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CHAPTER 4

Harmonic functions and Dirichlet Problem

Introduction

Harmonic functions are solutions of the Laplate equation $\Delta\mu = 0$. No other single partial differential equation is encountered in so many different situations and exihibits such depth and variety. One runs into the Laplate equation in many branches of applied physics: Electrostatics, stationary heat flow etc. Directly or indirectly the Dirichlet problem has influenced many branches of Analysis: Integral equations, special functions, Calculus of variations etc.

In §1, Dynkin's formula is proved and some applications are given. In §2, the Dirichlet problem is introduced. §3 deals with the Kelvin transformation. Some applications are found in the exercises and in Chapter 6. In §4 we prove the Fatou limit theorem and derive the existence of the Hilbert transform. §5, dealing with spherical harmonics can be considered an application of the Poisson integral formula. The original idea was to give applications in representation theory but we content ourselves with a reference.

Notation

In this Chapter X_t will denote the d-dimentional Brownian motion as introduced in Chapter 2. If D is an open set the exit time T from D is the stopping time

$$T = \inf(t\colon t > 0,\ X_t \notin D)$$

$$= \infty \text{ if there is no such } t.$$

§1. Dynkin's formula

If u and all its first and second partials are bounded

(1) $$E_a[u(X_t)] - u(a) = \tfrac{1}{2}E_a\left[\int_0^t \Delta u(X_s)\,ds\right], a \in \mathbb{R}^d, t > 0.$$

The verification of (1) is simple integration by parts. The most general conditions under which (1) is valid need not concern us. For example if u and all its first and second partials have at most polynomial growth (1) is still valid.

Markov property and (1) imply that

$$M_t = u(X_t) - u(X_0) - \tfrac{1}{2}\int_0^t \Delta u(X_s)\,ds$$

is a martingale (relative to P_a for every a). Since u and Δu are bounded M_t is bounded by a constant times t for $t \geq 1$. It is then simple to check that $E[M_T] = 0$ provided $E_a[T] < \infty$, where T is a stopping time. We have thus

<u>Proposition 1</u>. (Dynkin's formula). Let u and all its first two partials be bounded. If T is a stopping time such that $E_a[T] < \infty$ then

(2) $$E_a[u(X_T)] - u(a) = \tfrac{1}{2}E_a\left[\int_0^T \Delta u(X_s)\,ds\right]$$

Suppose now that u is continuous on $\bar{D}$, Δu exists and is continuous and bounded in D, where D is a relatively compact open set. We may suppose that u in fact is continuous on R^d with compact support. Let $0 \leq \varphi_n$ be C^∞, have support in $B(0,\frac{1}{n})$ and $\int \varphi_n(x)\,dx = 1$. $u_n = u * \varphi$ will be C^∞ with compact support. $u_n \to u$ on R^d and $\Delta u_n \to \Delta u$ in D boundedly. If T = exit time

from D, $E_a[T] < \infty$ for all a. (2) is valid with u replaced by u_n and letting n tend to infinity we get

<u>Theorem 2</u>. Let D be relatively compact open. If u is continuous on $\bar{D}$, Δu exists, is continuous and bounded in D then for all $a \in \bar{D}$

(3) $$E_a[u(X_T)] - u(a) = \tfrac{1}{2}E_a\left[\int_0^T \Delta u(X_s)\,ds\right]$$

where T = exit time from D.

<u>Some consequences</u>.

A function u is called harmonic in an open set U if u is C^2 in U and $\Delta u = 0$ in U. Let us show that a locally integrable function u is harmonic (in U) iff it has the mean value property:

(4) $$\int_{S(a,r)} u(b)\,p_a(db) = u(a)$$

where $S(a,r)$ is the surface of the ball $B(a,r)$ of radius r and centre a completely contained in U and $p_a(db)$ is the uniform distribution on $S(a,r)$. Let u be harmonic in U and $B(a,r)$ a ball completely contained in U. If T denotes the exit time from $B(a,r)$, we know from Chapter 2 that relative to P_a, X_T is univormly distributed on $S(a,r)$ and $E_a[T] < \infty$. (4) is thus a consequence of (3). Conversely suppose u is locally integrable and has the mean value property. If further u is C^2, Δu must be zero; because if $\Delta u(a_0) > 0$ for some a_0, in $B(a_0,r)$ for small r, $\Delta u > 0$ and for T = exit time from $B(a_0,r)$ we get a contradiction using (3). Finally we claim that a locally integrable u having the mean value property is necessarily C^∞ in U. Indeed let D be any relatively compact open subset of U. For r sufficiently small $B(a,r)$ is completely contained in U for all a in $\bar{D}$. If φ is any radial (i.e. depending only on distance C^∞ function with support in $B(o,r)$ such that $\int \varphi(b)\,db = 1$, integration using polar co-ordinates and the mean value property of u imply that $\varphi * u = u$ in D; in forming the convulution $\varphi * u$, define for definiteness $u = 0$ off U. Since $\varphi * u$ is C^∞ in R^d we have shown that u is C^∞ in D and D is any relatively compact open subset of U.

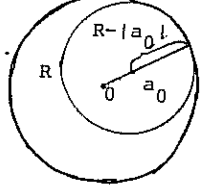
The mean velue property is very useful. We have the following corollaries:

Liouville's Theorem. A non-negative harmonic function on R^d is constant.

Indeed let a_0 be any point in R^d. The mean value property implies that the volume average of u over any ball $B(a_0,R)$ is $u(a_0)$. From the figure it is clear that

$$u(a) = \text{volume average of } u \text{ over } B(a_0, R-|a_0|)$$

$$\le \frac{\text{volume } B(0,R)}{\text{volume } B(a_0,R-|a_0|)}\, u(0).$$



Letting R tend to infinity we find $u(a) \le u(0)$. For a more general statement see Exercise 4.

Harnack's Theorem. Let u_n be harmonic and increase in a connected open set D. Then $u = \lim_n u_n$ is harmonic in D unless it is identically infinite.

Indeed by considering $u_n - u_0$ instead of u_n we may assume $u_n \ge 0$. u satisfies the mean value property. If $u(a_0) < \infty$ u is integrable in any ball $B(a_0,r)$ completely contained in D. But then the mean value property would imply that u is harmonic and hence finite in such a ball. By connectedness for any a in D there is a finite chain of intersecting balls, the first containing a_0 and the last containing a.

Exercise 5 gives an example of a non-harmonic function having the one-circle mean value property. See also Exercise 7.

An immediate consequence of (3) is: If u is continuous on $\bar{D}$ and harmonic in a bounded open set D then

$$u(a) = E_a[u(X_T)], \qquad a \in \bar{D}. \tag{5}$$

where T = exit time from D. Since $X_T \in \partial D$ the boundary of D we obtain

The maximum principle. If u is harmonic in a bounded open set D and continuous on $\bar{D}$

$$|u(a)| \le \sup_{b \in \partial D} |u(b)|, \qquad a \in \bar{D}. \tag{6}$$

Let us work out another consequence of (5). Direct calculation shows that for $d \ge 3$, $u(a) = |a|^{-d+2}$ is harmonic in the complement of $\{0\}$. Take in (5) for D the domain bounded by two concentric spheres: $D = (a: r < |a| < R)$. Since $|X_T|$ is either R or r and $P_a[|X_T| = R] = 1 - P_a[|X_T| = r]$ we get for $d \ge 3$, $r < |a| < R$

(7) $$P_a[|X_T| = r] = \frac{|a|^{-d+2} - R^{-d+2}}{r^{-d+2} - R^{-d+2}}, \qquad d \geq 3$$

where T = exit time from D. When $d = 2$, $u(a) = \log|a|$ is harmonic in the complement of $\{0\}$. By the same argument as above

(8) $$P_a[|X_T| = r] = \frac{\log R - \log|a|}{\log R - \log r} \qquad d = 2.$$

The event $(|X_T| = r)$ occurs iff at the time T of exit from the shell $(r < |x| < R)$ the Brownian path findts itself on the sphere $S(0,r) = (x: |x| = r)$. Letting R tend to infinity we find from (7) and (8) that for $|a| > r$

(9) $$P_a[T_r < \infty] = \begin{cases} 1 & \text{if } d = 2 \\ r^{d-2}|a|^{2-d} & \text{if } d \geq 3 \end{cases}$$

where T_r is the hitting time to $S(o,r)$:

$$T_r = \inf(t: t > 0, |X_t| = r)$$
$$= \infty \text{ if there is no such } t.$$

The above results imply that the d-dimensional Brownian motion is "recurrent" if $d \leq 2$ and "transient" if $d \geq 3$; if $d \geq 2$ $P_a[T_0 < \infty] = 0$ for all a where T_0 is the hitting time to zero: $T_0 = \inf(t: t > 0, X_t = 0)$. See Exercises 2 and 3.

Exercises to section 1

1. Let T = exit time from the ball $B(a_0,r)$. Show that

$$dE_a[T] = r^2 - |a-a_0|^2.$$

Hint. Take $u(a) = |a-a_0|^2$ in (5).

2. Let $d \geq 2$ and T_0 = hitting time to 0. Show that

$$P_a[T_0 < \infty] = 0 \qquad \text{for all } a.$$

Hint. Suppose $a \neq 0$. Let $r \to 0$ in (7) and (8) and then let R tend to infinity. One obtains $P_a[T_0 < \infty] = 0$ for all $a \neq 0$. Deduce using Markov property that $P_0[T_0(\vartheta_t) < \infty] = 0$ for all $t > 0$. Conclude that $P_0[T_0 < \infty] = 0$.

3. Show that the Brownian motion is recurrent for $d = 2$ and transient for $d \geq 3$: For all a

$$P_a[\lim_{t\to\infty}|X_t| = \infty] = 1 \qquad \text{if } d \geq 3$$
$$P_a[(X_s: s \geq t) \text{ is dense in } R^2 \text{ for all } t)] = 1, \qquad d = 2.$$

Hint. Define τ_n, T_n by

$$T_n = \inf(t: t > 0, |X_t| \leq n)$$

$$\tau_n = \inf(t: t > 0, |X_t| \geq n^3)$$

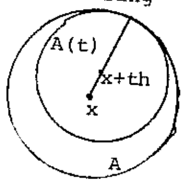
with the convention that the infimum over an empty set is ∞. For all n, $P_a[\tau_n < \infty] = 1$ (Exercise 1, above). Use strong Markov property and (9):

$$P_a[T_n(\vartheta_{\tau_n}) < \infty] = \begin{cases} n^{-2d+4} & d \geq 3 \\ 1 & d = 2. \end{cases}$$

Thus for $d \geq 3$, $\sum_1^\infty P_a[|X(t+\tau_n)| \leq n$ for some $t] = \sum_1^\infty P_a[T_n(\vartheta_{\tau_n}) < \infty] < \infty$. Now use Borel-Cantelli lemma.

