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CHAPTER 7

Introduction

In this chapter W will be a Greenian open set in $\mathbb{R}^d$, with Green function G.

In §1 we shall state and prove some of the main principles of potential theory. §2 is devoted to the celebrated capacity theorem some of whose applications are found in §3. §4 deals with the balayage procedure. In §5 we give the rudiments of Dirichlet spaces. Constraints of resources like time and space have forced us to drop subjets like Additive functionals, Martin Boundary, Fine topology etc.

§1. Some Potential Theoretic Principles

Let s be excessive in a Green domain W, with Green function G.

(1) $$h(x) = \lim_{D \uparrow W} E_x[s(X_T)],$$

D relatively compact open in W

T = exit time from D.

is locally integrable and satisfies the mean value property and hence is harmonic in W. Also if u is harmonic in W and $u \leq s$ then clearly $u \leq E_{\cdot}[s(X_T)]$ for all ralatively compact open D so that $u \leq h$:

(2) h <u>defined by</u> (1) <u>is the largest harmonic minorant of</u> s.

$s-h$ is excessive in W and its largest harmonic minorant must be zero. <u>An excessive function whose largest harmonic minorant is zero is called a potential</u>. Thus

(3) Every excessive function can be written in a unique way as the sum of a harmonic function and a potential.

To see uniqueness suppose $s = h_1 + p_1 = h_2 + p_2$, where h_i are harmonic and p_i potentials. Then $p_1 = p_2 + (h_2 - h_1) \geq h_2 - h_1$. Since p_1 is an potential $h_2 - h_1 \leq 0$ and by symmetry $h_1 - h_2 \leq 0$. Thus $h_i = h_2$ and $p_1 = p_2$.

There are no potentials on $\mathbb{R}^2$. Every excessive function on $\mathbb{R}^d, d \geq 3$ is the sum of a constant and a potential, because non-negative harmonic functions on $\mathbb{R}^d$ are constants.

If s is excessive then by Theorem 6 §1, Chapter 6

(4)
$$s = \frac{1}{A_d} Gm + h$$

where m is the Riesz-measure of s, $h \geq 0$ is harmonic and

(5)
$$Gm(x) = \int G(x,y)\, m(dy)$$

It is now clear that s is a <u>potential iff</u>

(6)
$$s = \frac{1}{A_d} Gm$$

Indeed if s is a potential (6) is a consequence of (4) since

h has to be zero. Conversely, the Riesz-measure of the excessive function $s = \frac{1}{A_d} Gm$ is m (as we saw before Theorem 6, §1, Chapter 6) and from (3), $s = p + h$ with h harmonic and p a potential. From what we said above $p = \frac{1}{A_d} Gm$ and hence h must be zero and $s = p$.

We will be needing

(7) $G(x,y) = E_x[G(X_H, y) : H < R]$, $y \in D$, $x \in W$

where D is an open set, $H =$ hitting time to D and $R =$ exit time from W.

To prove (7) start with the fact that $G(\cdot, y)$ is excessive harmonic except at y. So if y is fixed the right side of (7) cannot become larger as $D \ni y$ becomes smaller. So assume that D is relatively compact in W. For any relatively compact open A containing $\bar{D}$, $G(\cdot, y)$ is harmonic in the open set $A \setminus \bar{D}$ and continuous on its closure. Therefore for all x $\bar{A} \setminus D$

$$G(x,y) = E_x[G(X_H, y) : H < S] + E_x[G(X_S, y) : S < H]$$

where $S =$ exit time from A. Let A increase to W. Using (2), because $G(\cdot, y)$ is a potential we obtain (7) for all $x \notin D$. For $x \in D$, (7) is trivial.

Integrating both sides of (7) withrespect to a measure m living on D

(8)
$$\begin{cases} \text{If } Gm = p \text{ for a measure } m,\ D \text{ open and} \\ m(W \setminus D) = 0 \text{ then} \\ \qquad p = E_{\cdot}[p(X_H) : H < R] \\ H = \text{hitting time to } D. \end{cases}$$

In this chapter we shall be concerned mostly with potentials. We start with some of the main principles of potential theory. Note that the statements of these principles (except the domination principle of Maria-Frostman) are desined to make sense if the Green function G is replaced by any "Kernel" i.e. a positive lower semi continuous function on $W \times W$.

The main Principle of potential theory

In the following W is a Green domain with Green function G and for a measure m, Gm is the potential of m:

$$Gm(x) = \int G(x,y)m(dy).$$

The continuity principle. Gm is continuous on support m then Gm is continuous everywhere.

The minimum principle. The minimum of two potentials Gm_1 and Gm_2 is a potential Gm.

The uniqueness principle. If $Gm = Gn \not\equiv \infty$ then $m = n$.

The first or Frostman maximum principle. For all $x \in W$

$$Gm(x) \leq \sup gm(y)$$

where the supremum is over the support of m.

Domination or second maximum principle. If Gm is m-almost everywhere finite and

$$Gm \leq Gn, \quad m\text{-almost everywhere}$$

then $Gm \leq Gn$ everywhere.

The last two principles are formulations, for measures, the "principle of positive maximum" of §3 Chapter 3; and the last as the reader might have guessed says that "$\Delta u \leq 0$ at a maximum of u". The last two principles are inplied by the Domination principle of Maria-Frostman. If u is exessive, $Gm < \infty$ m-almost everywhere and $u \geq Gm$, m-almost everywhere then $u \geq Gm$ everywhere.

Proof. Let $p = Gm$. The hypothesis imply that we can find compacts F_n such that $u \geq p$ on F_n and the measures $m_n = m|F_n$ increase to m. Therefore these is no less of generality in assuming that m has compact support F, that $u \geq p$ on F, and that $p < \infty$ m-almost everywhere.

We claim

(9) $$p(\cdot) = E_{\cdot}[p(X_H) : H < R], \quad \begin{array}{l} H = \text{hitting time to } F \\ R = \text{exit time from } W. \end{array}$$

Indeed if D is relatively compact open and contains F and $D_1 = D \setminus F$ by Lemma 1, §5, Chapter 5

$$p(\cdot) = E_{\cdot}[p(X_T)], \qquad T = \text{exit time from } D_1.$$

Since $T = H$ or $T = S$, where S = exit time from D

$$p(\cdot) = E_{\cdot}[p(X_H) : H < S] + E_{\cdot}[p(X_S) : S < H]$$

$$\leq E_{\cdot}[p(X_H) : H < R] + E_{\cdot}[p(X_S)].$$

Since p is a potential the last term in the above inequality tends to zero as D increases to W by (2) and we obtain (9).

Since $X_H \in F$ if $H < \infty$, $u \geq p$ on F and $u \geq E_.[u(X_H):H<R]$ the proof now follows using (9). Q.e.d.

For the next principle we need some preparation. We say that a Borel set A is quasi null if the following holds:

(10) $m(A) = 0$ for every measure m such that Gm is uniformly bounded.

Clearly a Borel set is quasi mull iff every compact subset is and a countable union of quasi null sets is quasi null. We shall see in §3 that quasi null and polar are the same. In view of the maximum principle a set A is quasi null iff $m(A) = 0$ for every measure m such that $Gm < \infty$ m-almost everywhere.

Let F be a compact set and H = hitting time to F. A point $x \in F$ is called regular (irregular) for F if

$$P_x[H=0] = 1 \quad (=0).$$

Proposition 1. The set of irregular points in a compact set F is quasi null.

Proof. Just as in Proposition 2, §2, Chapter 4 $P_.[H \geq t]$ is seen to be upper semi-continuous. The set of irregular points in F is the union

(11) $$\bigcup_n \{x : x \in F,\ P_.[H \geq \tfrac{1}{n}] \geq \tfrac{1}{n}\}$$

and each set of the union is compact and consists entirely of irregular points.

Now suppose A is compact and not quasi null.

There then exists a measure m whose potential $p = Gm$ is m-almost everywhere finite and $m(A) > 0$. By restricting m to A if necessary we may as well suppose that m lives on A. From (9) (H denotes hitting time to A and R = exit time from W)

$$\int G(x,y)m(dy) = p(x) = E_x[p(X_H):H < R]$$

$$= \int E_x[G(X_H,y):H < R]m(dy)$$

showing that for m-almost all y in A

(12) $$G(x,y) - E_x[G(X_H,y):H > R] = 0.$$

Just as in the "barrier" argument of Property 2 §1, Chapter 6 one concludes from (12) that m-almost all $y \in A$ are regular for A.

Finally since each set of the union (11) consists entirely of irregular points each of these sets and hence the union is quasi null. Q.e.d.

As an application let us take up the matter of representing measures for $D(X)$ the set of continuous functions on the compact set X which are harmonis in the interior of X. Suppose a measure m on ∂X represents a point $x_0 \in X$ i.e. $\int f(y)m(dy) = f(x_0)$ for all $f \in D(X)$. We show that if m has no mass at x_0 then m must be the harmonic measure at x_0. Consider $Km(\cdot) = \int K(\cdot - y)m(dy)$. If $x \notin X$, $K(x-\cdot)$ is harmonic in a neighbourhood of X. Hence $Km(x) = K(x-x_0)$. So as $x \notin X$

tends to a point $z \in \partial X$, by Fatou, $Km(z) \leq K(z-x_0)$. If m has no mass at x_0, Km is thus finite m-almost everywhere and hence cannot charge the set of irregular points in ∂X i.e. it must live on the regular points in ∂X. That such a measure is the harmonic measure has already been shown in §5, Chapter 5. An example of a harmonic measure with infinite energy (this notion is introduced later in this section) is given in Example 4, §2, Chapter 4.

Proposition 2. Let Gn be n-almost everywhere finite. If $Gm \leq Gn$ then Gm is m-almost everywhere finite.

Proof. Put $q = Gn$. If the m-measure of the set $(q = \infty)$ is not zero, by restriction if necessary let us assume that m lives on a compact subset F of $(q = \infty)$. For any open set D containing F, if $p = Gm$, from (8)

(13) $$p(x) = E_x[p(X_H):H < R] \leq E_x[q(X_H):H < R]$$

That the last quantity in (13) tends to zero as D decreases to F for any x such that $q(x) < \infty$ is shown as follows:

If $\bar{D}$ is contained in the open set $(q > N)$

(14) $$NP_x[H < R] \leq E_x[q(X_H):H < R] \leq q(x)$$

The best quantity in (13) is equal to

$$\int E_x[G(X_H,y):H < R]n(dy)$$

which decreases to zero as D decreases to F because: The integral is bounded above by the n-integrable function $G(x,\cdot)$, for every y not in F it decreases to zero by (14), and n does not charge F. Thus p must be zero, which can only be true if $m(F) = 0$. That proves the Proposition.

The Balayage Principle

Let $F \subset W$ be compact. For any measure m such that $p = Gm$ is m-almost everywhere finite, then exists a unique measure n, with support in F such that $Gm \geq Gn$ in W, while $Gm = Gn$ quasi everywhere on F (i.e. except for a quasi nullsubset of F).

If H = hitting time to F

$$q(\cdot) = E_\cdot[p(X_H):H < R]$$

is excessive in W, harmonic off F, is less or equal to p and equals p at every point which is regular for F. If q_1 is any other potential with the properties $q_1 \leq p$ and $q_1 = p$ quasi everywhere on F then, $q_1 = q$ quasi everywhere on F, because by Proposition 1 the set of irregular points in F is quasi null. The Domination Principle together with Proposition 2 above implies that $q \equiv q_1$. Clearly the Riesz measure of q is the required measure. n is called the Balayage of m onto F.

The Equillibrium Principle

Any compact set F containing at least one regular point contains the support of a unique measure m whose potential

$p = Gm \le 1$ everywhere in W and $p = 1$ quasi everywhere on F. m is called the equillibrium distribution on F and p its equillibrium potential.

If R = exit time from W and H = hitting time to F

$$p(\cdot) = P_{\cdot}[H < R]$$

is easily seen to be a potential [if $d = 2$, use the fact that $P_{\cdot}[R < \infty] \equiv 1$. If $d \ge 3$ use the transcience of the Brownian motion, c.f. Exercise 3, §1, Chapter 4]. p is then the equillibrium distribution. The uniqueness follows as before from the domination principle.

In physics it is known that charges in a charged body redistribute thenselves into an equillibrium state and in this state (since there is no movement of charges) all points of the body are at the same potential. To find thsi distribution of charges is the equillibrium problem. The equillibrium principle asserts the existence and uniqueness of the equillibrium distribution.

The probabilistic meaning of the equillibrium distribution is explained by what is known as Chung's formula which we now describe. For a compact set F the last exit time γ is defined as follows:

$$\gamma = \sup(t: 0 < t < R,\ X_t \in F)$$

$$= 0 \quad \text{if there is no such } t$$

γ is not a stopping time but γ is measurable because $(\gamma \le t) = (H(\theta_t) \ge R)$ where H = hitting time to F and the shift

operators θ are defined in §1, Chapter 2. The last exit time satisfies $t + \gamma(\theta_t) = \gamma$ on the set $\gamma \ge t$ and the set $(0 < \gamma)$ is the same as the set $(H < R)$. For each subset $A \subset F$, the function $L(\cdot, A) = P_{\cdot}(X_\gamma \in A, 0 < \gamma)$ is excessive (use Markov Property) and (being less or equal to the equillibrium potential of F) is in fact a potential. Also, if A is compact, the hitting time to A is less or equal to γ on the set $(X_\gamma \in A)$. This fact and strong Markov property show that $L(\cdot, A)$ is harmonic off A i.e. the Riesz measure of $L(\cdot, A)$ is concentrated on A if A compact. The sum of the potentials $L(\cdot, A)$ and $L(\cdot, F \setminus A)$ is the hitting potential p of F. Therefore the sum of their Riesz measures is the Riesz measure of p = the equillibrium measure m of F. In particular $m(A, dy)(= \text{Riesz measure of } L(\cdot, A)) \le m(dy)$ and, because the former is concentrated on A we indeed have $m(A, dy) \le 1_A(y) m(dy)$. We have thus shown that for every compact subset A of F

$$L(\cdot, A) \le \int G(\cdot, y) 1_A(dy) m(dy)$$

But the above being an identity when $A = F$, it must be an identity for all compact (and hence Borel) A contained in F. The proof of the identity

$$L(x, dy) = G(x, y) m(dy)$$

known as Chungs formula, is thus complete.

