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Problem GO

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1 The problem

1A Problem GO The problem as asked in my problem sheet was:

Let μ be a Radon probability measure on $X = \mathbb{R}^r$, where $r \ge 1$, and $\lambda = \mu^{\mathbb{N}}$ the product measure on $X^{\mathbb{N}}$. Suppose that $F \subseteq X$ is a closed set. For $\omega \in X^{\mathbb{N}}$, $n \ge 1$ and $y \in X$, set $\zeta_{\omega yn} = \min_{i<n} \|\omega(i)-y\|$, $k_{\omega yn} = \min\{i : i < n, \|\omega(i)-y\| = \zeta_{\omega yn}\}$; set $F_{\omega n} = \{y : \omega(k_{\omega yn}) \in F\}$.
Is it always the case that $\lim_{n\to\infty} \mu(F \triangle F_n(\omega)) = 0$ for λ-almost every ω?

In this form, the question remains open, though affirmative answers in special cases are given in Theorems 3B and 5B below. I will look also at generalizations, based on the following definitions.

1B Definitions (a) Let (X, ρ) be a metric space.

(i) For $n \ge 1$, $\varpi \in X^n$ and $z \in \varpi[n]$, set

$$V(\varpi, z) = \{x : \rho(x, z) = \rho(x, \varpi[n]) \text{ and if } i < j < n \text{ and } z = \varpi(j) \ne \varpi(i) \text{ then } \rho(x, z) < \rho(x, \varpi(i))\};$$

that is, $z = \varpi(j)$ where $j < n$ is minimal subject to $\rho(x, \varpi(j)) = \rho(x, \varpi[n])$. Now the partition $\langle V(\varpi, z)\rangle_{z\in\varpi[n]}$ is the **Voronoi tesselation** defined by ϖ.

(ii) If f is any function defined on X, and $\varpi \in \bigcup_{n\ge1} X^n$, we have a function $F(\varpi, f)$ defined by setting

$$F(\varpi, f)(x) = f(z) \text{ whenever } z \text{ is a value of } \varpi \text{ and } x \in V(\varpi, z).$$

(b) Now suppose that μ is a topological probability measure on X. In this context, Σ will always be the domain of μ, λ the product measure $\mu^{\mathbb{N}}$ on $\Omega = X^{\mathbb{N}}$, and Λ the domain of λ. I will say that μ is **Mycielski-regular** if, for every Σ-measurable $f : X \to \mathbb{R}$, $\langle F(\omega\restriction n, f)\rangle_{n\ge1}$ converges in measure to f for λ-almost every ω.

1C Example Suppose that X is a set and μ is a probability measure with domain $\mathcal{P}X$ such that $\mu\{x\} = 0$ for every $x \in X$ and there is a set $H \subseteq X$ such that $0 < \mu H < 1$. (Such a measure exists whenever $\#(X)$ is not measure-free in the sense of Fremlin 03, §438. In this case, taking disjoint subsets X_0, X_1 of X of the same size as X, we have probability measures μ_0, μ_1 with domains $\mathcal{P}X_0$, $\mathcal{P}X_1$ and both zero on singletons, and we can set $\mu E = \frac{1}{2}(\mu_0(E \cap X_0) + \mu_1(E \cap X_1))$ for every $E \subseteq X$, $H = X_0$.)

Now let ρ be the zero-one metric on X. In this case, for any $n \in \mathbb{N}$ and $\varpi \in X^n$,

$$V(\varpi, \varpi(0)) = (X \setminus \varpi[n]) \cup \{\varpi(0)\}$$

has measure 1, while

$$V(\varpi, z) = \{z\}$$

is negligible for $z \in \varpi[n] \setminus \{\varpi(0)\}$. So $\int F(\varpi, f)d\mu = f(\varpi(0))$ for every $f : X \to \mathbb{R}$. In particular, for $\omega \in \Omega$,

$$\lim_{n\to\infty} \int F(\omega\restriction n, \chi H)d\mu = 1 \text{ if } \omega(0) \in H,$$
$$= 0 \text{ otherwise,}$$

and $\langle F(\omega\restriction n, \chi H)\rangle_{n\ge1}$ is never convergent in measure to χH. Thus μ is not Mycielski-regular.

1D Example Let $\langle n_k\rangle_{k\in\mathbb{N}}$, $\langle M_k\rangle_{k\in\mathbb{N}}$, $\langle \delta_k\rangle_{k\in\mathbb{N}}$ be sequences such that, for each $k \in \mathbb{N}$,

$$M_k = \prod_{j<k} 2n_j + 1, \quad \delta_k = \prod_{j<k} \frac{1}{3n_j},$$

$$n_k \in \mathbb{N}, \quad n_k \geq 2^{k+2}, \quad M_k(1 - \frac{\delta_k}{2^{k+3}})^{n_k} \leq \frac{1}{2^k},$$

starting from $\delta_0 = M_0 = 1$. For each k, let X_k be a set with $2n_k + 1$ elements, and t_k a point of X_k. Set $X = \prod_{k\in\mathbb{N}} X_k$ and define $\rho : X \times X \to [0,1]$ by setting

$$\begin{aligned}\rho(x, y) &= \frac{1}{2k+1} \text{ if } x{\restriction}k = y{\restriction}k \text{ and } x(k),\ y(k) \text{ are different members of } X_k \setminus \{t_k\},\\ &= \frac{1}{2k+2} \text{ if } x{\restriction}k = y{\restriction}k \text{ and just one of } x(k),\ y(k) \text{ is equal to } t_k,\\ &= 0 \text{ if } x = y.\end{aligned}$$

Then (X, ρ) is a compact metric space. For each $k \in \mathbb{N}$, let μ_k be the probability measure on X_k such that $\mu_k\{t_k\} = \frac{1}{2^{k+3}}$ and $\mu_k\{t\} = \frac{1}{2n_k}(1 - \frac{1}{2^{k+3}})$ for $t \in X_k \setminus \{t_k\}$; let μ be the product measure $\prod_{k\in\mathbb{N}} \mu_k$, so that μ is a Radon probability measure on X.

μ is not Mycielski-regular. **P** Set

$$G = \{x : x \in X,\ x(k) = t_k \text{ for some } k \in \mathbb{N}\},$$

so that

$$\mu G \leq \sum_{k=0}^{\infty} \frac{1}{2^{k+3}} = \frac{1}{4}.$$

For $k \in \mathbb{N}$ and $\sigma \in \prod_{j<k} X_j$, set $E_\sigma = \{x : x \in X,\ x{\restriction}k = \sigma,\ x(k) = t_k\}$. Note that as $\mu_j\{t\} \geq \frac{1}{3n_j}$ for every $j < k$ and $t \in X_j$, $\mu E_\sigma \geq \frac{\delta_k}{2^{k+3}}$. Set

$$W_k = \{\omega : \omega \in \Omega,\ \omega[n_k] \cap E_\sigma \neq \emptyset \text{ for every } \sigma \in \prod_{j<k} X_j\};$$

then

$$\lambda(\Omega \setminus W_k) \leq \sum_{\sigma\in\prod_{j<k} X_j} (1 - \mu E_\sigma)^{n_k} \leq M_k(1 - \frac{\delta_k}{2^{k+3}})^{n_k} \leq \frac{1}{2^k}.$$

If $\omega \in W_k$, consider $F(\omega{\restriction}n_k, \chi G)$. Set $T = X_k \setminus (\{t_k\} \cup \{\omega(i)(k) : i < n_k\})$. Then $\#(T) \geq n_k$ so

$$\mu_k T \geq \tfrac{1}{2}(\mu X_k \setminus \{t_k\}) \geq \frac{7}{16}.$$

So if we set $H = \{x : x \in X,\ x(k) \in T\}$, $\mu H \geq \frac{7}{16}$. Now if $x \in H$, there is a first $i < n_k$ such that $\omega(i) \in E_{x{\restriction}k}$, but there is no $j < n_k$ such that $\omega(j){\restriction}k + 1 = x{\restriction}k + 1$. This means that

$$\rho(x, \omega(i)) = \frac{1}{2k+2} = \rho(x, \omega[n_k])$$

and $\rho(\omega(j), x) \geq \frac{1}{2k+1}$ for $j < i$. Accordingly

$$F(\omega{\restriction}n_k, \chi G)(x) = \chi G(\omega(i)) = 1.$$

Thus we see that $\mu\{x : F(\omega{\restriction}n_k, \chi G)(x) = 1\} \geq \frac{7}{16}$ and $\mu\{x : |F(\omega{\restriction}n_k, \chi G)(x) - \chi G(x)| = 1\} \geq \frac{3}{16}$.

It follows that if $\omega \in \bigcup_{m\in\mathbb{N}} \bigcap_{k\geq m} W_k$, $\langle F(\omega{\restriction}n, \chi G)\rangle_{n\geq 1}$ does not converge in measure to χG. Since this is true for almost every ω, μ is not Mycielski-regular. **Q**

1E Problem Which topological probability measures on metric spaces are Mycielski-regular?

We have just seen two non-Mycielski-regular examples. There is an easy positive result in 2C, and more interesting ones in 3B and 5B. In §§4 and 6 I offer general classes of space in which we can hope for further positive results.

Several of the results below are stated for topological probability measures on separable metric spaces. In view of Proposition 2N, these will in fact apply to any topological probability measure on a metric space for which there is a separable subset of full outer measure (equivalently, a closed separable conegligible set); it is easy to see that these are just the τ-additive topological probability measures.

2 General remarks

2A Elementary facts Let (X,ρ) be a metric space.

(a) If $n \ge 1$, and $\varpi \in X^n$ and f is any function with domain X, then $F(\varpi,f)[X] \subseteq f[X]$. If $f : X \to Y$ and $g : Y \to Z$ are functions, then $F(\varpi, gf) = gF(\varpi,f)$. More generally, if $Y_0,\dots,Y_m,Z$ are sets and $f_j : X \to Y_j$, $\phi : \prod_{j\le m} Y_j \to Z$ are functions, then $\phi(F(\varpi,f_0)(x),\dots,F(\varpi,f_m)(x)) = F(\varpi,\phi(f_0,\dots,f_m))(x)$ for every $x \in X$.

$V(\varpi,z)$ is a Borel set (actually, the intersection of a closed set with an open set) for every $z \in \varpi[n]$; so if f is real-valued, then $F(\varpi,f)$ is Borel measurable.

(b) Now suppose that μ is a topological probability measure on X, as in 1Bb. If f, g are functions defined on X and equal μ-almost everywhere, then for λ-almost every $\omega \in \Omega$ we shall have $f\omega = g\omega$, so $F(\omega{\restriction}n, f) = F(\omega{\restriction}n, g)$ for every n.

2B It is sometimes convenient to interpolate an extra definition into the description of the functions $F(\varpi,f)$ in 1B(a-ii): for $n \in \mathbb{N}$, $\varpi \in X^n$ and $x \in X$, let $k(\varpi,x)$ be the least j such that $\rho(x,\varpi[n]) = \rho(x,\varpi(j))$, so that $x \in V(\varpi,\varpi(k(\varpi,x)))$ and $F(\varpi,f) = f(\varpi(k(\varpi,x)))$ for any function f.

Lemma Let (X,ρ) be a metric space, μ a topological probability measure on X with domain Σ, $f : X \to \mathbb{R}$ a Σ-measurable function, and $n \ge 1$.

(a) The functions $(\varpi,x) \mapsto k(\varpi,x) : X^n \times X \to n$, $(\varpi,x) \mapsto \varpi(k(\varpi,x)) : X^n \times X \to X$ are Borel measurable.

(b) The function $(\omega,x) \mapsto F(\omega{\restriction}n,f)(x)$ is $\mathcal{B}(\Omega)\widehat{\otimes}\Sigma$-measurable, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra of Ω.

(c)(i) $\omega \mapsto \int F(\omega{\restriction}n,f)d\mu$ is $\mathcal{B}(\Omega)$-measurable.

(ii) $\omega \mapsto \int \chi X \wedge |F(\omega{\restriction}n,f) - f|d\mu$ is $\mathcal{B}(\Omega)$-measurable.

proof (a)(i) For $i < n$,

$$\{(\varpi,x) : k(\varpi,x) = i\} = \bigcap_{j<n}\{(\varpi,x) : \rho(\varpi(j),x) - \rho(\varpi(i),x) \ge 0\}$$
$$\setminus \bigcup_{j<i}\{(\varpi,x) : \rho(\varpi(j),x) - \rho(\varpi(i),x) = 0\}$$

is Borel measurable because all the functions $(\varpi,x) \mapsto \rho(\varpi(j),x)$ are continuous.

(ii) For any $\alpha \in \mathbb{R}$ and $i < n$, $E = \{x : f(x) \le \alpha\}$ belongs to Σ and the Borel set $\{(\omega,x) : k(\omega{\restriction}n,x) = i\}$ belongs to $\Lambda\widehat{\otimes}\Sigma$, so

$$\{(\omega,x) : F(\omega{\restriction}n,f)(x) \le \alpha\} = \textstyle\bigcup_{i<n}\{(\omega,x) : k(\omega{\restriction}n,x) = i,\ \omega(i) \in E\} \in \Lambda\widehat{\otimes}\Sigma;$$

as α is arbitrary, $(\omega,x) \mapsto F(\omega{\restriction}n,f)(x)$ is $\Lambda\widehat{\otimes}\Sigma$-measurable.

(c)(i) Apply Fubini's theorem in the form Fremlin 01, 252P to the positive and negative parts of f.

(ii) Of course $(\omega,x) \mapsto \min(1,|F(\omega{\restriction}n,f)(x) - f(x)|)$ is also $\mathcal{B}(\Omega)\widehat{\otimes}\Sigma$-measurable, so we can use the same form of Fubini's theorem.

2C Proposition Let (X,ρ) be a metric space, and μ a point-supported probability measure on X. Then μ is Mycielski-regular.

proof If $f : X \to \mathbb{R}$ is any function and $x \in X$ is such that $\mu\{x\} > 0$, then for λ-almost every $\omega \in \Omega$, $x \in \omega[\mathbb{N}]$. So for such ω there is an m such that $x = \omega(m)$, and for any $n > m$ we shall have $x \in V(\omega{\restriction}n,x)$ and $F(\omega{\restriction}n,f)(x) = f(x)$. Since the set $X_0 = \{x : \mu\{x\} > 0\}$ is countable and μ-conegligible, $X_0 \subseteq \omega[\mathbb{N}]$ for almost every ω, and for such ω, $\langle F(\omega{\restriction}n,f)\rangle_{n\ge1}$ converges μ-almost everywhere to f.

2D Proposition Let (X,ρ) be a separable metric space, and μ a topological probability measure on X.

(a) If X_0 is the support of μ, then for every $k\in\mathbb{N}$, $X_0=\overline{\omega[\mathbb{N}\setminus k]}$ for λ-almost every ω.

(b) If $f:X\to\mathbb{R}$ is continuous, there are conegligible sets $\Omega_0\subseteq\Omega$, $X_0\subseteq X$ such that $\lim_{n\to\infty}F(\omega{\restriction}n,f)(x)=f(x)$ whenever $x\in X_0$ and $\omega\in\Omega_0$.

proof (a) Then X_0 is separable, so its topology has a countable base $\mathcal{U}$. For each $U\in\mathcal{U}\setminus\{\emptyset\}$, $\mu U>0$, so $\{\omega:\omega[\mathbb{N}\setminus k]\cap U\neq\emptyset\}$ is conegligible; accordingly

$$\begin{aligned}\Omega_0'&=\{\omega:\overline{\omega[\mathbb{N}\setminus k]})\supseteq X_0\}\\&\supseteq\bigcap_{U\in\mathcal{U}\setminus\{\emptyset\}}\{\omega:U\cap\omega[\mathbb{N}\setminus k]\neq\emptyset\}\end{aligned}$$

is conegligible. On the other hand, X_0 is μ-conegligible, so $X_0^{\mathbb{N}}$ and $\Omega_0=\Omega_0'\cap X_0^{\mathbb{N}}$ are λ-conegligible.

(b) If $\omega\in\Omega_0$ and $x\in X_0$, then $\omega[\mathbb{N}]$ meets every neighbourhood of x, so $\lim_{n\to\infty}\rho(x,\omega[n])=0$ and

$$\lim_{n\to\infty}F(\omega{\restriction}n,f)(x)=\lim_{n\to\infty}f(\omega(k(\omega{\restriction}n,x))=f(x).$$

2E Zero-one law: Proposition Let (X,ρ) be a separable metric space and μ a topological probability measure on X.

(a) There is a λ-conegligible set $\Omega_0\subseteq\Omega$ such that if ω, $\omega'\in\Omega_0$ are eventually equal, then for μ-almost every $x\in X$ there is an $n\in\mathbb{N}$ such that $F(\omega{\restriction}m,f)(x)=F(\omega'{\restriction}m,f)(x)$ for every $m\geq n$ and every function f defined on X.

