REFERENCES

1. Karlin, S., Taylor, H.M., A first Course in Stochastic Processes. Academic Press. (1975).

2. Meyer, P.A., Probability and potentials. Blaisdell. (1966).

For more on Martingales consult the above book of Meyer plus Strasbourg Lecture Notes.

CHAPTER 2

The d-dimensional Brownian motion

Introduction and summary

In this chapter we introduce the d-dimensional Brownian motion, assuming as known the 1-dimensional Brownian motion starting at 0. Then we consider Markov times in more detail than is absolutely necessary, in the hope that this will better the understanding of this important notion. Strong Markov property is proved and a few applications considered.

§1. The d-dimensional Brownian motion

As proved, for instance in [3] pp. 12 - 16, there is a real valued stochastic process $\xi(t)$, $0 \leq t < \infty$ (on some probability space), such that $\xi(t)$ is continuous, $\xi(0) \equiv 0$ and with the finite dimensional distributions

$$(1) \quad P[\xi(t_i) \in E_i,\ 1 \leq i \leq n] = \int_{E_1} p'(t_1, 0, da_1) \int_{E_2} p'(t_2 - t_1, a_1, da_2) \ldots \int_{E_n} p'(t_n - t_{n-1}, a_{n-1}, da_n)$$

where $0 < t_1 < \ldots < t_n$, $p'(t,a,db) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(b-a)^2}{2t}\right) db$ and $E_1, \ldots, E_n$ are Borel subsets of the real line. See also pp. 5 - 8 of [5] for another proof of this fact.

Now let us define the d-dimensional Brownian motion. There is a probability space $(\Omega, \mathbb{B}, P)$ and a d-dimensional stochastic process $X(t) = (X_1(t),\dots,X_d(t))$ such that $X_1(\cdot),\dots,X_d(\cdot)$ are independent stochastic processes and each is a copy of $\xi(\cdot)$. Let W be the space of continuous paths $t \to w(t) \in R^d$, $t \geq 0$. In W consider the smallest Borel field $\underline{\underline{B}}_\infty$ which makes all the coordinate maps measurable: $\underline{\underline{B}}_\infty$ is the smallest Borel field relative to which all the maps $w \to w(t)$ are measurable for all $t \geq 0$. The map

$$\Omega \ni w \xrightarrow{\varphi_a} x_t^a(w) = a + X_t(w) \in W, \qquad a \in R^d,$$

is clearly measurable and induces probability P_a on $(W, \underline{\underline{B}}_\infty)$.

$(W, \underline{\underline{B}}_\infty, P_a,\ a \in R^d,\ x_t)$ is called the <u>standard path space realization</u> of the Brownian motion, or simply, the standard d-dimensional Brownian motion, where x_t is now the coordinate mapping of the path space W:

$$x_t(w) = x(t,w) = w(t)$$

The standard Brownian motion is a collection of individual stochastic processes, each starting at a point $a \in R^d$ knitted together in a certain manner: this is the so-called (<u>simple</u>) <u>Markov property</u>.

(2)
$$P_a\Big[\bigcap_{k \leq m} (x(t_k) \in E_k)\Big] = \int_{E_1}\cdots\int_{E_m} p(t_1,a,dx_1)\,p(t_2-t_1,x_1,dx_2)\cdots p(t_m-t_{m-1},x_{m-1},dx_m),$$

where $E_1,\dots,E_m$ are Borel subsets of R^d and $0 < t_1 < t_2 < \dots < t_m$, $m \geq 1$, and $p(t,a,db) = P_a[x(t) \in db]$. We also have

(3)
$$P_a(B) = P_0(w+a \in B)$$
$$P_a(-w \in B) = P_{-a}(B), \qquad B \in \underline{\underline{B}}_\infty,$$

(4)
$$P_a(x(0) = a) = 1.$$

$P_a(B)$ is to be thought of as <u>the chance that the event</u> $B \in \underline{\underline{B}}_\infty$ occurs for the Brownian path starting at $a \in R^d$. (2) is easy to verify using (1).

In general, an event $B \in \underline{\underline{B}}_\infty$ depends on an infinite set of parameter values t. However, certain statements which are true for all events depending on finitely many parameters also hold for all events in $\underline{\underline{B}}_\infty$. We will counter several examples in the sequel. The following theorem will be found useful in reaching such a conclusion.

<u>Theorem 1</u>. Let H be a vector space of bounded real valued functions defined on a set X, which contains the constant 1, is closed under uniform convergence, and is such that for every increasing uniformly bounded sequence f_n of non-negative functions $f_n \in H$, the function $f = \lim f_n \in H$. Let C be a subset of H, closed under multiplication. Then the space H contains all bounded functions measurable with respect to the Borel field generated by the elements of C.

<u>Proof</u>. If $f_1,\dots,f_n \in C$ any polynomial in $f_1,\dots,f_n$ belongs to H. The conditions on H together with Stone-Weierstrass Theorem imply that for any continuous function φ on R^n, $\varphi(f_1,\dots,f_n) \in H$. H is closed under uniformly bounded increasing limits; since H contains constants, it is closed under uniformly bounded decreasing limits. The set of φ on R^n for which $\varphi(f_1,\dots,f_n) \in H$, contains the bounded continuous functions and is closed under uniformly bounded monotone limits. It therefore contains all bounded Borel measurable functions. We leave the rest of the proof to the reader.

As a first example of the way Theorem 1 is used, let us show that $E_a[f]$ is a-measurable on R^d for each bounded $\underline{\underline{B}}_\infty$-measurable function f. The set of such functions clearly satiesfies the conditions of Theorem 1 and contains the multiplicative class of $\underline{\underline{B}}_\infty$-measurable bounded functions depending on finitely many parameters, i.e. functions f of the form $f(w) = \varphi(x_{t_1}(w),\dots,x_{t_n}(w))$, where φ is bounded and measurable on R^{nd}. By Theorem 1 it contains all $\underline{\underline{B}}_\infty$-measurable bounded functions.

The reader will notice that sometimes we need to conclude the joint measurability of some stochastic processes. The following theorem will be found useful.

Theorem 2. Let $X(t,w)$ be a stochastic process, $t\in[0,\infty)$. If $X(t,w)$ is measurable in w for each t, and is right continuous in t for each w, then $X(t,w)$ is measurable in the pair (t,w).

