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CHAPTER 5

Superharmonic functions

Introduction

Superharmonic functions can be regarded as generalizations of concave functions and bear the same relationship to harmonic functions as concave functions do to linear functions.

The theory of superharmonic functions was started by F. Riesz. For a little history we refer the reader to Rado [5] and the references cited here. In Rado [5] there is also a discussion of a wide range of applications of superharmonic functions. We shall only give some applications. In any case the reader will see in the sequel that the study of superharmonic functions more than justifies itself by its applications to the theory of harmonic functions alone.

In §1 we define and investigate some general properties of superharmonic functions. §2 is devoted to applications and §3 deals with Riesz-measures associated with superharmonic functions. §4 is concerned with continuity properties of superharmonic functions and finally in §5 we show the uniqueness of solution of the modified Direchlet problem.

§1. Superharmonic functions

$B(a,r)$ will denote a ball with center a and radius r and $S(a,r) = \partial B(a,r)$ its topological boundary.

Let G be an open set in R^d. A function $f: G \to (-\infty,\infty]$ is called superharmonic on G if

(S.1) f is lower semi-continuous on G

(S.2) $f(a) < \infty$ at a dense set of points in G

(S.3) For every a in G and every ball $B(a,r) \subset G$, $E_a[f(X_T)] \leq f(a)$ where T is the exit time from $B(a,r)$.

f is called sub-harmonic if $-f$ is superharmonic.

If only (S.1) and (S.3) hold, we say that f is hyperharmonic. If G is a domain, the only function that is hyperharmonic and not superharmonic is the function identically equal to ∞. See Exercise 1 and the remark thereafter. It is easy to see that if f is hyperharmonic, so is $f \wedge n$ for all real n. Thus every hyperharmonic function is an increasing limit of a sequence of superharmonic functions that are bounded above.

Condition (S.3) can be rewritten as $\int_{S(a,r)} f(b)\,d\sigma(b) \leq f(a)$ where σ is the uniform distribution on $S(a,r)$. Integration using polar coordinates then implies

$$\text{(1)} \qquad \frac{1}{P_r}\int_{B(a,r)} f(b)\,db \leq f(a), \qquad P_r = \text{volume of } B(a,r).$$

By lower semi-continuity of f and (1)

$$\text{(2)} \qquad \lim_{r\to 0} \frac{1}{P_r}\int_{B(a,r)} f(b)\,db = f(a).$$

Indeed from (1) $\limsup_{r\to 0}\int_{B(a,r)} f(b)\,db \leq f(a)$; given a number $c < f(a)$, by lower semi-continuity we can find r so that

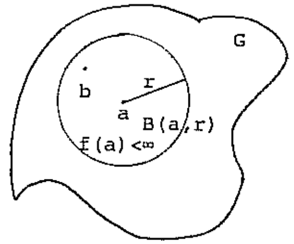
$\inf_{b\in B(a,r)} f(b) > c$ and this at once gives the above claim. Thus two superharmonic functions which are equal almost everywhere are identical.

Superharmonicity for smooth functions is explained by:

A C^2-function in an open set G is superharmonic iff $\Delta f \leq 0$ in G.

This fact is a very simple consequence of Theorem 2, §1, Chapter 4, and we leave it as an exercise.

A function f which is superharmonic in an open set G is locally integrable: As is clear from the figure, every point $b \in G$ has a neighbourhood in which f is integrable. Note that f being lower semi-continuous, it is bounded below on compact subsets of G and therefore f is integrable in $B(a,r)$ if $f(a) < \infty$.



A superharmonic function can be quite bad as we shall see later. However, it is a very useful fact that it can be approximated by smooth superharmonic functions:

Let f be superharmonic in an open set G and D a relatively compact open subset of G. Let $0 \leq \varphi$ be radial (i.e. $\varphi(x) = \varphi(|x|)$) and C^∞ with support in $B(0,1)$ such that $\int\varphi = 1$. Then $\varphi_n = n^d\varphi(nx)$ has support in $B(0,\frac{1}{n})$ and $\int\varphi_n = 1$. For all large n, $B(a,\frac{1}{n}) \subset G$ for all $a \in \overline{D}$ and by Fubini

$$f_n = f * \varphi_n \qquad (\text{define } f = 0 \text{ off } G)$$

is superharmonic in D for all large n. Using polar coordinates and superharmonicity of f,

$$f_n \leqq f \qquad \text{in } \bar{D} \text{ all large } n.$$

And (as in the proof of (2)) by lower semi-continuity of f,

$$\lim f_n = f \qquad \text{in } \bar{D}.$$

Thus in each relatively compact open subset D of G, f is a limit of C^∞-functions which are superharmonic in D.

Let us retain the above notation. For large n, f_n are superharmonic in D and C^∞ in R^d. Hence $\Delta f_n \leqq 0$ in D. Let T be the exit time from D and $S_1 \leqq S_2$ any stopping times. Proposition 1, §1, Chapter 4, is applicable to $T \wedge S_i$, $i = 1,2$. Since $\Delta f_n \leqq 0$,

$$E_a[f_n(X_{T\wedge S_1})] - f_n(a) \geqq E_a[f_n(X_{T\wedge S_2})] - f_n(a)$$

Fatou's lemma is applicable since f and hence f_n are uniformly bounded below on $\bar{D}$. Letting n tend to infinity,

$$E_a[f(X_{T\wedge S_2})] \leqq E_a[f(X_{T\wedge S_1})], \qquad a \in \bar{D}.$$

If also $f \geqq 0$, from the last inequality (note that since $S_1 \leqq S_2$ the event $(S_2 \geqq T) \supset (S_1 \geqq T)$)

$$E_a[f(X_{S_2}): S_2 < T] \leqq E_a[f(X_{S_1}): S_1 < T], \qquad a \in \bar{D},$$

which as D increases to G yields

$$E_a[f(X_{S_2}): S_2 < R] \leqq E_a[f(X_{S_1}): S_1 < R], \qquad a \in G,$$

where R = exit time from G.

Let us collect the above in

Theorem 1. Let f be superharmonic in G. Then for every relatively compact open subset of G:

(3) $$E_a[f(X_T)] \leqq f(a), \qquad T = \text{exit time from } D.$$

If further $f \geqq 0$, then for any stopping times $S_1 \leqq S_2$

(4) $$E_a[f(X_{S_2}): S_2 < R] \leqq E_a[f(X_{S_1}): S_1 < R] \leqq f(a), \quad a \in G.$$

where R = exit time from G.

The following proposition gives us an alternative and useful definition of superharmonic functions.

Proposition 2. Let f be lower semi-continuous in an open set G. Then f is superharmonic in G iff it has the following property:

> For every relatively compact open $D \subset G$ and every function u which is continuous on $\bar{D}$ and harmonic in D, $u \leqq f$ on ∂D implies $u \leqq f$ on D.

Proof. That a superharmonic function satisfies the property follows from the first part of Theorem 1 above and (5), §1, Chapter 4. Conversely, suppose the lower semi-continuous function f satisfies the above condition and let $B(a,r) \subset G$. If g is continuous on $S(a,r)$, its Poisson integral (see Example 1, §2, Chapter 4) is continuous on $B(a,r)$, harmonic on its interior and $\leqq f$ on $S(a,r)$. By our hypothesis

$$E_a[g(X_T)] \leqq f(a), \qquad T = \text{exit time from } B(a,r).$$

Letting g increase to f, we see that f is superharmonic.

The above proposition easily implies that <u>superharmonicity is a local property</u>:

<u>Proposition 3</u>. Let $f: G \to (-\infty,\infty]$ be lower semi-continuous. Suppose for each $a \in G$ there exists $r(a)$ such that

$$E_a[f(X_{T_r})] \leqq f(a)$$

for $r \leqq r(a)$, where T_r = exit time from $B(a,r)$. Then f is superharmonic in G.

Indeed, let u be harmonic in a relatively compact open subset D of G. Suppose u is continuous on $\bar{D}$ and $u \leqq f$ on ∂D. $f-u$ is lower semi-continuous on $\bar{D}$. Let α be the minimum of $f-u$ on $\bar{D}$. Now (1) is valid for f and each a in G provided $r \leqq r(a)$. It follows that if $f(a_0)-u(a_0) = \alpha$, for a point a_0 in D, the same holds almost everywhere, and hence everywhere in a neighbourhood of a_0, because the set $(f-u=\alpha)$ is closed. Thus if $f-u=\alpha$ at any interior point a_0 in D, then the same holds in the component of D containing a_0. Since $f-u \geqq 0$ on ∂D, we must have $\alpha \geqq 0$. This proves the claim.

<u>Excessive functions</u>

This notion was introduced by G. Hunt. A function $f: G \to [0,\infty]$ is called excessive if

$$E_\cdot[f(X_t): t<R] \leqq f(\cdot), \qquad R = \text{exit time from } G$$

and tends to f as t tends to zero.

