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Problem DU

D.H.Fremlin

University of Essex, Colchester, England

1 The problem

1A Notation (a) I will call a family S of sets **hereditary** if $\mathcal{P}I \subseteq S$ for every $I \in S$.

(b) Let T be a set and S a family of sets. I say that S **is** $\frac{1}{2}$**-filling over** T if S is hereditary and for every $I \in [T]^{<\omega}$ there is a $J \subseteq I$ such that $\#(J) \geq \frac{1}{2}\#(I)$ and $J \in S$.

For cardinals λ, κ I write $P(\kappa, \lambda)$ for the statement

whenever $S \subseteq [\kappa]^{<\omega}$ is $\frac{1}{2}$-filling over κ, there is a set $A \in [\kappa]^\lambda$ such that $[A]^{<\omega} \subseteq S$.

1B The problem For which cardinals is $P(\kappa, \lambda)$ true?

1C Two examples (a) $P(\omega, \omega)$ is false. **P** Set $S = \{\emptyset\} \cup \{I : I \subseteq \mathbb{N},\, I \neq \emptyset,\, \#(I) \leq 1 + \min I\}$. Then S is $\frac{1}{2}$-filling over $\mathbb{N}$ and there is no infinite $A \subseteq \mathbb{N}$ such that $[A]^{<\omega} \subseteq S$. **Q** (This is a version of the **Schreier family**.)

(b) If $\omega_1 = \mathfrak{c}$ then $P(\omega_1, \omega_1)$ is false. **P** Enumerate $[0,1]$ as $\langle t_\xi \rangle_{\xi<\omega_1}$. For each $\xi < \omega_1$ choose a compact set $K_\xi \subseteq [0,1] \setminus \{x_\eta : \eta \leq \xi\}$ of Lebesgue measure at least $\frac{1}{2}$. Set $S = \{I : I \subseteq \omega_1,\, \bigcap_{\xi \in I} K_\xi \neq \emptyset\}$. Then S is $\frac{1}{2}$-filling over ω_1 but there is no uncountable set $A \subseteq \omega_1$ such that $[A]^{<\omega} \subseteq S$. **Q**

2 General theory

2A Compact families (a) I will say that a family S of sets is **compact** if it is closed in $\mathcal{P}T$ where $T = \bigcup S$.

(b) If S is a hereditary family of finite sets, the following are equiveridical: (i) S is compact; (ii) there is no strictly increasing sequence in S; (iii) $[A]^{<\omega} \not\subseteq S$ for any infinite A.

(c) So if S is a compact hereditary family of finite sets, then any hereditary $S' \subseteq S$ is compact.

2B Derivations (a) Let S be a hereditary family of sets and $\mathcal{I}$ an ideal of sets. Set

$$\partial_{\mathcal{I}} S = \{I : \{t : t \notin I,\, I \cup \{t\} \in S\} \notin \mathcal{I}\}.$$

Then $\partial_{\mathcal{I}} S$ is hereditary and included in S.

(b) Again supposing that S is a hereditary family of sets and $\mathcal{I}$ an ideal of sets, define $\langle \partial^\alpha_{\mathcal{I}} S \rangle_{\alpha<\mathrm{On}}$ inductively:

$\partial^0_{\mathcal{I}} S = S$;

for any ordinal α, $\partial^{\alpha+1}_{\mathcal{I}} S = \partial_{\mathcal{I}}(\partial^\alpha_{\mathcal{I}} S)$;

for non-zero limit ordinals α, $\partial^\alpha_{\mathcal{I}} S = \bigcap_{\beta<\alpha} \partial^\beta_{\mathcal{I}} S$.

Then $\langle \partial^\alpha_I S \rangle_{\alpha<\mathrm{On}}$ is a non-increasing family of hereditary sets.

Write $\operatorname{rank}_{\mathcal{I}} S$ for the smallest ordinal γ such that $\partial^{\gamma+1}_{\mathcal{I}} S = \partial^\gamma_{\mathcal{I}} S$. Note that if S is a compact hereditary family of finite sets then this kernel $\partial^\gamma_{\mathcal{I}} S$ must be empty; so that for any ordinal α we have

$$\alpha < \operatorname{rank}_{\mathcal{I}}(S) \iff \partial^\alpha_{\mathcal{I}} S \neq \emptyset \iff \emptyset \in \partial^\alpha_{\mathcal{I}} S.$$

(c) If S, S' are hereditary families of sets and $\mathcal{I}$ is an ideal of sets, then $\partial_{\mathcal{I}}(S \cup S') = \partial_{\mathcal{I}} S \cup \partial_{\mathcal{I}} S'$. Consequently $\partial^\alpha_{\mathcal{I}}(S \cup S') = \partial^\alpha_{\mathcal{I}} S \cup \partial^\alpha_{\mathcal{I}} S'$ for every α. If S and S' are compact hereditary families of finite sets, then $\operatorname{rank}_{\mathcal{I}}(S \cup S') = \max(\operatorname{rank}_{\mathcal{I}}(S), \operatorname{rank}_{\mathcal{I}}(S'))$.