The sequence $X_n(t,w)$ of stochastic processes defined by

$$X_n(t,w) = X(\frac{i+1}{2^n},w), \quad \frac{i}{2^n} \leqq 1 < \frac{i+1}{2^n}, \qquad i = 0,1,2,\dots$$

for $n = 1,2,\dots$ are jointly measurable and

$$\lim_n X_n(t,w) = X(t,w). \qquad \text{Q.e.d.}$$

With the help of (2), it is seen that

(5) $$P_a[x(t)\in A; \bigcap_{k\leqq m}(x(t_k)\in E_k) = E_a[\int_A p(t-t_m,x(t_m),b)db : \bigcap_{k\leqq m}(x(t_k)\in E_k)]$$
$$= E_a[P_{x(t_m)}[x(t-t_m)\in A : \bigcap_{k\leqq m}(x(t_k)\in E_k)]$$
$$t_1 < t_2 < \dots < t_m < t.$$

We deduce

(6) $$P_a[x(t+s)\in A \mid \underline{\underline{B}}_s] = P_{x(s)}(x(t)\in A)$$

where $\underline{\underline{B}}_s = \underline{\underline{B}}[x(t_1);\ t_1 \leqq s]$ is the smallest Borel field generated by the variables $x(t_1), t_1 \leqq s$.

That (5) implies (6) is an example of the use of Theorem 1.

Markov property

Markov property has several different expressions which are all equivalent but each has some technical advantage. We will be concerned with the standard dimensional Brownian motion. Before discussing Markov property, let us introduce maps $W \to W$:

the shift operators θ_t $(t \geqq 0)$ defined by $\theta_t w(s) = w(s+t)$, $s \leqq 0$,

the stopping operator α_t $(0 \leqq t \leqq \infty)$ defined by $\alpha_t w(x) = w(s\wedge t)$, $s \geqq 0$, so that $\alpha_\infty w = w$.

It is easy to verify that both are measurable maps of $(W,\underline{\underline{B}}_\infty)$ into $(W,\underline{\underline{B}}_\infty)$. We leave it to the reader to verify the following fact:

The smallest Borel field $\underline{\underline{B}}_t$ relative to which α_t is measurable is precisely the Borel field of events depending on the sample path up to time t, i.e. $\underline{\underline{B}}_t = \underline{\underline{B}}(x_s : s \leqq t)$ = the least Borel field relative to which all maps $w \to w(s)$ are measurable for all $s \leqq t$.

The standard Brownian motion has the Markov property:

(7) $$P_a[\theta_s^{-1}A) \cap B] = E_a[P_{x_s}(A);B], \qquad A\in\underline{\underline{B}}_\infty, \quad B\in\underline{\underline{B}}_s$$
$$P_a[\theta_s^{-1}A \mid \underline{\underline{B}}_s] = P_{x_s}(A), \qquad A\in\underline{\underline{B}}_\infty.$$

(8) $$E_a[G\circ\theta_t\cdot F] = E_a[E_{x_t}(G)\cdot F]$$

$$E_a[G\circ\theta_t \mid \underline{\underline{B}}_t] = E_{x_t}(G)$$

for any G,F respectively bounded $\underline{\underline{B}}_\infty$- and $\underline{\underline{B}}_t$-measurable functions.

(9) $$E_a[f(x_t(\theta_s w) \mid \underline{\underline{B}}_s] = E_{x_s}(f(x_t))$$

for any bounded Borel-measurable function f on R^d.

That (7) and (8) are equivalent is general measure theory. (9) is a particular case of (8) and (9) is equivalent to equation (6). Let us show that (9) implies (8). Use induction and suppose G has the form

$$G(w) = f_1(x_{t_1}(w)) \ldots f_m(x_{t_m}(w)),$$

where $t_1 < t_2 < \ldots < t_m$ and $f_1,\ldots,f_m$ are bounded Borel measurable functions on R^d. $G\circ\theta_t(w) = f_1(x_{t_1+t}(w))\ldots f_m(x_{t_m+t}(w))$. And

$$E_a\Big[\prod_1^m f_i(x_{t_i+t}) \mid \underline{\underline{B}}_t\Big] = E_a\Big[E_a[\prod_1^m f_i(x_{t_i+t}) \mid \underline{\underline{B}}_{t_{m-1}+t}] \mid \underline{\underline{B}}_t\Big]$$

$$= E_a\Big[\prod_1^{m-1} f_i(x_{t_i+t})E_a\big[f_m(x_{t_m+t}) \mid B_{t_{m-1}+t}\big] \mid \underline{\underline{B}}_t\Big]$$

(since $\prod_1^{m-1} f_i(x_{t_i+t})$ is $B_{t_{m-1}+t}$-measurable)

$$= E_a\Big[\prod_1^{m-1} f_i(x_{t_i+t})E_{x_{t+t_{m-1}}}(f_m(x_{t_m-t_{m-1}})) \mid \underline{\underline{B}}_t\Big]$$

(since $E_a\big[f_m(x_{t_m+t}) \mid \underline{\underline{B}}_{t_{m-1}+t}\big] = E_{x_{t+t_{m-1}}}(f_m(x_{t_m-t_{m-1}}))$)

$$= E_{x_t}\Big[\prod_1^{m-1} f_i(x_{t_i})E_{x_{t_{m-1}}}(f_m(x_{t_m-t_{m-1}}))\Big]$$

(by induction assumption applied to G_1 where $G_1(w) = f_1(x_{t_1}(w))\ldots f_{m-1}(w))E_{x_{t_{m-1}}(w)}(f_m(x_{t_m-t_{m-1}}))$)

$$= E_{x_t}\Big[\prod_1^m f_i(x_{t_i})\Big]$$

(apply (9) with $s = t_{m-1}$, $t = t_m - t_{m-1}$ and $a = x_t(w)$). Now Theorem 1 takes over.

§2. Strong Markov property

The Brownian motion also starts afresh at certain random times, such as the hitting time

$$T_U = \inf(t>0;\ x(t)\in U), \quad U \text{ is an open set in } R^d$$
$$(= +\infty \text{ if } x(t) \text{ never hits } U),$$

instead of a constant time t:

$$E_a[f(x_s(\theta_T w)) \mid \underline{\underline{B}}_T] = E_{x_T}[f(x_s)].$$

This property was familiar and extensively used to derive deep results by, for example, P. Lévy. The complete statement of this feature of the Brownian motion, however, was discovered by Hunt and Dynkin independently in the early 1950's.