4. Let u be harmonic on R^d and $u(x) \leq \alpha|x|+\beta$ for all x where α and β are constants. Then u is affine i.e. $u-u(0)$ is linear.

<u>Hint</u>. Let $x \in R^d$ be fixed. Let A and $A(t)$ denote the balls $B(x,R)$ and $B(x+th,R)$ where h is a unit vector. Using spatial mean values, ($|B|$ denotes volume of B)



$$u(x+th) - u(x) = \left(\frac{1}{|A(t)|} - \frac{1}{|A|}\right)\int_{A(t)} u - \frac{1}{|A|}\int_{A\setminus A(t)} u$$

which, using $u(y) \leq \alpha|y|+\beta$, shows that the derivative of u in the direction h is at least equal to $\frac{u(x)}{R} - \frac{\alpha|x|+\alpha R+\beta}{R}$ Let R tend to infinity to conclude, using Liouville's theorem that $\frac{\partial u}{\partial h}$ is a constant.

5. The Bessel function $J_0(\xi)$ of order 0 may be defined by

$$J_0(\xi) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\xi \sin\theta} d\theta.$$

Show that if $\xi = a+ib$ is a root of $J_0(z) = 1$ and $u(x,y) = $ $= \operatorname{Re} e^{i\xi y} = e^{-by}\cos ay$ then u has the "one circle mean value property" i.e. for each $z_0 = (x_0, y_0)$

$$u(z_0) = \int_{S(z_0,1)} u(z)\sigma(dz)$$

where $S(z_0,1)$ is the circle of radius 1 centre z_0. u is not harmonic unless $\xi = a+ib = 0$.

<u>Hint</u>. That u has the one circle mean value property follows from the definition. Since u is independent of x it cannot be harmonic unless it is a constant. That $J_0(\xi) = 1$ has infinitely many roots follows from Exercise 6 below.

6. Let f be entire and $|f(z)| \leq e^{A|z|}$ for all large enough $|z|$ for some constant A. If f has only finitely many zeroes then $f(z) = P(z)e^{\alpha z}$ for some α and polynomial P. In particular, f entire, $|f(z)| \leq e^{A|z|}$, f bounded on the real axis implies $f(z) - a = 0$ has infinitely many roots for every complex a for which the equation has at least one root.

<u>Hint</u>. If f has only finitely many zeroes we can write $f(z) = p(z)e^{G(z)}$ where G is entire and p is a polynomial. If $G = u+iv$ the conditions on f imply $u(z) \leq A|z|$ for all large $|z|$. By Exercise 4 u is linear and this implies $G(z) = \alpha z$. If f is bounded on the real axis, p must reduce to a constant and α must be purely imaginary, i.e. $f(z) = Ae^{i\alpha z}$, α real. But then f never vanishes. For any a, we can apply the above reasoning to $f(z) - a$ to conclude that if $f(z) - a = 0$ has one solution it has infinitely many. Since $J_0(0) = 1$ and $|J_0(x)| \leq 1$ for x real we conclude $J_0(x) = 1$ has infinitely many roots.

Theorem 1, §2 is needed for Exercise 7 below.

7. Let D be a bounded domain in $\mathbb{R}^d$ for which every point of ∂D is regular. If u is continuous on $\bar{D}$, for each $x \in D$ there is a ball (depending on x), $B(x,r)$ such that $u(x) = \int_{S(x,r)} u(z)\sigma(dz)$ then u is harmonic in D.

Hint. Let v be harmonic with boundary values $u|\partial D$. Then v is continuous on $\bar{D}$; $w = u - v$ is continuous on $\bar{D}$ and vanishes on ∂D. Since for each x there is a ball such that the mean value over it of w is equal to $w(x)$ w cannot attain its maximum or minimum inside D. This implies $w \equiv 0$.

§2. Dirichlet Problem

Given a domain D and a continuous function f on the boundary ∂D, the Dirichlet problem consists in finding a function u which is continuous on $\bar{D}$ and harmonic in D such that $u = f$ on ∂D. If D is bounded, we know from the maximum principle that there can be at most one such harmonic function. In other words if D is bounded the solution of the Dirichlet problem is unique, if it exists.

The Dirichlet problem as stated above does not always have a solution. Weshall soon give mecessary and sufficient conditions for a solution to exist. These conditions involve individual points in the boundary ∂D. Roughly speaking the complement of D should not be too small in any neighbourhood of any point of ∂D.

Let T denote the exit time from D (this is the same as saying T is the hitting time to the complement of D):

$$T = \inf(t:\ t > 0, X_t \notin D)$$

the infimum over an empty set is always by definition ∞. T is a Markov time. From the zero-one law in Chapter 2, for any a, $P_a(T = 0) = 1$ or 0.

Definition. A point $a \in \partial D$ is called regular for D^c = complement of D if $P_a(T = 0) = 1$. Otherwise it is called irregular.

Starting at a regular point the Browinian path hits the complement immediately. In this sense the complement of D is not too thin at any regular point. We have

Theorem 1. Let D be a bounded open set. The Dirichlet problem is solvable for D for all continuous f on ∂D iff every point of ∂D is regular.

For the proof we need

Proposition 2. Let D be an open set and T = exit time from D. For each $t > 0$ the function

$$P_a(T \geq t)$$

is upper semi-continuous on R^d.

Proof. It is easy to show that the stopping times $s + T(\theta_s)$ decrease to T as s decreases to zero. Thus

$$P_a(T \geq t) = \inf_{s>0} P_a(s + T(\theta_s) \geq t)$$

The function $P_a(s + T(\theta_s) \geq t) = E_a(P_{X_s}(T \geq t-s))$ is continuous on R^d for each $s > 0$, because $P_a(T \geq t-s)$ is bounded and measurable in a.

Proof of Theorem 1. Suppose every point in ∂D is regular and f any continuous function on ∂D. Let

$$u(a) = E_a(f(X_T)), \qquad a \in \bar{D},$$

where T = exit time from D. Clearly $u = f$ on ∂D. If $a \in D$, $B(a,r)$ a ball contained in D and S = exit time from $B(a,r)$ then $T = S + T(\theta_S)$ and using strong Markov property

$$u(a) = E_a(u(X_S)).$$

Since X_S is uniformly distributed on $\partial B(a,r)$ relative to P_a u has the mean value property in D. To check continuity of u at a point $a_o \in \partial D$ we need only show that for a close enough to a_o, X_T is close to a_o or just that X_T is close to a with large P_a-probability. Now given $\epsilon > 0$,

(1) $$P_a\left(\sup_{0 \le s \le t} |X_s - a| \ge \epsilon\right) \text{ is uniformly small}$$

provided t is small enough (Exercise 1). For this t, by Proposition 2, $P_a(T \ge t)$ is small when a is close to a_o. By (1) the P_a-probability that $T \le t$ and $|X_T - a| \ge \epsilon$ is small. Thus X_T is close to a (with large P_a-probability) if a is close to a_o.

Conversely suppose that the Dirichlet problem is solvable for D for all continuous boundary data. Let $a_o \in \partial D$ and f be any non-negative continuous function on ∂D which vanishes exactly at a_o. By assumption there is a continuous function u on $\bar{D}$ which is harmonic in D and whose restriction to ∂D is f. From (5) §1 we have

(2) $$u(a) = E_a(f(X_T)), \qquad a \in \bar{D}.$$

Taking $a = a_0$ in (2) we conclude $P_{a_0}(T = 0) = 1$.

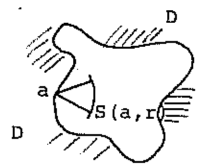
In view of the above theorem it is natural to seek conditions for regularity of boundary points. But first a few remarks.

Remarks. In the first part of the proof of the above theorem we have infact shown: for any open set D and any bounded continuous function f on ∂D the function u defined on $\bar{D}$ by $u(a) = E_a(f(X_T);\ T < \infty)$ is harmonic in D and continuous at every regular point in ∂D. Of course $u(a) = f(a)$ if $a \in \partial D$ is regular. Our second remark is that regularity is a local property:

A point $a \in \partial D$ is regular for D^c iff it is regular for $(D \cap U)^c$ where U is any open neighbourhood of a. Indeed if a is irregular the Brownian path starting at a remains in D for a short length of time so that it remains in $D \cap U$ for a short length of time. And conversely. Exercise 4 claims that the set of regular points is a non-empty G_δ-subset of ∂D.

The following is a nice geometrical condition guaranteeing regularity. It is called the Poincare cone condition.

Proposition 3. Let D be open and $a \in \partial D$. If there is a cone with vertex a contained in D^c then a is regular.



Proof. If $B(a,r)$ is the ball with centre a and radius r and T = exit time from $B(a,r)$ $P_a(X(T_r) \in D^c \cap S(a,r))$ = the uniform measure of $D^c \cap S(a,r) \ge$ the uniform measure of the part of $S(a,r)$ in the cone; and this measure is indpendent of r. Now T = the exit time from D, clearly cannot exeed T_r on the set $X(T_r) \in D^c$. Thus $P_a(T \leq T_r)$ is bounded below by a fixed constant. Letting r tend to zero we see that $P_a(T=0) > 0$. Q.e.d.

The above cone condition shows that if the boundary of an open set D is piecewise smooth then every point of ∂D is regular. In particular the Dirichlet problem is solvable for open sets with piecewise smooth boundaries.

The union of two open sets with regular boundaries need not have regular boundary: It is easy to find finite number of balls whose union is the unit ball punctured at the origin. However every open set is easily seen to be an increasing union of open sets with piecewise smooth boundaries.

Because the Dirichlet problem is not always solvable, Wiener formulated the socalled modified Dirichlet problem: Given a bounded open set D and a continuous function f on ∂D find a function u which harmonic in D and

$$\lim_{D \ni x \to b} u(x) = f(b)$$

for every $b \in \partial D$ that is regular for D^c.