(b) If $f:X\to\mathbb{R}$ is a bounded Σ-measurable function, then there is an $\alpha\in\mathbb{R}$ such that $\limsup_{n\to\infty}\int F(\omega{\restriction}n,f)d\mu=\alpha$ for λ-almost every ω.

proof (a) Let Y be $\{x:x\in X,\ \mu\{x\}>0\}$, so that Y is countable. Set

$$\Omega_0=\{\omega:Y\subseteq\omega[\mathbb{N}\setminus k]\text{ and }\mu(\overline{\omega[\mathbb{N}\setminus k]})=1\text{ for every }k\in\mathbb{N}\}.$$

If ω, $\omega'\in\Omega_0$ and $\omega(m)=\omega'(m)$ for every $m\geq l$, set $I=\omega[k]\cup\omega'[l]\setminus Y$; then $\mu I=0$, so $X_0=\overline{\omega[\mathbb{N}\setminus l]}\setminus I$ is μ-conegligible. If $x\in X_0$, then either $x\in Y$ and there is an $n\geq k$ such that $x\in\omega[n\setminus l]$, or $x\notin Y$ and there is an $n\geq l$ such that $\rho(x,\omega[n\setminus l])<\rho(x,I)$. The same will now be true for every $m\geq n$. So for any $m\geq n$ we either have $x\in Y$ and $\omega(k(\omega{\restriction}m,x))=x=\omega'(k(\omega{\restriction}m,x))$, or $x\notin Y$ and $k(\omega{\restriction}m,x)=k(\omega'{\restriction}m,x)\in m\setminus l$. In either case, $F(\omega{\restriction}m,f)(x)=F(\omega'{\restriction}m,f)(x)$ for any f.

(b) Set $h(\omega)=\limsup_{n\to\infty}\int F(\omega{\restriction}n,f)d\mu$ for $\omega\in\Omega$. Then h is Borel measurable, by 2B(c-i). If ω, $\omega'\in\Omega_0$ are eventually equal, then $\lim_{n\to\infty}(F(\omega{\restriction}n,f)-F(\omega'{\restriction}n,f))=0$ almost everywhere, so

$$\lim_{n\to\infty}(\int F(\omega{\restriction}n,f)d\mu-\int F(\omega'{\restriction}n,f)d\mu)=0$$

and $h(\omega)=h(\omega')$. By Fremlin 01, 254Sb, $\{\omega:\omega\in\Omega_0,\ h(\omega)>\alpha\}$ has measure either 0 or 1 for every α, so there is an α such that $h(\omega)=\alpha$ for almost every ω.

2F Proposition Let (X,ρ) be a separable metric space and μ a topological probability measure on X.

(a) We have a functional $\theta:\Sigma\to[0,1]$ such that, for any $E\in\Sigma$, $\limsup_{n\to\infty}\int F(\omega{\restriction}n,\chi E)d\mu=\theta E$ for λ-almost every $\omega\in\Omega$.

(b)(i) θ is a unital submeasure.

(ii) $\theta H\leq\mu H$ for every closed $H\subseteq X$, and $\theta G\geq\mu G$ for every open $G\subseteq X$.

(iii) If $E\in\Sigma$ is such that its topological boundary is μ-negligible, then $\theta E=\mu E$.

Remark For the basic properties of submeasures, see Fremlin 02, chap. 39.

proof (a) This is just a re-statment of 2Eb.

(b)(i) Elementary.

(ii)(α) Let $H\subseteq X$ be closed, and $\epsilon>0$. Then there is a continuous $f:X\to\mathbb{R}$ such that $\chi H\leq f$ and $\int fd\mu\leq\mu H+\epsilon$. Now, for almost every $\omega\in\Omega$, $\langle F_n(\omega{\restriction}n,f)\rangle_{n\geq1}$ converges to f μ-almost everywhere, by 2D. It follows at once that, for almost every ω,

$$
\begin{aligned}
\theta H &= \limsup_{n\to\infty} F(\omega\restriction n, \chi H)d\mu \\
&\le \limsup_{n\to\infty} F(\omega\restriction n, f)d\mu = \int f d\mu \le \mu H + \epsilon;
\end{aligned}
$$

as ϵ is arbitrary, $\theta H \le \mu H$.

(β) If now $G \subseteq X$ is open, then

$$\mu G = 1 - \mu(X \setminus G) \le \theta X - \theta(X \setminus G) \le \theta G$$

because θ is a unital submeasure.

(iii) We have

$$\mu E = \mu(\operatorname{int} E) \le \theta(\operatorname{int} E) \le \theta E \le \theta\overline{E} \le \mu\overline{E} = \mu E.$$

2G Theorem Let (X, ρ) be a separable metric space, μ a topological probability measure on X and $\theta : \Sigma \to [0, 1]$ the functional described in 2F. Then the following are equiveridical:

(i) μ is Mycielski-regular;

(ii) θ is absolutely continuous with respect to μ;

(iii) $\theta = \mu$.

proof (i)$\Rightarrow$(iii)$\Rightarrow$(ii) are immediate.

(ii)$\Rightarrow$(i) Suppose that (ii) is true. Given a measurable $f : X \to \mathbb{R}$, then for each $k \in \mathbb{N}$ take $\delta_k > 0$ such that $\theta E \le 2^{-k}$ whenever $\mu E \le \delta_k$. Then there is a continuous $g_k : X \to \mathbb{R}$ such that $E_k = \{x : g_k(x) \ne f(x)\}$ has measure at most $\min(2^{-k}, \delta_k)$ (Fremlin 03, 418Xq). Now $\{x : F(\varpi, f)(x) \ne F(\varpi, g_k)(x)\} \subseteq \{x : F(\varpi, \chi E_k)(x) = 1\}$ for every $\varpi \in \bigcup_{n\ge1} X^n$. Let W_k be the set of ω such that $\lim_{n\to\infty} F(\omega\restriction n, g_k)(x) = g_k(x)$ for almost every x and $\limsup_{n\to\infty} \int F(\omega\restriction n, \chi E_k)d\mu \le 2^{-k}$. Then W_k is conegligible. Set $\Omega_0 = \bigcap_{k\in\mathbb{N}} W_k$. For any ϖ,

$$
\begin{aligned}
|F(\varpi, f) - F(\varpi, g_k)| \wedge \chi X &= F(\varpi, |f - g_k|) \wedge F(\varpi, \chi X) \\
&= F(\varpi, |f - g_k| \wedge \chi X) \le F(\varpi, \chi E_k),
\end{aligned}
$$

and

$$
\begin{aligned}
&|F(\varpi, f) - f| \wedge \chi X \\
&\quad \le |F(\varpi, f) - F(\varpi, g_k)| \wedge \chi X + |F(\varpi, g_k) - g_k| \wedge \chi X + |g_k - f| \wedge \chi X \\
&\quad \le F(\varpi, \chi E_k) + |F(\varpi, g_k) - g_k| \wedge \chi X + \chi E_k.
\end{aligned}
$$

So if $\omega \in W$,

$$
\begin{aligned}
&\limsup_{n\to\infty} \int |F(\omega\restriction n, f) - f| \wedge \chi X \, d\mu \\
&\quad \le \limsup_{n\to\infty} \int F(\omega\restriction n, \chi E_k)d\mu + \limsup_{n\to\infty} \int |F(\omega\restriction n, g_k) - g_k| \wedge \chi X d\mu + \mu E_k \\
&\quad \le 2^{-k} + 0 + 2^{-k} = 2^{-k+1}
\end{aligned}
$$

for every k, and $\langle F(\omega\restriction n, f)\rangle_{n\ge1}$ converges to f in measure. As f is arbitrary, μ is Mycielski-regular.

2H Remark In this context, note that $F(\varpi, \chi E)$ will always be $\{0, 1\}$-valued (see the second remark in 2A); if $\varpi \in X^n$,

$$\int F(\varpi, \chi E)d\mu = \sum_{z\in E\cap\varpi[n]} \mu V(\varpi, z).$$

2I Moderated Voronoi tesselations Let X, ρ, μ, Ω and λ be as in 1B. I will say that μ has **moderated Voronoi tesselations** if for every $\epsilon > 0$ there is an $M \ge 0$ such that

$$\sum_{n=1}^{\infty}\lambda(\{\omega:\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\})\ge\epsilon\})$$

is finite, where here each $V'(\omega\restriction n,z)$ is the punctured Voronoi tile $V(\omega\restriction n,z)\setminus\{z\}$.

2J Theorem Let (X,ρ) be a separable metric space, and μ a topological probability measure on X which has moderated Voronoi tesselations. Then μ is Mycielski-regular.

proof Let θ be the submeasure described in 2E-2F. Then θ is absolutely continuous with respect to μ. **P** Let $\epsilon>0$. Let $M\ge 0$ be such that

$$\sum_{n=1}^{\infty}\lambda(\{\omega:\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\})\ge\frac{1}{3}\epsilon\})$$

is finite, defining $V'(\omega\restriction n,z)$ as in 2I. Let Ω_1 be the set of those ω such that $\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\})$ is less than $\frac{1}{3}\epsilon$ for all but finitely many n; then Ω_1 is λ-conegligible.

Let $\delta>0$ be such that $2M\delta\le\frac{1}{3}\epsilon$, $\delta\le\frac{1}{3}\epsilon$ and $\delta\le\frac{1}{2}$. Suppose that $\mu E\le\delta$, and let Ω_2 be the set of those ω such that $\{n:\#(\{i:i<n,\,\omega(i)\in E\})>2\delta n\}$ is finite; by the strong law of large numbers, Ω_2 is λ-conegligible. Take any $\omega\in\Omega_1\cap\Omega_2$. Let n be such that

$$\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\})\le\frac{1}{3}\epsilon,$$

$$\#(\{i:i<n,\,\omega(i)\in E\})\le 2\delta n\},$$

and set

$$I=E\cap\omega[n],\qquad J=\{z:z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\}.$$

We have

$$\begin{aligned}\int F(\omega\restriction n,\chi E)d\mu&=\sum_{z\in I}\mu V(\omega\restriction n,z)=\mu I+\sum_{z\in I\cap J}\mu V'(\omega\restriction n,z)+\sum_{z\in I\setminus J}\mu V'(\omega\restriction n,z)\\&\le\mu E+\frac{1}{3}\epsilon+\#(I\setminus J)\cdot\frac{M}{n}\le\frac{2}{3}\epsilon+\frac{M\#(I)}{n}\le\frac{2}{3}\epsilon+\frac{2M\delta n}{n}\le\epsilon.\end{aligned}$$

This is true for all but finitely many n, so $\limsup_{n\to\infty}\int F(\omega\restriction n,\chi E)d\mu\le\epsilon$; as this is true for almost every ω, $\theta E\le\epsilon$. As ϵ is arbitrary, θ is absolutely continuous with respect to μ. **Q**

By Theorem 2G, μ is Mycielski-regular.

2K Irregular measures The discussion above refers to 'topological measures', meaning measures which measure all Borel sets (Fremlin 03, 411A). This allows for the possibility that some highly irregular sets are measurable. However in the context of this note this is a bit beside the point. Let us say that a topological probability measure μ on a topological space X is **almost a Borel measure** if for every set E which is measured by μ there is a Borel set H such that $\mu(E\triangle H)=0$. (Obvious examples are completions of Borel measures, like Radon probability measures; another is the image measure $\mu_L f^{-1}$ on the split interval $I^{\|}$ (Fremlin 03, 419L), where μ_L is Lebesgue measure on $[0,1]$ and $f(t)=t^-$ for $t\in[0,1]$.) Now we have the following.

2L Proposition Let (X,ρ) be a separable metric space and μ a topological probability measure on X.

(a) If μ is Mycielski-regular, it is almost a Borel measure.

(b) If μ is almost a Borel measure, then μ is Mycielski-regular iff its restriction $\mu_{\mathcal{B}}$ to the Borel σ-algebra of X is Mycielski-regular.

proof (a) Suppose that $E\in\Sigma$. Then there is an $\omega\in\Omega$ such that $\langle F(\omega\restriction n,\chi E)\rangle_{n\ge1}$ converges in measure to E. Now there is a sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\langle F(\omega\restriction n_k,\chi E)\rangle_{k\in\mathbb{N}}\to\chi E$ μ-a.e. But all the functions $F(\omega\restriction n_k,\chi E)$ are Borel measurable (2A above), so if we set

$$H=\{x:\lim_{k\to\infty}F(\omega\restriction n_k,f)(x)=1\},$$

H is a Borel set and $\mu(E\triangle H)=0$.

(b) Let $\lambda_{\mathcal{B}}$ be the product measure $\mu_{\mathcal{B}}^{\mathbb{N}}$. Because μ extends $\mu_{\mathcal{B}}$, λ extends $\lambda_{\mathcal{B}}$ (Fremlin 01, 254H).

(i) If μ is Mycielski-regular and $E\subseteq X$ is Borel, then

$$\limsup_{n\to\infty}\int F(\omega\restriction n,\chi E)d\mu=\mu E$$

for λ-almost every $\omega\in\Omega$. But $F(\varpi,\chi E)$ is Borel measurable for every ϖ, so

$$\limsup_{n\to\infty}\int F(\omega\restriction n,\chi E)d\mu_{\mathcal{B}}=\mu_{\mathcal{B}}E$$

for λ-almost every $\omega\in\Omega$. Moreover, because E is Borel measurable,

$$\omega\mapsto\int F(\omega\restriction n,\chi E)d\mu_{\mathcal{B}}$$

is $\Lambda_{\mathcal{B}}$-measurable for every $n\in\mathbb{N}$ (2B). But this means that the λ-negligible set

$$\{\omega:\limsup_{n\to\infty}\int F(\omega\restriction n,\chi E)d\mu_{\mathcal{B}}\ne\mu_{\mathcal{B}}E\}$$

belongs to $\Lambda_{\mathcal{B}}$ and is $\lambda_{\mathcal{B}}$-negligible. As E is arbitrary, $\mu_{\mathcal{B}}$ is Mycielski-regular, by 2G.

(ii) If $\mu_{\mathcal{B}}$ is Mycielski-regular and $E\in\Sigma$, there is a Borel set H such that $\mu(E\triangle H)=0$. Now

$$W=\{\omega:\omega[\mathbb{N}]\cap(E\triangle H)\ne\emptyset\}$$

is λ-conegligible, while

$$W'=\{\omega:\limsup_{n\to\infty}\int F(\omega\restriction n,\chi H)d\mu_{\mathcal{B}}=\mu_{\mathcal{B}}H\}$$

is $\lambda_{\mathcal{B}}$-conegligible, therefore λ-conegligible, and $W\cap W'$ is λ-conegligible. But if $\omega\in W\cap W'$, $F(\omega\restriction n,\chi H)=F(\omega\restriction n,\chi E)$ for every n, so

$$\begin{aligned}\limsup_{n\to\infty}\int F(\omega\restriction n,\chi E)d\mu&=\limsup_{n\to\infty}\int F(\omega\restriction n,\chi H)d\mu\\&=\limsup_{n\to\infty}\int F(\omega\restriction n,\chi H)d\mu_{\mathcal{B}}=\mu_{\mathcal{B}}H=\mu E.\end{aligned}$$

As E is arbitrary, μ is Mycielski-regular.

2M Proposition Let (X,ρ) be a metric space. Then the set of Mycielski-regular topological probability measures on X is convex.

proof (a) Let μ_0, μ_1 be Mycielski-regular topological measures on X, with domains Σ_0 and Σ_1 respectively, and $\alpha_0\in\,]0,1[$; set $\alpha_1=1-\alpha_0$ and $\mu=\alpha_0\mu_0+\alpha_1\mu_1$, so that $\Sigma=\Sigma_0\cap\Sigma_1$. Let λ_0, λ_1 and λ be the corresponding product measures on $\Omega=X^{\mathbb{N}}$. Let ν be the product measure on $Q=\{0,1\}^{\mathbb{N}}$ where each factor $\{0,1\}$ is given the point-supported measure $\alpha_0\delta_0+\alpha_1\delta_1$, and let θ be the product measure $\nu\times\lambda_0\times\lambda_1$ on $Z\times\Omega\times\Omega$.

(b) Define $\phi:Q\times\Omega\times\Omega\to\Omega$ by setting

$$\phi(q,\omega_0,\omega_1)(n)=\omega_{q(n)}(k_{q(n)}(q,n))$$

for $n\in\mathbb{N}$, where

$$k_i(q,n)=\#(\{j:j<n,\,q(j)=i\})$$

for $q\in Q$, $n\in\mathbb{N}$ and $i\in\{0,1\}$. It is easy to check that ϕ is inverse-measure-preserving for θ and λ. (Use Fremlin 01, 254G.)

