CHAPTER 3

Semi groups

Introduction

In this chapter we shall give a brief and elementary exposition of semi-group theory. Whereas it is not absolutely essential to know semi-group theory to read the rest of this book a knowledge of this theory is indispensable if one wishes to proceed to more general Markov processes.

We shall give som examples which throw some light on the unifying aspect of the subject. And the patient reader may get an idea of the meaning of boundary conditions in partial differential equations.

Just to start off the following rough argument tells us how semi-groups naturaaly enter into considerations of partial differentialequations. Suppose we know that for each f in a Banach space of continuous functions on R^d the equation

$$\frac{\partial u}{\partial t} = Au, \quad u(0,x) = f(x)$$

where A is a differential operator depending only on x has a unique solution. If we put $T_t f(x) = u(t,x)$, then T_t is necessarily a semi-group. Indeed if $v(t,x) = T_{t+s} f(x) = u(t+s,x)$ then $\frac{\partial v}{\partial t} = Av$ and $v(0,x) = T_s f(x)$. We must thus have, by

uniqueness $T_{t+s} f(x) = T_t[T_s f](x)$.

As said the above is just a rough argument and is not intended to give any idea that miracles will follow. However the semi-group property of the Brownian kernel can already be used to give a simple proof of the

<u>Reisz composition formula</u>: If $0 < \alpha, \beta, \alpha+\beta < d$, then

$$I_\alpha * I_\beta = I_{\alpha+\beta}$$

where for $0 < \theta < d$ the Riesz kernel I_θ is defined by

$$I_\theta(x) = \Pi^{-\frac{d}{2}} \frac{\Gamma(\frac{d-\theta}{2})}{\Gamma(\frac{\theta}{2})} |x|^{-d+\theta}$$

with $|x|$ denoting the norm of $x \in R^d$.

It is simple to show that $I_\alpha * I_\beta = I_{\alpha+\beta} \cdot$ (constant). The nontrivial part lies in evaluating the constant. M. Riesz evaluated it first and a little later J. Deny used Fourier transforms to give a slightly simpler proof.

<u>Proof of the Reisz composition formula</u>. Writing $p(t,x) = (2\Pi t)^{-\frac{d}{2}} \exp(-\frac{1}{2t}|x|^2)$ we have for $0 < \alpha < d$

$$\int_0^\infty t^{\frac{\alpha}{2}-1} p(t,x)\,dt = c_\alpha |x|^{-d+\alpha}$$

where $c_\alpha = 2^{-\frac{\alpha}{2}} \Pi^{-\frac{d}{2}} \Gamma(\frac{d-\alpha}{2})$. Using the semi-group property of $p(t,.)$ we have thus

$$c_\alpha |x|^{-d+\alpha} * c_\beta |x|^{-d+\beta} = \int_0^\infty t^{\frac{\alpha}{2}-1} dt \int_0^\infty s^{\frac{\beta}{2}-1} p(s+t,x)\,ds$$

$$= \int_0^\infty p(s,x)\,ds \int_0^\infty t^{\frac{\alpha}{2}-1}(s-t)^{\frac{\beta}{2}-1}\,dt$$

$$= \frac{\Gamma(\frac{\alpha}{2})\,\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})} \int_0^\infty s^{\frac{\alpha+\beta}{2}-1} p(s,x)\,ds$$

A rearrangement of this equality gives us the composition formula.

§ 1.

As avery special case of the markov property we have

(1) $$E_a[f(X(t+s))] = E_a[E_{X_t}(f(X_s))]$$

for all bounded Borel measurable functions f on R^d. Let T_t denote the operator

$$T_t f(a) = E_a[f(X_t)], \qquad t \geq 0.$$

Equation (1) can then be written

$$T_{t+s} = T_t T_s.$$

i.e. the operators T_t form a semi-group. Written in terms of measures the above semi-group property is a special case of the so-called Chapman-Kolmogorov equation.

Let B be a Banach space. If $u(t)$, $a \leq t \leq b$ is a map on $[a, b]$ into B we denote by $\int_a^b u(t)dt$ the limit when it exists of $\sum_0^n u(t_k)(t_k - t_{k-1})$ where $a = t_0 < t_1 < \ldots < t_n = b$ and as max $|t_k - t_{k-1}|$ tends to zero; limit here means strong limit, i.e. limit in norm. This definition paralles the definition of the Reiman integral of a function on $[a,b]$. The integral over an infinite interval is then defined in the usual way i.e. the integral over an arbitrary compact interval in the complement

of $[a,b]$ is small provided a and b are chosen suitably. Let us simply write down the properties of integral we will need. These are all simple to verify.

1. If $u(t)$ is continuous in a compact interval $[a,b]$ then $\int_a^b u(t)dt$ exists.
2. $\| \int_a^b u(t)dt \| \leq \int_a^b \| u(t) \| dt$ ($\| x \|$ is the norm of the element $x \in B$).
3. If L is in B^*, i.e. L is continuous linear on B then $L(\int_a^b u(t)dt) = \int_a^b L(u(t))dt$.

Let $T_t, t \geq 0$ be a semi-group of linear operators on B. We will assume that T_t is <u>strongly continuous</u>:

$$\lim_{t \to t_0} \| T_t x - T_{t_0} x \| = 0. \qquad T_0 = I = \text{identify, for all}$$

$x \in B$ and $t_0 \geq 0$. It can be shown that (Exercise 8) a strongly continuous semi-group T_t satisfies $\| T_t \| \leq M e^{\beta t}$ for some $M, \beta > 0$. By considering $e^{-\beta t} T_t$ in place of T_t we may and do assume from now that $\sup_t \| T_t \| < \infty$.

Examples.

1. Using the Browian motion semi-group it is simple to derive several examples of semi-groups. For example let X_t be the one dimentional Browian motion. If f is uniformly continuous and bounded on $[0,\infty]$ define

$$S_t f(a) = E_a[f(|X_t|)]$$

S_t is then a semi-group on the Banach space of such functions and corresponds to the so-called reflected Brownian motion. As another example let f be continuous on the interval $[0,a]$

with $f(0) = f(a)$. Extend f to $(-a,a)$ to be even: $f(-x) = f(x)$ and now extend it periodically (with period 2a) to all of $\mathbb{R}$ and then define

$$H_t f(a) = E_a[f(X_t)].$$

H_t is a semigroup on the Banach space of all those continuous functions f on $[0,a]$ such that $f(0) = f(a)$. This is the semi-group corresponding to the Brownian motion "reflected at 0, and at a".

If f is continuous on $(0,\infty)$ with $f(0) = 0$, we can extend f to an odd function by defining $f(-x) = -f(x)$. Define A_t on such function by

$$A_t f(a) = E_a[F(X_t)]$$

where F is the odd extension of f. A_t is then a semi-group on the Banach space of continuous functions f on $[0,\infty]$ such that $f(0) = 0$. This is the semigroup that corresponds to "absorbing barrier at 0". Similarly if f is a function on $(0,a)$ such that $f(0) = f(a) = 0$. We can extend f to $(-a,a)$ by setting $f(-x) = -f(x)$. Then extend f periodically with period 2a. And define

$$B_t f(b) = E_b[F(X_t)], \qquad 0 < b < a$$

where F is the periodic function with period 2a obtained as above. B_t is the semigroup that corresponds to absorbing barriers at 0 and a.

2. Let H be a separable Hilbert space $\{e_n, n \in N\}$ a complete orthonormal system where N denotes either the set of all integers

or the set of non-negative integers. If λn, $n \in N$ are complex numbers with non-negative real parts define

$$T_t x = \sum_{n \in N} e^{-\lambda_n t}(x, e_n) e_n$$

where (x,y) denotes the scalar product in H. As special cases of this one may take the trigonometric system in $(-\Pi,\Pi)$, the Hermite polynomials in $(-\infty,\infty)$ (with the weightfunction $e^{-x^2/4}$), the Legendre polynomials in $(-1,1)$ etc.