A random variable $T: W \to [0,\infty]$ is said to be a Markov time (or stopping time) if

$$(w:\ T(w) < t) \in \underline{\underline{B}}_t \qquad t \geq 0.$$

The hitting time T_U is a Markov time since

$$(T_U < t) = \bigcup_{r \text{ rational} < t} (x_r \in U) \in \underline{\underline{B}}_t.$$

A constant time $T \equiv t$ is trivially a Markov time, but a last exit time such as $\sup(t \leq 1:\ x(t) = 0)$ is not.

Define $\underline{\underline{B}}_{T+}$ to be the class of sets $B \in \underline{\underline{B}}_\infty$ such that

$$B \cap (T < t) \in \underline{\underline{B}}_t, \qquad t \geq 0.$$

$\underline{\underline{B}}_{T+}$ is a Borel algebra and $(T < t) \in \underline{\underline{B}}_{T+}$ for each $t \geq 0$. $\underline{\underline{B}}_{T+}$ is to be thought of as measuring the Brownian path up to time $t = T+$ beacuse

$$\underline{\underline{B}}_{T+} = \bigcap_{\varepsilon>0} \underline{\underline{B}}[x(t \wedge (T+\varepsilon)) : t \geq 0].$$

We shall show this a little later.

Dynkin-Hunt's statement of the <u>strong</u> <u>Markov</u> <u>property</u> is that, conditional on the present position $x(T)$, the future path $x(t+T)$, $t \geq 0$, is a standard Brownian motion starting at $x(T)$, and this Brownian motion is independent of $\underline{\underline{B}}_{T+}$.

Before going into precise mathematical statements, let us note some facts. By Theorem 2, §1, $x(t,w)$ is measurable in the pair (t,w). Hence for any non-negative measurable function b on W, the function $x(b(w),w)$ is measurable. In particular for a Markov time T, x_T is measurable. We already defined the shift operators θ_t and stopping operators α_t for all $t \geq 0$. θ_T and α_T make sense (if $T < \infty$) and are easily seen to be measurable maps on W into W.

<u>Theorem 1</u>. The Brownian motion has the strong Markov property:

$$(1) \qquad E_a[G \circ \theta_T : B \cap (T < \infty)] = E_a[E_{x(T)}(G) : B \cap (T < \infty)]$$

for all bounded $\underline{\underline{B}}_\infty$-measurable functions G and all $B \in \underline{\underline{B}}_{T+}$.

<u>Proof</u>. We need only prove (1) for functions G of the form

$$G(w) = f_1(x(t_1)) \ldots f_m(x(t_m)),$$

where $f_1, \ldots, f_m$ are continuous bounded functions on R^d. Once we do this, Theorem 1, §1, takes over.

Define a sequence (of stopping times) T_n as follows:

$$T_n(w) = k2^{-n} \quad \text{if} \quad (k-1)2^{-n} \leq T < k2^{-n}, \qquad k \geq 1,$$
$$= \infty \quad \text{if} \quad T = \infty.$$

Noting the following facts (we leave the proofs to the reader):

i) $\quad G(\theta_T w) = \lim G(\theta_{T_n} w)$

ii) $\quad E_a[G]$ is a continuous function of a

iii) $\quad B \cap (T_n = k2^{-n}) = B \cap ((k-1)2^{-n} \leq T < k2^{-n}) \in \underline{\underline{B}}_{k2^{-n}}, \quad k \geq 1$ for all $B \in \underline{\underline{B}}_{T+}$,

we write

$$\begin{aligned} E_a[G \circ \theta_T : B \cap (T<\infty)] &= \lim_{n\to\infty} E_a[G \circ \theta_{T_n} : B \cap (T<\infty)] \\ &= \lim_{n\to\infty} \left[\sum_{k=1}^{\infty} E_a[G \circ \theta_{k2^{-n}} : B \cap (T_n = k2^{-n})] \right] \\ &= \lim_{n\to\infty} \sum_{k=1}^{\infty} E_a[E_{x(k2^{-n})}(G) : B \cap (T_n = k2^{-n})] \\ &\quad \text{(this is the simple Markov property)} \\ &= \lim_{n\to\infty} E_a[E_{x(T_n)}(G) : B \cap (T<\infty)] \\ &= E_a[E_{x(T)}(G) : B \cap (T<\infty)]. \end{aligned}$$

This proves the theorem.

(1) is equivalent to

$$(2) \qquad E_a[G \circ \theta_T \mid \underline{\underline{B}}_{T+}] = E_{x(T)}(G)$$

P_a - almost everywhere on the set $(T < \infty)$.

Proceeding as in Exercise 3, we can show that

$$P_a[\theta_T^{-1}A) \cap B \mid x(T)] = P_a[\theta_T^{-1}A \mid x(T)]P_a(B \mid x(T)), \quad A \in \underline{\underline{B}}_\infty,$$
$$B \in \underline{\underline{B}}_{T+}, \quad \text{almost everywhere on the set} \quad (T<\infty).$$

Stated otherwise, this means that conditional on $x(T)$, the motion $x(T+t)$ is independent of $\underline{\underline{B}}_{T+}$.

Markov times

Recall the definition of the stopping operator α_t:

$$\alpha_t w(s) = w(t \wedge s), \qquad t,s \geqq 0.$$

α_t has the following important property:

(3) $$\alpha_t[\alpha_t w] = \alpha_t w.$$

The reader will presently see the use of the following proposition in the study of Markov times.

Proposition 2. $B \in \underline{\underline{B}}_t$ if and only if

$$B = \alpha_t^{-1}(B).$$

Proof is immediate: Since $\underline{\underline{B}}_t$ is the smallest Borel field relative to which α_t is measurable, there exists a set $A \in \underline{\underline{B}}_\infty$ such that

$$B = \alpha_t^{-1}(A).$$

We have $\alpha_t^{-1}(B) = \alpha_t^{-1}\alpha_t^{-1}(A) = \alpha_t^{-1}(A) = B$ from (3).