(c) ? Suppose, if possible, that μ is not Mycielski-regular. Let $f:X\to\mathbb{R}$ be a Σ-measurable function such that

$$\begin{aligned}&\{\omega:\langle F(\omega\restriction n,f)\rangle_{n\ge1}\text{ does not converge in measure to }f\}\\&\quad=\bigcup_{m\in\mathbb{N}}\bigcap_{k\ge1}\bigcup_{n\ge k}\{\omega:\int\chi X\wedge|F(\omega\restriction n,f)-f|d\mu\ge2^{-m}\}\end{aligned}$$

is not λ-negligible. We therefore have an $\epsilon > 0$ such that

$$W = \bigcap_{k\ge 1}\bigcup_{n\ge k}\{\omega : \int \chi X \wedge |F(\omega\restriction n, f) - f|d\mu > 2\epsilon\}$$

is not λ-negligible, while $W \in \Lambda$, by 2B. Accordingly $\theta\phi^{-1}[W] > 0$. Setting $Q_0 = \{q : q \in Q$ is not eventually constant$\}$, Q_0 is ν-conegligible, so there is a $q \in Q_0$ such that $V = \{(\omega_0, \omega_1) : (q, \omega_0, \omega_1) \in \phi^{-1}[W]\}$ is not $(\lambda_0 \times \lambda_1)$-negligible.

However, because μ_0 and μ_1 are both Mycielski-regular, and f is Σ_i-measurable for both i,

$$V_i = \{\omega : \lim_{n\to\infty} \int \chi X \wedge |F(\omega\restriction n, f) - f|d\mu = 0\}$$

is λ_i-conegligible for both i, and $V_0 \times V_1$ is $\lambda_0 \times \lambda_1$-conegligible. There must therefore be a $k \in \mathbb{N}$ such that

$$\{(\omega_0, \omega_1) : \int \chi X \wedge |F(\omega_0\restriction n, f) - f|d\mu \le \epsilon,$$
$$\int \chi X \wedge |F(\omega_1\restriction n, f) - f|d\mu \le \epsilon \text{ for every } n \ge k\}$$

is non-empty. Take any point (ω_0, ω_1) in the intersection. Let $l \ge 1$ be such that $k_i(q, l) = \#(\{j : j < l, q(j) = i\}) \ge k$ for both i. Then $\omega = \phi(q, \omega_0, \omega_1)$ belongs to W, so there is an $n \ge l$ such that

$$\int \chi X \wedge |F(\omega\restriction n, f) - f|d\mu > 2\epsilon.$$

However, if we look at $\omega[n]$, we see that it is precisely $\omega_0[k_0(q, n)] \cup \omega_1[k_1(q, n)]$. If $z \in \omega[n]$ and $x \in V(\omega\restriction n, z)$, then if $z \in \omega_0[k_0(q, n)]$ we must have $x \in V(\omega_0\restriction k_0(q, n), z)$ and

$$F(\omega\restriction n, f)(x) = f(z) = F(\omega_0\restriction k_0(q, n), f)(x),$$

while if $z \in \omega_1[k_1(q, n)]$ we must have $x \in V(\omega_1\restriction k_1(q, n), z)$ and

$$F(\omega\restriction n, f)(x) = f(z) = F(\omega_1\restriction k_1(q, n), f)(x).$$

But this means that

$$|F(\omega\restriction n, f) - f| \le |F(\omega_0\restriction k_0(q, n), f) - f| \vee |F(\omega_1\restriction k_1(q, n), f) - f|,$$

while $k_0(q, n) \ge k_0(q, l)$ and $k_1(q, n) \ge k_1(q, l)$ are both at least k; so

$$2\epsilon < \int \chi X \wedge |F(\omega\restriction n, f) - f|d\mu$$
$$\le \int (\chi X \wedge |F(\omega_0\restriction k_0(q, n), f) - f|) + (\chi X \wedge |F(\omega_1\restriction k_1(q, n), f) - f|)d\mu \le 2\epsilon$$

which is absurd. **X**

(d) So μ is Mycielski-regular, as claimed.

2N Proposition Let (X, ρ) be a metric space and μ a topological probability measure on X. Let $Y \subseteq X$ be a set of full outer measure; set $\rho_Y = \rho\restriction Y \times Y$ and let μ_Y be the subspace measure on Y. Then μ_Y is Mycielski-regular iff μ is.

proof (a) Set $\Omega_Y = Y^{\mathbb{N}}$; then Ω_Y has full outer measure for $\lambda = \mu^{\mathbb{N}}$, and the subspace measure λ_Y induced by λ on Ω_Y is the product measure $\mu_Y^{\mathbb{N}}$ (Fremlin 01, 254L). Note that if $\varpi \in Y^n$, then the Voronoi tesselation of Y corresponding to ϖ is precisely $\langle Y \cap V(\varpi, z)\rangle_{z\in\varpi[n]}$ where $\langle V(\varpi, z)\rangle_{z\in\varpi[n]}$ is the Voronoi tesselation of X corresponding to ϖ. So if we have a function $f : X \to \mathbb{R}$, and we write $F_Y(\varpi, f\restriction Y) : Y \to \mathbb{R}$ for the function defined on Y by the formulae of 1Ba, then $F_Y(\varpi, f\restriction Y) = F(\varpi, f)\restriction Y$.

If $f : X \to \mathbb{R}$ is Σ-measurable and $n \in \mathbb{N}$, set

$$h_{fn}(\omega) = \int_X \chi X \wedge |F(\omega\restriction n, f) - f|d\mu$$

for $\omega \in \Omega$. Then for any $\omega \in \Omega_Y$ we shall have

$$h_{fn}(\omega) = \int_X \chi X \wedge |F(\omega\restriction n, f) - f|d\mu = \int_Y \chi Y \wedge |(F(\omega\restriction n, f)\restriction Y) - (f\restriction Y)|d\mu_Y$$

(Fremlin 01, 214F)

$$= \int_Y \chi Y \wedge |F_Y(\omega{\restriction}n, f_Y - f_Y| d\mu_Y$$

if we write f_Y for $f{\restriction}Y$.

(b) Suppose that μ is Mycielski-regular. Let $g : Y \to \mathbb{R}$ be Σ_Y-measurable, where $\Sigma_Y = \operatorname{dom}\mu_Y$ is the subspace σ-algebra induced by Σ on Y. Then there is a Σ-measurable $f : Y \to \mathbb{R}$ extending g (Fremlin 00, 121I). Now we know that

$$\{\omega : \omega \in \Omega,\ \lim_{n\to\infty} h_{fn}(\omega) = 0\}$$

is λ-conegligible, so

$$\{\omega : \omega \in \Omega_Y,\ \lim_{n\to\infty} h_{fn}(\omega) = 0\}$$

is λ_Y-conegligible, that is,

$$\{\omega : \omega \in \Omega_Y,\ \lim_{n\to\infty} \int_Y \chi Y \wedge |F_Y(\omega{\restriction}n, g - g| d\mu_Y = 0\}$$

is λ_Y-conegligible. As g is arbitrary, μ_Y is Mycielski-regular.

(c) Suppose that μ_Y is Mycielski-regular, and $f : X \to \mathbb{R}$ is Σ-measurable. In this case,

$$W = \{\omega : \omega \in \Omega,\ \lim_{n\to\infty} \int_X \chi X \wedge |F(\omega{\restriction}n, f - f| d\mu = 0\}$$

belongs to Λ, so

$$\lambda W = \lambda_Y(W \cap \Omega_Y) = \lambda_Y\{\omega : \omega \in \Omega_Y,\ \lim_{n\to\infty} \int_X \chi X \wedge |F(\omega{\restriction}n, f - f| d\mu = 0\}$$

$$= \lambda_Y\{\omega : \omega \in \Omega_Y,\ \lim_{n\to\infty} \int_Y \chi Y \wedge |F_Y(\omega{\restriction}n, f_Y - f_Y| d\mu_Y = 0\}$$

(where $f_Y = f{\restriction}Y$, as before)

$$= 1.$$

As f is arbitrary, μ is Mycielski-regular.

3 The one-dimensional case

3A Theorem Let μ be a topological probability measure on $\mathbb{R}$. Then μ has moderated Voronoi tesselations.

proof Let $\epsilon > 0$.

(a) Set $p = \max(2, \lceil\frac{5}{\epsilon}\rceil)$. Let $M \ge 1$ be an integer such that $e(p+1)(1-\frac{1}{2p})^M \le \frac{1}{2}$. By Stirling's formula (Fremlin 01, 252Yu), there is an $m_0 \in \mathbb{N}$ such that $m! \ge m^m e^{-m}$ for every $m \ge m_0$.[1] Set $\gamma = \frac{1}{2^{1/2Mp}}$, $n_0 = Mp\max(m_0, p) + 1$.

(b) For the time being (down to the end of (e) below) fix on an $n \ge n_0$. Set $l = \lfloor\frac{n-1}{M}\rfloor$ and $m = \lfloor\frac{l}{p}\rfloor$, so that $m \ge \max(m_0, p)$. For $1 \le i \le l$, set

$$\alpha_i = \sup\{\alpha : \mu\,]{-\infty}, \alpha[\, \le \frac{Mi}{n}\},$$

so that

$$\mu\,]{-\infty}, \alpha_i[\, \le \frac{Mi}{n} \le \mu\,]{-\infty}, \alpha_i].$$

Set $J_0 = \,]{-\infty}, \alpha_1]$, $J_l = [\alpha_l, \infty[$ and for $0 < i < l$ set $J_i = [\alpha_i, \alpha_{i+1}]$.

[1] I suppose that actually $m! \ge m^m e^{-m}$ for every $m \in \mathbb{N}$.

(c) (The fiddly bit.)

(i) For $i < l$, $\mu J_i \ge \frac{M}{n} \ge \mu(\operatorname{int} J_i)$. **P** J_i is a closed interval of one of the forms

$$]-\infty, \alpha_1], \quad [\alpha_i, \alpha_{i+1}],$$

so that μJ_i is one of

$$\mu\,]-\infty, \alpha_1], \quad \mu\,]-\infty, \alpha_{i+1}] - \mu\,]-\infty, \alpha_i[,$$

and in either case has measure at least $\frac{M}{n}$. On the other hand, $\operatorname{int} J_i$ is of one of the forms

$$]-\infty, \alpha_1[, \quad]\alpha_i, \alpha_{i+1}[,$$

and its measure is one of

$$\mu\,]-\infty, \alpha_1[, \quad \mu\,]-\infty, \alpha_{i+1}[- \mu\,]-\infty, \alpha_i],$$

which can be at most $\frac{M}{n}$. **Q**

(ii) $\mu(\operatorname{int} J_l) \le \frac{M}{n}$. **P**

$$\mu(\operatorname{int} J_l) = 1 - \mu\,]-\infty, \alpha_l] \le 1 - \frac{Ml}{n} \le \frac{M}{n}$$

because $Ml \le n - 1 < M(l+1)$ so $n - Ml \le M$. **Q**

(iii) If $C \subseteq \mathbb{R}$ is an interval (bounded or unbounded, open, closed, or half-open) and $\mu C > \frac{2M}{n}$, then there is an $i < l$ such that $J_i \subseteq C$. **P** As $\mu C > \mu(\operatorname{int} J_l)$, there is a first $i < l$ such that $C \cap J_i$ is non-empty. If $J_i \subseteq C$ we can stop. Otherwise, C cannot contain $\inf J_i$ (because if this is finite it belongs to J_{i-1}), and $\mu C > \mu(\operatorname{int} J_i)$, so C contains the upper endpoint of J_i, which is α_{i+1}. Now set $C' =]-\infty, \alpha_i]$ if $i > 0$, $\emptyset$ if $i = 0$; $C \cap C' = \emptyset$ and

$$\tfrac{Mi}{n} \le \mu C' \le 1 - \mu C < \tfrac{n-2M}{n} \le \tfrac{M(l-1)}{n}$$

because $n \le M(l+1)$. Thus $i + 1 < l$. At the same time,

$$\mu\,]-\infty, \alpha_{i+2}[\le \tfrac{M(i+2)}{n} < \mu(C \cup C') \le \mu\,]-\infty, \sup C]$$

so $\alpha_{i+2} \le \sup C$. If $\sup C \in C$, then $\alpha_{i+2} \in C$; if $\sup C \notin C$ then

$$\mu\,]-\infty, \alpha_{i+2}[\le \tfrac{M(i+2)}{n} < \mu(C \cup C') \le \mu\,]-\infty, \sup C[$$

and $\alpha_{i+2} < \sup C$, so again $\alpha_{i+2} \in C$. Thus we have $J_{i+1} \subseteq C$, which will do very well. **Q**

(iv) If $C \subseteq \mathbb{R}$ is an interval, then $\mu C \le \frac{2M}{n} + \frac{M}{n}\#(\{i : i < l,\ J_i \subseteq C\})$. **P** The set $K = \{i : i < l,$ $J_i \subseteq C\}$ is of the form $i_1 \setminus i_0$ where $i_0 \le i_1 \le l$. Set $i'_0 = \max(0, i_0 - 1)$ and $i'_1 = i_1 + 1$; then $C \subseteq \big]\alpha_{i'_0}, \alpha_{i'_1}\big[$ (counting α_0 as $-\infty$ and α_{l+1} as ∞), and

$$\mu C \le \mu\,\big]-\infty, \alpha_{i'_1}\big[- \mu\,\big]-\infty, \alpha_{i'_0}\big] \le \tfrac{Mi'_1}{n} - \tfrac{Mi'_0}{n} \le \tfrac{M(\#(K)+2)}{n}. \ \textbf{Q}$$

(v) If $K \subseteq l$ then $\mu(\bigcup_{i \in K} J_i) \ge \frac{M\#(K)}{n}$. **P** If K is of the form $i_1 \setminus i_0$, where $i_0 \le i_1 \le l$, then

— if $i_0 = 0$,

$$\mu(\textstyle\bigcup_{i \in K} J_i) = \mu(]0, \alpha_{i_1}]) \ge \tfrac{Mi_1}{n} = \tfrac{M\#(K)}{n};$$

— if $i_0 > 0$,

$$\mu(\bigcup_{i\in K} J_i) = \mu([\alpha_{i_0}, \alpha_{i_1}]) = \mu\,]-\infty, \alpha_{i_1}] - \mu\,]-\infty, \alpha_{i_0}[\, \geq \frac{Mi_1}{n} - \frac{Mi_0}{n} = \frac{M\#(K)}{n}.$$

Generally, we can induce on $\#(K)$, as follows. The case $K = \emptyset$ is trivial. For the inductive step to $\#(K) \geq 1$, set $j = \max K$ and consider the component C of $\bigcup_{i\in K} J_i$ including J_j. This must be of the form $\bigcup_{i\in i_1\setminus i_0} J_i$ where $i_0 \leq j < i_1 \leq l$. So

$$\mu C \geq \tfrac{M(i_1-i_0)}{n} \geq \tfrac{M\#(K\cap i_1\setminus i_0)}{n} = \tfrac{M\#(K\setminus i_0)}{n}.$$

On the other hand, $\bigcup_{i\in K} J_i \setminus C = \bigcup_{i\in K\cap i_0} J_i$ is disjoint from C, and $\#(K \cap i_0) < \#(K)$, so, using the inductive hypothesis,

$$\begin{aligned}\mu(\bigcup_{i\in K} J_i) &= \mu(\bigcup_{i\in K\cap i_0} J_i) + \mu(\bigcup_{i\in K\cap i_1\setminus i_0} J_i)\\ &\geq \tfrac{M\#(K\cap i_0)}{n} + \tfrac{M\#(K\setminus i_0)}{n} = \tfrac{M\#(K)}{n}. \text{ **Q**}\end{aligned}$$

(d) For $\omega \in \Omega$, set

$$H_n(\omega) = \bigcup\{V'(\omega{\restriction}n, z) : z \in \omega[n],\, \mu V'(\omega{\restriction}n, z) \geq \tfrac{4M+1}{n}\},$$

$$K_\omega = \{i : i < l,\, J_i \subseteq \bigcup_{z\in\omega[n]} V'(\omega{\restriction}n, z)\}.$$

Then $\mu H_n(\omega) \leq \frac{5M}{n}\#(K_\omega)$. **P** If $z \in \omega[n]$ and the punctured tile $V'(\omega{\restriction}n, z)$ has measure greater than $\frac{4M}{n}$, then $V'(\omega{\restriction}n, z)$ has two components, one on each side of z, both intervals, and at least one has measure greater than $\frac{2M}{n}$, so must include some J_i for $i < l$, by (c-iii). Thus the number of components of $H_n(\omega)$ is at most $2\#(K_\omega)$. Let $\mathcal{C}$ be the set of components of $H_n(\omega)$. For each $C \in \mathcal{C}$, $\mu C \leq \frac{2M}{n} + \frac{M}{n}\#(\{i : i < l,$ $J_i \subseteq C\})$, by (c-iv). So

$$\begin{aligned}\mu H_n(\omega) &= \sum_{C\in\mathcal{C}} \mu C \leq \sum_{C\in\mathcal{C}} \frac{M}{n}\#(\{i : J_i \subseteq C\}) + \frac{2M}{n}\#(\mathcal{C})\\ &\leq \frac{M}{n}\#(\{i : J_i \subseteq H_n(\omega)\}) + \frac{4M}{n}\#(K_\omega) \leq \frac{5M}{n}\#(K_\omega) \text{ **Q**}\end{aligned}$$