We have seen (Remark after the proof of Theorem 1) that such a function always exists. Wiener established the existence of a solution by a limit procedure: He wrote the open set as the increasing union of open sets with smooth boundaries and showed that solutions for these open sets converged to a solution for the union. The problem of uniqueness seems to have defeated Wiener. The maximum principle cannot be applied and a much more delicate technique is needed. O.D. Kellog established uniqueness for $d=2$ and G. Evans for $d \geq 3$. We shall return to this in Chapter 5.

Examples

1. Poisson integral formula. Direct calculation shows that u defined by

$$u(b) = \int_{S(a,r)} P_r(b,z) f(z) p_r(dz) \tag{3}$$

solves the Dirichlet problem for the ball $B(a,r)$ with boundary data a continuous function f on $S(a,r)$. Here the Poisson kernel $P_r(b,z)$ is

$$P_r(b,z) = r^{d-2} \frac{r^2 - |b-a|^2}{|b-z|^d}, \qquad d \geq 2 \tag{4}$$

and $p_r(dz)$ is the uniform distribution on $S(a,r)$. See Exercise 6 for a verification. The maximum principle of course implies that a function u which is continuous on $\bar{B}(a,r)$ and harmonic in its interior is given by (3).

The formula (3) is very useful. As an illustration let us show that u is positive and harmonic in open ball $B(0,1)$ iff

$$u(b) = \int P(b,z) m(dz), \qquad |b| < 1 \tag{5}$$

for a unique positive bounded measure m on $S(0,1)$, here $p(b,z)$ is given by (4) with $r=1$ and $a=0$. Indeed for $t<1$, the mean value property applied to $B(0,t)$ shows that $u(z)p_t(dz)$ are uniformly bounded measures on the closed ball $B(0,1)$, $p_t(dz)$ being the uniform measure on $S(0,t)$. Let m denote any weak limit. Apply the formula (3) for $B(0,t)$ and let t tend to 1 to get the above representation (5). For uniqueness see Exercise 12. For other applications on the formula see Exercises 8 and 11.

Poisson formula for the half-space. If f is bounded and cnontinuous on R^d

$$(6)\qquad u(x,t) = \Gamma(\tfrac{d+1}{2})\pi^{-\frac{d+1}{2}}\int_{R^d}\frac{t}{(|x-y|^2+t^2)^{\frac{d+1}{2}}}f(y)\,dy$$

is harmonic in the half space $((x,t): x\in R^d, t>0)$ continuous on its closure and equals f on its boundary. The constant in (6) is chosen so that $u=1$ when $f=1$. The verification is similar to that in Exercise 6. For a "derivation" of the formula (6) see Excercise 7.

If we replace $f(z)p(dz)$ and $f(y)dy$ by $m(dz)$ where m is a positive in (3) and (6) the resulting functions are harmonic in the ball and half space respectively unless they are identically infinite. This is easy to see. Harnack's theorem in §1 shows that if D is a connected open set and f is non-negative measurable on ∂D then u defined in D by $u(a) = E_a[f(X_T): T<\infty]$ is harmonic in D provided it is finite at one point; T is as usual the exit time from D.

For any constant c, $ct+u(x,t)$ with u defined by (6) is harmonic in the half space $((x,t): x\in R^d, t>0)$ and assumes the same boundary values as u, namely f. However the only bounded solution is given by (6); we shall see this in Corollary 3 §3. Granting this consider the following application: For all $\alpha\in R^d$ the function $\exp(-|\alpha|t+i\alpha\cdot x)$ is harmonic on R^{d+1} and in particular on the half space $((x,t): t>0)$ with boundary values $e^{i\alpha\cdot x}$; here $\alpha\cdot x$ denotes inner product. We must have

$$\exp(-|\alpha|t+i\alpha\cdot x) = \Gamma(\tfrac{d+1}{2})\Pi^{-\frac{d+1}{2}}\int_{R^d}\frac{t\,e^{i\alpha\cdot y}}{(|x-y|^2+t^2)^{\frac{d+1}{2}}}\,dy$$

Taking $x=0$ we see that the probability measures

$$F_t(dy) = \pi^{-\frac{d+1}{2}}\Gamma(\tfrac{d+1}{2})\,t(|y|^2+t^2)^{-\frac{d+1}{2}}\,dy$$

on R^d form a semigroup (under convolution). These are the socalled symmetric Cauchy distributions on R^d. We also see that the Fourier transform of F_t is $e^{-|\alpha|t}$. See p.p. 69-72 Feller [3].

As another application let Z_k be an infinite sequence of complex numbers with $\operatorname{Im}Z_k\neq 0$. Assume that $\lim Z_k = Z_0$ exists and $\operatorname{Im}Z_0\neq 0$. Let S be the set of functions on R^1, $S=(\frac{1}{x-Z_k},\frac{1}{x-\bar{Z}_k}, k\geq 1)$. We claim that the linear span of S is dense (with regard to uniform norm) in the space of all complex continuous functions on R^1 which vanish at ∞. To show this let m be any real bounded measure for which

$$\int\frac{m(dx)}{x-Z_k} = 0 \qquad k\geq 1.$$

The function $F(Z) = \int\frac{m(dx)}{x-Z}$ is holomorphic on $\operatorname{Im}(Z)>0$. (Assuming $\operatorname{Im}Z_0>0$, otherwise look at $\bar{Z}_k$), $F(Z)=0$ for an infinite set of values Z with limit point, namely the set $\{Z_k, Z_0\}$. It follows that $F\equiv 0$. Taking real parts,

$$\int\frac{t}{(x-y)^2+t^2}m(dy) = 0, \qquad t>0.$$

This implies $m=0$. Indeed if f is continuous and bounded, using Fubini

$$\int m(dy)\int\frac{t}{(x-y)^2+t^2}f(x)\,dx = 0, \quad t>0.$$

As t tends to zero the inner integral tends to $\pi f(y)$ proving that $\int f(y)m(dy)=0$ i.e. $m=0$.

2. Coordinate maps on R^d are clearly harmonic. In R^1 the only harmonic functions are linear. The real part of any holomorfic function on R^2 is harmonic, so is the logarithm of its modulus provided it does not vanish. In general solutions of the Dirichlet problem are given in series using the method of separation of variables as the following example illustrates.

3. The function $\exp(\pm|n|y)\sin nx$ is harmonic on

$$R^d = \{(x,y): x \in R^{d-1},\ y \in R^1\}.$$

Here $n = (n_1,\ldots,n_{d-1})$, $x = (x_1,\ldots,x_{d-1})$, $\sin nx = \prod_1^{d-1} \sin n_i x_i$, $|n|^2 = \Sigma n_i^2$.

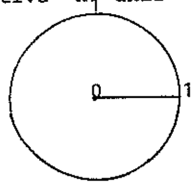
Superposition of such functions solves the Dirichlet problem for the cube. Thus

$$u(x,y) = \Sigma (A_n e^{-|n|y} + B_n e^{ny}) a_n \sin nx$$

is harmonic in the domain $\{(x,y): 0 < x_i < \pi,\ 0 < y < 1\}$ and assumes the value 0 on the boundary except on the side $(y = 0)$ where it assumes the value $h(x)$; A_n, B_n are found from: $A_n + B_n = 1$, $A_n e^{-|n|} + B_n e^{|n|} = 0$ and a_n are the Fourier coefficients of h:

$$h(x) = \Sigma a_n \sin nx.$$

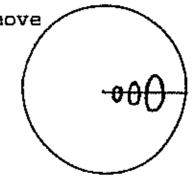
4. A very trivial example of an irregular boundary point is obtained when we remove zero from the plane. A more serious example is the following: Let $D = B(0,1) \setminus \{x: x \in R^3,\ x_1 \ge 0,\ x_2 = x_3 = 0\}$ i.e. D is the domain obtained by removing the non-negative x_1-axis from the unit ball in R^3. See figure. We claim that 0 (in fact every point $(x_1,0,0)$, $0 \le x_1 < 1$) is irregular for D^c. Indeed writing the



3-dimensional Brownian motion $X(t) = (X_1(t), X_2(t), X_3(t))$, $P_0(X_t \in x_1$-axis for some $t) = P_{(0,0)}((X_2(t), X_3(t)) = (0,0)$ for some $t) = 0$ since $(X_2(t), X_3(t))$ is the 2-dimensional Brownian motion and by Exercise 2, section 1, it never hits any point.

The same example works in any dimension $d \ge 3$. The case $d = 2$ is very different and much harder. Here is an example.

From the opne unit desc in the plane we will remove small discs along the x-axis so that for the resulting open set 0 becomes irregular.



To see how this can be done note that the probability of hitting, starting at 0, a disc $B(a,r)\ (r < |a|)$ before exiting from the unit disc is, since $B(a,|a|+1)$ contains the unit disc), less or equal to the probability, starting at 0, of hitting $B(a,r)$ before exiting from $B(a,|a|+1)$. This last is by (8) section 1,

$$\frac{\log(1+|a|) - \log|a|}{\log(1+|a|) - \log r}$$

Let a_n be a sequence, $a_n \downarrow 0, r_n < a_n$ so that say, $[\frac{1}{a_n}+1]^{2^n} \le \frac{1}{r_n} + \frac{a_n}{r_n}$. Let $D = B(0,1) \setminus \{\bigcup_n B(a_n,r_n) \cup \{0\}\}$. From the above we see

$$\sum_n P_0[\text{hitting } B(a_n,r_n) \text{ before exiting from } B(0,1)] < \infty.$$

That is, with P_0-probability 1 the Brownian path hits only finitely many $B(a_n,r_n)$ before hitting $\{x: |x| = 1\}$. This clearly means that 0 is irregular for D^c.

The above is aolso an example of an irregular point whose "harmonic measure has infinite energy". We shall return to this in Chapter 7.

Exercises to §2.

1. Show that $\frac{1}{t}P_a(\sup_{0\le s\le t}|X_s - a| \ge \epsilon)$ tends uniformly to zero as t tends to zero.

Hint. $P_a(\sup_{0\le s\le t}|X_s - a| \ge \varepsilon) = P_0(\sup_{0\le s\le t}|X_s| \ge \varepsilon)$. Now consider the fourth moment of X_t.

2. Let D be open and T = exit time from D. For every compact $K \subset D$, $\lim_{t\to 0}\sup_{a\in K} P_a(T \le t) = 0$.

Hint. Let ϵ = distance of K to the boundary of D in Exercise 1.

