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FAMILIES OF RANDOM TRIPLES

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I discuss some problems in combinatorial measure theory arising from work of P. Erdös and A. Hajnal ([1]).

1. DEFINITION : *Let* θ , λ , κ *be cardinals,* $1\leqslant\theta\leqslant\lambda\leqslant\kappa$, *and* $u \in [0,1]$.

I say that

$$([\kappa]^\theta, u) \Rightarrow [\lambda]^\theta$$

if for every family $\langle E_I\rangle_{I\in[\kappa]^\theta}$ *of measurable subsets of* $[0,1]$ *such that* $\mu E_I \geqslant u$ *for every* $I \in [\kappa]^\theta$, *where* μ *is Lebesgue measure, there is a* $K \in [\kappa]^\lambda$ *such that* $\bigcap_{I\in[K]^\theta} E_I \neq \emptyset$.

2. The case $\theta = 1$, λ countable is simple. We see easily that $([\kappa]^1, u) \Rightarrow [\lambda]^1$ iff for every family $\langle E_\xi\rangle_{\xi<\kappa}$ of measurable set of measure $\geqslant u$, there is a $K \in [\kappa]^\lambda$ such that $\bigcap_{\xi\in K} E_\xi \neq \emptyset$, that is to say, there is an $s \in [0,1]$ such

that $\#(\{\xi : s \in E_\xi\}) \geq \lambda$. So if $1 \leq \lambda \leq \min(\kappa,\omega)$, $([\kappa]^1,u) \Rightarrow [\lambda]^1$ iff

either (α) κ is finite and $\lambda < \kappa u + 1$

or (β) κ is infinite and $u > 0$.

3. The case $\theta = 2$, λ countable, κ infinite has been resolved.

If $2 \leq \lambda \leq \omega \leq \kappa$, $([\kappa]^2,u) \Rightarrow [\lambda]^2$ iff

either (α) $\lambda < \kappa = \omega$, $u > \frac{\lambda-2}{\lambda-1}$ ([1], [2])

or (β) $\kappa > \omega$, $u > 0$

or (γ) $\kappa = \omega$, $u = 1$.

The case principally of interest to us here is when $\theta = 3$. The remaining results of this note are proved in [3].

4. LEMMA : *If* $1 \leq k \leq r < \omega$ *and* $u \in [0,1]$ *and* $([\omega]^2,u) \Rightarrow [r]^k$, *then there are a* $u' < u$ *and an* $m < \omega$ *such that* $([m]^2,u') \Rightarrow [r]^k$.

Proof : A compactness argument ; see [3], Lemma 2.

5. PROPOSITION : *If* $1 \leq k \leq r < \omega \leq \kappa$ *and* $u \in [0,1]$ *and* $([\kappa^+]^{k+1},u) \Rightarrow [r+1]^{k+1}$ *then* $([\kappa]^k,u) \Rightarrow [r]^k$.

Proof : The contrapositive is a straighforward construction ; see [3], Proposition 3.

6. THEOREM : *If* $1 \leq k \leq r < \omega$ *and* $u \in [0,1]$, *then*

(a) *if* $([\omega]^k,u) \Rightarrow [r]^k$ *then* $([\omega_1]^{k+1},u) \Rightarrow [r+1]^{k+1}$ *and* $([\mathfrak{c}^+]^{k+2},u) \Rightarrow [r+2]^{k+2}$;

(b) *if* $\kappa \geq \mathfrak{c}$ *and* $([\kappa^+]^k,u) \Rightarrow [r]^k$ *then* $([(2^\kappa)^+]^{k+1},u) \Rightarrow [r+1]^{k+1}$.

Proof : This is more complicated ; it uses the ideas of the Erdös-Rado Stepping-up Lemma. See [3], Theorem 5.

7. PROPOSITION : *(a)* $([\omega]^3, \frac{5}{9}) \not\rightarrow [4]^3$.

(b) $([\omega_1]^3, u) \Rightarrow [4]^3$ *iff* $u > \frac{1}{2}$.

(c) $([\mathbb{R}]^3, \frac{1}{4}) \not\rightarrow [4]^3$.

(d) $([\mathbb{R}^+]^3, u) \rightarrow [4]^3$ *iff* $u > 0$.

Proof : (a) Let $X = \{0,1,2\}^\omega$ and let ν be the standard measure on X . For $I = \{i,j,k\} \in [\omega]^3$ set

$$E_I = \{x : x \in X \text{ , \underline{either} } x(i) , x(j) , x(k) \text{ are all different}$$
$$\text{\underline{or} } x(i) + x(j) + x(k) \equiv 1 \mod 3\}.$$

Then $\nu E_I = \frac{5}{9}$ for every I but $\bigcap_{I \in [K]^3} E_I = \emptyset$ for every $K \in [\omega]^4$. Since (X,ν) is isomorphic to $([0,1],\mu)$, this shows that $([\omega]^3, \frac{5}{9}) \not\rightarrow [4]^3$.

(b) Use Prop. 5 (with $\kappa = \omega$, $r = 3$, $k = 2$), Theorem 6a (with $r = 3$, $k = 2$) and 3α (with $\lambda = 3$).

(c) Let X be $\{0,1\}^\omega$ and ν the standard measure on X . Let $\leqslant$ be the lexicographic ordering of X and $+$ the group operation on X , identified with $\mathbb{Z}_2^\omega$. If $I = \{x,y,z\} \in [X]^3$, where $x < y < z$ in X , set

$$E_I = \{w : w \in X , w + x < w + y , w + z < w + y\} .$$

Then $\nu E_I = \frac{1}{2}$ if there is an n such that $x \upharpoonright n = y \upharpoonright n = z \upharpoonright n$ while $x(n) = z(n) \neq y(n)$, and $\frac{1}{4}$ otherwise. But if $x < y < z < t$ in X , then $E_{\{x,y,z\}} \cap E_{\{y,z,t\}} = \emptyset$. So $\bigcap_{I \in [K]^3} E_I = \emptyset$ whenever $K \in [X]^4$.

(d) Use 2β and 6a (with $r = 2$, $k = 1$).

8. REMARKS AND PROBLEMS : (a) Is there a $u > \frac{5}{9}$ such that $([\omega]^3, u) \not\twoheadrightarrow [4]^3$?

By analogy with the results of [2], it is possible that this comes to the same thing as asking :

> are there an integer $p \geq 1$ and a set $H \subseteq p^3$ such that $\#(H) > 5p^3/9$ and for every $f : 4 \to p$ there is an
>
> $I = \{i,j,k\} \in [4]^3$ such that $(f(i),f(j),f(k)) \notin H$?

The point is that such a p , H could be used to construct an example along the lines of 7a, which can be got by taking $p = 3$,

$$H = \{(i,j,k) : \text{either } i, j, k \text{ are all different or } i+j+k \equiv 1 \mod 3\} .$$

In [1] and [2] a large variety of similar problems, in two rather than three dimensions, are reduced to similar combinatorial questions.

(b) Is it consistent to suppose that $([\mathfrak{c}]^3, \frac{1}{2}) \twoheadrightarrow [4]^3$ or that $([\omega_2]^3, \frac{1}{2}) \not\twoheadrightarrow [4]^3$?

The example of 7c seems somehow less economical than that of 7a, so there may be room for improvement in it.

9. ACKNOWLEDGEMENTS : Prop. 7b is an answer to a question of P. Erdös. Prop. 7a was known to Erdös ; I understand it to be due to P. Turan.

R E F E R E N C E S

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