Corollary 3. Let $\{B_i, \ i \in I\}$ be a collection of sets in $\underline{\underline{B}}_t$. If $B = \bigcup_{i\in I} B_i \in \underline{\underline{B}}_\infty$, then $B \in \underline{\underline{B}}_t$.

Indeed, from the above proposition

$$\alpha_t^{-1}(B) = \cup_i \alpha_t^{-1}(B_i) = \cup_i B_i = B.$$

Introduce the Borel fields $\underline{\underline{B}}_{t+}$:

$$\underline{\underline{B}}_{t+} = \bigcap_{s>t} \underline{\underline{B}}_s, \qquad t \geqq 0.$$

An immediate corollary of Corollary 3 is

Corollary 4. Let $\{B_i, \ i \in I\}$ be a collection (not necessarily countable) of sets in $\underline{\underline{B}}_{t+}$. If $B = \bigcup_{i\in I} B_i \in \underline{\underline{B}}$, then $B \in \underline{\underline{B}}_{t+}$.

Our definition of a Markov time can be rewritten as:

$$(T \leqq t) \in \underline{\underline{B}}_{t+}, \qquad t \geqq 0.$$

Proposition 5. Let $T: W \longrightarrow [0,\infty]$ be $\underline{\underline{B}}_\infty$-measurable. Then T is a Markov time iff

$$(T = t) \in \underline{\underline{B}}_{t+}, \qquad t \geqq 0.$$

Indeed $(T \leqq t) = \bigcup_{s \leqq t} (T = s)$ and Corollary 4 takes over.

Using Proposition 5, it is a simple matter to check that the class of Markov times is closed under the operations:

$$T_1 \wedge T_2, \qquad T_1 \vee T_2$$
$$T_n \downarrow T, \qquad T_n \uparrow T$$
$$T_1 + T_2$$
$$T_1 + T_2(\theta_{T_1} w).$$

Consider, as an example, the proof that $S = T_1 + T_2(\theta_{T_1} w)$ is a Markov time whenever T_1, T_2 are. First S is $\underline{\underline{B}}_\infty$-measurable. And

$$(S = t) = \bigcup_{\substack{r,s \\ r+s=t}} (T_1 = r) \cap (T_2(\theta_r) = s).$$

Now note that for all $r, s, \theta_r^{-1}(A) \in \underline{\underline{B}}_{(r+s)+}$ for all $A \in \underline{\underline{B}}_{s+}$ (first prove that $\theta_r^{-1}(A) \in \underline{\underline{B}}_{r+s}$ for all $A \in \underline{\underline{B}}_s$ for all $r, s \geq 0$). Thus $(S = t) \in \underline{\underline{B}}_{t+}$ and the proof is complete.

Before examining Markov times further, let us introduce strict stopping times:

A non-negative function $T \leq \infty$ is a <u>strict Markov time</u> if for all $t \geq 0$,

$$(T \leq t) \in \underline{\underline{B}}_t.$$

Clearly every strict Markov time is also a Markov time; if T is a Markov time, $T + \varepsilon$ is a strict Markov time for all $\varepsilon > 0$. In particular, every Markov time is a limit of a decreasing sequence of strict Markov times.

We can easily show (cf. Proposition 5) that a <u>non-negative measurable function</u> $T \leq +\infty$ is a strict Markov time iff $(T = t) \in \underline{\underline{B}}_t$ for all $t \geq 0$.

For a strict Markov time T, the <u>Borel field</u> $\underline{\underline{B}}_T$ is defined as the Borel field of all sets $B \in \underline{\underline{B}}_\infty$ such that

$$B \cap (T \leq t) \in \underline{\underline{B}}_t \quad \text{for all} \quad t \geq 0.$$

If $T \leq S$ are (strict) Markov times and $B \in \underline{\underline{B}}_{T+}$ $(\in \underline{\underline{B}}_T)$, we have

$$B \cap (S < t) = B \cap (T < t) \cap (S < t) \in \underline{\underline{B}}_t$$

$$(B \cap (S \leq t) = B \cap (T \leq t) \cap (S \leq t) \in \underline{\underline{B}}_t)$$

for all $t \geq 0$, i.e. $B \in B_{S+}$ $(B \in B_S)$. Thus $\underline{\underline{B}}_{T+} \subset \underline{\underline{B}}_{S+}$ $[\underline{\underline{B}}_T \subset \underline{\underline{B}}_S]$. Using this, it is simple to show that if a sequence of Markov times T_n decreases to a Markov time T,

$$\underline{\underline{B}}_{T+} = \bigcap_n \underline{\underline{B}}_{T_n+}$$

and

$$\underline{\underline{B}}_{T+} = \bigcap_{\varepsilon>0} \underline{\underline{B}}_{T+\varepsilon}.$$

(Note that $T+\varepsilon$ is a strict Markov time for all $\varepsilon > 0$.)

Suppose T is a Markov time. Then

$$(T = t) \in \underline{\underline{B}}_{t+} \subset \underline{\underline{B}}_s \qquad \text{for all } s > t,$$

i.e.

$$\alpha_s^{-1}(T = t) = (T = t) \qquad \text{for all } s > t,$$

i.e.

$$T(w) = t \text{ implies } T(\alpha_s w) = t \quad \text{for all } s > t.$$

Thus

(4) $$T(w) = T(\alpha_{T+\varepsilon} w) \qquad \text{for all } \varepsilon > 0.$$

Similarly for a strict Markov time T we can verify

(5) $$T(w) = T(\alpha_T w).$$

A little more careful analysis leads to Galmarino's characterization of Markov and strict Markov times.

<u>Galmarino's Theorem</u>. <u>A non-negative Borel function</u> $T \leq \infty$ <u>is a Markov time (a strict Markov time) iff</u>

$$\alpha_t w = \alpha_t v, \quad Tw < t \quad \text{implies} \quad Tw = Tv.$$

$$(\alpha_t w = \alpha_t v, \quad Tw \leq t \quad \text{implies} \quad Tw = Tv).$$

Proof. If $A \in \underline{\underline{B}}_t$, $w \in A$, $\alpha_t w = \alpha_t v$ imply $v \in A$. (This is bacause $A = \alpha_t^{-1}(A)$.) Thus if T is a Markov time, $(T < t) \in \underline{\underline{B}}_t$. Therefore $T(w) < t$, $\alpha_t w = \alpha_t v$ imply $T(v) < t$. Hence $T(v) = T(\alpha_t v) = T(\alpha_t w) = T(w)$.

