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CHAPTER 6

Green functions

Introduction

Consider a domain G and the inhomogeneous equation $\Delta u = f$, called the Poisson equation. The solution is not unique without further conditions: if u is a solution, so is $u + w$ for any harmonic function w. If G is bounded and we impose the "boundary condition" that u vanish on the boundary, the maximum principle guarantees that there can be at most one solution. In simple terms, Green function for G is the "inverse of Δ together with this boundary condition". In order to better understand what follows, recall the definition and properties of Green functions for ordinary differential operators (see e.g. pp. 39 - 47 of Birkhoff [1]).

We will see that knowing the Green function for a domain is equivalent (theoretically) to solving the Dirichlet problem.

In § 1 we shall define Green functions and point out some of their most important properties. § 2 and § 3 are devoted to a discussion of Green functions for unbounded open sets in R^2. § 4 contains some examples and in § 5 an expression for the Green function in terms of relative transition densities is given.

§ 1. Green functions for bounded open sets.

Let D be a bounded open set and $K(x)$ be defined as in (3) § 3 Chapter 5. By Theorem 3, § 3 Chapter 5, for any c'-function f with compact support

(1) $$F(x) = \int K(x-y)f(y)\,dy$$

is a solution of the Poisson equation.

(2) $$\Delta F = -A_d f$$

where A_d are constants defined in (5) § 3 Chapter 5. This solution, however, does not satisfy our "boundary condition" of tending to zero at the boundary. In order to secure this, we solve the Dirichlet problem with boundary values F and subtract it from F; namely, if we define

(3) $$u(x) = F(x) - E_x[F(X_T)]$$

where T = exit time from D, then u satisfies $\Delta u = -A_d f$ in D and $u(b)$ tends to zero as b tends to any regular point in ∂D. From what we have seen in Chapters 4 and 5 one can not expect more. Equation (3) can be rewritten

(4) $$u(x) = \int G_D(x,y)f(y)\,dy$$

where

(5) $$G_D(x,y) = K(x-y) - E_x[KX_T - y)].$$

T = exit time from D.

Caution. The right side of (5) does not make sense if e.g. $x = y \in \partial D$ and regular for D^c. However for each x, $K(x-\cdot)$ and $E_x[K(X_T-\cdot)]$ are both superharmonic on $\mathbb{R}^d$ and hence locally

integrable. Therefore, even if the right side of (5) does not make sense for some x and y, for each fixed x the right side of (4) makes sense for every bounded Borel measurable f with compact support. Note however, that for each $x \in D$, the right side of (5) is well defined for all $y \in \mathbb{R}^d$; indeed for $y \in D$, $E_x[K(X_T-y)]$ is clearly finite, and for any y, $E_x[K(X_T-y)] \leq K(x-y)$.

Proposition 1. For each $x_0 \in D$, $E_{x_0}[K(X_T-y)] = K(x_0-y)$ for almost all $y \notin D$. To show this, let $x_0 \in D_n$ be open with $\bar{D}_n \subset D_{n+1} \subset D$ and $\cup D_n = D$. Let T_n = exit time from D_n. Then $P_{x_0}[T_n \uparrow T] = 1$, where T = exit time from D. The functions s_n defined by

(6) $$s_n(y) = E_{x_0}[K(X_{T_n} - y)], \quad y \in \mathbb{R}^d$$

are superharmonic on $\mathbb{R}^d$ and decrease (because $E_{x_0}[K(X_{T_{n+1}}-y)] = E_{x_0}[E_{X_{T_n}}(K(X_{T_{n+1}}-y))] \leq E_{x_0}[K(X_{T_n}-y)]$). If

(7) $$s(y) = \lim s_n(y)$$

and f is bounded Borel with compact support

$$\begin{aligned}\int (s(y)f(y)\,dy &= \lim_n \int s_n(y)f(y)\,dy \\ &= \lim_n E_{x_0}\left[\int K(X_{T_n}-y)f(y)\,dy\right] \\ &\quad \text{(Fubini's theorem is applicable)} \\ &= E_{x_0}\left[\int K(X_T-y)f(y)\,dy\right] \\ &\quad \text{(since } \textstyle\int K(z-y)f(y)\,dy \text{ is continuous on } \mathbb{R}^d \text{ by Exercise 2 § 3 Chapter 5)} \\ &= \int f(y)E_{x_0}[K(X_T-y)]\,dy\end{aligned}$$

proving that $s(y) = E_{x_0}[K(X_T - y)]$ for almost all y. Now $y \notin D$ implies $y \notin \bar{D}_n$ implies $s_n(y) = K(x_0 - y)$ for all n, i.e. $s(y) = K(x_0 - y)$, $y \notin D$. Thus $K(x_0 - y) = E_{x_0}[K(X_T - y)]$ for almost all y not in D, as claimed.

This shows that as far as (4) is concerned, for each $x \in D$ we can safely redefine $G_D(x,y) = 0$ for all $y \notin D$ and we can state: <u>For any</u> c'-<u>function with compact support (support not necessarily in</u> D) <u>the function</u> u <u>defined in</u> D <u>by</u>

$$u(x) = \int_D G_D(x,y) f(y) dy, \qquad x \in D$$

satisfies $\Delta u = -A_d f$ in D and

$$\lim_{x \to b} u(x) = 0, \quad b \in \partial D, \qquad b \text{ regular for } D^c.$$

$G_D(x,y)$ <u>defined on</u> $D \times D$ <u>by</u> (5) <u>is called the Green function for the bounded open set</u> D <u>and we define for convenience</u> $G_D = 0$ <u>outside</u> $D \times D$.

The following is a short list of properties of G_D:

<u>Property 1</u>. $G_D(\cdot,\cdot)$ is excessive in D in each variable when the other is fixed. $G_D(x_0,\cdot) - K(x_0 - \cdot)$ is harmonic in D for each $x_0 \in D$; $G_D(\cdot,x_0) - K(x_0 - \cdot)$ is harmonic in D for each $x_0 \in D$.

<u>Property 2</u>. Let D be connected open. For each $y \in D$ and $b \in \partial D$

$$\limsup_{D \ni x \to b} G_D(x,y) = 0$$

iff b is regular for D.

<u>Proof</u>. $E_\cdot[K(X_T - y)]$ is by Proposition 4, § 1 Chapter 5, superharmonic on $\mathbb{R}^d$ (in particular lower semi-continuous), $\leq K(\cdot - y)$ and equals $K(b-y)$ for each $b \in \partial D$ that is regular. In the other direction we can show the following:

<u>Let</u> h <u>be positive superharmonic in an open set</u> D <u>and</u> $b \in \partial D$. <u>If</u> $\lim_{D \ni x \to b} h(x) = 0$ <u>then</u> b <u>is regular for</u> D^c.

Indeed, assuming b is not regular, we can find $t > 0$ small enough so that $P_b(T>t) > 0$ where T is the exit time from D. Consider the process $h(X_t)1_{t<T}$. If $s < t$, on the set $t < T$, $t = s + T(\theta_s)$. By Markov property

$$E_b[h(X_t)1_{t<T}] = E_b[E_{X_s}(h(X_{t-s})1_{t-s<T}) : s<T].$$

$h > 0$ is superharmonic on D, i.e. h is excessive on D, i.e. for each $a \in D$ and each $r > 0$, $E_a[h(X_r) : r<T] \leq h(a)$. For $s < T$, $X_s \in D$. We get

$$E_b\, h(X_t) : t<T] \leq E_b[h(X_s) : s<T].$$

As $s \downarrow 0$, $X_s \to b$ and $h(X_s) \to 0$ (no loss of generality in assuming h is bounded) giving

$$E_b[h(X_t) : t<T] = 0.$$

This is a contradiction and b therefore must be regular.

A function h is a <u>barrier at</u> $x \in \partial D$ if h is defined, positive and superharmonic on $V \cap D$ for some neighbourhood V of x and $\lim_{V \cap D \ni y \to x} h(y) = 0$. Since regularity is a local property, we have in fact shown above that for any open set D, $b \in \partial D$ <u>is regular</u> iff there is a barrier at b. (If b is regular for D, V a bounded neighbourhood of b, b is regular for $V \cap D$.

If S is the exit time from $V \cap D$, $E_x(S)$ is positive superharmonic on $V \cap D$ and tends to zero as $x \to b$.)

Property 3. If $U \subset V$ are bounded open, then $G_U \leqq G_V$. Indeed, the first exit time from U is less or equal to the first exit time from V and (4) § 1 Chapter 5, applies. (In case $d \leqq 2$, we add a constant to K to make it excessive in a ball containing the closure of V.)

Property 4. If D is bounded open and D_n are open and increase to D, then $G_{D_n}(x,y)$ increases to $G_D(x,y)$ for all $(x,y) \in D \times D$. This is clear from (5) since the exit times from D_n increase to the exit time from D.

Property 5. Let D be bounded open and $y_0 \in D$. If v is non-negative superharmonic on D and is the sum of $K(\cdot - y_0)$ and a superharmonic function, then $v(x) \geqq G_D(x,y_0)$ for all $x \in D$. Indeed, let $B \ni x$ be an open set whose closure is contained in D and let S be the exit time from B. Writing $v = K(\cdot - y_0) + u$, the non-negativity of v implies

$$E_x[K(X_S - y_o) + u(X_S)] \geqq 0,$$

i.e. (u being superharmonic) that $u(x) \geqq E_x[u(X_S)] \geqq -E_x[K(X_S - y_0)]$. Therefore $v(x) \geqq K(x-y_0) - E_x[K(X_S - y_0)]$, which implies, by letting B increase to D, that $v(x) \geqq G_D(x,y_0)$.

Property 6. Let D be bounded open. G_D is symmetric on $D \times D$:

(8) $$G_D(x,y) = G_D(y,x) \qquad (x,y) \in D \times D.$$

More generally,

(9) $$E_x[K(X_T - y)] = E_y[K(X_T - x)], \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Proof. Denote the let side of (9) by $u(x,y)$ so that the right side is $u(y,x)$. Fix $x_0 \in D$. $u(x_0,\cdot)$ and $u(\cdot,x_0)$ are both superharmonic on R^d and harmonic in D. If G is an open set with $\bar{G} \subset D$ and S = exit time from G

$$u(x_0,y) = E_y[u(x_0,X_S)] \leq E_y[K(x_0 - X_S)], \quad y \in G$$

because $u(x_0,z) \leq K(x_0 - z)$ for all z. Letting G increase to D, we obtain by bounded convergence

$$u(x_0,y) \leq E_y[K(x_0 - X_T)] = u(y,x_0), \qquad y \in D.$$

Since $x_0 \in D$ is arbitrary this last inequality implies (9) for all $(x,y) \in D \times D$.

We now show that (9) holds for all $(x,y) \in R^d \times R^d$. By Proposition 1, $u(x_0,y) = K(x_0 - y) \geqslant u(y,x_0)$ for almost all $y \notin D$. We have just shown that $u(x_0,y) = u(y,x_0)$ for $y \in D$. Thus $u(x_0,y) \geqslant u(y,x_0)$ for almost all y and by superharmonicity for all y:

$$u(x_0,y) \geqslant u(y,x_0), \quad y \in R^d, \ x_0 \in D.$$

If $x \notin D$ or $x \in \partial D$ and is regular, $u(x,y) = K(x-y) \geqslant u(y,x)$. Thus for all y and all x, except perhaps for $x \in \partial D$ and irregular

$$u(x,y) \geqslant u(y,x), \quad y \in R^d, \ x \in \partial D, \ x \text{ not irregular.}$$

We shall presently show that the set of irregular points in ∂D has Lebesgue measure zero. Since for any $y, u(\cdot,y)$ and $u(y,\cdot)$ are both superharmonic the last inequality holding off the set of irregular points in ∂D implies $u(x,y) \geq u(y,x)$ for all x. Thus $u(x,y) \geq u(y,x)$ for all (x,y) which is (9).