The Energy Principle

Physically speaking $Gm(x)$ is the potential at x due to a distribution m of charges i.e. $Gm(x)$ is the work needed to bring a point charge from infinity to x. Thus $\int Gm(x)m(dx)$ is the total work need to assemble this distribution of charges; in other words it is the energy contained in this distribution of charges.

If m and n are (positive) measures their <u>mutual energy</u> (m,n) is defines by

$$(15) \qquad (m,n) = \int G(x,y)m(dx)n(dy) = (Gm,n) = (Gn,m).$$

where the symbol (Gm,n) has the obvious meaning that the potential Gm is integrated relative to n. We will write $\|m\|^2$ for (m,m); this quantity is called the energy of m. The energy principle then states

(16) If m and n have finite energy

$$(m,m) + (n,n) \geq 2(m,n)$$

with equality only if $m = n$.

Only strict positivety, symmetry and lower semi-continuity of G is needed to establich the following useful proposition.

<u>Proposition 3</u>. (Gauss-Frostman). Let F be a compact subset of W and f continuous on F. There is a measure m which minimizes

$$(17) \qquad Q(n) = (Gn,n) - 2(f,n)$$

among all positive measures n on F. $Gm = f$ on the support

of m and the set of $x \in F$ where $Gm(x) < f$ has n-measure zero for every n on F of finite energy.

<u>Proof</u>. Since G is strictly positive and lower semi-continuous its minimum on $F \times F$ is strictly positive. Therefore (Gn,n) is of the order of magnitude $n(F)^2$ if $n(F)$ is large, whereas (f,n) is bounded by $(\sup|f|)\ n(F)$. Thus $\inf Q(n) = \inf\{Q(n), n(F) \leq \Lambda\}$ where Λ is some large number. The set $(n:n(F) \leq \Lambda)$ is compact in the weak topology on measures on $F(C(F)$ in a Banach space and its dual is the set of all signed measures of finite total variation). G is lower semi-continuous and f is continuous so Q is a lower asemi-continuous function of n and such a function attains its minimum on a compact set. Tnus there is a measure m minimizing Q.

Let m' be the restriction of m to the set $(x; x \in F, Gm > f)$. For $0 < \varepsilon < 1$, $m - \varepsilon m'$ is a (non-negative) measure on F. By choice of m, $Q(m-\varepsilon m') - Q(m) \geq 0$ which is easily reduced to $\varepsilon^2(Gm',m') + 2\varepsilon(f,m') - 2\varepsilon(Gm,m') \geq 0$. Divide by ε and let ε tend to zero to get $(f,m') - (Gm,m') \geq 0$, which cannot be true unless $m' = 0$, because m' lives on the set $(Gm > f)$. Thus m-almost everywhere, and hence by lower semi-continuity, on the support of m, $Gm \leq f$. In particular m has finite energy.

Let now the measure n on F have finite energy and n' its restriction to the set $(Gm < f)$. n' has finite energy. The inequality $Q(m+\varepsilon n') \geq Q(m)$ for all $\varepsilon > 0$, leads as before to $(Gm - f, n') \geq 0$ which is false unless $n' = 0$. Q.e.d.

Proof of the Energy Principle

It is enough to prove (16) under the assumption that m,n live on a compact set F. Let $f \leq Gm$ be continuous on F. By Proposition 3, there is a measure μ on F such that $G\mu = f$ on support μ and

(18) $$(Gn,n) - 2(f,n) \geq (G\mu,\mu) - 2(f,\mu) = -(G\mu,\mu)$$

because $G\mu = f$ on support μ. $G\mu = f \leq Gm$ on support μ, implies by the domination principle, $G\mu \leq Gm$ everywhere. Thus $(G\mu,\mu) \leq (Gm,\mu) = (G\mu,m) \leq (Gm,m)$, which by (18) gives $(Gn,n) + (Gm,m) \geq 2(f,n)$. The validity og this for all continuous $f \leq Gm$ is just (16).

Thus (16) is established except for showing that in case of equality m must equal n. For this note that a measure having c'-density with compact support in W has finite energy. Let b be a measure with compact support with smooth density f. Then $m+b$ has finite energy ($(m,b) < \infty$ becuase Gm is locally integrable). Applying (16) to $m+b$ and n, and assuming that equality obtains in (16) for m and n, we get

$$(b,b) + 2(m,b) - 2(n,b) \geq 0$$

Replacing b by εb:

$$\varepsilon(b,b) + 2(m,b) - 2(n,b) \geq 0$$

and letting $\varepsilon \to 0$ we get

$$\int Gm(x)f(x)dx \geq \int Gn(x)f(x)dx$$

for all non-negative smooth functions f with compact supports.

This implies $Gm \geq Gn$ almost everywhere and hence everuwhere. By symmetry $Gm = Gn$ i.e. $m = n$. Thus (16) is completely established.

The above proof of the energy principle uses the domination principle which is very hard to check. Let us now indicate briefly another proof, which uses the result of §5, Chapter 6 but not the domination principle. If q is the relative transition density (notation of §5, Chapter 6), $\int q(t,\cdot,\cdot)$ is symmetric and has the semigroup property. Put for positive measures m and n, $Q(m,t,x) = \int q(t,x,y)m(dy)$ and $I(t,m,n) = \int q(t,x,y)m(dx)n(dy)$. Using the above properties of q it is easily checked that

$$I(t,m,n) = \int Q(m,t/2,x)Q(n,t/2,x)dx$$

which in turn implies: $I(t,m,m) + I(t,n,n) \geq 2I(t,m,n)$ i.e. $q(t)$ is a definite Kernel for each t by which we mean $I(t,\mu,\mu) \geq 0$ for any signed measure μ. Since $G = \int_0^\infty q(t)dt$, (16) is immediate.

The second method of proof gives us many Kernels satisfying the energy principle. For example $\int_0^\infty q(t,x,y)L(dt)$ satisfies the energy principle for almost any measure L. It would thus appear that this principle by itself is too general. However together with the minimum principle - which is hard to check - it implies all the other principles. Take for example the dimonation principle. Suppose m and n have compact support and $Gm \leq Gn$ on support m. $q = Gm \wedge Gn$ is of the form Gl for some measure l. $(G(m-l),m-l) = (Gm,m) - (Gl,m) - (Gm,l) + (Gl,l) \leq 0$ because on support m, $Gm = Gl$ and $Gm \geq Gl$ everywhere. By the energy pronciple $m = l$ and domination follows.

If in (16) we take tn instead of n and let $\|n\| t = \|m\|$ we get the <u>energy inequality</u>

(19) $$(m,n) \leq \|m\| \, \|n\|$$

og which an immediate consequence is

(20) $$\|m+n\| \leq \|m\| + \|n\|.$$

Let ξ_+ denote the set of all measures of finite energy. For $m_1, m_2 \in \xi_+$, $m = m_1 - m_2$ can be regarded as a signed Radon measure i.e. as a linear functional on the continuous functions of compact supports in W. $Gm = Gm_1 - Gm_2$ makes sense as a n-integrable function for every $n \in \xi_+$. Now we let ξ denote the set of all signed Radon measures $m = m_1 - m_2$ with $m_1, m_2 \in \xi_+$. (19) and (20) make ξ a pre-Hilbert space with inner product $(m,n) = \int Gm\,dn$. ξ <u>is not complete</u>. The completion of ξ can be identified with aspace of distributions; see §5. However a result of Cartan states that ξ_+ is complete. We will now prove this and indicate two consequences.

<u>Lemma 4</u>. Let $\{s_i\}$ be a sequence of excessive functions and $s = \inf_i s_i$. Let $\hat{s}$ deonote the lower semi-continuous regularization of s:

$$\hat{s}(x) = \liminf_{y \to x} s(y)$$

Then $\hat{s}$ is excessive and the set $(\hat{s} < s)$ is quasi-null.

<u>Proof</u>. Let $a > 0$. We shall show that every compact subset F of the set $(s > \hat{s} + 2a)$ consists entirely of irregular points.

By Proposition 1, F and hence the set itself will be quasi-null.

Let $z \in F$. $\hat{s}$ being lower semi-continuous there is a neighbourhood V of z such that for all $x \in \bar{V}$

(21) $$\hat{s}(x) > \hat{s}(z) - a, \quad x \in \bar{V}$$

Then for all $x \in \bar{V} \cap F$ and all i

(22) $$s_i(x) \geq \hat{s}(x) + 2a > \hat{s}(z) + a = A \text{ say}, \; x \in \bar{V} \cap F.$$

If H = hitting time to $\bar{V} \cap F$, from (22) for all x

(23) $$s_i(x) \geq E_x[s_i(X_H) : H < R] \geq AP_x[H < R].$$

The right side of (23) is independent of i and is lower semi-continuous. From the definion of $\hat{s}$

$$\hat{s}(x) \geq AP_x[H < R] = (\hat{s}(z) + a)P_x[H < R], \; x \in W.$$

The last inequality when $x = z$ excludes the possibility that z is regular for $\bar{V} \cap F$ and regularity being a local property, z cannot be regular for F.

To show that $\hat{s}$ is excessive, appeal to Proposition 2, §1, Chapter 5: If u is harmonic in a relatively compast open set D, ceontinuous on $\bar{D}$, $u \leq \hat{s}$ on ∂D then $u \leq s_i$ in D for all i. Since it is continuous in the open set D, $u \leq \hat{s}$ as well. Q.e.d.

<u>Lemma 5</u>. Let m_i be a sequence which is bounded in ξ_+ and $p_i = Gm_i$. Then there exists $m \in \xi_+$ whose potential p equals $\liminf p_i$ quasi everywhere.

Proof. By Lemma 4, for each i $\inf(p_j,\ j \geq i)$ is equal to a potential q_i quasi everywhere. $q_i \leq q_{i+1}$. $p = \lim q_i$ is then excessive and equals $\liminf p_i$ quasi everywhere. Let $x_0 \in W$, D be a relatively compact open set and T = exit time from D. Now $p \leq \liminf p_i$. So to show that p is a potential we need only show that

$$E_{x_0}[p_i(X_T)] \tag{24}$$

is uniformly small if D is large. If n is the harmonic measure at x_0 relative to D i.e. $n(dy) = P_{x_0}[X_T \in dy]$ the quantity in (24) is simply $\int p_i dn = (m_i, n)$. m_i being bounded, by (19) it is sufficient to show that the energy of n is small if D is large. The following shows precisely this:

$$\int G(x,y)\,n(dy) = E_{x_0}[G(X_T,x)] \leq G(x_0,x).$$

$$(n,n) \leq \int G(x_0,y)\,n(dy) = E_{x_0}[G(x_0,X_T)]$$

and the last quanlity tends to zero as D increases to W because $G(x_0,\cdot)$ is a potential.

Finally it remains to show that the Riesz measure m of p is in ξ_+. By Fatou $\int p\,dm \leq \liminf_i \int p_i\,dm$ and for each i $\int p_i\,dm = \int p\,dm_i \leq \liminf_j \int p_j\,dm_i = \liminf_j (m_j, m_i)$ and (m_i, m_j) is uniformily bounded by (19) and the boundedness of m_i in ξ_+. Q.e.d.

Theorem 6. ξ_+ is complete.

Proof. Let $m_i \in \xi_+$ be a Cauchy sequence. Choosing a sub-sequence if necessary assume $\sum \| m_i - m_{i+1} \| < \infty$. Put $p_i = Gm_i$. For $n \in \xi_+$

$$\int |p_i - p_{i+1}|\,dn = \int (p_i - p_{i+1})\,dn_1 + \int (p_{i+1} - p_i)\,dn_2 \tag{25}$$

$$= (m_i - m_{i+1}, n_1) + (m_{i+1} - m_i, n_2)$$

where n_1 and n_2 are the restrictions of n to the sets $(p_i \geq p_{i+1})$ and $(p_{i+1} > p_i)$ respectively. $n_1, n_2 \in \xi_+$ and their energies are dominated by the energy of n. From (19) and (25)

$$\sum \int (p_i - p_{i+1})\,dn \leq 2\|n\| \sum \| m_i - m_{i+1} \| . < \infty.$$

In particular: For any $n \in \xi_+$, $\lim p_i$ exists n-almost every-where and, putting $s = \liminf p_i$

$$\lim \int |s - p_i|\,dn = 0. \tag{26}$$

But by Lemma 5 there is $m \in \xi_+$ such that $s = Gm$, quasi every-where. (26) then implies (m_i, n) tends to (m,n) i.e. m_i tends weakly (in the Hilbert space sense) to m. Being a Cauchy sequence it tends strongly to m. Q.e.d.

Before proceeding for applications note the following. If a sequence $m_i \in \xi_+$ tends weakly to $m \in \xi_+$ then m_i tends Vangely to m. Indeed if φ is C^2 with compact support

$$-A_d \int \varphi\,dm_i = (\Delta\varphi, m_i) \rightarrow (\Delta\varphi, m) = -A_d \int \varphi\,dm \tag{27}$$

because $\Delta\varphi$ defines a signed measure of inite energy and $\int G(\cdot,y)\Delta\varphi(y)dy = -A_d\varphi$. Also the energies of m_i have to be uniformly bounded and so (G is strictly positive and lower semi-continuou), $m_i(F)$ is uniformly bounded for each compact set F. Vague convergence of m_i to m is thus clear from (27). <u>In particular the set of measures of finite energy living on a closed set is a closed convex subset of</u> ξ_+.

