CHAPTER 1

Martingales

§1 Optional sampling, inequalities, and convergence

We will not be needing too much martingale theory. What little is done is to make the notes more complete. In the following we assume given a fixed probability space $(\Omega, \mathbb{B}, P)$. All σ-fields considered are sub-σ-fields of $\mathbb{B}$. In some of the examples some knowledge of the relevant concepts is needed.

Let $F_0, F_1, \ldots$ be a (finite or infinite) sequence of σ-fields and $X_0, X_1, \ldots$ a sequence of random variables. $\{X_i\}$ is said to be adapted to F_i if X_i is F_i-measurable. An integer valued (possibly ∞) random variable T is called a stopping time relative to F_i if $(T = i) \in F_i$ for all i. For a stopping time T, the σ-field F_T consists of the events A for which $A \cap (T = i) \in F_i$ for all i. If $F_0 \subset F_1 \subset \ldots$ is increasing and $T \geq S$ are stopping times $F_T \supset F_S$; if F_i are decreasing and $T \geq S$ are stopping times $F_T \subset F_S$. These are easily verified. Until further notice F_i will be an increasing sequence of σ-fields.

A sequence $\{X_i\}$ of random variables having expectations and adapted to $\{F_i\}$ is called a super martingale if

(1) $$E[X_{i+1} | F_i] \leq X_i, \quad i = 0,1,2,\ldots,$$

a <u>sub-martingale</u> if $\{-X_i\}$ is a super martingale, and a <u>martingale</u> if both $\{X_i\}$ and $\{-X_i\}$ are super martingales. Thus for a sub-martingale the inequality in (1) is reversed and for a martingale the inequality in (1) is actually an equality. The expression "let $\{X_i\}$ be a super martingale" will mean that $\{X_i\}$ is adapted to $\{F_i\}$, X_i have expectations and $\{X_i\}$ is a super martingale. A super martingale is called <u>non-negative if all</u> X_i <u>are non-negative</u>.

Denoting by d_i the difference $X_i - X_{i-1}$ and defining $X_{-1} = 0$, (1) says that the conditional expectation of d_i given F_i is less or equal to zero. Therefore if $a_0 = b_0$ and for $i \geq 1$, $a_i \leq b_i$ are bounded and F_{i-1} measurable

(2) $$E[a_i d_i] \geq E[b_i d_i].$$

we get

<u>Proposition 1</u>. Let $\{X_i, i \geq 0\}$ be a super martingale. If $a_0 = b_0$ and for $i \geq 1$, $a_i \leq b_i$ are bounded F_{i-1}-measurable functions then for any N

(3) $$E[\sum_0^N b_i d_i] \leq E[\sum_0^N a_i d_i]$$

where $d_i = X_i - X_{i-1}$ and $X_{-1} = 0$.

Suppose now $T \leq S$ are stopping times. Taking a_i = indicator of $(T \geq i) \leq b_i$ = indicator of $(S \geq i)$ we get from (3)

<u>Corollary 2</u>. (<u>Optional sampling Theorem</u>). Let $\{X_i, i \geq 0\}$ be a super martingale. If $T \leq S$ are stopping times then for any N

(4) $$E[X_T : T \leq N] \geq E[X_S : S \leq N] + E[X_N : T \leq N < S]$$

In case of a martingale there is equality in (4).

<u>Remark</u>. There is a simple way of getting a more general conditional inequality from (4). For $A \in F_T$ let T_A be the stopping time which is T on A and infinite ortherwise. Similarly S_A. An application of (4) to these stopping times leads to the said conditional form of (4). This is clearly a general trick and it is useful to make a note of it. For some applications of the optional sampling theorem see Karlin-Taylor [1]. p.p. 263-272.