(d) If S is a hereditary family of sets and $\mathcal{I}$, $\mathcal{J}$ are ideals of sets with $\mathcal{I} \subseteq \mathcal{J}$, then $\partial_{\mathcal{J}} S \subseteq \partial_{\mathcal{I}} S$; consequently $\partial^{\alpha}_{\mathcal{J}} S \subseteq \partial^{\alpha}_{\mathcal{I}} S$ for every α, and if S is a compact family of finite sets then $\mathrm{rank}_{\mathcal{I}}(S) \ge \mathrm{rank}_{\mathcal{J}}(S)$.

(e) If S is a family of sets of ordinals, set

$$\tilde{\partial} S = \{I : I \in S,\, I \cup \{\xi\} \in S \text{ for some ordinal } \xi \text{ such that } I \subseteq \xi\},$$

and for ordinals α define $\tilde{\partial}^{\alpha} S$ by setting

$$\tilde{\partial}^0 S = S, \quad \tilde{\partial}^{\alpha} S = \bigcap\nolimits_{\beta<\alpha} \tilde{\partial}(\tilde{\partial}^{\beta} S)$$

if $\alpha > 0$. If S is hereditary, then every $\tilde{\partial}^{\alpha} S$ is hereditary and $\langle \tilde{\partial}^{\alpha} S \rangle_{\alpha \in \mathrm{On}}$ is non-increasing.

2C Lemma Suppose that S is a hereditary family of sets, and that $\mathcal{I}$ is an ideal of subsets of $T \supseteq \bigcup S$. For $t \in T$ set $S_t = \{I : I \cup \{t\} \in S,\, t \notin I\}$.

(a) $\partial^{\alpha}_{\mathcal{I}}(S_t) = (\partial^{\alpha}_{\mathcal{I}} S)_t$ for every ordinal α and every $t \in T$.

(b) If $S \subseteq [T]^{<\omega}$ is compact and not empty, then $\mathrm{rank}_{\mathcal{I}}(S) = (\min_{A \in \mathcal{I}} \sup_{t \in T \setminus A} \mathrm{rank}_{\mathcal{I}}(S_t)) + 1$.

proof (a) Induce on α. The induction starts with $\alpha = 0$, $S_t = S_t$. For the inductive step to a successor ordinal $\alpha + 1$, if $I \subseteq T$ then

$$\begin{aligned} I \in \partial^{\alpha+1}_{\mathcal{I}} S_t &\iff \{s : s \in T \setminus I,\, I \cup \{s\} \in \partial^{\alpha}_{\mathcal{I}} S_t\} \notin \mathcal{I} \\ &\iff \{s : s \in T \setminus I,\, I \cup \{s\} \in (\partial^{\alpha}_{\mathcal{I}} S)_t\} \notin \mathcal{I} \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &\iff t \notin I \text{ and } \{s : s \in T \setminus (I \cup \{t\}),\, I \cup \{t\} \cup \{s\} \in \partial^{\alpha}_{\mathcal{I}} S\} \notin \mathcal{I} \\ &\iff t \notin I \text{ and } I \cup \{t\} \in \partial^{\alpha+1}_{\mathcal{I}} S \\ &\iff I \in (\partial^{\alpha+1}_{\mathcal{I}} S)_t, \end{aligned}$$

so the induction proceeds. For the inductive step to a non-zero limit ordinal α, if $I \subseteq T$ then

$$\begin{aligned} I \in \partial^{\alpha}_{\mathcal{I}} S_t &\iff I \in \partial^{\beta}_{\mathcal{I}} S_t \text{ for every } \beta < \alpha \\ &\iff I \in (\partial^{\beta}_{\mathcal{I}} S)_t \text{ for every } \beta < \alpha \\ &\iff t \notin I \text{ and } I \cup \{t\} \in \partial^{\beta}_{\mathcal{I}} S \text{ for every } \beta < \alpha \\ &\iff t \notin I \text{ and } I \cup \{t\} \in \partial^{\alpha}_{\mathcal{I}} S \\ &\iff I \in (\partial^{\alpha}_{\mathcal{I}} S)_t, \end{aligned}$$

and again we can continue.

(b) In this case $S_t = \{I \setminus \{t\} : t \in I \in S\}$ is also compact, for every $t \in T$. Set $\gamma_t = \mathrm{rank}_{\mathcal{I}}(S_t)$ for every $t \in T$, $\gamma = \min_{A \in \mathcal{I}} \sup_{t \in T \setminus A} \gamma_t$.