With this preliminary we can give a geometrical meaning to belayage: Let m be any element of ξ_+ and F a closed set. By elementary Hilbert space theory there is a unique measure n on F - the projection of m on the closed convex set of measures of finite energy living on F - such that $\|m-n\|^2 \leq \|m-\mu\|^2$ for all $\mu \in \xi_+$ which live on F. Take $n+t\mu$ instead of μ and reduce to get

(28) $$-2(m-n,t\mu) + t^2\|\mu\|^2 \geq 0$$

for all such μ. This can only be true if $(m-n,\mu) \leq 0$ and (taking $t=1$, $\mu = n$ in (28)) $(m-n,n) = 0$. As in the proof of Proposition 3 this implies $Gm \leq Gn$ quasi everywhere on F. But then by $(m-n,n) = 0$ we must have $Gm = Gn$, n-almost everywhere. By the domination principle $Gm \geq Gn$ everywhere on W. Thus $Gm \geq Gn$ everywhere on W and equality holds quasi everywhere on F i.e. n is the belaye of m.

Similarly the equillibrium destribution has the following interprelation. Let F be a compact set. The set of probability measures on F of finite energy is a closed convex subset ξ_+. The projection of the zero-measure on this set is simply the unique probability measure m on F of minimal energy. $\|m\|^{-2}\cdot m$ is

the equillibrium distribution for F.

The existence parts of balayage and the equillibrium principles can be directly deduced from Proposition 3. See Exercise 6 and 7.

<u>Remark</u>. The reader might question the wisdom of proving a difficult theorem like the completeness of ξ_+ just to be able to interpret balayage as a projection. The point is, with this method a sort of balayage can be worked out even when the domination principle fails. A case in point in this: The M. Riesz Kernels $|x-y|^{-d+\alpha}$ satisfy the domination principle only when $0 < \alpha \leq 2$. But for all $0 < \alpha < d$, the corresponding ξ_+ is complete. Thus for $2 < \alpha < d$ the Hilbert space method gives a solution to balayage problem whereas other methods fail. We shall discuss these matters in a latter section.

<u>Exercises to §1</u>

1. Show that the first maximum principle implies the continuity principle.

<u>Hint</u>. See the hint to Exercise 2, §4, Chapter 5.

2. Show that the second maximum principle implies the first maximum principle.

<u>Hint</u>. Suppose m has compact support. F $Gm \leq 1$ on F. If D is an open set containing F the hitting potential of D is equal to 1 on F.

Remark. That the continuity principle implies the Maria - Frostman maximum principle is shown in the test. Thus these principles are equivalent.

3. Let q be a potential and $A = q^{-1}(\infty)$. Show that A is quasi null.

Hint. If F is a compact subset of A, m lives on F and $p = Gm$ is m-almost everywhere finite then (9) holds so that the P_x-probability of the event $(H < R)$ cannot be zero. On the other hand if $q(x) < \infty$,

$$E_x[q(X_H):H < R] \leq q(x) < \infty \quad \text{and} \quad q = \infty \quad \text{on} \quad F.$$

4. Let q and A be as in Exercise 3 above. Then for every measure n living on A, $Gn = \infty$ n-almost everywhere.

Hint. If F is compact $\subset A$, $n(F) > 0$, $Gn < \infty$, on F, and $m = n|F$ then $Gm < \infty$, m-almost everywhere. Now use Exercise 3 above.

5. Show that the set $m \in \xi$ such that m and Gm both are compactly supported is dense in ξ.

Hint. Let $m \in \xi_+$ have compact support, D relatively compact open, T = exit time form D. IF $p = Gm$, then $q = E_\cdot[p(X_T)]$ is Gn for some n. Since $q \leq p$, $\|n\|^2 \leq \int p\,dn = \int q\,dm \leq \int p\,dm = \|m\|^2 < \infty$. As D increases to W, the m-integrable functions q decrease to 0 and are bounded by the m-integrable function p. So $\|n\|$ is small for large D. $m-n$ is the required signed measure. See also Lemma 1, §5.

6. From Proposition 3 and the domination principle derive the balayage principle.

Hint. Let F be compact and f continuous on F with $f \leq Gm$. There is a measure n on F such that $Gn = f$ on support n, $Gn \geq f$ quasi everywhere on F and $Gn \leq Gm$ everywhere. Let f increase to Gm.

7. Using Proposition 3 and the first maximum principle derive the equillibrium principle.

Hint. Take $f = 1$ in Proposition 3. For uniqueness we need also the domination principle.

§2. Capacity

Let F be compact $\subset W$ (our Green domain) and p its equillibrium potential:

$$p(\cdot) = P_\cdot[H < R]$$

where H = hitting time to F and R = exit time from W. The Riesz measure of p is concentrated on ∂F because $p \equiv 1$ in the interior of F and is obviously harmonic off F. We call $m(F)$ the (Newtonian) capacity of F:

$$N(F) = m(F) = m(W) = m(\partial F).$$

Just as in Measure theory we define the inner and outer capacities of our arbitrary subset A of W as follows:

$$\text{The inner Capacity } N_*(A) = \sup N(F)$$

where the supremum is over all compact subsets F of A.

$$\text{The outer Capacity } N^*(A) = \inf N_*(D)$$

where the infimum is over all open sets $D \supset A$. A set E is called capacitable if its inner and outer capacities coincide and $N(E)$ will denote this commen value and is called its capacity.

Proposition 1. Let a and b be Radon measures with potentials u and v. If $u \leq v$ then

$$a(W) \leq b(W).$$

Proof. Let F be any compact subset of W and p its equillibrium potential and m its equillibrium measure. Then if F^o = interior of F

$$a(F^o) \leq \int pda = \int udm \leq \int vdm = \int pdb \leq b(W)$$

because $p=1$ on F^o and ≤ 1 everywhere. The proposition follows by letting F increase to W. Q.e.d.

Observation. The domination principle of §1 and Proposition 1 above permit us to identify the capacity of a compact set as:

$$N(F) = \sup m(F)$$

where the sup is over all measures m on F whose potential $Gm \leq 1$ everywhere. In particular a compact set has capacity zero iff it is quasi null. Now a Borel set is quasi null iff every compact subset is quasi null. And we obtain:

A Borel set is quasi null iff its inner capacity is zero.

That a compact set is capacitable is seen as follows: Let A be compact and D a relatively compact open set containing A. Suppose D_n are open $\bar{D}_n \subset D$ and D_n decrease to A. Denote by p the relative hitting potential of D:

$$p(\cdot) = P_\cdot[H < R],\ H = \text{hitting time to } D.$$

and let p_n similarly denote the hitting potentials to D_n. The Riesz measure m of p is concentrated on ∂D and p_n decrease to the hitting potential q of A at all points of ∂D (in fact $p_n(x)$ decreases to $q(x)$ for all x, except when $x \in \partial A$ is irregular). If m_n are the Riesz measures of p_n, because $p=1$ on $\bar{D}_n$ and m_n live on ∂D_n,

$$m_n(W) = \int pdm_n = \int p_n dm \downarrow \int qdm = \int pda = a(W)$$

where a is the Riesz measure of q i.e. the equillibrium destribution for A. On the other hand it is clear from Proposition 1 that $m_n(W) \geq N_*(D_n)$. Therefore

$$(1) \qquad N^*(A) = N(A) = N_*(A),\quad A \text{ compact}.$$

Theorem 2. The outer capacity N^* has the following properties:

i) $N^*(A) \leq N^*(B)$ if $A \subset B$

ii) $N^*(A_n) \downarrow N^*(A)$ if $A_n \downarrow A$ strongly i.e. in such a way that every open set containing A contains some A_n

iii) $N^*(A \cup B) + N^*(A \cap B) \leq N^*(A) + N^*(B)$. This property is called strong subadditivity.

iv) $N^*(An) \uparrow N^*(A)$ if $A_n \uparrow A$.

v) Borel sets (even analytic sets) are capacitable.

Proof. i) and ii) are clear from the difinition. Let us prove iii). If A and B are compact and H_1, H_2 the hitting times to A and B respectively, $H_1 \wedge H_2$ is the hitting time to $A \cup B$ and $H_1 \vee H_2$ is less or equal to the hitting time H to $A \cap B$. If $R =$ exit time from W (recall $H \geq H_1 \vee H_2$) the trivial inequality

$$P_x[H_1 \wedge H_2 < R] + P_x[H < R] \leq P_x[H_1 < R] + P_x[H_2 < R]$$

leads by Proposition 1 to

(2) $$N(A \cup B) + N(A \cap B) \leq N(A) + N(B).$$

If D_1 and D_2 are open

(3) $$N_*(D_1 \cup D_2) + N_*(D_1 \cap D_2) \leq N_*(D_1) + N_*(D_2)$$

follows from the definition of N_*, (1) and the elementary topological fact that if A and B are compact subsets of $D_1 \cup D_2$ and $D_1 \cap D_2$ respectively then we can find compact sets $A_1 \subset D_1$ and $A_2 \subset D_2$ such that $A_1 \cup A_2 \supset A$ and $A_1 \cap A_2 \supset B$. [Write $D_1 = \cup A_n$, $D_2 = \cup B_n$ with A_n, B_n increasing open and relatively compact in D_1, D_2 respectively. For some n, $A_n \cup B_n$ will contain A and $A_n \cap B_n$ will contain B]. Now the proof of iii) should create no problems.

To prove iv) we may and do assume that $\sup_n N^*(A_n) < \infty$. Let $a > 0$. We shall find inductively an increasing sequence of open sets $D_n \supset A_n$ such that $N_*(D_n) < N^*(A_n) + a$. Let D_1 be open, $D_1 \supset A_1$ and $N_*(D_1) < N^*(A_1) + C\ a/2$. Suppose $D_1 \subset D_2 \subset \cdots \subset D_n$ have been found such that $D_n \supset A_n$ and $N_*(D_n) < N^*(A_n) + (1-2^{-n})a$. Let D^1_{n+1} be open with $N_*(D^1_{n+1}) < N^*(A_{n+1}) + a2^{-n-1}$. Put $D_{n+1} = D_n \cup D^1_{n+1}$. $D_{n+1} \supset A_{n+1}$. From (3)

$$N_*(D_{n+1}) + N_*(D^1_{n+1} \cap D_n) \leq N_*(D_n) + N_*(D^1_{n+1})$$

$$\leq N^*(A_n) + N^*(A_{n+1}) + a(1-2^{-n}+2^{-n-1}).$$

Since $D^1_{n+1} \cap D_n \supset A_n$ we get from the above inequality

(4) $$N_*(D_{n+1}) \leq N^*(A_{n+1}) + a(1-2^{-n-1})$$

If $D = \cup D_n$, then D contains A. And $N_*(D) = \sup N_*(D_n)$ because any compact subset of D is contained in some D_n. iv) now follows from (4).

v) is proved by using the famous Capacity Theorem which follows:

The Capacity theorem

We need to introduce some terminology. Let us denote by N the topological product of countably many copies of the set of natural numbers $(1,2,3,\ldots)$ with the discrete topology. It is well known that N is homeomorphic to the set of irrationals in the interval $[0,1]$ but we do not need this. See Kuralowski [1].

We need the following facts about N.

i) If $\underline{n} = (n_1, n_2 \ldots) \in N$ then the sets $N(n_1,\ldots,n_k) = (\underline{m};\ \underline{m} \in N,\ m_i = n_i,\ 1 \leq i \leq k)$ form a base of neighbourhoods of $\underline{n}$ and each of these sets is open and closed.

ii) A countable union of disjoint copies of N is homeomorphic to N. Denoting the disjoint copies by (k,N) the map which sends $(k,\underline{n})$ into the element of N with first coordinate k and whose $(i+1)$-st coordinate is the i-th coordinate of $\underline{n}$ is a homeomorphism of $\bigcup_k (k,N)$ onto N.

iii) Countable product of copies of N (being still a countable product of the natural numbers) is homeomorphic to N.

A Hausdorff topological space is called a <u>Polish space</u> if there is a metric on it - consistent with the given topology - making it complete and separable. A countable product of Polish spaces is Polish. A Hausdorff topological space is called <u>analytic</u> if it is the contonuous image of N. The set of natural numbers being a Polish space, N and also any closed subset of N is Polish.

With these preliminasies we have

<u>Lemma 3</u>. A Polish space is analytic. The class of analytic subsets of an analytis space is closed under coutnable unions and countable intersections and contains all Borel sets (i.e. sets in the σ-field generated by closed sets.)

<u>Proof</u>. Let (Y,d) be a complete separable metric space. For each finite set $(n_1,n_2,\ldots,n_k)$ of integers we will find non-empty closed balls of diameter $\leq \frac{1}{k}$ in the following way: Let $F_1,F_2,\ldots F_n,\ldots$ be a cover of Y with closed balls of diameter ≤ 1, in case of finite cover we repeat the last one. If $F_{n_1,n_2 \ldots n_k}$ has been found, we cover $F_{n_1 n_2 \ldots n_k}$ by closed balls $F_{n_1 n_2 \ldots n_k j}$ of diameter $\leq \frac{1}{k+1}$, (repeating the last if necessary). $F_{n_1 \ldots n_k}$ thus found has the properties:

(5) $$\text{diamter } F_{n_1 \ldots n_k} \leq \frac{1}{k}$$

$$\bigcup_{n_1 \ldots n_k} F_{n_1 \ldots n_k} = Y.$$

If $\underline{n} = (n_1,n_2,\ldots,n_k \ldots) \in N$ we define $f(\underline{n}) = \bigcap_k F_{n_1 \ldots n_k}$. This intersection is nonempty and contains exactly one point by completeness and Hausdorffness. That it is onto follows from (5). The set $N(n_1,\ldots n_k) = (q : g = (q_1,\ldots q_k,\ldots),\ q_i = n_i, i \leq k)$ is open in N and $f^{-1}(F_{n_1 \ldots n_k})$ contains $N(n_1,\ldots n_k)$. This proves continuity. Thus Y is the dontinuous image of N. Thus a continuous image of a Polish space is analytic.