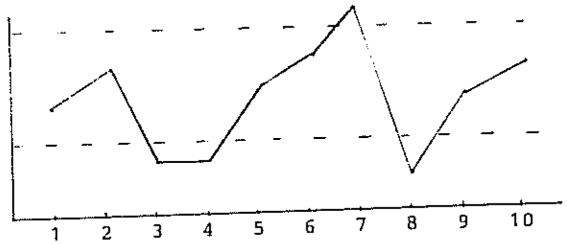
If X_i is non-negative, the last term in (4) may be omitted. Letting N tend to infinity we obtain by monotone converge theorem $E[X_T] \geq E[X_S]$ where we put $X_\infty = 0$. This already implies that a non-negative super martingale converges almost surely. See Exercises 4 and 5.

The case of a decreasing sequence of σ-fields is equally important. Suppose $F_0 \supset F_1 \supset F_2 \ldots$ is a decreasing sequence of σ-fields. A super martingale relative to F_n will be a sequence $X_0, X_1, \ldots$ where X_n is F_n-measurable, has expectation and $E[X_n | F_{n+1}] \leq X_{n+1}$. Exactly as before we obtain an analogue of Corollary 2: If $T \leq S$ are stopping times for any N

(5) $$E[X_T : T \leq N] \leq E[X_S : S \leq N] + E[X_N : T \leq N < S]$$

We now give another famous inequality of Doob originally proved to establish the martingale convergence theorem. Before we do this, however we must establish some notation.

Let $\alpha_1,\dots,\alpha_n$ be an ordered set of numbers and $r<s$ numbers. We say that the set $\{\alpha_1,\dots,\alpha_n\}$ experiences at least k upcrossings of the interval $[r,s]$ if there exist indices $1\le i_1<j_1<i_2<j_2<\dots<i_k<j_k\le n$ such that $\alpha_{i_1}<r$, $\alpha_{j_1}>s$, $\alpha_{i_2}<r$, $\alpha_{j_2}>s$, etc. $\alpha_{i_k}<r$, $\alpha_{j_k}>s$.
Graphically this means the following: suppose we join the successive points (i,α_i) by a broken line, the graph look like this:



and there must be k points above the line s and k points below the line r.

Suppose now that $(X_n, n\ge 0)$ is a super martingale relative to the increasing sequence $(F_n, n\ge 0)$ og σ-fields. If $\gamma<s$ are numbers let for $i\ge 1$

$$b_i = \begin{cases} 1 & \text{if there is } j<i \text{ such that } X_j<\gamma \text{ and} \\ & X_{j+1},\dots,X_{i-1}\le s \\ 0 & \text{otherwise} \end{cases}$$

If $U(N)$ is the number of uperostrings of $[\gamma,s]$ by $X_0,\dots,X_n$, the reader is invited to convince himself that

$$\sum_1^N b_j d_j \ge (s-\gamma)U(N) + X_n - a \ge (s-\gamma)U(N) - (X_N-a)^-.$$

Taking $a_i\equiv 0=b_0$ in (3) we obtain

(6) $$(s-\gamma)E(U(N)) \le E[(X_N-\gamma)^-]$$

Now if $\{F_n, n\le 0\}$ is decreasing the σ-fields $G_n = F_{N-n}$ are increasing for $n\le N$. The following Corollary is thus clear from (6) as N tends to infinity.

<u>Corollary 3</u>. (Doob's upcrossing inequality). Let X_n be a super martingale relative a monotone sequence of σ-fields. If U is the number of upcrossing of $[\gamma,s]$ by the sequence X_n then

$$(s-\gamma)E[U] \le \sup_N E[|X_n-\gamma|]$$

In particular the number of upcrossings is finite with probability 1 provided $\{X_n\}$ is L^1-bounded i.e. $\sup_n E[|X_n|]<\infty$.

As a simple corollary we have the famous

<u>Theorem 4</u>. (Martingale convergence theorem). Let X_n be an L^1-bounded super martingale relative to a monotone sequence of σ-fields. Then $\lim X_n$ exists almost surely.