It follows that if $\mu H(\omega) \geq \epsilon$,

$$\#(K_\omega) \geq \frac{n\epsilon}{5M}\epsilon \geq \frac{\epsilon l}{5} \geq \frac{l}{p} \geq m.$$

(e) We find that

$$\lambda\{\omega : \mu H_n(\omega) \geq \epsilon\} \leq \gamma^n.$$

P By the last remark in (d), $\{\omega : \mu H_n(\omega) \geq \epsilon\} \subseteq \{\omega : \#(K_\omega) \geq m\}$ has measure at most

$$\begin{aligned}\lambda\{\omega : \#(K_\omega) \geq m\} &\leq \sum_{K\in[l]^m} \lambda\{\omega : \omega[n] \text{ does not meet } \bigcup_{i\in K} J_i\}\\ &\leq \#([l]^m)(1 - \frac{Mm}{n})^n\end{aligned}$$

(because by (c-vi), $\bigcup_{i\in K} J_i$ has measure at least $\frac{Mm}{n}$ for every $K \in [l]^m$)

$$\leq \#([m(p+1)]^m)(1 - \frac{Mm}{n})^n$$

(because $m \geq p$ and $l \leq (m+1)p \leq m(p+1)$)

$$\leq \frac{(m(p+1))^m}{m!}(1-\frac{1}{2p})^{Mmp}$$

(because $Mmp \leq Ml < n \leq M(l+1) \leq M(m(p+1)+1) \leq 2Mmp$)

$$= \frac{e^{-m}m^m}{m!}(e(p+1)(1-\frac{1}{2p})^{Mp})^m \leq \frac{1}{2^m} \leq \frac{1}{2^{n/2Mp}} = \gamma^n. \text{ Q}$$

(f) At last we are ready to vary n. Since (b)-(e) apply to all $n \geq n_0$, and $\gamma < 1$, we have

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu(\bigcup\{V'(\omega\restriction n, z) : z \in \omega[n],\, \mu V'(\omega\restriction n, z) \geq \frac{4M+1}{n}\}) \geq \epsilon\}$$

$$\leq n_0 + \sum_{n=n_0}^{\infty} \lambda\{\omega : \mu H_n(\omega) \geq \epsilon\} \leq n_0 + \sum_{n=n_0}^{\infty} \gamma^n < \infty.$$

As ϵ is arbitrary, the definition in 2I is satisfied and μ has moderated Voronoi tesselations.

3B Corollary If μ is a topological probability measure on $\mathbb{R}$, it is Mycielski-regular.

proof Put 2J and 3A together.

3C Remark The definition in 1Bb refers to convergence in measure of sequences $\langle F(\omega\restriction n, f)\rangle_{n\geq 1}$. The corresponding question for almost-everywhere convergence has a positive answer for separable X and continuous f (see 2D) but not for general measurable f, even in the one-dimensional case; when (X, ρ, μ) is the unit interval with its usual metric and Lebesgue measure, there is a construction in Mycielski p10, §4 of an open set $G \subseteq X$ such that, for λ-almost every ω, $\langle F(\omega\restriction n, \chi G)\rangle_{n\geq 1}$ does not converge μ-almost everywhere to χG.

4 The homogeneous case

4A(a) In this section, I will suppose that the metric space (X, ρ) of 1B is compact and that its isometry group G is **sesquitransitive**, that is, for any x, $y \in X$ there is an $R \in G$ such that $Rx = y$ and $Ry = x$. Give G its topology of pointwise convergence, so that G is a compact Hausdorff topological group (Fremlin 03, 441G), and has a unique Haar probability measure ν. In this case there is exactly one isometry-invariant Radon probability measure on X (Fremlin 03, 441H and 443Ud); I will suppose that μ is that invariant measure.

(b)(i) For any $T \in G$, the map $\omega \mapsto T\omega : \Omega \to \Omega$ is an isomorphism of the measure space (Ω, λ), just because $x \mapsto T(x)$ is an isomorphism of (X, μ).

(ii) Note that $(T, \omega) \mapsto T\omega : G \times \Omega \to \Omega$ is continuous (Fremlin 03, 441Ga, or otherwise), and also that it is inverse-measure-preserving for the product measure $\nu \times \lambda$ and λ. **P** If $W \subseteq \Omega$ is a Borel set, then $W' = \{(T, \omega) : T\omega \in W\}$ is a Borel set, so

$$(\nu \times \lambda)(W') = \int \lambda\{\omega : T\omega \in W\}\nu(dT) = \int \lambda W \nu(dT) = \lambda W.$$

As λ is inner regular with respect to the Borel sets, this is enough to show that $(T, \omega) \mapsto T\omega$ is inverse-measure-preserving (Fremlin 03, 412K). **Q**

(iii) Another useful fact: if $z \in X$, then $T \mapsto T(z) : G \to X$ is inverse-measure-preserving for ν and μ. **P** It is continuous, so defines an image Radon probability measure ν' on X by the formula $\nu' E = \nu\{T : T(z) \in E\}$ for every Borel set $E \subseteq X$. Now, for any $S \in G$, $\nu' S^{-1}[E] = \nu\{T : ST(z) \in E\} = \nu\{T : T(z) \in E\}$ because ν is translation-invariant; but this means that $\nu' S^{-1}[E] = \nu' E$ for every Borel set E, so that ν' is S-invariant. As S is arbitrary, ν' is isometry-invariant; but μ is the only isometry-invariant Radon probability measure on X, so $\nu' = \mu$ and $T \mapsto T(z)$ is (ν, μ)-inverse-measure-preserving. **Q**

(c) For examples of the situation in (a) above, we have a sphere (in any Euclidean space), with either the Euclidean metric or the great-circle metric, and with a normalized Hausdorff measure; also (X, ρ, μ) as described in Example 2D, and the tori considered in 5A below.

4B Proposition In the context of 1B/4A,

$$(\lambda \times \mu)\{(\omega, x) : F(\omega\restriction n, f)(x) \ge \alpha\} = \mu\{x : f(x) \ge \alpha\}$$

whenever $n \ge 1$, $f : X \to \mathbb{R}$ is Σ-measurable and $\alpha \in \mathbb{R}$.

proof Set $\gamma = \mu\{x : f(x) \ge \alpha\}$.

(a) Consider first the case in which f is Borel measurable, so that the functions $(\omega, x) \mapsto F(\omega\restriction n, f)(x)$ and $(T, \omega, x) \mapsto F(T\omega\restriction n, f)(x)$ are Borel measurable. For fixed $\varpi \in X^n$, consider $k(T\varpi, x)$ as defined in 2B. We have

$$\begin{aligned} \rho(x, T\varpi[n]) &= \rho(x, T\varpi[n]) = \rho(x, T[\varpi[n]]) = \rho(T^{-1}(x), \varpi[n]), \\ k(T\varpi, x) &= \min\{i : \rho(x, T\varpi(i)) = \rho(x, T\varpi[n])\} \\ &= \min\{i : \rho(T^{-1}(x), \varpi(i)) = \rho(T^{-1}(x), \varpi[n]))\} = k(\varpi, T^{-1}(x)), \\ F(T\varpi, f)(x) &= f(T\varpi(k(T\varpi, x))) = f(T\varpi(k(\varpi, T^{-1}(x)))) = F(\varpi, fT)(T^{-1}(x)) \end{aligned}$$

for every $T \in G$ and $x \in X$. Next, for any $y \in X$, $T \mapsto T(y)$ is inverse-measure-preserving, so

$$\nu\{T : f(T(y)) \ge \alpha\} = \mu\{x : f(x) \ge \alpha\} = \gamma.$$

Consequently

$$\begin{aligned} (\nu \times \mu)\{(T, x) : F(T\varpi, f)(x) \ge \alpha\} &= (\nu \times \mu)\{(T, x) : F(\varpi, fT)(T^{-1}(x)) \ge \alpha\} \\ &= \int \mu\{x : F(\varpi, fT)(T^{-1}(x)) \ge \alpha\}\nu(dT) \\ &= \int \mu(T[\{x : F(\varpi, fT)(x) \ge \alpha\}])\nu(dT) \\ &= \int \mu\{x : F(\varpi, fT)(x) \ge \alpha\}\nu(dT) \end{aligned}$$

(because each $T \in G$ is an isomorphism of (X, μ))

$$\begin{aligned} &= (\nu \times \mu)\{(T, x) : F(\varpi, fT)(x) \ge \alpha\} \\ &= (\nu \times \mu)\{(T, x) : fT(\varpi(k(\varpi, x))) \ge \alpha\} \\ &= \int \nu\{T : fT(\varpi(k(\varpi, x))) \ge \alpha\}\mu(dx) \\ &= \int \gamma\, \mu(dx) = \gamma. \end{aligned}$$

Applying this with $\varpi = \omega\restriction n$,

$$(\lambda \times \mu)\{(\omega, x) : F(\omega\restriction n, f)(x) \ge \alpha\} = (\nu \times \lambda \times \mu)\{(T, \omega, x) : F(T\omega\restriction n, f)(x) \ge \alpha\}$$

(because the map $(T, \omega) \to T\omega$ is inverse-measure-preserving)

$$\begin{aligned} &= \int (\nu \times \mu)\{(T, x) : F(T\omega\restriction n, f)(x) \ge \alpha\}\lambda(d\omega) \\ &= \int \gamma\, \lambda(d\omega) = \gamma, \end{aligned}$$

as required.

(b) For general bounded measurable f, we have Borel measurable functions f_0, f_1 such that $f_0 \le f \le f_1$ and

$$\mu\{x : f_0(x) \ge \alpha\} = \mu\{x : f_1(x) \ge \alpha\} = \gamma;$$

now $\{(\omega, x) : F(\omega\restriction n, f)(x) \ge \alpha\}$ includes $\{(\omega, x) : F(\omega\restriction n, f_0)(x) \ge \alpha\}$ and is included in $\{(\omega, x) : F(\omega\restriction n, f_1)(x) \ge \alpha\}$, so also has measure γ.

(c) Finally, for general measurable f, set $g = \arctan f$; then $F(\varpi, g) = \arctan F(\varpi, f)$ for all ϖ, so

$$\{(\omega, x) : F(\omega\restriction n, f)(x) \ge \alpha\} = \{(\omega, x) : F(\omega\restriction n, g)(x) \ge \arctan(\alpha)\}$$

has measure $\mu\{x : g(x) \ge \arctan(\alpha)\} = \gamma$.

Remark Note that an alternative expression of this result is

$\lambda\{\omega : F(\omega\restriction n, f)(\omega(n)) \ge \alpha\} = \mu\{x : f(x) \ge \alpha\}$ whenever $n \ge 1$, $f : X \to \mathbb{R}$ is Σ-measurable and $\alpha \in \mathbb{R}$

(compare MYCIELSKI p10, Theorem 1).

4C Proposition In the context of 1B/4A,

(a) $\int F(T\varpi, f)d\mu = \int F(\varpi, fT)d\mu$ whenever $T \in G$, $\varpi \in \bigcup_{n\ge1} X^n$ and f is μ-integrable.

(b) $\iint F(T\varpi, f)d\mu\,\nu(dT) = \int f d\mu$ whenever $\varpi \in \bigcup_{n\ge1} X^n$, $x \in X$ and $f : X \to \mathbb{R}$ is μ-integrable.

proof (a) As in part (a) of the proof of 4B, $F(T\varpi, f)(x) = F(\varpi, fT)(T^{-1}(x))$ for all x and T. So

$$\int F(T\varpi, f)d\mu = \int F(\varpi, fT)(T^{-1}x)\mu(dx) = \int F(\varpi, fT)(x)\mu(dx)$$

because T is an automorphism of (X, μ).

(b)(i) Suppose that f is bounded and Borel measurable. Note that $\int f(T(y))\nu(dT) = \int f d\mu$ for every $y \in X$, because $T \mapsto T(y)$ is inverse-measure-preserving. So

$$\begin{aligned}\iint F(T\varpi, f)d\mu\,\nu(dT) &= \iint F(\varpi, fT)(x)\mu(dx)\nu(dT)\\ &= \iint fT(\varpi(k(\varpi, x)))\nu(dT)\mu(dx)\\ &= \int (\int f d\mu)\mu(dx) = \int f d\mu.\end{aligned}$$

(ii) If f is bounded and integrable, let f_0, f_1 be Borel measurable functions such that $f_0 \le f \le f_1$ and both are equal almost everywhere to f. Then

$$F(T\varpi, f_0) \le F(T\varpi, f) \le F(T\varpi, f_1)$$

for every T, and

$$\iint F(T\varpi, f_0)d\mu\,\nu(dT) = \iint F(T\varpi, f_1)d\mu\,\nu(dT) = \int f d\mu,$$

so $\iint F(T\varpi, f)d\mu\,\nu(dT)$ is defined and equal to $\int f d\mu$.

(iii) If f is non-negative and integrable, set $f_l = f \wedge l\chi X$ for $l \in \mathbb{N}$. Then $\langle F(T\varpi, f_l)\rangle_{l\in\mathbb{N}}$ is a non-decreasing sequence with supremum $F(T\varpi, f)$ for every T, so

$$\begin{aligned}\iint F(T\varpi, f)d\mu\,\nu(dT) &= \sup_{l\in\mathbb{N}} \iint F(T\varpi, f_l)d\mu\,\nu(dT)\\ &= \sup_{l\in\mathbb{N}} \int f_l d\mu = \int f d\mu.\end{aligned}$$

(iv) Finally, for general integrable f,

$$\begin{aligned}\iint F(T\varpi, f)d\mu\,\nu(dT) &= \iint F(T\varpi, f^+)d\mu\,\nu(dT) - \iint F(T\varpi, f^-)d\mu\,\nu(dT)\\ &= \int f^+ d\mu - \int f^- d\mu = \int f d\mu.\end{aligned}$$

4D Example There is a compact metric space (X, ρ) with a sesquitransitive isometry group such that its invariant Radon measure does not have moderated Voronoi tesselations.

proof Choose integers $M_0, n_0, m_0, M_1, n_1, m_1, \ldots$ such that, for every $k \in \mathbb{N}$, $M_k = \prod_{j<k} m_j$ (starting with $M_0 = 1$), $n_k \ge 1$, $n_k \ge 2kM_k$, $M_k(1 - \frac{1}{M_k})^{n_k} \le \frac{1}{2}$ and $m_k = 2n_k$.

For $k \in \mathbb{N}$, let Y_k be a set with m_k elements. Set $X = \prod_{k\in\mathbb{N}} Y_k$, and $Z_k = \prod_{j<k} Y_j$ for $k \in \mathbb{N}$ (starting with $Z_0 = \{\emptyset\}$), so that $\#(Z_k) = M_k$. For x, $y \in X$, set

$$\rho(x, y) = \inf\{\frac{1}{k+1} : k \in \mathbb{N},\, x\restriction k = y\restriction k\}.$$

Then (X, ρ) is a compact metric space. For $k \in \mathbb{N}$ and $\sigma \in Z_k$ set $X_\sigma = \{x : \sigma \subseteq x \in X\}$; then $X_\sigma = B(x, \frac{1}{k+1})$ for every $x \in X_\sigma$. The isometry group of X contains all functions of the form

$$x \mapsto \langle \pi_k(x(k))\rangle_{k\in\mathbb{N}}$$

where $\pi_k : Y_k \to Y_k$ is a permutation for each k, so is sesquitransitive. Let μ be the invariant Radon probability measure; then $\mu X_\sigma = \mu X_\tau = \dfrac{1}{M_k}$ whenever $k \in \mathbb{N}$ and σ, $\tau \in Z_k$, so μ is the product of the uniform probability measures on the factors Y_k.