Markov property easily implies that $E_\cdot[f(X_t): t<R]$ increases as t decreases. Now for $t>0$, $E_\cdot[f(X_t): t<R]$ is lower semi-

continuous being the increasing limit of continuous functions $E_\cdot[f(X_t)\wedge n: t<R]$ (Exercise 3, §2, Chapter 4). Since $E_\cdot[f(X_t): t<R]$ increases to f as t tends to zero:

<u>Every excessive function is lower semi-continuous</u>.

By ordinary Markov property $f(X_t)1_{t<R}$ is a super martingale relative to P_a provided $f(a)<\infty$. Therefore for discrete stopping times T,

(5) $$E_a[f(X_T): T<R] \leqq f(a).$$

Since any stopping time is a limit of discrete ones, lower semi-continuity of f ensures that (5) is valid for any stopping time. Taking T = exit time from $B(a,r) \subset G$, we get the defining property of super harmonic functions. Note that the case $f(a)=\infty$ is trivial.

If $0 \leqq f$ is superharmonic, the second part of Theorem 1 (with $S=t$) and lower semi-continuity of f shows that f is excessive. Thus

$0 \leqq f$ <u>is hyperharmonic iff</u> f <u>is excessive</u>.

<u>Proposition 4</u>. Let f be superharmonic in an open set G. Let D be a relatively compact open subset of G. Then

$$u(\cdot) = E_\cdot[f(X_T)], \quad T = \text{exit time from } D$$

is superharmonic in G and harmonic in D.

<u>Proof</u>. Since $u=f$ off $\bar{D}$, by Proposition 3 it is sufficient to show that f is superharmonic in a neighbourhood of $\bar{D}$. f is bounded below in a neighbourhood of $\bar{D}$. Replacing G by

a neighbourhood of $\bar{D}$ if necessary, we may assume that $f \geqq 0$ in G and hence excessive. The continuous bounded excessive functions

$$f_n(\cdot) = E_\cdot[(f\wedge n)(X_{\frac{1}{n}}) : \tfrac{1}{n} < R], \quad R = \text{exit time from } G.$$

increase to f. So by monotone convergence we may further suppose that f is bounded, continuous and excessive in G. Then $f(X_t)1_{t<R}$ is a continuous bounded super martingale relative to $P_\cdot$. Since $t+T(\Theta_t)$ decreases to T as t rends to zero, by Markov property and optional sampling Theorem for super martingales

$$E_a[u(X_t): t<R] = E_a[f(X_{t+T(\Theta_t)}): t+T(\Theta_t) < R]$$

$$\leqq E_a[f(X_T)] = u(a).$$

That the left side of the above converges to $u(a)$ as t tends to zero follows from bounded convergence and the continuity of f. Thus u is excessive and hence superharmonic. That it is harmonic in D is clear.

Remark. In Proposition 4, suppose f were harmonic in D and superharmonic in G. Can we conclude that $u = f$? We shall see that the answer to this is intimately connected with the problem of uniqueness of solution of the modified Dirichlet problem (§2, Chapter 4).

Examples

1. If u is harmonic in an open set, $|u|^p$ is subharmonic for all $p \geq 1$ as is easily seen using Holder's inequality. If u is subharmonic, $\exp(u)$ is subharmonic. To see this, if u is also C^∞ then $\Delta(e^u) = (\Delta u + |\text{grad}\, u|^2)e^u \geqq 0$. In the general case u is the decreasing limit (on any compact subset) of a sequence of C^∞ subharmonic functions. More generally, use of Jensen's inequality (Rudin [6], p.61) shows that $A(u)$ is subharmonic if u is subharmonic and A is convex and increasing on the line. If u is harmonic, we need only assume that A is convex.

2. If f is holomorphic in an open set of the plane, $\log|f|$ is harmonic in the open set $(f \neq 0)$, as is seen by Cauchy-Riemann equations. By propositions 3, $\log|f|$ is subharmonic. It follows that if f and g are analytic, $p,q > 0$, $p\log|f| + q\log|g| = \log|f|^p|g|^q$ is subharmonic. From Example 1 above, $|f|^p|g|^q$ is subharmonic. This is a very useful fact in complex function theory.

Taking $f(x) = x$, we see that $-\log|x|$ is superharmonic on R^2 and harmonic off the origin. If $d \geq 3$, the function $|x|^{-d+2}$ is superharmonic off the origin (just differentiate). From Proposition 3 $|x|^{-d+2}$ is superharmonic on R^d, $d \geq 3$. These are the socalled fundamental harmonic functions. A function which is harmonic and radial in a neighbourhood of the origin is of the form $aK(x)+b$, $a>0$ and b are constants and where

$$K(x) = -\log|x| \text{ if } d=2, \qquad K(x) = |x|^{-d+2} \text{ if } d \geq 3.$$

3. There are more subharmonic functions than is evident at first sight. Every polynomial on R^d is the difference of two subharmonic polynomials. This need only be shown for homogeneous polynomials. According to Exercise 3, §5, Chapter 4, every homo-

geneous polynomial degree n is a linear combination of polynomials of the form $(\sum_1^d a_i x_i)^n$ where a_i are reals. If n is even, $(\Sigma a_i x_i)^n$ is subharmonic since $\Sigma a_i x_i$ is harmonic. Write $(\Sigma a_i x_i)^{2n+1} = \Sigma a_i x_i (\Sigma a_j x_j)^{2n}$ and note that each of the summands is a difference of two subharmonic polynomials as seen by the formula $2uv = (u+v)^2 - u^2 - v^2$.

4. We have seen that excessive functions are the same as non-negative hyperharmonic functions. If $d \geq 3$, $K(x) = |x|^{-d+2}$ is excessive on R^d and hence by restriction every subdomain of R^d, $d \geq 3$ has non-constant excessive functions. However, there are no non-constant excessive functions on R^2:

Suppose u is excessive on R^2. Let $a, b \in R^2$, and S = hitting time to $B(b,r)$. Because the 2-dimensional Brownian motion is recurrent (Exercise 3, §1, Chapter 4), $P_a(S < \infty) = 1$. Using (4)

$$u(a) \geq E_a[u(X_S)].$$

As r tends to zero, by lower semi-continuity of u, $u(a) \geq u(b)$, i.e. u is constant.

An open subset $G \subset R^2$ is called Greenian if there is a non-constant excessive function defined on it. Question: Which open subsets of R^2 are Greenian? Kakutani has the answer.

__Theorem 5__. Let G be a domain in $\mathbb{R}^2$ and T = exit time from G. Then $P_a[T < \infty] \equiv 1$ or $\equiv 0$ in $\mathbb{R}^2$.

G is Greenian iff $P_a[T < \infty] \equiv 1$.

__Proof__. Put $u(a) = P_a[T < \infty]$. u is easily seen to be excessive on $\mathbb{R}^2$; so it is a constant. Clearly $P_a[T < \infty, T > n]$ tends to zero as $n \to \infty$ and

$$P_a[T < \infty,\ T > n] = E_a[P_{X_n}(T < \infty) : T > n]$$

$$= cP_a[T > n] = c(1 - P_a[T \leq n])$$

where $c \equiv u$ is a constant. As $n \to \infty$, we get $c(1-c) = 0$, i.e. $c = 1$ or 0. If $P_a(T < \infty) \equiv 1$, the function $u(a) = P_a(T > t)$ is excessive and non-constant on G for each $t > 0$. Indeed, if D_n increase to G with $\bar{D}_n \subset G$ and T_n = exit time from D_n, then $T_n \uparrow T$ and since $T < \infty, T(\theta_{T_n}) \downarrow 0$. It follows that $P_a[T(\theta_{T_n}) > t]$ tends to zero. If $P_a(T > t)$ were a constant, this constant must be zero which is absurd (why?). On the other hand, let u be a positive superharmonic function on G. Assume $P_a(T < \infty) \equiv 0$, i.e. the Brownian path starting at any point $a \in \mathbb{R}^2$ immediately enters G and then never exits from G. For any $t > 0$, $X_t \in G$, and we may easily verify that $E_a[u(X_t)]$ is superharmonic and non-negative on $\mathbb{R}^2$ so it must be a constant. As $t \to 0$, for each $a \in G$, $E_a(u(X_t)) \to u(a)$. So u is constant.

Exercises to §1

1. Let f be hyperharmonic in an open set G. The set of points $a \in G$ such that there exists a neighbourhood $U \ni a$ with $\int_U |f| < \infty$ is open and closed in G.

Remark. If G is connected, this implies that $f(a) < \infty$ for one $a \in G$. Then f is locally integrable in G.

2. An increasing limit of hyperharmonic functions is hyperharmonic.

3 (The minimum principle). Let $0 \leq f$ be superharmonic in a domain G. Then either $f > 0$ or $f \equiv 0$.

Hint. The set $(f = 0)$ is closed. It is also open because by (1), $f(a) = 0$ implies f is almost everywhere zero in a neighbourhood of a.

4 (Boundary minimum principle). Let f be superharmonic in an open set G. Suppose f is bounded below in G and

$$\liminf_{G \ni y \to x} f(y) \geq 0, \qquad x \in \partial G.$$

(If G is unbounded, the point at infinity is considered an element of ∂G.) Then $f \geq 0$ in G.