For an ordinal α,

$$\alpha + 1 < \mathrm{rank}_{\mathcal{I}}(S) \iff \emptyset \in \partial^{\alpha+1}_{\mathcal{I}} S$$

(2Bb)

$$\begin{aligned} &\iff \{t : \{t\} \in \partial^{\alpha}_{\mathcal{I}} S\} \notin \mathcal{I} \\ &\iff \{t : \emptyset \in (\partial^{\alpha}_{\mathcal{I}} S)_t\} \notin \mathcal{I} \\ &\iff \{t : \emptyset \in \partial^{\alpha}_{\mathcal{I}}(S_t)\} \notin \mathcal{I} \\ &\iff \{t : \alpha < \gamma_t\} \notin \mathcal{I} \\ &\iff \alpha < \gamma. \end{aligned}$$

It follows at once that $\mathrm{rank}_{\mathcal{I}}(S) \le \gamma + 1$. In the other direction, if $\gamma = \beta + 1$ is a successor ordinal, then $\beta < \gamma$, $\emptyset \in \partial^{\beta+1}_{\mathcal{I}} S$ and $\gamma < \mathrm{rank}_{\mathcal{I}}(S)$; if γ is a non-zero limit ordinal, then $\emptyset \in \partial^{\alpha}_{\mathcal{I}} S$ for every $\alpha < \gamma$, so again

$\emptyset \in \partial_{\mathcal{I}}^{\gamma} S$ and $\gamma < \operatorname{rank}_{\mathcal{I}}(S)$; finally, if $\gamma = 0$ then $\gamma < \operatorname{rank}_{\mathcal{I}}(S)$ because $S \neq \emptyset$. So in all cases $\operatorname{rank}_{\mathcal{I}}(S)$ is greater than γ and must be exactly $\gamma + 1$, as claimed.

2D Lemma Suppose that $S \subseteq [T]^{<\omega}$ is hereditary and $\frac{1}{2}$-filling over a non-empty set T, and that $\mathcal{I}$ is a proper ideal of subsets of T containing singletons. Set $\kappa = \operatorname{add} \mathcal{I}$. Then $\emptyset \in \partial_{\mathcal{I}}^{\kappa} S$.

Remark Here $\operatorname{add} \mathcal{I}$ is the **additivity** of $\mathcal{I}$, the least cardinal of any subset of $\mathcal{I}$ with no upper bound in $\mathcal{I}$ (see Fremlin 08, §511).

proof (a) Suppose first that $\kappa \geq \omega_1$, that is, that $\mathcal{I}$ is a σ-ideal.

(i) In this case, I show by induction on α that for every $\alpha < \kappa$ there is an $M_\alpha \in \mathcal{I}$ such that $\partial_{\mathcal{I}}^{\alpha} S$ is $\frac{1}{2}$-filling over $T \setminus M_\alpha$. The induction starts with $M_0 = \emptyset$.

(ii) For the inductive step to $\alpha + 1$, given that $\partial_{\mathcal{I}}^{\alpha} S$ is $\frac{1}{2}$-filling over $T \setminus M_\alpha$, **?** suppose, if possible, that $\partial_{\mathcal{I}}^{\alpha+1} S$ is not $\frac{1}{2}$-filling over $T \setminus M$ for any $M \in \mathcal{I}$. Because $\mathcal{I}$ is a σ-ideal containing singletons, we can choose inductively families $\langle I_\xi \rangle_{\xi < \omega_1}$ and $\langle N_\xi \rangle_{\xi < \omega_1}$ such that

every I_ξ is a finite subset of $T \setminus M_\alpha$,

every N_ξ belongs to $\mathcal{I}$,

$I_\xi \cap \bigcup_{\eta < \xi} N_\eta = \emptyset$ and $\#(J) < \frac{1}{2}\#(I_\xi)$ whenever $J \in \partial_{\mathcal{I}}^{\alpha+1} S$ and $J \subseteq I_\xi$

(because we are supposing that $\partial_{\mathcal{I}}^{\alpha+1} S$ is not $\frac{1}{2}$-filling over $T \setminus (M_\alpha \cup \bigcup_{\eta < \xi} N_\eta)$),

$N_\xi = I_\xi \cup \bigcup_{J \subseteq I_\xi, J \notin \partial_{\mathcal{I}}^{\alpha+1} S} \{t : J \cup \{t\} \in \partial_{\mathcal{I}}^{\alpha} S\}$

(which belongs to $\mathcal{I}$ by the definition of $\partial_{\mathcal{I}}^{\alpha+1} S$). Now there is some $k \in \mathbb{N}$ such that $A = \{\xi : \#(I_\xi) = k\}$ is infinite; of course $k \geq 1$. Let m be the greatest integer less than $\frac{k}{2}$, and let $r \geq 1$ be such that $mr + k < \frac{1}{2}k(r+1)$. Take $\xi_0 < \ldots < \xi_r \in A$ and consider $I = \bigcup_{i \leq r} I_{\xi_i}$. This is a finite subset of $T \setminus M_\alpha$ so there is a $J \subseteq I$ such that $J \in \partial_{\mathcal{I}}^{\alpha} S$ and $\#(J) \geq \frac{1}{2}\#(I) = \frac{1}{2}k(r+1)$. There must therefore be some first j such that $\#(J \cap I_{\xi_j}) \geq \frac{1}{2}k$. In this case, by the choice of the I_ξ, $J \cap I_{\xi_j} \notin \partial_{\mathcal{I}}^{\alpha+1} S$ and N_{ξ_j} includes $\{t : (J \cap I_{\xi_j}) \cup \{t\} \in \partial_{\mathcal{I}}^{\alpha} S\} \supseteq J$, so that $I_{\xi_i} \cap J$ must be empty for every $i > j$. What this means is that

$$\#(J) = \sum_{i=0}^{r} \#(J \cap I_{\xi_i}) \leq mj + k,$$

and

$$\frac{1}{2}k(r+1) \leq mj + k \leq mr + k,$$

which is impossible, by the choice of r. **X**

Thus the induction proceeds to $\alpha + 1$.