For each $k \in \mathbb{N}$, set

$$H_k = \{\omega : \mu(\bigcup\{V'(\omega\restriction n_k, z) : z \in \omega[n_k], \mu V'(\omega\restriction n_k, z) \ge \frac{k}{n_k}\}) \ge \frac{1}{2}\},$$

$$H'_k = \{\omega : \omega[n_k] \cap X_\sigma \ne \emptyset \text{ for every } \sigma \in Z_k\}.$$

Then $H'_k \subseteq H_k$. **P** If $\omega \in H'_k$, then for each $\sigma \in Z_k$ let $i_\sigma < n_k$ be minimal such that $\omega(i_\sigma) \in X_\sigma$. If we look at $V(\omega\restriction n_k, \omega(i_\sigma))$, we see that this includes

$$X_\sigma \setminus \bigcup_{i<n_k} X_{\omega(i)\restriction k+1},$$

because if $x \in X_\sigma$ then i_σ is the first i such that $\rho(x, \omega(i)) \le \frac{1}{k+1}$, while if $x \notin \bigcup_{i<n_k} X_{\omega(i)\restriction k+1}$ then $\rho(x, \omega[n_k]) \ge \frac{1}{k+1}$. So

$$\mu V'(\omega\restriction n_k, \omega(i_\sigma)) = \mu V(\omega\restriction n_k, \omega(i_\sigma)) \ge \frac{1}{M_k} - \frac{n_k}{M_{k+1}}$$
$$= \frac{1}{M_k}(1 - \frac{n_k}{m_k}) = \frac{1}{2M_k} \ge \frac{k}{n_k},$$

so

$$V'(\omega\restriction n_k, \omega(i_\sigma)) \subseteq \bigcup\{V'(\omega\restriction n_k, z) : z \in \omega[n_k], \mu V'(\omega\restriction n_k, z) \ge \frac{k}{n_k}\}.$$

Note also that the calculations just above show that

$$\mu V'(\omega\restriction n_k, \omega(i_\sigma)) \cap X_\sigma \ge \frac{1}{2}\mu X_\sigma.$$

This is true for each $\sigma \in Z_k$. So

$$\mu(\bigcup\{V'(\omega\restriction n_k, z) : z \in \omega[n_k], \mu V'(\omega\restriction n_k, z) \ge \frac{k}{n_k}\}) \ge \sum_{\sigma\in Z_k} \mu V'(\omega\restriction n_k, \omega(i_\sigma)) \cap X_\sigma$$
$$\ge \sum_{\sigma\in Z_k} \frac{1}{2}\mu X_\sigma = \frac{1}{2},$$

and $\omega \in H_k$. **Q**

On the other hand, $\lambda H'_k \ge \frac{1}{2}$. **P** For each $\sigma \in Z_k$,

$$\lambda\{\omega : \omega[n_k] \cap X_\sigma = \emptyset\} = (1 - \frac{1}{M_k})^{n_k},$$

so

$$\lambda(\Omega\setminus H'_k)\le M_k(1-\frac{1}{M_k})^{n_k}\le\frac{1}{2}.\ \mathbf{Q}$$

Thus $\lambda H_k\ge\frac{1}{2}$ for every k. Now if we take any $M\in\mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty}&\lambda(\{\omega:\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n],\,\mu V'(\omega\restriction n,z)\ge\frac{M}{n}\})\ge\frac{1}{2}\}\\ &\ge\sum_{k=M}^{\infty}\lambda(\{\omega:\mu(\bigcup\{V'(\omega\restriction n,z):z\in\omega[n_k],\,\mu V'(\omega\restriction n_k,z)\ge\frac{k}{n}_k\})\ge\frac{1}{2}\}\\ &=\sum_{k=M}^{\infty}\lambda H_k=\infty,\end{aligned}$$

and Definition 2I is not satisfied.

5 Lebesgue measure We can hope to apply results proved in the context of §4 to Lebesgue measure on $[0,1]^r$, through the following device.

5A Let $S^1\subseteq\mathbb{C}$ be the unit circle, regarded as a group under complex multiplication. For ξ, $\eta\in S^1$, set $\sigma_0(\xi,\eta)=|\arg(\xi^{-1}\eta)|$. Let $r\ge1$ be an integer, and set $Y=(S^1)^r$ with the ℓ^2-metric

$$\sigma(y,y')=\sqrt{\textstyle\sum_{j=0}^{r-1}\sigma_0(y(j),y'(j))^2}$$

for y, $y'\in Y$. Then S^1 and Y are compact metric spaces and their isometry groups are sesquitransitive. The isometry-invariant Radon probability measure μ_Y on Y is its Haar measure. Set $X=[0,1]^r$, with the Euclidean metric ρ, and define $\phi:X\to Y$ by setting

$$\phi(x)(j)=e^{2\pi ix(j)}$$

for $x\in X$ and $j<r$. Then ϕ is inverse-measure-preserving for Lebesgue measure μ_X on X and μ_Y. Write Σ_X, Σ_Y for the domains of μ_X, μ_Y respectively. The set $X_0=\,]0,1[^r$ is a conegligible subset of X, and for any $x\in X_0$ there is a $\delta>0$ such that $\sigma(\phi(x),\phi(x'))=\rho(x,x')$ and $x'\in X_0$ whenever either $\rho(x,x')\le\delta$ or $\sigma(\phi(x),\phi(x'))\le\delta$. The map $\omega\mapsto\phi\omega:X^{\mathbb{N}}\to Y^{\mathbb{N}}$ is inverse-measure-preserving for the product measures $\lambda_X=\mu_X^{\mathbb{N}}$ and $\lambda_Y=\mu_Y^{\mathbb{N}}$. If $f:X\to\mathbb{R}$ is Σ_X-measurable, there is a Σ_Y-measurable $g:Y\to\mathbb{R}$ such that $g\phi=_{\text{a.e.}}f$. In this case, for any $x\in X_0$, there is a λ_X-conegligible subset W_x of $X^{\mathbb{N}}$ such that for every $\omega\in W_x$ there is an $n\in\mathbb{N}$ such that $F(\phi\omega\restriction m,g)(\phi(x))=F(\omega\restriction m,f)(x)$ for every $m\ge n$. $\mathbf{P}$ Take $\delta>0$ such that $\sigma(\phi(x),\phi(x'))=\rho(x,x')$ whenever either $\rho(x,x')\le\delta$ or $\sigma(\phi(x),\phi(x'))\le\delta$. Set $W_x=\{\omega:\rho(x,\omega[\mathbb{N}])<\delta\}$. Because μ_X is strictly positive, W_x is λ_X-conegligible. If $\omega\in W_x$, there is an $n\in\mathbb{N}$ such that $\rho(x,\omega[n])<\delta$; now $\rho(\phi(x),\phi\omega[m])=\rho(x,\omega[m])$, $k(\phi\omega\restriction m,\phi x)=k(\omega\restriction m,x)$, $F(\phi\omega\restriction m,g)(\phi x)=F(\omega\restriction m,f)(x)$ for every $m\ge n$. $\mathbf{Q}$

It follows that if g is such that $\langle F(\omega\restriction n,g)\rangle_{n\in\mathbb{N}}$ converges to g in measure for almost every $\omega\in Y^{\mathbb{N}}$, then $\langle F(\omega\restriction n,f)\rangle_{n\in\mathbb{N}}$ will converge to f is measure for almost every $\omega\in X^{\mathbb{N}}$.

5B Theorem Let $r\ge1$ be an integer. Let (X,ρ,μ) be $[0,1]^r$ with its Euclidean metric and Lebesgue measure. Then μ has moderated Voronoi tesselations, so is Mycielski-regular.

proof (a) Take $\epsilon\in\,]0,1]$. Set $q=(2+2\sqrt{r})^r$. Let m_0 be such that $m!\ge e^{-m}m^m$ for $m\ge m_0$. Set $M=1+\lceil\ln\frac{10}{\epsilon}\rceil$. Set $\gamma=\frac{1}{2^{\epsilon/4M}}$. Set

$$n_0=\lceil\max(\frac{2M}{(2^{1/r}-1)^r},\frac{4Mm_0}{\epsilon},\frac{20}{\epsilon})\rceil.$$

(b) Take $n\ge n_0$. Set $l=\lfloor(\frac{n}{M})^{1/r}\rfloor\ge1$. Observe that $n\le M(l+1)^r$ so $l+1\ge\frac{2^{1/r}}{2^{1/r}-1}$, $l\ge\frac{1}{2^{1/r}-1}$, $l+1\le2^{1/r}l$ and $n\le2Ml^r$. Set $m=\lfloor\frac{\epsilon n}{4M}\rfloor\ge m_0$; then $\frac{\epsilon n}{5M}\le\frac{\epsilon n}{4M}-1\le m$ and $\frac{l^r}{m}\le\frac{n}{Mm}\le\frac{5}{\epsilon}$; also $m\le\frac{n}{4M}\le l^r$.

Let $\mathcal{J}$ be the set of hypercubes of the form $\prod_{j<r}[\frac{i_j}{l},\frac{i_j+1}{l}]$ where $i_j<l$ for $j<r$, so that $\#(\mathcal{J})=l^r$, $\bigcup\mathcal{J}=X$ and $\mu J=\frac{1}{l^r}$ and $\operatorname{diam}J=\frac{\sqrt{r}}{l}$ for $J\in\mathcal{J}$.

(c) Now the key to the proof is the following elementary fact. Suppose that $V\subseteq[0,1]^r$ is convex and $x\in V$. Set

$$\tilde{V}=V\cap\bigcup_{y\in V}\{J:J\in\mathcal{J},\ J\subseteq\operatorname{int}B(y,\rho(y,x))\}.$$

Then $\mu(V\setminus\tilde{V})\le\frac{2Mq}{n}$. **P** Let $\mathcal{K}$ be the set of members of $\mathcal{J}$ meeting $B(x,\frac{\sqrt{r}}{l})$; then $\#(\mathcal{K})\le q$, because each projection of $B(x,\frac{\sqrt{r}}{l})$ onto a coordinate has length at most $\frac{2\sqrt{r}}{l}$ and meets at most $2+2\sqrt{r}$ intervals of the form $[\frac{i}{r},\frac{i+1}{r}]$. If $y\in V\setminus\bigcup\mathcal{K}$, let $J\in\mathcal{J}$ be such that $y\in J$. Then $J\subseteq B(y,\frac{\sqrt{r}}{l})$, while $\rho(y,x)>\frac{\sqrt{r}}{r}$, so $J\subseteq\operatorname{int}B(y,\rho(x,y))$ and $y\in\tilde{V}$. Accordingly $V\setminus\tilde{V}\subseteq\bigcup\mathcal{K}$ is covered by q members of $\mathcal{J}$, and has measure at most $\frac{q}{l^r}\le\frac{2Mq}{n}$. **Q**

I should perhaps remark at this point that because half-spaces in $\mathbb{R}^r$ are convex, all sets $V(\varpi,z)$, as defined in 1B(a-i) from the Euclidean metric ρ, will be convex.

(d) For $\omega\in\Omega$, set

$$H_n(\omega)=\bigcup\{V(\omega\restriction n,z):z\in\omega[n],\ \mu V(\omega\restriction n,z)>\frac{4Mq}{n}\},$$

$$\mathcal{K}_\omega=\{J:J\in\mathcal{J},\ J\cap\omega[n]=\emptyset\}.$$

Then $\mu H_n(\omega)\le\frac{4M}{n}\#(\mathcal{K}_\omega)$. **P** If $z\in\omega[n]$ and $V(\omega\restriction n,z)$ has measure greater than $\frac{4Mq}{n}$, then for every $y\in V(\omega\restriction n,z)$, $\operatorname{int}B(y,\rho(y,z))$ does not meet $\omega[n]$, and every member of $\mathcal{J}$ included in $\operatorname{int}B(y,\rho(y,z))$ belongs to $\mathcal{K}_\omega$. By (c), $V(\omega\restriction n,z)\setminus\bigcup\mathcal{K}_\omega$ has measure at most $\frac{2Mq}{n}$. Consequently

$$\mu(V(\omega\restriction n,z))\le 2\mu(V(\omega\restriction n,z)\cap\bigcup\mathcal{K}_\omega).$$

Summing over the relevant z,

$$\mu H_n(\omega)\le 2\mu(H_n(\omega)\cap\bigcup\mathcal{K}_\omega)\le\frac{2}{l^r}\#(\mathcal{K}_\omega)\le\frac{4M}{n}\#(\mathcal{K}_\omega).\ \mathbf{Q}$$

It follows that if $\mu H_n(\omega)\ge\epsilon$, $\#(\mathcal{K}_\omega)\ge\frac{\epsilon n}{4M}\ge m$.

(e) We find that

$$\lambda\{\omega:\mu H_n(\omega)\ge\epsilon\}\le\gamma^n.$$

P By the last remark in (d), $\{\omega:\mu H_n(\omega)\ge\epsilon\}\subseteq\{\omega:\#(\mathcal{K}_\omega)\ge m\}$ has measure at most

$$\lambda\{\omega:\#(\mathcal{K}_\omega)\ge m\}\le\sum_{\mathcal{K}\in[\mathcal{J}]^m}\lambda\{\omega:\omega[n]\text{ does not meet }\bigcup\mathcal{K}\}$$

$$\le\#([\mathcal{J}]^m)(1-\frac{m}{l^r})^n\le\frac{l^{rm}}{m!}(1-\frac{m}{l^r})^{Ml^r}$$

(because $Ml^r\le n$)

$$\le\frac{e^m l^{rm}}{m^m}(1-\frac{1}{l^r})^{Mml^r}$$

(because $m\ge m_0$ and $1-\frac{m}{l^r}\le(1-\frac{1}{l^r})m$)

$$\le\frac{e^m l^{rm}}{m^m}(\frac{1}{e})^{Mm}$$

(because $\ln(1-x)\le -x$, so $(1-x)^{1/x}\le\frac{1}{e}$ for every $x>0$)

$$\leq \Big(\frac{el^r}{me^M}\Big)^m \leq \Big(\frac{5e}{\epsilon e^M}\Big)^m \leq \frac{1}{2^m}$$

(by the choice of M)

$$\leq \frac{1}{2^{\epsilon n/4M}} = \gamma^n.$$

(f) At last we are ready to vary n. Since (b)-(e) apply to all $n \geq n_0$, and γ is less than 1, we have

$$\sum_{n=1}^{\infty} \lambda\{\omega : \mu(\bigcup\{V(\omega\restriction n, z) : z \in \omega[n],\ \mu V(\omega\restriction n, z) \geq \frac{4Mq}{n}\}) \geq \epsilon\}$$

$$\leq n_0 + \sum_{n=n_0}^{\infty} \lambda\{\omega : \mu H_n(\omega) \geq \epsilon\} \leq n_0 + \sum_{n=n_0}^{\infty} \gamma^n < \infty.$$

As ϵ is arbitrary, the definition in 2I is satisfied and μ has moderated Voronoi tesselations. By Theorem 2J, it follows at once that μ is Mycielski-regular.

5C Proposition In 1B, let (X, ρ, μ) be the unit interval with its usual metric and Lebesgue measure. Let $\langle \beta_n \rangle_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}$ converging to 0. Then there are a continuous function $g : X \to [0, 1]$ and a sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ in $\mathbb{N} \setminus \{0\}$ such that $\int \min(1, |F(\omega\restriction n_k, g) - g|)d\mu \geq \beta_{n_k}$ for every $k \in \mathbb{N}$ and every $\omega \in \Omega$.

proof Let $\langle n_k \rangle_{k\in\mathbb{N}}$ be such that $\beta_{n_k} \leq 2^{-2k-6}$ for every k. Set

$$\begin{aligned} g(x) &= 2^{-k} \sin(2^{k+4} n_k \pi x) \text{ if } k \in \mathbb{N} \text{ and } 2^{-k-1} < x \leq 2^{-k}, \\ &= 0 \text{ if } x = 0. \end{aligned}$$

Take $k \in \mathbb{N}$ and $\omega \in \Omega$. Set $\delta = \frac{1}{2^{k+3} n_k}$, so that if $4n_k \leq j < 8n_k$, then $g(x) = \sin(\frac{2\pi x}{\delta})$ for $j\delta \leq x \leq (j + 1)\delta$. Set

$$K = \{i : 2n_k \leq i < 4n_k,\ \omega[n_k] \cap]2i\delta, (2i + 2)\delta[= \emptyset\}.$$

Then $\#(K) \geq n_k$, and for $i \in K$, there can be at most two values of $z \in \omega[n_k]$ such that $V(\omega\restriction n_k, z)$ meets $]2i\delta, (2i + 2)\delta[$. For these i, therefore, at least one of $]2i\delta, (2i + 1)\delta[$, $](2i + 1)\delta, (2i + 2)\delta[$ is included in a single tile $V(\omega\restriction n_k, z)$, and accordingly $F(\omega\restriction n_k, g)$ must be constant on that interval; call it I_i. Now since g runs through a full cycle in the interval I_i, with magnitude 2^{-k}, the subintervals $\{x : x \in I_i,\ g(x) \geq 2^{-k-1}\}$ and $\{x : x \in I_i,\ g(x) \leq -2^{-k-1}\}$ both have length $\frac{\delta}{3}$. But this means that

$$\int_{I_i} |F(\omega\restriction n_k, g) - g|d\mu \geq \frac{\delta}{3} \cdot 2^{-k-1}.$$

Summing over $i \in K$,

$$\int \min(1, |F(\omega\restriction n_k) - g|)d\mu = \int |F(\omega\restriction n_k) - g|d\mu \geq \frac{\delta\#(K)}{3} \cdot 2^{-k-1}$$

$$\geq 2^{-k-3} n_k \delta = 2^{-2k-6} \geq \beta_{n_k},$$

as required.