3. Let D and T be as in Exercise 2. Show that for each bounded Borel measurable f on D and $t > 0$, $E_a(f(X_t): t < T)$ is a continuous function of a in D.

Hint. $T = s + T(\theta_s)$ on the set $T > s$. So for all $s < t$

$$E_a(f(X_t): t < T) = E_a[E_{X_s}(f(X_{t-s}): t-s < T): s < T]$$

$$= E_a[E_{X_s}(f(X_{t-s}): t-s < T)] - E_a[E_{X_s}(f(X_{t-s}): t-s < T): T \le s]$$

The first term is continuous on R^d and the last term tends uniformly to zero on compact subsets of D from Exercise 2.

4. Let D be an open set. Show that the set of regular points in ∂D is a nonempty G_δ-subset.

Hint. The set $\bigcap_n (a: a \in \partial D,\ P_a(T < \frac{1}{n}) > 1 - \frac{1}{n})$ is the set of regular points. Each set of the intersection is open in ∂D by

Proposition 2. To establish nonemptiness we may assume D is bounded. Let T, S be the exit times from D and $\bar{D}$ respectively and $a \in D$. $T \le S$ and $S(\theta_S) = 0$ if $S < \infty$ so that every point X_S is regular for D, P_a-almost surely.

5. Let D be any open set and $a_0 \in \partial D$ be irregular. Then a_0 is in boundary of a connected component of D.

Hint. Starting at a_0 the continuous Brownian path is in D for a short length of time.

6. Verify that the Poisson integral formula does indeed solve the Dirichlet problem for the ball.

Hint. Assume $a = 0$. For fixed z, $|b-z|^{-d+2}$ is harmonic in b except at z. By differentiation $(z_i - b_i)|b-z|^{-d}$ is harmonic except at z. Multiply by $2z_i$ and add: $(2|z|^2 - 2(z,b))|b-z|^{-d}$ is harmonic except at z. Now subtract $|b-z|^{-d+2}$ to see that $(|z|^2 - |b|^2)|z-b|^{-d}$ is harmonic except at z. Thus u defined by (3) is certainly harmonic in $B(0,r)$. Improving the continuity of u on the closed ball the only non-routine verification is that $u = 1$ if $f = 1$. This is verified by noting that u is then rotation invariant and that a function which is harmonic in a neighbourhood of 0 and which depends only on distance is a constant: $\Delta u = \frac{d^2u}{d\gamma^2} + \frac{d-1}{\gamma}\frac{du}{d\gamma}$ if $u(a) = u(|a|)$, $|a| = r$.

7. Let D be the half spece $((a,b): a \in R^{d-1}, b > 0)$ and T = exit time from D. Show that the distribution of X_T relative to $P_{a,b}$ has density $\pi^{-d/2}\,\Gamma(d/2)\,b(|a-x|^2+b^2)^{-d/2}$.

Hint. Write the d-dimensional Brownian motion as (X_t, Y_t) where X_t is (d-1)-dimensional and Y_t 1-dimensional. The exit time T from the half space is simply $\inf(t: t>0, Y_t=0)$. Relative to $P_{(a,b)}$ the processes $(X.),(Y.)$ are independent. In Chapter 2 §2 the destribution of T has been found using the first passage time relation. Thus the variable (X_T, Y_T) has, relative to $P_{(a,b)}$, the (d-1)-dimensional density

$$\int_0^\infty \frac{1}{(2\pi t)^{\frac{d}{2}-1}} \exp\left(-\frac{|x-a|^2}{2t}\right)\frac{1}{\sqrt{2\pi}}\int_0^b \left(\frac{1}{t^{3/2}}-\frac{y^2}{t^{5/2}}\right)e^{-y^2/2t}\,dy\,dt$$

Integrate by parts the integral $\int_0^b \exp\left(-\frac{|x-a|^2+y^2}{2t}\right)dy$ to see taht the required density is equal to

$$b(2\pi)^{-d/2}\int_0^\infty t^{-\frac{d+2}{2}}\exp\left(-\frac{|x-a|^2+b^2}{2t}\right)dt$$

which is easily evaluated.

8. (Harnack's inequalities) Let u be positive and harmonic in $B(o,\gamma)$ Then for $|a|<\gamma$

$$\gamma^{d-2}\frac{\gamma^2-|a|^2}{(\gamma+|a|)^d}u(0)\le u(a)\le \gamma^{d-2}\frac{\gamma^2-|a|^2}{(\gamma-|a|)^d}u(0).$$

Hint. Use the Poisson integral formula.

9. (Generalised Harnack inequality). Let D be connected and open and K a compact subset of D. There exists a number M depending only on K and D such that for all positive harmonic functions u on D

$$\frac{u(a)}{u(b)}\le M, \quad a,b\in K.$$

Hint. Use Exercise 7 above and cover K suitably by balls.

10. Let u_n be positive harmonic functions in a connected open set D. If for some $x_0\in D$ $\lim u_n(x_0)=0$ then u_n converges to zero uniformly on compact subsets of D.

Hint. Use Exercise 9.

11. Show that a harmonic function is real analytic.

Hint. Use Poisson integral formula.

12. If for a signed measure m on $S(0,1)$

$$v(b) = \int P(b,z)m(dz) = 0 \qquad |b|<1$$

then $m=0$. $P(b,z)$ is defined by (4) with $a=0$ and $r=1$.

Hint. If f is continuous on $S(0,1)$ and $u(a)=\int P(a,z)f(z)p(dz)$ $p=$ uniform measure on $S(0,1)$ then $\lim_{r\to 1} u(ra)=f(a)$ boundedly for all $a\in S(0,1)$. Now use Fubini, dominated convergence and $p(a,rz)=P(ra,z)$ to get

$$\int f(a)m(da) = \lim_{\gamma\to 1}\int f(z)p(dz)v(\gamma z) = 0.$$

13. Let u be harmonic in the ball $B(0,1)$. u is the difference of two non-negative harmonic functions v and w in $B(0,1)$ iff

$$\sup_{r<1}\int |u(z)|p_r(dz) < \infty$$

$p_r(dz)=$ uniform distribution on $S(0,r)$.

Hint. Let m be the weak limit of $|u(z)|P_r(dz)$ as r tends to 1, there can only be one by Exercise 12. $P_r(b,z)$ tends uniformly to $P(b,z)$ as r tends to 1. Use this to show that

$$|u(b)| \leq \int_{S(0,1)} P(b,z)m(dz).$$

14. Let D be an open set. A uniformly bounded family of harmonic functions on D is equicontinuous on every compact subset K i.e. given $\epsilon > 0$ there exists a $\delta > 0$ such that $|u(a) - u(b)| \leq \epsilon$ for all $a, b \in K, |a-b| < \delta$.

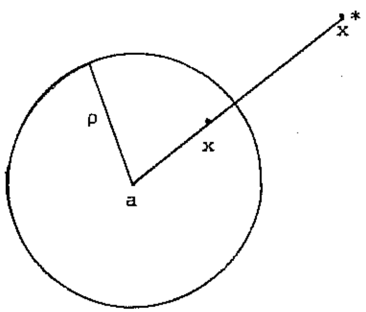
Hint. Use the Poisson integral formula and the boundedness of the family to see that the family must bo locally equicontinuous.

§3. The Kelvin transformation

Let $B(a,\rho)$ be the ball with centre a and radius ρ. For $x \in \mathbb{R}^d - a$ the point x^* defined by

$$x^* = a + \frac{\rho^2}{|x-a|^2}(x-a)$$

is called the *inverse* of x relative to $\partial B(a,\rho) = S(a,\rho)$. x^* lies on the line joining a and x and $|x^*-a| \cdot |x-a| = \rho^2$. The map $x \to x^*$ is a homomorphism on $\mathbb{R}^d - a$ onto $\mathbb{R}^d - a$. If f and g are related by



$g(x) = f(x^*)$ elementary calculations show that

$$\Delta g(x) = \frac{4}{r^4}(\Delta f)(x^*) + \frac{2\rho^2}{r^4}(2-d)\sum_j \frac{\partial f}{\partial x_j}(x^*)(x_j - a_j),$$

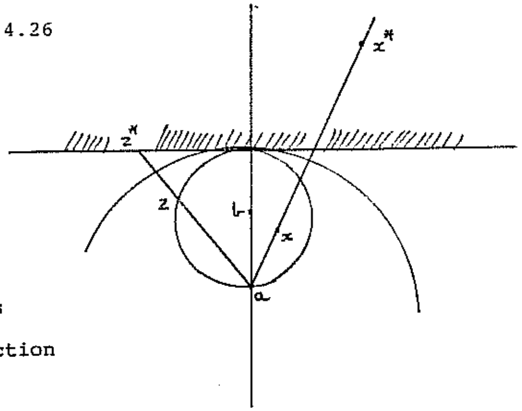
where $r = |a-x|$. Thus if $d = 2$ harmonicity of f in a neighbourhood of x^* implies that of g in a neighbourhood of x. For $d > 2$ this is false. However, the function $\frac{\rho^{d-2}}{r^{d-2}}g$ is harmonic in a neighbourhood of x if f is harmonic in a neighbourhood of x^*. The map $f \to \frac{\rho^{d-2}}{r^{d-2}}g$ is called the *Kelvin transformation* relative to $S(a,\rho)$. The factor ρ^{d-2} ensures that $f = g$ on $S(a,\rho)$. This transformation is very useful in considerations of the Dirichlet problem for unbounded domains. The following result is an example of its application. See also Exercise 4.

Theorem 1. A positive function u on the half space $D = \{(x_1, \ldots, x_d): x_d > 0\}$ is harmonic iff

(1) $$u(x) = Ax_d + x_d \int \frac{1}{|x-z|^d} m(dz)$$

for a unique positive measure m on ∂D and a non-negative constant A.

Proof. It is simple to see that u defined by (1) is harmonic if it is finite at one point. Let $a = (0,0,\ldots,0,-2)$ and consider inversion relative to $\partial B(a,2)$. Let $b = (0,0,\ldots,0,-1)$. From the figure it is clear that the half space D is mapped onto the interior of the ball $B(b,1)$, with centre b and radius 1.



The point a goes into the "point at infinity" while the rest of $\partial B(b,1)$ is mapped onto ∂D. The Kelvin transformation gives us a positive harmonic function h in the ball $B(b,1)$.