It remains to show that the set of irregular points in ∂D has Lebesgue measure zero. It is enough to show this assuming that D is connected. Indeed every irregular point is an irregular boundary point of a connected component of D (see Exercise 5 § 2, Chapter 4) and there ar only countably many connected components. If $x_0 \in D$

$$h(y) = K(x_0-y) - E_{x_0}[K(X_T-y)]$$

is superharmonic and strictly positive in D. For $b \in \partial D$

$$\limsup_{D \ni y \to b} h(y) = 0 \quad \text{iff } b \text{ is regular}$$

as we saw in Property 2. But from Proposition 1 for almost all $b \in \partial D$

$$K(x_0-b) = E_{x_0}[K(X_T-b)] \leq \liminf_{D \ni y \to b} E_{x_0}[K(X_T-y)]$$

by lower semi continuity. Thus for almost all $b \in \partial D$

$$\limsup_{D \ni y \to b} h(y) = 0.$$

Thus (9) is completely established.

<u>Property 7</u>. Let D be bounded, V open $\subset D$, $U \subset D \setminus \bar{V}$ and S = exit time from U. Then for all $x \in V$, $y \in U$

(10) $$G_D(x,y) = E_y[G_D(X_S,x) : S < T].$$

In particular if $x \in V$ and $D \ni y \notin \bar{V}$

$$G_D(x,y) \leq \sup_{z \in D \cap \partial V} G_D(x,z).$$

Indeed, for $x \in V$ and $y \in U$, $K(x-y) = E_y[K(X_S-x)]$.

$$\begin{aligned} G_D(x,y) &= K(x-y) - E_y[K(X_T-x)] \\ &= E_y\{K(X_S-x) - E_{X_S}[K(X_T-x)] : S < T\} \\ &= E_y[G_D(X_S,x) : S < T]. \end{aligned}$$

If $U = D \setminus \bar{V}$, for $S < T$, $X_S \in D \cap \partial V$ and the last inequality claimed above follows.

Now let us briefly see how the knowledge of the Green function for D (a bounded open set) allows us to solve, theoretically, the generalized Dirichlet problem.

Because any function in $C(\partial D)$ can be approximated uniformly on ∂D by a c^∞-function with compact support, it is sufficient to know the solution of the generalized Dirichlet problem with c^∞-boundary values. For a c^∞-function f with compact support

$$u(x) = f(x) - \frac{1}{A_d} \int_D G_D(x,y) \Delta f(y)\, dy, \qquad x \in D,$$

is the solution of the generalized Dirichlet problem with boundary function f. If furthermore ∂D is smooth by an application of Green's identity (recall $G_D(x,\cdot) = -A_d \delta_x$, δ_x the unit mass at x)

$$u(x) = \frac{1}{A_d} \int_{\partial D} D_n G_d(x,y) f(y) \sigma(dy), \qquad x \in D,$$

where σ is the area measure on ∂D and D_n the normal derivative.

Unbounded open sets.

We have now seen that the Green function for a bounded open set is related to the Dirichlet problem and the Poisson equation. The uniqueness of solutions in these considerations was a consequence of the maximum principles. In the formulation and deduction of maximum principles for bounded open sets the compactness of $\bar{D}$ was a key factor. For an unbounded open set D, compactness of $\bar{D}$ requires the addition of the point at infinity. If the point at infinity is considered an element of ∂D, its role has to be investigated. If D is an unbounded open set in $\mathbb{R}^d$ and ∂D is compact, Proposition 2 below shows that an arbitrary positive harmonic function in D has a limit at infinity and in particular is bounded at infinity, provided $d \geq 3$. This is false when $d = 2$ as shown by the example $D = \{x: x \in \mathbb{R}^2, |x| > 1\}$ and $u(x) = \log|x|$, $x \in D$. Proposition 3 below, on the other hand, shows that for an arbitrary open set $D \subset \mathbb{R}^2$ and any harmonic function in D which is bounded and continuous on $\bar{D}$ the formula

$$u(x) = E_x[u(X_T)], \quad x \in \bar{D}, \quad T = \text{exit time from } D$$

holds; in particular u harmonic in D, bounded and continuous on $\bar{D}$, $u = 0$ on ∂D imply $u \equiv 0$. Such a conclusion is invalid in $d \geq 3$ dimensions if D is unbounded as shown by the example $D = \{x: x \in \mathbb{R}^d, |x| > 1\}$ and $u(x) = 1 - K(x)$, $x \in D$. Thus in considerations of the Dirichlet problem for unbounded open sets the dimension plays an important role.

Proposition 2. Let u be positive and harmonic in $D = \{x: x \in \mathbb{R}^d, |x| > 1\}$. If $d \geq 3$, $\lim_{x\to\infty} u(x)$ exists. If $D = 2$,

$$u(x) = c \log|x| + h(x), \quad c \geq 0,$$

where h is positive harmonic in D and $\lim_{x\to\infty} h(x)$ exists. Further, if $d = 2$ and u is positive superharmonic in D and $\lim_{x\to\infty} u(x) = 0$, then $u \equiv 0$.

Proof. The Kelvin transformation relative to $S(0,1) = \partial B(0,1)$ gives a function which is positive and harmonic in the ball $B(0,1)$ punctured at 0. Use a result of Bocher (Proposition 4, § 3, Chapter 5) and note that Kelvin transformation is idempotent. Recall also that a positive superharmonic function vanishing at a point vanishes in its connected component.

Proposition 3. Let $D \subset \mathbb{R}^2$ be open. If u is continuous and bounded on $\bar{D}$ and harmonic in D, then $u(x) = E_x[u(X_T)]$, $x \in \bar{D}$, where T = exit time from D. In $d \geq 3$ dimensions the same conclusion obtains, provided we assume, in addition, that $u(x)$ tends to zero as $D \ni x \to \infty$.

Proof. Recall that $P_{\cdot}(T<\infty) \equiv 1$ or $\equiv 0$ on $\mathbb{R}^2$, by Theorem 5, § 1, Chapter 5, and in the latter case every bounded (indeed positive) harmonic function on D is a constant. Put

$$D_n = D \cap \{x: |x| < n\}.$$

u is harmonic on the bounded open set D_n and continuous on $\bar{D}_n$. So

$$u(x) = E_x[u(X_{T_n})], \; x \in \bar{D}_n, \quad T_n = \text{exit time from } D_n.$$

If $x \in \bar{D}$, $x \in \bar{D}_n$ from and after some n. Clearly $T_n = R_n \wedge T$, where R_n is the exit time from the disc $B_n = \{x: |x| < n\}$. On the set $T_n < T$, $R_n < T$. $P_x(R_n \uparrow \infty) = 1$, $P_x(T < \infty) = 1$, u bounded imply by letting $n \to \infty$ in

$$u(x) = E_x[u(X_{T_n})] = E_x[u(X_T) : T = T_n] + E_x[u(X_{T_n}) : R_n < T]$$

that

$$u(x) = E_x[u(X_T)], \qquad x \in \bar{D}.$$

That proves the Proposition.

Let us return to the discussion of the Green functions for unbounded open sets. One would like that the Green function for an unbounded open set satisfy as many of the Properties 1 - 7 as possible. Property 4 or 5 can be used for this purpose. We shall use Property 4. Let D be open and D_n an increasing sequence of bounded open sets with union D. By Property 3, $\lim G_{D_n}$ exists for all $(x,y) \in D \times D$. Now if D_n increase to D and A is a bounded open subset of D, $D_n \cap A$ increases to A. By Property 3 and 4

$$\lim G_{D_n} \geq \lim G_{D_n \cap A} \geq G_A$$

showing that the limit $\lim G_{D_n}$ is independent of the exhausting sequence $\{D_n\}$. In $d \geq 3$ dimensions $K \geq 0$ and a glance at (5) shows that for $(x,y) \in D \times D$

$$G_D(x,y) = \lim G_{D_n}(x,y) = K(x-y) - E_x[K(X_T-y)] \tag{11}$$

where T = exit time from D (when $T = \infty$, $K(X_T-y) = 0$ by definition). The situation when $d = 2$ is more complicated. In this case, as the following Proposition shows, $\lim G_{D_n}(x,y) < \infty$ off the diagonal of $D \times D$ iff it is finite at one point. When this limit is finite we will say that D has a Green function and

$$G_D(x,y) = \lim G_{D_n}(x,y), \quad (x,y) \in D \times D$$

will be its Green function. Whenever we write G_D we assume that D is Greenian i.e. that D has a Green function. By Property 6 G_D is symmetric on $D \times D$.

Proposition 4. Let D be open $\subset R^2$ and D_n be bounded open and increase to D. If T_n = exit time from D_n

$$s(x,y) = \lim E_x[K(X_{T_n}-y)] \qquad x,y \in D$$

is either $\equiv -\infty$ or is separately harmonic in D.

Proof. Suppose first that D is connected. Since $s(\cdot,\cdot)$ is independent of the exhausting sequence $\{D_n\}$, we may assume that D_n are connected and increase to D. For each m, $E_x[K(X_{T_n}-y)]$ is separately harmonic in the connected set D_m, for all $n \geq m$. And they decrease. The limit is either $\equiv -\infty$ or is separately harmonic in D_m and m is arbitrary.

Now suppose D is not connected. If x and y belong to different components of D, they belong to different components of D_n say A_n, B_n for large n. $K(\cdot - y)$ is then harmonic in A_n and continuous on $\bar{A}_n$ so that

$$E_x[K(X_{T_n}-y)] = K(x-y) \qquad \text{for all large } n$$

i.e. $s(x,y) = K(x-y)$. The case x,y in the same component is to be treated.

If D is not connected, the closure of any component of D can not be R^2. So suppose D itself is connected and $\bar{D} \neq R^2$. Let $z \notin \bar{D}$ and $x,y \in D_1$, say where D_n increase to D. Clearly there is a positive number α such that $|z-\xi| \cdot |y-\xi|^{-1} \geqq \alpha$ as ξ varies on ∂D_n, uniformly in n. Hence

$$(12) \qquad E_x[K(X_{T_n}-y)] - E_x[K(X_{T_n}-z)] \geqq \log \alpha$$

uniformly in n. Since $K(z-\cdot)$ is harmonic in D_n and continuous on its closure the second term in (12) is just $K(z-x)$. Thus $E_x[K(X_{T_n}-Y)]$ are bounded below by a harmonic function in D. $s(\cdot,\cdot)$ is therefore separately harmonic in D.