3. Let H be a separable Hilbert space and A a self-adjoint operator (bounded or unbounded) with domain $D(A) \subset H$. If $E_\lambda, -\infty < \lambda < \infty$ is the resolution of identity determined by A the operators $T_t, t \geq 0$ defined by

$$T_t = \int e^{it\lambda}\, dE_\lambda$$

form a semigroup on H. (If A is bounded $T_t = e^{tA}$).

4. <u>The Kac's semi-group</u>. Let X_t be the d-dimensional Brownian motion. $K \geq 0$ a bounded measurable function on R^d. Let $B = \{f : f$ uniformly continuous bounded on $R^d\}$. B provided with the uniform norm is a Banach space. Define for $t \geq 0$

$$T_t f(a) = E_a[f(X_t) \exp(-\int_0^t K(X_s)\, ds)]$$

T_t is a semi-group on B. It is a little tricky to show that T_t maps B into B. See Exercise 8 section 2.

<u>Excercises to section 1.</u>

1. Show that Examples 1 are semi-groups and find their "transition densities".

<u>Hint</u>. If $T_t f(x) = \int f(y)p(t,x-y)\,dy$ where $p(t,z) = \frac{1}{\sqrt{2\pi t}}\exp(-z^2/2t)$ show that f even (odd) implies $T_t f$ is even (odd) and that if f has period a so does $T_t f$ for all $t \geq 0$. With notations in Examples 1 $S_t f(x) = \int_0^\infty f(y)q(t,x,y)\,dy$ where

$$q(t,x,y) = p(t,x-y) + p(t,x+y), \quad x > 0$$

$H_t f(x) = \int_0^a f(y)h(t,x,y)\,dy$ where

$$h(t,x,y) = \sum_{-\infty}^{\infty} (p(t,x-y-2ka) + p(t,x+y-2ka))$$

$$A_t f(x) = \int_0^\infty f(y)a(t,x,y)\,dy$$

$$a(t,x,y) = p(t,x-y) - p(t,x+y)$$

$$B_t f(x) = \int_0^a f(y)b(t,x,y)\,dy$$

$$b(t,x,y) = \sum_{-\infty}^{\infty} (p(t,x-y-2ka) - p(t,x+y-2ka))$$

2. Let B be the Banach space of continuous functions on $[0,1]$ and let for $f \in B$ and $t > 0$

$$T_t f(x) = \frac{1}{\Gamma(t)}\int_0^x (x-y)^{t-1} f(y)\,dy, \quad t > 0.$$

Show that $T_{t+s} = T_t T_s$, $\|T_t f - T_s f\| \to 0$ as $s \to t > 0$ but that $T_t f$ does not tend to f in B as t tends to 0 unless $f(0) = 0$.

3. Complete details in the following. Let T_t be the d-dimensional Brownian semi-group i.e. $T_t f(a) = E_a[f(X_t)]$ where X_t is the d-dimensionalBrownian motion. If f depends only on distance i.e. $f(x) = f(|x|)$, $T_t f$ also depends only on distance. Therefore the definition

$$B_t f(a) = E_a[f(|X_t|)]$$

where a denotes both a and the vector $(a,0,0\ldots,0)$, defines a strongly continuous semi-group on the set of bounded uniformly continuous functions on $[0,\infty)$.

4. There is a well known procedure to construct new semi-groups from given ones, the so called subordination procedure. Let F_t be a semi-group of probability measures on $[0,\infty)$: $F_{t+s} = F_t * F_s$, $*$ denoting convolution. Assume that $F_t \to \delta_0$ weakly as $t \to 0$. If T_t is a strongly continuous semi-group on a Banach space the "subordinated semi-group" defined by

$$S_t = \int_0^\infty T_s\, F_t(ds)$$

is also strongly continuous. As a special case let $p(t,\cdot)$ be the density of the d-dimensional Brownian semi-group and $F_t(ds)$ the semi-group of Γ-distributions:

$$F_t(ds) = 2^{-\frac{t}{2}} \frac{1}{\Gamma(\frac{t}{2})} e^{-\frac{1}{2}s} s^{\frac{t}{2}-1}\,ds$$

we obtain the densities

$$S_t(a) = \frac{|a|^{\frac{t-d}{2}}\, \pi^{-\frac{d}{2}}\, 2^{-\frac{d+t-2}{2}}}{\Gamma(\frac{t}{2})} K_{\frac{t-d}{2}}(a)$$

where $K_\nu(x)$ is the modified Hankel function $= \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$ and I_ν is the modified Bessel function

$$I_\nu(x) = \sum_0^\infty \frac{1}{n!}\frac{1}{\Gamma(\nu+n+1)}\left(\frac{x}{2}\right)^{\nu+2n}$$

The densities S_t were first introduced by Aronszajn and Smith under the name of Bessel Potentials in connection with differential problems.

Hint. From Tables Of Laplace Transforms Roberts and Kaufman Saunders Co. 1966

$$\int_0^\infty e^{-\alpha t} t^\nu e^{-a/t} dt = 2\left(\frac{a}{\alpha}\right)^{\frac{\nu+1}{2}} K_{\nu+1}(2a^{\frac{1}{2}}\alpha^{\frac{1}{2}})$$

Re $a > 0$, Re $\alpha > 0$.

5. If m is a probability measure on R^d show that $m*f(x) = \int f(x-y)\,m(dy)$ exists in L_p, $1 \le p \le \infty$ for all Borel measurable $f \in L_p$ and ($\|\ \|_p$ denoting norm) $\|m*f\|_p \le \|f\|_p$.

Hint. Suppose $f \ge 0$. For all $g \in L_q$

$$\int g(x)dx \int f(x-y)m(dy) = \int m(dy)\int g(x) f(x-y) ds$$

$$\le \|g\|_q \|f\|_p$$

proving that $\int f(x-y)m(dy) \in L_p$.

6. Let F_t, $t \ge 0$ be probability measures on R^d such that $F_t * F_s = F_{t+s}$ and $\lim_{t\to 0} F_t * f = f$, pointwise for all bounded continuous f. $T_t f = F_t * f$ then defines a strongly continuous semi-group on $L_p(R^d)$ for $1 \le p < \infty$.

Hint. If g is continuous, has compact support and $\|f-g\|_p$ is small then $\|T_t(f-g)\|_p$ is small for all t. $T_t g \to g$, pointwise as $t \to 0$ and $\|T_t g\|_p \le \|g\|_p$. Fatou implies $\lim_{t\to 0} \|T_t g\|_p = \|g\|_p$. Now use Exercise 7 below.

7. Let $f_n \in L_p$. Suppose $f_n \to f$ almost everywhere, $f \in L_p$ and $\|f_n\|_p \to \|f\|_p$. Then f_n tends to f in L_p.

Hint. Let A_n be the set where $|f_n| \le 2|f|$ and B_n the complement of A_n. If $g_n = f_n$ on A_n and zero elsewhere g_n tends to f in L_p by dominated convergence. Therefore $\int_{B_n} |f_n|^p dx$ tends to zero. Finally

$$\int |f_n - f|^p \, dx \le \int |g_n - f|^p + 2 \int_{B_n} |f_n|^p.$$

8. Let T_t, $t \ge 0$ be a strongly continuous semi-group on a Banach space B. Show that

(a) $\|T_t\|$ is bounded in every compact interval.

(b) $\lim_{t\to\infty} \|T_t\|^{\frac{1}{t}} = \inf_t \|T_t\|^{\frac{1}{t}}$.