From (5) §2 there is a unique positive measure ν_1 on $S(b,1)$ whose Poisson integral equals h. Letting $C = \nu_1(a)$ and $\nu = \nu_1 - C\delta_a$ (δ_a = point mass at a) we can write

$$h(x) = \frac{1-|x-b|^2}{|x-a|^d}C + (1-|x-b|^2)\int \frac{1}{|x-z|^d}\nu(dz).$$

The map $x \to x^* = a + \frac{4}{|x-a|^2}(x-a)$ takes the measure ν into a measure μ on ∂D. Since $(x^*)^* = x$ and $h(x) = \frac{4}{|x-a|^{d-2}}u(x^*)$ we get

$$u(x^*) = \frac{1-|x-b|^2}{4|x-a|^2}C + \frac{1-|x-b|^2}{4|x-a|^2}\int \frac{|x-a|^d}{|x-z^*|^d}\mu(dz)$$

or that

$$u(x) = \frac{1-|x^*-b|^2}{4|x^*-a|^2}C + \frac{1-|x^*-b|^2}{4|x^*-a|^2}\int \frac{|x^*-a|^d}{|x^*-z^*|^d}\mu(dz).$$

Using the definition of x^*,

$$|x^*-z^*|^2 = |(x^*-a)-(z^*-a)|^2 = \frac{16}{|x-a|^2} + \frac{16}{|z-a|^2} - \frac{32}{|x-a|^2|z-a|^2}(x-a,z-a)$$

$$= \frac{16}{|x-a|^2|z-a|^2}|(x-a)-(z-a)|^2 = \frac{16}{|x-a|^2|z-a|^2}|x-z|^2.$$

And

$$|x^*-b|^2 = |x^*-a|^2 + |b-a|^2 - 2(x^*-a,b-a)$$

$$= |x^*-a|^2 + 1 - \frac{8}{|x-a|^2}(x-a.b-a)$$

$$= |x^*-a|^2 + 1 - \tfrac{1}{2}|x^*-a|^2(x_d+2)$$

giving $1-|x^*-b|^2 = \frac{1}{2}x_d|x^*-a|^2$. Thus

$$u(x) = \frac{c}{8}x_d + \frac{1}{8}x_d\int \frac{1}{|x-z|^d}|z-a|^d\mu(dz)$$

which is of the form claimed. The uniqueness follows from Proposition 2 below. Q.e.d.

<u>Proposition 2</u>. Let m be a signed measure on R^d such that

(2) $$\int \frac{1}{(1+|x|^2)^{\frac{d+1}{2}}}|m|(dx) < \infty$$

where $|m|$ is the total variation measure of m. Then

(3) $$v(x,t) = ct\int \frac{1}{(|x-y|^2+t^2)^{\frac{d+1}{2}}}m(dx), \quad c = \frac{\Gamma(\frac{d+1}{2})}{\pi^{\frac{d+1}{2}}}$$

is harmonic in the half space $\{(x,t): x \in R^d,\ t > 0\}$ and m is the weak limit of $v(x,t)dx$ i.e.

(4) $$\lim_{t\to 0}\int f(x)v(x,t)dx = \int f(x)m(dx).$$

for every continuous function f with compact support.

Proof. Note that the Poisson integral $P(t,x,f)$ of f i.e. the expression given by (6) of §2 converges to f boundedly, and for x far from the support of f, easy estimate on the denominator shows that $P(t,x,f)$ is comparable to $t|x|^{-d-1}$. The integral on the left side of (4) is by Fubini $\int m(dx)P(t,x,f)$ Because of (2) and dominated convergence the limit in (4) can be taken inside. Q.e.d.

v is called the Poisson integral of m and denoted $P(t,x,m)$.

Corollary 3. Let v be bounded and harmonic in the half space $\{(x,t): x \in R^d, t>0\}$. Then there is a unique measure m satisfying (2) such that v is given by (3). In particular if v is bounded and $\lim_{t\to 0} v(x,t) = 0$ for almost all x then $v \equiv 0$.

Proof. Suppose $|v| \le 1$. Then $1-v$ is positive and harmonic in the half space so by Theorem 1 there is a unique positive measure n such that

$$1-v(x,t) = At + P(t,x,n)$$

Since $1-v$ is bounded and n positive, A must be zero. Also $1 = P(t,x,dy)$; dy denotes Lebesque measure on R^d. Since the Poisson integral of n is finite, n satisfies (3); so does ofcourse dy. v is the Poisson integral of $dy-n$. The rest follows from Proposition 2.

We leave it to the reader to derive the Poisson integral formula for a ball using the Kelvin transformation and the Poisson integral formula for a half space (Exercise 7, section 2).

Exercises to §3

1. Show that inversion takes spheres into spheres, hyper-planes being spheres of infinite radius. Specifically show that inversion in $S(a,\rho)$ takes the sphere $S(Q,r)$ into the sphere with centre $a + \rho^2 \frac{Q-a}{|Q-a|^2-r^2}$ and radius $\frac{r\rho^2}{||Q-a|^2-r^2|}$

2. Show that two spheres one contained in the other can be transformed by inversion into concentric spheres. Specifically let $S(a,r) \subset S(0,1)$. If $Q = \alpha a$ where α is a root of

$$\alpha^2|a|^2 + (r^2-|a|^2-1)\alpha + 1 = 0,$$

then inversion in $S(Q,\rho)$, $\rho > 0$ takes $S(a,r)$ and $S(0,1)$ into concentric shperes with centre

$$Q - \rho^2 \frac{Q}{|Q|^2-1}$$

3. Show that inversion takes a line into a line iff the line passes through the centre of inversion.

4. Let $D = \{x: |x| > 1\}$. If u is harmonic in D, continuous on $\bar{D}$ and $\lim_{x\to\infty} u(x) = a$ then

$$u(x) = \int \frac{|x|^2-1}{|x-z|^d} u(z)\sigma(dz) + a\left[1 - \frac{1}{|x|^{d-2}}\right], |x| > 1$$

where σ = uniform distribution on $S(0,1)$.

Hint. Use Kelvin transformation relative to $S(0,1)$ to get a function h, which is harmonic in the unit ball punctured at 0, satisfying $\lim_{x\to 0}|x|^{d-2}h(x) = a$. For each $0 < r < 1$, h is harmonic in the shell $\{x : r < |x| < 1\}$ and continuous on its closure. Now use (7), (8) of §1 and the Poisson integral formula of Example 1, §2 to see that

$$h(x) = a[|x|^{-d+2}-1] + \int \frac{1-|x|^2}{|x-z|^d} h(z)\sigma(dz), \quad 0 < |x| < 1.$$

Transform back to arrive at the formula for u.

§4. Boundary limit theorems of Fatou

Let m be a positive measure on R^d satisfying (2) of §3. Then $P(t,x,m)$, the Poisson integral of m, defined by (3) of §3 is harmonic in the upper half space $\{(x,t) : x \in R^d,\ t > 0\}$.

Define the upper and lower derivatives of m at x by

$$\bar{D}m(x) = \limsup_{r\to 0} \frac{m(B(x,r))}{|B(x,r)|}$$

$$\underline{D}m(x) = \liminf_{r\to 0} \frac{m(B(x,r))}{|B(x,r)|}$$

where $|B(x,r)|$ denotes the volume of the ball $B(x,r)$ with centre x and radius r. The common value of the upper and lower derivatives (when they are equel) is denoted by $Dm(x)$. It is known $Dm(x)$ exists for almost all x, relative to Lebesgue measure on R^d. For a proof see Theorem 8,6 p.p. 154 of Rudin [5]. The following result is known as the "radial limit theorem" of Fatou.

Theorem 1. With the above notation and $u(x,t) = P(t,x,m)$

(1) $$\underline{D}m(x) \le \liminf_{t\to 0} u(x,t) \le \limsup_{t\to 0} u(x,t) \le \bar{D}m(x).$$

In particular for almost all x, $\lim_{t\to 0} u(t,x)$ exists and equals $Dm(x)$.

Proof. Let $x \in R^d$ be fixed. If $F(r) = m(B(x,r))$ we can write

$$u(x,t) = c\int_{[0,\infty)} \frac{t}{(r^2+t^2)^{\frac{d+1}{2}}} dF(r)$$

which when integrated by parts yields

$$u(x,t) = (d+1)c\int_{[0,\infty)} \frac{tr}{(r^2+t^2)^{\frac{d+3}{2}}} F(r)\,dr$$

(2) $$= (d+1)c\int_{[0,s]} \frac{F(\gamma)}{\gamma^d} \frac{t\gamma^{d+1}}{(\gamma^2+t^2)^{\frac{d+3}{2}}} d\gamma + (d+1)c\int_s^\infty \frac{t\gamma}{(\gamma^2+t^2)^{\frac{d+3}{2}}} F(\gamma)\,d\gamma$$

for an arbitrary but fixed $s > 0$. The last integral tends to zero ad t tends to zero because $\int_s^\infty \gamma^{-d-1} F(\gamma)\,d\gamma$ is finite as is seen from $u(x,1) < \infty$ And

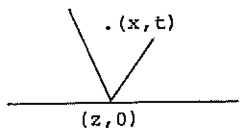
$$\lim_{t\to 0} c(d+1)\int_0^s \frac{t\gamma^{d+1}}{(\gamma^2+t^2)^{\frac{d+3}{2}}} d\gamma = c(d+1)\int_0^\infty \frac{\gamma^{d+1}}{(\gamma^2+1)^{\frac{d+3}{2}}} d\gamma = \frac{d}{\sigma}$$

where σ = surface area of the unit sphere in R^d. Thus from (2) for all $s > 0$

$$\limsup_{t\to 0} u(x,t) \le \sup_{0<r\le s} F(\gamma)\cdot\frac{d}{\sigma r^d}$$

Since $|B(x,\gamma)| = \frac{\sigma\gamma^d}{d}$, we have proved the last part of (1). The first part is similar.

We can strengthen the above theorem to include non-tangential limits: Let $K > 0$ be given. If $(x,t) \to (z,0)$ subject only to the condition that $|x-z| \leq Kt$ then for almost all z, $\lim u(x,t)$ exists and equals $Dm(z)$. In the figure one sees the restriction imposed on the mode of convergence of (x,t) to $(z,0)$ and the reason is now clear for the name "non-tangential limit".



