zeta+1$, **?** suppose, if possible, that $\#(F_\zeta) < \kappa$ but $\#(F_{\zeta+1}) \ge \kappa$. For each $h \in F_{\zeta+1}$ choose an increasing family $\langle f_{h\xi}\rangle_{\xi<\omega_1}$ in F_ζ with supremum h. The set $h[I]$ of values of h is a countable subset of $Y = \bigcup_{i\in I}\alpha_i$. By 5A, taking $X = F_{\zeta+1}$ and μ a non-trivial κ-additive measure on X, there is a set $H \subseteq F_{\zeta+1}$, of cardinal κ, such that $M = \bigcup_{h\in H} h[I]$ is countable. Now, for each $h \in H$, there is a $\gamma(h) < \omega_1$ such that whenever $i \in I$ and $\beta \in M$ then $h(i) > \beta$ iff $f_{h,\gamma(h)}(i) > \beta$. If g, $h \in H$ and $i \in I$ and $g(i) < h(i)$, then $f_{g,\gamma(g)}(i) \le g(i) < f_{h,\gamma(h)}(i)$, because $g(i) \in M$. Thus $h \mapsto f_{h,\gamma(h)} : H \to F_\zeta$ is injective; but $\#(F_\zeta) < \kappa = \#(H)$. **X**

Thus $\#(F_{\zeta+1}) < \kappa$ if $\#(F_\zeta) < \kappa$. At limit ordinals ζ the induction proceeds without difficulty because $\mathrm{cf}(\kappa) > \zeta$. **Q**

So $\#(F_{\omega_2}) < \kappa$ and we may take $F = F_{\omega_2}$.

S7B Theorem Let κ be a real-valued-measurable cardinal.

(a) For any cardinal θ, $\mathrm{cf}([\kappa]^{<\theta}) \le \kappa$.

(b) For any cardinal $\lambda < \kappa$, any θ, $\mathrm{cf}([\lambda]^{<\theta}) < \kappa$.

proof (a)(i) Consider first the case $\theta = \omega_1$. Write G_1 for the set of ordinals less than κ of cofinality less than or equal to ω_1; for $\delta \in G_1$ let $\psi_\delta : \mathrm{cf}(\delta) \to \delta$ enumerate a cofinal subset of δ. Next, write G_2 for $\kappa \setminus G_1$, and for every countable set $A \subseteq G_2$ let $F(A) \subseteq \prod_{\alpha\in A}\alpha$ be a cofinal set, of cardinal less than κ, closed under suprema of increasing families of length ω_1; such exists by S7A above.

(ii) It is worth observing at this point that if $\langle A_\zeta\rangle_{\zeta<\omega_1}$ is any family of countable subsets of G_2, $D = \bigcup_{\zeta<\omega_1} A_\zeta$, and $g \in \prod_{\alpha\in D}\alpha$, then there is an $f \in \prod_{\alpha\in D}\alpha$ such that $f \ge g$ and $f\restriction A_\zeta \in F(A_\zeta)$ for every $\zeta < \omega_1$. **P** Let $\langle\phi(\xi)\rangle_{\xi<\omega_1}$ run over ω_1 taking every value uncountably often. Choose an increasing family $\langle f_\xi\rangle_{\xi<\omega_1}$ in $\prod_{\alpha\in D}\alpha$ in such a way that $f_0 = g$ and $f_{\xi+1}\restriction A_{\phi(\xi)} \in F(A_{\phi(\xi)})$ for every ξ; this is possible because $F(A)$ is cofinal with $\prod_{\alpha\in A}\alpha$ for every A. Set $f = \sup_{\xi<\omega_1} f_\xi$; this works because every $F(A)$ is closed under suprema of increasing families of length ω_1. **Q**

(iii) We can now find a family $\mathcal{A}$ of countable subsets of κ such that

(α) $\{\alpha\} \in \mathcal{A}$ for every $\alpha < \kappa$;

(β) whenever A, $A' \in \mathcal{A}$, $\zeta < \omega_1$ then $A \cup A'$, $A \cap G_2$, $\{\psi_\alpha(\xi) : \alpha \in A \cap G_1, \xi < \min(\zeta, \mathrm{cf}(\alpha))\}$ all belong to $\mathcal{A}$;

(γ) whenever $A \in \mathcal{A}$, $A \subseteq G_2$ then $f[A] \in \mathcal{A}$ for every $f \in F(A)$;

(δ) $\#(\mathcal{A}) \le \kappa$.

(iv) **?** Suppose, if possible, that $\mathrm{cf}([\kappa]^{\le\omega}) > \kappa$. Because $[\kappa]^{\le\omega} = \bigcup_{\lambda<\kappa}[\lambda]^{\le\omega}$, there is a cardinal $\lambda < \kappa$ such that $\mathrm{cf}([\lambda]^{\le\omega}) > \kappa$. We can therefore choose inductively a family $\langle a_\xi\rangle_{\xi<\kappa}$ of countable subsets of λ such that

$$a_\xi \not\subseteq \bigcup_{\eta\in A\cap\xi} a_\eta$$

whenever $\xi < \kappa$, $A \in \mathcal{A}$. By 5A, there is a set $W \subseteq \kappa$, of cardinal κ, such that $\bigcup_{\xi\in W} a_\xi$ is countable. Let $\delta < \kappa$ be such that $W \cap \delta$ is cofinal with δ and of order type ω_1.