Hint. In the first part of Theorem 1, let D increase to G and use Fatou.

5. Show by an example that the lim inf condition in Exercise 4 above cannot be replaced by a lim sup condition.

Hint. Let (a_n) be dense in $S(0,1)$, T = exit time from $B(0,1)$. Then $P_0[X_T \in (a_n)] = 0$. Let U be open $\supset (a_n)$ such

that $P_0[X_T \in U] < \frac{1}{3}$ say. Let $f = 1$ on U and -1 elsewhere on $S(0,1)$. $u(\cdot) = E_\cdot[f(X_T)]$ is harmonic in $B(0,1)$ and tends to $f(b)$ for all $b \in U$ because f is continuous on U (see remark after the proof of Theorem 1, §2, Chapter 4). $u(0) < 0$ and $\limsup u = 1$ on $S(0,1)$.

6. Let G be open and R = exit time from G. For every non-negative Borel function f and every open or closed subset A, the functions

$$E_\cdot\left[\int_0^R f(X_t)\,dt\right] \quad \text{and} \quad P_\cdot[T < R]$$

are excessive in G. T = hitting time to A.

7. Let m be any measure on R^d, $d \geq 3$. Then $\int |\cdot - y|^{-d+2} m(dy)$ is hyperharmonic on R^d.

Hint. Fatou and Fubini.

8. Let m be any Radon measure on R^2. Then $v(\cdot) = \int |\log|\cdot - y|| \, m(dy)$ is locally integrable unless $\equiv \infty$. In the former case $\int \log^+|x-y| m(dy) < \infty$ for all x and $u(\cdot) = \int \log|\cdot - y| \, m(dy)$ is subharmonic on R^2.

Hint. If $v < \infty$ at one point, m is necessarily a finite measure. Using $\log^+|x+y| < \log 2 + \log^+|x| + \log^+|y|$ show that $\int \log^+|x-y| m(dy)$ is (finite and) subharmonic. Since $\log|x-y| \leq \log^+|x-y|$ and the latter is m-integrable, Fubini and Fatou can be used to show that u is subharmonic.

9. Let s be superharmonic in a neighbourhood of $\overline{D}$, D bounded open. If $a \in \partial D$ is irregular,

$$s(a) = \liminf_{D \ni b \to a} s(b).$$

Hint. s is bounded below in a neighbourhood of $\overline{D}$ so assume s is excessive. The Brownian path starting at a must remain in D for a positive length of time. Now use excessivity and lower semi-continuity of s.

10. Let D be bounded open, T = exit time from D and $a \in \partial D$ irregular. Then there exists a sequence $a_n \in D$, $a_n \to a$, such that the harmonic measures $P_{a_n}(X_T \in dz)$ tend weakly to the harmonic measure $P_a(X_T \in dz)$.

Hint. Let B be a ball containing $\overline{D}$. The minimum of two continuous and excessive functions in B is continuous and excessive in B. The differences of functions which are continuous and excessive in B is thus a lattice and is therefore uniformly dense in $C(\overline{D})$. It is thus enough to show that there exists a sequence a_n in D converging to a such that $E_{a_n}[s(X_T)]$ tends to $E_a[s(X_T)]$ for every s which is continuous and excessive in B. For this let $\{s_n\}$ be a dense subset of the set

$$J = \{s:\ 0 \leq s \leq 1,\quad s \text{ continuous and excessive in } B\}$$

considered as a subset of the separable Banach space $C(\overline{D})$. Put $s = \Sigma 2^{-n} s_n$. $u = E_\cdot[s(X_T)]$ is then excessive in B. Using Exercise 9 above, let $a_m \in D$ be such that

$$\lim_m u(a_m) = u(a).$$

This implies using semi-continuity

$$\lim_m E_{a_m}[s_n(X_T)] = E_a[s_n(X_T)]$$

for every n.

2. Applications

The fact that $\log|f|$ is subharmonic for f analytic has many applications in complex function theory. We will consider one application in H^p-spaces. For more on this subject, consult Duren [2].

Let f be analytic in the unit disk $B(0,1)$. f is said to be in H^p, $p > 0$, if

(1) $$\sup_{0<r<1} \int |f(rz)|^p \sigma(dz) < \infty.$$

where σ is the uniform distribution on $S(0,1)$. The real and imaginary parts of f are harmonic. By the Poisson formula

(2) $$f(x) = \int_{|y|=1} P_r(x,y) f(ry) \sigma(dy), \qquad |x| < r$$

where

$$P_r(x,y) = (r^2 - |x|^2)|x-y|^{-2}, \qquad |x| < r, \qquad |y| = 1.$$

Standard proofs of the following theorem use Blaschke products.

Theorem 1. Let $p \geq 1$ and $f \in H^p$. There exists $f^* \in L^p(\sigma)$ such that

(3) $$f(x) = \int P_1(x,y) f^*(y) \sigma(dy), \qquad |x| < 1.$$

Proof. We prove this for $p = 1$. For $p > 1$, the argument is similar and simpler. Consider the functions $g_r(z) = |f(rz)|^{\frac{1}{2}}$. (1) with $p = 1$ implies that the functions g_r form a bounded subset of $L^2(\sigma)$. So there is a sequence $r_n \to 1$ and $g_1 \in L^2$ such that g_{r_n} tends weakly to g_1 in L^2. Recalling $g = |f|^{\frac{1}{2}}$ is subharmonic in $|x| < 1$, from the first part of Theorem 1, §1 (and Poisson integral formula),

$$g(x) \leqq \int P_{r_n}(x,y) g_{r_n}(y)\sigma(dy), \qquad |x| < r_n.$$

By weak convergence the above inequality holds with r_n replaced by 1. Since $P_1(x,y)\sigma(dy)$ is a probability measure

$$g^2(x) \leqq \int P_1(x,y) g_1^2(y)\sigma(dy). \tag{4}$$

Integrating both sides of the above on $(|x| = r)$ (remember $g(rx) = g_r(x)$),

$$\|g_r\|_2 \leqq \|g_1\|_2$$

$\| \ \|_2$ denoting L^2-norm. Since $g_{r_n} \to g_1$ weakly, (4) shows that g_{r_n} converges to g_1 in L^2. But then $g_{r_n}^2$ converges to g_1^2 in L^1, i.e. that $|f(r_n\cdot)|$ is a convergent sequence in L^1 and in particular it is uniformly integrable. Thus the sequence $f(r_n\cdot)$ is also uniformly integrable. There exists an $f^* \in L^1$ such that a subsequence of $f(r_n\cdot)$ converges weakly to f^*. See Meyer [3] p. 20. It is now obvious that (3) holds.

Remark. The Fatou boundary limit theorem (§4, Chapter 4) and Theorem 1 above show that $f(r\cdot)$ actually converges in L^1 to f^*. If $f \in H^p$ for $0 < p < 1$, it is still true that $f(r\cdot)$ converges in L^p to a function $f^* \in L^p$. This needs a little more work. See the exercises.

The following Corollary, which is equivalent to Theorem 1, is important in prediction theory.

Corollary 2 (F. and M. Riesz). Let m be a complex measure on $S(0,1)$ for which $\int z^n m(dz) = 0$ for $n = 1,2,\ldots$. Then m is absolutely continuous relative to σ.

Proof. Using the condition it is easily seen that the Poisson integral of m belongs to H^1. Now use Theorem 1 and the uniqueness of Poisson integrals.

Theorem (F. and R. Nevanlinna). Let N be the class of functions analytic in the unit disk for which

$$\sup_{0<r<1} l(+,r,f) < \infty,$$

where $l(+,r,f)$ is the L^1-norm of $\log^+|f(r\cdot)|$. $f \in N$ iff it is the quotient of two bounded analytic functions.

Proof. Suppose $f \in N$. $w = \log^+|f|$ is subharmonic in the unit disk. Let T_r denote the exit time from $B(0,r)$. Then

$$w(\cdot) \leqq E_\cdot[w(X_{T_r})], \qquad |x| < r.$$

The right is positive and harmonic for $|x| < r$. It increases as r increases (simply use strong Markov property). The limit u is either $\equiv \infty$ or is harmonic. The condition implies that $u(0) < \infty$. Thus u is a positive harmonic function that dominates w. Let v be conjugate harmonic so that $g = u+iv$ is analytic in the disk. $h = \exp g$ is analytic and $|h| = \exp u > 1$. $|f| \leqq h$ because $\log^+|f| < u$. (f/h) and h^{-1} are both analytic and bounded by 1, $f = (f/h)h$.

Conversely, suppose $f = (a|b)$ where a and b are analytic and bounded in the disk. We may assume that $|a| \leqq 1$, $|b| \leqq 1$, and that $b(0) \neq 0$. $\log^+|f| \leqq -\log|b|$:

$$-\infty < \log|b(0)| \leqq E_0[\log|b(X_{T_r})|] = E_0[\log^+|b(X_{T_r})|] - E_0[\log^-|b(X_{T_r})|],$$

$\log^+|b(X_{T_r})| \leqq |b(X_{T_r})|$ and b is bounded. Thus we must have $\sup_r E_0[|\log|b(X_{T_r})||] < \infty$. Q. E. D.