To prove the non-tangential limit result let $m = m_a + m_s$ where m_a and m_s are the absolutely continuous and the singular parts of m. If Pm is the Poisson integral of m $Pm = Pm_a + Pm_s$. Consider first Pm_s. Because

$$(|z-y|^2+t^2)^{\frac{1}{2}} \leq (|x-y|^2+t^2)^{\frac{1}{2}} + |x-z|$$

if $|x-z| \leq Kt$

$$(|z-y|^2+t)^{\frac{d+1}{2}} (|x-y|^2+t^2)^{-\frac{d+1}{2}} \leq \left[1 + \frac{|x-z|}{(|x-y|^2+t^2)^{\frac{1}{2}}}\right]^{d+1}$$

$$\leq \left[1 + \frac{|x-z|}{t}\right]^{d+1} (1+K)^{d+1} = M \quad \text{say}$$

and we get $(Pm_s)(x,t) \leq M(Pm_s)(z,t)$. $Dm_s(z) = 0$ for almost all z. Therefore (1) implies that the non-tangential limit $(Pm_s)(x,t)$ exists and equals zero for almost all z.

Now consider Pm_a. If f denotes the density of m_a, f is locally integrable. It is known that (see Theorem 8.8. p.p. 158 of Rudin [5]) for almost all z

(3) $$\lim_{r\to 0} \frac{1}{|B(z,r)|}\int_{B(z,r)} |f(y)-f(z)|dy = 0.$$

Points for which (3) holds are called <u>Lebesque points</u> for f. If z_0 is a Lebesque point for f and we define the measure a by

$$a(dy) = |f(y)-f(z_0)|dy$$

$Da(z_0)$ exists and equals 0. As in the case of Pm_s, we have

$$(Pa)(x,t) \leq M(Pa)(z,t)$$

for any (x,t), (z,t) satisfying $|x-z| \leq Kt$. One concludes from (1) that

$$\lim (Pa)(x,t) = 0, \text{ as } (x,t) \to (z_0,0) \text{ and } |x-z_0| \leq Kt,$$

i.e. that $\lim (Pm_a)(x,t)$ exists and equals $f(z_0)$ if (x,t) tends to $(z_0,0)$ subject only to the condition $|x-z_0| \leq Kt$. This proves

<u>Theorem 2</u>. (Non-tangential limit theorem of Fatou). Let u be the Poisson integral of m. For almost all z, the following statement holds: if (x,t) tends to $(z,0)$ subject only to the condition $|x-z| \leq Kt$ where K is some positive number (which may depend on z) $\lim u(x,t)$ exists and equals $Dm(z)$.

The above non-tangential result together with a Kelvin transformation can be uset to get corresponding non-tangential limit theorems for harmonic functions in a ball. We leave the details to the reader.

An Application. Let $p \geq 1$ and $f \in L^p(-\infty,\infty)$; for each $\varepsilon > 0$ let

(4) $$H_\varepsilon(x,f) = \frac{1}{\pi}\int_{|x-t| \geq \varepsilon} \frac{f(t)}{x-t}dt.$$

We will show below that

(5) $$H(x,f) = \lim_{\varepsilon \to 0} H_\varepsilon(x,f)$$

exists almost everywhere. $H(x,f)$ is called the Hilbert transform of f and written

$$H(x,f) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt.$$

For an account of the importance of Hilbert transform in Fourier series refer to Chapter 4 of Garsia [4]. We may assume that $f \geq 0$. The Poisson integral

(6) $$u(x,y) = \frac{1}{\pi}\int \frac{f(t)\cdot y}{(x-t)^2+y^2}dt$$

is harmonic in the upper half plane $(y > 0)$ and tends as $y \to 0$ to $f(x)$ at every Lebesque point x of f. The function

(7) $$v(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{x-t}{(x-t)^2+y^2}dt$$

is called the conjugate Poisson integral of f; the nomenclature is justified because $u + iv$ is the analytic (in the half plane) function

$$\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{z-t}dt, \quad z = x + iy, \quad y > 0.$$

The conjugate Poisson kernel $\frac{1}{\pi}\frac{x-t}{(x-t)^2+y^2}$ is not integrable. However, it is bounded and is in L^q for all $q > 1$. The integral

in (7) thus makes sense for all $f \in L^p$, $p \geq 1$. Since

$$\int_{|x-t| > y} \frac{1}{(x-t)((x-t)^2+y^2)}dt = \int_{|x-t| \leq y} \frac{(x-t)}{(x-t)^2+y^2}dt = 0$$

we easily see that

$$|H_y(x,f) - v(x,y)| \leq \frac{1}{\pi}\int_{|x-t| > y} |f(t)-f(x)| \frac{y}{|x-t|[(x-t)^2+y^2]}dt$$

$$+\frac{1}{\pi}\int_{|x-t| \leq y} |f(t)-f(x)| \frac{|x-t|}{(x-t)^2+y^2}dt$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} |f(t)-f(x)| \frac{y}{(x-t)^2+y^2}dt.$$

We have shown that the right side tends as $y \to 0$ to zero provided x is a Lebesgue point for f. Thus for almost all x, $|H_y(x,f)-v(x,y)| \to 0$ as $y \to 0$. Hence we need only show that $\lim_{y \to 0} v(x,y)$ exists for almost all x. The function $e^{-(u+iv)}$ is analytic in the upper half plane and bounded by 1 since $u \geq 0$. It follows that $\lim_{y \to 0} e^{-(u+iv)}$ exists almost everywhere (the real and imaginary parts of $e^{-(u+iv)}$ are bounded harmonic functions in the upper half plane, see Exercise 1. Since $\lim_{y \to 0} u(x,y)$ exists and is finite almost everywhere one deduces that $\lim_{y \to 0} e^{iv(x,y)}$ exists almost everywhere. Since for each x, $v(x,y)$ is continuous in y, $\lim_{y \to 0} v(x,y)$ exists almost everywhere. [See Exercise 3]. This establishes the existence of the Hilbert transform. If $p > 1$, the Hilbert transform H maps L^p onto L^p is continuous and $H^2 = -I$ $(I = \text{identity})$ and for $p = 2$ it is an isometry. These non-trivial results follow from maximal inequalities. For real variables proofs see Chapter 4 og Garsia [4].

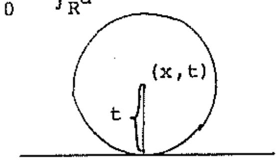
Exercises to §4

1. Let $u(x,t)$ be bounded and harmonic in the half space $\{(x,t): x \in R^d, t > 0\}$. Then $g(x) = \lim_{t \to 0} u(x,t)$ exists almost everywhere and u is the Poisson integral of g.

Hint. By Corollary 3 §3, u is the Poisson integral of a measure m. By Theorem 1 of section 4 $g(x) = \lim_{t \to 0} u(x,t)$ exists almost everywhere. Now use last part of Proposition 2 §3 and bounded convergence.

2. Let $u(x,t)$ be harmonic in the half space. u is the Poisson integral of a function $f \in L^p, 1 < p < \infty$ iff $\sup_t \|u(.,t)\|_p < \infty$, $\| \ \|_p$ denoting L^p-norm. u is the Poisson integral of a finite signed measure iff $\sup_t \|u(.,t)\|_1 < \infty$.

Hint. By the mean value property $|u(x,t)|^p \leq$ the volume average of $|u(y,s)|^p$ on the ball of radius t and centre (x,t). Therefore (see figure) $|u(x,t)|^p \leq \text{const} \int_0^{2t} ds \int_{R^d} |u(y,s)|^p dy \leq$ const t^{-d}. Hence for each $t > 0$, $u(x,t+s)$ is bounded and harmonic on the half space $\{(x,s): s > 0\}$. By Exercise 1



$$u(x,t+s) = c \int \frac{s}{(|x-y|^2+s^2)^{\frac{d+1}{2}}} u(y,s)\,dy.$$

If $\sup_s \|u(\cdot,s)\|_1 < \infty$ let m be a weak limit of the measures $u(x,s)dx$ (as s tends to zero). If $p > 1$ and $\sup_s \|u(\cdot,s)\|_p < \infty$ let f be a weak limit of $u(\cdot,s)$ as s tends to zero. Note taht the Poisson kernel is continuous, vanishes at ∞ and $\in L^q$ for all $q \geq 1$.

3. Let $-\infty \leq \lambda < \mu \leq \infty$ and f be real and continuous on the open interval (λ, μ). If $\lim_{x \to \lambda} e^{if(x)}$ exists so does $\lim_{x \to \lambda} f(x)$ exist as a finite limit.

§5. Spherical harmonics

To a function on the circle corresponds its Fourier series. The functions $e^{in\theta}$ are the restrictions to the unit circle of z^n whose real and imaginary parts are homogeneous harmonic polynomials. Any homogeneous polynomial of degree n in $x_1, \ldots, x_d$ is called a n'th solid harmonic of order n in d-dimensions. Its restriction to the unit sphere is called a n'th surface or spherical harmonic of order n. For example a constant is a solid harmonic of order 0, $\Sigma a_i x_i$ (a_i constant) is the most general solid harmonic of order 1, $\Sigma a_{ij} x_i x_j$ with $\Sigma a_{ii} = 0$ is the most general solid harmonic or order 2. and so on. In this section we shall breifly look at some properties of surface harmonics and expand a function in terms of surface harmonics analogous to Fourier expansion on the circle.

Our starting is the Poisson Kernel for the unit ball:

(1) $$P(x,z) = (1-|x|^2)|x-z|^{-d}, \quad |x| < 1, |z| = 1.$$

Writing $\mu = \frac{x}{|x|} \cdot z$ = inner product of $\frac{x}{|x|}$ and z, and $\gamma = |x|$, $P(x,z)$ takes the form

(2) $$(1-\gamma^2)[1-2\mu\gamma + \gamma^2]^{-d/2}$$

If $|2\mu\gamma - \gamma^2| < 1$, we can expand the function in (2) as a power series in $(2\mu\gamma - \gamma^2)$. It is however more useful to write this a power series in γ. This involves rearrangement of terms, which is justifiable if the series is absolutely convergent. Absolutely convergence is clearly guaranteed if $2|\mu\gamma| + |\gamma^2| < 1$ which $(|\mu| \le 1)$ is valid if $|\gamma| < \sqrt{2} - 1$.