<u>Proof</u>. If $Z=\limsup X_n$ and $Y=\liminf X_n$ then $P[\lim X_n \text{ exists}]=1$ iff $P[Y<\gamma<s<Z]=0$ for all rationals $\gamma<s$. Since the event $(Y<\gamma<s<Z)$ implies an infinite number of upcrossings of $[\gamma,s]$ by the sequence $\{X_n\}$ the result is clear from Corollary 3. Q.e.d.

There are many applications of the martingale convergence theorem. For a nice accountof these we refer to Chapter VIII of Meyer [2].

Example 1. Consider the probability space $[0,1)$ with the lebesgue measure. Let F_n = the field generated by the sets $A_{i,n} = [i2^{-n}, (i+1)2^{-n})$, $i = 0,1,2,\ldots, 2^n-1$. $F_n \subset F_{n+1}$ and $\cup F_n$ generates the Borel field in $[0,1)$. Let m be a probability measure on $[0,1)$. Put $X_n = \sum_i 2^n m(A_{i,n}) A_{i,n}$, where $A_{i,n}$ denotes also the indicator function of the set $A_{i,n}$ X_n is a martingale relative to F_n and is L^1-nounded. $Y = \lim X_n$ exists almost everywhere by Theorem 4.

For any $k > 0$ let $B_k = \{\sup X_n \le k\}$. Let us show that on B_k, $m = Ydx$, where dx is the lebesgue measure. If $D_i = \{X_j \le k, j \le i\}$ then $m(D_i) = \int_{D_i} X_i dx$. On $D_i, X_i \le k$ so that by dominated convergence, $m(B_k) = \int_{B_k} Ydx$. On the other hand for any $A \in F_n$ and all $i \ge n$, $m(A) = \int_A X_i^k dx$ so that by Fatou, $m(A) \ge \int_A Ydx$. Thus $m = Ydx$ on $(\sup_n X_n < \infty) = \cup B_k$; because X_n converges almost everywhere this set has probability 1. The measure $m-Ydx$ sits on a set of measure zero, so it is singular. We have obtained Lebesgue's decomposition. $m = Ydx + s$, where s is singular.

The following example assumes some acquaintence with some relevant concepts.

Example 2. Let D be a bounded domain in R^d and let us show that a positive harmonic function in D which is extreme among positive harmonic functions in D cannot be bounded. Let u be a bounded positive harmonic function in D. Let D_n be open relatively compact subsets of D increasing to D. X_t will denote the Brownian motion in R^d and T_n, T the exit times from D_n and D respectively. $u(X_{T_n})$ is then a bounded martingale relative to P_a and B_{T_n} (the stopped Borel fields) for every $a \in D$. If $F = \limsup u(X_{T_n})$, $\lim u(X_{T_n}) = F$, P_a-almost everywhere for each

$a \in D$ by Theorem 4.

$$u(a) = E_a[u(X_{T_n})] = E_a[F].$$

Let $0 \le f \le 1$ be continous on R^d. The functions $E_\cdot[fu(X_{T_n})]$ are harmonic in D_n and converge. The limit $v = E_\cdot[f(X_T)F]$ is positive harmonic in D. For the same reason $w = E_\cdot[(1-f)(X_T)F]$ is positive harmonic in D and $u = \frac{1}{2}2v + \frac{1}{2}2w$. For $0 \le f < \frac{1}{2}$ it is clear that $2v$ cannot be equal to u. i.e. u is not extreme.

If in Example 1 m is singular relative to Lebesgue measure, Y must vanish almost everywhere. In particular X_n cannot converge in L^1. Therefore L^1 convergence is not implied by Theorem 4.

Definition. A family $\{X_i\}$ of random variables is called uniformly integrable if

(7) $\quad E[|X_i| : |X_i| \ge N]$ is uniformly small

provided N is large enough. For an equivalent defintion see Exercise 6.

Proposition 5. Let $\{X_n\}$ be a sequence of random variables. Suppose that $\lim X_n = X$ almost everywhere or just in probability. Then X_n tends to X in L^1 iff $\{X_n\}$ is uniformly integrable.