Properties. The function G_D defined on $D \times D$ by

$$G_D(x,y) = \lim G_{D_n}(x,y), \qquad x,y \in D,$$

where D_n is any sequence of bounded open sets increasing to D clearly satisfies Properties 1, 3, 4, 5, 6 and 7. That it also satisfies Property 2 is shown as follows:

Let $a \in \partial D$ be regular, $x_0 \in D$. We must show that $G_D(x_0,y)$ tends to zero as $y \to a$. Let V be a ball centre x_0, with closure contained in D and $U = D \cap (x: |x-a| < r)$. For small r, U will not intersect $\bar{V}$. From (10)

$$G_D(x_0,y) = E_y[G_D(x_S,x_0) : S < T], \qquad y \in U,$$

where S = exit time from U. $G_D(x_0,\cdot)$ is continuous on $\partial V \subset D$. By the second statement of Property 7, for all $y \notin V$, $G_D(x_0,y) \leqq \sup_{z \in \partial V} G_D(x_0,z) = M$, say. Therefore $G_D(x_0,y) \leqq M P_y[S < T]$, for $y \in U$. If S_1 = exit time from the ball $(x: |x-a| < r)$, then $S = S_1 \wedge T$ and $P_y[S < T] = P_y[S_1 < T]$. It is

thus enough to show that $P_y(S_1 \leqq T)$ tends to zero as $y \to a$. But this is simple. Indeed, given $\varepsilon > 0$, for small t,

$$P_y(S_1 \leqq t) < \varepsilon \qquad \text{for all } y \in B(a,\tfrac{r}{2}).$$

And

$$P_y[S_1 \leqq T] \leqq P_y[S_1 \leqq t] + P_y[t \leqq T].$$

As $y \to a$, $P_y[t \leqq T]$ tends to zero because a is regular.

<u>Thus in Properties</u> 1 <u>through</u> 7, D <u>can be any open set having a Green function</u>.

As to the Poisson equation, let D be open and unbounded $\subset \mathbb{R}^d$, $d \geq 3$, and f be c' on $\mathbb{R}^d$ with compact support. Then there is one and only one c^2-function u defined on D such that $\Delta u = -A_d f$ in D, $u(x)$ tends to zero as x tends to any regular point in ∂D and $u(x)$ tends to zero as x tends to infinity. And this function u is given by

$$u(x) = \int_D G_D(x,y) f(y)\,dy, \qquad x \in D.$$

The uniqueness claim above follows from Theorem 3, § 5, Chapter 5.

The discussion of the Poisson equation for unbounded open subsets (having Green functions) of $\mathbb{R}^2$ seems a little more involved. If $D \subset \mathbb{R}^2$ has a Green function and f is c' on $\mathbb{R}^2$ with compact support, it is possible to show the following: There is one and only one bounded c^2-function u defined on D such that $\Delta u = -A_d f$ in D and $u(x)$ tends to zero x tends to any regular point in ∂D. And this function u is given by

$$u(x) = \int_D G_D(x,y) f(y)\,dy, \qquad x \in D.$$

We shall discuss this in § 3.

The following is a characterization of Green sets. See also Theorem 5, § 1, Chapter 5.

<u>Proposition 5</u>. An open set $\Omega \subset \mathbb{R}^2$ is Greenian iff there exists a non-constant excessive function on Ω.

<u>Proof</u>. If Ω is Greenian, for any $x_0 \in \Omega$, $G_\Omega(x_0,\cdot)$ is non-constant excessive on Ω. Conversely suppose h is a non-constant excessive function on Ω. By considering $\sqrt{h}$ if necessary, we may assume that the Riesz measure m of h is not zero. Let D be a relatively compact open subset of Ω. By Theorem 5, § 3, Chapter 5

(13) $$\frac{1}{A_d}\int_D K(x-y)\,m(dy) + g(D,x) = h(x), \quad x \in \Omega$$

If T = exit time from D, we get from (13) (recall $g(D,\cdot)$ is superharmonic in Ω)

(14) $$\frac{1}{A_d}\int_D \{K(x-y) - E_x[K(X_T-y)]\}\,m(dy) \le h(x) - E_x[h(X_T)] \le h(x).$$

As D increases to Ω, the integrand in (14) increases to G_Ω:

(15) $$\frac{1}{A_d}\int_\Omega G_\Omega(x,y)\,m(dy) \le h(x).$$

For any x_0 with $h(x_0) < \infty$, since $m \neq 0$, (15) shows that $G_\Omega(x_0,\cdot) < \infty$ m-almost everywhere. Thus Ω is Greenian by Proposition 4.

Let Ω be a Greenian open set in $\mathbb{R}^d$ (if $d \ge 3$, this simply means an open subset). $G_\Omega(\cdot,\cdot)$ is separately excessive in Ω. It follows that for any measure m

(16) $$G_\Omega m(x) = \int G_\Omega(x,y)\,m(dy)$$

is excessive in Ω. If Ω is connected $G_\Omega m$ is either superharmonic or is $\equiv\infty$ in Ω. An easy consequence of Proposition 4 is:

(17) The Riesz measure of $\frac{1}{A_d}G_\Omega m$ is m.

Indeed suppose $\varphi \in C^\infty$ and has compact support in Ω. On the support of φ, $G_\Omega(x,\cdot)$ and $K(x-\cdot)$ are both integrable. Also $G_\Omega(x,y) = K(x-y) - s(x,y)$, where $s(\cdot,\cdot)$ is separately harmonic in Ω, by Proposition 4.

(18) $$\int (G_\Omega m)\,\Delta\varphi = \int m(dx)\int K(x-y)\,\Delta\varphi(y)\,dy - \int m(dx)\int s(x,y)\,\Delta\varphi(y)\,dy$$
$$= \int m(dx)\int K(x-y)\,\Delta\varphi(y)\,dy = -A_d\int \varphi(x)\,m(dx)$$

because $\int s(x,y)\,\Delta\varphi(y)\,dy = 0$. (18) implies (17).

Suppose now h is excessive in Ω and has Riesz measure m. As we saw in Proposition 5, (15) is valid. From (17) $h = \frac{1}{A_d}G_\Omega m + u$. where $0 \le u$ is harmonic in Ω. We have thus

<u>Theorem 6</u>. (F. Riesz). Let Ω be a Greemian open subset of $\mathbb{R}^d$. If h is excessive in Ω and has Riesz measure m

(19) $$h = \frac{1}{A_d}G_\Omega m + u$$

where $0 \le u$ is harmonic in Ω and, $G_\Omega m$ is defined by (16).

<u>Kelvin transformation and Green functions</u>.

Consider inversion relative to the sphere of radius ρ centre 0: $S(0,\rho) = \partial B(0,\rho)$. If $x \neq 0$, its inverse $x^* = \rho^2 \frac{x}{|x|^2}$. Let D be open $\subset \mathbb{R}^d \setminus \{0\}$ and D_1 its image under inversion.

If f is C^2 on D_1 and g its Kelvin transform:

$$g(x) = \frac{\rho^{d-2}}{|x|^{d-2}} f(x^*),$$

then

$$(\Delta g)(x) = \frac{\rho^{d+2}}{|x|^{d+2}} (\Delta f)(x^*).$$

This (together with the approximation of superharmonic functions by smooth ones, § 1, Chapter 5) shows that <u>Kelvin transformation preserves superharmonicity</u>. Let $x_0 \in D$. The function

$$u(x) = \frac{\rho^{d-2}}{|x|^{d-2}} G_{D_1}(x_0^*, x^*)$$

is positive superharmonic on D. Since $G_{D_1}(x_0^*, y) - K(x_0^* - y)$ is harmonic for $y \in D_1$, its Kelvin transform is harmonic in D:

$$u(x) - \frac{\rho^{d-2}}{|x|^{d-2}} K(x_0^* - x^*)$$

is harmonic in D. The relation $|x_0^* - x^*| = \frac{\rho^2}{|x_0||x|}|x - x_0|$ shows that

$$u(x) = \begin{cases} u(x) - \frac{\rho^{d-2}}{|x|^{d-2}} K(x_0^* - x^*) + \frac{|x_0|^{d-2}}{\rho^{d-2}} K(x_0 - x), & d \geq 3 \\ u(x) - K(x_0^* - x^*) + K(x_0 - x) + \log|x_0||x| - \log \rho^2, & d = 2 \end{cases}$$

Thus the function $\frac{\rho^{d-2}}{|x_0|^{d-2}} u(x)$ which is positive and superharmonic in D is the sum of $K(x_0 - x)$ and a superharmonic function. By Property 5 one concludes that

$$\frac{\rho^{(d-2)}}{|x_0|^{d-2}} u(x) \geq G_D(x_0, x).$$

The Kelvin transformation being idempotent

$$G_D(x_0, x) = \frac{\rho^{2(d-2)}}{|x|^{d-2}|x_0|^{d-2}} G_{D_1}(x_0^*, x^*)$$

<u>Green functions and holomorphic transformations</u>.

If $w(x) = u(x) + iv(x)$ is a holomorphic map on an open set D onto open set D_1, f is C^2 in D_1 and $g(x) = f(w(x))$ then

$$\Delta g(x) = |\text{grad } u(x)|^2 (\Delta f)(w(x))$$

as is seen by using Cauchy-Riemann equations. This, as in the case of the Kelvin transformation, shows that whenever s is superharmonic on D_1, the composed function $h(x) = s(w(x))$ is superharmonic on D. <u>D_1 has a Green function thus implies that D has a Green function</u> by Proposition 5. Suppose then that D_1 has a Green function. For any $z_0 \in D$

$$a(z) = G_{D_1}(w(z_0), w(z))$$

is non-negative superharmonic on D. Since

$$G_{D_1}(w(z_0), x) + \log|w(z_0) - x|$$

is harmonic on D_1, so is

$$b(z) = a(z) + \log|w(z_0) - w(z)|$$

on D. We have

$$a(z) = b(z) - \log\left|\frac{w(z_0) - w(z)}{z - z_0}\right| + K(z_0 - z).$$

$\frac{w(z_0) - w(z)}{z_0 - z}$ being holomorphic on D, its logarithm is subharmonic in D. The positive superharmonic function a is thus the sum of a superharmonic function and $K(z_0 - z)$. By Property 5, $a(z) \geq G_D(z_0, z)$. In particular, if w is simple, i.e. 1-1 homomorphic on D onto D_1

$$G_{D_1}(w(z_0), w(z)) = G_D(z_0, z),$$

i.e. <u>Green function is a 1-1 conformal invariant</u>.

Exercises to § 1.

1. Let D be a bounded open set and T = exit time from D. Then

$$L(x,y) = E_x[K(X_T-y)]$$

is lower semi continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

2. Let D be bounded open in $\mathbb{R}^d$, u positive harmonic in D and $z_0 \in \partial D$ regular. Assume that $u(z) \le O(K(z-z_0)) + O(1)$ and $\lim_{z\to b} u(z) = 0$ for all $z_0 \neq b \in \partial D$ regular. Then $u \equiv 0$.

Hint. Let $x_0 \in D$. By (9), $E_{x_0}[K(X_T-z_0)] = K(z_0-x_0)$. Let $\bar{D}_n \subset D$ increase to D and T_n = exit time from D_n. $K(x_0-z_0) = E_{x_0}[K(X_{T_n}-z_0)] \ge E_{x_0}[K(X_T-z_0)] = K(x_o-z_o)$. Since $K(X_{T_n}-z_0)$ tends to $K(X_T-z_0)$ and all the functions are bounded below, $K(X_{T_n}-z_0)$ thus converges in L^1 (relativ to P_{x_0}) to $K(X_T-z_0)$. In particular $K(X_{T_n}-z_0)$ and hence also $u(X_{T_n})$ is uniformly integrable. Now $u(x_0) = E_{x_0}[u(X_{T_n})]$ and X_{T_n} tends to X_T. Now use Lemma 2, § 5, Chapter 5.