Thus there exists M and β such that $\|T_t\| \le Me^{\beta t}$, $t \ge 0$.

Hint. (a) Put $q(x) = \sup_{t\in K} T_t x$, where K is compact. Then q is a lower semi-continuous semi-norm. Use Baire Category theorem to conclude that $q(x) \le p\|x\|$ for some p. (b) If a is larger than the inf, choose t_0 so that

$$a > \|T_{t_0}\|^{\frac{1}{t_0}}. \quad \text{For } nt_0 \le t \le (n+1)t_0$$

$$\|T_t\|^{\frac{1}{t}} \le \|T_{t-nt_0}\|^{\frac{1}{t}} a^{\frac{nt_0}{t_0}}.$$

Now use the fact from (a) that $\|T_t\|$ is bounded for $0 \le t \le 1$.

§ 2. The infinitesimal generator.

The semi-group property clearly makes semi-groups of operators hard to come by. One has to replace it by a simpler object. And this is the infinitesimal generator.

Let T_t be a strongly continuous semi-group on a Banach space B. We will assume that $\|T_t\| \le M$ for all t.

Definition. The infinitesimal generator A of T_t is defined to be the operator whose domain $D(A)$ of definition is the set of $x \in B$ such that

$$D(A) = (x\colon \lim_{t\to 0} \frac{T_t x - x}{t} \text{ exists})$$

and this limit is by definition Ax.

If $x \in D(A)$ it is easily verified that $T_t x \in D(A)$ for all t and $\frac{d}{dt} T_t x = AT_t x = T_t Ax$.

This suggests that in some sense $T_t = \exp(tA)$. If A were bounded the right side is meaningful. However this is still true in a limiting sense. See Exercise 2.

In the investigation of semi-groups a fundamental role is played by the Resolvent Operator which is defined for $\lambda > 0$ by

$$R_\lambda x = \int_0^\infty e^{-\lambda t} T_t x\, dt.$$

From our condition that $\|T_t\| \le M$ for all $t, \lambda\|R_\lambda\| \le M$. An induction argument shows that

$$R_\lambda^{n+1} x = \frac{1}{n!}\int_0^\infty t^n e^{-\lambda t} T_t x\, dt, \qquad n \ge 0$$

so that $\lambda^{n+1}\|R_\lambda^{n+1}\| \le M$ for $n \ge 0$. It is very simple to show that R_λ satisfies the resolvent equation:

$$R_\lambda - R\mu + (\lambda-\mu) R_\lambda R_\mu = 0$$

The resolvent equation shows that the range of R_λ is independent of λ. Also $R_\lambda x = 0$ for some λ implies, using the resolvent equation that $R_\mu x = 0$ for all μ. and since $\mu R_\mu x$ tends to

x as μ tends to infinity, $x = 0$.

The relation between the infinitesimal generator and the resolvent operators is contained in the folowing theorem.

Theorem 1. $D(A) = R_\lambda(B)$ and $A(R_\lambda x) = \lambda R_\lambda x - x$.

Proof. Let $u = R_\lambda x$. Let us show that $u \in D(A)$. We have

$$T_t u - u = \int_0^\infty e^{-\lambda s} T_{t+s} x\, ds - \int_0^\infty e^{-\lambda s}\, T_s x\, ds$$

$$= (e^{\lambda t} - 1)\int_t^\infty e^{-\lambda s} T_s x\, ds - \int_0^t e^{-\lambda s}\, T_s x\, ds.$$

Thus

$$\lim_{t\to 0} \frac{T_t u - u}{t} = \lambda u - x.$$

Conversly let $u \in D(A)$ and $x = \lim_{t\to 0} \frac{T_t u - u}{t}$. Since R_λ is a bounded operator and R_λ commutes with T_t we get

$$R_\lambda x = \lim_{t\to 0} \frac{T_t R_\lambda u - R_\lambda u}{t} = \lambda R_\lambda u - u$$

from what we just proved, namely that $R_\lambda u \in D(A)$ and $AR_\lambda u = \lambda R_\lambda u - u$. This shows that $u = R_\lambda(\lambda u - Au)$. Q.e.d.

It is useful to note the following facts which are completely contained in the proof.

1. $\lambda u = Au$ for some $\lambda > 0$ implies $u = 0$.
   In other words $\lambda I - A$, defined on $D(A)$ is 1-1, $I =$ identity.

2. $\lambda I - A$ is onto. Indeed for any $x \in B$, $u = R_\lambda x \in D(A)$ and $\lambda u - Au = x$.
3. $u = R_\lambda(\lambda u - Au)$ i.e. $(\lambda I - A)^{-1}$ exists and equals R_λ and we know that $\|R_\lambda^n\| \leq \lambda^{-n}M$.
4. Two semi-groups with the same infinitesimal generator are identical. Indeed from 3. the semi-groups must have the same resolvent and therefore the uniqueness of Laplace transforms implies the identity of the semi-groups.
5. A is a closed operator: $D(A) \ni u_n \to u, Au_n \to y$ umply $u \in D(A)$ and $Au = y$. Indeed if $u_n = R_1 x_n$, $Au_n = u_n - x_n$. Our assumptions say that u_n and x_n conver Now recall that R_1 is bounded. In particular we see from the closed theorem that $D(A) = B$ iff A is bounded.

Theorem 2. Let B be a Banach space and A a linear operato defined on a dense subspace of B. A is the infinitesimal generator of a semi-group T_t, $\|T_t\| \leq M$ iff for all $\lambda > 0$, $(\lambda I - A)$ maps $D(A)$ onto $B, R_\lambda = (\lambda I - A)^{-1}$ exists and $\|R_\lambda^n\| \leq \lambda^{-n}M$. See Exercise 6 for a proof.

It is not always easy nor essential to know the precise domain of generator. It is sufficient most often to know "enough" elements of the domain. The examples below will illustrate this.

Examples.

1. In the Examples 1 of § 1 the semi-groups were constructed using the Brownian semi-group. It is natural that their generators should be expressible in terms of the generator of the Brownian

semi-groups plus some conditions at the boundary points. It is very simple to show (use Taylors expansion) that every c^2-function u with comapct support belongs to the domain of generator of the Brownian semi-group and $Au = \frac{1}{2}\frac{d^2u}{dx^2}$.

Retain the notation of Examples 1 §1. f belongs to the domain of generator of S_t iff $f(|\cdot|)$ (defined on all R^1) belongs to the domain of generator of the Brownian semi-group. In particular if f has compact support and $f(|\cdot|)$ is c^2 it will belong to the domain of generator of S_t. Thus the generator can loosely be described by $\frac{1}{2}u''$ with the boundary condition $u'(0) = 0$.

Similar reasoning applies to the other examples.

2. If T_t and S_t are commuting strongly continuous semi-groups i.e. $T_t S_s = S_s T_t$ for all $0 \leq s,t, C_t = T_t S_t$ is also a strongly continuous semi-group. The generator C of C_t has domain $D(C) \supset D(T) \cap D(S)$, T,S generators of T_t, S_t respectively. And for $u \in D(T) \cap D(S)$

$$Cu = Tu + Su.$$

This simple fact can be used as follows. Let $Z(t) = (x(t), y(t))$ denote the 2-dimensional Brownian motion. Let B = set of bounded uniformly continuous functions on R^2. Let T_t, S_t be the semi-groups on B:

$$T_t f(x,y) = E_x[f(x_t,y)]$$

$$S_t f(x,y) = E_y[f(x,y_t)]$$

It is easy to show that T_t and S_t commute. $C_t = T_t S_t$ is simply the 2-dimensional Brownian semi-group. Thus $u \in c^2$ with compact support impies that u is in the domain of generator of C_t and

$$Cu = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Similarly if u is c^2 with compact support in R^d, then $u \in D(A)$, A = generator of d-dimentional Brownian motion semi-group and $Au = \frac{1}{2}\Delta u$, Δ = Laplacian.