Proof. We will be brief. Suppose $\{X_n\}$ is uniformly integrable. This implies in particular that $E[|X_n|]$ is uniformly bounded.

By Fatou X is integrable. X_n truncated at N tends to X truncated at N almost everywhere and in L^1 by dominated convergence. The rest is uniformly small by (7) for large enough N.

The other direction is equally simple. Q.e.d.

In view of the above Proposition we can say: <u>L^1-convergence obtains in Theorem 4 iff $\{X_n\}$ is uniformly integrable</u>.

It is easy to see that any one of the following implies uniform integrability of a family $\{X_i\}$:

X_i are bounded by a integrable function.

The p-th moment of X_i is uniformly bounded for some $p > 1$.

There is a random variable X with finite expectation such that each X_i is the conditional expectation (relative to some σ-field) of X.

Super martingales relative to increasing and decreasing sequences of σ-fields are not completely similar as the following simple proposition shows. We shall also be using this proposition in the next section.

<u>Proposition 5</u>. Let X_n be a super martingale relative to a decreasing sequence of σ-fields F_n. Then X_n is uniformly integrable iff $\sup E[X_n] < \infty$.

<u>Proof</u>. $E[X_n]$ is increasing with n. And if $m \geq n$, $Y_{n,m} = X_m - E[X_n | F_m] \geq 0$. Also $E[Y_{n,m}] = E[X_m] - E[X_n]$ tends to zero as

$n \leq m$ tend to infinity. We have if $A = (|X_m| \geq \lambda)$

$$E[|X_m| : A] \leq E[Y_{n,m}] + E[|E(X_n | F_m)| : A] \leq E[Y_{n,m}] + E[|X_n| : A]$$

$E[Y_{n,m}]$ is small if n is large enough, uniformly in $m \geq n$. $P(A)$ will be small for large λ, if we show that $\{X_m\}$ is L^1-bounded i.e. if $\sup E[|X_m|] < \infty$ and then the last quantity in the above inequality will also be small uniformly in $m \geq n$. Now $E[X_0 : X_m < 0] \leq E[X_m : X_m < 0]$ so that $E[X_m^-]$ is bounded by $E[|X_0|]$. Hence $\sup E[X_m] < \infty$ is equivalent to saying that $\{X_m\}$ is L^1-bounded. Q.e.d.

The following "maximal" inequality extends the well known Kolmogorov inequality.

<u>Lemma 6</u>. Let X_n, $n = 0,1,2,\ldots$ be a non-negative sub-martingale. Then for any $a > 0$

(8) $$aP(\sup X_i \geq a) \leq \sup_n E[X_n].$$

<u>Proof</u>. Put $T = \min(n : X_n \geq a)$, $T = \infty$ if there is no such n. Take $S = \infty$ in Corollary 2 and remember that we now have a sub-martingale: For any N

$$E[X_T : T \leq N] \leq E[X_N : T \leq N]$$

Since $X_T \geq a$, if $T < \infty$ the above inequality implies that

(9) $$aP(T \leq N) \leq E[X_N : T \leq N]$$

(8) is a consequence of (9), because the set $(\sup X_i \geq a)$ is the same as the set $(T < \infty)$. Q.e.d.

Exercises to §1

1. If X_i and Y_i are super martingales so is $X_i + Y_i$. If X_i is a martingale $|X_i|$ is a sub-martingale. If X_i is a non-negative sub-martingale so is X_i^p, $p \geq 1$.

Hint. Hölders inequality.

2. A super martingale X_i is called L^1-bounded if $\sup E[|X_i|] < \infty$. Show that a martingale is the difference of two non-negative martingales iff it is L^1-bounded.

Hint. Suppose X_i is L^1-bounded. For n fixed, $Y_{p,n} = E[|X_{p+n}||F_n]$ dominates $|X_n|$ and increases with p. Then $Y_n = \lim Y_{p,n}$ is a martingale.

