

AARHUS UNIVERSITET

BROWNIAN MOTION
AND
CLASSICAL POTENTIAL THEORY

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February 1977

Lecture Notes Series

NO. 47.

MATEMATISK INSTITUT

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P R E F A C E

In these notes we present some basic classical potential theory. Although language of Brownian motion is used the discussion is more analytic than probabilistic.

Chapters 4 to 7 describe the principal results whereas the first three chapters are of a more or less general character. Ample hints are provided for almost all the exercises.

Senior under graduate level mathematics is necessary prerequisites for reading this set of notes.

Murali Rao

C O N T E N T S

<u>Chapter 1.</u> Martingales.	P-P.
§ 1. Optimal sampling.	1. 1-1.12
§ 2. Continuous parameter.	1. 13-1.17
<u>Chapter 2.</u> The d-dimensional Brownian motion.	
§ 1. d-dimensional Brownian motion.	2.1-2.7
§ 2. Strong Markov property.	2.7-2-20
<u>Chapter 3.</u> Semi Groups.	
§ 1. Semi Groups.	3.3-3.10
§ 2. Infinitesimal generator.	3.10-3.21
§ 3. Potential operators.	3.23-3.34
<u>Chapter 4.</u> Harmonic functions.	
§ 1. Dynkin's formula.	4.2-4.10
§ 2. Dirichlet problem.	4.10-4.24
§ 3. The Kelvin transformation.	4.24-4.30
§ 4. Fatou Theorems.	4.30-4.37
§ 5. Spherical harmonics.	4.37-4.49
<u>Chapter 5.</u> Superharmonic functions.	
§ 1. Superharmonic functions.	5.2-5.14
§ 2. Applications.	5.15-5.20
§ 3. Riesz measure.	5.21-5.34
§ 4. The continuity Principle.	5.34-5.39
§ 5. The Dirichlet problem revisited.	5.39-5.46
<u>Chapter 6.</u> Green functions.	
§ 1. Bounded open sets.	6.2-6.21
§ 2. Unbounded open subsets of R^2 .	6.21-6.29
§ 3. Unbounded open sets (continued).	6.30-6.36
§ 4. Examples.	6.36-6.44
§ 5. Relative transition.	6.45-6.51

Chapter 7. Potential Theory.

- § 1. Principles.
- § 2. Capacity.
- § 3. Applications.
- § 4. Balayage.
- § 5. Dirichlet Spaces.

p.p.
7.1-7.23
7.23-7.33
7.33-7.40
7.40-7.49
7.49-7.71

CHAPTER 1

Martingales

§1 Optional sampling, inequalities, and convergence

We will not be needing too much martingale theory. What little is done is to make the notes more complete. In the following we assume given a fixed probability space (Ω, \mathcal{B}, P) . All σ -fields considered are sub- σ -fields of \mathcal{B} . In some of the examples some knowledge of the relevant concepts is needed.

Let F_0, F_1, \dots be a (finite or infinite) sequence of σ -fields and X_0, X_1, \dots a sequence of random variables. $\{X_i\}$ is said to be adapted to F_i if X_i is F_i -measurable. An integer valued (possibly ∞) random variable T is called a stopping time relative to F_i if $(T=i) \in F_i$ for all i . For a stopping time T , the σ -field F_T consists of the events A for which $A \cap (T=i) \in F_i$ for all i . If $F_0 \subset F_1 \subset \dots$ is increasing and $T \geq S$ are stopping times $F_T \supset F_S$; if F_i are decreasing and $T \geq S$ are stopping times $F_T \subset F_S$. These are easily verified. Until further notice F_i will be an increasing sequence of σ -fields.

A sequence $\{X_i\}$ of random variables having expectations and adapted to $\{F_i\}$ is called a super martingale if

$$(1) \quad E[X_{i+1} | F_i] \leq X_i, \quad i = 0, 1, 2, \dots,$$