(v) I choose a family $\langle A_{k\zeta}\rangle_{\zeta<\omega_1,k\in\mathbb{N}}$ in $\mathcal{A}$ as follows. Start by setting $A_{0\zeta}=\psi_\delta[\zeta]$ for every $\zeta<\omega_1$; then $A_{0\zeta}\in\mathcal{A}$ by (iii)(α-β). Given $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$, set $A'_{k\zeta}=A_{k\zeta}\cap G_2$ for each $\zeta<\omega_1$. For $\alpha\in D_k=\bigcup_{\zeta<\omega_1}A'_{k\zeta}$, set $g_k(\alpha)=\sup(\alpha\cap W\cap\delta)<\alpha$; choose $f_k\in\prod_{\alpha\in D_k}\alpha$ such that $g_k\le f_k$ and $f_k\restriction A'_{k\zeta}\in F(A'_{k\zeta})$ for every ζ; this is possible by (ii) above. Set

$$A_{k+1,\zeta}=A_{k\zeta}\cup f_k[A'_{k\zeta}]\cup\{\psi_\alpha(\xi):\alpha\in A_{k\zeta}\cap G_1,\ \xi<\min(\zeta,\mathrm{cf}(\alpha))\}\in\mathcal{A}$$

for each $\zeta<\omega_1$, and continue. An easy induction on k shows that $\langle A_{k\zeta}\rangle_{\zeta<\omega_1}$ is increasing for every k; also, every $A_{k\zeta}$ is a subset of δ.

(vi) Set $V_k=\bigcup_{\zeta<\omega_1}A_{k\zeta}$, $b_k=\bigcup\{a_\xi:\xi\in W\cap V_k\}$; then b_k is countable and there is a $\beta(k)<\omega_1$ such that $b_k=\bigcup\{a_\xi:\xi\in W\cap A_{k,\beta(k)}\}$. Now $\bigcup_{k\in\mathbb{N}}A_{k,\beta(k)}$ is a countable subset of δ, so there is a member γ of $W\cap\delta$ greater than its supremum. We have

$$a_\gamma\not\subseteq\bigcup\{a_\eta:\eta\in A_{k,\beta(k)}\}$$

for every k, so $a_\gamma\not\subseteq b_k$ and $\gamma\notin V_k$, for each k.

Set $V=\bigcup_{k\in\mathbb{N}}V_k$, $\gamma_0=\min(W\cap\delta\setminus V)$. Because $V_0=\psi_\delta[\omega_1]$ is cofinal with δ, $V\setminus\gamma_0\ne\emptyset$; let γ_1 be its least member. Then $\gamma_1>\gamma_0$. Suppose $\gamma_1\in A_{k\zeta}$. Observe that if $\alpha\in V\cap G_1$ then $V\cap\alpha$ is cofinal with α; but $V\cap\gamma_1\subseteq\gamma_0$, so $\gamma_1\notin G_1$ and $\gamma_1\in D_k$. But now $f_k(\gamma_1)\in A_{k+1,\zeta}\subseteq V$ and $\gamma_0\le g_k(\gamma_1)\le f_k(\gamma_1)<\gamma_1$, so $\gamma_1\ne\min(V\setminus\gamma_0)$. **X**

(vii) This contradiction shows that $\mathrm{cf}([\kappa]^{\le\omega})\le\kappa$. Now consider $\mathrm{cf}([\kappa]^{\le\delta})$, where $\delta<\kappa$ is an infinite cardinal. Then

$$\mathrm{cov}_{\mathrm{Sh}}(\kappa,\delta^+,\delta^+,\omega_1)=\sup_{\lambda<\kappa}\mathrm{cov}_{\mathrm{Sh}}(\lambda,\delta^+,\delta^+,\omega_1)\le\kappa$$

by 7Kd. (See A1J for the definition of $\mathrm{cov}_{\mathrm{Sh}}$.) So there is a family $\mathcal{B}\subseteq[\kappa]^{\le\delta}$, of cardinal at most κ, such that every member of $[\kappa]^{\le\delta}$ is covered by a sequence in $\mathcal{B}$. But now there is a family $\mathfrak{C}$ of countable subsets of $\mathcal{B}$ which is cofinal with $[\mathcal{B}]^{\le\omega}$ and of cardinal at most κ; setting $\mathcal{D}=\{\bigcup\mathcal{C}:\mathcal{C}\in\mathfrak{C}\}$, we have $\mathcal{D}$ cofinal with $[\kappa]^{\le\delta}$ and of cardinal at most κ. So $\mathrm{cf}([\kappa]^{\le\delta})\le\kappa$.

Finally, of course, $[\kappa]^{<\theta}=\bigcup_{\delta<\theta}[\kappa]^{\le\delta}$, so $\mathrm{cf}([\kappa]^{<\theta})\le\sup_{\delta<\theta}\mathrm{cf}([\kappa]^{\le\delta})\le\kappa$ whenever $\theta\le\kappa$. For $\theta>\kappa$ we have $\mathrm{cf}([\kappa]^{<\theta})=1$, so $\mathrm{cf}([\kappa]^{<\theta})\le\kappa$ for every θ.

(b) If $\mathcal{A}$ is cofinal with $[\kappa]^{<\theta}$ then $\{A\cap\lambda:A\in\mathcal{A}\}$ is cofinal with $[\lambda]^{<\theta}$, so $\mathrm{cf}([\lambda]^{<\theta})\le\mathrm{cf}([\kappa]^{\theta})\le\kappa$, by (a). But by 7Id and 7Jb, $\mathrm{cf}([\lambda]^{<\theta})\ne\kappa$. Thus $\mathrm{cf}([\lambda]^{<\theta})<\kappa$, as claimed.

Remark This result is taken from Shelah #430.

S7C Corollary Let κ be a real-valued-measurable cardinal. Let $\langle P_\zeta\rangle_{\zeta<\lambda}$ be a family of partially ordered sets such that $\lambda<\mathrm{add}(P_\zeta)\le\mathrm{cf}(P_\zeta)<\kappa$ for every $\zeta<\kappa$. Then $\mathrm{cf}(\prod_{\zeta<\lambda}P_\zeta)<\kappa$.

proof For each $\zeta<\lambda$ let Q_ζ be a cofinal subset of P_ζ of cardinal less than κ. Set $P=\prod_{\zeta<\kappa}P_\zeta$, $Z=\bigcup_{\zeta<\lambda}Q_\zeta$; then $\#(Z)<\kappa$ so $\mathrm{cf}([Z]^{\le\lambda})<\kappa$, by S7Bb. Let $\mathcal{A}$ be a cofinal subset of $[Z]^{\le\lambda}$ with $\#(\mathcal{A})<\kappa$. For each $A\in\mathcal{A}$ choose $f_A\in P$ such that $f_A(\zeta)$ is an upper bound for $A\cap P_\zeta$ for every ζ; this is possible because $\mathrm{add}(P_\zeta)>\#(A)$. Set $F=\{f_A:A\in\mathcal{A}\}$.