3. Show that a super martingale is the difference of two non-negative super martingales iff it is L^1-bounded.

Hint. If X_i is a L^1-bounded super martingale, for fixed n, $Y_{p,n} = E[X_{n+p}|F_n]$ is decreasing in p, is dominated by X_n. $Y_n = \lim Y_{p,n}$ is a L^1-bounded martingale. Now use Exercise 2.

4. If X_i is a non-negative super martingale then $\lim X_i$ exists almost everywhere.

Hint. Given $0 < a < b$ define the sequences T_i, S_i of stopping times by

$$T_1 = \inf\{n: X_n < a\}$$
$$S_1 = \inf\{n: n > T_1, X_n > b\}$$

and inductively

$$T_{n+1} = \inf\{m: m > S_n, X_m < a\}$$
$$S_{n+1} = \inf\{m: m > T_{n+1}, X_m > b\}$$

the infimum over an empty set being always defined ∞. Then $T_n \leq S_n \leq T_{n+1}$, $T_n < \infty$ implies $X_{T_n} < a$ and $S_n < \infty$ implies $X_{S_n} > b$. And

$$A = \cap(S_i < \infty) = \cap(T_i < \infty) = (\liminf X_i < a < b < \limsup X_i)$$

Finally $bP[S_i < \infty] \leq E[X_{S_i}] \leq E[X_{T_i}] \leq aP[T_i < \infty]$. Let i tend to ∞ to conclude $bP(A) \leq aP(A)$.

5. An L^1-bounded super martingale converges almost surely.

Hint. Use Exercises 3 and 4.

6. Show that uniform integrability of a family $\{X_\alpha\}$ is equivalent to the two conditions: a) $E[|X_\alpha|]$ is uniformly bounded and, b) to every $\varepsilon > 0$ there corresponds a $\delta > 0$ such that $P(A) < \delta$ implies $E[|X_\alpha|:A] < \varepsilon$ for all α. If the probability space is non-atonic the first condition follows from the second.

Hint. Under condition a) forlarge N, by Chebyschev $P[|X_\alpha| > N]$ is uniformly small. Therefore by condition b) we get 7). If there are no atoms, for $\varepsilon = 1$ we can write the space as a union of a finite number of sets each of which has probability $\leq \delta$ such that the integral of $|X_\alpha|$ on each of these sets ≤ 1 for all α. Hence condition a) is automatic.

7. A martingale $\{X_n\}$ is uniformly integrable iff these exists X such that $X_n = E[X|F_n]$.

Hint. Let $X = \lim X_n$.

8. Let F_n be a sequence of σ-fields. F_n is said to "converge" to a σ-field F, written $\lim F_n = F$ if $\bigcap_n G_n = F =$ σ-field generated by $\bigcup_n H_n$ where $G_n =$ σ-field generated by $\bigcup_{m \geq n} F_m$ and $H_n = \bigcap_{m \geq n} F_m$. For integrable X, $E[X|F_n]$ converges in L^1 to $E[X|F]$.

9. Let F_n be an increasing sequence of σ-fields and X_n integrable random variables such that $[X_n] \leq \varphi$ where $\varphi \in L_1$ and $\lim X_n = X$ almost surely. Then $\lim Y_n = E[X_\infty|F_\infty]$ where $F_\infty =$ σ-field generated by $\bigcup_n F_n$ and $Y_n = E[X_n|F_n]$. A similar result holds for a decreasing sequence of σ-fields.

10. Let F and G be σ-fields. F and G are said to be <u>conditionally independent</u> given $F \cap G$ if for all F-measurable X and G-measurable Y, $E[XY|F \cap G] = E[X|F \cap G] \cdot E[Y|F \cap G]$. Show that F and G are donditionally independent given $F \cap G$ iff for all Z, $E[E[Z|F]|G] = E[E[Z|G]|F]$.

11. Let F_n be an increasing sequence of σ-fields. For stop rules T, S, F_T and F_S are conditionally independent given $F_{T \wedge S}$.