(iii) For the inductive step to a non-zero limit ordinal $\alpha < \kappa$, we need only set $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$, which belongs to $\mathcal{I}$ because $\mathcal{I}$ is κ-additive.

(iv) Now because $T \notin \mathcal{I}$ it follows that $\emptyset \in \partial_{\mathcal{I}}^{\alpha} S$ for every $\alpha < \kappa$, so that $\emptyset \in \partial_{\mathcal{I}}^{\kappa} S$, as claimed.

(b) Now suppose that $\kappa = \omega$. Fix a non-zero $m \in \mathbb{N}$ for the moment. Choose a sequence $\langle t_n \rangle_{n \in \mathbb{N}}$ of distinct elements of T so that

whenever $j \leq m$ and $J \subseteq \{t_i : i < n\}$ and $J \notin \partial_{\mathcal{I}}^{j+1} S$, then $J \cup \{t_n\} \notin \partial_{\mathcal{I}}^{j} S$;

this is possible because when we come to choose t_n only a set belonging to $\mathcal{I}$ is forbidden to us. Now look at $K = \{t_n : n < 2m\}$. There is an $I \in S$ such that $I \subseteq K$ and $\#(I) = m$; express I as $\langle t_{n_i} \rangle_{i<m}$ where $n_0 > n_1 > \ldots > n_{m-1}$. Set $I_j = \{t_{n_i} : j \leq i < m\}$ for each j. Now $I_j \in \partial_{\mathcal{I}}^{j} S$ for every $j \leq m$. **P** Induce on j. $I_0 = I \in S = \partial_{\mathcal{I}}^{0} S$. If $I_j \in \partial_{\mathcal{I}}^{j} S$, then $I_{j+1} \subseteq \{t_i : i < n_j\}$ and $I_{j+1} \cup \{t_{n_j}\} = I_j \in \partial_{\mathcal{I}}^{j} S$, so $I_{j+1} \in \partial_{\mathcal{I}}^{j+1} S$. **Q**

In particular, $\emptyset = I_m \in \partial_{\mathcal{I}}^{m} S$. As m is arbitrary, $\emptyset \in \partial_{\mathcal{I}}^{\omega} S$. So we have the result in this case too.

2E Minimal families (a) If S is $\frac{1}{2}$-filling over T, there is a minimal $S' \subseteq S$ which is $\frac{1}{2}$-filling over T. (For the intersection of any downwards-directed family of $\frac{1}{2}$-filling sets is again $\frac{1}{2}$-filling.)

(b) If $S \subseteq [T]^{<\omega}$ is a minimal $\frac{1}{2}$-filling set, then for every maximal $J \in S$ there must be a finite $I \supseteq J$ such that J is the unique member of S included in I such that $\#(J) \ge \frac{1}{2}\#(I)$. (For otherwise we could delete J.)

(c) If $S \subseteq [T]^{<\omega}$ is compact and hereditary and $\frac{1}{2}$-filling, there is a compact minimal $\frac{1}{2}$-filling $S' \subseteq [T]^{<\omega}$.

(d) The Schreier family S of 1Ca is minimal $\frac{1}{2}$-filling. **P** If $I \in S$ is maximal, set $n = \min I$, $J = I \cup n$; then $\#(I) = n + 1$, $\#(J) = 2n + 1$ and I is the only member of $[J]^{n+1}$ belonging to S. **Q**

2F Lemma Let T be a set and $S \subseteq [T]^{<\omega}$ a compact $\frac{1}{2}$-filling family. Then there is a finite $I \subseteq T$ such that for every non-empty $J \in [T \setminus I]^{<\omega}$ there is a $K \in S \cap \mathcal{P}J$ such that $\#(K) > \frac{1}{2}\#(J)$.

proof By 2Ea, there is a minimal $\frac{1}{2}$-filling family $S' \subseteq S$. Because S is compact, so is S', and S' has a maximal element J_0. By 2Eb, there is an $I \in [T]^{<\omega}$ such that J_0 is the unique member of $S' \cap \mathcal{P}I$ with $\#(J_0) \ge \frac{1}{2}\#(I)$. Let $J \subseteq T \setminus I$ be a non-empty finite set. If $\#(J) = 1$, then of course $J \in S'$ and $\#(J) > \frac{1}{2}\#(J)$. If $\#(J) \ge 2$, there is a $K \in S' \cap (J \cup I)$ such that

$$\#(K) \ge \lceil \tfrac{1}{2}\#(J \cup I) \rceil \ge 1 + \lceil \tfrac{1}{2}\#(I) \rceil > \#(J_0).$$

As J_0 is maximal, $K \not\supseteq J_0$ and $K \cap I$ is a member of $S' \cap \mathcal{P}I$ other than J_0. It follows that $\#(K \cap I) < \frac{1}{2}\#(I)$ so $\#(K \setminus I) > \frac{1}{2}\#(J)$, while $K \setminus I \in S' \subseteq S$.