If $g\in P$, then there is an $h\in\prod_{\zeta<\lambda}Q_\zeta$ such that $g\le h$. Now $h[\lambda]\in[Z]^{\le\lambda}$ so there is an $A\in\mathcal{A}$ such that $h[\lambda]\subseteq A$. In this case $h\le f_A$. Accordingly F is cofinal with P and $\mathrm{cf}(P)\le\#(F)<\kappa$, as required.

S7D The techniques of 7C can be adapted to give a further combinatorial result.

Theorem Let κ be an atomlessly-measurable cardinal, and $\lambda<\kappa$ an infinite cardinal. Then there is a function $f:[\lambda^+]^2\to\mathbb{N}$ such that whenever $\mathcal{I}$ is a disjoint family of finite subsets of λ^+ with $\#(\mathcal{I})=\lambda^+$, and $k\in\mathbb{N}$, there are distinct I, $J\in\mathcal{I}$ such that $f(\{\xi,\eta\})=k$ for every $\xi\in I$, $\eta\in J$.

proof (a) For $\xi<\lambda^+$ let $e_\xi:\xi\to\lambda$ be an injective function. Give $\mathbb{N}$ the Radon measure μ_0 defined by saying that $\mu_0\{n\}=2^{-n-1}$ for every $n\in\mathbb{N}$, and give $\mathbb{N}^\lambda$ the product measure μ. Then μ is an atomless Radon probability measure with Maharam type λ, so there is a κ-additive measure ν, defined on every subset of $\mathbb{N}^\lambda$, extending μ (3I).

For $w\in\mathbb{N}^\lambda$, define $f_w:[\lambda^+]^2\to\mathbb{N}$ by setting

$$f_w(\{\xi,\eta\})=w(e_\eta(\xi))$$

whenever $\xi < \eta < \lambda^+$.

(b) **?** Suppose, if possible, that no f_w witnesses the truth of the theorem. Then for every $w \in \mathbb{N}^\lambda$ there are a $k_w \in \mathbb{N}$ and a disjoint $\mathcal{I}_w \subseteq [\lambda^+]^{<\omega}$ such that $\#(\mathcal{I}_w) = \lambda^+$ and whenever I, J are distinct members of $\mathcal{I}_w$ there are $\xi \in I$, $\eta \in J$ such that $f_w(\{\xi, \eta\}) \neq k_w$. Take $k \in \mathbb{N}$ such that $\nu R > 0$, where

$$R = \{w : w \in \mathbb{N}^\lambda,\, k_w = k\}.$$

For every $\alpha < \lambda^+$, $w \in \mathbb{N}^\lambda$ there is an $I \in \mathcal{I}_w$ such that $I \cap \alpha = \emptyset$; because $\#([\lambda^+ \setminus \alpha]^{<\omega}) < \kappa$, there is an $I \subseteq \lambda^+ \setminus \alpha$ such that $\nu\{w : w \in R,\, I \in \mathcal{I}_w\} > 0$. We may therefore choose $\langle I_\alpha \rangle_{\alpha < \lambda^+}$ inductively so that

$$\xi < \eta \text{ whenever } \xi \in I_\alpha,\, \eta \in I_\beta \text{ and } \alpha < \beta < \lambda^+,$$

$$\nu S_\alpha > 0 \text{ for every } \alpha < \lambda,$$
$$\text{where } S_\alpha = \{w : w \in R,\, I_\alpha \in \mathcal{I}_w\}.$$

Take $\delta > 0$, $n \in \mathbb{N}$ such that $\#(A) = \lambda^+$, where

$$A = \{\alpha : \alpha < \lambda^+,\, \#(I_\alpha) = n,\, \nu S_\alpha \geq \delta\}.$$

Take $l \in \mathbb{N}$ such that $(1 - 2^{-(k+1)n^2})^l \leq \frac{1}{3}\delta$.

(b) By A2S, there is a set $B \subseteq A$, of cardinal λ^+, such that $\mu^*(\bigcap_{\alpha \in L} S_\alpha) \geq \frac{2}{3}\delta$ whenever $L \subseteq B$ and $\#(L) \leq l + 1$. Take $\beta \in B$ such that $B \cap \beta$ is infinite. Now for each $\eta \in I_\beta$, $e_\eta : \eta \to \lambda$ is injective. Accordingly we can find a set $L \subseteq B \cap \beta$ such that $\#(L) = l$ and

$$e_\eta[I_\alpha] \cap e_{\eta'}[I_{\alpha'}] = \emptyset$$

whenever α, α' are distinct members of L and η, $\eta' \in I_\beta$. For $\alpha \in L$ set

$$K_\alpha = \{e_\eta(\xi) : \xi \in I_\alpha,\, \eta \in I_\beta\};$$

then $\langle K_\alpha \rangle_{\alpha \in L}$ is disjoint and $\#(K_\alpha) \leq n^2$ for every α.

Consider

$$E = \{w : w \in \mathbb{N}^\lambda,\, \forall\, \alpha \in L\, \exists\, \zeta \in K_\alpha \text{ such that } w(\zeta) \neq k\}.$$

Then

$$\mu E = \prod_{\alpha \in L}(1 - 2^{-(k+1)\#(K_\alpha)}) \leq \frac{1}{3}\delta.$$

On the other hand, if $w \in S_\beta \cap \bigcap_{\alpha \in L} S_\alpha$, $I_\beta \in \mathcal{I}_w$ and $I_\alpha \in \mathcal{I}_w$ for every $\alpha \in L$. So for every $\alpha \in L$ there are $\xi \in I_\alpha$, $\eta \in I_\beta$ such that $f_w(\{\xi, \eta\}) \neq k_w$, that is, $w(e_\eta(\xi)) \neq k$; and we have a $\zeta \in K_\alpha$ such that $w(\zeta) \neq k$. Thus

$$\bigcap_{\alpha \in L \cup \{\beta\}} S_\alpha \subseteq E,$$

and (by the choice of B) we get

$$\frac{2}{3}\delta \leq \mu^*(\bigcap_{\alpha \in L \cup \{\beta\}} S_\alpha) \leq \mu E \leq \frac{1}{3}\delta,$$

which is absurd. **X**

(d) Thus there is a $w \in \mathbb{N}^\lambda$ for which f_w is a suitable function.

