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Puzzles

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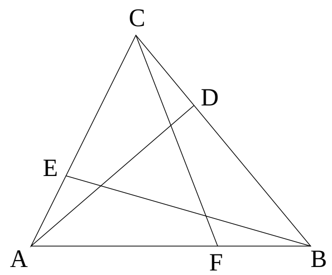
I collect here some questions – all classics – which I have enjoyed answering.

ZR Problem Let ABC be a triangle. Let D be the point which is two thirds of the way from B to C, E the point which is two thirds of the way from C to A, and F the point which is two thirds of the way from A to B, as in the diagram:



Consider the inner triangle formed by the lines AD, BE and CF. (i) Show that its area is one seventh of the area of the triangle ABC. (ii) Show that its centroid is the same as the centroid of ABC. (iii) Find the ratios of the lengths of the three segments of the line AD.

ZS Problem Let $X_0, X_1, \ldots$ be independent random variables uniformly distributed in $[0, 1]$. Set $S_n = \sum_{i=0}^{n} X_i$ for each n. Let τ_1, τ_2 be the arrival times for the process $\langle S_n \rangle_{n\in\mathbb{N}}$ in $]1, \infty[$, $]2, \infty[$ respectively. What is the probability that $X_{\tau_1} \geq X_{\tau_2}$?

(If you have not seen 'stopping times' before, think of τ_k as being equal to m if $S_m > k$ but $S_{m-1} \leq k$, so that X_{τ_k} is the sample which pushes the sum over k. With probability 1 there is such a sample, by Chebyshev's inequality or otherwise.)

ZT Problem For each of the five Platonic solids, determine (i) its chromatic number n (that is, the smallest number of colours which can be used to colour its faces in such a way that no two faces with an edge in common have the same colour) (ii) the number of different ways of colouring the faces with colours $1, \ldots, n$, counting colourings as different if neither can be obtained from the other by rotating the solid (iii) the number of different ways of colouring the faces with colours $1, \ldots, n$, counting colourings as different if neither can be obtained from the other by rotating or reflecting the solid (iv) the number of different ways of colouring the faces with colours $1, \ldots, n$, counting colourings as different if neither can be obtained from the other by rotating or reflecting the solid and permuting the colours.

What about the cuboctahedron, icosidodecahedron, rhombic dodecahedron and rhombic triacontahedron?

ZU Problem Let H be a regular hexagon in the plane. Divide it into $6k^2$ equilateral triangles with sides parallel to the sides of H. Consider partitions of H into lozenges each of which is a pair of these triangles joined at a common edge. Each such lozenge has one of three orientations. Show that there are k^2 lozenges with each orientation.

What about other shapes than regular hexagons?

ZV Problem Let ABC be a triangle in the plane. Trisect each of the angles. For each of the three sides of ABC, look at the trisectors of the angles at the ends of that side which are nearer to the side, and take their intersection; this gives a new triangle DEF inside ABC. Show that DEF is always equilateral.

ZW Problem You have a cylindrical well exactly 1 metre in diameter, and a stock of long thin planks each 11 cm wide. How many of the planks do you need to cover the well completely without leaving any vertical path into the well? (You have no way of cutting the planks.)

Hint: eureka!

ZX Problem ('Gossiping Dons') n individuals have information to contribute. They share this information by making simple telephone calls in which all information possessed by either participant is passed to the other. How many calls are needed for all information to reach every participant?

ZY Problem Suppose that there are n students wishing to study medicine, and n places available at medical schools. Each student has a partial preference order among the medical schools, and each school has a partial preference order among the students. Show that there is a 'stable matching', that is, an assignment of students to schools in such a way that there is no pair (x, y) where x is a student, y is a school, x strictly prefers y to the school to which she is assigned, and y strictly prefers x to one of the students assigned to y.

ZZ Problem Show that for any $n \geq 1$ the equation

$$x^n + y^n \equiv z^n \mod p$$

has non-trivial solutions for all but finitely many primes p.

Acknowledgments Problems ZZ and ZX are from Imre Leader's collection of problems for students; I heard of them from A.Zsak and T.D.Austin respectively. Problem ZU came to me from Bruce Anderson. Problem ZS was one which my son J.T.Fremlin set me. Problem ZR is in K.L.Chung's book *Green, Brown and Probability.*