Let G be an open set. A subset A of G is called polar if there is a superharmonic function s on G such that $A \subset s^{-1}(\infty)$. Because s is locally integrable, a polar set is of Lebesgue measure zero. Actually a polar set is much thinner as we shall see below.

Let F be a relatively closed polar subset of G. Suppose s is superharmonic on $G \setminus F$ and is locally bounded below (this means that for each compact $K \subset G$, s is restricted to $K \setminus F$ is bounded below). Extend s in a lower semi-continuous fashion, i.e. define

$$(5) \qquad s(a) = \liminf_{G \setminus F \ni y \to a} s(y).$$

Clearly $s: G \to (-\infty, \infty]$ is unaltered on $G \setminus F$. We claim that s is superharmonic on G. To this end we use Proposition 2, §1:

Suppose u is harmonic in a relatively compact open set $D \subset G$, continuous on $\bar{D}$, and $u \leq s$ on ∂D.

Let f be superharmonic on G such that $F \subset A = f^{-1}(\infty)$. Since f is bounded below on D, we may assume that $f \geq 0$ on D. For any $\varepsilon > 0$, $F = s + \varepsilon f$ is identically ∞ on A. Using Proposition 3, §1, it is clear that F is superharmonic on G. Also $u \leq F$ on ∂D and hence $u \leq F$ on D. Letting ε tend to zero, $u \leq s$ on $D \setminus A$. Since A has measure zero, $u \leq s$ on $D \setminus F$ (use (2), §1). From (5) $u \leq s$ on D. Thus

Proposition 3. Let F be a relatively closed polar subset of an open set G. If s is superharmonic on $G \setminus F$ and locally bounded below, there is a superharmonic function on G which agrees with s on $G \setminus F$.

Local boundedness is essential as shown by the example: $G = R^3$, $F = \{0\}$, $s(x) = -\frac{1}{|x|}$.

As a corollary we have the following

Theorem 4 (Theorem of Rado). Let f be a continuous complex valued function on an open set G of the plane. Suppose that f is holomorphic at every point of G at which f is not zero. Then f is holomorphic in G.

Proof. There is no loss of generality in assuming that G is connected. Proposition 3, §1, implies that $\log|f|$ is subharmonic in G. Therefore the set $(f = 0) = (\log|f| = -\infty)$ is polar. If $f = u+iv$, both u and v must be harmonic in G, i.e. f is holomorphic in G.

Exercises to §2

1. Let $0 \neq f \in H^1$ and f^* be as in Theorem 1. Show that

$$\log|f|(x) \leq \int P_1(x,y)\log|f^*|(y)\sigma(dy), \qquad |x| < 1.$$

In particular, $\log|f^*| \in L^1$ and so f^* cannot vanish on a set of positive measure.

Hint. In the proof of Theorem 1 it was shown that $|f(r_n\cdot)|$ converged to $|f^*|$ in L^1. Assume $|f(r_n\cdot)|$ converges to $|f^*|$, σ-almost every where. Using $|z| - \log|z| \geq 0$, and Fatou

$$\limsup_{r_n \to 1} \int P_{r_n}(x,y)\log|f(ry)|\sigma(dy) \leq \int P_1(x,y)\log|f^*(y)|\sigma(dy)$$

and the terms on the left dominate $\log|f(x)|$ as soon as $|x| < r_n$ because $\log|f|$ is subharmonic. $\log|f^*| \leq |f^*|$ so that only $\log^-|f^*|$ can have a divergent integral. Choosing x so that $f(x) \neq 0$, one concludes that $\log|f^*| \in L^1(\sigma)$.

2. Let $f \in H^p$, $p > 0$. Show that there exists $f^* \in L^p(\sigma)$ such that $f(r\cdot)$ tends to f^* in L^p as r tends to 1.

Hint. f is necessarily in N. $f = g/h$ where h and g are bounded analytic. The radial limits of g and h exist and are non-zero almost everywhere. So the radial limit f^* of f exists almost everywhere and is in L^p. Just as in the proof of Theorem 1, $|f(r\cdot)|$ converges in L^p to $|f^*|$. Now use Exercise 3 below.

3. Let $f_n \in L^p$, $p > 0$. Suppose f_n converges to f almost everywhere and $\int |f_n|^p$ tends to $\int |f|^p$. Then f_n converges to f in L^p.

Hint. $g_n = |f_n|^p \wedge |f|^p$ is bounded by an integrable function. $|f_n|^p - g_n$ are non-negative and their integrals converge to zero. Thus $|f_n|^p$ is uniformly integrable. So is $|f_n - f|^p \leq 2^p(|f_n|^p + |f|^p)$.

4. Show that a countable union of polar sets is polar.

Hint. Suppose $E_n \subset s_n^{-1}(\infty)$, with s_n superharmonic in G. If D_n open relatively compact with $\bigcup_n D_n = G$, s_n is bounded below on $\bar{D}_n$: say $s_n \geq b_n$ on $\bar{D}_n$. Let $a_n > 0$ such that $\Sigma a_n |b_n| < \infty$. $s = \Sigma a_n s_n$ is superharmonic on G: For $n \geq m$, $s_n - b_n$ is non-negative and superharmonic on D_m. $s \equiv \infty$ on $\bigcup E_n$.

§ 3. Riesz measure.

We shall associate, with a superharmonic function f, a measure, called its Riesz measure. In the language of Schwartz distributions this measure in simply $(-\Delta)f$. We shall see that this will have important consequences.

Let us start with the simple identity

$$\int F(x)\Delta G(x)\,dx = \int G(x)\Delta F(x)\,dx. \tag{1}$$

where G is a C^2-function with compact support and F is C^2 in a neigbourhood of the support of G.

Lemma 1. Let f be locally integrable in an open set Ω. Suppose for every $F \in C_0^\infty(\Omega)$ (C^∞-functions with compact support contained in Ω)

$$\int f\Delta F = 0$$

Then there exists a harmonic function g in Ω such that $f = g$ almost everywhere.

Proof. We will show that f is equal almost everywhere (in Ω) to a C^2-function; such a function must be harmonic by (1).

Let D be a relatively compact open set with $\bar{D} \subset \Omega$. Let V be a neighbourhood of 0 such that $\bar{D} + V \subset \Omega$. Let A be any C^∞-function with support contained in V. If F is C^∞ and has support contained in D, for every $x \in V$, $F(x-y)$ as a function of y is C^∞ with support in Ω. From our condition on f

$$\int f(x-y)\Delta F(y)\,dy = \int f(y)\Delta F(x-y)\,dy = 0$$

for all $x \in V$, implying

$$\int A(x)\,dx \int f(x-y)\,\Delta F(y)\,dy = \int \Delta F(y)\,dy \int A(x) f(x-y)\,dx = 0.$$

$\int A(x) f(x-y)\,dx$ is c^∞ in D and the above equation holds for all c^∞-functions F with support in D. This implies that $g(y) = \int A(x)\, f(x-y)\,dx$ is harmonic in D and so has the mean value property. Let D_1 be open with $\bar{D}_1 \subset D$ and V_1 a neighbourhood of 0 for which $\bar{D}_1 + V_1 \subset D$. If B is c^∞ with support in V_1 and B depends only on distance i.e. $B(x) = B(|x|)$, the mean value property of g implies that $g * B = g$ in D_1 i.e. $A * f * B = A * f$. This holds for all c^∞ functions with support in V implying that $f * B = f$ almost everywhere in D_1. That is to say that f is almost everywhere equal in D_1 to a c^∞-function. That completes the proof.

__Lemma 2.__ Let f be superharmonic in an open set Ω. There exists a unique measure m such that

$$\int f \Delta\varphi = -\int \varphi\, dm \tag{2}$$

for every φ in c^2 with compact support contained in Ω. $m = 0$ iff f is harmonic.

__Proof.__ Uniqueness is clear. If $m = 0$, by Lemma 1 f is equal almost everywhere to a harmonic function but then f must itself be harmonic. (That two superharmonic functions equal almost everywhere are identical follows from the remarks made just after the definition of a superharmonic function).

f being locally integrable, we can define a linear functional L on the set of c^2-functions with compact support in Ω by

$$LF = -\int f \Delta F.$$

We claim $LF \geq 0$ if $F \geq 0$. To see this let $D \subset \bar{D} \subset \Omega$ be an open set with compact closure such that the support of F is contained in D. If f_n is the sequence of smooth superharmonic functions we constructed in the beginning of § 1 (to approximate f)

$$LF = -\int f \Delta F = \lim_n -\int f_n \Delta F = \lim_n \int (-\Delta) f_n F \geq 0$$

since $f_n \leq 0$ and $F \geq 0$. By the Riesz representation theorem a non-negative linear functional on the set of c^2-functions with compact support in Ω is given by a positive Radon measure.