Let Y be analytic and A_i analytic subsets of Y. $\bigcup_i A_i$ will then be a continuous image of disjoint copies of N and union of countable disjoint copies of N is homeomorphic to N.

To show that $\bigcap A_i$ is analytic note first that a closed subset of an analytic space is analytic because it is a continuous images of a closed subset of N and the latter is polish. Now $\bigcap_i A_i$ can be identified with a closed subset of the product $\prod_i A_i$, namely the set of $x=(x_1,x_2,\dots)$ with $x_i=x_j$ for all i and j. And the latter is analytic being the continuous image of the product $\prod_i N$ (which is homeomorphic to N).

We have also shown above that Borel sets are analytic because a family closed under countable unions and intersections and containing closed sets contains all Borel sets. Q.e.d.

Some more information on analytic spaces is in Exercises.

Let Y be an analytic space. By a <u>Choquet capacity on</u> Y <u>we shall mean a nonnegative set function</u> C <u>defined on all subsets of</u> Y <u>such that</u>

1) $C(A) \le C(B)$, if $A \subset B$
2) $C(A_n) \uparrow C(A)$ if $A_n \uparrow A$.
3) $C(A) = \lim C(A_n)$ if A_n <u>decreases strongly to</u> A i.e. $A = \bigcap A_n$ and each open set containing A contains some A_n.

We have the fundamental

<u>Theorem 4</u>. (<u>Capacity Theorem</u>). Let Y be an analytic space and C a Choquet Capacity on Y. Then for any number $a < C(Y)$ there is a compact subset $A \subset Y$ such that $a \le C(A)$.

<u>Proof</u>. Let f be continuous on N onto Y. Let V_k denote

the set of $\underline{n} \in N$ whose first coordinate $\le K$:

$$V_K = (\underline{n} \in N,\ \underline{n} = (n_1,\dots,),\ n_1 \le K).$$

Since $V_K \uparrow N$ as $K\uparrow\infty$, $f(V_K)\uparrow Y$. We can find k_1 such that $C(f(V_{k_1})) > a$. Let $V_{k_1 j}$ be the set of $\underline{n} \in V_{k_1}$ whose second coordinate is less or equal to j. $V_{k_1 j}$ increases to V_{k_1}. We can find k_2 such taht $C(f(V_{k_1k_2})) > a$. This procedure determines for each j a set $V_{k_1\dots k_j}$.

$$V_{k_1\dots k_j} = (\underline{n}: (n_1,\dots,n_k\dots) \in N,\ n_1 \le k_1,\dots,n_j \le k_j)$$

such that $C(f(V_{k_1\dots k_j})) > a$. Put

$$V = \bigcap_j V_{k_1 k_j} = (\underline{n}:\ n_i \le k_i).$$

That V is compact and $V_{k_1\dots k_j}$ strongly decrease to V are easily shown. It follows that $A = f(V)$ is compact and $f(V_{k_i\dots k_j})$ strongly decreases to A. We conclude $C(A) \ge a$. Q.e.d.

<u>Exercises to §2</u>.

1. Show that a G_δ-subset of a Polish space is Polish.

<u>Hint</u>. As we saw in the proof of Lemma 3, intersection can be identified with a closed subset of a product. An open subset U a Polish space is complete with the metric:

$$d(x,y) + \left|\frac{1}{d(x,F)} - \frac{1}{d(y,F)}\right|$$

where d is the metric in the ambient space and F = the complement of U.

2. A topological space is a Lindelöf space if every open cover has a countable subcover. Show that every analytic space is a Lindelöf space.

Hint. Continuous image of a Lindelöf space i Lindelöf.

3. Let A and B be analytic. Show that the Borel field in $A \times B$ is the product Borel field.

Hint. $A \times B$ is analytic so is every open subset. An open set 0 in $A \times B$ is a union of open sets of the form $U \times V$ where U is open in A and V is open in B. There is a countable subcover by Exercise 2 and $U \times V$ belongs to the product Borel field.

Remark. Some result clearly holds for countable product of analytic spaces.

4. Let A and B be analytic and $f: A \to B$, Borel measurable. Show that graph f is a measurable subset of $A \times B$.

Hint. The map $B \times A \xrightarrow{g} B \times B$ given by $g(b,a,)=(b,f(a))$ is measurable and the inverse image of the diagonal in $B \times B$ (which is measurable relative to the product Borel field by Exercise 3) is simply the graph of f.

5. Show that images and inverse images under Borel maps of analytic sets are analytic.

Hint. $f: X \to Y$ is Borel, $A \subset Y$ analytic, then

$$f^{-1}(A) = \Pi_x[(\text{graph } f) \cap (X \times A)]$$

and if $A \subset X$ is analytic then

$$f(A) = \Pi_y[(\text{graph } f) \cap (A \times Y)]$$

where Π_x and Π_y are projections onto X and Y.

6. If D is an open relatively compact subset of W, the hitting function $u = P.[H < R]$, H = hitting time to D is necessarily a potential. For D not relatively compact give examples where u is harmonic, harmonic + potential or just potential.

7. Let D be any open subset of W and u as in Exercise 6 above. If u is a potential then $N(D)$ is the total mass of the Riesz measure of u. If u is not a potential then $N(D) = \infty$.

Hint. The energy of the equillibrium distribution of every compact subset of D is bounded by $N(D)$. So, if $N(D) < \infty$, by Lemma 5 §1 u must be a potential. Of course the same holds for analytic sets.

§3 Applications

Our first application of the capacity theorem is to V) of Theorem 2. i), ii) and iv) of Theorem 2 say taht N^* is a Choquet Capacity. We have already shown that for a compact set

$N^* = N$. Theorem 4 then guaranties that all analytic sets are capacitable.

Out next application is to showing that a Borel set i quasi null (definition in §1) iff it is polar. Recall the definition of a polar set given in §2, Chapter 5: A set is called polar if it is contained in the set of poles i.e. infinities of a super-harmonic function. We need a small proposition in which we prove more (for future reference) than is immediately needed.

Proposition 1. Let $F \subset W$ be polar and $x_0 \notin F$. Then there exists a potential p with finite Riesz measure such that $p \equiv \infty$ on F and $p(x_0) \leq 1$.

Proof. Suppose first that F is relatively compact. F being polar there is a superharmonic function s which is identically infinition on F. Let n_1 be the Riesz measure of s. If D is relatively compact open neighbourhood of $\bar{F}$ then $n_1(D) < \infty$. If $n = n_1|D$, the potential g of n and s have the same Riesz measure on D implying that $s = g + h$, with h harmonic in D. g is then infinite on F.

In the general case $F = \bigcup_n F_n$ with F_n relatively compact. If g_n are determined as above, for suitable constants a_n, $g = \Sigma a_n g_n$ will be a potential with finite Riesz measure which is infinte on F.

Finally let $x_0 \notin F$ and $B_n = B(x_0 1/n)$ be balls with centre x_0 and radius $1/n$. With $T_n =$ exit time from B_n,

$$p_n(\cdot) = E.[g(X_{T_n})]$$

are potentials, are finite at x_0 and equal g off B_n. In particular $p_n \equiv \infty$ on $F \setminus B_n$. For suitable constants b_n, $p = \Sigma b_n p_n$ is a potential with finite Riesz measure, $p = \infty$ on F and $p(x_0) \leq 1$. Q.e.d.

Now it is easy to show that a polar Borel set A is quasi null. If not, there would exist a measure m on A whose potential $p = Gm$ is bounded. Let g be a potential with finite Riesz measure n such that $g = \infty$ on A. We arrive at the contradiction

$$\infty = \int g dm = \int p dn < \infty.$$

In view of the observation made after Proposition 1 §2, Theorem 4 §2, asserts that a Borel set is quasi null iff it has capacity zero. Let us now show that a set A of capacity zero is polar.

We may assume A is relatively compact. Let $A \subset D_n$ be a sequence of relatively compact open sets, all contained in a fixed compact set, such that $N(D_n) \leq 2^{-n}$. Now $N(D_n)$ (c.f. Exercise 7 §2) is simply the total mass of the Riesz measure m_n of the hitting potential p_n of D_n; because the supremum of the hitting potentials of comapct subsets of D_n is simply the hitting potential of $D_n \cdot p = \Sigma p_n$ is a potential, because $p = Gm$ where $m = \Sigma m_n$ is a finite measure and all the m_n live on the compact set in which all the D_n are contained. Clearly on A, $p_n = 1$ for all n. This proves that A is contained in the poles of p. We have proved

Theorem 2. A set is polar iff it has capacity zero iff it is quasi null. In particular the set of irregular points in a compact set (being quasi null by Proposition 1,§1) is polar

Application in Measure Theory

Using the Capacity theorem we will now show that hitting time to an analytic set is measurable provided the fields are complete. Precise formulations will follow.

Let us first discuss some application of the Capacity theorem to Measure theory. Let Y be an analytic space and P a Borel probability measure on Y i.e. P is a probability measure defined on all Borel subsets of Y. For an arbitrary subset A contained in Y define

$$C(A) = \inf P(U) \tag{1}$$

where the infimum is over all open sets containing A. C has properties 1) and 3) of the definition of Choquet capacity given before Theorem 4,§2. It also has property 2) because P being a measure iii) of Theorem 2,§2 is clear for open sets A and B and hence also for arbitrary sets:

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B). \tag{2}$$

The argument leading from iii) of Theorem 2,§2 to iv) of Theorem 2,§2 alså applies here. Thus C defined by (1) is a Choquet capacity on all subsets of Y.

Now we show that C agrees with P on all Borel sets. To this end let us show that C is additive on disjoint analytic sets:

$$C(A \cup B) = C(A) + C(B), \quad A,B \text{ disjoint analytic.} \tag{3}$$

From (2), the left side of (3) is less or equal to the right side. It is sufficient, by Theorem 4, §2 to prove reverse inequality for disjoint compact sets A and B. This is immediate from the following three observations: C agrees with P on open sets; P is additive on disjoint open sets; given any open set U containing the compact set $A \cup B$ we can find disjoint open neighbourhoods of A and B with union contained in U. Thus (3) is established. Finally let A be any Borel set and B its complement. C clearly dominates P on Borel sets. The equality

$$1 = P(A) + P(B) \leq C(A) + C(B) = C(Y) = 1$$

shows that C and P agree on Borel sets.

Observe also that for an analytic set A, there is an increasing sequence K_n of compact subsets of A and a decreasing sequence V_n of open sets containing A such that $C(A)$ agrees with both $P(\bigcup_n K_n)$ and $P(\bigcap_n V_n)$. In other words A is "measurable" relative to P.

We collect all this in

Theorem 3. Every Borel measure on an analytic space Y is regular i.e. the measure of every Borel set is approximable from within by the measures of compact subsets. Every analytic subset of Y is universally measurable i.e. measurable relative to every Borel probability measure on Y.

Now let us look at hitting times. Fix $t > 0$ and let Z denote the separable Banach space of continuous functions on $[0,t]$ into R^d. $Y = (0,t] \times Z$ is a Polish space and the map X

$$X(s,w) = w(s) \in R^d$$

is continuous on Y. The inverse image of any analytic subset of R^d is analytic in Y: Indeed the graph of X: grX is a closed subset of $Y \times R^d$, because X is continuous. If A is analytic subset of R^d, $X^{-1}(A)$ is simply the Y-projection of the analytic set $(\mathrm{gr}X) \cap (Y \times A)$. Projection is continuous, and continuous image of an analytic set (being the image under a composite map of N) is analytic. The set

(4) $\{w: w \in W,\ w(s) \in A \text{ for some } 0 < s \le t\}$

being the Z-projection of the set $X^{-1}(A)$ is analytic and therefore universally measurable as a subset of Z by Theorem 3 [See also the Exercises to §2].

Now the Borel field of Z is the Borel field generated by open balls. By continuity the norm $\|w-w_0\|$ is the suprenum, over the rationals γ in $[0,t]$, of $|w(\gamma) - w_0(\gamma)|$. The norm is thus measurable relative to Borel field $\mathbb{B}$ generated by the coordinate maps: $w \to w(s)$, $0 \le s \le t$. The Borel sets in Z are thus simply the elements of $\mathbb{B}$. The set (4) is thus measurable relative to every probability measure on $\mathbb{B}$. The relation between this Borel field $\mathbb{B}$ and the stopped Borel field B_t introduced in §1, Chapter 2 is clear. Now the reader should have no difficulty in

Theorem 4. Let A be an analytic subset of R^d and H its hitting time

$$H = \inf\{s: s > 0,\ X_s \in A\}.$$

the infimum over an empty set being defined ∞. Then the set $(H \le t)$ is measurable relative to every probability measure on B_t.

Remark. The hitting time to an analytic set can be approximated by the hitting times to compact subsets. Briefly the details are as follows. Let P be any probability measure on B_∞ = the Borel field introduced in §1, Chapter 2. Given $t > 0$, we can regard P as a probability measure on the stopped Borel field B_t which can be identified with the Borel field of Z. Let Π be the projection of Y onto Z:

$$\Pi(s,w) = w.$$

For any subset $B \subset Y$, let

$$C(B) = P^*(\Pi(B))$$

where P^* is the outer measure corresponding to P. Since π is continuous, C is seen to be a Choquet capacity on all subsets of Y - note that if B_n decrease strongly to B then $\Pi(B_n)$ decreases strongly to $\Pi(B)$.

Now let A be an analytic subset of R^d and $B = X^{-1}(A)$. B is analytic in Y. $C(B)$ is the supremum of $C(F)$ with F compact in B. In particular, X being continuous: $C(B) = \sup C(X^{-1}(K))$ K compact in A.