5D Corollary If (X, ρ, μ) is $[0, 1]$ with its usual metric and Lebesgue measure, there is a continuous function $g : X \to [0, 1]$ such that $\langle \|F(\omega\restriction n, g) - g\|_1 \rangle_{n\geq 1}$ does not converge to 0 at a geometric rate for any $\omega \in \Omega$.

proof Take a sequence $\langle \beta_k \rangle_{k\in\mathbb{N}} \to 0$ such that $\lim_{k\to\infty} e^{\gamma k} \beta_k = \infty$ whenever $\gamma > 0$, and apply 5C.

6 The Lebesgue density property

6A Definition I will say that a locally finite topological measure μ on a metric space (X, ρ) has the **Lebesgue density property** if μ has a support and for any set $E \subseteq X$ measured by μ we have

$$\lim_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta)}{\mu B(x,\delta)}) = 1$$

for μ-almost every $x \in E$, where $B(x,\delta)$ is the closed ball with centre x and radius δ. (Saying that μ is 'locally finite' amounts to saying that for every $x \in X$ there is a $\delta > 0$ such that $\mu B(x,\delta) < \infty$. Saying that 'μ has a support' amounts to saying that, for μ-almost every x, $\mu B(x,\delta) > 0$ for every $\delta > 0$.) By Besicovitch's density theorem (Fremlin 03, 472D), every Radon measure on Euclidean space has the Lebesgue density property.

6B Theorem (Mycielski p10, Theorem 1) Suppose that (X,ρ) is a separable metric space and μ is a topological probability measure on X with the Lebesgue density property. Let $f : X \to \mathbb{R}$ be a $\operatorname{dom}\mu$-measurable function. For $n \geq 1$, define $g_n : \Omega \times X \to \mathbb{R}$ by setting $g_n(\omega,x) = F(\omega\restriction n, f)(x) - f(x)$. Then $\langle g_n\rangle_{n\geq 1}$ converges in measure (for the product measure $\lambda \times \mu$) to the zero function on $\Omega \times X$.

6C Besides convergence in measure, as considered in 1Bb and 6B, and pointwise convergence, as in 3C, we can ask for convergence of Cesaro limits, calculated as follows. Let μ be a topological probability measure on a metric space (X,ρ). For $n \geq 1$, $\varpi \in X^n$ and $x \in X$, set $J(\varpi,x) = \{i : i < n,\ \rho(x,\varpi(i)) < \rho(x,\varpi(j))$ for $j < i\}$. Now, for $f : X \to \mathbb{R}$, set $\overline{F}(\varpi,f)(x) = \frac{1}{\#(J(\varpi,x))}\sum_{i\in J(\varpi,x)} f(\varpi(i))$. Under what circumstances shall we have $\langle\overline{F}(\omega\restriction n,f)\rangle_{n\geq 1}$ converging almost everywhere to f, for almost every ω? The interesting case is when X is separable, and now we also want μ to be atomless, since if $\mu\{x\} > 0$ then $\langle\overline{F}(\omega\restriction n,f)(x)\rangle_{n\geq 1}$ will be eventually constant for almost every ω.

6D Theorem (Mycielski p10, Theorem 2) If (X,ρ) is a metric space and μ is an atomless topological probability measure on X with the Lebesgue density property, then for any bounded $\operatorname{dom}\mu$-measurable function $f : X \to \mathbb{R}$, $\langle\overline{F}(\omega\restriction n,f)\rangle_{n\geq 1} \to f$ μ-a.e. for λ-almost every ω.

6E Let (X,ρ,μ) be the unit interval with its usual metric and Lebesgue measure. In Mycielski p10, §4, there is an example of a measurable function $f : X \to [0,\infty[$ such that

$$\langle\overline{F}(\omega\restriction n,f)\rangle_{n\geq 1}$$

does not converge μ-almost everywhere to f, for λ-almost every ω.

However the question remains open for integrable f (8E).

7 L^1-convergence

So far, we have mostly been considering convergence in measure. If we have an integrable function f, we can also ask whether $\langle F(f,\omega\restriction n)\rangle_{n\geq 1}$ converges to f for $\|\ \|_1$ for many ω. This seems to be a hard question in general. However I can give positive answers in a couple of cases. The first is straightforward, in view of the results so far.

7A Proposition Let (X,ρ) be a separable metric space, and μ a Mycielski-regular topological probability measure on X with domain Σ. Let $f : X \to \mathbb{R}$ be a bounded Σ-measurable function. Then $\lim_{n\to\infty}\int|f - F(f,\omega\restriction n)|d\mu = 0$ for λ-almost every $\omega \in \Omega$.

proof (a) Suppose that $f = \chi E$ where $E \in \Sigma$. By Proposition 2F and Theorem 2G, $\limsup_{n\to\infty}\int F(f,\omega\restriction n)d\mu = \mu E = \int f d\mu$ for almost every ω; we also know that $\langle F(f,\omega\restriction n)\rangle_{n\geq 1}$ converges in measure to f, for almost every ω; by Fremlin 01, 245H(a-ii), $\lim_{n\to\infty}\int|f - F(f,\omega\restriction n)|d\mu = 0$ for almost every ω.

(b) It follows at once that $\lim_{n\to\infty}\int|f - F(f,\omega\restriction n)|d\mu = 0$ for almost every ω whenever $f : X \to \mathbb{R}$ is a simple function.

(c) In general, given a bounded measurable $f : X \to \mathbb{R}$ and $\epsilon > 0$, there is a simple function $g : X \to \mathbb{R}$ such that $|g - f| \leq \epsilon\chi X$. In this case,

$$|f - F(f,\omega\restriction n)| \leq |g - F(g,\omega\restriction n)| + 2\epsilon\chi X$$

for every ω, so $\limsup_{n\to\infty}\int|f - F(f,\omega\restriction n)|d\mu \leq 2\epsilon$ for almost every ω; as ϵ is arbitrary, $\lim_{n\to\infty}\int|f - F(f,\omega\restriction n)|d\mu = 0$ for almost every ω.

7B Not every Mycielski-regular measure will give much more than this, as the following example (based on the same idea as Example 1D) shows.

Example There are a compact metric space (X,ρ), a point-supported probability measure μ on X, and a function $f : X \to [0,\infty[$ such that $\int f^p d\mu < \infty$ for every $p \in [1,\infty[$ but $\limsup_{n\to\infty} \int F(f,\omega\restriction n)d\mu = \infty$ for λ-almost every $\omega \in \Omega$.

proof (a) Choose α_n, δ_n, k_n, η_n and m_n, for $n \in \mathbb{N}$, such that

$$\alpha_n = 2^{n+2}(n+1), \quad \alpha_n\delta_n^{1/(n+1)} = 2^{-n},$$

$$(1-\delta_n)^{k_n} \le 2^{-n-1},$$

$$k_n\eta_n \le 2^{-n-2}, \quad \delta_n + (m_n-1)\eta_n = 2^{-n-1}.$$

(b) Set $X = \{0\} \cup \{(n,i) : n \in \mathbb{N},\ i < m_n\}$ and define $\rho : X \times X \to [0,\infty[$ by setting

$\rho(0,(n,i)) = \rho((n,i),0) = \frac{1}{n+1}$ whenever $n \in \mathbb{N}$, $i < m_n$,

$\rho((n,i),(n',i')) = \frac{1}{n+1} + \frac{1}{n'+1}$ whenever n, $n' \in \mathbb{N}$ are different, $i < m_m$ and $i' < m_{n'}$,

$\rho((n,0),(n,i)) = \rho((n,i),(n,0)) = \frac{1}{n+2}$ whenever $n \in \mathbb{N}$ and $0 < i < m_n$,

$\rho((n,i),(n,j)) = \rho((n,j),(n,i)) = \frac{1}{n+3}$ whenever $n \in \mathbb{N}$ and $0 < i < j < m_n$,

$\rho(x,x) = 0$ for every $x \in X$.

It is easy to see that (X,ρ) is a compact metric space.

(c) Let μ be the point-supported measure on X such that $\mu\{0\} = 0$ and $\mu\{(n,0)\} = \delta_n$ and $\mu\{(n,i)\} = \eta_n$ whenever $n \in \mathbb{N}$ and $0 < i < m_n$. Because $\sum_{n=0}^{\infty} \delta_n + (m_n-1)\eta_n = 1$, μ is a probability measure.

(d) Define $f : X \to [0,\infty[$ by setting

$$f(n,0) = \alpha_n \text{ for } n \in \mathbb{N}, \quad f(x) = 0 \text{ for other } x \in X.$$

If $1 \le p < \infty$, then

$$\|f\|_p \le \sum_{n=0}^{\infty} \alpha_n\delta_n^{1/p} < \infty$$

because

$$\alpha_n\delta_n^{1/p} \le \alpha_n\delta_n^{1/(n+1)} \le 2^{-n}$$

whenever $n \ge p-1$.

(e) Suppose that $n \in \mathbb{N}$, $\varpi \in X^{k_n}$ and $(n,0) \in \varpi[k_n]$; then $(n,0) \in V(\varpi,(n,0))$ and

$$\{i : i < m_n,\ (n,i) \notin V(\varpi,(n,0))\} = \{i : i < m_n,\ (n,i) \in \varpi[k_n]\}$$

has at most k_n members, so $\mu V(\varpi,(n,0)) \ge 2^{-n-1} - k_n\eta_n \ge 2^{-n-2}$ and $\int F(\varpi,f)d\mu \ge 2^{-n-2}\alpha_n \ge n$. At the same time,

$$\mu^{k_n}\{\varpi : \varpi \in X^{k_n},\ (n,0) \notin \varpi[k_n]\} = (1-\delta_n)^{k_n} \le 2^{-n}$$

for each n, so for almost every $\omega \in \Omega$ there are infinitely many n such that $(n,0) \in \omega[k_n]$, and $\limsup_{n\to\infty} F(\omega\restriction n, f) = \infty$.

7C I now turn to a partial result concerning Lebesgue measure on the unit interval, based on ideas in Grahl 07.

Lemma (Evans & Humke 07) Let (X,Σ,μ) be a probability space and $\langle E_i\rangle_{i\in I}$ a disjoint family in Σ with $\gamma_i = \mu E_i > 0$ for every $i \in I$. Set $Z = \prod_{i\in I} E_i$ with the product probability measure $\theta = \prod_{i\in I} \frac{1}{\gamma_i}\mu_{E_i}$, where μ_{E_i} is the subspace measure on E_i for each i. Suppose that $m \in \mathbb{N}$, $\alpha \ge 0$ and that $f : X \to [0,\alpha[$ is a Σ-measurable function. For $\varpi \in Z$ set

$$h(\varpi) = \int_E f d\mu - \sum_{i\in I} \gamma_i f(\varpi(i))$$

where $E = \bigcup_{i\in I} E_i$. Then

$$\int |h|^{2m} d\theta \le (2m)! 4^m \alpha^{2m} \gamma^m,$$

where $\gamma = \max_{i\in I} \gamma_i$.

proof (a) Let Q be the set of pairs (J, f) where $J \in [2m]^m$ and $f : J \to 2m \setminus J$ is a function such that $f(i) < i$ whenever $i \in J$. Then $\#(Q) \le (2m)!$. **P**

$$\#(Q) \le m^m \#([2m]^m) = \frac{m^m}{(m!)^2} \cdot (2m)!.$$

Now

$$\frac{(k+1)^{k+1}}{((k+1)!)^2} = \frac{k^k}{(k!)^2} \cdot \frac{1}{k+1}(1 + \frac{1}{k})^k \le \frac{k^k}{(k!)^2} \cdot \frac{e}{k+1} \le \frac{k^k}{(k!)^2}$$

if $k \ge 2$, while

$$\frac{0^0}{(0!)^2} = \frac{1^1}{(1!)^2} = \frac{2^2}{(2!)^2} = 1,$$

so $m^m \le (m!)^2$ and $\#(Q) \le (2m)!$. **Q**

(b) Let K be the set of functions $k : I \to \{0, \dots, 2m\}$ such that $\sum_{i\in I} k(i) = 2m$, and $N_k = \frac{(2m)!}{\prod_{i\in I} k(i)!}$ for $k \in K$; let K^* be $\{k : k \in K,\, k(i) \ne 1 \text{ for every } i\}$. Then $\sum_{k\in K^*} N_k \prod_{i\in I} \gamma_i^{k(i)} \le (2m)!\gamma^m$. **P** Take any $i_\infty \notin I$, set $I_\infty = I \cup \{i_\infty\}$, and let ν be the probability measure on I_∞ such that $\nu\{i\} = \gamma_i$ for each $i \in I$. Let ν^{2m} be the product measure on I_∞^{2m}. Then

$$\sum_{k\in K^*} N_k \gamma_i^{k(i)} \le \nu^{2m} W$$

where

$$W = \{w : w \in I^{2m},\ w \text{ takes at most } m \text{ values}\}.$$

Now for any $w \in W$ the set

$$\{j : j < 2m,\ w(j) = w(i) \text{ for some } i < j\}$$

must have at least m members. So $W = \bigcup_{(J,f)\in Q} W_{Jf}$, where $W_{Jf} = \{w : w \in I^{2m},\ w(j) = w(f(j)) \text{ for every } j \in J\}$. On the other hand, if $(J, f) \in Q$ and we identify ν^{2m} with the product measure $\nu^{2m\setminus J} \times \nu^J$ on $I_\infty^{2m\setminus J} \times I_\infty^J \cong I_\infty^{2m}$, we have

$$\begin{aligned}
\nu^{2m} W_{Jf} &= \int_{I^{2m\setminus J}} \nu^J\{v : u \cup v \in W_{Jf}\} \nu^{2m\setminus J}(du) \\
&= \int_{I^{2m\setminus J}} \nu^J\{v : v(j) = u(f(j))\ \forall\, j \in J\} \nu^{2m\setminus J}(du) \\
&= \int_{I^{2m\setminus J}} \prod_{j\in J} \gamma_{u(f(j))} \nu^{2m\setminus J}(du) \\
&\le \int \gamma^m \nu^{2m\setminus J}(du) = \gamma^m.
\end{aligned}$$

So

$$\sum_{k\in K^*} N_k \gamma_i^{k(i)} \le \nu^{2m} W \le \gamma^m \#(Q) \le (2m)!\gamma^m. \text{ **Q**}$$

(c) For $i \in I$ set

$$h_i(\varpi) = \int_{E_i} f d\mu - \gamma_i f(\varpi(i))$$

for $\varpi \in Z$, so that $\langle h_i \rangle_{i\in I}$ is an independent family of random variables with zero expectation, and $h = \sum_{i\in I} h_i$. Now

$$|h|^{2m} = h^{2m} = \textstyle\sum_{k\in K} N_k \prod_{i\in I} h_i^{k(i)}.$$

So

$$\int |h|^{2m} d\theta = \sum_{k\in K} N_k \int \prod_{i\in I} h_i^{k(i)} d\theta = \sum_{k\in K} N_k \prod_{i\in I} \int h_i^{k(i)} d\theta$$

(because the h_i are independent)

$$= \sum_{k\in K^*} N_k \prod_{i\in I} \int h_i^{k(i)} d\theta$$

(because $\int h_i d\theta = 0$ for every i)

$$\le \sum_{k\in K^*} N_k \prod_{i\in I} \|h_i\|_\infty^{k(i)} \le \sum_{k\in K^*} N_k \prod_{i\in I} (2\gamma_i\alpha)^{k(i)}$$
$$= 2^{2m}\alpha^{2m} \sum_{k\in K^*} N_k \gamma_i^{k(i)} \le (2m)! 2^{2m}\alpha^{2m}\gamma^m$$

by (b).