Remark. The example: D = unit disc in $\mathbb{R}^2$ punctured at the origin, $u = K$ shows that the restriction in Exercise 2, that $z_0 \in \partial D$ be regular is not redundant.

3. Let D be an unbounded open set in $\mathbb{R}^d$. Suppose u is bounded harmonic in D and $\lim_{D\ni x\to b} u(x) = 0$ for every $b \in \partial D$ that is regular. If $D = 2$, $u \equiv 0$. If $d \ge 3$ and further u tends to zero at infinity then $u \equiv 0$.

4. Let D be Greenian and s superharmonic in D. Suppose the Reisz measure of s is finite. Then

$$s(x) = \int G(x,y)m(dy) + h, \quad h \text{ harmonic in } D.$$

Hint. $G(x,\cdot)$ behaves like $K(x-\cdot)$ near x. and $G(x,\cdot)$ is bounded off any neighbourhood of x (Property 7). Thus $\int G(x,y)m(dy)$ can not be identically infinite in any connected component of D.

§ 2. Unbounded open subsets of $\mathbb{R}^2$.

We have defined the Green function for unbounded open subsets of $\mathbb{R}^2$ as a limit of Green functions for bounded open sets. Even in very simple cases it is difficult to apply this definition to find Green functions. By far the most powerful method is to find a "mapping function" to map a given domain D onto another domain D_1 whose Green function is known. This will then allow us to find the Green function for D.

In this and the next section we shall investigate the extent to which "formula" (5), § 1 holds; it does not hold in general. For example, if $D = (x:|x| > 1)$

$$G_D(x,y) = E_x[\log|X_T-y|] - \log|x-y| + \log|x|$$

T = exit time from D, as is seen by taking $D_n = (x:1 < |x| < n)$ and using (9), § 1, Chapter 4. We shall find a necessary and sufficient condition for formula (5) to hold but first some propositions. The discussion will also throw some light on the role of the point at infinity.

Until further notice, D will denote a Greenian open set, i.e. an open set having a Green function. Given D, let $D_n = B_n \cap D$ where B_n is the disc of radius n, centre 0: $B_n = (x:|x| < n)$. T_n = exit time from $\bar{D}_n$ and T = exit time from D. Put

$$s_n(x,y) = E_x[K(X_{T_n}-y)], \qquad x,y \in \mathbb{R}^2. \tag{1}$$

$s_n(x,\cdot)$ are superharmonic on $\mathbb{R}^2$ harmonic in D_n and decrease. Let

(2) $$s(x,y) = \lim s_n(x,y).$$

Since $s_n(x,y) \leq K(x-y)$, $s(x,y) \leq K(x-y)$. So $s(x,y) < \infty$ unless $x=y$. $s(x,y)$ may apriori be $-\infty$. s is symmetric because s_n are (Property 6, Section 1).

Suppose $x_0 \in D$, $y_0 \in D$. D being Greenian $s(x_0,y_0) > -\infty$. From an n on $|X_{T_n} - y_0|$ is clearly bounded away from zero (since $X_{T_n} \in \partial D_n$) and $K(X_{T_n} - y_0)$ is then bounded above. Fatou's Lemma is applicable:

$$-\infty < s(x_0,y_0) \leqq E_{x_0}[\limsup K(X_{T_n} - y_0)]$$
$$= E_{x_0}[K(X_T - y_0)].$$

$K(X_T - y_0)$ being bounded above, $E_{x_0}[K(X_T - y_0)]$ makes sense and from the last inequality we conclude that $E_{x_0}[|K(X_T - y_0)|] < \infty$. By exercise 8, § 1, Chapter 5, $E_{x_0}[K^-(X_T - y)]$ is finite for all y. For any y

(3) $$E_{x_0}[K(X_{T_n} - y)] = E_{x_0}[K^+(X_T - y) : T_n = T]$$
$$- E_{x_0}[K^-(X_T - y) : T_n = T] + E_{x_0}[K(X_{T_n} - y) : T_n < T].$$

$P_{x_0}[T_n < T]$ tends to zero. On the set $T_n < T$, $|X_{T_n}| = n$ and $|X_{T_n} - y|/n$ tends boundedly to 1. Putting $y = y_0$ in (3) and taking limits shows (since $E_{x_0}[|K(X_T - y_0)|] < \infty$ and $s(x_0,y_0) > -\infty$) that $\lim_{n\to\infty} (\log n) P_{x_0}(T_n < T)$ exists and is finite. From (3), letting $n \to$, for any y

$$s(x_0,y) = E_{x_0}[K^+(X_T - y)] - E_{x_0}[K^-(X_T - y)] - \lim (\log n) P_{x_0}(T_n < T).$$

The left side of the last equality can at worst be $-\infty$ while the right side at worst $+\infty$. One concludes that $s(x_0, y) > -\infty$ for all y and $E_{x_0}[|K(X_T - y)|] < \infty$ for all y. s is symmetric. So $s(x,y) > -\infty$ for all x, provided $y \in D$.

Let ξ be arbitrary and $y_0 \in D$. From the last paragraph $s(\xi, y_0) > -\infty$. As before (since $|X_{T_n} - y_0|$ is bounded away from zero etc.), $E_\xi[|K(X_T - y_0)|] < \infty$, $\lim \log n \, P_\xi(T_n < T)$ exists and is finite and $E_\xi[K(X_T - \cdot)]$ is superharmonic. In their turn the last two facts imply, as before that, $s(\xi, y) > -\infty$ for all y and

$$s(\xi,y) = E_\xi[K(X_T - y)] - \lim (\log n) P_\xi(T_n < T).$$

Let us collect all the above in

<u>Proposition 1</u>. With above notation $E_x[K(X_T - y)]$ is superharmonic in y for every $x \in \mathbb{R}^2$. And

(4) $$\begin{cases} s(x,y) = E_x[K(X_T - y)] - a_D(x), & x,y \in \mathbb{R}^2 \\ a_D(x) = \lim_n (\log n) P_x[T_n < T], & x \in \mathbb{R}^2. \end{cases}$$

<u>Remarks</u>. From (4), for each x, $s(x,\cdot)$ is superharmonic and by symmetry, $s(\cdot,\cdot)$ is separately superharmonic. It is also easy to see that

(5) $$\lim_{n\to\infty} E_x[|K(X_{T_n} - y)|] = E_x[|K(X_T - y)|] + a_D(x), \quad x,y \in \mathbb{R}^2.$$

Since $s(\cdot,\cdot)$ is separately harmonic in D, (4) allows that a_D is positive and harmonic in D.

Corollary 2. For all $x \in \mathbb{R}^2$ and $b \in \partial D$

(6) $$\liminf_{D \ni y \to b} E_x[K(X_T - y)] = E_x[K(X_T - b)], \quad x \in \mathbb{R}^2.$$

If $b \in \partial D$ is regular, the lim inf in (6) can be replaced by a limit. If $b \notin \bar{D}$ or $b \in \partial D$ regular

(7) $$E_x[K(X_T - b)] = K(x-b) + a_D(x), \qquad x \in \mathbb{R}^2.$$

when a_D is defined in (4).

Proof. If $b \in \partial D$ is irregular (6) follows from Exercise 9, § 1, Chapter 5. Now suppose $b \in \partial D$ is regular. By lower simicontinuity, the left side in (6) is at last equal to the right side. Since $s(x,y) \leq K(x-y)$, we obtain by (4)

$$\limsup_{D \ni y \to b} [E_x\, K(X_T - y)] \leq K(x-b) + a_D(x)$$

and the right side of the above inequality is just $E_x[K(X_T - b)]$, as is seen by using (4) and recalling that $s(x,b) = K(x-b)$ if $b \in \partial D$ is regular. (7) follows from (4). Q.e.d.

Clearly Corollary 2 above says nothing if $x \notin \bar{D}$ or $x \in \partial D$ is regular. It is not difficult to show that a_D is non-negative harmonic in D and

$$\lim_{D \ni x \to b} a_D(x) = 0 \quad \text{if } b \in \partial D \text{ is regular.}$$

Theorem 3. Let m be a probability measure on $\mathbb{R}^2$ such that

(8) $$s(x) = \int K(x-y)\, m(dy)$$

is superharmonic. Then $E_x[s^-(X_T)] < \infty$ for all x and

(9) $$E_x[s(X_T)] \leq s(x) + a_D(x), \qquad x \in \mathbb{R}^2.$$

If m does not charge D or the set of irregular points in ∂D then equality obtains in (9).

Proof. We claim that

(10) $$\int E_x[K^-(X_T - y)]\, m(dy) < \infty, \qquad x \in \mathbb{R}^2.$$

which of course implies that $E_x[s^-(X_T)] < \infty$. For any y_0

$$K^-(X_T - y_0) \leq K^-(2(X_T - y_0)) + K^-(2(y-y_0))$$

$$\leq \log 4 + K^-(X_T - y_0) + K^-(y-y_0).$$

Since $E_x[K(X_T - \cdot)]$ and $\int K(\cdot - z)\, m(dz)$ are superharmonic

$$E_x[K^-(X_T - y_0)] < \infty \quad \text{and} \quad \int K^-(y_0 - z)\, m(dz) < \infty$$

for all y_0 (Exercise 8, § 1, Chapter 5). (10) is thus established and because of (10) use of Fubini is permitted below. Using (4) and that $s(x,y) \leq K(x-y)$ we obtain

$$E_x[s(X_T)] = \int E_x[K(X_T - y)]\, m(dy)$$

$$\leq \int K(x-y)\, m(dy) + a_D(x) = s(x) + a_D(x).$$

Similarly for the last statement use (7). Q.e.d.

Examples.

In very simple cases (7) can be used to find a_D. If D is the complement of the closed unit disc and $b = 0$, we get at once from (7) $a_D(x) = \log|x|$ for $|x| > 1$. If D is the upper half plane, X_T lies on the real axis. For any b of the form $b = (0,-n)$ and z real, clearly $|z-b| \geq n$. It follows that $E_x[K(X_T-b)] \leq \log\frac{1}{n}$. Using (7) and letting $n \to \infty$, we get $a_D(x) \leq 0$, i.e. $a_D \equiv 0$. Consider again the strip $(0 < \operatorname{Im} z < 1)$ For $z \in \partial D$, $|z-n| \geq n-1$ so that for $x \in D$, $E_x[K(X_T-n)] \leq \log\frac{1}{n-1}$ Use (7) and let $n \to \infty$ to get $a_D = 0$. Similar arguments can be used to show that $a_D = 0$ for a wedge, a quadrant etc. One may suspect that simple connectivity of a Green domain D is sufficient to guarentee that $a_D \equiv 0$. This will turn out to be correct.