Calling the set in (4) $R(t,A)$, what has shown is this: For each $t > 0$, tere is anincreasing sequence $K_j(t)$ of compact subsets of A such that $P(R(t,K_j(t)))$ increases to $P(R(t,A))$. Enumerate the non-negative rationals $\{\gamma_n\}$. $K_n = K_n(\gamma_1) \cup \ldots \cup K_n(\gamma_n)$ gives an increasing sequence of compact subsets of A for which, for every rational γ, $P(R(\gamma,K_n))$ increases to $P(R(\gamma,A))$. Now it is easy to show that the hitting times to K_n decrease to the hitting time to P-almost everywhere.

§4. Balayage.

Physically the problem of balayage is the following: Given a compact set F and a spatial distribution of charges m can we find a distribution of charges n on F such that the potential on F is unaltered. We have seen in §1 that given a measure m and a compact set F we can find a measure n with support F such that the potentials of m and n agree on F except on a subset of capacity zero (c.f. Theorem 2, §3). The Maria-Frostman maximum principle of §1 shows that any non-negative superharmonic function which dominates the potential of m quasi everywhere on F (i.e. except for a set of Capacity zero), dominates the potential of n everywhere. Since the potential of n is also superharmonic we can state this as: the potential of n = infimum of all non-negative superharmonic functions which dominate, quasi everywhere on F, the potential of m.

We now describe the well-known balayage technique. As before W will denote a Green domain $\subset R^d, d \geq 2$.

For a real valued function f its <u>lower-semicontinuous regularization</u>, denoted $\hat{f}$, is defined by

$$\hat{f}(x) = \liminf_{y \to x} f(y)$$

$\hat{f}$ is the largest lower semicontinuous function less or equal to f. Let $E \subset W$ and φ a non-negative function on E. The <u>reduit</u> or <u>reduced function</u> R_φ^E is defined to be the infimum of all non-negative hyperharmonic functions (on W) which dominate φ on E:

$$R_\varphi^E = \inf\{u\colon u \geq 0 \text{ hyperharmonic } u \geq \varphi \text{ on } E\}.$$

The lower semicontinuous regularization of R_φ^E, denoted $\hat{R}_\varphi^E$ is called the <u>balayage of</u> φ <u>relative to</u> E. This is standard notation. However we will some times write $R(E,\varphi)$ and $\hat{R}(E,\varphi)$ respectively for reduit and balayage.

Now we need to generalize Lemma 4, §1 to an arbitrary family of excessive functions. To this end we need the following Lemma.

<u>Lemma 1</u>. Let $\{f_i, i \in I\}$ be a family of extended real valued functions on W. For $J \subset I$ put $f_J(x) = \inf_{i \in J} f_i(x)$, $x \in W$. Then there exists a countable subset $I_0 \subset I$ such that $\hat{f}_{I_0} = \hat{f}_I$.

<u>Proof</u>. Replacing f_i by arctan f_i if needed assume that f_i are uniformly bounded. Let $\{U_j\}$ be a countable base for the topology on W. Let $x_j \in U_j$ satisfy

$$f_I(x_j) < \inf_{x \in U_j} f_I(x) + \frac{1}{2^j}$$

and f_{i_j} be such that $f_{i_j}(x_j) < f_I(x_j) + \frac{1}{2j}$. Then

$$\inf_{x \in U_j} f_{i_j}(x) \le f_{i_j}(x_j) < \inf_{x \in U_j} f_I(x) + \frac{1}{j}.$$

Let I_0 be the set $\{i_1, i_2, \ldots\}$. The last inequality implies that for all j

$$\inf_{x \in U_j} f_{I_0}(x) < \inf_{x \in U_j} f_I(x) + \frac{1}{j}.$$

i.e. that $\hat{f}_{I_0} \le \hat{f}_I$. Q.e.d.

Now the following Theorem is immediate from the above Lemma, Lemma 4, §1, and Theorem 2, §3:

Theorem 2. Let $\{s_i, i \in I\}$ be a family of excessive functions and $s = \inf s_i$. Then $\hat{s}$ is excessive and the set $(\hat{s} < s)$ is polar.

From the above Theorem the reduit and balayage differ at most on a polar set. And balayage of φ is superharmonic.

We can define balayage directly as follows: Suppose $F \subset E$ is polar, u hyperharmonic and $\ge \varphi$ on $E \smallsetminus F$. $x_0 \notin F$. By Proposition 1, §3, there is a potential p with $p(x_0) \le 1$ and $p = \infty$ on F. $u + \varepsilon p \ge \varphi$ on E and hence $u + \varepsilon p \ge \hat{R}(E,\varphi)$. Letting ε tend to zero we see $u(x_0) \ge \hat{R}(E,\varphi)(x_0)$. i.e. $u \ge R(E,\varphi)$ except perhaps on F. F being polar it has measure zero: $u \ge \hat{R}(E,\varphi)$ everywhere. Thus we can also define (since $\hat{R}(E,\varphi) \ge \varphi$ on E except for a polar set)

(1) $\hat{R}(E,\varphi) = \inf\{u : u$ hyperharmonic, $u \ge \varphi$ quasi everywhere on $E\}$.

When $E = W$, reduit and balayage will sumply be denoted by R_φ and $\hat{R}_\varphi$ respectively. Clearly $R_\varphi^E = R_{1_E \varphi}$.

The following remark will be useful in the proof of Lemma 3 below.

Remark. For any x, $R_\varphi(x) > \varphi(x)$ implies $\hat{R}_\varphi(x) = R_\varphi(x)$. This is the same as saying $\hat{R}_\varphi(x) < R_\varphi(x)$ implies $R_\varphi(x) = \varphi(x)$. Indeed let $F = (\hat{R}_\varphi < R_\varphi)$. F is polar. Let $x_0 \in F$. If $\varphi(x_0) = \infty$ there is nothing to show. If $\varphi(x_0) < \infty$, let p be a potential, $p(x_0) \le 1$ and $p = \infty$ on $F \smallsetminus \{x_0\}$. $\hat{R}_\varphi + \varepsilon p \ge \varphi$ everywhere for any ε such that $\hat{R}_\varphi(x_0) + \varepsilon p(x_0) \ge \varphi(x_0)$. Were $\hat{R}_\varphi(x_0) \ge \varphi(x_0)$ we could let ε tend to zero to get $\hat{R}_\varphi(x_0) = R_\varphi(x_0)$ i.e. $x_0 \notin F$. We must therefore have $\hat{R}_\varphi(x_0) < \varphi(x_0)$. But in this case for suitable ε $\hat{R}_\varphi(x_0) + \varepsilon p(x_0) = \varphi(x_0)$ i.e. $R_\varphi(x_0) = \varphi(x_0)$.

Lemma 3. Let $0 \le \varphi_n$ increase to φ. Then

(2) $$\sup_n R_{\varphi_n} = R_\varphi, \quad \sup_n \hat{R}_{\varphi_n} = \hat{R}_\varphi.$$

Proof. Since a countable union of polar sets is polar the second equality in (2) follows from (1). Using the second equality in (2) and the above Remark, $\hat{R}_\varphi(x_0) < R_\varphi(x_0)$ implies $R_\varphi(x_0) = \varphi(x_0)$ i.e. $R_{\varphi_n}(x_0) < \varphi_n(x_0)$ from some n on. Again by the same Remark, this in turn replies that $R_{\varphi_n}(x_0) = \varphi_n(x_0)$ for those n for which $\hat{R}_{\varphi_n}(x_0) < \varphi_n(x_0)$. The first part of 2) thus follows for those x for which $\hat{R}_\varphi(x) < R_\varphi(x)$. For other x's this is a consequence of the second part of 2). Q.e.d.

Lemma 4. Let A_n decrease to A strongly. i.e. each open set containing A contains som A_n. If $\varphi \geq 0$ is finite and upper semi-continuous then

$$\lim R(A_n,\varphi) = R(A,\varphi). \tag{3}$$

Proof. If s is excessive and $\geq \varphi$ on A then for any $\varepsilon > 0$. $s + \varepsilon > \varphi$ in one open set containing A. Such an open set will contain some A_n. And for such n, $s + \varepsilon > R(A_n,\varphi)$ proving that $\varepsilon + R(A,\varphi) \geq \inf_n R(A_n,\varphi)$. The reverse inequality being clear (3) follows. Q.e.d.

The above two Lemmas assert that for at finite upper semi-continuous φ, for every x, $R(A,\varphi)(x)$ is a Choquet capacity on all subsets of our Green domain W.

Theorem 5. Let $\varphi \geq 0$ be finite and upper semi-continuous. Then

$$R(E,\varphi) = \inf R(D,\varphi) \tag{4}$$

where the infimimum is over all open sets D containing E. If E is analytic

$$R(E,\varphi) = \sup R(A,\varphi) \tag{5}$$

where the supremum is over all compact subsets of E and there exists an increasing sequence A_n of compact subsets of E such

that

$$\hat{R}(E,\varphi) = \sup_n \hat{R}(A_n,\varphi). \tag{6}$$

Proof. If s is excessive dominates φ on E, then for every $\varepsilon > 0$, $s + \varepsilon$ dominates φ is an open set containing E and 4) follows. 5) is a consequence of the capacity theorem.

To prove (6) note taht the family $\{\hat{R}(A,\varphi),\ A \text{ compact} \subset E\}$ is fittering to the right i.e. any two members are both deminated by a third. For any compact set $K \subset W$,

$$\sup_{\text{compact } A \subset E} \int_K \hat{R}(A,\varphi)(x)\,dx = \int_K \hat{R}(E,\varphi)(x)\,dx \tag{7}$$

because the balayage is almost everywhere equal to the reduit and we are using (5). Now let K_n be compacts increasing to W and choose compact subsets $A_n \subset E$ so that the integrals in (7) differ by at most $\frac{1}{n}$. If $B_n = A_1 \cup \ldots \cup A_n$, $\sup_n \hat{R}(B_n,\varphi)$ has the same integral as $\hat{R}(E,\varphi)$ on all compact subsets of W and hence they are identical. Q.e.d.

Interpretation

For a compact set F, as is clear from (1) $\hat{R}(F,1)$ is simply the hitting potential of F. The same holds for an analytic set. Indeed if H is the hitting time to the analytic set A, using (4), $P.[H < R] \leq \hat{R}(A,1)$, R = exit time from W. The reverse inequality follows from (6).

Thinness

A set A is called thin at a point x if $x \notin \bar{A}$ or if $x \in \bar{A}$ and there is a potential p on W such that

$$(8) \qquad \liminf_{A \setminus \{x\} \ni y \to x} p(y) > p(x)$$

Thinness can also be defined as follows: A is thin at $x \in \bar{A}$ iff there is a neighbourhood U of x such that for $B = A \cap U$

$$(9) \qquad \hat{R}(B,1)(x) < 1.$$

Indeed if p satisfies (8), we can find a neighbourhood U of x such that the potential $q = p|p(x)$ is larger than or equal to b in $U \cap A = B$, where b is a number larger than 1 and $q(x) = 1$. The potential $b^{-1}q$ dominated 1 on B and is strictly less than 1 at x. (9) must therefore be true. Conversely suppose (9) holds. If $E = B \setminus \{x\}$ then $\hat{R}(E,1) = \hat{R}(B,1)$. (From (1) it should be clear that polar sets can be added or subtracted without affecting balayage). Since x is not in E, we have $\hat{R}(E,1)(x) = R(E,1)(x)$. This is easy to see, c.f. Exercise 3. And the first term is less than 1. Now use the definition of $R(E,1)$ to find a potential p, dominating 1 on E such that $p(x) < 1$. This p satisfies (8).

Probabilistically thinness is explained as follows. Let A be thin at $x \in \bar{A}$ and p a potential satisfying (8). If α is a number between the quantities in (8), the open set $D = (p > \alpha)$ contains $A \setminus \{x\}$ and $x \in \partial D$. p is excessive and $p(x) < \alpha$. The hitting time to D, starting at x cannot therefore be zero. Thus the Brownian path, starting at x remains in the complement of A for a positive time. It is clear that D itself is thin at x.

If F is compact, it is clear from (9) that F is thin at a point $x \in F$ iff x is irregular for F. We know that the set of irregular points in a compact set is polar (Theorem 2, §3). More generally we have

Proposition 6. For any set A the set of points $x \in A$ at which A is thin is polar. In particular a set is polar iff it is thin at each of its points.

Proof. Since a countable union of polar sets is polar, by (9) we need only show the following: For each $a > 0$, the set E of $x \in A$ for which $\hat{R}(V_x,1)(x) < 1$ where $V_x = B(x,3a) \cap A$, is polar. The open cover of E consisting of ball of radius a around each point of E has a countable subcover: There is a countable set $I \subset E$ such that every $y \in E$ belongs to $U_x = B(x,a) \cap A$ for some $x \in I$. But if $x \in I$ and $y \in U_x$ then $B(y,3a) \supset B(x,a)$ so that by the definition of E $\hat{R}(U_x,1)(y) < 1$ for all $y \in U_x \cap E$. i.e. $U_x \cap E$ is polar. Thus E is a countable union of polar sets.

That a polar set is thin at each of its points is contained in Proposition 1, §3. Q.e.d.

Exercises to §4

1. Show that $R(E,\varphi)$ is Lebesgue measurable.
2. Show that $R(E,\varphi)$ is harmonic off $\bar{E}$ unless it is identically infinite.

Hint. If $B(a,r)$ is disjoint from $\bar{E}$, T = exit time from $B(a,r)$ and s is excessive and dominates φ on E then $E.[s(X_T):T<R]$

does the same. R = exit time from W.

3. $R(E,\varphi) = \hat{R}(E,\varphi)$ off E. If φ is continuous, equality holds also in interior of E.

Hint. For the first use the Remark before Lemma 1.

For the second note that balayage is the largest lower semi-continuous function less or equal to the reduit.