Conversely, suppose T has the above property. Then $(T < t) \in \underline{\underline{B}}_\infty$ and $\alpha_t^{-1}(T < t) = (T < t)$, i.e. $(T < t) \in \underline{\underline{B}}_t$.

Galmarino's theorem gives us a nice intuitive idea of what a Markov time really is.

We will now look more closely at the Borel fields $\underline{\underline{B}}_{T+}$ and $\underline{\underline{B}}_T$ (for strict Markov times). First let T be a strict Markov time. By definition $B \in \underline{\underline{B}}_T$ iff

$$B \cap (T \leqq t) \in \underline{\underline{B}}_t$$

for all $t \geqq 0$. This can also be defined as

$$B \in \underline{\underline{B}}_\infty$$

and

$$B \cap (T = s) \in \underline{\underline{B}}_s \quad \text{for all } s \geqq 0.$$

Now we claim that $B \in \underline{\underline{B}}_T$ iff $B \in \underline{\underline{B}}_\infty$ and

(6) $$B = \alpha_T^{-1}(B).$$

If $B = \alpha_T^{-1}(B)$ and $B \in \underline{\underline{B}}_\infty$, we get

$$B \cap (T = s) = \alpha_T^{-1}(B) \cap (T = s) = \alpha_s^{-1}(B) \cap (T = s) \in \underline{\underline{B}}_s$$

since $\alpha_s^{-1}(B) \in \underline{\underline{B}}_s$, and $(T = s) \in \underline{\underline{B}}_s$. Thus $B \in \underline{\underline{B}}_T$. Conversely, if $B \in B_T$, for all s

$$B \cap (T = s) \in \underline{\underline{B}}_s,$$

i. e.

$$\alpha_s^{-1}(B) \cap (T = s) = B \cap (T = s) [\alpha_s^{-1}(T = s) = (T = s)].$$

Taking union over all s, we get

$$\bigcup_s (\alpha_s^{-1}(B) \cap (T = s)) = B$$

i.e.

$$\alpha_T^{-1}(B) = B,$$

For any $A \in \underline{\underline{B}}_\infty$, $B = \alpha_T^{-1}(A)$ has the property $\alpha_T^{-1}(B) = B$. We thus see that $\underline{\underline{B}}_T$ <u>is the smallest Borel field relative to which</u> α_T <u>is measurable</u>, i.e.

$$\underline{\underline{B}}_T = \underline{\underline{B}}(x(s \wedge T),\ s \geqq 0).$$

If T is a Markov time, $T + \varepsilon$ is a strict Markov time for all $\varepsilon > 0$ and $\underline{\underline{B}}_{T+} = \bigcap_{\varepsilon > 0} \underline{\underline{B}}_{T+\varepsilon}$. Thus

(7) $$B \in \underline{\underline{B}}_{T+} \text{ iff } B \in \underline{\underline{B}}_\infty \text{ and } B = \alpha_{T+\varepsilon}^{-1}(B) \text{ for all } \varepsilon > 0.$$

And

$$\underline{\underline{B}}_{T+} = \bigcap_{\varepsilon > 0} \underline{\underline{B}}(x(s \wedge (T+\varepsilon)),\ s \geqq 0).$$

If $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are Borel fields, we denote by $\underline{\underline{A}}_1 \vee \underline{\underline{A}}_2$ the least Borel field containing $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$.

Proposition 6. If T is a Markov time,

$$\underline{\underline{B}}_\infty = \underline{\underline{B}}_{T+} \vee \underline{\underline{B}}[1_{(T<\infty)} x(t+T),\ t \geqq 0].$$

If T is a strict Markov time,

$$\underline{\underline{B}}_\infty = \underline{\underline{B}}_T \vee \underline{\underline{B}}[1_{(T<\infty)} x(t+T),\ t \geqq 0]$$

where

$$1_{(T<\infty)} \begin{cases} = 1 & \text{if } T < \infty \\ = 0 & \text{if } T = \infty. \end{cases}$$

Proof. We need only show that for every t, $x(t)$ is measurable relative to $\underline{\underline{B}}_{T+} \vee \underline{\underline{B}}[1_{T<\infty)} x(t+T), \quad t \geq 0]$,

$$x(t) = x(t)1_{(T \geq t)} + x(t)1_{(T<t)}.$$

We can rewrite (7) in the form: a $\underline{\underline{B}}_\infty$-measurable function f is $\underline{\underline{B}}_{T+}$-measurable iff

$$f(w) = f(\alpha_{T+\varepsilon} w) \qquad \text{for all } \varepsilon > 0.$$

We then see that $x(t)1_{(T \geq t)}$ is $\underline{\underline{B}}_{T+}$-measurable (use (4)). And

$$x(t)1_{(T<t)} = \lim_{n\to\infty} \sum_{k2^{-n}<t} x(t-(k-1)2^{-n}+T)1_{((k-1)2^{-n} < T \leq k2^{-n})}$$

The indicator function in the above sum is $\underline{\underline{B}}_{T+}$-measurable and

$$1_{(T<2^{-n}k)} x(t-(k-1)2^{-n}+T)$$

is $\underline{\underline{B}}[1_{(T<\infty)} x(t+T), \ t \geq 0]$-measurable, and we are done.

One important consequence of the Strong Markov Property is

Proposition 7. $\underline{\underline{B}}_{0+}$ is trivial, i.e. $A \in B_{0+}$ implies

$$P_a(A) = 0 \quad \text{or} \quad 1, \qquad \forall a \in R^d.$$

Indeed: $P_a[A] = P_a[A;A] = P_a[\theta_0^{-1}(A);A] = E_a[P_{x(0)}(A);A] = P_a(A) \cdot P_a(A)$ since θ_0 is the identity and $P_a[x(0) = 1] = 1$.

Since for any Markov time T, the set $(T=0) \in \underline{\underline{B}}_{0+}$, we see that $P_a[T=0] = 1$ or 0.