Again in some simple cases (7) can be used to compute the harmonic measure at x, i.e. the distribution of X_T relative to P_x. As an example, consider D = the upper half plane. Denote the points on the plane by (x,y) If $b > 0$, the pont $(a,-b) \notin \bar{D}$. From (7)

$$\int \log[(z-a)^2 + b^2] P_{(x,y)}(dz) = \log[(x-a)^2 + (y+b)^2]$$

where $P_{(x,y)}(dz) = P_{(x,y)}(X_T \in dz)$. Differentiate both sides relative to b:

$$\int \frac{b}{(z-a)^2 + b^2} P_{(x,y)}(dz) = \frac{y+b}{(x-a)^2 + (y+b)^2}.$$

Multiply both sides by $e^{i\alpha a}$ and integrate relative to da from $-\infty$ to ∞.

$$e^{-|\alpha| b} \hat{P}_{(x,y)}(\alpha) = e^{i\alpha x} e^{-|\alpha|(y+b)}$$

where $\hat{P}_{(x,y)}(\alpha) = \int e^{i\alpha z} P_{(x,y)}(dz)$. Inversion shows that $P_{(x,y)}(dz)$ has density

$$\frac{1}{\Pi} \frac{y}{(x-z)^2 + y^2}.$$

We have thus found the Poisson kernel for the half plane $((x,y): y > 0)$ (See § 2, Chapter 4.)

As another example, consider the strip $((x,y): 0<x<1) = D$. Fix $(x,y) \in D$. $P_{(x,y)}(X_T \in dz)$ lives on the lines $x = 0$ and $x = 1$. Denote these parts by Q_0 and Q_1 respectively. If $a > 1$, the point $(a,b) \notin \bar{D}$. From (7)

$$\int \log[a^2 + (b-z)^2] Q_0(dz) + \int \log[(a-1)^2 + (b-z)^2] Q_1(dz)$$
$$= \log[(x-a)^2 + (y-b)^2].$$

As before, differentiating relative to a then multiplying both sides by $e^{i\alpha b}$ and integrating relative to db from $-\infty$ to ∞:

$$\hat{Q}_0(\alpha) + e^{|\alpha|}\hat{Q}_1(\alpha) = e^{|\alpha| x} e^{i\alpha y}$$

where $\hat{Q}_j(\alpha) = e^{i\alpha z} Q_j(dz)$, $j = 0,1$.

Also the point $(-a,b) \notin \bar{D}$ if $a > 0$. Repetition of the above leads to

$$\hat{Q}_0(\alpha) + e^{-|\alpha|}\hat{Q}_1(\alpha) = e^{-|\alpha| x} e^{i\alpha y}.$$

We must thus have

$$e^{|\alpha|}\left[1 - e^{-2|\alpha|}\right]\hat{Q}_1(\alpha) = e^{i\alpha y}\left[e^{|\alpha| x} - e^{-|\alpha| x}\right]$$

which expands into

$$\hat{Q}_1(\alpha) = \sum_0^\infty e^{i\alpha y}\left\{e^{-(2n+1-x)|\alpha|} - e^{-(2n+1+x)|\alpha|}\right\}.$$

$Q_1(dz)$ thus has density

$$\frac{1}{\pi}\sum_0^\infty\left\{\frac{2n+1-x}{(2n+1-x)^2+(y-z)^2} - \frac{2n+1+x}{(2n+1+x)^2(y-z)^2}\right\} .$$

and $Q_0(dz)$ has density

$$\frac{1}{\pi}\sum_0^\infty\left\{\frac{2n+x}{(2n+x)^2+(y-z)^2} - \frac{2n+2-x}{(2n+2-x)^2+(y-z)^2}\right\} .$$

The above can be interpreted in the "image method". See Remark after Example 2, Section 4.

Exercises to § 2.

1. Show that

$$\lim_{D\ni x\to b} a_D(x) = 0. \quad \text{if} \quad b\in\partial D \text{ is regular}$$

Hint. Let $y_0\in D$. Then

$$G_D(x,y_0) = K(x-y_0) - E_x[K(X_T-y_0)] + a_D(x).$$

As $x\to b$, $\lim G_D(x,y_0) = 0$. Thus as $D\ni x\to b$

$$\limsup a_D(x) + \liminf \{K(x-y_0) - E_x[K(X_T-y_0)]\} \le 0.$$

If $m = K(b-y_0)$, $K(\cdot-y_0)\vee m$ is bounded and continuous on ∂D. So

$$\limsup_{x\to b} E_x[K(X_T-y_0)] \le \limsup_{x\to b} E_x[K(X_T-y_0)\vee m]$$

$$= K(b-y_0)\vee m = K(b-y_0).$$

2. Show that $a_D(x) \le \log|x| + 0(1)$

Hint. Let $y_0\in D$. Then

$$G_D(x,y_0) = K(x-y_0) - E_x[K(X_T-y_0)] + a_D(x)$$

is bounded off any neighbourhood of y_0. Also since $|X_T-y_0|$ is bounded away from zero, $E_x[K(X_T-y_0)]$ is bounded above. Finally a_D being harmonic in D is bounded in a neighbourhood of y_0.

3. Show that $\limsup_{x\to\infty}\frac{a_D(x)}{\log|x|} = 1$ unless $a_D\equiv 0$.

Hint. a_D is harmonic in D and bounded in D_n. (by the above Exercise). Hence

$$a_D(x) = E_x[a_D(X_{T_n}) : T_n < T]$$

Therefore $$a_D(x) \le \left(\sup_{|x|=n}\frac{a_D(x)}{\log n}\right)[\log n\; P_x[T_n<T]$$

i.e. $$a_D(x) \le \left(\limsup_{|x|\to\infty}\frac{a_D(x)}{\log|x|}\right) a_D(x).$$

§3. Unbounded open subsets of R^2 (continued)

Let D be an open set haveing a Green function. Retaining the notion of §2, we have the following expression for G_D

(1) $$G_D(x,y) = K(x-y) - E_x[K(X_T - y)] + a_D(x), \qquad x,y \in D.$$

Thus G_D is given by (5), §1, iff $a_D \equiv 0$.

Proposition 1. For any probability measure m on $\mathbb{R}^2$ satisfying

(2) $$\int \log(1+|y|)m(dy) < \infty$$

we have

$$\limsup_{x\to\infty} \left\{ \int \log|x-y|m(dy) - \log|x| \right\} = 0.$$

Proof. If f(x) is the function in the brackets in (3)

(4) $$f(\tfrac{1}{x}) = \int \log|1-xy|m(dy), \qquad x \neq 0$$

and the right side of (4) is subharmonic on $\mathbb{R}^2$, under (2). (Indeed, $\log|1-xy|$ is subharmonic in x because $1-yx$ is analytic in x. If $|x| \leq n$, $\log|1-xy|$ is bounded above by the m-integrable function $n\log(1+|y|)$, Fatou and Fubini are therefore applicable.) The lim sup in (3) is thus the value of this subharmonic function at 0, namely 0. Q.E.D.

Remark. Note that (2) is necessary and sufficient that

$$\int \log|x-y|m(dy)$$

be subharmonic on $\mathbb{R}^2$.

The following theorem gives necessary and sufficient conditions that $a_D \equiv 0$, i.e. (5), §1, be valid.

Theorem 2. Let D be Greenian. Then

(5) $$a_D(x) = \limsup_{D\ni y\to\infty} G_D(x,y), \qquad x \in D.$$

Proof. From the above remark and Proposition 1, for any $x \in D$,

(6) $$\limsup_{y\to\infty} \left\{ K(x-y) - E_x[K(X_T - y)] \right\} = 0.$$

If $a_D(x) = 0$, we see from (1) that

$$0 \leq \limsup_{D\ni y\to\infty} G_D(x,y) = \limsup_{D\ni y\to\infty} \{K(x-y) - E_x(K(X_T - y)\} \leq 0.$$

The last inequality follows from (6).

Now suppose $a_D(x) > 0$. If $y \notin \bar{D}$, by (7), §2, the quantity in brackets of (6) is just $-a_D(x)$. Also (6), §2, says that for any $b \in \partial D$, $E_x[K(X_T - b)]$ is a limit point of $E_x[K(X_T - y)]$ as $D \ni y$ tends to b. Thus the lim sup in (6) can be taken as y tends to infinity in D. Taking lim sup in (1) gives us (5). Q. E. D.

The following proposition gives a simple geometrical condition for regularity and can also be proved using the fact that the Brownian path winds around its starting point infinitely often. See H.P.McKean, Jr.[4].

Proposition 3. Let D be an open set. If $b \in \partial D$ is contained in a continuum completely contained in the complement of D, then b is regular.

Proof. Let $b \in F \subset \mathbb{R}^2 \setminus D$ be a continuum. If r is small, $F \cap (x: |x-b| \geq r) \neq \emptyset$. Let D_1 be the open set

$$D_1 = (x: |x-b| < r) \setminus F.$$

D_1 is bounded open, $D_1 \supset D \cap (x: |x-b| < r)$, $b \in \partial D$ and ∂D_1 is connected. D_1 is simply connected because ∂D_1 is connected. By the Riemann mapping theorem there is a 1-1 holomorphic map f defined on D_1 such that $f(D_1)$ is the open unit disc. It is easily seen that as $x \in D_1$ tends to any point in ∂D_1, $|f(x)|$ tends to 1. The function $h(x) = 1 - |f(x)|$ is thus a barrier at all points of ∂D_1. Every point in ∂D_1, in particular b, is thus regular for D_1^c. Since $b \in \partial D$ and $D_1 \supset D \cap (x: |x-b| < r)$, b is regular for $(D \cap (x: |x-b| < r))^c$ and hence for D^c. That proves the proposition.

Remark. Let D be an unbounded Green domain. Assume $0 \notin D$ and consider the map $x \to \frac{1}{x}$. If D_1 is the image of D under this map, $0 \in \partial D$, and $G_{D_1}(x,y) = G_D(\frac{1}{x},\frac{1}{y})$. $\lim_{D \ni y \to \infty} G_D(x,y) = 0$ is equivalent to $\lim_{D_1 \ni y \to 0} G_{D_1}(x,y) = 0$, i.e. that (Property 2, §1) 0 is regular for the complement of D_1. Proposition 3 thus implies: D is an unbounded domain such that $\mathbb{R}^2 \setminus D$ has an unbounded component then $a_D \equiv 0$. In particular, if $D \neq \mathbb{R}^2$ is simply connected, then $a_D \equiv 0$.

The case of D with compact complement

When the complement of D is compact, a_D can be expressed in a nice way as follows. From (4), §2, and the symmetry of s

$$a_D(x) = \left\{E_x[K(X_T - y)] + a_D(y)\right\} - E_y[K(X_T - x)], \qquad x,y \in \mathbb{R}^2. \tag{7}$$

Let B be a disc containing the complement of D and let S = hitting time to B. Then

$$E_y[K(X_T - x] = E_y[E_{X_S}(K(X_T - x))]. \tag{8}$$

For a sequence y_n tending to infinity, the probability measures $P_{y_n}[X_S \in dz]$ converge to a probability measure μ, say, on ∂B. Since $E.[K(X_T - x)]$ is continuous on ∂B, as y_n tends to infinity, the left side of (8) tends to

$$\int E_z[K(X_T - x)]\mu(dz) = \int K(z-x)m(dz) \tag{9}$$

where

$$m(dz) = \int P_y[X_T \in dz]\mu(dy). \tag{10}$$

It follows that the term in brackets in (7) converges as $y_n \to \infty$; and, because $\log\frac{|z-y_n|}{|y_n|}$ tends to zero uniformly for $z \in \partial D$, this limit is simply

$$\lim_{y_n \to \infty} [a_D(y_n) - \log|y_n|] = \gamma, \text{ say}. \tag{11}$$

Thus we obtain from (7)

$$a_D(x) = \gamma - \int K(x-z)m(dz), \qquad x \in \mathbb{R}^2, \tag{12}$$

where m is a probability measure on ∂D as is clear from (10)

Remark. Given a compact set C, the Robin problem is to find a measure m on C whose logarithmic potential, i.e. $\int K(\cdot - z)$, is constant on $C \setminus$ (irregular points). If C = the complement of D, we see from (12) that m solves the Robin problem for C.