Thus if $\gamma < \sqrt{2} - 1$, we can expand the function in (2), as a power series in $\gamma(2\mu-\gamma)$ and rearrange the series as a power series in γ. It is clear that the coefficient of γ^n is a polynomial in μ of degree exactly n:

(3) $$\frac{1-\gamma^2}{(1-2\mu\gamma+\gamma^2)^{d/2}} = \sum_0^\infty P_n(\mu)\gamma^n, \quad |\gamma| < \sqrt{2}-1, \quad |\mu| \le 1$$

P_n are called Gegenbauer or hyperspherical polynomials.

Let us emphasize taht the series in (3) converges uniformly in μ for $|\mu| \le 1$ if we replace each coefficient of the polynomial P_n by its absolutely value. This fact easily justifies term by term differentiation relative to μ.

Thus writing: $\mu = \frac{x}{|x|} z$

(4) $$\frac{1-|x|^2}{|x-z|^d} = \sum_0^\infty |x|^n P_n(\mu)., \quad |x| < \tfrac{1}{3}, |z| = 1$$

We shall soon show taht the above is valid for $|x| < 1$. For fixed $z, |z| = 1$, the left side in (4) is harmonic for $|x| < 1$. Since P_n is a polynomial $|x|^n P_n(\mu)$ is infinitely diffenrentiable except perhaps at $x = 0$. Term by term differentiation is justified $(\Delta(|x|^n P_n(\mu)) = |x|^{n-2}\{(1-\mu^2)P_n''(\mu) - (d-1)\mu P_n'(\mu) + n(n+d-2)P_n(\mu)$ except perhaps at $x = 0)$.

We get

$$\sum_0^\infty \Delta[|x|^n P_n(\mu) = 0, \quad |x| < \tfrac{1}{3}, \ x \neq 0.$$

$|x|^n P_n(\mu)$ is homogeneous of degree n, $\Delta(|x|^n P_n(\mu))$ is homogeneous of degree $n-2$ $(n \ge 2;\ P_0(\mu) = 1,\ |x|P_1(\mu)$ is easily seen to be a polynomial of degree 1 and hence harmonic). Such a sum cannot be identically zero unless each term is zero (because for $\frac{x}{|x|}$ and z fixed it becomes a power series in $r = |x|$). Thus for each n, $|x|^n P_n(\mu)$ is harmonic for $|x| < \frac{1}{3}$ except perhaps at 0; since it is continuous at 0 it must be harmonic at 0 at well. Since it is homogeneous of order n it must be a homogeneous polynomial of degree n. See Exercise 1 and 2. Thus we have shown

[A] For each $z \in \partial B(0,1)$, $|x|^n P_n(\frac{x}{|x|} \cdot z)$ is a solid harmonic of order n and for each $z \in \partial B(0,1)$, $P_n((x,z)$ $(|x| = 1)$ is a spherical harmonic of order n.

[A] clearly implies (s9nce the space of solid harmonics or order n is finite dimensional):

[B] For each finite measure μ on $\partial B(0,1)$

$$|x|^n \int P_n\left(\frac{x}{|x|} \cdot z\right)\mu(dz)$$

is a solid harmonic of order n and

$$\int P_n(\xi \cdot z)\mu(dz), \quad |\xi| = 1$$

a spherical harmonic of order n.

If u is continuous on the closed unit ball and harmonic inside

$$u(x) = \int_{|x|=1} \frac{1-|x|^2}{|x-z|^d} u(z)p(dz)$$

p being the uniform distribution on $|x| = 1$. Using (4)

(5) $$u(x) = \sum_0^\infty |x|^n \int P_n(\frac{x}{|x|}\cdot z)u(z)p(dz), \quad |x| < \frac{1}{3}$$

the n'th term is a solid harmonic or order n. In particular, if u is a solid harmonic of order n equation (5) expresses a homogeneous function of order n as a sum, for $|x| < \frac{1}{3}$, of homogeneous functions of order $0,1,2$, etc. We conclude that all terms except the n'th must vanish. We have thus shown

[C] If H is a solid harmonic of order n then
$$H(x) = |x|^n \int_{|x|=1} P_n(\frac{x}{|x|}\cdot z)H(z)p(dz)$$

(6) $$0 = \int_{|z|=1} P_m(\frac{x}{|x|}\cdot z)H(z)p(dz), \quad m \neq n.$$

The second identity in (6) is identical to saying that a spherical harmonic of order n is orthogonal (relative to p) to $P_m(\xi\cdot z)$ (regarded as a function of z) for each $|\xi| = 1$ and $m \neq n$ while the first identity says taht if S_n is a surface harmonic of order n

(7) $$S_n(x) = \int_{|z|=1} P_n(x\cdot z)S_n(z)p(dz), \quad |x| = 1.$$

The set of all solid harmonics of order n is finite dimensional. The first formula in (6) shows that the solid harmonics

$$|x|^n P_n(\frac{x}{|x|}\cdot z)$$

span this space as z varies on $\partial B(0,1)$. Thus we have shown

[D] There is a finite subset $F \subset \partial B(0,1)$ such that every surface harmonic S_n of order n can be written

$$S_n(x) = \sum_{z\in F} c_z P_n(x\cdot z), \quad |x| = 1$$

where c_z are constants.

We have shown in [C] that a surface harmonic of order n is orthogonal to $P_m(x,z)$ for each $|x| = 1$ and $m \neq n$. This together with [D] gives us

[E] Two surface harmonics of distinct degrees are orthogonal (with respect to p).

Let $S_{n,j}$ be a total orthonormal set of n'th spherical harmonics (i.e., $S_{n,j}$ are orthonormal relative to p and span the space of all n-spherical harmonics). For each ξ, $|\xi| = 1$, $P_n(\xi\cdot\eta)$ is as a function of η a spherical harmonic of order n. We can therefore write

$$P_n(\xi\cdot\eta) = \sum_j a_j(\xi)S_{n,j}(\eta).$$

The coefficients $a_j(\xi)$ are given by

$$a_j(\xi) = \int P_n(\xi\cdot\eta)S_{n,j}(\eta)p(d\eta).$$

(7) implies that $a_j(\xi) = S_{n,j}(\xi)$. Thus we have proved

[F] (<u>Addition theorem</u>). Let $S_{n,j}$ be a total orthonormal set of spherical harmonics of order n. Then

(8) $$\sum_j S_{n,j}(\xi)S_{n,j}(\eta) = P_n(\xi\cdot\eta), \quad |\xi| = |\eta| = 1.$$

Putting $\xi = \eta$ in (8) we get

(9) $$\sum_j (S_{n,j}(\xi))^2 = P_n(1).$$

Integrating with respect to σ and remenbering that $\int (S_{n,j}(\xi))^2\sigma(d\xi) = 1$ we get

G. $P_n(1)$ = dimension of the space of spherical harmonics of order n.

$P_n(1)$ can easily be calculated as follows. Writing

$$(1-2\mu r+r^2)^{-\frac{d}{2}} = \sum_0^\infty Q_n(\mu)r^n$$

we get

$$(1-r^2)(1-2\mu r+r^2)^{-\frac{d}{2}} = Q_0(\mu) + rQ_1(\mu) + \sum_2^\infty r^n(Q_n(\mu)-Q_{n-2}(\mu))$$

so that $P_n(\mu) = Q_n(\mu) - Q_{n-2}(\mu)$ for $n \geq 2$. $Q_n(1)$ is the coefficient of r^n in $(1-2r+r^2)^{-\frac{d}{2}}$, i.e.

$$(1-r)^{-d} = \sum Q_n(1)r^n,$$

giving $Q_n(1) = \binom{n+d-1}{n}$ and therefore

$$P_n(1) = \binom{n+d-1}{n} - \binom{n+d-3}{n-2} \qquad n \geq 2$$

$$P_1(1) = d$$

$$P_0(1) = 1.$$

Thus we have shown

H. The space of spherical harmonics of order n has dimension

$$\binom{n+d-1}{n} - \binom{n+d-3}{n-2}, \qquad (n \geq 2).$$

(8) and (9) imply, using Schwar's inequality,

$$|P_n(\xi\cdot\eta)| \leq \sum |S_{n,j}(\xi)S_{n,j}(\eta)|$$

$$\leq (\sum (S_{n,j}(\xi))^2)^{\frac{1}{2}}(\sum S_{n,j}(\eta)^2)^{\frac{1}{2}} = P_n(1)$$

proving that

(10) $$|P_n(t)| \leq P_n(1) \qquad -1 \leq t \leq 1.$$

Combining (10) and the value of $P_n(1)$ found in G we see that the series in (4) is uniformly convergent on compact subsets of the open ball $B(0,1)$. It therefore represents a harmonic function in $B(0,1)$. Thus (4) is valid for $|x| < 1$ (harmonic functions are analytic!).

One easily deduces from the above taht if u is harmonic in the ball $B(0,R)$ and continuous on its closure then for $|x| < R$

(11) $$u(x) = \sum_0^\infty \frac{|x|^n}{R^n} \sum_1^{P_n(1)} S_{n,j}\left(\frac{x}{|x|}\right) \int_{(|z|=1)} S_{n,j}(z)u(Rz)p(dz)$$

(use (8) and G). This is the expansion of u in harmonics analogous to the Fourier expansion in 2 dimensions.

As a simple application we have

Theorem 1. ("Liouville's Theorem"). Let u be harmonic on R^d and suppose that for some N, $u(x) \leq |x|^N$ for all large $|x|$. Then u is a polynomial of degree at most N.

Proof. In (11) the coefficient

$$a_{n,j} = \frac{1}{R^n}\int_{|z|=1} S_{n,j}(z)u(Rz)p(dz)$$

does not depend on R. Indeed writing rx $(r < R)$ instead of x in (11), with $|x| = 1$, multiplying by $S_{n,j}$ and integrating we get

$$\int_{|x|=1} u(rx)S_{n,j}(x)p(dx) = \frac{r^n}{R^n}\int_{|z|=1} S_{n,j}(z)u(Rz)p(dz)$$

as asserted. Let $n > N$ and $b_{n,j} = \sup_{|x|=1} |S_{n,j}(x)|$. We get

$$\int_{|x|=1} (b_{n,j} \pm S_{n,j}(x))u(rx)p(dx) = b_{n,j}u(0) \pm a_{n,j}r^n.$$

Since $b_{n,j} \pm S_{n,j}(x) \geq 0$ and $u(rx) \leq r^N$,

$$b_{n,j}u(0) \pm a_{n,j}r^n \leq b_{n,j}r^N$$

since $\int S_{n,j}(x)p(dx) = 0$ (because $\int_{|x|=1} S_{n,j}(x)\sigma(dx) =$ the value at 0 of a homogeneous polynomial of degree n).
THis inequality for all large r implies $a_n = 0$ for all $n > N$.
Q.e.d.