S7E Corollary Let κ be an atomlessly-measurable cardinal, and $\lambda < \kappa$ an infinite cardinal. Then there is a function $g : [\lambda^+]^2 \to \mathbb{N}$ such that whenever $\mathcal{I}$ is a disjoint family of finite subsets of λ^+ with $\#(\mathcal{I}) = \lambda^+$, and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function, there are distinct I, $J \in \mathcal{I}$ such that $g(\{\xi, \eta\}) = h(\#(I \cap \xi), \#(J \cap \eta))$ whenever $\xi \in I$, $\eta \in J$ and $\xi < \eta$.

proof Take $f : [\lambda^+]^2 \to \mathbb{N}$ from S7D. Enumerate as $\langle h_k \rangle_{k \in \mathbb{N}}$ the set of all maps from finite products $\{0,1\}^m \times \{0,1\}^m$ to $\mathbb{N}$; take the domain of h_k to be $\{0,1\}^{m_k} \times \{0,1\}^{m_k}$. Choose a family $\langle w_\xi \rangle_{\xi < \lambda^+}$ of distinct elements of $\{0,1\}^{\mathbb{N}}$. Now, given $\xi < \eta < \lambda^+$, set $k = f(\{\xi, \eta\})$ and try $g(\{\xi, \eta\}) = h_k(w_\xi \restriction m_k, w_\eta \restriction m_k)$.

Suppose that $\mathcal{I}$ is a disjoint family of finite subsets of λ^+ with cardinal λ^+, and that $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is any function. Let m, n be such that

$$\mathcal{J} = \{I : I \in \mathcal{I},\ \#(I) = n,\ w_\xi{\restriction}m \ne w_\eta{\restriction}m \text{ for all distinct } \xi,\ \eta \in I\}$$

has cardinal λ^+, and $\langle v_i \rangle_{i<n}$ such that

$$\mathcal{K} = \{I : I \in \mathcal{J},\ w_\xi{\restriction}m = v_i \text{ whenever } \xi \in I,\ \#(\xi \cap I) = i < n\}$$

has cardinal λ^+. Take $k \in \mathbb{N}$ such that $m_k = m$ and $h_k(v_i, v_j) = h(i, j)$ for all i, $j < n$. (Note that $v_i \ne v_j$ if $i \ne j$, because $w_\xi{\restriction}m \ne w_\eta{\restriction}m$ if ξ, η are distinct members of any $I \in \mathcal{J}$.) Then there are distinct I, $J \in \mathcal{K}$ such that $f(\{\xi, \eta\}) = k$ for $\xi \in I$, $\eta \in J$. So if $\xi \in I$, $\eta \in J$, $\xi < \eta$ we have

$$g(\{\xi, \eta\}) = h_k(w_\xi{\restriction}m, w_\eta{\restriction}m) = h_k(v_i, v_j) = h(i, j),$$

where $i = \#(I \cap \xi)$, $j = \#(J \cap \eta)$. And this is what we need.

S7F Corollary If there is an atomlessly-measurable cardinal, there is a ccc Hausdorff abelian topological group whose square is not ccc.

proof (a) Let $g : [\omega_1]^2 \to \mathbb{N}$ be such that whenever $\mathcal{I}$ is an uncountable disjoint family of finite subsets of ω_1, and $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a function, there are distinct I, $J \in \mathcal{I}$ such that $g(\{\xi, \eta\}) = h(\#(I \cap \xi), \#(J \cap \eta))$ whenever $\xi \in I$, $\eta \in J$ and $\xi < \eta$. Set $X = \omega_1 \times \{0, 1\}$, and let G be the family of finite subsets of X with the group operation $a + b = a \triangle b$. For $\beta < \omega_1$ let H_β be the set of those $a \in G$ such that

$(\beta, 0)$, $(\beta, 1) \notin a$,

if $\alpha < \beta$ and $g(\{\alpha, \beta\}) = i \le 1$ then $(\alpha, i) \notin a$,

if $\alpha < \beta$ and $g(\{\alpha, \beta\}) \ge 2$ then $(\alpha, 0) \in a$ iff $(\alpha, 1) \in a$.

Note that H_β is a subgroup of G. Give G the topology generated by the subgroups H_β and their cosets for $\beta < \omega_1$. This is a topological-group topology, and because $\bigcap_{\beta<\omega_1} H_\beta = \{\emptyset\}$ it is Hausdorff.

(b) It may make the idea behind this definition clearer if I show first that $G \times G$ is not ccc. **P** Consider

$$W_\xi = (\{(\xi, 0)\} + H_\xi) \times (\{(\xi, 1)\} + H_\xi)$$

for $\xi < \omega_1$.

If $\xi < \eta < \omega_1$ and $a \in (\{(\xi, 0)\} + H_\xi) \cap (\{(\eta, 0)\} + H_\eta)$, then $a + \{(\xi, 0)\} \in H_\xi$, so $(\xi, 0) \in a$ and $(\xi, 1) \notin a$. But now $(\xi, 0) \in a + \{(\eta, 0)\}$ and $(\xi, 1) \notin a + \{(\eta, 0)\}$; since $a + \{(\eta, 0)\} \in H_\eta$, we must have $g(\{\xi, \eta\}) = 1$.

If $\xi < \eta < \omega_1$ and $b \in (\{(\xi, 1)\} + H_\xi) \cap (\{(\eta, 1)\} + H_\eta)$, then $b + \{(\xi, 1)\} \in H_\xi$, so $(\xi, 0) \notin b$ and $(\xi, 1) \in b$. But now $(\xi, 0) \notin b + \{(\eta, 1)\}$ and $(\xi, 1) \in b + \{(\eta, 1)\}$; since $b + \{(\eta, 1)\} \in H_\eta$, we must have $g(\{\xi, \eta\}) = 0$.