The third term in (12), being superharmonic on $\mathbb{R}^2$, is bounded below on compact sets. a_D is thus bounded on compact sets. It is also clear from (12) that

(13) $$\lim_{x\to\infty}(a_D(x) - \log|x|) = \gamma.$$

γ is called the Robin constant for D.

Poisson Equation

If D is Greenian and f is C^1 on R^2 with bounded support,

$$u(x) = \int G_D(x,y)f(y)dy, \qquad x \in D,$$

is obviously C^2 on D and satisfies $\Delta u = -2\pi f$. This is so because $G_D - K$ is harmonic in D. Let us show that u is bounded in D and $u(x)$ tends to zero as $x \to b \in \partial D$ regular. This we do by reducing to the case where ∂D is compact as follows: Let $b \in \partial D$ be regular and $x_0 \in D$. Let A be a compact subset of ∂D, containing b such that $P_{x_0}[X_T \in A] > 0$. If B is a disc containing A and $W = D \cup B^c$, then W^c is compact, $b \in \partial W$, and is regular for $(B \cap D)^c = (B \cap W)^c$ and regularity is a local property. In particular, by Theorem 5, §1, Chapter 5, W is Greenian. Since $W \supset D$, $G_W \geq G_D$.

Thus it is sufficient to show the following: Let f be bounded measurable with bounded support and D Greenian with D^c compact. Then

(14) $$u(x) = \int G_D(x,y)f(y)dy$$

is bound in D and $u(x) \to 0$ as $x \to b \in \partial D$ regular. If $F(y) = \int K(y-z)f(z)dz$, u of (14) has the form

(15) $$u(x) = F(x) - E_x[F(X_T)] + a_D(x)\cdot\left(\int f(y)dy\right).$$

F is continuous on $\mathbb{R}^2$ so F is bounded and continuous on ∂D. Therefore the second term in (15) tends to $F(b)$ as $x \to b \in \partial D$ regular. And $a_D(x)$ tends to zero as $x \to b$ (Exercise 1, §2).

Finally, since $F(x) = \log(x)\cdot\left(\int f(y)dy\right) + O(1)$, boundedness of u is clear from (13).

Exercises to §3

1. Let D have a Green function. For any bounded subset A of D and any $\varepsilon > 0$, show that

$$\sup G_D(x,y) < \infty,$$

where the sup is over all $x \in A$ and $y \in D$ such that $|x-y| \geq \varepsilon$.

Hint. Reduce to the case where D^c is compact.

2. In the notation of §2, show that $P_x[T_n < T]$ tends to zero uniformly on compact subsets of $\mathbb{R}^2$.

Remark. If we reduce to the case where D^c is compact, this is clear from (13) and the expression of a_D given in (4), §2. But show this directly.

3. In the notation of §2, show that $s_n(\cdot,\cdot)$ decreases uniformly on bounded subsets of $\mathbb{R}^2 \times \mathbb{R}^2$ to $s(\cdot,\cdot)$.

Hint. For $n < m$,

$$s_n(x,y) = s_m(x,y) + E_x[G_{D_m}(X_{T_n},y) : T_n < T].$$

Note that $G_{D_m} \leq G_D$. Now use Exercises 1 and 2 above.

Remark. This provides another prrof that $s(\cdot,\cdot)$ is separately superharmonic and lower semi-continuous in both variables.

§4. Examples

Example 1. $\mathbb{R}^1$ does not have a Green function but any open set $D \subset \mathbb{R}^1$, whose complement is not empty, has a Green function. In fact $u(x) = |x|$ is positive and harmonic in the open set $\mathbb{R}^1 \setminus \{0\}$.

The Green function for a finite open interval (a,b) is

(1) $$G(x,y) = -|x-y| + \frac{(a+b)(x+y) - 2(xy+ab)}{b-a}$$

and of a half-line (a,∞) is

(2) $$G(x,y) = -|x-y| - 2a + (x+y).$$

Indeed (by looking at the harmonic function with boundary value $= 1$ at a, and $= 0$ at b) it is seen that $P_x(X_T = a) = \frac{b-x}{b-a}$. Thus the Green function of (a,b) is

$$G(x,y) = -|x-y| + E_x[|X_T - y|]$$
$$= -|x-y| + (y-a)P_x(X_T = a) + (b-y)P_x(X_T = b)$$

which is the expression given in (1). Letting $b \to \infty$ in (1), one gets the Green function for (a,∞).

(2) resembles (1), §3, with $a_D(x) = x-a$ which is positive harmonic in (a,∞). It follows that formula (5), §1, holds for an

unbounded open set in $\mathbb{R}^1$ iff it has no unbounded components, i.e. iff it is the disjoint union of bounded open intervals.

Example 2. The Green function of the ball $D = B(0,r)$ in $\mathbb{R}^d$ with centre 0 and radius r is

(3) $$G_D(x,y) = \log\left|\frac{r^2 - x\bar{y}}{r(x-y)}\right|, \qquad |x|,|y| < r, \quad d = 2,$$
$$G_D(x,y) = \frac{1}{|x-y|^{d-2}} - \frac{r^{d-2}}{|y|^{d-2}}\frac{1}{|x-y^*|^{d-2}}, \qquad d \geq 3$$

where $\bar{y}$ = the conjugate of y and y^* the inverse of y relative to $\partial B(0,r)$: $y^* = \frac{r^2}{|y|^2}y$.

To show (3) we evaluate $E_x[K(X_T - y)]$, T = exit time from D. For $z \in \partial B(0,r)$

$$\frac{|z-y^*|}{|z-y|} = \frac{r}{|y|}.$$

So $K(X_T - y) = \log\frac{r}{|y|} + K(X_T - y^*)$ if $d = 2$ and $K(X_T - y) = \frac{r^{d-2}}{|y|^{d-2}}K(X_T - y^*)$ if $d \geq 3$. $K(\cdot - y^*)$ is continuous on $\bar{D}$ and harmonic in D giving

$$E_x[K(X_T - y)] = \log\frac{r}{|y|} + K(x-y^*), \qquad d = 2,$$

$$E_x[K(X_T - y)] = \frac{r^{d-2}}{|y|^{d-2}}K(x-y^*), \qquad d \leq 3.$$

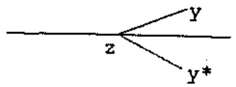
Remark. The method in Example 2 is sometimes referred to as the "method of images" due to Lord Kelvin. In this method one tries to guess, ujsing the geometry of the domain, at a suitable "distribution of charges" outside the domain whose potential on the boundary equals that of a point charge inside the domain; this means that we try to guess at a function which is harmonic in D and whose values are equal to $K(x-.)$ on ∂D. Its applicability is limited; nontheless it is very ingeneous. When it is applicable, it allows us to find the Green function almost without computation. Let us give a few examples to make the idea a little clearer.

Example 3. The Green function of the half space $D = (x: x = (x_1,\dots,x_d), x_d > 0)$ is

$$G_D(x,y) = K(x-y) - K(x-y^*), \quad (4)$$

where y^* is the image of y in the plane ∂D: If $y = (y_1,\dots,y_d)$, $y^* = (y_1,\dots,y_{d-1}, -y_d)$.

Clearly, for $z \in \partial D$, $|y-z| = |y^*-z|$ so that $K(X_T - y) = K(X_T - y^*)$, where T = exit time from D. $K(\cdot - y^*)$ is harmonic in D and continuous on $\bar{D}$. If $d \geq 3$, further $K(z-y^*)$ tends to zero as $z \to \infty$. We must have



$$K(x-y^*) = E_x[K(X_T - y)] \quad (5)$$

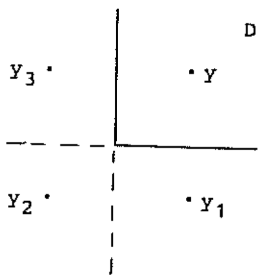
giving (4) for $d \geq 3$. When $d = 2$, $K(\cdot - y^*)$ is not bounded and we cannot at once claim (5). The results in §3, however, guarantee that (4) and (5) obtain for $d = 2$ as well.

Example 4. Let us use the method of images to find the Green function of a quadrant. For simplicity we take the case $d = 2$; the method is almost exactly the same for $d \geq 3$. Let D denote the quadrant

$$D = (z: \operatorname{Re} z > 0, \operatorname{Im} z > 0).$$

We know from §3 that (5), §1) is valid in this case, and the problem is to find $E_x[K(X_T - y)]$.

If y_1 $(= \bar{y})$ is the image of y in $(\operatorname{Im} z = 0)$, y_2 the image of y_1 in $(\operatorname{Re} z = 0)$ and y_3 the image of y_2 in $(\operatorname{Im} z = 0)$, then



$$K(z-y_1) - K(z-y_2) + K(z-y_3) = K(z-y)$$

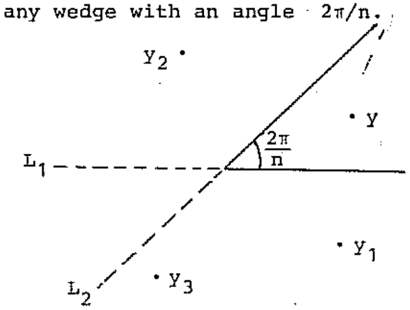
for all $z \in \partial D$ as is easily seen. See Figure. From §3 follows

$$E_x[K(X_T - y)] = K(x-y_1) - K(x-y_2) + K(x-y_3).$$

Since $y_1 = \bar{y}$, $y_2 = -y$, $y_3 = -\bar{y}$, we find that the Green function of D is

$$\log\left|\frac{x^2 - \bar{y}^2}{x^2 - y^2}\right|.$$

The above argument extends to any wedge with an angle $2\pi/n$. For $y \in D$ (see Figure), let y_1 be the image of y in L_1, y_2 the image of y_1 in the line L_2, y_3 the image of y_2 in the L_1, etc. The Green function of D is then



$$K(x-y) - K(x-y_1) + K(x-y_2) - K(x-y_3) + \text{etc.}$$

The sum has only n terms.

Example 5. The method of images can be used to find the Green function of the region between two parallel hyperplanes. One needs an infinite series of reflections and has to watch out when summing the resulting series. The Green function of the strip $D = (0 < \operatorname{Im} z < \pi)$ can be computed to be

$$\sum_{-\infty}^{\infty} \{K(x-y-2ni\pi) - K(x-\bar{y}+2ni\pi)\} = \sum_{-\infty}^{\infty} \log\left|\frac{x-\bar{y}+2ni\pi}{x-y-2ni\pi}\right|. \quad (6)$$

In Courant-Hilbert [2, p. 378] the Green function of a rectangular paralleliped is computed using the image method and is given by an infinite series which has to be grouped correctly for convergence.