__Definition.__ The measure m given by Lemma 2 is called the __Riesz measure of superharmonic function__ f.

Note that no claim is made that m is a finite measure; m however, is finite on compact subsets of Ω. We will write down some properties of the Riesz measure m corresponding to a superharmonic function.

__Properties of m__

1. For any open subset U of Ω, the Riesz measure of f/U is m/U. This is obvious from (2). The Riesz measure of $f_1 + f_2$ is $m_1 + m_2$.

2. For any open subset U of Ω, $m(U) = 0$ iff f is harmonic in U. Use property 1 above and (2).

3. If f_n are superharmonic with Riesz measures m_n and

$f_n \uparrow f$ then m_n tend weakly to m. Indeed for any c^2-function φ with compact support in Ω,

$$\lim_n \int \varphi dm_n = \lim_n -\int f_n \Delta\varphi = -\int f\Delta\varphi = \int \varphi dm.$$

We will now apply Lemma 2 to a special superharmonic function. Let

(3) $$K(x) = \begin{cases} -|x| & \text{if } d = 1 \\ -\log|x| & \text{if } d = 2 \\ |x|^{-d+2} & \text{if } d \geq 3 \end{cases}$$

K is superharmonic on $\mathbb{R}^d$ and is harmonic except at the origon. By property 2 above the Riesz measure of K must be concentrated at the origin:

For any c^2-function F with compact support

(4) $$\int K(x)\ \Delta F(x)dx = -A_d F(0).$$

where A_d is a constant. Taking

$$F(x) = \begin{matrix} (1-|x|^2)^3, & |x| \leq 1 \\ 0 & |x| \geq 1 \end{matrix}$$

in (4) gives us the value of A_d:

(5) $$A_d = \begin{cases} 2, & d = 1 \\ 2\pi, & d = 2 \\ (d-2)\,2\pi^{d/2}/\Gamma(d/2), & d \geq 3 \end{cases}$$

because the area of the unit sphere in R^d is

$$2\pi^{d/2}\Gamma(d/2).$$

If F is c^2 and has compact support the function $F(x-\cdot)$

has the same properties. From (4) we get the first part of

<u>Theorem 3</u>. For any c^2-function F with compact support

(6) $$\int K(y)\,\Delta F(x-y)\,dy = -A_d F(x).$$

If f is c' with compact support the function

(7) $$u(x) = \int K(x-y)\ f(y)dy$$

is c^2 and satisfies the Poisson equation

(8) $$\Delta u = -A_d f.$$

The constants A_d are given in (5).

<u>Proof</u>. That u given by (7) is c^2 if f is c' with compact support is routine. To prove (8), multiply both sides of (7) by a c^2-function F with compact support, use (1), Fubini and (6) to get

$$\int F\Delta u = -A_d \int fF$$

And this is equivalent to (8).

Suppose s_1 and s_2 are superharmonic in an open set G with Riesz measures m_1 and m_2. Suppose $m_1 = m_2$ in an open set $D \subset G$. $s_1 - s_2$ is defined almost everywhere in D and is locally integrable. From the definition of Riesz measures and Lemma 1, there is a function h, harmonic in D such that $s_1 - s_2 = h$ i.e. $s_1 = s_2 + h$ almost everywhere and hence everywhere, because both sides are superharmonic. As a corollary of this observation we have

Proposition 4 (Bôcher). Every function f which is positive and harmonic in the punctured ball $V = B(0,1) \setminus \{0\}$ is of the form

$$f = cK + h$$

where the constant $c \geq 0$ and h is harmonic in $B(0,1)$.

Proof. Because $\{0\}$ is clearly polar, by Proposition 3, § 2 there is no loss of generality in assuming that f is superharmonic in $B(0,1)$. Its Riesz measure can only be concentrated at the origin. This follows from the above observation.

We shall use this proposition in Chapter 6. See also Exercise 5.

Theorem 5. (F. Riesz). Let f be superharmonic in an open set G with Riesz measure m. To any relatively compact open subset D of G, corresponds a function $g(D,\cdot)$ which is harmonic in D, superharmonic in G and satisfies

$$\frac{1}{A_d}\int_D K(x-y)\, m(dy) + g(D,x) = f(x),\ x \in G. \tag{9}$$

The constants A_d are defined in (5).

Proof. Using the observation made before Proposition 4, there is a function $g(D,\cdot)$ harmonic in D such that (9) holds in D. Off $\bar{D}$, the first term in (9) is harmonic. Define $g(D,\cdot)$ off $\bar{D}$ by (9). To define $g(D,\cdot)$ on ∂D: let D_1 be any open relatively compact set with $D_1 \supset \bar{D}$ and define

$$g(D,x) = g(D_1,x) + \frac{1}{A_d}\int_{D_1 \setminus D} K(x-y)\, m(dy),\ x \in D_1.$$

$g(D,\cdot)$ is then superharmonic in D_1. It is easy to see that this definition of $g(D,\cdot)$ on ∂D does not depend on D_1.

Remark. If $\int |K(x_0-y)|\, m(dy) < \infty$ for one x_0 then

$$f(x) = \frac{1}{A_d}\int K(x-y)\, m(dy) + h \tag{10}$$

with h harmonic in G. Indeed the first term on the right side of the above equality is then superharmonic on R^d with Riesz measure m. In particular we see that if m is finite with bounded support then (10) holds. This is not true in general. For example in R^2, the function $f(x) = |x|$ is subharmonic with Riesz measure $-\frac{1}{|y|}dy$ (see examples below) and $\int |\log|x-y||\, \frac{1}{|y|}dy = \infty$ for every x.

Examples.

1. If u is harmonic, $|u|^p$ is subharmonic for $p \geq 1$. If $p = 1$, the Riesz measure of $|u|$ is clearly concentrated on the set $u = 0$. Using Green's second identity (see for instance Buck [1] pp. 443) we can show that the Riesz measure of $|u|$ is a multiple of $\frac{\partial u}{\partial n}\, d\sigma$ where $d\sigma$ is the surface element on $(u = 0)$ and $\frac{\partial u}{\partial n}$ is the normal derivative; the point is that for a harmonic function level sets (i.e. sets $u = a$, a constant) are not too bad but we will not go into this. If $p > 1$, the Riesz measure of $|u|^p$ is absolutely continuous relative to Lebesgue measure with density a multiple of $|u|^{p-2}|\text{grad}\, u|^2$.

If $f(z)$ is analytic, $|f|^p$ is subharmonic for all $p > 0$. It can be seen using Theorem 5 that the Riesz measure of $|f|^p$ has no atoms. Outside of its zero set $|f(z)|^p$ is differentiable so that its Riesz measure has density a multiple of $\Delta|f(z)|^p$.

On the other hand $\log|f|$ is also subharmonic, and clearly its Riesz measure must be concentrated on the zero set of f. If a is a zero of f of multiplicity n, $(z-a)^{-n} f(z)$ is holomorphic at a and does not vanish at a. So the Riesz measure of $\log|f|$ has an atom of weight $2\pi n$ at the point a.

2. The function K defined in (3) is superharmonic on $\mathbb{R}^d$. If $B = B(0,1) = (x: |x| < 1)$ the function v defined by

$$v(x) = K(x) \wedge K(1) = \begin{cases} K(1) & \text{if } x \in B \\ K(x) & \text{if } x \notin B \end{cases}$$

is superharmonic on $\mathbb{R}^d$. v is clearly invariant under rotations. Therefore the Riesz measure of v (which must of course be concentrated on ∂B) is a multiple of the uniform distribution σ on B. By the Remark after Theorem 5

$$v(x) = \frac{c}{A_d} \int_{|y|=1} K(x-y)\sigma(dy) + h, \quad c = \text{constant}$$

where h is harmonic in $\mathbb{R}^d$. But it is also rotation invariant. It must therefore be a constant. (If u is harmonic, $u(x) = f(|x|)$ where f is defined on $(0,\infty)$, $0 = \Delta u(x) = f''(|x|) + \frac{d-1}{|x|} f'(|x|)$ implying for $d \geq 2$ that f is unbounded near 0 unless constant.) For $|x| > 1$, $K(x-z)$ as a function of z is harmonic for $|z| < |x|$. The mean value property gives

$$K(x) = \int K(x-y)\sigma(dy), \quad |x| > 1.$$

All the above facts imply that for $|x|>1$ (note $v(x) = K(x)$ for $|x| \geq 1$) $(1-\frac{c}{A_d}) K(x)$ is equal to a constant i.e. that the constant is zero and $c = A_d$. Therefore

$$v(x) = \int_{|y|=1} K(x-y)\sigma(dy), \quad x \in \mathbb{R}^d$$

σ = uniform distribution on $|y| = 1$. As a simple application this last equality for $d = 2$ and $x = 1$ clearly implies $\frac{1}{2\pi}\int_0^{2\pi} \log|1-\cos\theta|d\theta = 1$.

The following generalizes Example 2.