Since these cannot happen together, $W_\xi \cap W_\eta = \emptyset$ whenever $\xi < \eta < \omega_1$, and we have an uncountable disjoint family of open subsets of $G \times G$. **Q**

(c) G is ccc. **P** Let $\langle U_\xi \rangle_{\xi<\omega_1}$ be a family of non-empty open subsets of G. For each $\xi < \omega_1$ take $a_\xi \in U_\xi$, $I_\xi \in [\omega_1]^{<\omega}$ such that $a_\xi + \bigcap_{\beta \in I_\xi} H_\beta \subseteq U_\xi$. Let $A \subseteq \omega_1$ be an uncountable set such that $\langle a_\xi \rangle_{\xi \in A}$, $\langle I_\xi \rangle_{\xi \in A}$ are Δ-systems with roots a, I respectively; set $a'_\xi = a_\xi \setminus a$ for $\xi \in A$. Set

$$J = I \cup \{(\alpha, 0) : \alpha \in a\} \cup \{(\alpha, 1) : \alpha \in a\},$$

$$J_\xi = (I_\xi \setminus I) \cup \{\alpha : (\alpha, 0) \in a'_\xi\} \cup \{\alpha : (\alpha, 1) \in a'_\xi\},$$

$$m_\xi = \#(J_\xi),$$

and enumerate J_ξ in ascending order as $\langle \alpha_{\xi i} \rangle_{i<m_\xi}$. Define $h_\xi : m_\xi \to \{0, 1, 2\}$ by setting

$$\begin{aligned} h_\xi(i) &= 0 \text{ if } (\alpha_{\xi i}, 0) \notin a'_\xi,\ (\alpha_{\xi i}, 1) \in a'_\xi, \\ &= 1 \text{ if } (\alpha_{\xi i}, 0) \in a'_\xi,\ (\alpha_{\xi i}, 1) \notin a'_\xi, \\ &= 2 \text{ otherwise.} \end{aligned}$$

Let m, $h : m \to \{0, 1, 2\}$ be such that

$$B = \{\xi : \xi \in A,\ m_\xi = m,\ h_\xi = h\}$$

is uncountable. Because any $\alpha < \omega_1$ can belong to at most three of the J_ξ, there is an uncountable $C \subseteq B$ such that (i) $\alpha < \beta$ whenever $\alpha \in J$ and $\beta \in J_\xi$ for some $\xi \in C$ (ii) $\alpha < \beta$ whenever ξ, $\eta \in C$, $\xi < \eta$,

$\alpha \in J_\xi$ and $\beta \in J_\eta$. Now, by the choice of g, there must be $\xi, \eta \in C$ such that $\xi < \eta$ and $g(\{\alpha_{\xi i}, \beta\}) = h(i)$ whenever $i < m$, $\beta \in J_\eta$.

What this means is that for $\alpha \in J_\xi$, $\beta \in J_\eta$,

$$\begin{aligned} g(\{\alpha, \beta\}) &= 0 \text{ if } (\alpha, 0) \notin a'_\xi,\ (\alpha, 1) \in a'_\xi, \\ &= 1 \text{ if } (\alpha, 0) \in a'_\xi,\ (\alpha, 1) \notin a'_\xi, \\ &= 2 \text{ otherwise.} \end{aligned}$$

Set

$$b = a_\xi \cup a_\eta = a_\xi + a'_\eta = a_\eta + a'_\xi.$$

I seek to show that $b \in U_\xi \cap U_\eta$; it will be enough if I can show that $a'_\eta \in H_\beta$ for every $\beta \in I_\xi$ and $a'_\xi \in H_\beta$ for every $\beta \in I_\eta$.

(i) Suppose $\beta \in I$. In this case $\beta \in J$ so $\beta < \alpha$ whenever either $(\alpha, 0)$ or $(\alpha, 1)$ belongs to $a_\xi \triangle a_\eta = a'_\xi \cup a'_\eta$; consequently a'_ξ, a'_η belong to H_β.

(ii) Suppose $\beta \in I_\xi \setminus I$. In this case $\beta < \alpha$ whenever either $(\alpha, 0)$ or $(\alpha, 1)$ belongs to a'_η. So $a'_\eta \in H_\beta$.

(iii) Suppose $\beta \in I_\eta \setminus I$. In this case $\alpha < \beta$ whenever either $(\alpha, 0)$ or $(\alpha, 1)$ belongs to a'_ξ. In particular, neither $(\beta, 0)$ nor $(\beta, 1)$ belongs to a'_ξ. If $\alpha < \beta$ and $g(\{\alpha, \beta\}) = i \le 1$, then either $\alpha \notin J_\xi$, in which case (α, i) certainly does not belong to a'_ξ, or $\alpha \in J_\xi$, in which case again $(\alpha, i) \notin a'_\xi$. While if $\alpha < \beta$ and $g(\{\alpha, \beta\}) \ge 2$, then either $\alpha \notin J_\xi$ and neither $(\alpha, 0)$ nor $(\alpha, 1)$ belongs to a'_ξ, or $\alpha \in J_\xi$ and $(\alpha, 0) \in a'_\xi$ iff $(\alpha, 1) \in a'_\xi$. Thus $a'_\xi \in H_\beta$.

Putting these together, we see that all the requirements for $b \in U_\xi \cap U_\eta$ are met, so that $U_\xi \cap U_\eta \ne \emptyset$. As $\langle U_\xi \rangle_{\xi < \omega_1}$ is arbitrary, G is ccc. **Q**

Remark S7D-S7F are taken from §2 of TODORČEVIĆ 93.

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S9. Quasi-measurable cardinals

S9A Theorem Let κ be a quasi-measurable cardinal.

(a) $\mathrm{cf}([\kappa]^{<\theta}) \le \kappa$ for every cardinal θ.

(b) If $\lambda < \kappa$ then $\mathrm{cf}([\lambda]^{<\theta}) < \kappa$ for every cardinal θ.