§2. Continuous parameter

Let $F(t), 0 \leq t < \infty$ be an increasing family of σ-fields. Assume also that the <u>family is right continuous</u>:

$$F(t) = \bigcap_{s>t} F(s)$$

As in the discrete case a family $X(t)$ of integrable random variables is called a super martingale if $X(t)$ is $F(t)$-measurable and

$$E[X(t)|F(s)] \leq X(s) \qquad s \leq t,$$

a sub-martingale if $-X(t)$ is a super martingale and a martingale if it is both sub and super.

A super martingale is called <u>right continuous if for almost all</u> w, $t \to X(t)$ <u>is right continuous</u>.

A <u>stopping time</u> T is a non-negative random variable such that

$$(T \leq t) \in F(t) \quad \text{for all } t.$$

Each stopping time T is the limit of a decreasing sequence of <u>descrete stopping times</u>: Define T_n by

(1) $$T_n = (i+1)2^{-n} \quad \text{if} \quad i2^{-n} \leq T < (i+1)2^{-n}$$
$$= \infty \quad \text{if} \quad T = \infty.$$

It is easy to check that T_n are stopping times and decrease to T.

Thus if $X(\cdot)$ is right continuous $X(T)$ is a random variable for any stopping time T.

For a stopping time T the σ-field $F(T)$ is defined by:

$$F(T) = \{A : A \cap (T \leq t) \in F(t) \quad \text{for all} \quad t\}.$$

Then, if $T \leq S$, $F(T) \subset F(S)$ and if $X(\cdot)$ is right continuous, $X(T)$ is $F(T)$-measurable.

The right continuity of the σ-fields $F(t)$ implies that $F(T_n)$ decreases to $F(T)$ whenever the stopping times T_n decrease to T.

All the above statements are easily proved. We will not do this here because they are done for the Brownian motion in Chapter 2 and the proofs in the general case are essentially the same.

Proposition 1. (Optional sampling theorem). Let $X(t)$ be a right continuous super martingale. If $T \leq S$ are bounded stopping times then

(2) $$E[X(T)] \geq E[X(S)]$$

And there is equality in case of a martingale.

Proof. If T and S are descrete, this is just Corollary 2, §1; (4), §1 is simply another way of writing (2) if $S \leq N$. Now suppose T_n is as in (1) and decreases to T. By the right continuity $X(T_n)$ tends to $X(T)$. That $X(T_n)$ is a super martingale relative to the decreasing sequence $F(T_n)$ of σ-fields follows from (2) (for descrete stopping times) and the remark after Corollary 2, §1. Since $E[X(T_n)] \leq E[X(0)]$, Proposition 5, §1 applies and we can conclude that $X(T_n)$ tends to $X(T)$ in L^1. Q.e.d.

Corollary 2. Let $X(t)$ be a non-negative right continuous sub-martingale. Then for any $a > 0$,

$$aP[\sup_t X(t) \geq a] \leq \sup_t E[X(t)].$$

Proof is exactly as that of Lemma 6, §1.

We shall use the following simple proposition in Example 2 below.

Proposition 3. Let $X(t)$ be a martingale such that $|X(t)| \leq Mt$ for som constant M. If T is a stoppingtime such that $E[T] < \infty$ then $E[X(T)] = E[X(0)]$.

Proof. $T < \infty$ almost everywhere. Apply (2) with $S = 0$ and $T \wedge n$, instead of T. The variables $X(T \wedge n)$ tend to $X(T)$ and are bounded by the integrable function $M.T.$ and dominated converges can be used to conclude the proof.

In the example below a knowledge of relevant terms will be assumed.

Example 1. This example illustrates a simple application of the martingale convergence theorem to excessive functions. We will show that if f and g are excessive (see §1, Chapter 5) so is $h = f \wedge g$.