2G Monotonicity I spell out an obvious fact: if $P(\kappa, \lambda)$ is true, then $P(\kappa', \lambda')$ is true whenever $\lambda' \le \lambda$ and $\kappa' \ge \kappa$.

3 A connexion with bases in Banach spaces

3A Theorem Suppose that κ is an uncountable cardinal. Then the following are equiveridical:

(i) $P(\kappa, \omega)$;

(ii) If X is a Banach space and $\langle e_\xi \rangle_{\xi<\kappa}$ is a family of unit vectors in X such that every weak neighbourhood of 0 in X contains all but finitely many of the e_ξ, there is a sequence $\langle \xi_i \rangle_{i\in\mathbb{N}}$ of distinct elements of κ such that $\inf_{n\in\mathbb{N}} \|\frac{1}{n+1}\sum_{i=0}^{n} e_{\xi_i}\| = 0$;

(iii) If X is a Banach space and $\langle e_\xi \rangle_{\xi<\kappa}$ is a family of unit vectors in X such that every weak neighbourhood of 0 in X contains all but finitely many of the e_ξ, there is a sequence $\langle \xi_i \rangle_{i\in\mathbb{N}}$ of distinct elements of κ such that $\lim_{n\to\infty} \|\frac{1}{n+1}\sum_{i=0}^{n} e_{\xi_i}\| = 0$.

proof (i)⇒(iii) Suppose that $P(\kappa, \omega)$ is true. Let X be a Banach space and $\langle e_\xi \rangle_{\xi<\kappa}$ is a family of unit vectors in X such that every weak neighbourhood of 0 in X contains all but finitely many of the e_ξ.

(α) For $\epsilon > 0$, consider the set

$$S_\epsilon = \{I : I \subseteq \kappa, \text{ there is some } f \in X^* \text{ such that } \|f\| \le 1 \text{ and } f(e_\xi) \ge \epsilon \text{ for every } \xi \in I\}.$$

If $A \subseteq \kappa$ and $[A]^{<\omega} \subseteq S_\epsilon$, then $A \in S_\epsilon$. **P** For each $I \in [A]^{<\omega}$ there is an f_I in the unit ball of X^* such that $f_I(e_\xi) \ge \epsilon$ for every $\xi \in I$. Let f be a weak* limit of the f_I as I increases through the finite subsets of A; then $f(e_\xi) \ge \epsilon$ for every $\xi \in A$, so f witnesses that $A \in S_\epsilon$. **Q**

Since every weak neighbourhood of 0 contains all but finitely many of the e_ξ, $S_\epsilon \subseteq [\kappa]^{<\omega}$, and there is no infinite set $A \subseteq \kappa$ such that $[A]^{<\omega} \subseteq S_\epsilon$. We are supposing that $P(\kappa, \omega)$ is true. So if $A \in [\kappa]^{<\omega}$, $\epsilon > 0$ and $\delta > 0$, S_ϵ is not δ-filling over $\kappa \setminus A$.

(β) ? Suppose, if possible, that there is an $\epsilon > 0$ such that

$$\forall\, n \in \mathbb{N}\ \exists\, I \in [\kappa]^{<\omega}\ \forall\, J \in [\kappa \setminus I]^n,\ \|\textstyle\sum_{\xi\in J} e_\xi\| \ge \epsilon n.$$

Then there is an $A \in [\kappa]^{\le\omega}$ such that $\|\sum_{\xi\in J} e_\xi\| \ge \epsilon\#(J)$ for every finite $J \subseteq \kappa \setminus A$. For each $J \in [\kappa \setminus A]^{<\omega}$, choose $f_J \in X^*$ such that $\|f_J\| \le 1$ and $f_J(\sum_{\xi\in J} e_\xi) \ge \epsilon\#(J)$, and set $I_J = \{\xi : \xi \in J,\ f_J(e_\xi) \ge \frac{\epsilon}{2}\}$. Then

$$\epsilon\#(J) \le \textstyle\sum_{\xi\in J} f_J(e_\xi) \le \#(I_J) + \frac{\epsilon}{2}\#(J),$$

so $\#(I_J) \ge \frac{\epsilon}{2}\#(J)$, while $I_J \in S_{\epsilon/2}$. As J is arbitrary, $S_{\epsilon/2}$ is $\frac{\epsilon}{2}$-filling over $\kappa \setminus A$; but this is supposed to be impossible. **X**