4. If u and v are continuous and excessive

$$R(E,u) + R(E,v) = R(E,u+v).$$

Hint. By 4) may assume $E = D$ is open. If H is the hitting time to D, $R(D,u) = E.[u(X_H): H < R]$. Note that by exercise 3, balayage and reduit are the same for open sets.

5. If u be continuous and excessive then

$$R(A \cup B,u) + R(A \cap B,u) \leq R(A,u) + R(B,u).$$

This is strong subadditivety.

Hint. By 4) sufficient to show this assuming A and B are open. If H_1 and H_2 are hitting times to A and B, $H = H_1 \wedge H_2$ is the hitting time to $A \cup B$ and $H_1 V H_2 = S$ is less or equal to the hitting time I to $A \cap B$. Sum on the right side is simply $E.[u(X_H):H < R] + E.[u(X_S):S < R]$. The first term is just $R(A \cup B,u)$. The second term is deminates $R(A \cap B,u) = E.[u(X_I):I < R]$ because u is excessive and $I \geq S$.

6. E is polar iff $\hat{R}(E,1) \equiv 0$.

Hint. Suppose $\hat{R}(E,1) \equiv 0$ and $x_0 \notin E$. There exists excessive functions s_n such that $s_n(x_0) < 2^{-n}$ and $s_n \geq 1$ on E.

7. If E is relatively compact, $\hat{R}(E,\varphi)$ is a potential unless it is identically infinite.

Hint. If u is excessive, not identically infinite, dominates φ on E, D relatively compact open and contains E then $p(\cdot) = E.[u(X_H):H < R]$ is a potential and equals u on E. R = exit time from the Green domain W.

8. A is thin at $x \in \bar{A}$ iff there is a super harmonic function s such that $\liminf_{A \ni y \to x} s(y) > s(x)$.

Hint. Consider the potential of the restriction of the Riesz measure of s to a relatively compact neighbourhood of x.

9. Let A and B be thin at x. Then $A \cup B$ is thin at x.

§5. A little somthing on Dirichlet Spaces

In the following W is a Green domain in R^d, $d \geq 2$. A complex Randon measure m will be said to have finite energy if $|m|$ = total variation measure of m has finite energy. If m and n are complex Radon measures of finite energy, the expression

(1) $$(m,n) = \int G(x,y)m(dx)\bar{n}(dy)$$

clearly makes sense; $\bar{n}$ is the complex conjugate of n. And from the energy principle $\|m\|^2 = (m,m)$ is positive unless $m = 0$. The space ξ of all complex Radon measures of finite energy is a pre-Hilbert space with the inner product (1). For each $m \in \xi$, the function Gm, which is defined except for a set of capacity zero will be called the potential of m. Since $|Gm| \leq G|m|$, if h is complex harmonic and $|h| \leq |Gm|$ then $h \equiv 0$; indeed if u is the real part of h, then $u \leq G|m|$ so that $u \leq 0$ and for the same reason $-u \leq 0$. The name complex potential is thus not unjustified. The complex potential Gm completely determines m by

(2) $$\int \Delta\varphi Gm = -A_d \int \varphi dm$$

for every c^∞-complex function φ with compact support. m is thus the <u>Riesz measure</u> of Gm. IF f and g are complex potentials with Riesz measures m and n

(3) $$(f,g) = (m,n) = \int f d\bar{n} = \int \bar{g} dm$$

defines an inner product. (f_i) is a Cauchy sequence iff the corresponding Riesz measures (m_i) is a Cauchy sequence. Now the energy of a complex Radon measure is the sum of the energies of its real and imaginary parts. It follows that the real and imaginary parts of (f_i) are themselves Cauchy sequences. An argument as in the proof of Theorem 6, §1 shows that there is a function f such that f_i

tends to f in $L^1)dn)$ for every $n \in \xi$. Indeed a candidate for f can be found as follows: For a subsequence, we call (l_i), of (m_i), $\Sigma\|l_{i+1} - l_i\| < \infty$. If $u_j + iv_j = Gl_j$, $f = \liminf u_j + i \liminf v_j$ is such a function. Obviously any two such functions f can differ at most on a set of capacity zero. Thus the completion of the pre-Hilbert space of complex potentials can be identified with a set of functions; two of these functions considered the same if they agree guasi-everywhere. <u>These are the socalled BLD functions</u>.

There is another way of difining the class of BLD functions. The claim is that the completion of the pre-Hilbert space of all complex valued c^∞-functions with compact supports in W, provided with the inner product

(4) $$(\varphi,\psi) = \int (\text{grad } \varphi, \overline{\text{grad } \psi})$$

is simply the class of BLD functions.

Every complex c^∞-function φ with compact support is a complex potential whose Riesz measure is (upto a constant) $\Delta\varphi$. The inner product in (4) is up to a constant $\int \bar{\psi}\Delta\varphi$ as is seen by partial integration. Thus the inner product in (4) is upto a constant the inner product of complex potentials given in (3). The claim will therefore be shown by the following Lemma 1.

Let us collect here a few elementary properties of BLD functions. This will be useful in the proof of Lemma 1. All these are immediate from the definition.

1. If f is a BLD function and g a complec potential with Riesz measure n then $f \in L^1(dn)$ and the scalar product

(5) $$(f,g) = \int f d\bar{n} \quad \text{and} \quad \left|\int f d\bar{n}\right| \leq \|f\| \, \|n\|, \quad n \in \xi.$$

2. The restriction of the Lebesgue measure to any compact set being of finite energy every BLD function is locally integrable.

3. If $m_i \in \xi$ converges to $m \in \xi$ then $\int f dm_i$ converges to $\int f dm$ for every BLD function f.

Lemma 1. The set of all c^∞-functions with compact supports is dense in the space of complex potentials.

Proof. Step 1. The set of $m \in \xi$ of compact support is dense in ξ ; the general case follows from the case of positive measures. For $m \in \xi_+$ this is obvious from: $\int_F\int_F G(x,y)m(dx)m(dy)$ increases to (m,m) as the compact set F increases to W.

Step 2. The set of m with compact support and smooth density is dense in ξ. Suppose $m \in \xi_+$ has compact support and $p = Gm$.. Let $0 \le \varphi_i$ be c^∞ and radial (i.e. depends only on distance) and have support in $B(0,1/i)$ = the ball with center zero and radius $1/i$, and $\int \varphi_i = 1$. If p_i is the potential of $m_i = m*\varphi_i$

(6) $$p_i = \int m(dy)\int G(\cdot,z)\varphi_i(z-y)dz \le p$$

for all large i and tends to p because the inner integral is less or equal to and tends to $G(\cdot,y)$ on support m as soon as $B(y,1/i) \subset W$ for all y in support m (see the discussion on approximation of super harmonic functions by smooth ones in the beginning of §1, Chapter 5). Now

(7) $$0 \le \|m-m_i\|^2 = \int p dm - 2\int p_i dm + \int p_i dm_i \le \int p dm - \int p_i dm$$

because $\int p_i dm_i \le \int p dm_i = \int p_i dm$. Since $p_i \le p$ and tends to p the last term in (7) tends to zero. Thus m_i tends to m in the energy norm.

Step 3. If the claim in the Lemma were false there would exist a BLD function f such that for every c^∞-function φ with compact support $\int f\Delta\varphi = 0$. By Lemma 1, §3, Chapter 5 there is a (complex) harmonic function h such that $f = h$ almost everywhere. Now let $m \in \xi_+$ have compact support. If φ_i are as in Step 2, $m_i = m*\varphi_i$ tends to m in ξ_+ and m_i tends to m vaguely.

For all large enough i the supports of m_i are contained in a fixed compact subset of W. Also m_i has smooth density.

(8) $$\left|\int h dm\right| = \lim\left|\int h dm_i\right| = \lim\left|\int f dm_i\right| = \left|\int f dm\right|$$

Let D be open relatively compact and $a \in D$. The harmonic measure at a relative to D has small energy if D is large (see the proof of Lemma 5,§1). h being harmonic the first term in (8) is $|h(a)|$ if m is the harmonic measure at a relative to D. And the last term in (8) is small if D is large. Thus $h \equiv 0$. But then the last term in (8) is zero for all $m \in \xi_+$ with compact support and hence for all $m \in \xi$. Thus $f = 0$ quasi everywhere. See also Exercise 2. Q.e.d.

Lemma 1 easily implies the following result about continuity properties of BLD functions: If f is BLD and $\varepsilon > 0$ there is an open set of capacity smaller than ε such that the restriction of f to the complement of this open set is continuous. This is Exercise 3.

For the next corollary we need a difinition. A locally integrable function g in W is said to have a generalized gradient grad g =

$(g_1,\dots,g_d)$ if for every c^∞-function φ with compact support (in W)

(9) $$\int g\frac{\partial\varphi}{\partial x_i} = -\int g_i\varphi, \quad 1 \le i \le d.$$

We will say that grad $g \in L^2$ if $g_i \in L^2$ for $1 \le i \le d$. With this definition we have

<u>Corollary 2</u>. If f is BLD then the generalized grad f exists and $\|f\|^2 = \int |\text{grad } f|^2$.

Conversely if g has a generalized gradient in L^2 then there exists an f in BLD and h harmonic such that $g = f + h$ almost everywhere.

<u>Proof</u>. By Lemma 1 if f is BLD then there exist c^∞-functions φ_i with compact supports such that $\|\varphi_i - f\|$ tends to zero. In particular φ_i is a Cauchy sequence. As we have remarked the (energy) norm $\|\varphi_i - \varphi_j\|$ is, upto a constant the L^2-norm of grad $(\varphi_i - \varphi_j)$. So grad φ_i converges in L^2 (and in particular locally in L^1) to say $(f_1,\dots,f_d)$. If ψ is c^∞ with compact support (recall that f is locally integrable and that φ_i converges to f in $L^1(dn)$ for every $n \in \xi_+$, in particular locally in L^1)

$$\int f\frac{\partial\psi}{\partial x_1} = \lim\int\varphi_i\frac{\partial\psi}{\partial x_1} = -\lim\int\frac{\partial\varphi_i}{\partial x_1}\psi = -\int f_1\psi$$

same comclusion holding for other partials. Thus $(f_1,\dots,f_d)$ is the generalized gradient of f.

Conversely suppose g has generalized gradient grad $g =$

$= (g_1,\dots,g_d) \in L^2(W)$. For every φ which is c^∞ with compact support

(10) $$\int g\Delta\varphi = -\int (\text{grad } g, \text{grad } \varphi)$$

as is clear from (9). Since grad g is in L^2, and the L^2-norm of grad φ is upto a constant the energy norm of $\Delta\varphi$, (10) shows by Lemma 1 that g defines a continuous linear functional on the Hilbert space of BLD functions. So there is an f in BLD which determines the same linear functional as g: $\int f\Delta\varphi = \int g\Delta\varphi$ for every c^∞-function φ with compact support. g being locally integrable, $g - f$ is thus almost everywhere equal to a harmonic function h. Q.e.d.

<u>Remark</u>. A BLD function f is absolutely continuous on almost all lines parallel to the coordinate axes and the gradient in the usual sende exists almost everywhere. See Exercise 5. In any case the gradient of the potential of a positive measure m exists almost everywhere (Exercise 1, §3, Chapter 5) and is locally integrable. Thus the energy norm of a positive measure is upto a constant the L^2-norm of its gradient. See Exercise 4. The same of course is true for any $m \in \xi$.

The space of BLD functions has one more important property. To describe this we need a difinition. A function B on the complex plane into itself is called a <u>normal contraction</u> if $B(0) = 0$ and

(11) $$|B(x) - B(y)| \le |x-y|, \quad x,y \text{ complex}.$$

<u>Theorem 3</u>. Normal contractions operate on the space of BLD functions: If f is BLD so is Bf for any normal contraction B and $\|Bf\| \leq \|f\|$.

<u>Proof</u>. A Lipschitz function (i.e. a function f which satisfies $|f(x) - f(y)| \leq M|x-y|$) is absolutely continuous on every line. Therefore its restriction to any line is differentiable almost everywhere with a bounded derivative; see Rudin [4] p.p. 165. In particular if a Lipschitz function f has compact support, it has a generalized gradient which is in L^2. If $0 \leq \varphi_i$ are c^∞ with small supports around the origin and $\int \varphi_i = 1$ then $f_i = f*\varphi_i$ will be c^∞ with compact support (in W); it converges to f and grad f_i = grad $f*\varphi_i$ converges in L^2 to grad f. Hence f is BLD.

Next if φ is c^∞ with compact support and B a normal contraction then the support of $B\varphi$ is contained in the support φ and, because φ is clearly Lipschitz, (11) shows that $B\varphi$ is also Lipschitz. Also $|B\varphi(x) - B\varphi(y)| \leq |\varphi(x) - \varphi(y)|$ shows that $|\text{grad } B\varphi|^2 \leq |\text{grad } \varphi|^2$. Thus $B\varphi$ is BLD and its BLD norm does not exceed the BLD norm of φ[Recall that the BLD norm of a BLD function is upto a constant the L^2-norm of its (generalized) gradient].

Finally let g be a BLD. Then there is a sequence g_i of c^∞ functions with compact supports such that g_i tends to g in $L^1(dn)$ for every $n \in \xi_+$ and hence also for all $n \in \xi$. By (11) Bg_i tends to Bg in $L^1(dn)$ for all $n \in \xi$. And

$$\left|\int Bg dn\right| = \lim \left|\int Bg_i dn\right| \leq \lim \|g_i\| \, \|n\| = \|g\| \, \|n\|$$

where $\|\ \|$ denotes the energy or BLD norm. Se there is an $f \in$ BLD such that $\int Bg\, dn = \int f\, dn$ for all $n \in \xi$. But then $Bg = f$ quasi-everywhere. That proves the Theorem.

Let $\mathbb{D}$ denote a class of functions on W having the following four properties. That the space of BLD functions has these properties has been established.