As another example of the same idea consider

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - v\frac{\partial u}{\partial x}, \quad u(0,x) = f(x)$$

where v is a constant. This is the equation of diffusion in a rod which moves with velocity v along the x-axis. The semi-group $S_t f(a) = f(a-vt)$ has infinitesimal generator $-v\frac{d}{dx}$ in the sense that if u is differentiable and belongs to the domain of generator A of S_t, then $Au = -v\frac{du}{dx}$. S_t commutes with the 1-dimensional Brownian semi-group T_t whose generator is $\frac{1}{2}\frac{d^2}{dx^2}$. Thus

$$u(t,a) = T_t S_t f(a) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(b-vt)\exp(-\frac{(b-a)^2}{2t})\, db$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(b)\exp(-\frac{(b+vt-a)^2}{2t})\, db$$

satisfies the above diffusion equation.

3. Let A be the generator of the Kac's semi-group of Example 4 § 1. Assume K (with the notation of the cited example) $\in B$. Using the computation of Exercise 8, if $v = G_\alpha f$, then $v = R_\alpha(f-Kv)$. Therefore $Av = \alpha v - f = \alpha R_\alpha(f-Kv) - (f-Kv) = \frac{1}{2}\Delta v - Kv$, where Δ denotes the generator of the Brownian semi-group.

4. THis example illustrates how semi-groups help in solving perturbed equations. Let T_t be a strongly continuous semi-group on a Banach space B. Let A denote the generator of T_t. Suppose $f:[0,\infty) \to B$ is a strongly diffentiable map. Using routine computation for each $x_0 \in D(A)$ the function

$$y(t) = T_t x_0 + \int_0^t T_{t-s} f(s)\, ds$$

satisfies $\frac{dy}{dt}(t) = Ay(t) + f(t)$. Apply this to the d-dimensional Brownian semi-group: The solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + g(t,a), \quad \lim_{t\to 0} u(t,a) = f(a)$$

is given by (for nice $g(t,a)$)

$$u(t,a) = E_a(f(X_t)) + \int_0^t E_a(g(s,X_{t-s}))\, ds$$

where X_t denotes the d-dimensional Brownian motion.

Exercises to § 2.

1. Let A be a bounded operator on a Banach space. $T_t = \exp(tA) = \sum_0^\infty \frac{t^n}{n!}A^n$ is then a strongly continuous semi-group with infinitesimal generator A.

2. Show that

$$Af(x) = \int_0^x [f(y) - f(x)]\, dy,$$

f continuous on $[0,1]$ generator a semi-group T_t. Find T_t.

Hint. A is bounded. $T_t f(x) = e^{-tx} f(x) + t\int_0^x e^{-ty} f(y)\, dy$.

3. Let S_t, T_t be commuting strongly continuous semi-groups ($S_s T_t = T_t S_s$ is assumed) with infinitesimal generators A and B. Assume that

$$\| T_t \| \le M, \| S_t \| \le M$$

for all t. Then for $x \in D(A) \cap D(B)$

$$\| S_t x - T_t x \| \le M^2 t \| Ax - Bx \|$$

Hint. $S_t - T_t = S^n_{\frac{t}{n}} - T^n_{\frac{t}{n}} = (S_{\frac{t}{n}} - T_{\frac{t}{n}})(S^{n-1}_{\frac{t}{n}} + \dots + T^{n-1}_{\frac{t}{n}})$

so that $\| S_t x - T_t x \| \le n M^2 \| S_{\frac{t}{n}} x - x - (T_{\frac{t}{n}} x - x) \|$

because $\| S^i_{\frac{t}{n}} T^j_{\frac{t}{n}} \| = \| S_{\frac{it}{n}} T_{\frac{jt}{n}} \| \le M^2$.

Now use $\frac{S_u x - x}{u}$ tends to Ax as u tends to zero.

4. Let $\| T_t \| \le M$. Show that $S_{t,h} x = \exp\left(t \frac{T_h - I}{h}\right) x$ converges strongly to $T_t x$ as $h \to 0$.

Hint. $\| S_{t,h} \| \le e^{-\frac{t}{h}} \sum \frac{t^n}{h^n n!} \| T_h{}^n \| \le M$.

$S_{t,h}$ is a semi-group with generator $\frac{T_h - I}{h}$. From Exercise 3 $S_{t,h} x$ tends to $T_t x$ (uniformly in compact intervals) for all $x \in D(A)$. And D(A) is dense.

5. Let R_λ be the resolvent of T_t. Where $\| T_t \| \le M$. Show that $T_{t,\lambda}(x) = \exp(t\lambda(\lambda R_\lambda - I))x$ tends to $T_t x$ as λ tends to infinity.

Hint. Using $\| (\lambda R_\lambda)^n \| \le M$, as in Exercise 4, $\| T_{t,\lambda} \| \le M. T_{t,\lambda}$ is a semi-group with generator $\lambda(\lambda R_\lambda - I)$. For $x \in D(A), \lambda R_\lambda Ax = \lambda A(R_\lambda x) =$

$= \lambda[\lambda R_\lambda x - x]$. And $\lambda R_\lambda Ax$ tends to Ax as λ tends to infinity. Rest as in Excercise 4.

6. Prove Theorem 2 in the following steps.

Step 1. If $R_\lambda = (\lambda - A)^{-1}$, R_λ, R_μ commute. Indeed if $x \in D(A)$ $(\lambda I - A) R_\lambda x = x$ and $(\mu I - A) R_\mu y = y$. Operate on both sides of the first equality by R_μ and let $y = R_\lambda x$ in the second. Subtracting the resulting equations one obtains $(R_\lambda - R_\mu + (\lambda - \mu) R_\mu R_\lambda x = 0$. D(A) is dense.

Step 2. Since $\| R_\lambda x \| \le M\lambda^{-1} \| x \|$ the equality $(\lambda I - A) R_\lambda x = x$, shows that $\lambda R_\lambda x$ tends to x for all x in D(A) and hence for all x.

Step 3. $\lambda R_\lambda Ax = \lambda A(R_\lambda x) = \lambda[\lambda R_\lambda - I]x$. So the semi-group $T_{t,\lambda} = \exp(t\lambda R_\lambda Ax)$ has norm $\le M$. $T_{t,\lambda}$ and $T_{s,\mu}$ commute from Step 1. And $\| T_{t,\lambda} x - T_{t,\mu} x \| \le M^2 t \| \lambda R_\lambda Ax - \mu R_\mu Ax \|$. Conclude from Step 2 that $\lim_{\lambda \to \infty} T_{t,\lambda} x = T_t x$ exists uniformly for t in a compact interval. T_t is then a strongly continuous semi-group. Letting μ tends to infinity we get for all $x \in D(A)$ $\| T_{t,\lambda} x - T_t x \| \le M^2 t \| \lambda R_\lambda Ax - Ax \|$.

Step 4. Let A_1 be the generator of T_t. Use the last inequality in Step 3 to show that $x \in D(A)$ implies $x \in D(A_1)$ and $A_1 x = Ax$.

Step 5. Conclude $D(A_1) = D(A)$. Indeed if $R^1_\lambda = (\lambda - A_1)^{-1}$, $x = (\lambda - A_1) R^1_\lambda x$, $x = (\lambda - A_1) R_\lambda x$, since by Step 4 $A_1 = A$ for $y \in D(A)$. By uniqueness $R_\lambda x = R^1_\lambda x$.