(c) If $\langle P_\zeta \rangle_{\zeta < \lambda}$ is a family of partially ordered sets with $\lambda < \mathrm{add}(P_\zeta) \le \mathrm{cf}(P_\zeta) < \kappa$ for every $\zeta < \lambda$, then $\mathrm{cf}(\prod_{\zeta < \kappa} P_\zeta) < \kappa$.

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Appendix: Useful Facts

SA2. Measure theory

SA2A τ-additive Borel measures Let X be a regular topological space and μ a totally finite τ-additive Borel measure on X. Then the completion $\hat{\mu}$ of μ is a quasi-Radon measure on X. **P** If $H \subseteq X$ is open, then (because X is regular) $H = \bigcup \mathcal{G}$, where $\mathcal{G} = \{G : G \subseteq X \text{ is open}, \overline{G} \subseteq H\}$. Consequently

$$\mu H = \sup_{G \in \mathcal{G}} \mu G = \sup\{\mu L : L \subseteq H \text{ is closed}\}.$$

Now let $\mathcal{F}$ be the family of closed sets in X and write $\phi F = \mu F$ for $F \in \mathcal{F}$. Then (i) $\phi\emptyset = 0$ (ii) if E, F are closed subsets of X and $E \subseteq F$,

$$\begin{aligned} \phi F - \phi E &= \mu F - \mu E \\ &= \mu((X \setminus E) \cap F) \\ &= \sup\{\mu(L \cap F) : L \subseteq X \setminus E \text{ is closed}\} \\ &= \sup\{\phi L : L \subseteq F \setminus E \text{ is closed}\} \end{aligned}$$

(iii) whenever $\mathcal{K} \subseteq \mathcal{F}$ is non-empty and downwards-directed and $\bigcap \mathcal{K} = \emptyset$, then $\inf_{K\in\mathcal{K}} \phi K = \mu X - \sup_{K\in\mathcal{K}} \mu(X \setminus K) = 0$ because $\{X \setminus K : K \in \mathcal{K}\}$ is an upwards-directed family of open sets with union X. By FREMLIN 74, Theorem 72A, there is a complete measure μ_1 on X, extending ϕ, which is inner regular for the closed sets. Now for any Borel set $E \subseteq X$,

$$\begin{aligned}\mu_1 E &= \sup\{\mu_1 F : F \subseteq E \text{ is closed}\}\\ &= \sup\{\phi F : F \subseteq E \text{ is closed}\}\\ &= \sup\{\mu F : F \subseteq E \text{ is closed}\}\\ &\leq \mu E;\end{aligned}$$

similarly, $\mu_1(X \setminus E) \leq \mu(X \setminus E)$; but $\mu_1 X = \phi X = \mu X$, so in fact $\mu_1 E = \mu E$, and μ_1 extends μ. It follows at once that μ_1 is actually the completion of μ. Now $\mu_1 = \hat{\mu}$ is τ-additive and inner regular for the closed sets, therefore quasi-Radon. **Q** (Compare GARDNER & PFEFFER 84, .)

SA2B Disintegrations Let $(X, \mathfrak{T})$ be a compact Hausdorff space and (Y, T, ν) a complete probability space. Let μ be a probability on $X \times Y$ such that (i) $\mu(E \times F)$ is defined whenever $E \in \Sigma$, the Baire σ-algebra of X, and $F \in \mathrm{T}$ (ii) $\mu(X \times F) = \nu F$ for every $F \in \mathrm{T}$. Then there is a family $\langle \mu_y \rangle_{y\in Y}$ of Radon probabilities on X such that $\int_F \mu_y E \nu(dy)$ exists and is equal to $\mu(E \times F)$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$.

proof Write $\mathcal{L}^\infty$ for the space of bounded T-measurable functions from Y to $\mathbb{R}$. Because (Y, T, ν) is complete, there is a 'linear lifting' $\rho : \mathcal{L}^\infty \to \mathcal{L}^\infty$ such that

$\rho(f + g) = \rho(f) + \rho(g)$, $\rho(\alpha f) = \alpha\rho(f)$ for all f, $g \in \mathcal{L}^\infty$ and $\alpha \in \mathbb{R}$;
if $f = 0$ ν-a.e. then $\rho(f) = 0$;
$\rho(f) = f$ ν-a.e. for every $f \in \mathcal{L}^\infty$;
if $f \geq 0$ then $\rho(f) \geq 0$;
if f is constant then $\rho(f) = f$

(IONESCU TULCEA & IONESCU TULCEA 69, p. 46, Theorem 3, or FREMLIN 89,). Write $C(X)$ for the set of continuous functions $g : X \to \mathbb{R}$. For each $g \in C(X)$, we have a functional $\nu_g : \mathrm{T} \to [0, 1]$ given by the formula $\nu_g F = \int_{X\times F} g(x)\mu(dxdy)$ for every $F \in \mathrm{T}$; now ν_g is additive and dominated by ν, so by the Radon-Nikodým theorem (FREMLIN 74, 63J) there is a bounded T-measurable function $f_g : Y \to \mathbb{R}$ such that $\nu_g F = \int_F f_g d\nu$ for every $F \in \mathrm{T}$. Set $f'_g = \rho(f_g)$. If g, $h \in C(X)$, then $\nu_{g+h} = \nu_g + \nu_h$ so $f_{g+h} = f_g + f_h$ ν-a.e. and $f'_{g+h} = f'_g + f'_h$. Similarly, $f'_{\alpha g} = \alpha f_g$ for every $g \in C(X)$, $\alpha \in \mathbb{R}$. Also, if $g \geq 0$ in $C(X)$, then $\nu_g \geq 0$ so $f_g \geq 0$ a.e. and $f'_g \geq 0$.