That P_n has degree n implies an interesting result on

harmonic functions with polynomial values on $\partial B(0,1)$. Since P_n has degree n for each n, we can write t^n as a linear combination of polynomials P_k, $0 \leq k \leq n$, i.e. we can write

$$t^n = \sum_0^n a_k P_k(t)$$

implying

(12) $$(x\cdot y)^n = \sum_{k\leq n} a_k P_k(x\cdot y), x,y \in \mathbb{R}^d.$$

For $|x| = 1$, $P_k(x\cdot y)$ (as a function of x on $S(0,1)$) is a spherical harmonic of order k for each y with $|y| = 1$. Equation (12) says that for each y with $|y| = 1$ (and hence for any $y \in R^d$) the homogeneous polynomial $Q(x) = (x\cdot y)^n$ agrees with a harmonic polynomial of degree at most n on the surface of the unit ball. It can be shown (see Exercise 3) that every homogeneous polynomial of degree n is a linear combination of polynomials of the form $Q(x) = (x\cdot y)^n$ as y ranges over R^d. We have thus shown

[I] To every polynomial of degree n corresponds a unique harmonic polynomial of degree at most n which agrees with the given polynomial on $S(0,1)$.

See Exercise 4 for the so-called Funk-Hecke formula. In the Exercises some properties of P_n as well as an application is given.

Example. When $d = 2$ from [G] there are only 2 linearly independent solid harmonics of degree n namely the real and imaginary parts of z^n.

Exercises to §5.

1. If u is bounded and harmonic in the ball $B(0,1)$ punctured at 0, it can be difined to be harmonic at 0 as well.

Hint. Let $r<1$ be fixed. For $s<|x|<r$

$$u(x) = E_x[u(X_{T_r}): T_r < T_s] + E_x[u(X_{T_s}): T_s < T_r]$$

where T_s and T_r are the hitting times to $B(0,s)$ and $\partial B(0,r)$ respectively. As s tends to zero the second term tends to zero.

2. Let u be homogeneous of degree n. If u is C^n in a neighbourhood of 0 then u is a polynomial of degree n.

Hint. $\frac{\partial u}{\partial x_i}$ is homogeneous of degree $n-1$ and is C^{n-1} in a neighbourhood of 0. Use induction.

3. (Using induction) show that every homogeneous polynomial of degree m in $(x_1,\ldots,x_d)$ is a linear combination of polynomilas of the form $((\xi\cdot\eta))^m$ as ξ varies in $\mathbb{R}^d$.

Hint. Induction will be on the number of variables. So assume the result true for all homogeneous polynomials of arbitrary degree in $d-1$ variables. It is clearly sufficient to show the result for any polynomial of the form $x_1^{i_1}\ldots x_{d-1}^{i_{d-1}}\, x_d^{i_d}$ with $i_1+\ldots+i_d=m$. By induction assumption $x_1^{i_1}\ldots x_{d-1}^{i_{d-1}}$ is a linear combination of expressions of the form $((x\cdot\xi))^{m-i_d}$, ξ ranging over R^{d-1} or ξ ranging over R^d with $\xi_d=0$.

Thus it suffices to show the result for each polynomial of the form $((x\cdot\xi))^i x_d^j$, $i+j=m$, $\xi_d=0$. If $\eta_k=\xi_k$, $k\le d-1$ and $\eta_d=\alpha_k$, $(x\cdot\eta)=(x\cdot\xi)+\alpha x_d$. We have

$$((x\cdot\xi)+\alpha_k x_d)^m = \Sigma \binom{m}{i}(x,\xi)^{m-1}\alpha_k^i x_d^i.$$

If the α's are distinct the Vandermonde determinant

$$\begin{vmatrix} 1 & \ldots & \alpha_1 & \ldots & \alpha_1^m \\ \vdots & & & & \\ 1 & & \alpha_{m+1} & \ldots & \alpha_{m+1}^m \end{vmatrix}$$

is not zero (see Bourbaki, N [2] Chapter III p.p. 99) and therefore each $\binom{m}{i}(x\cdot\xi)^{m-i}x_d^i$ is a linear combination of $((x\cdot\xi)+\alpha_k x_d)^m = (x\cdot\eta)^m$.

4. (Funk-Hecke formula). Let S_n be a spherical harmonic of order n and f continuous on $[-1,1]$. Show that

$$\int f((\xi\cdot\eta))S_n(\eta)p(d\eta) = \lambda S_n(\xi)$$

for some constant λ.

Hint. Let m be the measure on $[-1,1]$ induced by the map $\eta\to\xi\cdot\eta, \eta\in S(0,1)$. From [C], $\int_{-1}^{1} P_n(t)P_m(t)m(dt)=0$ if $n\neq m$. Since t^n is a linear combination of $P_K, k\le n$ $\{P_K(t)\}$ is total in $L^2(dm)$. If $f\in L^2(dm)$

$$f = \Sigma a_n P_n \qquad (L^2\text{-sense})$$

i.e. $f(\xi\cdot\eta) = \Sigma a_K P_K(\xi\cdot\eta)$. Now use (7) and [E]

5. Show that P_n satisfy

$$(1-t^2)P_n''(t) + (1-d)tP_n'(t) + n(n+d-2)P_n(t) = 0.$$

<u>Hint</u>. Use that $|x|^n P_n(\frac{x}{|x|}\cdot y)$ is harmonic.

<u>Remark</u>. The differential equation satisfied by P_n may be written

$$\frac{d}{dt}[(1-t^2)^{\frac{d-1}{2}} P_n'] + n(n+d-2)(1-t^2)^{\frac{d-3}{2}} P_n = 0.$$

This has the familar Strum-Liouville form and it is general knowledge (one can verify this directly, see for instance Birkhoff [1] Chapter X) that P_n must be orthogonal on $[-1,1]$ relative to the weight function $(1-t^2)^{\frac{d-3}{2}}$. If m is the probability measure given by the map $y \to x\cdot y$ of $S(0,1)$ onto $[-1,1]$, we also know that P_n are orthogonal relative to m. This implies m has density $c(1-t^2)^{\frac{d-3}{2}}$, c being chosen to make the integral 1; indeed one sees inductively that $\int_{-1}^{1} t^n dm = c\int_{-1}^{1}(1-t^2)^{\frac{d-3}{2}} t^n dt$.

6. Show that every function u which is harmonic in a shell $(x: r < |x| < R)$ can be written $u = v + W$ where v is harmonic in $(x: |x| < R)$ and w is harmonic in $(x: |x| > r)$.

<u>Hint</u>. Using Kelvin transformation we see that $|x|^{-n-d+2} S_{i,n}(\frac{x}{|x|})$ is harmonic except at the origin, where $S_{i,n}$ is a spherical harmonic of order n. For the exercise it is clearly enough to assume that u is continuous in the closed shell (otherwise just look at smaller shells) with boundary values $f(x)$ on

$|x| = r$ and $g(x)$ on $|x| = R$. Consider formally the series

$$\sum_{n=1}^{\infty} \sum_i (a_{i,n}|x|^n + b_{i,n}|x|^{-n-d+2}) S_{i,n}(\frac{x}{|x|})$$

where for each n, $S_{i,n}$ form a complete set of orthonormal spherical hermonics of degree n. Compute the "Fourier coefficients" $a_{i,n}$ and $b_{i,n}$ by

$$a_{i,n}r^n + b_{i,n}r^{-n-d+2} = \int_{|x|=1} f(rx) S_{i,n}(x)\, p(dx)$$

$$a_{i,n}R^n + b_{i,n}R^{-n-d+2} = \int_{|x|=1} g(Rx) S_{i,n}(x)\, p(dx)$$

where p is the uniform distribution on the unit sphere $(x: |x| = 1)$. It is simple to show that the series

$$v(x) = \sum_{n\geq 1} |x|^n \sum_i a_{i,n} S_{i,n}(\frac{x}{|x|})$$

converges uniformly on compact subsets of $(x: |x| < R)$, and

$$h(x) = \sum_{n\geq 1}^{\infty} |x|^{-n-d+2} \sum_i b_{i,n} S_{i,n}(\frac{x}{|x|})$$

converges uniformly on compact subsets of $|x| > r$. Thus $v(x) + h(x)$ is harmonic in $(r < |x| < R)$. SInce $S_{i,n}, n = 0,1,2,\ldots$ form a complete set in $L^2(dp)$ the difference $u - h$ can only assume constant values on $|x| = r$ and $|x| = R$ and the most general such function is

$$a + b \log|x|, \qquad d = 2$$

$$a + b\,|x|^{-d+2}, \qquad d \geq 3.$$

Thus u has the expansion

$$u(x) = a + b\log|x| + \sum_{n\geq 1}^{\infty} (\sum_i a_{i,n}|x|^n + b_{i,n}|x|^{-n}) S_{i,n}(\frac{x}{|x|}), \quad d = 2$$

$$u(x) = \sum_{n=0}^{\infty} (\sum_i a_{i,n}|x|^n + b_{i,n}|x|^{-n-d+2}) S_{i,n}(\frac{x}{|x|}), \quad d \geq 3$$

R E F E R E N C E S

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For more on spherical harmonics consult:
McRobert T.M., Spherical Harmonics. Pergamon (1967)

For applications to representation theory:
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CHAPTER 5

Superharmonic functions

Introduction

Superharmonic functions can be regarded as generalizations of concave functions and bear the same relationship to harmonic functions as concave functions do to linear functions.

The theory of superharmonic functions was started by F. Riesz. For a little history we refer the reader to Rado [5] and the references cited here. In Rado [5] there is also a discussion of a wide range of applications of superharmonic functions. We shall only give some applications. In any case the reader will see in the sequel that the study of superharmonic functions more than justifies itself by its applications to the theory of harmonic functions alone.

In §1 we define and investigate some general properties of superharmonic functions. §2 is devoted to applications and §3 deals with Riesz-measures associated with superharmonic functions. §4 is concerned with continuity properties of superharmonic functions and finally in §5 we show the uniqueness of solution of the modified Direchlet problem.