7. If f is C^∞ on the real line with compact support in the open interval $(0,\infty)$,

$$u = \int_0^\infty f(t)\, T_t\, x\, dt \in D(A)$$

and

$$Au = -\int_0^\infty f'(t)\, T_t\, x\, dt.$$

In particular $D_\infty(A)$ is dense in B where $D_1(A) = D(A)$ and for $k \geq 2$, $D_k(A) = \{u : u \in D_{k-1}(A),\ Au \in D(A)\}$ and $D_\infty(A) = \bigcap_k D_k(A)$.

8. Show that the Kac's semi-group – the semi-group of Example 4, § 1 maps B into B.

<u>Hint</u>. That T_t is a semi-group follows from Markov property. Write

$$R_\alpha f(a) = E_a[\int_0^\infty e^{-\alpha t} f(x_t)\, dt]$$

$$G_\alpha f(a) = \int_0^\infty e^{-\alpha t} E_a[f(x_t) \exp(-\int_0^t K(x_s)\, ds)]dt$$

Using $$1 - \exp(-\int_0^t K(x_s)\, ds) = \int_0^t K(x_s) \exp(-\int_s^t K(x_\theta)\, d\theta)\, ds$$

and Fubini $$R_\alpha f(a) - G_\alpha f(a) = E_a[\int_0^\infty K(x_s)\, ds \int_s^\infty e^{-\alpha t} f(x_t) \exp(-\int_s^t K(x_\theta)\, d\theta)$$

(change of variables and use Markov property)

$$= E_a[\int_0^\infty K(x_s) e^{-\alpha s} v(x_s)\, ds] = R_\alpha[Kv]$$

where $v(a) = G_\alpha f(a)$. Now both $R_\alpha f$ and $R_\alpha[Kv]$ are in B. [If $\delta > 0$, $|E_a(g(x_t)) - E_{a+h}(g(x_t))| < \varepsilon$ for all $a \in R^d$ and $t \geq \delta$ provided h is small enough showing that $R_\alpha g \in B$ for all bounded Borel g]. $G_\alpha f$ is therefore in B. $\alpha G_\alpha f$ converges

uniformly to f if $f \in B$. By the Hille-Yosida Theorem, Theorem 2, the resolvent G_α determines a semi-group S_t whose Laplace transform is G_α. Because $T_t f$ is continuous in t for $f \in B$, we must have $T_t f = S_t f$.

9. Let R_λ be operators on a Banach space B such that $\|\lambda R_\lambda\| \leq 1$. Suppose R_λ satisfies the resolvent equation and the range of R_λ is dense in B. Show that there is a unique contraction semi-group T_t on B whose resolvent is R_λ.

10. Let B be a Banach space and A an operator with dense domain $D(A)$. A is the infinitesimal generator of a strongly continuous semi-group iff there is a $\lambda_0 \geq 0$ such that for $\lambda > \lambda_0$, $R_\lambda = (\lambda I - A)^{-1}$ exists and $\|R_\lambda^n\| \leq M(\lambda - \lambda_0)^{-n}$ for all n.

<u>Hint</u>. Apply the Hille-Yosida Theorem to $A - \lambda_0 I$.

11. Let Δ denote the generator of the Brownian semi-group on $B = C_0(\mathbb{R}^d)$. = space of continuous functions vanishing at ∞. Let p be a bounded continuous function of $\mathbb{R}^d$. Show that $pI + \Delta$ generates a semi-group on B.

<u>Hint</u>. Let R_λ denote the resolvent of the Brownian kernel. $u \to R_\lambda(pu)$ is an operator on B of norm < 1 for $\lambda > \|p\|$. So $S_\lambda = (I - R_\lambda p)^{-1} R_\lambda$ exists in B for $\lambda > \|p\|$ and $\|S_\lambda^n\| \leq (\lambda - \|p\|)^{-n}$. Finally $u = S_\lambda f$ solves $\lambda u - pu - \Delta u = f$. Now use Exercise 10 above.

12. Let $p_t, t \geq 0$, be probability measures on $\mathbb{R}^d$ such that $p_t * p_s = p_{t+s}$ and $\lim_{t\to 0} p_t = \delta_0$. Then ψ defined by $e^{-t\psi(\alpha)} = \int \exp(i(\alpha,x))p_t(dx)$ is continuous and satisfies $|\psi(2\alpha)| \leq 4|\psi(\alpha)|$.

In particular $|\psi(\alpha)| = 0(|\alpha|^2)$ as $|\alpha|$ tends to infinity.

Hint. Let $\varphi(\alpha) = \int e^{i(\alpha,x)}F(dx)$ be any characteristic function. Use the identity $(1-a) + (1-b) - (1-a)(1-b) = 1-ab$ and

$$\left(\int |1-e^{i(\alpha,x)}|F(dx)\right)^2 \leq 2\int (1-\cos(\alpha,x))F(dx) \leq 2|1-\varphi(\alpha)|$$

to show that $p(\alpha) = \sqrt{|1-\varphi(\alpha)|}$ is sub-additive: $p(\alpha+\beta) \leq p(\alpha) + p(\beta)$. Use this to show that $\sqrt{|\psi(\alpha)|}$ is sub-additive. Continuity of ψ follows from: $\frac{1}{1+\psi}$ is the characteristic function of the probability measure $m(dx) = \int_0^\infty e^{-t}p_t(dx)dt$.

13. Let T_t be a convolution semi-group on $C_0(\mathbb{R}^d): T_t f = p_t * f$ where p_t are as in Exercise 12 above. Show that every c^3-functionwith compact support is in the domain of generator of T_t.

Hint. Retain the notation of Exercise 12 above. The domain of generator is precisely the set $\{m*f;\ f \in C_0(\mathbb{R}^d)\}$.

Now use Exercise 12 above and Fourier transforms

For more on convolution semi-groups see Berg and Forst [1].

14. Let T_t be a positive semi-group on $C_0(\mathbb{R}^d)$ with generator A. Suppose $D(A) \supset \mathcal{D}$ = the set of c^∞-functions with compact support and that support $(Au) \subset$ support (u) for all $u \in \mathcal{D}$. Show that $A/\mathcal{D}$ is a differential operator of order at most 2.

Hint. If $u \in \mathcal{D}$ vanishes together with all its first two partials at a point x_0 then $(Au)(x_0) = 0$. Indeed if $0 \leq P_\gamma \leq 1$ are in $\mathcal{D}$ such that $P_\gamma = 1$ in say $B(x_0,\gamma)$ = ball of radius γ and centre x_0 and with $P_\gamma = 0$ off $B(x_0, 2\gamma)$. Since $u = u\,P_1 P_\gamma$ in $B(x_0,\gamma)$,

$$|Au(x_0)| = |A(uP_1P_\gamma)(x_0)| = \lim_{t\to 0} \left|\frac{T_t(u\,P_1\,P_\gamma)(x_0)}{t}\right|$$

$$\leq 2M\gamma \lim_{t\to 0} \frac{T_t((x-x_0)^2 P_1(\cdot))(x_0)}{t}$$

since $|u| \leq M|x-x_0|^3$. The last limit is finite because $(x-x_0)^2 P_1 \in \mathcal{D}$.