3. Let f be superharmonic on an open set G and $D \subset \bar{D} \subset G$ a bounded open set. The function g defined by

$$g(x) = \begin{cases} f(x) & x \notin \bar{D} \\ E_x(fX_T)), & x \in \bar{D} \end{cases}$$

where T = exit time from D is superharmoni in G by Proposition 4, § 1. We claim that the Riesz measure n of g is

$$n(dz) = \int P_a(X_T \in dz)\, m(da)$$

where m = Riesz measure of f and $P_a(X_T \in dz)$ = the Dirac measure at a for $a \notin \bar{D}$. We need the following symmetry result which will be shown in Chapter 6:

$$\int K(x-y)\, P_a(X_T \in dy) = \int K(a-y)\, P_x(X_T \in dy)$$

for all $a, x \in D$. Assuming this let A be a bounded open set with $\bar{D} \subset A \subset \bar{A} \subset G$.

$$f(x) = \frac{1}{A_d} \int_A K(x-y)\, m(dy) + h$$

where h is harmonic in A, because m is the Riesz measure of f. Since $E_a(h(X_T)) = h(a)$ for all $a \in A$

$$g(x) = \frac{1}{A_d}\int_A m(dy) \int K(z-y) P_x(X_T \in dz) + h = \frac{1}{A_d}\int_A K(x-z) n(dz) + h,$$

using the above symmetry result. But this simply means that n is the Riesz measure of g as claimed.

Remark. Example 3 says that the Riesz measure of g is obtained as follows: The measure m is left undisturbed outside $\bar{D}$. The part of m in $\bar{D}$ is "swept out" onto ∂D by use of the harmonic measures $P.(X_T \in dy)$. This operation is the "sweeping out" or balayage method and n is called the "balayage" of m relative to D. We will return to his in Chapter 7.

Exercises to § 3.

1. Show that the gradient of a superharmonic function (defined on an open set) exists almost everywhere.

Solution. By Theorem 5 it is clearly sufficient to show the following:

If m is a positive measure with compact support in $\mathbb{R}^d$ the function

$$u(x) = \int K(x-y)\, m(dy)$$

has gradient almost everywhere.

The case $d = 1$ is simple. If $d \geq 2$, $a < b$ reals and $x = (x_1,\dots,x_d)$, $y = (y_1,\dots,y_d)$

$$\int_a^b \frac{x_1-y_1}{|x-y|^d}\, dx_1 = K(b-y_1,\ x_2-y_2,\dots,x_d-y_d) - K(a-y_1,\ x_2-y_2,\dots,x_d-y_d)$$

in the sense that if one side is meaningful the other side is too and equality holds. For any $M < \infty$,

$$\int_{|x|\leq M} dx \int_{|y|\leq N} \frac{|x_1-y_1|}{|x-y|^d}\, m(dy)$$

$$= \int_{|y|\leq N} m(dy) \int_{|x|\leq M} \frac{|x_1-y_1|}{|x-y|^d}\, dx$$

$$\leq \int_{|y|\leq N} m(dy) \int_{|z|\leq M+N} \frac{1}{|z|^{d-1}}\, dz$$

where the ball $|y| \leq N$ contains the support of m. It follows that for almost all $(x_2,\dots,x_d)$ the integral

$$\int_a^b dx_1 \int \frac{|x_1-y_1|}{|x-y|^d}\, m(dy) < \infty.$$

For any such $(x_2,\dots,x_d)$,

$$\int_a^b dx_1 \int \frac{x_1-y_1}{|x-y|^d}\, m(dy) = \int m(dy) \int_a^b \frac{x_1-y_1}{|x-y|^d}\, dx_1$$

$$= u(b,x_2,\dots,x_d) - u(a,x_2,\dots,x_d).$$

That proves the result.

2. Let u be superharmonic in an open set. If the Riesz measure of u has bounded density then $u \in C^1$.

Hint. By Theorem 5 it is enough to show that if g bounded measurable with compact support the function $u(x) = \int K(x-y)g(y)\,dy$ is continuously differentiable. If φ is locally in L^1 and $\psi \in L^\infty$ with compact support $\varphi * \psi$ is continuous. So the result follow since K and grad K are locally in L^1.

3. Let g be Hölder continuous with compact support. Then $u(x) = \int K(x-y)g(y)\,dy$ is C^2 on $\mathbb{R}^d$. (A function g is Hölder continuous if there are numbers $\alpha > 0, M > 0$ such that $|g(x)-g(y)| < M|x-y|^{\alpha}$).

Solution. From Exercise 2 above u is continuously differentiable and the partial $u_1 = \frac{\partial u}{\partial x_1}$ is given by

$$u_1(x) = \int \frac{x_1-y_1}{|x-y|^d} g(y)\,dy.$$

Let A be a C^1-function with compact support, which is equal to 1 on the support of g. We assume that $|g(x)-g(y)| \leq |x-y|^\alpha$. and $0 \leq A \leq 1$. There is no loss of generality in this. We shall show that the second partial u_{11} is given by

$$u_{11}(x) = \int \frac{1}{|x-y|^d}(g(y)-g(x))A(y)\,dy$$

$$(*) \qquad - d\int \frac{(x_1-y_1)^2}{|x-y|^{d+2}}(g(y)-g(x))A(y)\,dy$$

$$+ g(x)\int \frac{x_1-y_1}{|x-y|^d} A_1(y)\,dy$$

where $A_1 = \frac{\partial A}{\partial y_1}$. By the Hölder continuity of g the function

$$\frac{1}{|x-y|^d}|g(y)-g(x)|\,A(y) \leqq \frac{1}{|x-y|^{d-\alpha}} A(y)$$

and

$$(**) \qquad \sup_z \int \frac{1}{|z-y|^{p(d-\alpha)}} A(y)^p\,dy < \infty$$

Provided $p(d-\alpha)-(d-1)<1$. Therefore the integrands $\frac{1}{|x-y|^d}(g(y)-g(x))A(y)$ are uniformly integrable showing that the first integral on the right side of (*) is continuous in x. Similar conclusion obtains with the other two integrals in (*).

Let e_1 denote the unit vector $(1,0,0,\ldots,0)$. The difference quotient

$$\frac{1}{h}[u_1(x+he_1) - u_1(x)]$$

$$(***) \qquad = \frac{1}{h}\int \left(\frac{x_1+h-y_1}{|x+he_1-y|^d} - \frac{x_1-y_1}{|x-y|^d}\right)(g(y)-g(x))A(y)\,dy$$

$$+ g(x)\,\frac{1}{h}\int \left(\frac{x_1+h-y_1}{|x+he_1-y|^d} - \frac{x_1-y_1}{|x-y|^d}\right)A(y)\,dy$$

After a change of variable the second term on the right side of (***) becomes $g(x)\frac{1}{h}\int \frac{y_1}{|y|^d}(A(x+he_1-y)-A(x-y))dy$ which tends as $h \to 0$ to

$$g(x)\int \frac{y_1}{|y|^d} A_1(x-y)\,dy.$$

To show that lim and $\int$ can be interchanged in the first integral in (***) we show that the integrands in the first term of (***) has a bounded L^p norm if $p(d-\alpha)-(d-1)<1$. Write $P = |x+he_1-y|$, $Q = |x-y|$.

$$\frac{1}{|h|}\left|\frac{x_1+h-y_1}{P^d} - \frac{x_1-y_1}{Q^d}\right| \leq \frac{1}{|h|}\,\frac{|P^d-Q^d|}{P^{d-1}Q^d} + \frac{1}{Q^d}$$

since $|x_1+h-y_1| \leq P$. $Q^{-d}(g(y)-g(x))A(y)$ is the first integrand on the right side of (*) and has already been shown to have a bounded L^p-norm. The function

$$\frac{1}{|h|}\,\frac{|P^d-Q^d|}{P^{d-1}Q^d}|g(y)-g(x)|\,A(y) \leqq \frac{1}{|h|}\,\frac{|P^d-Q^d|}{P^{d-1}Q^{d-\alpha}}A(y) = B(h,x,y)$$

say; $|P^d-Q^d| \leq |P-Q|\cdot d\cdot(\max(P,Q))^{d-1}$ and $|P-Q| \leq |h|$. On the set $P \geq Q$, $B(h,x,y) \leq d\,Q^{-d+\alpha}.A(y)$. And on the set $P<Q$, $P<Q$, $B(h,xy) \leq P^{1-d}.A(y).Q^{\alpha-1} \leq P^{-d+\alpha}A(y)$. And in both cases (**) can be applied. That completes the proof.

For a simpler proof using Greens identity see Courant-Hilbert, Methods of Mathematical Physics Vol.II Interscience (1962) p.p. 248-250.

4. Let G be open and E a polar subset of G. Show that E is a polar subset of $\mathbb{R}^d$.

Hint. Use Theorem 5 and Exercise 4 §2.

5. Let D be open and F a compact polar subset of D. If u is positive harmonic in $D \setminus F$ then for a measure m on F $u(x) = \int K(x-y)m(dy) + h(x)$, where h is harmonic in D.