(γ) Accordingly

$$\forall\, \epsilon > 0\ \exists\, n \in \mathbb{N}\ \forall\, I \in [\kappa]^{<\omega}\ \exists\, J \in [\kappa \setminus I]^n,\ \|\textstyle\sum_{\xi\in J} e_\xi\| < \epsilon n.$$

For each $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ such that

$$\forall\, I \in [\kappa]^{<\omega}\ \exists\, J \in [\kappa \setminus I]^{n_k},\ \|\textstyle\sum_{\xi\in J} e_\xi\| < 2^{-k} n_k.$$

Of course no n_k can be 0. Choose $\langle m_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\textstyle\sum_{l<k} 2^{-l} m_l n_l \le 2^{-k} m_k n_k, \quad n_{k+1} \le 2^{-k} m_k n_k$$

for every k. Now we can choose J_{kr}, for $k \in \mathbb{N}$ and $r < m_k$, such that

$$J_{kr} \in [\kappa]^{n_k}, \quad J_{kr} \cap J_{ls} = \emptyset \text{ if either } l < k \text{ and } s < m_l \text{ or } l = k \text{ and } s < r,$$

$$\|\textstyle\sum_{\xi\in J_{nr}} e_\xi\| < 2^{-k} n_k.$$

Take a sequence $\langle \xi_i\rangle_{i\in\mathbb{N}}$ enumerating $\bigcup_{k\in\mathbb{N}, r<m_k} J_{kr}$ in such a way that if $\xi_i \in J_{ls}$, $\xi_j \in J_{kr}$ and $i \le j$, then either $l < k$ or $l = k$ and $r \le s$.

Set $M_k = \sum_{l<k} m_l n_l$ for $k \in \mathbb{N}$. Suppose that $k \ge 1$ and $M_k \le n < M_{k+1}$. Then $\|\sum_{i<n} e_{\xi_i}\| \le 7 \cdot 2^{-k} n$. **P** Express n as $M_k + m n_k + j$ where $m < m_k$ and $j < n_k$. Then

$$\begin{aligned}
\|\sum_{i<n} e_{\xi_i}\| &= \|\sum_{l<k}\sum_{s<m_l}\sum_{\xi\in J_{ls}} e_\xi + \sum_{r<m}\sum_{\xi\in J_{kr}} e_\xi + \sum_{M_k+mn_k\le i<n} e_\xi\| \\
&\le \sum_{l<k}\sum_{s<m_l} 2^{-l} n_l + \sum_{r<m} 2^{-k} n_k + j \\
&\le \sum_{l<k} 2^{-l} n_l m_l + 2^{-k} m n_k + n_k \\
&\le \sum_{l<k-1} 2^{-l} n_l m_l + 2^{-k+1} m_{k-1} n_{k-1} + 2^{-k} m n_k + 2^{-k+1} m_{k-1} n_{k-1} \\
&\le 2^{-k+1} n_{k-1} m_{k-1} + 2^{-k+1} m_{k-1} n_{k-1} + 2^{-k} m n_k + 2^{-k+1} m_{k-1} n_{k-1} \\
&\le 2^{-k+1} n + 2^{-k+1} n + 2^{-k} n + 2^{-k+1} n = 7 \cdot 2^{-k} n,
\end{aligned}$$

as claimed. **Q**

(δ) Thus $\lim_{n\to\infty} \|\frac{1}{n}\sum_{i<n} e_{\xi_i}\| = 0$ and (iii) is true.

(iii)$\Rightarrow$(ii) is trivial.

$\neg$(i)$\Rightarrow \neg$(ii) (α) Let $S \subseteq [\kappa]^{<\omega}$ be a $\frac{1}{2}$-filling family not including $[A]^{<\omega}$ for any infinite $A \subseteq \kappa$. Write c_{00} for the linear space of functions $x : \kappa \to \mathbb{R}$ such that $\{\xi : x(\xi) \ne 0\}$ is finite. For $x \in c_{00}$ set $\|x\| = \sup_{I\in S} \sum_{\xi\in I} |x(\xi)|$. Because all singleton subsets of κ belong to S, $\|\ \|$ is a norm on c_{00}. Let X be the completion of c_{00} for this norm, and set $e_\xi(\eta) = 1$ if $\eta = \xi$, 0 otherwise, so that $\|e_\xi\| = 1$ for every $\xi < \kappa$.

(β) We can identify the dual of X with a subspace of $c_0(\kappa)$. **P** Because c_{00} is dense in X, every $f \in X^*$ is defined by the family $\langle f(e_\xi)\rangle_{\xi<\kappa} \in \mathbb{R}^\kappa$. Consider the set $D = \{y \times \chi I : I \in S, \|y\|_\infty \le 1\}$. By §4C, this is a compact subset of $\mathbb{R}^\kappa$, and it is norm-bounded in $c_0(\kappa)$, so it is compact for the weak topology $\mathfrak{T}_s(c_0(\kappa), \ell^1(\kappa))$. By Kreĭn's theorem, the closed convex hull C of D in $c_0(\kappa)$ for $\mathfrak{T}_s(c_0(\kappa), \ell^1(\kappa))$ is compact. Being compact and balanced, C must also be the closed balanced convex hull of D in $\mathbb{R}^\kappa$ for $\mathfrak{T}_s(\mathbb{R}^\kappa, c_{00})$. So if $z \in C$, we have $(z|x) \le \|x\|$ whenever $x \in c_{00}$, and z represents a member of the unit ball of X^*. On the other hand, if $z \in \mathbb{R}^\kappa \setminus C$, there is an $x \in c_{00}$ such that $(z|x) > \sup_{y\in D}(y|x) = \|x\|$. So we can identify C with the unit ball of X^*, and every member of X^* with a multiple of a member of C, which will still lie in $c_0(\kappa)$. **Q**