Now we determine continuous functions a, a_i, a_{ij} as follows: Let φ_n be $\in \mathcal{D}$ such that $\varphi_n = 1$ in $B(0,n)$. Then $x_i\varphi_n, x_i x_j \varphi_n$ are in $\mathcal{D}$ where x_i are the coordinate functions. Define

$$a(x) = (A\varphi_n)(x),$$
$$a_i(x) = A(x_i\varphi_n)(x), \qquad x \in B(0,n), 1 \leq i,j \leq d.$$
$$a_{ij} = A(x_i x_j \varphi_n)(x).$$

Fix x_0 and let $u \in \mathcal{D}$. For all n such that support $u \subset B(0,n)$

$$u(x) = u(x_0)\varphi_n + \sum\left(\frac{\partial u}{\partial x_i}(x_0)\right)(x_i - x_{0i})\varphi_n(x)$$
$$+ \tfrac{1}{2}\sum_{i,j}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\right)(x_i - x_{0i})(x_j - x_{0j})\varphi_n(x) + u(x_0,x)\varphi_n(x)$$

where $u(x_0,\cdot) \in \mathcal{D}$ vanishes together with all its first and second partials at x_0. Therefore $A[u(x_0,\cdot)\varphi_n](x_0) = 0$ and

$$(Au)(x_0) = a(x_0)u(x_0) + \sum_i b_i(x_0)\frac{\partial u}{\partial x_i}(x_0) + \sum b_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)$$

where $\quad b_i(x) = a_i(x) - x_{0i}a(x)$

$$b_{ij}(x) = \tfrac{1}{2}[a_{ij}(x) - x_{0i}a_j(x) - x_{0j}a_i(x) + x_{0i}x_{0j}a(x)].$$

15. Incontrast to the above Exercise show that Δ^2 is the restriction to $\mathcal{D}$ of the generator of a semi-group on $C_0(\mathbb{R}^d)$.

Hint. $\exp(-t|x|^4)$ being rapidly decreasing is the Fourier transform of a rapidly decreasing function $F(t,x): \int \exp i(\xi,x)F(t,x)dx$ $\exp(-t\cdot|\xi|^4)$. The semigroup $T_t f(x) = \int f(y)F(t,x-y)dy$ has as generator a constant multiple of Δ^2.

§ 3. Potential operators.

Potential operators are in a general sense inverses of infinitesimal generators. Most important examples of these are Green functions which we will meet in Chapter 6.

Let T_t be a strongly continuous semi-group with $\|T_t\| \leq M$ on a Banach space B. R_λ will be the resolvent of T_t and A its infinitesimal generator. To investigate when A^{-1} exsists as a densely defined operator first consider the equation $Au = 0$. Using the fact that u is in the range of R_λ and the resolvent equation this is possible only if $u = \lambda R_\lambda u$ for all λ or equivalently $T_t u = u$ for all t. Next, suppose $v = Au$ is in the range of A. Then $\lambda R_\lambda v = \lambda R_\lambda Au = \lambda A(R_\lambda u) = \lambda[\lambda R_\lambda u - u]$ which clearly

tends to zero as λ tends to zero. Since $\|\lambda R_\lambda\| \leq M$, we also have: $v \in$ closure of the range of A iff $\lim_{\lambda \to 0} \lambda R_\lambda v = 0$. (Indeed if $\lambda R_\lambda v$ tends to zero as λ tends to zero, $A(R_\lambda v) = \lambda R_\lambda v - v$ so that $A(R_\lambda v)$ tends to $-v$ as λ tends to zero).

Thus A^{-1} has a densely defined domain iff $\lim_{\lambda \to 0} \lambda R_\lambda x = 0$ for all $x \in B$. Further if $v = Au$.

$$R_\lambda v = R_\lambda Au = AR_\lambda u = \lambda R_\lambda u - u$$

so that $\lim_{\lambda \to 0} R_\lambda v = -u$. Conversely if $\lim_{\lambda \to 0} R_\lambda v = u$ exsists, the resolvent equation $R_1 v - R_\lambda v + (1-\lambda)R_1 R_\lambda v = 0$ gives $R_1 v - u + R_1 u = 0$ i.e. $u \in D(A)$ and $Au = -v$. We have proved.

Proposition 1. $v \in$ closure of the range of A iff $\lim_{\lambda \to 0} \lambda R_\lambda v = 0$ A^{-1} exists as a densely defined operator iff $\lim_{\lambda \to 0} \lambda R_\lambda x = 0$ for all $x \in B$. If A^{-1} exists as a densely defined operator then $v \in D(A^{-1})$ iff $\lim_{\lambda \to 0} R_\lambda v = u$ exists and then $A^{-1}v = -u$.

Generally speaking potential operators are integral operators and thus are sometimes easier to handle. For example consider the d-dimensional Brownian semi-group as acting on the space $C_0(\mathbb{R}^d)$ of continuous functions vanishing at ∞. The above reasoning applies to this and we see that A^{-1} has a densely defined domain. It will be shown in Chapter 5 that $u \in c^2$ with compact support then

$$u(x) = A_d \int K(x-y)\Delta u(y)dy$$

where A_d are constants and $K(y) = -\log|y|$ if $d = 2$ and $K(y) = |y|^{-d+2}$ if $d \geq 3$. Thus at least A^{-1} restricted to $A(\mathcal{D})$ ($\mathcal{D}$ = all c^∞-functions with compact support) is given by an integral operator.

The most inportant theorem in this connection is a theorem of G. Hunt. To describe this we need a little termnology. Let $K(X)$ and $C_0(X)$ denote the space of all continuous functions with compact support, and all continuous functions vanishing at ∞ respectively, on the locally compact, σ-compact space X.A linear map $V: K(X) \to C_0(X)$ is said to satisfy the Principle of positive maximum if the following holds:

For every $f \in K(X)$ such that Vf attains strictly positive values

(1) $$\sup_x V f(x) = \sup\{V f(y) : f(y) \geq 0\}$$

(1) is equivalent to the following apparently stronger condition.

For every $f \in K(X)$ such that Vf attains strictly positive values

(2) $$\sup V f(x) = \sup\{Vf(y) : f(y) > 0\}.$$

Indeed suppose a and b denote the left and right sides of (2) and $b < c < a$. Put

$$A = \{x: Vf(x) \geq c\}.$$

A is compact since Vf tends to zero at infinity. Let g be any function in $K(X)$, which is strictly negative on A. For ε such that $\varepsilon \| Vg \| < \frac{a-c}{2}$, for all points $x \notin A, V(f+\varepsilon g) < \frac{a+c}{2}$, while at any $x \in A$ at which $Vf(x) = a, Vf(x) + \varepsilon Vg(x) > \frac{a+c}{2}$. This contradicts (1) since at all points of $A, f + \varepsilon g < 0$.

(2) shows that V must be a non-negative operator. For if $f \leq 0$. the right side of (4) is zero so that Vf cannot attain a strictly positive value.

Another condition equivalent to (1) is the following: For all $\alpha \geq 0$, $\lambda > 0$ and $f \in K(X)$

(3) $$\alpha + \lambda Vf + f \geq 0 \text{ implies } \alpha + \lambda Vf \geq 0.$$

Indeed if $\alpha + \lambda Vf(x) < 0$, $V(-f)$ would attain strictly positive values and, assuming (1)

(4) $$\sup \lambda V(-f) = \sup_{f \leq 0} \lambda V(-f) > \alpha$$

and from $\alpha + \lambda Vf + f \geq 0$,

$$\sup_{f \leq 0} V(-f) \leq \alpha$$

which contradicts (3).

Conversely suppose (3) holds and $\alpha = \sup(\lambda Vf + f)$. α is necessarily ≥ 0, since f and Vf tends to zero at infinity. $\alpha + \lambda V(-f) + (-f) \geq 0$, implying by (3), $\alpha \geq \lambda Vf$. In particular if x is a point at which $\alpha = \lambda Vf(x) + f(x)$ then $f(x) \geq 0$. Thus at every point at which $Vf + \frac{f}{\lambda}$ attains its' maximum, f is non-negative. Letting λ tend to infinity we obtain (1).

Theorem 2. Let V be as above. Assume that $VK(X)$ is dense in $C_0(X)$. There exists exactly one positive strongly continuous contraction semi-group P_t on $C_0(X)$ such that

$$Vf = \int_0^\infty P_t f \, dt, \ f \in K(X).$$

For every $f \in K(X)$, $Vf \in D(A)$ (A = infinitesimal generator of P_t) and $A V f = -f$.