More generally we can show that for any $t \geq 0$, $\underline{\underline{B}}_t$ and $\underline{\underline{B}}_{t+}$ are equivalent: every set in $\underline{\underline{B}}_{t+}$ differs from a set in $\underline{\underline{B}}_t$

at most in a set of measure zero, i.e. the completion of $\underline{\underline{B}}_t$ and $\underline{\underline{B}}_{t+}$ with respect to P_a for any a are identical. To see this, we use Proposition 6. This proposition implies that the set of functions of the form

$$f(\alpha_t w) g(\theta_t w),$$

with f, g bounded $\underline{\underline{B}}_\infty$-measurable functions. generate the Borel field $\underline{\underline{B}}_\infty$. We have

$$E_a[f(\alpha_t)g(\theta_t) \mid \underline{\underline{B}}_{t+}] = f(\alpha_t)E_{x(t)}(g) = E_a[f(\alpha_t)g(\theta_t) \mid \underline{\underline{B}}_t].$$

The validity of this for all f, g implies

$$E_a[F \mid B_{t+}] = E_a[F \mid \underline{\underline{B}}_t]$$

for all bounded $\underline{\underline{B}}_\infty$-measurable functions F, i.e. $\underline{\underline{B}}_{t+}$ and $\underline{\underline{B}}_t$ are equivalent.

More generally for a strict Markov time T, $\underline{\underline{B}}_T$ and $\underline{\underline{B}}_{T+}$ are equivalent where $\underline{\underline{B}}_{T+}$ is the intersection of the Borel fields $\underline{\underline{B}}_{T+(1/n)}$. We leave the proof of this as an exercise.

We have the following simple extension of the strong Markov property: <u>the time dependent strong Markov property</u>:

Theorem 8. Let $F(s,w)$ be bounded and measurable in (s,w). Then

$$E_a[F(T,\theta_T) \mid \underline{\underline{B}}_{T+}] = E_{x(T)}[F(s,w)]_{s=T} \quad \text{on the set } (T<\infty).$$

If $F(s,w)$ has the form $f(s)g(w)$, the above is a consequence of the strong Markov property. Now one uses the usual procedures.

Let us look at a particular case of the generalized strong Markov property. Let T be a Markov time and

$$\mu_a(ds,db) = P_a(T \in ds,\ X_T \in db),$$

i.e. μ_a is the joint distribution of (T, x_T). Then

(8) $$P_a[x_t \in E] = P_a[x_t \in E,\ T > t] + \int_{[0,t]\times R^d} P_b(x_{t-s} \in E)\mu_a(dsdb).$$

When T is the first passage time, this is known as the "first passage time relation". For the proof we let

$$F(s,w) = 1_{[0,t]}(s)1_E(x_{t-s}(w)).$$

Then $F(T,\theta_T w) = 1_{[0,t]}(T(w))1_E(x_t(w))$ so that

$$\begin{aligned} P_a[x_t \in E,\ T \leq t] &= E_a[F(T,\theta_T)] \\ &= E_a[E_{x_T}(F(s,w))_{s=T}] \\ &= \int_{[0,t]\times R^d} P_b(x_{t-s} \in E)\mu_a(dsdb), \end{aligned}$$

which is what we set out to show. Let us look at applications.

Applications

Consider the 1-dimensional Brownian motion, $a > 0$ and E a Borel subset of $(0,\infty)$ and define the Markov time T by

$$T = \inf\{t: x_t = 0\} = \infty \text{ if there is no such } t.$$

Then we have

$$P_a(x_t \in E,\ T > t) = \int_E \{p(t,a,b) - p(t,a,-b)\}db$$

where

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}}e^{-(x-y)^2/2t}.$$

Since $x_T = 0$ if $T < \infty$, we get from the first passage time relation

$$P_a(x_t \in E) = P_a(x_t \in E,\ T > t) + \int_0^t P_0(x_{t-s} \in E)\mu_a(ds)$$

where $\mu_a(ds) = P_a(T \in ds)$ and (using $-E$ instead of E)

$$P_a(x_t \in -E) = P_a[x_t \in -E, T > t] + \int_0^t P_0(x_{t-s} \in -E)\mu_a(ds).$$

Now $E \subset (0,\infty)$ so x_t cannot belong to $-E$ if $T > t$, i.e. $P_a[x_t \in -E,\ T > t] = 0$. Also $P_0(x_{t-s} \in -E) = P_0(x_{t-s} \in E)$. The last two equalities thus imply

$$P_a[x_t \in E,\ T > t] = P_a[x_t \in E] - P_a[x_t \in -E].$$

This is what we set out to show. Taking $E = (0,\infty)$, we get the distribution of T:

$$\begin{aligned} P_a[T > t] &= \int_0^\infty \frac{1}{\sqrt{2\pi t}}\left\{e^{-\frac{(x-a)^2}{2t}} - e^{-\frac{(x+a)^2}{2t}}\right\}dx. \\ &= \frac{2}{\sqrt{2\pi t}}\int_0^a e^{-x^2/2t}dx = P_0[|x_t| < a]. \end{aligned}$$

In particular, we see that $P_a[T < \infty] = 1$ and that

$$E_a(T) = \int_0^\infty P_a[T > t]dt = 2\int_0^a dx \int_0^\infty \frac{e^{-x^2/2t}}{\sqrt{2\pi t}}dt = \infty.$$

As another application, consider the d-dimensional Brownian motion. Equation (8) is clearly equivalent to

$$E_a[f(x_t)] = E_a[f(x_t): T > t] + \int_{[0,t]\times R^d} E_b(f(x_{t-s}))\mu_a(dsdb).$$

Let $f(b) = \sum_1^d b_i^2$. Take $a = 0$. We have

$$E_b[f(x_s)] = \|b\|^2 + d\cdot s.$$

Let T be the exit time through a sphere of radius r:

$$T = \inf\{t: \|x_t\| > r\} = \infty \text{ if no such } t \text{ exists.}$$

We get

$$d\cdot t = E_0[f(x_t): T>t] + \int_{[0,t]\times R^d} [\|b\|^2 + d(t-s)]\mu_0(ds\,db).$$

Since x_T is clearly on the surface of the sphere $\|b\|^2 = r^2$, we get

$$d\cdot t = E_0[f(x_t): T>t] + r^2 + d\,E_0[t-T: T \leqq t],$$

i.e.

$$d\cdot t\,P_0[T>t] + d\,E_0[T: T\leqq t] = E_0[\|x_t\|^2: T>t] + r^2 \leqq r^2 + r^2 = 2r^2$$

since $T>t$, $\|x_t\|^2 < r^2$. Letting $t\to\infty$, we see that $tP_0[T>t]$ is bounded, i.e. $P_0[T<\infty] = 1$. Then letting $t\to\infty$, we see that $E_0[T] < \infty$. Finally $t\to\infty$ gives $E_0[T] = \frac{r^2}{d}$. Thus the first exit time through a sphere of radius r has expectation $\frac{r^2}{d}$.