Hint. Same as Proposition 4.

§ 4. The Contiuity Principle.

Superharmonic functions can be very discontinuous. For example let $\{a_n\}$ be a sequence dense in R^3. We can choose positive numbers b_n sucb that

$$u(x) = \Sigma\, b_n \frac{1}{|x-a_n|}$$

defines a superharmonic function on R^3 (one need only choose b_n so that the series converges at one point). $u(a_n) = \infty$ for every n so that u is discontinuous at every point at which it is finite. Thus (recall u is lower semi continuous) the set of points of continuity of u (continuity in the generalised sense) coinsides with the set of infinities of u. This last set has measure zero. On the other hand u being lower semi continuous the set of points of continuity of u is a dense G_δ - set.

Thus it would appear that little can be said about the continuity properties of superharmonic functions in general. Theorem 5 § 3 shows that local properties of superharmonic functions can be investigated with the help of superharmonic functions of the form $\int K(x-y)\, m(dy)$ where m is a finite measure with compact

support. For example the gradient of a superharmonic function exists almost everywhere; see Exercise 1, §3. See also Exercise 2, §3.

The following theorem is very important.

Theorem 1. (Continuity Principle of Evans-Vascilesco). Let f be superharmonic in an open set G. If the restriction of f to the support of its Riesz measure is continuous then f is continuous on G.

Proof. By Theorem 5, §3, it is sufficient to show the following: Let m be a finite measure with compact support F. If the restriction of

(1) $$u(x) = \int K(x-y)\, m(dy)$$

to F is finite and continuous then u is continuous on R^d.

Let $x_0 \in F$. For each neighbourhood V of x_0

(2) $$u(x) = \int_V + \int_{F\setminus V} = s_V + s_{F\setminus V},$$

say. Both the summands on the right side of (2) are continuous on F because each is lower semi continuous and their sum is continuous on F. Also as V decreases to x_0, s_V decreases to zero because m has no atom at $x_0 (u(x_0) < \infty)$. Given $\varepsilon > 0$, using Dini we can find a V such that

(3) $$\sup_{x \in F} s_V(x) < \varepsilon, \quad m(V) < \varepsilon.$$

For any x let z_x be a point in F nearest to x.

$$|z_x - y| \le |z_x - x| + |x-y| \le 2|x-y|, \quad y \in F$$

so that for all $y \in F$,

$$K(x-y) \leq \log 2 + K(z_x - y), \qquad d = 2$$

$$K(x-y) \leq 2^{d-2} K(z_x - y), \qquad d \geq 3.$$

Therefore for all x

$$s_V(x) \leq \log 2 . m(V) + s_V(z_x), \quad d = 2$$

$$s_V(x) \leq 2^{d-2} s_V(z_x), \qquad d \geq 3.$$

In other words using (3)

(4) $\qquad \sup s_V(x)$ is small if V is small.

Thus since $s_{F \setminus V}$ is continuous at x_0

$$\limsup_{x \to x_0} u(x) \leq \sup_x s_V(x) + s_{F\setminus V}(x_0)$$

$$\leq \sup_x s_V(x) + u(x_0)$$

which, using (4) gives the continuity of u at x_0. qed.

<u>Theorem</u> 2. Let G be a domain and f excessive in G, with Riesz measure m. Suppose f is finite m-almost everywhere. Then

(5) $$f = \sum_n f_n + h$$

where f_n are finite continuous and excessive and h is harmonic and ≥ 0.

<u>Proof</u>. Let D_n be an increasing sequence of open relatively compact sets with union G. Let us show that there exists a continuous excessive function f_1 (on G) whose Riesz measure m_1 is concentrated on D_1 such that

$$f - f_1 \quad \text{is excessive in} \quad G$$

and $$m(D_1) - m_1(D_1) < \tfrac{1}{2}$$

By Theorem 5, § 3

$$s_1(x) + g(D_1, x) = f(x), \qquad x \in G$$

where $$s_1(x) = \frac{1}{A_d}\int_{D_1} K(x-y)\, m(dy)$$

Since f is m-almost everywhere finite, $s_1 < \infty$ m-almost everywhere. By Lusin's Theorem, [Rudin [6] pp 53], we can find a closed set $F \subset D_1$ such that, s_1/F is continuous and $m(D_1 \setminus F) < \tfrac{1}{2}$. Now

$$s_1 = \frac{1}{A_d}\int_F + \frac{1}{A_d}\int_{D_1 \setminus F}$$

Since s_1/F is continuous and each of the summands above is lower semi continuous $s_2(x) = \frac{1}{A_d}\int_F K(x-Y)\, m(dy)$ is continuous on F and hence on R^d by the continuity principle.

(6) $$s_2(x) + u(x) = f(x), \qquad x \in G$$

$$u(x) = \frac{1}{A_d}\int_{D_1 \setminus F} K(x-y)\, m(dy) + g(D_1, x), \qquad x \in G$$

Let T_n = exit time from D_n. Since u is superharmonic

(7) $$s_2(x) - E_x[s_2(X_{T_n})] \leq f(x) - E_x[f(X_{T_n})] \leq f(x).$$

$E_x[s_2(X_{T_n})]$ is harmonic in D_n and decreases as n increases. The limit say h_2 is either $\equiv -\infty$ or is harmonic in G. The former is excluded by (7). Writing $f_1 = s_2 - h_2$, f_1 is excessive and continuous, its Riesz measure m_1 is concentrated on D_1, $m(D_1) - m_1(D_1) < 2^{-1}$ and $0 \leq f - f_1 = u + h_2$ is excessive in G. Repeat the above argument with $f - f_1$ and D_2 to get a continuous excessive function f_2 whose Riesz measure m_2 is concentrated on D_2 such that

$$f - f_1 - f_2 \quad \text{is excessive in } G$$

$$m(D_2) - m_1(D_2) - m_2(D_2) < 2^{-2}.$$

In general we can find a continuous function f_n with Riesz measure m_n concentrated on D_n such that

$$f - \sum_1^n f_i \quad \text{is excessive in } G$$

$$m(D_n) - \sum_1^n m_i(D_n) < 2^{-n}$$

Clearly Σf_i is excessive in G with Riesz measure m. (5) then follows from the observations made after Theorem 3, § 3. qed.

Remark. The simple example $f(x) = K(x)$ shows that the assumptions in the above theorem are not superfluous.

Exercises to § 4.

1. Suppose u is given by (1) where m has support F. If the restriction of u to F is continuous at $x_0 \in F$ then u is continuous at x_0.

Hint. Using continuity of u/F at x_0, find a neighbourhood V of x_0 such that $s_V(x) = \int_V K(x-y)\, m(dy)$ and $m(V)$ are small for all $x \in V$. Then use (4).

2. Let L be a kernel on R^d i.e. L is defined on $R^d \times R^d$, $L > 0$, $L(x,x) \equiv \infty$ and $(x,y) \to L(x,y)^{-1}$ is finite and continuous. Suppose $F \subset R^d$ is compact and m a measure on F such that $Lm(x) = \int L(x,y)\, m(dy)$ is continuous when restricted to F. If Lm is not continuous then there exists a measure p on F such that $Lp \leq 1$ on F and $\sup_x Lp(x) = \infty$.

Hint. Suppose $x_0 \in F$ and $\limsup_{x \to x_0} Lm(x) > \varepsilon + Lm(x_0)$. Using Dini find neighbourhoods V_n of x_0 such that $m(V_n) < 2^{-2n}$, $Lm_n \leq 2^{-2n}$ on F where $m_n = m/V_n$. We must have $\limsup_{x \to x_0} Lm_n(x) > \varepsilon + Lm_n(x_0)$. $p = \Sigma 2^n m_n$ is then the required measure.

§ 5. The Dirichlet problem revisited.

Let D be a bounded open set. For any continuous function f on ∂D the function $u(a) = E_a[f(X_T)]$, T = exit time from D, is harmonic in D and $\lim_{b \to a} u(b) = f(a)$ for every regular point $a \in \partial D$. This we have seen in Chapter 4. It is far from obvious that u is uniquely determined by these properties. We shall show in this section that this is indeed the case.

Lemma 1. Let G be open and D an open relatively compact subset of G. Let f be superharmonic in G with Riesz measure m. If f is finite m-almost everywhere and harmonic in D then

$$(1) \qquad f(a) = E_a[f(X_T)], \quad T = \text{exit time from } D.$$

Proof. Since f is bounded below in a neighbourhood of $\bar{D}$, we may (by replacing G by this neighbourhood if needed) assume that f is excessive in G. Now appeal to Theorem 2 § 4 and note that for each n, f_n is continuous on G and harmonic in D so that (1) is true for each n by (5) § 1, Chapter 4.

Remark. The example $G = \mathbb{R}^3$, $D = (x: 0 < |x| < 1)$ and $f(x) = |x|^{-1}$, shows that, Lemma 1 above is not valid in general.

Lemma 2. Let D be a bounded open set and T = exit time from D. For every $a \in \bar{D}$

$$P_a(T(\theta_T) = 0) = 1.$$

In other words, with probability 1, X_T is regular.