1. $\mathbb{D}$ is a Hilbert space, with norm $\|\ \|$ and scalar product $(\ ,\ )$ of locally integrable functions on W.
2. For any compact set F

$$\int_F |f| \leq A(F) \|f\|, \quad f \in \mathbb{D}.$$

   where $A(F)$ is a constant depending only on F. To see this for BLD consider the Borel function g defined on F by $g = |f|/f$ if $f \neq 0$ and zero otherwise; $g = 0$ off F. The scalar product (f,Gg) is then simply the integral of $|f|$ on F. And the energy norm of g is less or equal to the energy norm of the indicator of F. This property for BLD is also an immediate consquence of property 4 below.
3. If $K(W)$ = set of continuous functions with compact support then $K(W) \cap \mathbb{D}$ is dense both in $\mathbb{D}$ and in $K(W)$. The density in $K(W)$ is understood as follows: For any $f \in K(W)$ and any neighbourhood U of the support of f and any $a > 0$ there is a g in $K(W) \cap \mathbb{D}$ with support in U satisfying $\sup|f - g| < a$.
4. For any normal contraction B and $f \in \mathbb{D}$, $Bf \in \mathbb{D}$ and $\|Bf\| \leq \|f\|$.

The above four properties characterise the so called <u>Dirichlet spaces</u>. We now give a rough sketch of the possibilities in the direction of a Kernel-free potential theory implicit in the above properties. The reader is invited to elaborate on these ideas himself.

If $f \in \mathbb{D}$, f and $\bar{f}$ are contractions of each other; so $\bar{f} \in \mathbb{D}$ and $\|f\| = \|\bar{f}\|$. $z \to |z|$ is a normal contraction so $f \in \mathbb{D}$ implies $|f| \in \mathbb{D}$.

$f \in \mathbb{D}$ is called a pure potential if there is a non-negative measure m such that

(12) $$(f,\varphi) = \int \bar{\varphi}\,dm$$

for every $\varphi \in K(W) \cap \mathbb{D}$. The reader is invited to check that in the case of BLD functions f is a pure potential iff $f = Gm$ almost everywhere. Note that the four properties by themselves do not permit us to distinguish between functions which are equal almost everywhere. A purely geometric characterization of a pure potential is given by the following important proposition.

Proposition 4. $f \in \mathbb{D}$ is a pure potential iff

(13) $$\|g + f\| \geq \|f\|$$

whenever $g \in \mathbb{D}$ and $\mathrm{Re}\, g \geq 0$. (13) is equivalent to

(14) $$\mathrm{Re}(f,g) \geq 0 \quad \text{provided} \quad \mathrm{Re}\, g \geq 0$$

Proof. To see (13) and (14) are equivalent take a.g. in stead of g in (13) where $a > 0$ and let a tend to zero. If f is a pure potential (14) is a consequence of (12) for $g \in K(W) \cap \mathbb{D}$ and the positivity of m; the general case then follows by continuity.

Suppose now that f satisfies (13). The closed convex set

$$F = \{f+g:\ g \in \mathbb{D},\quad \mathrm{Re}\, g \geq 0\}$$

contains a unique element of least norm, which by (13) must be f. Since $|f| \in F$ and its norm cannot exceed that of f, we must have $f = |f|$.

If $\varphi \in K(W) \cap \mathbb{D}$ so is $|\varphi|$. So there are enough positive functions in this intersection. The map $\varphi \to (f,\varphi)$ is a positive linear functional on $K(W) \cap \mathbb{D}$ by (14), and must be given by a unique measure m. Q.e.d.

Corollary 5. If u and v are pure potentials so is $u \wedge v$.

Proof. Since $f \in \mathbb{D}$ implies $|f| \in \mathbb{D}$, $w = u \wedge v = \frac{1}{2}[u+v - |u-v|] \in \mathbb{D}$ The closed convex set $F = \{f: \mathrm{Re}(f-w) \geq 0\}$ contains a unique element say h of least norm. h must be a pure potential because for any g with $\mathrm{Re}\, g \geq 0$, $h + g \in F$ and so $\|h+g\| \geq \|h\|$. Clearly $u \wedge h \in F$ and

$$4\|u \wedge h\|^2 = \|u+h\|^2 + \|\,|u-h|\,\|^2 - 2(u+h, |u-h|)$$

$$\leq \|u+h\|^2 + \|u-h\|^2 - 2(u+h, u-h) = 4\|h\|^2$$

because u being a pure potential, $(u, |u-h| - (u-h)) \geq 0$ by (14) and similarly for h. By uniqueness $u \wedge h = h$. i.e. $h \leq u$. For the same reason $h \leq v$ i.e. $h \leq u \wedge v = w$. Q.e.d.

Each bounded non-negative measurable function f with compact support determines a pure potential Gf (just notation) by the prescription

$$(Gf,v) = \int f\bar{v}, \quad v \in \mathbb{D}$$

Indeed by property 2 the map $v \to \int f\bar{v}$ is a continuous linear functional on $\mathbb{D}$. Thus there are lots of pure potentials.

Our properties by themselves do not distinguish functions which are equal almost everywhere. In order to get some of the deeper results of potential theory we will refine this equivalence relation as follows. Let us say that a <u>positive measure</u> m <u>is of finite energy</u> if it determines a pure potential, that is, if

$$(15) \qquad \left|\int \varphi dm\right| \leq \|\varphi\| \cdot \|m\|, \quad \varphi \in K(W) \cap \mathbb{D}$$

Where $\|m\|$ is simply the norm of the pure potential m determines. We write $\xi_+ =$ set of all positive measures of finite energy.

Let $f \in \mathbb{D}$. Then f is in fact an equivalence class of functions. We will select a representative (in fact a class of representatives) v with the following porperties

$v \in L^1(dm)$ for every $m \in \xi_+$; whenever $\varphi_n \in K(W) \cap \mathbb{D}$ converges in $\mathbb{D}$ to f, φ_n converges to v in $L^1(dm)$ for every $m \in \xi_+$

Clearly any two representatives with the above properties coincide quasi everywhere i.e. are equal m-almost everywhere for every $m \in \xi_+$ The procedure for such a selection is the following: Let $f \in \mathbb{D}$ and $\varphi_n \in \mathbb{D} \cap K(W)$ such that $\|\varphi_n - f\|$ tends to zero and

$$(16) \qquad \Sigma \|\varphi_n - \varphi_{n+1}\| < \infty.$$

Because $\|(|\varphi_n-\varphi_{n+1}|)\| \leq \|\varphi_n-\varphi_{n+1}\|$, from (15) and (16)

$$(17) \qquad \Sigma \int |\varphi_n - \varphi_{n+1}| dm < \infty, \quad m \in \xi_+.$$

In particular (because $|(\mathrm{Re}\ x)^+ - (\mathrm{Re}\ y)^+| \leq |x-y|$ etc.), $(\mathrm{Re}\varphi_n)^+$, $(\mathrm{Re}\varphi_n)^-$ etc. all converge in $L^1(dm)$ and m-almost everywhere for all $m \in \xi_+$. Define a representative v of f by:

$$v = \liminf(\mathrm{Re}\varphi_n)^+ - \liminf(\mathrm{Re}\varphi_n)^- + i \text{ (similarly)}$$

Clearly $v \in L^1(dm)$ for every $m \in \xi_+$. If ψ_n is a sequence in $\mathbb{D} \cap K(W)$ converging in $\mathbb{D}$ to f then by (15), ψ_n is a cauchy sequence in $L^1(dm)$. Also $\|\varphi_n - \psi_n\| \to 0$, which by (15) and what we have said above means that ψ_n tends to v in $L^1(dm)$.

From now on we shall assume that we have made such a selection. In particular we may assume that every $f \in \mathbb{D}$ is in $L^1(dm)$ for every $m \in \xi_+$. A simple consequence of all this is: Let $m \in \xi_+$ determine the pure potential Gm (just notation) then

$$(18) \qquad (Gm,f) = \int \bar{f}\, dm., \quad f \in \mathbb{D}$$

and in particular $\|Gm\|^2 = \int Gm\, dm$ (Recall $Gm = |Gm|$).

Now we can prove the main principles of potential theory in our setup. We give two illustrations.

<u>Corollary 6</u>. The domination principle is valid in $\mathbb{D}$: Let m and n be positive measures of finite energy with potentials u and v. If $u \leq v$, m-almost everywhere then the inequality holds everywhere.

Proof. By Cololläry 5, $w = u \wedge v$ is the potential of a measure 1. By (18)

$$\|u-w\|^2 = \int u\,dm - \int w\,dm - \int u\,dl + \int w\,dl \leq 0$$

because $u = w$ m-almost everywhere and $w \leq u$ everywhere. Thus $w = u$. q.e.d.

Corollary 7. The equillibrium principle is valid in $\mathbb{D}$: Let F be acompact set which supports a measure of finite energy. Then there is a measure m with support F whose potential $Gm = 1$ quasi everywhere on F and $Gm \leq 1$ everywhere on W.

Proof. The set of probability measures on F with finite energy is a closed conbex subset of $\mathbb{D}$ as is clear from (18). Let m be the probability measure with minimal energy with corresponding pure potential u.

The restriction of a measure $n \in \xi_+$ to any set A is itself in ξ_+: For every $v \in \mathbb{D}$ $|\int_A v\,dn| \leq \int |v|dn \leq \|(|v|)\|\ \|n\| \leq \|v\|\ \|n\|$. This and an argument very similar to that of Proposition 3, §1 shows that $u = \|u\|^2$ m-almost everywhere and $u \geq \|u\|^2$ quasi-everywhere on F. $\|u\|^{-2}$ m gives rise to a pure potential v which is equal to 1 m-almost everywhere and is larger than or equal to 1 quasi-everywhere on F. By the domination principle it is sufficient to show that $v \wedge 1$ is a pure potential.

$Bz = \inf\{1, (\mathrm{Re}\,z)^+\}$ being a normal contraction, $v \wedge 1 \in \mathbb{D}$. The set $X = \{h : h \in \mathbb{D}, \mathrm{Re}\, h \geq v \wedge 1\}$ is a closed convex set with a unique element f of minimal norm. Corollary 5 easily implies

that f is a pure potential. $f \wedge 1$ belongs to X and being a contraction of f its norm cannot exceed that of f. By minimality $f = f \wedge 1$. Q.e.d.

The following simple lemma will be found useful.

Lemma 8. In a Dirichlet space $\mathbb{D}$ the set of pure potentials is a closed convex set (in particular it is complete). And the linear span of pure potentials is dense in $\mathbb{D}$.

Proof. The first part is trivial from Proposition 4. For the second note that by property 2 of the definition of a Dirichlet space, every bounded non-negative measurable function h with compact support determines a pure potential Gh by the prescription $(f, Gh) = \int fh$. Therefore the only element in $\mathbb{D}$ orthogonal to all pure potentials is zero. Q.e.d.

Some knowledge of Fourier transforms will be assumed in the following example. An excellent reference (also for the terminology used here) is Chapter 7 of Rudin [5]. Thus $\mathcal{D}$ = set of C^∞-functions with compact supports.

Example. Let σ be a positive measure on R^d such that

$$\int (|x|^2 \wedge 1)\,\sigma(dx) < \infty \tag{19}$$

and let ψ be defined by

$$\psi(\alpha) = 2\int [1-\cos(\alpha, x)]\sigma(dx), \quad \alpha \in R^d. \tag{20}$$

Provide $\mathcal{D}$ with the norm

(21) $$\|f\|^2 = \int \sigma(dy) \int |f(x+y) - f(x)|^2 dx$$

It is clear how to define an inner product (f,g) so that $\|f\|^2 = (f,f)$. Denote by $\mathbb{D}$ the set of functions f such that f is the almost everywhere finite limit of a sequence of functions in $\mathcal{D}$, the said sequence beingat the same time a Cauchy sequence in the norm given by (21). With the understanding that two functions are considered equal if they are equal almost everywhere, $\mathbb{D}$ is a Dirichlet space provided ψ^{-1} is locally integrable, where ψ is defined in (20).

Let us first show that (21) defines a finite quality. This is easy to see derectly but we will use Fourier transforms because we will need this later. Applying Parseval to the inner integral in (21) we can write

(22) $$\begin{aligned}\|f\|^2 &= \text{const.}\int \sigma(dy) \int |\hat{f}(\alpha)|^2 |1-\exp\, i\,(\alpha,x)|^2 d\alpha \\ &= \text{const.}\int |\hat{f}|^2(\alpha)\psi(\alpha)d\alpha\end{aligned}$$

where the constant is $(2\pi)^{-d}$. Now ψ is easily seen to be continuous. The simple inequality $1-\cos(\alpha,x) \le |\alpha|^2|x|^2 \wedge 1 \le |\alpha|^2(|x|^2 \wedge 1)$ shows, using (19), that $\psi(\alpha) = 0(|\alpha|^2)$ at ∞. Since the Fourier transforms in (22) decrease rapidly at ∞, (f,g) is well defined.

More generally for $f \in \mathbb{D}$, (21) makes sense: Suppose $\varphi_n \in \mathbb{D}$ converge almost everywhere to f and is a Cauchy sequence in the norm given by (21). Then for every y, $\lim(\varphi_n(x+y) - \varphi_n(x)) = f(x+y) - f(x)$ almost all x or by Fubini, this limit relation holds for $(dx \times \sigma)$ - almost all pairs (x,y). Thus because φ_n is a Cauchy sequence, $\|f\| < \infty$ and $\|f - \varphi_n\|$ tends to zero.

With these simple perliminaries it is easy to show that normal contractions operate on $\mathbb{D}$: Let $f \in \mathbb{D}$ and T a normal contraction (see (11)). There is a Cauchy sequence $\varphi_n \in \mathbb{D}$ such that φ_n tends to f almost everywhere and $\{\varphi_n(x+y) - \varphi_n(x)\}$ tends $dx\,\sigma$-almost everywhere to $f(x+y) - f(x)$. Since T is continuous $T\varphi_n$ tends to Tf almost everywhere. Choose non-negative $a_n \in \mathcal{D}$ with $\int a_n = 1$ such that a subsequence of $b_n = (T\varphi_n)*a_n$ tends to Tf almost everywhere.