Let X_t be the Brownian motion (or any standard Markov process). We claim that if t_n is a sequence decreasing to zero then

(3) $$P_a[\lim f(X_{t_n}) = f(a)] = 1.$$

Indeed, for any positive number A, $f(X_{t_n}) \wedge A$ being a non-negative bounded super martingale relative to a decreasing sequence of σ-fields converges almost everywhere and in L^1. $\lim f(X_{t_n})$ thus exists almost everywhere, which by the zero one law must be a constant, say B.

By L^1-convergence of $f(X_{t_n}) \wedge A$

$$f(a) = \lim E_a[f(X_t)] \geq \lim E_a[f(X_{t_n}) \wedge A]$$
$$= B \wedge A \geq E_a[f(X_t) \wedge A]$$

because the last quantity is a decreasing function of t. Letting A tend to infinity and t tend to zero we get (3).

If f and g are excessive and $h = f \wedge g$ then for all t, $E_a[h(X_t)] \leq h(a)$. From (3) as t_n decreases to zero $h(X_{t_n})$ tends to $h(a)$ and so an appeal to Fatou shows that h is indeed exsessive-

Example 2. Let u be twice continuously differentiable in R^d and suppose that Δu is bounded, Δ denoting Laplacian. If X_t denotes Brownian motion

(4) $$u(X_t) - u(X_0) - \tfrac{1}{2}\int_0^t \Delta u(X_s)\,ds$$

is a martingale relative to P_a, for any $a \in R^d$. To see this denote by p the heat Kernel $p(t,x) = (2\pi t)^{-d/2} \exp(-|x|^2 |2t)$, which satisfies

$$\tfrac{1}{2}\Delta p = \frac{\partial}{\partial t}p$$

Then

$$\frac{\partial}{\partial t}u*p(t,\cdot) = u*\frac{\partial}{\partial t}p(t,\cdot)$$
$$= u*\tfrac{1}{2}\Delta p = \tfrac{1}{2}\Delta u*p$$

which when integrated leads to

$$E_\cdot[u(X_t)] - u(\cdot) = \tfrac{1}{2}E_\cdot[\int_0^t \Delta u(X_s)\,ds]$$

(4) follows easily from Markov property and the last identity. Proposition 3 now implies:

If T is a stopping time such that $E_a[T] < \infty$ then

$$E_a[u(X_T)] - u(a) = \tfrac{1}{2}E_a[\int_0^T \Delta u(X_s)\,ds]$$

This is known as Dynkins formula. More on this in Chapter 4.

REFERENCES

1. Karlin, S., Taylor, H.M., A first Course in Stochastic Processes. Academic Press. (1975).

2. Meyer, P.A., Probability and potentials. Blaisdell. (1966).

For more on Martingales consult the above book of Meyer plus Strasbourg Lecture Notes.

CHAPTER 2

The d-dimensional Brownian motion

Introduction and summary

In this chapter we introduce the d-dimensional Brownian motion, assuming as known the 1-dimensional Brownian motion starting at 0. Then we consider Markov times in more detail than is absolutely necessary, in the hope that this will better the understanding of this important notion. Strong Markov property is proved and a few applications considered.

§1. The d-dimensional Brownian motion

As proved, for instance in [3] pp. 12 - 16, there is a real valued stochastic process $\xi(t)$, $0 \leq t < \infty$ (on some probability space), such that $\xi(t)$ is continuous, $\xi(0) \equiv 0$ and with the finite dimensional distributions

$$P[\xi(t_i) \in E_i,\ 1 \leq i \leq n] \tag{1}$$
$$= \int_{E_1} p'(t_1,0,da_1) \int_{E_2} p'(t_2 - t_1, a_1, da_2) \ldots \int_{E_n} p'(t_n - t_{n-1}, a_{n-1}, da_n)$$

where $0 < t_1 < \ldots < t_n$, $p'(t,a,db) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(b-a)^2}{2t}\right) db$ and $E_1, \ldots, E_n$ are Borel subsets of the real line. See also pp. 5 - 8 of [5] for another proof of this fact.

Now let us define the d-dimensional Brownian motion. There is a probability space $(\Omega, \mathbb{B}, P)$ and a d-dimensional stochastic