For each $y \in Y$, define $\phi_y : C(X) \to \mathbb{R}$ by setting $\phi_y(g) = f'_g(y)$ for every $g \in C(X)$. Then ϕ_y is a positive linear functional. Also, writing $\mathbf{1}$ for the function with constant value 1, $\nu_{\mathbf{1}} = \nu$ so $f_{\mathbf{1}} = 1$ a.e. and $f'_{\mathbf{1}} = 1$ everywhere; thus $\phi_y(\mathbf{1}) = 1$ for every y. By the Riesz representation theorem (FREMLIN 74, 73D) there is for each $y \in Y$ a Radon probability μ_y on X such that $\int g d\mu_y = \phi_y(g)$ for every $g \in C(X)$. Consequently, if $g \in C(X)$ and $F \in \mathrm{T}$,

$$\int_F (\int g d\mu_y)\nu(dy) = \int_F f'_g(y)\nu(dy) = \int_F f_g d\nu = \nu_g F = \int_{X\times F} g(x)\mu(dxdy).$$

Consider the set H of functions $h : X \to [0, 1]$ such that

$$\int_F \Big(\int h(x)\mu_y(dx)\Big)\nu(dy) = \int_{X\times F} h(x)\mu(dxdy) \ \ \forall \ \ F \in \mathrm{T}.$$

H contains all continuous functions from X to $[0, 1]$ and $\lim_{n\to\infty} h_n \in H$ whenever $\langle h_n \rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in H. Consequently H contains the characteristic function of any set $E \in \Sigma$; that is,

$$\int_F \mu_y E \mu(dy) = \mu(E \times F)$$

for every $E \in \Sigma$, $F \in \mathrm{T}$, as required. (Compare IONESCU TULCEA & IONESCU TULCEA 69, chap. IX.)

SA2C Universally negligible sets If X is a Polish space (that is, a separable metrizable space in which the topology can be defined by a complete metric) a set $A \subseteq X$ is **universally negligible** if it is negligible for every Radon measure on X.

Appendix: Useful Facts

SA1 Combinatorics

SA1A Lemma There is a family $\langle e_\alpha\rangle_{\alpha<\omega_1}$ such that

(i) for each α, e_α is an injective function from α to $\mathbb{N}$;

(ii) if $\alpha \le \beta < \omega_1$, $\{\xi : \xi < \alpha,\ e_\alpha(\xi) \ne e_\beta(\xi)\}$ is finite.

proof Construct the e_α inductively. The inductive hypothesis must include a third condition

(iii) $\mathbb{N} \setminus e_\alpha[\alpha]$ is infinite for every α.

The induction starts trivially with $e_0 = \emptyset$. For the inductive step to $\beta+1$, take any $n \in \mathbb{N} \setminus e_\beta[\beta]$, and set

$$e_{\beta+1}(\xi) = e_\beta(\xi) \text{ if } \xi < \beta,\ e_{\beta+1}(\beta) = n.$$

For the inductive step to a non-zero limit ordinal $\beta < \omega_1$, take a strictly increasing sequence $\langle\beta_n\rangle_{n\in\mathbb{N}}$ with supremum β, starting with $\beta_0 = 0$. Observe that, for any n,

$$A_n = \mathbb{N} \setminus \bigcup_{i\le n} e_{\beta_i}[\beta_i] \supseteq (\mathbb{N} \setminus e_{\beta_n}[\beta_n]) \setminus \{e_{\beta_i}(\xi) : i \le n,\ \xi < \beta_i,\ e_{\beta_i}(\xi) \ne e_{\beta_n}(\xi)\}$$

is infinite. So we can choose a sequence $\langle m_n\rangle_{n\in\mathbb{N}}$ of distinct integers such that $m_n \in A_n$ for every n. Set $B = \{m_n : n \in \mathbb{N}\}$; then B is infinite and $B \cap e_{\beta_n}[\beta_n]$ is finite for every n.

Set $C_{mn} = \{\xi : \xi < \beta_m,\ e_{\beta_m}(\xi) \ne e_{\beta_n}(\xi)\}$ for $m \le n \in \mathbb{N}$. These sets are all finite, so

$$C_n = e_{\beta_{n+1}}^{-1}[B \cup \bigcup_{m\le n} e_{\beta_m}[C_{m,n+1}]]$$

is always finite. If $\xi \in D_n = \beta_{n+1} \setminus (\beta_n \cup C_n)$, then $e_{\beta_{n+1}}(\xi) \notin \bigcup_{m\le n} e_{\beta_m}[\beta_m]$. **P** If $m \le n$ and $\eta < \beta_m$, then either $\eta \notin C_{m,n+1}$, so that $e_{\beta_m}(\eta) = e_{\beta_{n+1}}(\eta) \ne e_{\beta_{n+1}}(\xi)$, or $\eta \in C_{m,n+1}$ so that $e_{\beta_{n+1}}(\xi) \ne e_{\beta_m}(\eta)$ because $\xi \notin e_{\beta_{n+1}}^{-1}[e_{\beta_m}[C_{m,n+1}]]$. **Q**

Set $C = \bigcup_{n\in\mathbb{N}} C_n \setminus \beta_n = \beta \setminus \bigcup_{n\in\mathbb{N}} D_n$, and let $f : C \to B$ be an injective function such that $B \setminus f[C]$ is infinite. Now define $e_\beta : \beta \to \mathbb{N}$ by setting

$$\begin{aligned} e_\beta(\alpha) &= e_{\beta_{n+1}}(\alpha) \text{ if } \alpha \in D_n,\\ &= f(\alpha) \text{ if } \alpha \in C. \end{aligned}$$

I check the three requirements of the inductive hypothesis in turn.

(i) Because f and all the e_{β_n} are injective and

$$e_{\beta_{n+1}}[D_n] \cap e_{\beta_m}[\beta_m] = e_{\beta_{n+1}}[D_n] \cap B = \emptyset$$

whenever $m \le n$, e_β is injective.

(ii) For $n \in \mathbb{N}$, set $E_n = \{\xi : \xi < \beta_n,\ e_{\beta_n}(\xi) \ne e_\beta(\xi)\}$. Then

$$E_{n+1} \subseteq \bigcup_{m\le n}(E_m \cup C_{m,n+1}) \cup C_n$$

for each n, so every E_n is finite. If $\alpha < \beta$, then there is some n such that $\alpha \le \beta_n$, so that

$$\{\xi : \xi < \alpha,\ e_\alpha(\xi) \ne e_\beta(\xi)\} \subseteq E_n \cup \{\xi : \xi < \alpha,\ e_\alpha(\xi) \ne e_{\beta_n}(\xi)\}$$

is finite.