Let us give one last example of the image method.

Example 6. Let D be the spherical shell

$$D = (x: r < |x| < R)$$

in $\mathbb{R}^d$, $d \geq 3$. Let $y = y_0 \in D$ and define y_n, $n = 1,2,\ldots$ inductively as follows: y_1 is the image of y_0 in the sphere $(x: |x| = R)$, i.e. $y_1 = y^* = R^2 \frac{y}{|y|^2}$. y_2 is the image of y_1 in the sphere $(x: |x| = r)$, y_3 is the image of y_2 in the sphere $(x: |x| = R)$ etc. Similarly define y_{-n}, $n = 1,2,\ldots$ as follows: y_{-1} is the image of y_0 in $(x: |x| = r)$, y_{-2} the image of y_{-1} in $(x: |x| = R)$ etc. If $a = \frac{R}{r}$, it is seen that

$$y_{2n} = a^{-2n} y, \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$y_{2n+1} = a^{2n} y^*, \qquad y^* = \frac{R^2 y}{|y|^2}, \qquad n = 0, \pm 1, \pm 2, \ldots .$$

And the Green function of D is

$$\sum_{-\infty}^{\infty} a^{-n(d-2)} K(x - y_{2n}) - \frac{R^{d-2}}{|y|^{d-2}} \sum_{-\infty}^{\infty} a^{n(d-2)} K(x - y_{2n+1}).$$

The series above converges uniformly and absolutely in $\bar{D}$. See also the Remark after Example 8.

Example 7. For plane regions conformal maps provide a very powerful method of determining Green functions. The Riemann mapping theorem in theory permits us to write down the Green function of any simply connected domain with at least two boundary points.

The function $w(z) = \frac{1+iz}{z+i}$ maps the upper half plane $(\operatorname{Im} z > 0)$ 1-1 conformally onto the unit disc. By Example 2, the Green function of the upper half plane is

(7) $$\log\left|\frac{x-\bar{y}}{x-y}\right|, \qquad \operatorname{Im} x > 0, \qquad \operatorname{Im} y > 0.$$

The function $w(z) = e^z$ maps the strip $(0 < \operatorname{Im} z < \pi)$ 1-1 conformally onto the upper half plane. From (7) the Green function of the strip $(0 < \operatorname{Im} z < \pi)$ is

(8) $$\log\left|\frac{e^{x-\bar{y}} - 1}{e^{x-y} - 1}\right|.$$

We leave it as an exercise to show that the expressions in (6) and (8) represent the same function; consult Konrad Knopp, Problem Book, vol. 11, Dover (1952), Exercise 11a, p 82, if needed.

Again the function $w(z) = e^z$ maps the strip $(0 < \operatorname{Im} z < 2\pi)$ conformally onto the plane cut along the positive real axis. Using (8), the Green function of the plane slit along the positive real axis is found to be

$$\log\left|\frac{\sqrt{x} - \sqrt{\bar{y}}}{\sqrt{x} - \sqrt{y}}\right| .$$

In the following example we use the method of separation of variables.

Example 8. Let us compute the Green function for the circular ring

$$D = (x: x \in \mathbb{R}^2, \ a < |x| < b).$$

The geometry of D suggests use of polar coordinates. In polar coordinates the Laplacian has the form

(9) $$\frac{\partial^2}{\partial \gamma^2} + \frac{1}{\gamma}\frac{\partial}{\partial \gamma} + \frac{1}{\gamma^2}\frac{\partial^2}{\partial \theta^2} .$$

An attempt at finding a harmonic function of the form $f(\gamma)g(\theta)$ leads us to the equations

$$f'' + \frac{1}{\gamma}f' + \frac{\lambda}{\gamma^2}f = 0$$
$$\frac{d^2g}{d\theta^2} = \lambda g,$$

where λ is a constant. Since g must be periodic with period 2π the possible values of λ (the eigenvalues) are $-n^2$, $n = 1,2,\ldots$ and the corresponding independent solutions are, for $n \neq 0$, $e^{in\theta}$ and $e^{-in\theta}$.

Now, let us attempt to solve the Poisson equation

$$\frac{\partial^2 u}{\partial\gamma^2} + \frac{1}{\gamma}\frac{\partial u}{\partial\gamma} + \frac{1}{\gamma^2}\frac{\partial^2 u}{\partial\theta^2} = -f(\gamma,\theta)$$

in D with boundary condition $u(a,\theta) = u(b,\theta) = 0$. Writing

$$u(\gamma,\theta) = \sum_{-\infty}^{\infty} u_n(\gamma)e^{in\theta}$$

$$f(\gamma,\theta) = \sum f_n(\gamma)e^{in\theta}$$

we arrive at

(10) $$u_n'' + \frac{1}{\gamma}u_n' - \frac{n^2}{\gamma^2}u_n = -f_n.$$

The unique solution of (10) satisfying $u_n(a) = u_n(b) = 0$ is given by

$$u_n(s) = \int_a^b g_n(\gamma,s)\gamma f_n(\gamma)d\gamma$$

where if $n \neq 0$

(11) $$g_n(\gamma,s) = \begin{cases} c_n\left(\frac{b^{2n}}{s^n} - s^n\right)\left(\gamma^n - \frac{a^{2n}}{\gamma^n}\right), & a \leq \gamma \leq s \\ c_n\left(s^n - \frac{a^{2n}}{s^n}\right)\left(\frac{b^{2n}}{\gamma^n} - \gamma^n\right), & s \leq \gamma \leq b \end{cases}$$

with $c_n^{-1} = 2n(b^{2n} - a^{2n})$ and

(12) $$g_0(\gamma,s) = \begin{cases} c_0 \log\frac{b}{s}\log\frac{\gamma}{a}, & a \leq \gamma \leq s \\ c_0 \log\frac{s}{a}\log\frac{b}{\gamma}, & s \leq \gamma \leq b \end{cases}$$

with $c_0^{-1} = \log\frac{b}{a}$.

Using $f_n(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}f(\gamma,\theta)d\theta$, we get an expression for u

(13) $$u(s,\theta) = \frac{1}{2\pi}\sum_{-\infty}^{\infty}\int_a^b d\gamma\int_0^{2\pi} d\varphi\, g_n(\gamma,s)e^{in(\theta-\varphi)}\gamma f(\gamma,\varphi)$$

$\gamma d\gamma d\varphi$ being the area element in polar coordinates, we obtain from (13) a tentative expression for the Green function for D:

(14) $$\sum_{-\infty}^{\infty} g_n(\gamma,s)e^{in(\theta-\varphi)}$$

where $x = (\gamma,\theta)$ and $y = (s,\varphi)$ and g_n are defined in (11) and (12). Since $g_n = g_{-n}$, (14) is in fact

(15) $$g_0(\gamma,s) + 2\sum_1^{\infty} g_n(\gamma,s)\cos n(\theta-\varphi).$$

We have yet to show that (15) in fact represents G_D. It can be verified that $\sum_1^{\infty}|g_n - g_{n+1}|$ converges uniformly in $a \leq \gamma$, $s \leq b$. By Theorem (2,6), p.4 of A.Zygmund [3], (15) represents a continuous function of (x,y) if $\theta \neq \varphi$. On the other hand, if $\gamma \neq s$ (i.e. $|x| \neq |y|$), $\Sigma g_n(\gamma,s)$ is dominated by a geometric series ($g_n(\gamma,s)$ is less or equal to a constant limes $(\frac{\gamma}{s})^n$ for $\gamma \leq s$) and so (15) represents a continuous function of (x,y) if $|x| \neq |y|$. Together these imply that (15) represents a continuous function of (x,y) outside of the diagonal $(x = y)$.

It can be checked that $\int_a^b g_n(\gamma,s)d\gamma = \hat{O}\left(\frac{1}{n^2}\right)$. Using this and denoting the continuous function given in (15) by $F(x,y)$ we see that for a c'-function h of the form $h(\gamma,\theta) = f(\gamma)e^{in\theta}$ the function

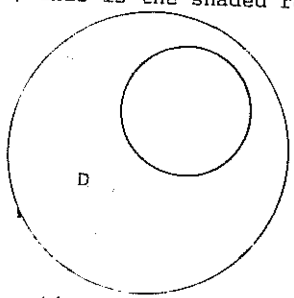
$$u(\gamma,\theta) = \int_D F(x,y)h(y)dy$$

$$= e^{in\theta}\int_a^b g_n(\gamma,s)f(s)sds$$

satisfies $\Delta u = -2\pi h$ and u vanishes on ∂D. We must have $u(x) = \int_D G_D(x,y)h(y)dy$. It follows that $F(x,y) = G_D(x,y)$ for almost all y and by continuity $F \equiv G_D$.

We have shown that (15) represents the Green function for the circular ring $\{x: a < |x| < b\}$.

Remark. Using Example 8 and inversion, one can determine the Green function for the region between two circles, one completely contained in the other; this is the shaded region in the figure below. See Exercise 2, §3, Chapter 4.



The method of separation of variables can be used whenever we have a product domain. For a more detailed account of this method consult any elementary book on partial differential equations; See Courant-Hilbert [2] for a complex function method of determining the Green function of a circular ring.

§5. The Green function and relative transition.

Let $W \subset R^d$ be open and T = exit time from W. The relative transition measure $Q(t,x,A)$ is

(1) $$Q(t,x,A) = P_x[X_T \in A,\ t < T], \quad A \text{ Borel}.$$

It is clear that $Q(t,x,\cdot)$ is absolutely continuous relative to Lebesgue measure. We shall find nice densities for Q and relate these to the Green function of W. $p(t,x,y)$ will denote the Gauss kernel:

$$p(t,x,y) = (2\pi t)^{-d/2}\exp\left(-\frac{|x-y|^2}{2t}\right), \quad t > 0.$$

We start with

Proposition 1. For all x and $t > 0$,

(2) $$P_x(T = t) = 0.$$

Proof. Suppose (2) is false. On the set $T > s$, $T = s+T(\theta_s)$. By Markov property we obtain for each $0 < s < t$, $E_x[P_{X_s}(T=t-s): T>s] > 0$. The measures $P_x(X_s \in \cdot)$ and $P_x(X_1 \in \cdot)$ are clearly equivalent so that $E_x[P_{X_1}(T=t-s)] > 0$, for $s < t$. This is impossible because $E_x[P_{X_1}(T \in \cdot)]$ is a finite measure. Q. E. D.

By the first time relation ((8), §2, Chapter 2)

(3) $$P_x(X_t \in A) = Q(t,x,A) + \int_{[0,t]\times\partial W} P_b(X_{t-s} \in A)\, P_x(T \in ds,\ X_T \in db).$$

If $q(t,x,y)$ denotes any density of $Q(t,x,\cdot)$, from (3) for almost all y,

(4) $$p(t,x,y) = q(t,x,y) + E_x[p(t-T,X_T,y): T < t].$$

In the last expectation, using (2) we have replaced $(T \leq t)$ by $(T < t)$. Now $p(\cdot,\cdot,\cdot)$ is lower semi-continuous, being the increasing limit of continuous functions

$$(2\pi t)^{-d/2} \exp\left(- \frac{|x-y|^2 + \varepsilon}{2t} \right)$$

Therefore the last term in (4) is lower semi-continuous by Fatou. q defined by (4) is therefore upper semi-continuous in $((t,x,y): t > 0)$. For fixed t,x it is non-negative almost everywhere (being a density); hence it is non-negative everywhere. We define q by (4) for all $t > 0$, $x,y \in R^d$.