7D Lemma (Grahl 07, quoting Gut 05) Suppose that $p > 1$ and that $\langle X_i\rangle_{i\in I}$ is a finite independent family of random variables such that $\mathbb{E}(|X_i|^p)$ is finite for every $i \in I$. Set $S = \sum_{i\in I} X_i$. Then

$$\mathbb{E}(|S|^p) \le \max(2^p \textstyle\sum_{i\in I} \mathbb{E}(|X_i|^p), 2^{p^2}(\sum_{i\in I}\mathbb{E}(|X_i|))^p).$$

proof It will be enough to deal with the case in which every X_i is non-negative. In this case, set $\alpha = \mathbb{E}(S^p)$ and $S_i = S - X_i$ for each i. Then

$$\alpha = \sum_{i\in I} \mathbb{E}(S^{p-1}X_i) = \sum_{i\in I} \mathbb{E}((X_i + S_i)^{p-1}X_i) \le 2^{p-1} \sum_{i\in I} \mathbb{E}((X_i^{p-1} + S_i^{p-1})X_i)$$
$$= 2^{p-1} \sum_{i\in I} \mathbb{E}(X_i^p) + \mathbb{E}(S_i^{p-1}X_i) = 2^{p-1} \sum_{i\in I} \mathbb{E}(X_i^p) + \mathbb{E}(S_i^{p-1})\mathbb{E}(X_i)$$

(because X_i and S_i are independent)

$$\le 2^p \max(\sum_{i\in I} \mathbb{E}(X_i^p), \sum_{i\in I} \mathbb{E}(S^{p-1})\mathbb{E}(X_i))$$
$$\le 2^p \max(\sum_{i\in I} \mathbb{E}(X_i^p), \sum_{i\in I} (\mathbb{E}(S^p))^{(p-1)/p}\mathbb{E}(X_i))$$

(Fremlin 01, 244Xd)

$$= 2^p \max(\sum_{i\in I} \mathbb{E}(X_i^p), \alpha^{(p-1)/p}\mathbb{E}(S)).$$

So if $\alpha > 2^p \sum_{i\in I} \mathbb{E}(X_i^p)$, we must have $\alpha \le 2^p\alpha^{(p-1)/p}\mathbb{E}(S)$ and $\alpha \le 2^{p^2}(\mathbb{E}(S))^p = 2^{p^2}(\sum_{i\in I}\mathbb{E}(X_i))^p$.

7E Lemma (see Grahl 07, Theorem 3.2) Let (X, Σ, μ) be a probability space and $p > 2$; set $m = \lceil \frac{p(p-1)}{p-2} \rceil$. Let $f : X \to [0, \infty[$ be a Σ-measurable function with $\int f^p d\mu < \infty$, and $\epsilon > 0$. Set $\beta = \int f d\mu$ and

$$C = \frac{2^p}{\epsilon^p} \max(2^p\|f\|_p^p, 2^{p^2}\|f\|_p^{p^2}) + \frac{2^{4m}(2m)!}{\epsilon^{2m}}.$$

Let $\langle E_i\rangle_{i\in I}$ be a finite disjoint family of measurable subsets of X with $\gamma_i = \mu E_i > 0$ for every $i \in I$. Set $Z = \prod_{i\in I} E_i$ with the product probability measure $\theta = \prod_{i\in I} \frac{1}{\gamma_i}\mu_{E_i}$, where μ_{E_i} is the subspace measure on E_i for each i. For $\varpi \in Z$ set $g(\varpi) = \sum_{i\in I} \gamma_i f(\varpi(i))$. Then

$$\theta\{\varpi : g(\varpi) \ge \beta + \epsilon\} \le C\gamma^{p-1}$$

where $\gamma = \max_{i\in I}\gamma_i$.

proof (a) For $x \in X$ and $\varpi \in Z$, set

$$\begin{aligned} f_0(x) &= f(x) \text{ if } f(x) \le \gamma^{-1/p}, \\ &= 0 \text{ otherwise}, \\ f_1(x) &= f(x) - f_0(x), \\ g_0(\varpi) &= \sum_{i\in I}\gamma_i f_0(\varpi(i)), \\ g_1(\varpi) &= \sum_{i\in I}\gamma_i f_1(\varpi(i)). \end{aligned}$$

(b) Set $\beta_0 = \int_E f_0 d\mu \le \beta$, where $E = \bigcup_{i\in I} E_i$. Then Lemma 7C tells us that

$$\int |\beta_0 - g_0(\varpi)|^{2m}\theta(d\varpi) \le (2m)!4^m\gamma^{-2m/p}\gamma^m = (2m)!4^m\gamma^{m(p-2)/p} \le (2m)!4^m\gamma^{p-1}.$$

So

$$\begin{aligned} \theta\{\varpi : g_0(\varpi) \ge \beta + \frac{1}{2}\epsilon\} &\le \theta\{\varpi : |\beta_0 - g_0(\varpi)| \ge \frac{1}{2}\epsilon\} \\ &\le \frac{2^{2m}}{\epsilon^{2m}}\int |\beta_0\chi Z - g_0|^{2m}d\theta \le \frac{2^{4m}(2m)!}{\epsilon^{2m}}\gamma^{p-1}. \end{aligned}$$

(c) By Lemma 7D,

$$\begin{aligned} \int g_1^p d\theta &\le \max(2^p\sum_{i\in I}\int_Z(\gamma_i f_1(\varpi(i)))^p\theta(d\varpi), 2^{p^2}(\sum_{i\in I}\int_Z \gamma_i f_1(\varpi(i))\theta(d\varpi))^p) \\ &= \max(2^p\sum_{i\in I}\frac{1}{\gamma_i}\int_{E_i}(\gamma_i f_1(x))^p\mu(dx), 2^{p^2}(\sum_{i\in I}\frac{1}{\gamma_i}\int_{E_i}\gamma_i f_1(x)\mu(dx))^p) \\ &= \max(2^p\sum_{i\in I}\gamma_i^{p-1}\int_{E_i}f_1(x)^p\mu(dx), 2^{p^2}(\sum_{i\in I}\int_{E_i}f_1(x)\mu(dx))^p) \\ &\le \max(2^p\sum_{i\in I}\gamma^{p-1}\int_{E_i}f_1(x)^p\mu(dx), 2^{p^2}(\int_X f_1(x)\mu(dx))^p) \\ &\le \max(2^p\gamma^{p-1}\int_X f_1(x)^p\mu(dx), 2^{p^2}(\gamma^{(p-1)/p}\int_X f_1(x)^p\mu(dx))^p) \end{aligned}$$

(because $f_1(x) \ge \gamma^{-1/p}$ whenever $f_1(x) \ne 0$, so $f_1 \le \gamma^{(p-1)/p}f_1^p$)

$$\begin{aligned} &= \max(2^p\gamma^{p-1}\int_X f_1(x)^p\mu(dx), 2^{p^2}\gamma^{p-1}(\int_X f_1(x)^p\mu(dx))^p) \\ &\le \gamma^{p-1}\max(2^p\int_X f(x)^p\mu(dx), 2^{p^2}(\int_X f(x)^p\mu(dx))^p) \\ &= \gamma^{p-1}\max(2^p\|f\|_p^p, 2^{p^2}\|f\|_p^{p^2}). \end{aligned}$$

So

$$\theta\{\varpi : g_1(\varpi) \ge \frac{1}{2}\epsilon\} \le \frac{2^p}{\epsilon^p}\int g_1^p d\theta \le \frac{2^p}{\epsilon^p}\max(2^p\|f\|_p^p, 2^{p^2}\|f\|_p^{p^2})\gamma^{p-1}.$$

(d) Putting these together,

$$\begin{aligned} \theta\{\varpi : g(\varpi) \ge \beta + \epsilon\} &\le \theta\{\varpi : g_0(\varpi) \ge \beta + \frac{1}{2}\epsilon\} + \theta\{\varpi : g_1(\varpi) \ge \frac{1}{2}\epsilon\} \\ &\le \frac{2^{4m}(2m)!}{\epsilon^{2m}}\gamma^{p-1} + \frac{2^p}{\epsilon^p}\max(2^p\|f\|_p^p, 2^{p^2}\|f\|_p^{p^2})\gamma^{p-1} \\ &= C\gamma^{p-1}, \end{aligned}$$

as claimed.

7F Lemma Let (X,μ) be $]0,1[$ with Lebesgue measure, and $n \ge 2$ an integer. Write μ^n, $\mu^n \times \mu^k$ for the product measures on X^n, $X^n \times X^k$ respectively. For $\varpi \in X^n$ define $\langle t_i(\varpi)\rangle_{i<n}$ as in B. Define $\phi : X^n \to [X]^{\le n}$ by setting $\phi(\varpi) = \varpi[n]$ for $\varpi \in X^n$, and let ζ be the image measure $\mu^n\phi^{-1}$ on $[X]^{\le n}$.

(a) Suppose that $n = 2k$ is even. Define $\psi_1 : X^n \times X^k \to [X]^{\le n}$ by setting

$$\psi_1(\varpi,\varpi') = \{t_{2i}(\varpi) : i < k\} \cup \{(1-\varpi'(i))t_{2i}(\varpi) + \varpi'(i)t_{2i+2}(\varpi) : i < k-1\}$$
$$\cup \{(1-\varpi'(k-1))t_{2k-2}(\varpi) + \varpi'(k-1)\}.$$

Then the image measure $\zeta_1 = (\mu^n \times \mu^k)\psi_1^{-1}$ is equal to ζ.

(b) Suppose that $n = 2k$ is even. Define $\psi_2 : X^n \times X^k \to [X]^{\le n}$ by setting

$$\psi_2(\varpi,\varpi') = \{t_{2i+1}(\varpi) : i < k\} \cup \{\varpi'(0)t_0(\varpi)\}$$
$$\cup \{(1-\varpi'(i))t_{2i-1}(\varpi) + \varpi'(i)t_{2i}(\varpi) : 1 \le i < k\}.$$

Then $(\mu^n \times \mu^k)\psi_2^{-1} = \zeta$.

(c) Suppose that $n = 2k+1$ is odd. Define $\psi_3 : X^n \times X^k \to [X]^{\le n}$ by setting

$$\psi_3(\varpi,\varpi') = \{t_{2i}(\varpi) : i \le k\} \cup \{(1-\varpi'(i))t_{2i}(\varpi) + \varpi'(i)t_{2i+2}(\varpi) : i < k\}.$$

Then $(\mu^n \times \mu^k)\psi_3^{-1} = \zeta$.

(d) Suppose that $n = 2k+1$ is odd. Define $\psi_4 : X^n \times X^{k+1} \to [X]^{\le n}$ by setting

$$\psi_4(\varpi,\varpi') = \{t_{2i+1}(\varpi) : i < k\} \cup \{\varpi'(0)t_1(\varpi)\}$$
$$\cup \{(1-\varpi'(i))t_{2i-1}(\varpi) + \varpi'(i)t_{2i+1}(\varpi) : 1 \le i < k\}$$
$$\cup \{(1-\varpi'(k-1))t_{2k-1}(\varpi) + \varpi'(k-1)\}.$$

Then $(\mu^n \times \mu^{k+1})\psi_4^{-1} = \zeta$.

Remark Maybe it will help if I try to explain what the functions $\psi_1,\dots,\psi_4$ are doing. Given a pair (ϖ,ϖ'), we take every second member of $\varpi[n]$ and discard the rest; then we use ϖ' to replace the discarded members of $\varpi[n] = \phi(\varpi)$ by random members of the intervals between the retained members of $\varpi[n]$. The four forms of the result correspond to whether n is even or odd and whether we are keeping the even members of $\varpi[n]$ or the odd members. Saying that we get the same image measure on $[X]^{\le n}$ in every case amounts to saying that we can generate our random set $K \in [X]^{\le n}$ in two stages, first fixing certain members and then filling in the gaps independently.

proof (a)(i) If we give $[X]^{\le 2k}$ its Fell topology (Fremlin 03, 4A2T), then ϕ and ψ_1 are both continuous, so ζ and ζ_1 are Radon probability measures (Fremlin 03, 418I). Set $W = \{\varpi : \varpi \in X^{2k}$ is injective$\}$; then W is μ^{2k}-conegligible and $W \times X^k$ is $\mu^{2k} \times \mu^k$-conegligible, while $\phi[W] = \psi_1[W \times X^k] = [X]^{2k}$. Accordingly $[X]^{2k}$, which is an open subset of $[X]^{\le 2k}$, is conegligible for both ζ and ζ_1.

(ii) It will help to note that if $x \in X$ then $F = \{K : x \in K \in [X]^{\le 2k}\}$ is negligible for both ζ and ζ_1. **P** $\{\varpi : x \in \varpi[2k]\}$ is μ^{2k}-negligible, so $\zeta F = 0$. On the other hand, if $\varpi \in X^{2k}$ and x is not a value of ϖ, then

$$\mu^k\{\varpi' : (1-\varpi'(i))t_{2i}(\varpi) + \varpi'(i)t_{2i+2}(\varpi) = x\}$$
$$= \mu^k\{\varpi' : (1-\varpi'(k-1))t_{2k-2}(\varpi) + \varpi'(k-1) = x\} = 0$$

for every $i < k-1$, so $\{\varpi' : x \in \psi_1(\varpi,\varpi')\}$ is negligible. Accordingly $\zeta_1 F = 0$. **Q**

(iii) Let $\epsilon > 0$.

(α) Let $\mathcal{U}_\epsilon$ be the family of subsets of $[X]^{2k}$ of the form

$$\{K : K \cap]\alpha_i,\beta_i[\ne \emptyset \text{ for } i < 2k\}$$

where $0 \le \alpha_i < \beta_i \le 1$ for each i, $\beta_i - \alpha_i \le \epsilon(\alpha_{i+1} - \beta_i)$ for $i \le 2k-2$, $\beta_i - \alpha_i \le \epsilon(\alpha_i - \beta_{i-1})$ for $i > 0$ and $\beta_{k-1} - \alpha_{k-1} \le \epsilon(1 - \beta_{k-1})$. If $U \in \mathcal{U}_\epsilon$ is in the above form,

$$\zeta U = \mu^{2k}\phi^{-1}[U] = (2k)!\textstyle\prod_{i<2k}(\beta_i - \alpha_i).$$

To estimate $(\mu^{2k}\times\mu^k)\psi_1^{-1}[U]$, set

$$\begin{aligned} V = \{\varpi : \varpi \in X^{2k},\ &\alpha_{2i} < t_{2i}(\varpi) < \beta_{2i} \text{ for } i < k, \\ &\beta_{2i} < t_{2i+1}(\varpi) < \alpha_{2i+2} \text{ for } i < k-1,\ \beta_{2k-2} < t_{2k-1}(\varpi)\}. \end{aligned}$$

Then

$$\mu^{2k}V = (2k)!\textstyle\prod_{i=0}^{k-1}(\beta_{2i}-\alpha_{2i})\cdot\prod_{i=0}^{k-2}(\alpha_{2i+2}-\beta_{2i})\cdot(1-\beta_{2k-2}).$$

For $\varpi \in V$,

$$\begin{aligned} \{\varpi' : \psi_1(\varpi,\varpi') \in U\} = \bigcap_{i<k-1} &\{\varpi' : \alpha_{2i+1} < (1-\varpi'(i))t_{2i}(\varpi) + \varpi'(i)t_{2i+2}(\varpi) < \beta_{2i+1}\} \\ &\cap \{\varpi' : \alpha_{2k-1} < (1-\varpi'(k-1))t_{2k-2}(\varpi) + \varpi'(k-1)\} \end{aligned}$$

has measure

$$\begin{aligned} &\prod_{i=0}^{k-2}\tfrac{\beta_{2i+1}-\alpha_{2i+1}}{t_{2i+2}(\varpi)-t_{2i}(\varpi)}\cdot\tfrac{\beta_{2k-1}-\alpha_{2k-1}}{1-t_{2k-2}(\varpi)} \\ &\qquad \ge \prod_{i=0}^{k-2}\tfrac{\beta_{2i+1}-\alpha_{2i+1}}{(1+2\epsilon)(\alpha_{2i+2}-\beta_{2i})}\cdot\tfrac{\beta_{2k-1}-\alpha_{2k-1}}{(1+\epsilon)(1-\beta_{2k-2})} \\ &\qquad \ge \tfrac{1}{(1+2\epsilon)^k}\prod_{i=0}^{k-2}\tfrac{\beta_{2i+1}-\alpha_{2i+1}}{\alpha_{2i+2}-\beta_{2i}}\cdot\tfrac{\beta_{2k-1}-\alpha_{2k-1}}{1-\beta_{2k-2}}. \end{aligned}$$

So

$$\begin{aligned} \zeta_1 U &= (\mu^{2k}\times\mu^k)\psi_1^{-1}[U] \\ &\ge \tfrac{1}{(1+2\epsilon)^k}\prod_{i=0}^{k-2}\tfrac{\beta_{2i+1}-\alpha_{2i+1}}{\alpha_{2i+2}-\beta_{2i}}\cdot\tfrac{\beta_{2k-1}-\alpha_{2k-1}}{1-\beta_{2k-2}}\mu^{2k}V \\ &= \tfrac{(2k)!}{(1+2\epsilon)^k}\prod_{i=0}^{k-2}\tfrac{\beta_{2i+1}-\alpha_{2i+1}}{\alpha_{2i+2}-\beta_{2i}}\cdot\tfrac{\beta_{2k-1}-\alpha_{2k-1}}{1-\beta_{2k-2}} \\ &\qquad\qquad \cdot\prod_{i=0}^{k-1}(\beta_{2i}-\alpha_{2i})\cdot\prod_{i=0}^{k-2}(\alpha_{2i+2}-\beta_{2i})\cdot(1-\beta_{2k-2}) \\ &= \tfrac{(2k)!}{(1+2\epsilon)^k}\prod_{i<2k}(\beta_i-\alpha_i) = \tfrac{\zeta U}{(1+2\epsilon)^k}. \end{aligned}$$