Proof. We give the proof in a series of steps. Any missing details can easily be supplied by the reader. The general idea is to define a resolvent and then use the Hille-Yosida theorem.

Step 1. For every $\lambda > 0$ and $f \in K(X)$

(5) $$\| \lambda Vf + f \| \geq \| \lambda Vf \| .$$

Indeed from (3) if $\alpha = \inf(\lambda Vf + f)$ and $\beta = \sup(\lambda Vf + f)$ then ($\alpha \leq 0$, $\beta \geq 0$ because both f, $Vf \in C_0(X)$

$$\alpha \leq Vf \leq \beta$$

and this implies (5).

Let $0 \leq a \leq 1$ be $\in K(X)$. Define

$$V_a f = V(af), \quad f \in C_0(X)$$

From (2), if $V_a(f)$ attains strictly positive values

$$\sup V_a f = \sup_{af>0} V(af) = \sup_{f>0} V_a(f)$$

i.e. V_a also satisfies the principle of positive maximum. Thus (5) is valid with V replaced by V_a and for all $f \in C_0(X)$. Also V_a is a bounded operator since V is positive.

Step 2. For all $\lambda > 0$

(6) $$\text{Range } (\lambda V_a + I) = C_0(X).$$

That the range is closed follows from (5) with V replaced by V_a. For small λ (6) is obvious by series expansion. Since the resolventset of V_a is open it is enough to show that if the claim is valid in the open interval $(0,\lambda_0)$ it is valid for λ_0. If $\lambda V_a f + f = g$ then from (5) with V replaced by V_a

$$\| f \| \leq \| g \| + \| \lambda V_a f \| \leq \| g \| + \| \lambda V_a f + f \| = 2 \| g \|$$

$$\| \lambda_0 V_a f + f - g \| = \| \lambda_0 V_a f + f - \lambda V_a f - f \| = |\lambda_0 - \lambda| \, \| V_a f \| \leq \frac{|\lambda_0 - \lambda|}{\lambda} \| g \|$$

showing that the range of $\lambda_0 V_a + I$ is dense.

Step 3. $\lambda V + I$ has dense range for all $\lambda > 0$. Let $L \in C_0(X)^*$ be such that $L(\lambda Vf + f) = 0$ for all $f \in K(X)$. In particular we have for all $0 \leq a \leq 1$, $a \in K(X)$, $L(\lambda V_a f + a.f) = 0$, $f \in C_0(X)$. From step 2, for all $g \in C_0(X)$ there exists $f \in C_0(X)$ such that $\| f \| \leq 2 \| g \|$ and $\lambda V_a f + f = g$. We have

$$|Lg| = |L(\lambda V_a f + f)| = |L((1-a)f)| \leq 2 \| g \| \, |L|(1-a)$$

where $|L|$ denotes the toal variation measure corresponding to L. Letting a increase to 1, leads to $L = 0$.

From step 3 and (5) there is a bounded operator J_λ defined on $C_0(X)$ such that

(7) $$J_\lambda(\lambda Vf + f) = f, \quad f \in K(X).$$

Define R_λ by

$$\lambda R_\lambda = I - J_\lambda .$$

Step 4. R_λ satisfies the resolvent equation:

(8) $$R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0.$$

Indeed let $h \in C_0(X)$. By step 3 there is a sequence $f_n \in K(X)$ such that $\lim(\mu V f_n + f_n) = h$. Then

(9) $$\lim f_n = \lim J_\mu(\mu V f_n + f_n) = J_\mu h.$$

Finally

$$J_\lambda h = \lim J_\lambda(\mu V f_n + f_n) = \lim \frac{\mu}{\lambda} J_\lambda(\lambda V f_n + f_n) + \frac{\lambda-\mu}{\lambda} \lim J_\lambda f_n = \frac{\mu}{\lambda} J_\mu h + \frac{\lambda-\mu}{\lambda} J_\lambda J_\mu h$$

by (7) and (9). Thus $\lambda J_\lambda - \mu J_\mu = (\lambda-\mu)J_\lambda J_\mu$ which is equivalent to (8).

Step 5. $R_\lambda \geq 0, \|\lambda R_\lambda\| \leq 1$ and the range of R_λ is dense. Indeed for $f \in K(X)$

$$(10) \qquad R_\lambda(\lambda Vf + f) = Vf$$

so that the range of R_λ contains the range of V. Also the same equation shows that $\|\lambda R_\lambda(\lambda Vf + f)\| = \|\lambda Vf\| \leq \|\lambda Vf + f\|$ which from step 3 is equivalent to $\|\lambda R_\lambda\| \leq 1$. That $R_\lambda \geq 0$ is equivalent to $J_\lambda h \leq h$ if $h \geq 0$. To show this let f_n be such that $\lim(\lambda Vf_n + f_n) = h$. For any $\epsilon > 0$, for all large n, $\epsilon + \lambda Vf_n + f_n \geq 0$, and using (3) $\epsilon + \lambda Vf_n \geq 0$ for all large n. But then, $J_\lambda h = \lim J_\lambda(\lambda Vf_n + f_n) = \lim f_n \leq \lim(\epsilon + \lambda Vf_n + f_n) \leq \epsilon + h$ for all $\epsilon > 0$.

An appeal to Exercise 9, § 2 gives us a strongly continuous positive semi-group P_t on $C_0(X)$ with resolvent R_λ.

Now by (10) for all $f \in K(X)$, Vf belongs to the domain D(A) of generator A of P_t and

$$AVf = \lambda Vf - (\lambda Vf + f) = -f.$$

From Proposition 1 $\lim \lambda R_\lambda h = 0$ for all $h \in C_0(X)$. A look at (10) then convinces us that $\lim_{\lambda\to 0} R_\lambda f = Vf$ for all $f \in K(X)$. Since P_t is non-negative this gives (first for non-negative and then general)

$$\int_0^\infty P_t f\, dt = Vf, \qquad f \in K(X).$$

That proves the theorem.

Examples.

1. Let S_λ be a sub-Markov resolvent on R^d: The operators S_λ map the set $B(R^d)$ of bounded measurable functions into itself, $S_\lambda \geq 0, \lambda S_\lambda 1 \leq 1$, each S_λ is given by a measure and S_λ satisfies the resolvent equation. Suppose for a set of $f \in D(V)$,

$$V(|f|) = \lim_{\lambda\to 0} S_\lambda(|f|)$$

exists. Then for any $\varepsilon > 0$, $\lambda > 0$, $f \in D(V)$

$$(13) \qquad \varepsilon + \lambda Vf + f \geq 0 \quad \text{implies} \quad \varepsilon + \lambda Vf \geq 0.$$

Indeed, $Vf = S_\mu f + \mu S_\mu Vf$ is valid for all $\mu > 0$ and $f \in D(V)$. Therefore operating on the first inequality in (13) we get the second because $0 \leq \lambda S_\lambda \leq 1$.

As a special case consider the resolvent of the Brownian semi-group. We see for $d \geq 3$ that the operator V defind on $K(R^d)$ by

$$Vf(x) = \int \frac{f(y)}{|x-y|^{d-2}} dy$$

satisfies (13) and hence the principle of positive maximum.

2. Let S_λ and V be as in Example 1 above. A non-negative Borel measurable function s is called supermedian relative to S_λ if $\lambda S_\lambda s \leq s$ for all $\lambda > 0$. Define

$$\tilde{V}f(a) = \begin{cases} (s(a))^{-1} V(sf)(a) & s(a) \neq 0 \\ 0 & s(a) = 0. \end{cases}$$

Then $\tilde{V}$ satisfies (13).