Now p is C^∞ in $(0,\infty) \times R^d \times R^d$. Since $X_T \in \partial W$ for each x, we find from (4)

$q(t,x,y)$ is C^∞ in (t,y) if $y \notin \partial W$.

Using the semigroup property of $p(t,\cdot,\cdot)$ we see from (4)

$$p(t,x,y) = \int q(t-\varepsilon,x,z)p(\varepsilon,z,y)dz + E_x[p(t-T,X_T,y): T < t-\varepsilon].$$

Comparing the above with (4),

(5) $$\int q(t-\varepsilon,x,z)p(\varepsilon,z,y)dz = q(t,x,y) + E_x[p(t-T,X_T,y): t-\varepsilon \leq T < t].$$

In particular

(6) $$\lim_{\varepsilon \to 0} \int q(t-\varepsilon,x,z)p(\varepsilon,z,y)dz = q(t,x,y), \qquad t > 0, \ x,y \in R^d.$$

By Markov property $Q(t,x,\cdot)$ is a semigroup of measures. In terms of densities this means: for each x and $t,s > 0$ for almost all y

(7) $$q(t+s,x,y) = \int q(t,x,z)q(s,z,y)dz.$$

But because of (6), it is immediately seen that (7) holds for all y.

(7) can be rewritten

$$q(t+s,x,y) = E_x[q(s,X_t,y): t < T]$$

which shows that $q(\theta,x,y)$ is continuous in x for $x \in W$ $(q(u,\cdot,\cdot) \leq p(u,\cdot,\cdot) \leq (2\pi u)^{-d/2}$. Now refer to Exercise 3, §2, Chapter 4). Using this and Proposition 1 we see further (since $X_t \in W$ for $t < T$) that

(8) for all x,y $q(\cdot x,y)$ is continuous in $(0,\infty)$.

Now we shall relate these relative transition densities q to the Green function of W as follows. We assume from now on that $d = 2$. As the reader will see, the case $d \geq 3$ is similar and simpler.

Let $0 \leq f$ be C^1 on R^2 with compact support, $D_n = W \cap (0 \leq |x| < n)$ and T_n = exit time from D_n. If $u = \int K(\cdot-y)f(y)dy$, then u is C^2 on R^2 and $u = -2\pi f$ by Theorem 3, §3, Chapter 5. By Dynkin's formula, Theorem 2, §1, Chapter 4,

(9) $$u(x) = E_x\left[u(X_{T_n})\right] + \pi E_x\left[\int_0^{T_n} f(X_t)dt\right].$$

Suppose now that W has a Green function. Let n tend to infinity in (9). With the notation of §2,

(10) $$\int K(x,y)f(y)dy = \pi E_x\left[\int_0^T f(X_t)dt\right] + \int s(x,y)f(y)dy$$

because $s(x,\cdot)$ being superharmonic on R^2 is bounded below on compact sets. Deduction of (10) from (9) is thus justified. (10) is equivalent to

(11) $$K(x-y) = \pi\int_0^\infty q(t,x,y)dt + s(x,y)$$

for almost all y. To show that (11) holds for all y, write for

$\varepsilon > 0$, $\int_0^\infty q(t,x,y)dt$ in the form $\int_\varepsilon^\infty q(t-\varepsilon,x,y)dt$ and use (5):

(12) $$\int K(x-z)p(\varepsilon,z,y)dz = \pi\int_\varepsilon^\infty q(t,x,y)dt + \int_0^\varepsilon E_x[p(s,X_T,y)]ds + \int s(x,z)p(\varepsilon,z,y)dz$$

<u>for all</u> y. To see what happens as ε tends to zero, we need

<u>Proposition 2</u>. For a constant A

(13) $$\int_0^1 p(s,z,y)ds \leqq AK(z-y) \quad \text{if} \quad |z-y| \leqq \tfrac{1}{2}.$$

If m is a finite measure on R^2 for which

(14) $$u(y) = \int K(y-z)m(dz)$$

is superharmonic, then

(15) $$\lim_{\varepsilon\to 0}\int u(z)p(\varepsilon,z,y)dz = u(y).$$

We postpone the proof and go ahead with our discussion. Since $p(s,z,y)$ is bounded on the set $(s>0,\ |y-z| \geqq \tfrac{1}{2})$, we see from (13) that $\int_0^1 E_x[p(s,X_T,y)]\,ds < \infty$ provided $E_x[|K(X_T-y)|] < \infty$ that is provided $E_x[K(X_T-y)] < \infty$ because by Proposition 1, §2, $E_x[K(X_T-\cdot)]$ is superharmonic and hence by Exercise 8, §1, Chapter 5, $E_x[K^-(X_T-y)] < \infty$ for all y. Thus if $s(x,y) < \infty$, the middle term on the right side of (12) tends to zero as ε tends to zero. Now use Proposition 2 together with Proposition 1, §2, to show that (11) is valid <u>for all</u> y such that $s(x,y) < \infty$. If $\infty = s(x,y)$ $(\leqq K(x-y))$, (11) is trivial. Since $s(x.y) = \infty$ only if $x = y$, we see from (11) that V defined by

(16) $$V(x,y) = \int_0^\infty q(t,x,y)dt, \qquad x,y \in R^2,$$

is symmetric in (x,y). As is clear from (11), for $x,y \in W$, V is its Green function.

<u>Proof of Proposition 2</u>. To prove (13) we must show $(|z-y|^2 = r)$

$$A\log\frac{1}{r} - \int_0^1 \frac{1}{\pi t}\exp\left(-\frac{r}{2t}\right)dt \geqq 0, \qquad r \leqq \tfrac{1}{2}$$

or calling $\frac{1}{r} = s$ and changing variables, it is needed to show

(17) $$A\log s - \int_0^s \frac{1}{\pi t}\exp\left(-\frac{1}{2t}\right)dt \geqq 0, \qquad s \geqq 2.$$

Differentiation of the above expression relative to s leads to

$$A/s - (1/\pi s)\exp(-1/2s)$$

which is certainly non-negative provided $A - 1/\pi \geqq 0$. So the left side of (17) is increasing in s. If we choose $A \geqq 1$ so that the left side of (17) is non-negative at $s = 2$, it will remain so for all $s \geqq 2$. For such A then (13) is established.

To prove (15), note first that

(18) $$\int_{B(y,1)} u(z)p(\varepsilon,z,y)dz \quad \text{tends to} \quad u(y)$$

as ε tends to zero where $B(y,1)$ is, as usual, the ball of radius 1 and centre y. This is so because u being locally integrable, the existence of the integral (18) is clear. Using superharmonicity of u and integrating relative to polar co-ordinates shows that the integral in (18) cannot exceed $u(y)$ because the integral of $p(\varepsilon,z,y)$ over $B(y,1)$ is at most one. On the other hand, lower semi-continuity of u and the fact that the integral of $p(\varepsilon,z,y)$ over $B(y,1)$ tends to 1 as ε tends to zero show that (18) is valid. Now

(19) $$\int_{|y-z|\geqq 1} p(\varepsilon,z,y)|u(z)|dz \leqq \int m(dx)\int_{|y-z|\geqq 1} |K(x-z)|p(\varepsilon,z,y)dz.$$

If $|z-y| \geq 1$, then $p(\varepsilon,z,y) \leq 2p(1,z,y)$ for all $\varepsilon > 0$. So the inner integral on the right side of (19) is dominated by

(20) $$2\int |K(a-z)|\,p(1,z,0)\,dz = I_1 + I_2 \quad \text{say}, \qquad a = x-y.$$

where I_1 is the integral over $B(a,1)$ and I_2 over its complement. Clearly

(21) $$I_1 \leq \int_{|b|\leq 1} |K(b)|\,db = O(1).$$

The estimate (recall $a = x-y$) $\log|a-z| \leq \log(1+|a|) + \log(1+|z|)$ gives

(22) $$I_2 \leq O(1) + \log(1+|x-y|).$$

Now we can show that the right side of (19) tends to zero as ε tends to zero. Indeed, the inner integral on the right side of (19) tends to zero for every x (y fixed) because the integrand does so for all $z \neq x$ and is bounded by the (obviously integrable) integrand on the left side of (20). Forther, the inner integral on the right side (19) is, as a function of x, bounded by $I_1 + I_2$ which is m-integrable as seen by the estimates (21) and (22). Recall that (2), §3 must be valid if u is to be superharmonic.

Q. E. D.

Symmetry of q

In Chapter 7 we shall be using that q is symmetric in x,y. We shall now prove this fact. Put for $\alpha \geq 0$

(23) $$R_\alpha(x,y) = \int_0^\infty e^{-\alpha t} q(t,x,y)\,dt$$

so that R_0 is symmetric. By the semigroup property of q,

(24) $$R_0 = R_\alpha + \alpha R_\alpha R_0 = R_\alpha + \alpha R_0 R_\alpha,$$

where $R_\alpha R_0(x,y) = \int R_\alpha(x,z)R_0(z,y)\,dz$ and similar meaning is given to R_0R_α. The resolvent equation (24) reveals a nice uni-
city property of R_0: If f and g are non-negative measurable and $\alpha > 0$,

(25) $$f + \alpha R_0 f = g + \alpha R_0 g$$

implies $R_0 f = R_0 g$ and hence $f = g$ at every point at which $R_0 f < \infty$. To see this, just operate by R_α both sides of (25) and use (24).

Now it is easy to prove the symmetry of q. Put $\hat{q}(t,x,y) = q(t,y,x)$ and define $\hat{R}_\alpha$ as in (23) replacing q by $\hat{q}$. The validity of (7) for all x,y shows that $\hat{q}$ also has the semi-group property. Using (24) and symmetry of R_0, i.e. that $R_0 = \hat{R}_0$,

$$R_\alpha + \alpha R_0 R_\alpha = R_0 = \hat{R}_0 = \hat{R}_\alpha + \alpha \hat{R}_0 \hat{R}_\alpha.$$

which by the unicity shown above leads to

(26) $$R_\alpha(x,y) = \hat{R}_\alpha(x,y) = R_\alpha(y,x)$$

at every point at which $R_0(x,y) < \infty$, i.e. for every x,y with $x \neq y$. For $x = y$ (26) being trivial, an appeal to the uniqueness of Laplace transform and a glance at (8) gives us the symmetry of q:

(27) $$q(t,x,y) = q(t,y,x), \qquad t > 0, \qquad x,y \in R^d.$$

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CHAPTER 7

Introduction

In this chapter W will be a Greenian open set in $\mathbb{R}^d$, with Green function G.

In §1 we shall state and prove some of the main principles of potential theory. §2 is devoted to the celebrated capacity theorem some of whose applications are found in §3. §4 deals with the balayage procedure. In §5 we give the rudiments of Dirichlet spaces. Constraints of resources like time and space have forced us to drop subjets like Additive functionals, Martin Boundary, Fine topology etc.

§1. Some Potential Theoretic Principles

Let s be excessive in a Green domain W, with Green function G.

(1) $$h(x) = \lim_{D \uparrow W} E_x[s(X_T)],$$

D relatively compact open in W

T = exit time from D.

is locally integrable and satisfies the mean value property and