Postscript to Fremlin 89

1. I note first that Problems 5.15(a) and 5.15(b) have been solved, negatively, by M.Talagrand (Talagrand 08; see Fremlin 12, §394). Concerning 5.15(c), I find that I don't know whether Talagrand's example satisfies the σ-bounded chain condition.

2. In 6.2(e) on p. 958 of Fremlin 89 I say that 'if a Boolean algebra A satisfies the countable chain condition and $\gamma^*_\omega(A) > \omega_1$ then $\gamma^*_\omega(A) = \gamma_\omega(A)$', without giving a proof. I expect the argument I had in mind was essentially as follows. Let A be a Boolean algebra. Repeating the definitions in 6.1, set

$$\gamma_\omega(A) = \min\{|X| : X \subseteq A^+ \text{ and there is no countable } D \subseteq A^+ \text{ such that every member of } X \text{ includes a member of } D\},$$

$$\gamma^*_\omega(A) = \min\{|X| : X \subseteq A^+ \text{ and there is no countable } D \subseteq A^+ \text{ such that } x = \sum\{d \in D : d \le x\} \text{ for every } x \in X\}.$$

Suppose that $\gamma^*_\omega(A) > \omega_1$ and that $X \subseteq A$ and $\#(X) < \gamma_\omega(A)$. Choose a non-decreasing family $\langle D_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $D_0 = \{1\}$. Given $\xi < \omega_1$ such that D_ξ has been defined, set

$$E_{x\xi} = \{y : y \in A^+,\ y \le x,\ y \cdot d = 0 \text{ whenever } d \in D_\xi \text{ and } d \le x\}$$

for $x \in X$, and $X_\xi = \{x : x \in X,\ E_{x\xi} \ne \emptyset\}$; choose $z_{x\xi} \in E_{x\xi}$ for $x \in X_\xi$; take a countable $D'_\xi \subseteq A^+$ such that every $z_{x\xi}$ includes a member of D'_ξ, and set $D_{\xi+1} = D_\xi \cup D'_\xi$. For non-zero countable limit ordinals ξ, set $D_\xi = \bigcup_{\eta<\xi} D_\eta$.

This construction ensures that every D_ξ is a countable subset of A^+. Set $D^* = \bigcup_{\xi<\omega_1} D_\xi$, so that $D^* \subseteq A^+$ and $|D| \le \omega_1$. Now $x = \sup\{d : d \in D^*,\ d \le x\}$ for every $x \in X$. For if $x \in X$ is such that x is not the supremum of $\{d : d \in D^*,\ d \le x\}$, then $E_{x\xi}$ is never empty, and $z_{x\xi}$ is defined for every $\xi < \omega_1$; but in this case $\{z_{x\xi} : \xi < \omega_1\}$ is a disjoint family in A^+, which is impossible.

At the same time, because $\gamma^*_\omega(A) > \omega_1 \ge |D^*|$, there is a countable set $D \subseteq A^+$ such that $z = \sup\{d : d \in D,\ d \le z\}$ for every $z \in D^*$. And now $x = \sup\{d : d \in D,\ d \le x\}$ for every $x \in X$. As X is arbitrary, $\gamma_\omega(A) \le \gamma^*_\omega(A)$; but the reverse inequality is trivial, as noted in Fremlin 89.

3. If A satisfies the countable chain condition and $\gamma_\omega(A) > \mathfrak{c}$ then $\pi(A) \le \omega$. **P?** Otherwise, choose a non-decreasing family $\langle D_\xi\rangle_{\xi<\omega_1}$ of subsets of A^+ of size at most $\mathfrak{c}$, as follows. Start with $D_0 = \emptyset$. Given D_ξ, where $\xi < \omega_1$, $D_{\xi+1} \supseteq D_\xi$ is to be a set of size at most $\mathfrak{c}$ such that (i) whenever $C \subseteq D_\xi$ is countable and has a non-zero lower bound in A, then it has a lower bound in $D_{\xi+1}$ (ii) whenever $C \subseteq D_\xi$ is countable, there is an element of $D_{\xi+1}$ not including any member of C; this is possible as D_ξ has at most $\mathfrak{c}$ countable subsets. At limit ordinals $\xi \le \omega_1$, set $D_\xi = \bigcup_{\eta<\xi} D_\eta$. At the end of the induction, $\#(D_{\omega_1}) \le \mathfrak{c} < \gamma_\omega(A)$ so there is a countable set $C \subseteq A^+$ such that every member of D_{ω_1} includes a member of C. Because A satisfies the countable chain condition, there is for each $c \in C$ a countable set $E_c \subseteq D_{\omega_1}$ with the same lower bounds as $\{d : c \le d \in D_{\omega_1}\}$. Now there is a $\xi < \omega_1$ such that $\bigcup_{c\in C} E_c \subseteq D_\xi$, and there is a $c' \in D_{\xi+1}$ which is a lower bound for E_c. Next, $C' = \{c' : c \in C\}$ is a countable subset of $D_{\xi+1}$, so there is a $d \in D_{\xi+2}$ not including any member of C'. However, there is a $c \in C$ such that $d \in E_c$ and $c' \le d$, which is impossible. **XQ**

4. So if the continuum hypothesis is true, $\gamma_\omega(A) = \gamma^*_\omega(A)$ for every Boolean algebra A satisfying the countable chain condition. (If $\gamma^*_\omega(A) > \omega_1$ use §2, and if $\gamma_\omega(A) > \omega_1$ then §3 tells us that $\pi(A) = \omega$ so $\gamma^*_\omega = \infty$.)

5. I still don't know whether $\gamma^*_\omega(A) = \gamma_\omega(A)$ whenever A is a Boolean algebra satisfying the countable chain condition if the continuum hypothesis is false.

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References

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