Indeed if $\varepsilon s + \lambda V(sf) + sf \geq 0$, operate by S_λ to get $\varepsilon S_\lambda s + V(sf) \geq 0$ and recall that $\lambda S_\lambda s \leq s$.

3. Let $0 < \alpha < 1$, $0 < T \le \infty$. For each continuous f define

$$u(t) = Vf(t) = \int_0^t f(t-s)s^{-\alpha}ds, \quad 0 \le t \le T.$$

We claim that V satisfies the principle of positive maximum in $[0,T]$. To see this suppose first that f is continuously differentiable. Then it can be seen that

$$f(t) = \frac{1}{A}t^{-\beta}u(t) + \frac{\beta}{A}\int_0^t [u(t) - u(t-s)]s^{-1-\beta}ds$$

where, $\beta = 1-\alpha$, $A = \int_0^1 s^{-\alpha}(1-s)^{-\beta}ds$. In particular if t_0 is a maximum point of u in $[o,T]$ then $f(t_0) \ge A^{-1}t_0^{-\beta}u(t_0)$. [Note that $\lim_{t\to 0} t^{-\beta}u(t)$ exists]. Now approximate. The semi-group corresponding to V when $T = \infty$ is the one associated with the stable distribution with Laplace transform $\exp(-\lambda^{\beta})$. See Feller [2] p.p. 424.

4. Let again $0 < \alpha < 1$. For f bounded and integrable on $[0,\infty)$ put

(14) $$u(t) = Vf(t) = \int_0^\infty f(t+s)s^{-\alpha}ds.$$

If Vf attains strictly positive values then

(15) $$\sup Vf = \sup_{(f>0)} Vf$$

This can be proved as in Example 3 above. A more general procedure is the following: For each $t > 0$ there is a probability distribution F_t on $[0,\infty)$ with Laplace transform $\exp(-t\lambda^{1-\alpha})$. See Feller [2] p.p. 424. The operators S_t defined by

$$S_tf(x) = \int_0^\infty f(x+y)F_t(dy)$$

form a semi-group and

$$Vf = \int_0^\infty S_tf\,dt$$

as is seen by using Laplace Transforms. Thus (13) is valid for V. For f bounded and integrable Vf is continuous and vanishes at infinity and in this case (13) can be seen to imply (15).

5. Let $d \ge 3$ and $0 < \alpha < 2$. For each bounded measurable function f with compact support on $\mathbb{R}^d$ define

(16) $$I_\alpha(f)(x) = \int |x-y|^{\alpha-d}f(y)\,dy.$$

Then

(17) $$\sup I_\alpha f = \sup_{f \ge 0} I_\alpha f$$

We have seen in the proof of the Riesz composition formula that, $I_\alpha f$ is equal except for a constant to

(18) $$Vf = \int_0^\infty t^{\frac{\alpha}{2}-1}T_tf\,dt$$

where T_t is the Brownian semi-group. Suppose that $\varepsilon + \lambda Vf + f \ge 0$. We then have since $T_t\varepsilon = \varepsilon$,

$$\varepsilon + \lambda VT_tf + T_tf \ge 0$$

for all t and x.

Fix x and let $g(t) = T_tf(x)$. Then $VT_tf(x) = \int_0^\infty s^{\frac{\alpha}{2}-1}g(t+s)ds$. By Example 4

$$\varepsilon + \lambda VT_tf(x) \ge 0. \quad t \ge 0.$$

Thus (13) is valid for V and this implies (17). The semi-group corresponding to V is the symmetric stable semi-group of exponent α and is obtained from the Brownian semi-group by the subordination procedure (Exercise 4, section 1) using the one-sided stable process on $[0,\infty)$ of exponent $\frac{\alpha}{2}$.

6. Let $N(x,dy)$ be probability measures on $\mathbb{R}^d$ such that $Nf = \int N(\cdot,dy)f(y)$ is measurable for every $f \in B(\mathbb{R}^d)$ = set of bounded measurable functions. Let $G = \sum_0^\infty N^n$, then G satisfies

(19) $$\varepsilon + Gf + f \geq 0 \text{ implies } \varepsilon + Gf \geq 0.$$

Note that if the first condition in (19) is satisfied for an $f \in B(\mathbb{R}^d)$, Gf is necessarily bounded below. Apply N to (19) to get $\varepsilon + NGf + Nf \geq 0$. Adding this to (19) and using the obvious identity $NG + I = G$

$$2\varepsilon + 2Gf + Nf \geq 0.$$

Operate by N on this last inequality and add to twice the first inequality in (19) to get

$$4\varepsilon + 4Gf + N^2 f \geq 0.$$

And in general $2^n\varepsilon + 2^n Gf + N^n f \geq 0$. Since f is bounded the second inequality in (19) must be valid. The corresponding semi-group is the "Compound Poisson Semi-group":

$$S_t = \sum_0^\infty e^{-t}\frac{t^n}{n!}N^n.$$

Exercises to § 3.

1. Let V satisfy the principle of positive maximum. Show that for $p \geq 0$, $I + pV$ also satisfies the same.

Hint. Suppose $\varepsilon + \lambda(I + pV)f + f \geq 0$. Then, since V satisfies (5), $\epsilon + \lambda pVf \geq 0$. Multiply the first $\frac{\lambda}{\lambda+1}$ and the second by $\frac{1}{\lambda+1}$ and add. The result is $\epsilon + \lambda(I + pV)f \geq 0$.

2. Let V be as above. Show that $Vf = 0$ implies $V(|f|) = 0$. In particular for $p \geq 0$, $f + pVf = 0$ implies $f = 0$.

Hint. Let $g \in K(X)$ with support in $(|f| \geq \epsilon)$. Since $\sup Vg = \sup(Vg: g > 0)$., there is a point x_0 with $f(x_0) \geq \varepsilon$ such that $Vg(x_0) = \sup Vg$. But for all α, x_0 is a maximum point of $V(g + \alpha f)$ so that $g(x_0) + \alpha f(x_0) \geq 0$. But then $f(x_0) = 0$. For second part use Exercise 1 above.

3. Let V_n be operators from $K(X)$ into $C_0(X)$, satisfying the principle of positive maximum. If $V_n f$ converges to Vf uniformly then V satisfies the same.

REFERENCES

1. Berg, C. and Forst, G., Potential Theory on Locally Compact Abelian Groups, Springer (Ergebnisse...) (1975)
2. Feller, W., An Introduction to Probability Theory and its Applications, John Wiley (1971)

Additional References.

Hille, E., and Phillips, R.S., Functional Analysis and Semi Groups, A.M.S. Publication (1957)

Yosida, K., Functional Analysis, Springer (1971)

CHAPTER 4

Harmonic functions and Dirichlet Problem

Introduction

Harmonic functions are solutions of the Laplate equation $\Delta\mu = 0$. No other single partial differential equation is encountered in so many different situations and exihibits such depth and variety. One runs into the Laplate equation in many branches of applied physics: Electrostatics, stationary heat flow etc. Directly or indirectly the Dirichlet problem has influenced many branches of Analysis: Integral equations, special functions, Calculus of variations etc.

In §1, Dynkin's formula is proved and some applications are given. In §2, the Dirichlet problem is introduced. §3 deals with the Kelvin transformation. Some applications are found in the exercises and in Chapter 6. In §4 we prove the Fatou limit theorem and derive the existence of the Hilbert transform. §5, dealing with spherical harmonics can be considered an application of the Poisson integral formula. The original idea was to give applications in representation theory but we content ourselves with a reference.

Notation

In this Chapter X_t will denote the d-dimentional Brownian motion as introduced in Chapter 2. If D is an open set the exit time T from D is the stopping time

$$T = \inf(t: t > 0, X_t \notin D)$$

$$= \infty \text{ if there is no such } t.$$