(β) Now let $G \subseteq [X]^{\le 2k}$ be an open set. For $m \in \mathbb{N}$, set $D_m = \{2^{-m}k : k \le 2^m\}$. If

$$F = \{K : K \in [X]^{2k},\ K \cap \textstyle\bigcup_{m\in\mathbb{N}} D_m \ne \emptyset\} \cup [X]^{<2k},$$

then F is ζ-negligible and ζ_1-negligible, by (i)-(ii). For each m, let $\mathcal{V}_m$ be the set of members U of $\mathcal{U}_\epsilon$, included in G, such that (when expressed in the form of (α) just above) α_i, β_i are successive members of D_m for each $i < 2k$. Note that if $m' \le m$, $U \in \mathcal{V}_m$, $U' \in \mathcal{V}_{m'}$ and $U \cap U' \ne \emptyset$, then $U \subseteq U'$; while

$$G \setminus F \subseteq \textstyle\bigcup_{m\in\mathbb{N}}\bigcup\mathcal{V}_m \subseteq G.$$

P Of course $\bigcup\mathcal{V}_m \subseteq G$ for every m. If $K \in G \setminus F$, let $\langle s_i\rangle_{i<2k}$ be the increasing enumeration of K, and $\delta > 0$ such that

$$\delta \le \epsilon(s_{i+1} - s_i - 2\delta) \text{ for every } i < 2k-1, \quad \delta \le \epsilon(1 - s_{2k-1} - \delta),$$

$$\{K' : K' \in [X]^{\le 2k},\ K' \cap\,]s_i - \delta, s_i + \delta[\, \ne \emptyset \text{ for every } i < 2k\} \subseteq G.$$

Then there is an $m \in \mathbb{N}$ such that $2^{-m} \le \delta$. For each $i < 2k$, $s_i \notin D_m$, so we can take successive α_i, $\beta_i \in D_m$ such that $\alpha_i < s_i < \beta_i$, and now

$$U = \{K' : K' \in [X]^{\le 2k},\ K' \cap \,]\alpha_i, \beta_i[\, \ne \emptyset \text{ for every } i < 2k\}$$

belongs to $\mathcal{V}_m$ and contains K. So $G \setminus F \subseteq \bigcup_{m\in\mathbb{N}} \bigcup \mathcal{V}_m$. **Q**

Now take

$$\mathcal{V} = \bigcup_{m\in\mathbb{N}} \{U : U \in \mathcal{V}_m,\ U \cap U' = \emptyset \text{ whenever } m' < m \text{ and } U' \in \mathcal{V}_{m'}\}.$$

Then $\bigcup \mathcal{V} = \bigcup_{m\in\mathbb{N}} \bigcup \mathcal{V}_m$. By (α), $\zeta U \le (1+2\epsilon)^k \zeta_1 U$ for every $U \in \mathcal{U}_\epsilon$, so

$$\zeta G = \zeta(\bigcup \mathcal{V}) = \sum_{U\in\mathcal{V}} \zeta U \le (1+2\epsilon)^k \sum_{U\in\mathcal{V}} \zeta_1 U = (1+2\epsilon)^k \zeta_1 G.$$

(γ) This is true for every open $G \subseteq [X]^{\le 2k}$. Since ζ and ζ_1 are Radon measures, it follows that $\zeta \le (1+2\epsilon)^k \zeta_1$ (Fremlin 03, 416Ea).

(iv) As ϵ is arbitrary, $\zeta \le \zeta_1$. But both are probability measures, so they must agree on the Borel sets and are identical.

(b)-(d) The arguments are elementary modifications of those above.

7G Lemma Let (X, μ) be $]0,1[$ with Lebesgue measure, and $n \ge 1$. For $\varpi \in X^n$, let $\mathcal{E}_\varpi$ be the family of components of $X \setminus \varpi[n]$, and $\gamma_\varpi = \max\{\mu E : E \in \mathcal{E}_\varpi\}$. Then

$$\mu^n\{\varpi : \gamma_\varpi \ge \tfrac{1+3\ln n}{n}\} \le \tfrac{1}{n^2}.$$

proof If $\varpi \in X^n$ is such that $X \setminus \varpi[n]$ has a component of length $\frac{1+3\ln n}{n}$ or more, there must be a $j \le n - 3\ln n$ such that $\varpi[n]$ does not meet the interval $I_j = \left]\frac{j}{n}, \frac{j+3\ln n}{n}\right[$. The probability of this happening, for any particular j, is $(1-\mu I_j)^n$; so the probability of it happening for some j is at most

$$n(1 - \frac{3\ln n}{n})^n \le n\exp(-3\ln n) = \frac{1}{n^2}.$$

7H Lemma Let (X, ρ, μ) be $]0,1[$ with its usual metric and Lebesgue measure, and $p > 2$. Let $f : X \to [0,\infty[$ be such that $\int f^p d\mu$ is finite, and $\epsilon > 0$. Set $\beta = \int f d\mu$, $m = \lceil \frac{p(p-1)}{p-2} \rceil$ and

$$C = \frac{2^p}{\epsilon^p} \max(2^p \|f\|_p^p, 2^{p^2} \|f\|_p^{p^2}) + \frac{2^{4m}(2m)!}{\epsilon^{2m}}.$$

If $n \ge 3$ and μ^n is the product measure on X^n, then

$$\mu^n\{\varpi : \int F(\varpi, f) d\mu \ge 2(\beta+\epsilon)\} \le 2(\frac{1}{n^2} + (\frac{2+6\ln n}{n})^{p-1} C).$$

proof (a) Note first that because $\mu\{x : \rho(x,z) = \rho(x,z')\} = 0$ for every z, $z' \in X$, we have a function $h : [X]^{\le n} \to [0,\infty[$ defined by saying that $h(K) = \int F(\varpi, f) d\mu$ whenever $\varpi \in X^n$ and $K = \varpi[n]$ (and $h(\emptyset) = 0$, if you like). Let ζ be the measure of Lemma 7F, so that $\varpi \mapsto \varpi[n]$ is inverse-measure-preserving for μ^n and ζ, and

$$\begin{aligned}\mu^n\{\varpi : \int F(\varpi, f) d\mu \ge 2(\beta+\epsilon)\} &= \zeta\{K : h(K) \ge 2(\beta+\epsilon)\} \\ &= \zeta\{K : K \in [X]^n,\ h(K) \ge 2(\beta+\epsilon)\}.\end{aligned}$$

Write γ for $\frac{1+3\ln n}{n}$.

(b) For the time being (down to the end of (d) below), suppose that $n = 2k$ is even. Define h_1, $h_2 : [X]^{2k} \to [0,\infty[$ by saying that if $K \in [X]^{2k}$ and $\langle t_i \rangle_{i<2k}$ is the increasing enumeration of K, then

$$h_1(K) = (1 - t_{2k-2})f(t_{2k-1}) + \sum_{i=0}^{k-2}(t_{2i+2} - t_{2i})f(t_{2i+1}),$$

$$h_2(K) = t_1 f(t_0) + \sum_{i=1}^{k-1}(t_{2i+1} - t_{2i-1})f(t_{2i}).$$

Then $h(K) \leq h_1(K) + h_2(K)$ for every $K \in [X]^{2k}$, so

$$\zeta\{K : K \in [X]^{2k},\, h(K) \geq 2(\beta + \epsilon)\} \leq \zeta\{K : K \in [X]^{2k},\, h_1(K) \geq \beta + \epsilon\} + \zeta\{K : K \in [X]^{2k},\, h_2(K) \geq \beta + \epsilon\}.$$

(c)(i) To estimate $\zeta\{K : K \in [X]^{2k},\, h_1(K) \geq \beta + \epsilon\}$, consider the function $\psi_1 : X^{2k} \times X^k \to [X]^{\leq 2k}$ described in Lemma 7F. This is inverse-measure-preserving for the product measure $\mu^{2k} \times \mu^k$ and ζ, so

$$\zeta\{K : K \in [X]^{2k},\, h_1(K) \geq \beta + \epsilon\} = (\mu^{2k} \times \mu^k)\{(\varpi, \varpi') : \varpi \text{ is injective}, h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\}$$

because μ^{2k}-almost every ϖ is injective, while $\phi_1(\varpi, \varpi')$ has $2k$ elements whenever ϖ is injective. Now set

$$F = \{\varpi : \varpi \in X^{2k} \text{ is injective, every component of } X \setminus \varpi[2k] \text{ has length at most } \gamma\};$$

by Lemma 7G, $\mu^{2k}(X^{2k} \setminus F) \leq \frac{1}{n^2}$. Accordingly

$$(\mu^{2k} \times \mu^k)\{(\varpi, \varpi') : \varpi \text{ is injective, } h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\} \leq \frac{1}{n^2} + \int_F \mu^k\{\varpi' : h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\}\mu^{2k}(d\varpi).$$

(ii) If $\varpi \in F$, then

$$\mu^k\{\varpi' : h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\} \leq (2\gamma)^{p-1}C.$$

P Enumerate $\varpi[2k]$ in increasing order as $\langle t_i \rangle_{i<2k}$, and set

$$K_0 = \{t_{2i} : i < k\},$$
$$E_i =]t_{2i}, t_{2i+2}[\text{ if } i < k-1,$$
$$=]t_{2k-2}, 1[\text{ if } i = k-1.$$

For any $\varpi' \in X^k$, $\phi_1(\varpi, \varpi') = K_0 \cup \{s_i(\varpi') : i < k\}$ where $s_i(\varpi') = (1 - \varpi'(i)) \inf E_i + \varpi'(i) \sup E_i$ for each i, and $h_1(\phi_1(\varpi, \varpi')) = \sum_{i=0}^{k-1} f(s_i(\varpi'))\mu E_i$. Now observe that $\varpi' \mapsto \langle s_i(\varpi') \rangle_{i<k}$ is an isomorphism between (X^k, μ^k) and $(\prod_{i<k} E_i, \theta)$, where θ is the product measure $\prod_{i<k} \frac{1}{\mu E_i}\mu_{E_i}$ as considered in Lemma 7E. So that lemma tells us that

$$\mu^k\{\varpi' : \varpi' \in X^k,\, h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\} = \theta\{\varpi' : \varpi' \in \prod_{i<k} E_i,\, \sum_{i=0}^{k-1} f(\varpi'(i))\mu E_i \geq \beta + \epsilon\} \leq (\max_{i<k} \mu E_i)^{p-1}C \leq (2\gamma)^{p-1}C$$

because each E_i is obtained as the union of two components of $X \setminus \varpi[2k]$ together with the point between them. **Q**

(iii) So we see that

$$\zeta\{K : K \in [X]^{2k},\, h_1(K) \geq \beta + \epsilon\} = (\mu^{2k} \times \mu^k)\{(\varpi, \varpi') : \varpi \text{ is injective}, h_1(\phi_1(\varpi, \varpi')) \geq \beta + \epsilon\} \leq \frac{1}{n^2} + (2\gamma)^{p-1}C.$$

(d) Similarly, using Lemma 7Fb and the function ψ_2 there,

$$\zeta\{K : K \in [X]^{2k},\, h_2(K) \geq \beta + \epsilon\} \leq \frac{1}{n^2} + (2\gamma)^{p-1}C.$$

So

$$\begin{aligned}\mu^n\{\varpi : \int F(\varpi, f)d\mu \geq 2(\beta+\epsilon)\} &= \zeta\{K : h(K) \geq 2(\beta+\epsilon)\}\\ &\leq \zeta\{K : K \in [X]^{2k},\, h_1(K) \geq \beta+\epsilon\}\\ &\qquad + \zeta\{K : K \in [X]^{2k},\, h_2(K) \geq \beta+\epsilon\}\\ &\leq 2(\frac{1}{n^2} + (2\gamma)^{p-1}C).\end{aligned}$$

(e) This deals with the case of even n. The argument for odd n is essentially the same, using parts (c) and (d) of Lemma 7F.

7I Theorem Let (X, ρ, μ) be $]0,1[$ with its usual topology and Lebesgue measure, and $p > 2$. If $f : X \to \mathbb{R}$ is such that $\int |f|^p d\mu$ is finite, then

$$\lim_{n\to\infty} \int |F(\omega\restriction n, f) - f| d\mu = 0$$

for λ-almost every ω.

proof (a) Note first that if $g : X \to [0,\infty[$ is pth-power-integrable,

$$\limsup_{n\to\infty} \int F(\omega\restriction n, g)d\mu \leq 2\int |g|d\mu$$

for almost every ω. **P** Let $\epsilon > 0$. As in Lemmas 7E and 7H, set $\beta = \int |g|d\mu$, $m = \lceil\frac{p(p-1)}{p-2}\rceil$ and

$$C = \frac{2^p}{\epsilon^p}\max(2^p\|g\|_p^p, 2^{p^2}\|g\|_p^{p^2}) + \frac{2^{4m}(2m)!}{\epsilon^{2m}}.$$

Then 7H tells us that, for any $n \geq 3$,

$$\lambda\{\omega : \int F(\omega\restriction n, g)d\mu \geq 2(\beta+\epsilon)\} \leq 2(\frac{1}{n^2} + (\frac{2+6\ln n}{n})^{p-1}C).$$

Since $\sum_{n=3}^{\infty} 2(\frac{1}{n^2} + (\frac{2+6\ln n}{n})^{p-1}C)$ is finite, $\{n : \int F(\omega\restriction n, g)d\mu \geq 2(\beta+\epsilon)\}$ is finite for almost every ω, and

$$\limsup_{n\to\infty} \int F(\omega\restriction n, g)d\mu \leq 2(\beta+\epsilon)$$

for almost every ω. As ϵ is arbitrary,

$$\limsup_{n\to\infty} \int F(\omega\restriction n, g)d\mu \leq 2\beta$$

for almost every ω. **Q**

(b) Next, given $\epsilon > 0$, we can express $|f|$ as $h + g$ where $h : X \to \mathbb{R}$ is bounded and continuous and $\int |g|d\mu \leq \epsilon$. In this case, for almost every ω, $\langle F(\omega\restriction n, h)\rangle_{n\geq 1}$ is uniformly bounded and converges pointwise to h, while $\limsup_{n\to\infty} \int F(\omega\restriction n, |g|)d\mu \leq 2\epsilon$; so

$$\begin{aligned}\limsup_{n\to\infty} \int F(\omega\restriction n, |f|)d\mu &\leq \limsup_{n\to\infty} \int F(\omega\restriction n, h)d\mu + \limsup_{n\to\infty} \int F(\omega\restriction n, |g|)d\mu\\ &\leq \int hd\mu + 2\epsilon \leq \int |f|d\mu + 3\epsilon.\end{aligned}$$

As ϵ is arbitrary,

$$\limsup_{n\to\infty} \int |F(\omega\restriction n, f)|d\mu = \limsup_{n\to\infty} \int F(\omega\restriction n, |f|)d\mu \leq \int |f|d\mu$$

for almost every ω.

(c) However, we already know that μ is Mycielski-regular, by Theorem 5B (or Corollary 3B) and Proposition 2N. So, for almost every ω, we know that $\langle F(\omega\restriction n, f)\rangle_{n\geq 1}$ converges in measure to f and also $\limsup_{n\to\infty}\int|F(\omega{\restriction}n,f)|d\mu\le\int|f|d\mu$. But now $\lim_{n\to\infty}\int|f-F(\omega{\restriction}n,f)|d\mu=0$ for all such ω, by Fremlin 01, 245H(a-ii), as in 7A.

8 Problems The following questions seem to remain open.

8A Is every Radon probability measure on every Euclidean space $\mathbb{R}^r$ Mycielski-regular? (This is J.Mycielski's original version of the problem, expressed in the language of this note.)

8B Is the measure of Example 4D Mycielski-regular? (I rather think it is.)

8C Is the invariant Radon probability measure on any compact metric space with sesquitransitive isometry group Mycielski-regular?

8D If (X,ρ) is a separable metric space and μ is an atomless topological probability measure on X with the Lebesgue density property, must μ be Mycielski-regular?

8E Let (X,ρ,μ) be $[0,1]$ with its usual metric and Lebesgue measure, and $f:X\to\mathbb{R}$ a μ-integrable function. Is it necessarily true that $\langle\overline{F}(\omega{\restriction}n,f)\rangle_{n\ge1}$, as defined in 6C, converges to f μ-a.e. for λ-almost every ω?

8F Let (X,ρ) be a metric space, μ a Mycielski-regular topological probability measure on X and $f:X\to\mathbb{R}$ a μ-integrable function. Under what circumstances do we have $\lim_{n\to\infty}\int F(\omega{\restriction}n,f)d\mu=\int fd\mu$ for almost every ω? (See §7.)

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