[This can be dome as follows. Let m be a finite measure equivalent to Lebesgue measure on R^d. Then $T\varphi_n$ tends to Tf in m-measure. There are a's in $\mathcal{D}$ such that $T\varphi_n * a$ is uniformly close to $T\varphi_n$. So for suitable a_n, b_n will tend to Tf in m-measure. A subsequence will then tend m-almost everywhere to Tf and m is equivalent to the Lebesgue measure]. b_n belongs to $\mathcal{D}$ and by Fubini $b_n(x+y) - b_n(x)$ tends $dx \times \sigma$ almost everywhere to $Tf(x+y) - Tf(x)$. Also

$$\|b_n\| \le \|T\varphi_n\| \quad \text{and} \quad |T\varphi_n(x+y) - T\varphi_n(x)| \le |\varphi_n(x+y) - \varphi_n(x)|$$

The first inequality is a simple consquence of Schwarz inequality (note that $0 \le a_n$ and $\int a_n = 1$) and Fubini, the second is just (11). Since φ_n is a Cauchy sequence, the second inequality above implies that $|T\varphi_n(x+y) - T\varphi_n(x)|^2$ is uniformly integrable (namely the integral over sets of small measure in uniformly small) and so by Exercise 7, §1, Chapter 3, $\|T\varphi_n - Tf\|$ tends to zero. But then Fatou Lemma, together with this last fact and the first of the above inequalities says that $\|b_n\|$ tends to $\|Tf\|$. And again by the same Exercise, $\|b_n - Tf\|$ tends to zero. In particular b_n is a Cauchy sequence. So $Tf \in \mathbb{D}$. Ofcourse by (11) and (21), $\|Tf\| \le \|f\|$.

To show that $\mathbb{D}$ is a Dirichlet space we need only prove taht the defining property 2 is valid and that $\mathbb{D}$ is complete. The latter follows from the former because it would imply that a Cauchy sequence in $\mathbb{D}$ converges locally in $L^1(R^d)$. Let us prove property 2. Let $A \in \mathcal{D}$ be such that its Fourier transform $\hat{A}$ is strictly positive [If $b \in \mathcal{D}$ is real and symmetric i.e. $b(x) = b(-x)$ and $a = b*b$ then $\hat{a} \geq 0$. If p is the standard Gauss Kernel, $p*a$ is strictly positive and its Fourier transform $= \hat{p}.\ a \in \mathcal{D}$]. Then for any $f \in \mathcal{D}$, if g is the Fourier transform of $|f|$

$$(23) \qquad \int |f|\hat{A} = \int gA \leq \left(\int |g|^2\psi\right)^{\frac{1}{2}} \left(\int \psi^{-1}|A|^2\right)^{\frac{1}{2}}$$

The last member of (23) is finite because A has compact support and ψ^{-1} is locally integrable. And the third member is simply the norm in $\mathbb{D}$ of $|f|$–because (22) is valid for any $f \in L^2(R^d)$ – and from whar we have already shown ($z \to |z|$ is a normal contraction) this norm does not exceed the norm of f. Since A is strictly positive, property 2 follows from (23). Thus $\mathbb{D}$ <u>is a Dirichlet space</u>.

By (22) the map $f \to \hat{f}$ is an isometry (upto a constant which in our context is unimportant) of $\mathcal{D}$ into $L^2(\psi)$. By the density of $\mathcal{D}$ in $\mathbb{D}$, this extends to an isometry of $\mathbb{D}$ into $L^2(\psi)$. To show that it is actually onto it is sufficient to show that the image of $\mathcal{D}$ is dense: namaly that the set of $\hat{f}$ with $f \in \mathcal{D}$ is dense in $L^2(\psi)$. If $g \in L^2(\psi)$ and $\int \bar{g}\hat{f}\psi = 0$ for all $f \in \mathcal{D}$, then if A is as in (23), $\int \bar{g}\hat{A}\hat{f}\psi = 0$ for all $f \in \mathcal{D}$. Since $\bar{g}\hat{A}\psi$ is in L^1, this is equivalent by Parseval to $\int \widehat{\bar{g}\hat{A}\psi} f = 0$ for all $f \in \mathcal{D}$ i.e. that $\bar{g}\hat{A}\psi = 0$ almost everywhere. But $\hat{A} > 0$ and from (20), ψ can only have isolated zeroes and so g must be zero. <u>Thus the map</u> $f \to \hat{f}$ <u>of</u> $\mathcal{D}$ <u>into</u> $L^2(\psi)$ <u>extends</u>

<u>to an isometry – upto a constant – of</u> $\mathbb{D}$ <u>onto</u> $L^2(\psi)$. This fact leads to the following useful description of $\mathbb{D}$:

<u>Theorem 9</u>. A locally summable function f belongs to $\mathbb{D}$ iff there is a $g \in L^2(\psi)$ such that for all $\varphi \in \mathcal{D}$ we have $\int f\varphi = (2\pi)^{-d}\int g\bar{\hat{\varphi}}$. The $\mathbb{D}$-norm of f is $(2\pi)^{-d\frac{1}{2}}$ times the $L^2(\psi)$-norm of g. We are justified in calling g the Fourier transform of f.

<u>Proof</u>. Note that the theorem implicitly claims that $g\hat{\varphi}$ is summable for every $g \in L^2(\psi)$ and $\varphi \in \mathcal{D}$. Let T be the map inverse to the map $\varphi \to \hat{\varphi}$ of $\mathcal{D}$ into $L^2(\psi)$. T is continuous on $L^2(\psi)$ onto $\mathbb{D}$. Let $a \in \mathcal{D}$. The map $f \to \int f\, a$ is continuous on $\mathbb{D}$ by property 2. So there is an element $b \in L^2(\psi)$ such that $\int Tg\, a = \int g\, \bar{b}\psi$ for all $g \in L^2(\psi)$. Now if $\varphi \in \mathcal{D}$ and $g = \hat{\varphi}$ then $Tg = \varphi$ so that $\int \varphi a = \int \hat{\varphi}\bar{b}\psi$ for all $\varphi \in \mathcal{D}$. By Parseval this is the same as $\int \hat{\varphi}\bar{\hat{a}} = (2\pi)^d \int \hat{\varphi}\bar{b}\psi$ for all $\varphi \in \mathcal{D}$. As before $\hat{a} = (2\pi)^d b\psi$. Because T is 1-1 onto, one part of Theorem 9 follows.

What we have proved above can be restated as: For every $a \in \mathcal{D}$, and $g \in L^2(\psi), g\hat{a}$ is summable (indeed $\hat{a} = b\psi$ for some $b \in L^2(\psi)$ as we saw above), the map $g \to \int g\hat{a}$ is continuous and $\int Tg\, a = (2\pi)^{-d}\int g\hat{a}$. So if f satisfies the conditions of the theorem f must equal Tg. Q.e.d.

<u>Remark</u>. That $\hat{a} = b\psi$ for some $b \in L^2(\psi)$, proved above says in particular that $\int |\hat{a}|^2\psi^{-1} < \infty$ for each $a \in \mathcal{D}$. This implies

a certain growth restriction on ψ^{-1} i.e. that it is a tempered distribution.

A special case. Riesz potentials.

We will describe the situation briefly and invite the reader to supply the details. For simplicity we will ignore constants i.e. equations will hold upto multiplicative constants.

Let $0<\theta<2$ and $\sigma(dy)=|y|^{-d-\theta}$. The function $\psi(\alpha)$ of (20) is a constant multiple of $|\alpha|^{\theta}$. Let $\mathbb{D}$ be the corresponding Dirichlet space. Consider the Riesz Kernels $|x|^{-d+\theta}$ introduced in the beginning of Chapter 3. Using the representation of these in terms of the Brownian semi-group it is seen that the Fourier transform (in the sense of distributions) og $|x|^{-d+\theta}$ is $|x|^{-\theta}$.

The θ-potential $Im=I(\theta,m)$ of a positive measure m is $Im=m*|x|^{-d+\theta}$; its θ-energy norm is $(\int Im\, dm)^{\frac{1}{2}}$. A complex Radon measure m will be said to have finite energy if the corresponding total variation measure $|m|$ has finite energy. Using the convolution relation between the Riesz Kernels established at the beginning of Chapter 3, it is seen that the energy norm of m is the L^2-norm of $I(\frac{\theta}{2},m)$. Because the Fourier transform (in the distribution sense) of the θ-potential of a finite complex measure is $\hat{m}|\alpha|^{-\theta}$ we see that a finite complex measure m has finite energy iff $I(\theta|2,m)\in L^2$ i.e. iff $\int|\hat{m}|^2|\alpha|^{-\theta}<\infty$ which is identical to $\hat{Im}\in L^2(\psi)$ where $\psi(\alpha)=|\alpha|^{\theta}$. By Theorem 9, $Im\in\mathbb{D}$. And the energy norm of m is the L^2-norm of $I(\theta/2,m)$ which by Parseval is $\int|\hat{m}|^2|\alpha|^{-\theta}$ i.e. the energy norm of m is the $\mathbb{D}$-norm of Im. It is a simple step now to remove the restriction of finiteness of m and we can say: A complex Radon measure m has finite energy iff its potential Im belongs to $\mathbb{D}$.

Now let us describe the pure potentials in $\mathbb{D}$. Suppose a finite positive measure m generates a pure potential f. This means $(f,A)=\int\bar{A}\,dm$ for all $A\in\mathbb{D}$ with $(\ ,\ )$ denoting inner product in $\mathbb{D}$; see (18). For any $\varphi\in\mathcal{D}$ we have $\int Im\bar{\varphi}=\int I\bar{\varphi}dm$ and, $I\varphi\in\mathbb{D}$ so that $\int Im\bar{\varphi}=(f,I\varphi)$. It is clear from (22) that scalar product in $\mathbb{D}$ corresponds to scalar product in $L^2(\psi)$. Transfering to $L^2(\psi)$-space this equality reads: if g corresponds to f then $\int Im\varphi=(f,I\varphi)=\int g\overline{\hat{I\varphi}}\psi=\int g\hat{\bar{\varphi}}$ because $\psi=|\alpha|^{\theta}$ and $\hat{I\varphi}=|\alpha|^{-\theta}\hat{\varphi}$. By Theorem 9 the last integral is just $\int f\varphi$. Thus $\int Im\varphi=\int f\varphi$ for all $\varphi\in\mathcal{D}$ i.e. $f=Im$. We have proved:

> $f\in\mathbb{D}$ is a pure potential iff $f=Im$ for a positive measure m of finite energy.

By Lemma 8 we have the following corollary:

Corollary 10. The set of positive measures of finite θ-energy is complete under the energy norm. The minimum of two θ-potentials is a θ-potential.

For different proofs of the first part of Corollary 10 see p.p. 80-94 of du Plessis [3] and p.p. 82-90 of Landkof [2] . For a different proof of the second part see p.p. 129 of Landkof.

Exercises to §5

1. Show that $f,g\in BLD$ and $f=g$ almost everywhere implies $f=g$ quasi everywhere.

<u>Hint</u>. The set of measure m having density is dense in the set of all measures of finite energy. So if $\int|f-g|dm=0$ for every m with density the same holds for all m of finite energy.

2. Show that a continuous function equal almost everywhere to a BLD function is itself BLD.

<u>Hint</u>. Follow the proof of step 3 of Lemma 1.

3. Let f be BLD and $\varepsilon>0$. Show that there is an open set U of capacity smaller than ε such that $f|F$ is continuous where $F=W\setminus U$.

<u>Hint</u>. Let a_n be c^∞ with compact supports such that (the energy norm) $\|a_n-f\|<n^{-2}$. Recall the following: The capacity of any Borel set is the supremum of the capacities of its compact sets; the capacity of a compact set = the total mass of its equillibrium distribution = the energy of its equillibrium distribution. Let $A_n=(|a_n-f|>n^{-1})$. For any compact subset F of A_n and m = the equillibrium distribution of F

$$n^{-1}m(F)\leq\int|a_n-f|dm\leq\|m\|\|a_n-f\|\leq n^{-2}\sqrt{m(F)}$$

showing that $N(A_n)$ (= capacity of A_n) $\leq n^{-2}$. Off $B_k=\bigcup_{n\geq k}A_n$, a_n converges uniformly to f, $N(B_k)$ is small for large k and the capacity of a Borel set is the infimum of the capacities of open sets containing it.

4. Show that the gradient of the potential of a positive measure m exists almost everywhere and is locally summable. The energy norm of a positive measure is, upto a constant the L^2-norm of the gradient of its potential.

<u>Hint</u>. That the gradient in the ordinary sense exists almost everywhere and is locally summable follows from Exercise 1, §3 Chapter 5. Energy norm of m is the L^2-norm of the generalized gradient of its potential.

5. Show that a BLD function f is absolutely continuous on almost all lines parallel to the coordinate axes.

6. Let $\mathbb{D}$ be as in the Example. Show that if f is integrable, has compact support and $\|f\|$ as defined in (21) is finite then $f\in\mathbb{D}$.

<u>Hint</u>. If $0\leq a_n\in\mathcal{D}$ has small support around 0, $\int a_n=1$ then $f*a_n\in\mathcal{D}$, tends to f almost everywhere and $\|f*a_n\|\leq\|f\|$. Now use Fatou and Exercise 7, §1, Chapter 3.

7. If $\mathbb{D}$ is as in the Example, $f\in\mathbb{D}$ and $\varphi\in\mathcal{D}$ then $\varphi*f\in\mathbb{D}$.

<u>Hint</u>. If $\|\varphi_n-f\|$ tends to zero so does $\|\varphi_n*\varphi-\varphi*f\|$.

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References for further study

These are references in addition to the ones already given.

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