(γ) It follows that every weak neighbourhood of 0 in X contains all but finitely many of the e_ξ. Now suppose that $\langle\xi_i\rangle_{i\in\mathbb{N}}$ is a sequence of distinct elements of κ. Then for every $n\in\mathbb{N}$ there is an $I\subseteq\{\xi_i:i\le n\}$ such that $I\in S$ and $\#(I)\ge\frac{n+1}{2}$. So $\|\frac{1}{n+1}\sum_{i=0}^n e_{\xi_i}\|\ge\frac{1}{2}$.

Thus X and $\langle e_\xi\rangle_{\xi<\kappa}$ witness that (ii) is false.

4 Other concepts of density

4A Weaker ideas of density Suppose that $\psi:\mathbb{N}\to[0,\infty[$ is a function. If T is a set and $S\subseteq[T]^{<\omega}$, we can say that S is **ψ-filling over** T if (i) $\mathcal{P}I\subseteq S$ for every $I\in S$ (ii) for every $I\in[T]^{<\omega}$ there is a $J\subseteq I$ such that $\#(J)\ge\psi(\#(I))$ and $J\in S$. For infinite cardinals κ and λ, let $P_\psi(\kappa,\lambda)$ be the statement

whenever $S\subseteq[\kappa]^{<\omega}$ is ψ-filling over κ, then there is an $A\in[\lambda]^\kappa$ such that $[A]^{<\omega}\subseteq S$.

Now we have the following.

4B Proposition If $\psi:\mathbb{N}\to[0,\infty[$ is any function such that $\psi(n)\le n$ for every n and $\lim_{n\to\infty}\frac{\psi(n)}{n}=0$, then $P_\psi(\mathfrak{c},\omega)$ is false.

proof Set $\epsilon_n=\sup_{i\ge n}\frac{\psi(i)}{i}$ for $n\ge1$, so that $\langle\epsilon_n\rangle_{n\ge1}$ is a non-increasing sequence in $[0,1]$ with limit 0. Set $T=\{z:z\in\mathbb{C},\ |z|=1\}$. Set

$$S=\bigcup_{n\ge1}\{I:I\in[T]^{\le n},\ \exists\, w\in T,\ |\arg\frac{z}{w}|\le\pi\epsilon_n \text{ for every } z\in I\}.$$

Then $S\subseteq[T]^{<\omega}$ and $J\in S$ whenever $J\subseteq I\in S$. If $I\in[T]^n$, where $n\ge1$, then for each $w\in T$ set $I_w=\{z:z\in I,\ |\arg\frac{z}{w}|\le\pi\epsilon_n\}\in S$. Writing μ for normalized Haar measure on T, then for each $z\in T$, $\{w:z\in I_w\}$ has measure ϵ_n, so $\int\#(I_w)\mu(dw)=\epsilon_n\#(J)=n\epsilon_n$ and there is a $w\in T$ with $\#(I_w)\ge n\epsilon_n\ge\psi(n)$. This shows that S is ψ-filling over T.

If $A\subseteq T$ is infinite, take any distinct z_1, $z_2\in A$, and let $n\ge2$ be such that $2\pi\epsilon_n<|\arg\frac{z_1}{z_2}|$. If $I\in[A]^n$ contains both z_1 and z_2, and $m\ge n$, then there can be no $w\in T$ such that $|\arg\frac{z_i}{w}|\le\pi\epsilon_m$ for both i, so $I\notin S$. Thus $[A]^{<\omega}\not\subseteq S$.

Since $\#(T)=\mathfrak{c}$ we have the result.

4C Proposition If $\lim_{n\to\infty}\psi(n)=\infty$ and S is a compact hereditary ψ-filling family over an infinite set T, and $\mathcal{I}$ is a proper ideal of subsets of T containing singletons, then $\operatorname{rank}_{\mathcal{I}}(S)>\omega$.

proof Use the argument of (b) of the proof of Lemma 2D; in place of $2m$, take some r such that $\psi(r)\ge m$.

4D If $\psi(n)/n$ is bounded away from zero, then we return to the original problem.