Proof. Let B be a ball containing $\bar{D}$ and S = exit time from B. $E.[S]$ is bounded and exessive in B.

$$v(a) = E_a[E_{X_T}(S)] = E_a[S] - E_a[T] \tag{2}$$

is also bounded and excessive in B and it is harmonic in D. By Lemma 1, $v(a) = E_a[v(X_T)]$, which, using (2) gives $E_a[T(\theta_T)] = 0$. q.e.d.

The following Theorem contains as a special case the uniquiness of the solution of the modified Dirichlet problem.

Theorem 3. (The strong boundary minimum principle).

Let D be a bounded open set. If u is superharmonic in D, is bounded below and

$$\liminf_{D \ni b \to a} u(b) \geq 0, \quad a \in \partial D \text{ regular}$$

Then $u \geq 0$ in D.

Proof. Let $a \in D$, and $a \in D_n$ are open sets whose closures are contained in D and which increase to D. T_n = exit time from D_n increases to T = exit time from D, P_a-almost surely.

$$u(a) \geq E_a[u(X_{T_n})].$$

An appeal to Fatou's lemma and Lemma 2 gives the result. q.e.d.

Remark. The example $D = (x: x \in \mathbb{R}^3, 0 < |x| < 1)$ and $u(x) = 1 - \frac{1}{|x|}$, $x \in D$ shows that the boundedness assumption in the above theorem is not superfluous. We have all the machinery now to prove the famous Kellog-Evans theorem: The set of irregular points in ∂D is polar. See exercises. We shall return to this in Chapter 7.

Let us consider another application of Theorem 2, § 4. Let $B = B(0,R)$. The minimum of two continuous excessive functions on B is continuous and excessive. It follows that the differences $s_1 - s_2$ with s_1 and s_2 continuous and excessive on B is a lattice and hence by the Stone-Weierstrass theorem any continuous function on any compact subset of B can be approximated on the set arbitrarily closely by such differences. Let now D be bounded open, $\bar{D} \subset B$ and f continuous on ∂D. For any $\varepsilon > 0$, we can find continuous excessive functions s_1, s_2 such that for $s = s_1 - s_2$

$$|f(x) - s(x)| < \varepsilon, \quad x \in \partial D.$$

If T = exit time from D

$$|E_a[f(X_T) - s(X_T)]| < \varepsilon, \; a \in \bar{D}.$$

By Theorem 2 § 4.

$$E.(s_i(X_T)) = \sum_n s_{i,n}, \qquad i = 1,2$$

where $s_{i,n}$ are continuous, excessive (in B) and harmonic in D. If $a \in \partial D$ is regular. $E_a[f(X_T)] = f(a)$. Since the partial sums $\sum_{n \leq N} s_{i,n}$ are continuous on $\bar{D}$ and harmonic in D and $s_1 - s_2$ is close to f on ∂D, we have proved

<u>Proposition 4</u>. Let D be bounded and open. Each continous function f on D is the bounded pointwise limit on $\gamma(D)$ (= the set of regular points in ∂D) of a sequence of functions f_n each of which is continuous on $\bar{D}$ and harmonic in D.

One comment on Proposition 4 is in order. For unexplained terminology consult Phelps [4]. One associates with each compact set $X \subset \mathbb{R}^d$ a subspace:

$D(X)$ = set of function in $C(X)$ which are harmonic in the interior of X.

The content of Proposition 4 together with Lemma 2 is then:

The Choquet boundary of $D(X)$ = set of points in ∂X that are regular for the complement of the interior of X.

and

$D(X)$ is simplicial, namely that the only signed measure living on the Choquet boundary and annihilating $D(X)$ is the zero measure.

A positive measure on ∂X is said to represent $x \in X$ if $\int f(y)\, m(dy) = f(x)$ for every $f \in D(X)$. That $D(X)$ is simplicial is equivalent to saying that each point $x \in X$ has a unique representing measure living on the Choquet boundary. We can say more: for each x in the interior of X there is only one measure for ∂X which represents x. We shall return to this point in Chapter 7.

We end this section by giving one last application of Theorem 2 § 4 to a Theorem of J.L. Doob.

<u>Theorem 5</u>. (J.L. Doob). Let s be superharmonic on an open set G and T = exit time form G. Then for every $a \in G$, P_a-almost surely $s(X_t)$ is continuous in the generalised sense on $0 \leq t < T$ and is finite for $0 < t < T$.

<u>Proof</u>. It is sufficient to prove this theorem by assuming s is excessive; indeed consider first an open relatively compact subset D of G and then let D increase to G. Assume also s is finite. By Theorem 2 § 4, $s = \sum s_i$ where s_i is continuous excessive in G. $\sum_n^{n+p} s_i(X_t) 1_{t<T}$ is then a right continuous supermartingale, which is continuous for $t < T$. By the supermartingale inequalities of Chapter 1

$$\varepsilon P_a[\sup_{0 \leq t < T} \sum_n^{n+p} s_K(X_t) > \varepsilon] \leq \sum_n^{n+p} s_K(a) \leq \sum_n^{\infty} s_K(a).$$

This shows the uniform convergence of $\sum_1^n s_K(X_t)$ to $\sum_1^\infty s_K(X_t)$ in the interval $0 \le t < T$.

Now let s be any excessive function. We claim for every $t_0 > 0$ and $a \in G$

(3) $$E_a[s(X_{t_0}): t_0 < T] < \infty.$$

Indeed let $a \in D$ be any open relatively compact subset of G and S = exit time from D. From (4), § 1,

(4) $$E_a[s(X_{t_0 \vee S}): t_0 < T] \le E_a[s(X_S)] < \infty$$

because from Proposition 4 § 1, $E.[s(X_S)]$ is harmonic in D. Now for $t_0 > 0$

$$P_a[X_{t_0} \in dy, t_0 < S] \le P_a[X_{t_0} \in dy] \le (2\pi t_0)^{-d/2} dy$$

so that

(5) $$E_a[s(X_{t_0}): t_0 < S] \le (2\pi t_0)^{-d/2} \int_D s(y)\,dy < \infty$$

because s is locally integrable. (3) is a consequence of (4) and (5).

Now $s(X_t) 1_{t<T}$ is a non-negative supermartingale. Since s is lower semi-continuous $(s > n)$ is open.

Hence

$$P_a[s(X_t) > n \text{ for some } t_0 < t < T] = P_a[s(X_r) > n$$

for some rational r with $t_0 < r < T] \le \frac{1}{n} E_a[s(X_{t_0}): t_0 < T]$ which implies that

$$P_a[(s(X_t) = (s \wedge n)(X_t) \text{ for all } t_0 < t < T) \text{ for some } n] = 1.$$

We will let the reader complete the rest.

Exercises to § 5.

1. Let D be a bounded open set. Show that the set of irregular points in ∂D is polar. (This is the famous Kellog-Evans Theorem).

Hint. By exercise 4, § 2 Chapter 4, the set of irregular points in ∂D is a countable union of compact sets. Therefore by Exercise 4, § 2 it is enough to show that a compact subset F of the set of irregular points is polar. Let B be a ball containing $\bar{D}$ and put $G = B \setminus F$. Then $F \subset \partial G$ and clearly every point in F is irregular for the complement of G. By Lemma 2, starting in G, it is impossible to hit F. Let V_n be a sequence og open sets with $V_n \supset \bar{V}_{n+1}$ and $\cap V_n = F$. Let R_n = hitting time to $\bar{V}_n$. The functions

$$p_n(a) = P_a(R_n < S), \quad S = \text{exit time from } B$$

are excessive in B, $p_n(a) = 1$ for $A \in V_n$ and $p_n(a)$ decreases to zero for every a in G, because $X_{R_n} \in \bar{V}_n$ on the set $R_n < S$ and therefore a path in $\bigcap_n (R_n < S)$ must hit F. Let $a_0 \in G$. By choosing a subsequence if necessary we may assume $\Sigma p_n(a_0) < \infty$. $p = \Sigma p_n$ is then excessive in B and infinite on F.

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CHAPTER 6

Green functions

Introduction

Consider a domain G and the inhomogeneous equation $\Delta u = f$, called the Poisson equation. The solution is not unique without further conditions: if u is a solution, so is $u + w$ for any harmonic function w. If G is bounded and we impose the "boundary condition" that u vanish on the boundary, the maximum principle guarantees that there can be at most one solution. In simple terms, Green function for G is the "inverse of Δ together with this boundary condition". In order to better understand what follows, recall the definition and properties of Green functions for ordinary differential operators (see e.g. pp. 39 - 47 of Birkhoff [1]).

We will see that knowing the Green function for a domain is equivalent (theoretically) to solving the Dirichlet problem.

In § 1 we shall define Green functions and point out some of their most important properties. § 2 and § 3 are devoted to a discussion of Green functions for unbounded open sets in R^2. § 4 contains some examples and in § 5 an expression for the Green function in terms of relative transition densities is given.