(iii) Because $e_{\beta_{n+1}}^{-1}[B] \subseteq C_n$, $e_{\beta_{n+1}}[D_n] \cap B = \emptyset$ for every n, so $e_\beta[\beta]$ does not meet $B \setminus f[C]$ is $\mathbb{N} \setminus e_\beta[\beta]$ is infinite.

Thus the induction continues.

Remark This is due to .

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SA2. Measure theory

SA2A τ-additive Borel measures Let X be a regular topological space and μ a totally finite τ-additive Borel measure on X. Then the completion $\hat\mu$ of μ is a quasi-Radon measure on X. **P** Fremlin p00*, 415Cb. **Q**

SA2B Disintegrations Let $(X,\mathfrak{T})$ be a compact Hausdorff space and (Y,T,ν) a complete probability space. Let μ be a probability on $X\times Y$ such that (i) $\mu(E\times F)$ is defined whenever $E\in\Sigma$, the Baire σ-algebra of X, and $F\in\mathrm{T}$ (ii) $\mu(X\times F)=\nu F$ for every $F\in\mathrm{T}$. Then there is a family $\langle\mu_y\rangle_{y\in Y}$ of Radon probabilities on X such that $\int_F\mu_yE\nu(dy)$ exists and is equal to $\mu(E\times F)$ whenever $E\in\Sigma$ and $F\in\mathrm{T}$.

proof Write $\mathcal{L}^\infty$ for the space of bounded T-measurable functions from Y to $\mathbb{R}$. Because (Y,T,ν) is complete, there is a 'linear lifting' $\rho:\mathcal{L}^\infty\to\mathcal{L}^\infty$ such that

$\rho(f+g)=\rho(f)+\rho(g)$, $\rho(\alpha f)=\alpha\rho(f)$ for all f, $g\in\mathcal{L}^\infty$ and $\alpha\in\mathbb{R}$;
if $f=0$ ν-a.e. then $\rho(f)=0$;
$\rho(f)=f$ ν-a.e. for every $f\in\mathcal{L}^\infty$;
if $f\ge 0$ then $\rho(f)\ge 0$;
if f is constant then $\rho(f)=f$

(Ionescu Tulcea & Ionescu Tulcea 69, p. 46, Theorem 3, or Fremlin p00*, §341). Write $C(X)$ for the set of continuous functions $g:X\to\mathbb{R}$. For each $g\in C(X)$, we have a functional $\nu_g:\mathrm{T}\to[0,1]$ given by the formula $\nu_gF=\int_{X\times F}g(x)\mu(dxdy)$ for every $F\in\mathrm{T}$; now ν_g is additive and dominated by ν, so by the Radon-Nikodým theorem (Fremlin 74, 63J) there is a bounded T-measurable function $f_g:Y\to\mathbb{R}$ such that $\nu_gF=\int_F f_gd\nu$ for every $F\in\mathrm{T}$. Set $f'_g=\rho(f_g)$. If g, $h\in C(X)$, then $\nu_{g+h}=\nu_g+\nu_h$ so $f_{g+h}=f_g+f_h$ ν-a.e. and $f'_{g+h}=f'_g+f'_h$. Similarly, $f'_{\alpha g}=\alpha f_g$ for every $g\in C(X)$, $\alpha\in\mathbb{R}$. Also, if $g\ge 0$ in $C(X)$, then $\nu_g\ge 0$ so $f_g\ge 0$ a.e. and $f'_g\ge 0$.

For each $y\in Y$, define $\phi_y:C(X)\to\mathbb{R}$ by setting $\phi_y(g)=f'_g(y)$ for every $g\in C(X)$. Then ϕ_y is a positive linear functional. Also, writing $\mathbf{1}$ for the function with constant value 1, $\nu_{\mathbf{1}}=\nu$ so $f_{\mathbf{1}}=1$ a.e. and $f'_{\mathbf{1}}=1$ everywhere; thus $\phi_y(\mathbf{1})=1$ for every y. By the Riesz representation theorem (Fremlin 74, 73D) there is for each $y\in Y$ a Radon probability μ_y on X such that $\int gd\mu_y=\phi_y(g)$ for every $g\in C(X)$. Consequently, if $g\in C(X)$ and $F\in\mathrm{T}$,

$$\int_F(\int gd\mu_y)\nu(dy)=\int_F f'_g(y)\nu(dy)=\int_F f_gd\nu=\nu_gF=\int_{X\times F}g(x)\mu(dxdy).$$

Consider the set H of functions $h:X\to[0,1]$ such that

$$\int_F\Bigl(\int h(x)\mu_y(dx)\Bigr)\nu(dy)=\int_{X\times F}h(x)\mu(dxdy)\ \ \forall\ \ F\in\mathrm{T}.$$

H contains all continuous functions from X to $[0,1]$ and $\lim_{n\to\infty}h_n\in H$ whenever $\langle h_n\rangle_{n\in\mathbb{N}}$ is a pointwise convergent sequence in H. Consequently H contains the characteristic function of any set $E\in\Sigma$; that is,

$$\int_F\mu_yE\mu(dy)=\mu(E\times F)$$

for every $E\in\Sigma$, $F\in\mathrm{T}$, as required. (Compare Ionescu Tulcea & Ionescu Tulcea 69, chap. IX, and Fremlin p00*, §452.)

SA2C Universally negligible sets If X is a Polish space (that is, a separable metrizable space in which the topology can be defined by a complete metric) a set $A\subseteq X$ is **universally negligible** if it is negligible for every Radon measure on X.

SA2D Theorem Let X be a Hausdorff space and suppose that there is a Radon probability measure μ on X with Maharam type $\kappa\ge\omega$. Then whenever $\omega\le\lambda<\kappa$ there is a Maharam homogeneous Radon probability measure μ_1 on X with Maharam type λ.

proof

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Index

This is a joint index to both the present supplement and the original 'Real-valued-measurable cardinals'. References in **bold** type indicate definitions; references in *italics* indicate passing references.

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