We note a fundamental property of the d-dimensional Brownian motion: <u>If T is the exit time from a sphere of radius r, centre zero, then x_T is uniformly distributed on the surface of the sphere</u>. To prove this, note that for any rotation 0 and $a \in R^d$

$$P_a[0x_{t_i} \in E_i,\ 1\leqq i\leqq n] = P_{0^{-1}a}[x_{t_i}\in E_i,\ 1\leqq i\leqq n]$$

showing that x_t and $0x_t$ have the same finite dimensional distributions relative to P_0. Therefore x_T and $0x_T$ have the same distribution and this means that x_T is uniformly distributed on the surface of the sphere.

<u>Exercises</u>

1. Show that θ_1 and α_1 are measurable.

2. Show that α_t generates the Borel field $\underline{\underline{B}}_t$: the smallest field relative to which α_t is measurable is $\underline{\underline{B}}_t$.

3. Show that the Borwnian motion has no memory (the past and future are independent given the present):

$$P_a[(\theta_t^{-1}A)(\alpha_t^{-1}B)\,|\,x_t] = P_a[(\theta_t^{-1}A)\,|\,x_t]\,P_a[\alpha_t^{-1}B\,|\,x_t]$$

for all $A, B \in \underline{\underline{B}}_\infty$.

<u>Hint</u>. If F and G are bounded $\underline{\underline{B}}_\infty$-measurable functions,

$$\begin{aligned} E_a[F\circ\theta_t \cdot G\circ\alpha_t\,|\,x_t] &= E_a[G\circ\alpha_t\, E_a[F\circ\theta_t\,|\,\underline{\underline{B}}_t]\,|\,x_t] \\ &= E_a[G\circ\alpha_t\cdot E_{x_t}(F)\,|\,x_t] \\ &= E_{x_t}(F)\,E_a[G\circ\alpha_t\,|\,x_t] \\ &= E_a[F\circ\theta_t\,|\,x_t]\cdot E_a[G\circ\alpha_t\,|\,x_t]. \end{aligned}$$

4. To prove Theorem 1, §2, it is sufficient to show that

$$E_a[f(x_{T+t}); A\cap(T<\infty)] = E_a[E_{x_t}[f(x_t)]; A\cap(T<\infty)]$$

$$a\in R^1,\quad A\in\underline{\underline{B}}_{T+},\quad 0\leqq f\leqq 1,\quad f\in C(R^1)\quad t\geqq 0.$$

<u>Hint</u>. Use induction as in the case of simple Markov property.

5. $(T_1 < T_2) \in \underline{\underline{B}}_{T_1+}$.

<u>Hint</u>. $(T_1<T_2)\cap(T_1<t) = \bigcup_{s<t}((T_1=s)\cap(s<T_2))]$.

6. Show that for a strict Markov time T, $\underline{\underline{B}}_T$ and $\underline{\underline{B}}_{T+}$ are equivalent where

$$\underline{\underline{B}}_{T+} = \bigcap_n \underline{\underline{B}}_{T+\frac{1}{n}} = \{E\colon E \cap (T \leq t) \in \underline{\underline{B}}_{t+},\ t \geq 0\}.$$

Hint. Use Proposition 6, §2.

7. Show that for any bounded $\underline{\underline{B}}_\infty$-measurable function f

$$E_a[f(\theta_s w) \mid \underline{\underline{B}}_{t+s}] = Y(\theta_s w)$$

where

$$Y(w) = E_a[f \mid \underline{\underline{B}}_t](w).$$

Hint. First let f have the form

$$f_1(x_{t_1}) \ldots f_n(x_{t_n}) g_{n+1}(x_{t_{n+1}}) \ldots g_m(x_{t_m})$$

with $t_1 < \ldots < t_n < t+s < t_{n+1} < \ldots < t_m$.

8. Let S be a strict Markov time and T a Markov time such that $T \geq S$. There exists a $\underline{\underline{B}}_S \times \underline{\underline{B}}_\infty$-measurable function $\overline{T}(w_1, w_2)$ on $W \times W$ such that

a) $T(w) = S(w) + \overline{T}(w, \theta_S w)$

b) $\overline{T}(w, \cdot)$ is a Markov time for each fixed w.

Solution. According to Proposition 6, §2, the map

$$w \to (\alpha_S w, \theta_S w) \in W \times W, \quad \underline{\underline{B}}_S \times \underline{\underline{B}}_\infty$$

defined on the set $(S < \infty)$ generates the Borel field $\underline{\underline{B}}_\infty$ restricted to $(S < \infty)$. Thus there exists a $\underline{\underline{B}}_S \times \underline{\underline{B}}_\infty$-measurable function $\overline{T}(w_1, w_2)$ such that

$$\overline{T}(w) - S(w) = \overline{T}(\alpha_S w, \theta_S w) \quad \text{if} \quad S(w) < \infty.$$

By redefining $\overline{T}(w_1, w_2) = \infty$ if $S(w_1) = \infty$ or if $w_2(0) \neq w_1(S)$, we do not change a) or the $\underline{\underline{B}}_S \times \underline{\underline{B}}_\infty$-measurability. This we do. Since $\overline{T}$ is $\underline{\underline{B}}_S \times \underline{\underline{B}}_\infty$-measurable,

$$\overline{T}(w_1, w_2) = \overline{T}(\alpha_S w_1, w_2).$$

Fix w_1. We must show that

$$(w_2 = \overline{T}(w_1, w_2) = t) \in \underline{\underline{B}}_u \quad \text{for all} \quad u > t,$$

i.e.

$$\overline{T}(w_1, w_2) = t \quad \text{iff} \quad \overline{T}(w_1, \alpha_u w_2) = t.$$

Define w and w' by

$$w(v) = w_1(v), \qquad v \leq S$$

$$w(v+S) = w_2(v), \qquad v \geq 0$$

$$w' = \alpha_{u+S(w_1)}(w).$$

Clearly $S(w) = S(w_1) = S(w')$. We have

$$t = \overline{T}(w_1, w_2) = \overline{T}(\alpha_S w_1, w_2) = \overline{T}(\alpha_S w, \theta_S w) = T(w) - S(w).$$

Thus

$$T(w) = S(w) + t < S(w) + u.$$

Hence

$$T(w') = T(w) = S(w) + t = S(w') + t,$$

i.e.

$$\overline{T}(\alpha_S w', \theta_S w') = t,$$

i.e.

$$\overline{T}(\alpha_S w_1, \alpha_u w_2) = t$$

since $\alpha_S w' = \alpha_S w_1$, $\theta_S w' = \alpha_u w_2$. Q.E.D.

9. Show that for any t, $0 \leq t < \infty$, the conditional probability of $\underline{\underline{B}}_\infty$ given by $\underline{\underline{B}}_t$ exists, i.e. there exists a function $P(w,B)$ such that

1) $P(w,B)$ is $\underline{\underline{B}}_t$-measurable for all $B \in \underline{\underline{B}}_\infty$.

2) $P(w,B)$ is a probability measure for all $w \in W$.

3) $P(w,B) = 1_B(w)$ if $B \in \underline{\underline{B}}_t$.

4) For any a, $B \in \underline{\underline{B}}_t$, $A \in \underline{\underline{B}}_\infty$

$$E_a[P(w,A): B] = P_a[B \cap A].$$

Solution. Consider the map

$$(*) \qquad w \to (\alpha_t w, \theta_t w) \in W \times W.$$

By Proposition 6, §2, the least Borel algebra relative to which this map is measurable is precisely $\underline{\underline{B}}_\infty$. Let Ω be the subset of $W \times W$ satisfying

$$(w_1, w_2) \in \Omega \text{ iff } w_1 = \alpha_t w_1, \quad w_2(0) = w_1(t).$$

Then the map (*) maps W onto Ω; indeed, if $(w_1, w_2) \in \Omega$, define $w \in W$ by

$$w(s) = w_1(s), \quad s \leq t$$

$$w(t+s) = w_2(s), \quad s \geq 0.$$

Also it is clear that Ω is a measurable subset of $W \times W$. It follows that given $B \in \underline{\underline{B}}_\infty$ there exists a unique $\tilde{B}$ (measurable) $\subset \Omega$ such that

$$B = \{w: (\alpha_t w, \theta_t w) \in \tilde{B}\}.$$

For each $w \in W$ define

$$B_w = \{w': (\alpha_t w, w') \in \tilde{B}\}.$$

Then $B_w \in \underline{\underline{B}}_\infty$. Finally put

$$P(w,B) = P_{x(t)}(B_w).$$

We note that if $B \in \underline{\underline{B}}_t$, then $B_w = \emptyset$ if $w \notin B$, $B_w = \{w': w'(0) = w(t)\}$ if $w \in B$. Thus

$$P(w,B) = P_{x(t)}(B_w) = P_{w(t)}\{w': w'(0) = w(t)$$

$$= 1_B.$$

Thus 3) is verified. 1) and 2) we leave to the reader. 4) is verified by looking first on sets determined by time points $s_1 < \ldots < s_n \leq t < t + t_1 < \ldots < t + t_m$ and then generalizing. Thus let A be the set

$$\{w: x_{s_i} \in A_i, \quad 1 \leq i \leq n, \quad x_{t+t_j} \in C_j, \quad 1 \leq j \leq m\}.$$

Then $\tilde{A} \subset \Omega$ is the set

$$(w_1, w_2) \text{ such that } w_1 = \alpha_t(w_1)$$

$$w_2(0) = w_1(t), \quad w_1(s_i) \in A_i, \quad 1 \leq i \leq n,$$

$$w_2(t_j) \in C_j, \quad 1 \leq j \leq m.$$

It follows that

$$A_w = \emptyset \text{ if } w \notin B_1 = (x_{s_i} \in A_i, \ 1 \leq i \leq n)$$

$$= B_2 \text{ if } w \in B_1$$

where

$$B_2 = \{w': w'(t_j) \in C_j, \quad 1 \leq j \leq m\}.$$

And if $B \in \underline{\underline{B}}_t$

$$E_a[P(w,A): B] = E_a[P_{x(t)}(B_2): B_1 \cap B] = P_a[\Theta_t^{-1}(B_2) \cap B_1 \cap B] = P_a[A \cap B]$$

since $\Theta_t^{-1}(B\ ) \cap B_1 = A$. Q.E.D.

10. If R,S are Markov times, then $R(\alpha_S) \geqq \min(R,S)$ and $R(\alpha_S)$ is a Markov time.

Hint. For $s > t$, $\underline{B}_s \supset (T \leqq t) = (T(\alpha_s) \leqq t)$ for any Markov time T, by Proposition 2, §2. Using this,

$$(R(\alpha_S) \leqq t) = \bigcup_{s \leqq t} (R(\alpha_s) \leqq t,\ S = s) \cup (R \leqq t,\ S > t),$$

which gives both the assertions.

11. For any set A, let

$$T = \inf\{t:\ t \geqq 0,\ x_t \in A\}$$
$$= \infty \text{ if no such } t.$$

Show that T is a strict Markov time if A is closed and not a strict Markov time if A is open.

Hint. Use (5), §2.

12. Let D be a bounded open subset of R^d and T the exit time from D:

$$T = \inf\{t:\ t > 0,\ x_t \notin D\},$$

the infimum over an empty set being always ∞ by definition.

Show that

$$\sup_{a \in D} E_a[e^{\varepsilon T}] < \infty \quad \text{for some } \varepsilon > 0.$$

Hint. Let $\varphi(t) = \sup_{a \in D} P_a[T > t]$. Using Markov property, show that $\varphi(t+s) \leqq \varphi(t)\varphi(s)$. So if $\varphi(t_0) < 1$ for one t_0, $\varphi(t) \leqq Ke^{-t\lambda}$ for some K, $\lambda > 0$. By considering a ball containing D, it is seen that $\sup_{a \in D} E_a[T] < \infty$, which implies $\varphi(t_0) < 1$ for large t_0.

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