Proposition Let $\psi:\mathbb{N}\to[0,\infty[$ be a function such that, for some $\epsilon>0$, $\epsilon n\le\psi(n)\le(1-\epsilon)n$ for every n. Then $P_\psi(\kappa,\lambda)\iff P(\kappa,\lambda)$ for all infinite cardinals κ and λ.

proof (a) Suppose that $\delta>0$ and $\delta'<1$ and ϕ, $\theta:\mathbb{N}\to[0,\infty[$ are such that $\theta(n)\ge\delta n$ and $\phi(n)\le\delta' n$ for every n. Let m be such that $(1-\delta)^m\le1-\delta'$. If $S\subseteq[\kappa]^{<\omega}$ is θ-filling over κ then $S'=\{I_0\cup I_2\cup\ldots\cup I_{m-1}:I_0,\ldots,I_{m-1}\in S\}$ is ϕ-filling over κ. **P** Take any $J\in[\kappa]^{<\omega}$. Choose I_i inductively such that $I_i\subseteq J\setminus\bigcup_{j<i}I_j$ and $I_i\in S$ and $\#(I_i)\ge\theta(\#(J\setminus\bigcup_{j<i}I_j))$ for each i. Inducing on i, we see that $\#(J\setminus\bigcup_{j<i}I_j)\le(1-\delta)^j\#(J)$ for each i, so that if we set $I=\bigcup_{i<m}I_i$ then $I\in S'$ and $I\subseteq J$ and $\#(I)\ge\delta'\#(J)\ge\phi(\#(J))$. **Q**

(b) It follows that $P_\phi(\kappa,\lambda)\Rightarrow P_\theta(\kappa,\lambda)$. **P** Let $S\subseteq[\kappa]^{<\omega}$ be θ-filling. Set $S'=\{I_0\cup I_2\cup\ldots\cup I_{n-1}:I_0,\ldots,I_{m-1}\in S\}$. Then S' is ϕ-filling over κ, so there is an $A'\in[\kappa]^\lambda$ such that $[A']^{<\omega}\subseteq S'$. Let $\mathcal{F}$ be an ultrafilter on $[A']^{<\omega}$ containing $\{J:\xi\in J\in[A']^{<\omega}\}$ for every $\xi\in A'$. For each $J\in[A']^{<\omega}$, there is a function $f_J:J\to m$ such that $f_J^{-1}[\{i\}]\in S$ for every $i<n$. Define $f:A'\to m$ by setting $f(\xi)=\lim_{J\to\mathcal{F}}f_J(\xi)$ for every $\xi\in A'$. Then there is some $j<m$ such that $A=f^{-1}[\{j\}]$ has cardinal λ. If $K\in[A]^{<\omega}$ the set $\{J:K\subseteq f_J^{-1}[\{j\}]\}$ belongs to $\mathcal{F}$, so is not empty, and $K\in S$. Thus $[A]^{<\omega}\subseteq S$. As S is arbitrary, we have $P_\theta(\kappa,\lambda)$. **Q**

(c) Applying this with $\delta = \min(\frac{1}{2}, \epsilon)$, $\delta' = \max(\frac{1}{2}, 1-\epsilon)$, $\phi = \psi$, $\theta(n) = \frac{1}{2}n$ and the other way about, we have the result.

4E Proposition There is a function $\psi : \mathbb{N} \to [0, \infty[$ such that $\lim_{n\to\infty} \psi(n) = \infty$ and $P_\psi(\kappa^+, \kappa^+)$ is false for every infinite cardinal κ.

proof For $n \in \mathbb{N}$ let $\psi(n)$ be the largest number m such that

whenever $D \subseteq [n]^3$ then either there is a set $I \in [n]^m$ such that $[I]^3 \subseteq D$ or there is a set $I \in [n]^m$ such that $[I]^3 \cap D = \emptyset$.

By the (finite) Ramsey theorem, $\lim_{n\to\infty} \psi(n) = \infty$.

Now suppose that κ is any cardinal. For each $\zeta < \kappa^+$ let $f_\zeta : \zeta \to \kappa$ be an injection. Let $D \subseteq [\kappa^+]^3$ be the set of triples $\{\xi, \eta, \zeta\}$ where $\xi < \eta < \zeta < \kappa^+$ and $f_\zeta(\xi) < f_\zeta(\eta)$. Let S be the family of all finite subsets I of κ^+ such that *either* $[I]^3 \subseteq D$ *or* $[I]^3 \cap D = \emptyset$. By the choice of ψ, S is ψ-filling.

Let $A \subseteq \kappa^+$ be such that $[A]^{<\omega} \subseteq S$. Then either $[A]^3 \subseteq D$ or $[A]^3 \cap D = \emptyset$.

case 1 If $[A]^3 \subseteq D$ then $\#(A) \le \kappa$. **P?** Otherwise, take $\xi \in A$ such that $\operatorname{otp}(A \cap \xi) = \kappa$ and $\zeta \in A$ such that $\xi < \zeta$. Then $f_\zeta \restriction A \cap \xi$ is an injection from $A \cap \xi$ to $f_\zeta(\xi) < \kappa$, which is impossible. **XQ**

case 2 If $[A]^3 \cap D = \emptyset$ then A is countable. **P?** Otherwise, take $\zeta \in A$ such that $\operatorname{otp}(A \cap \zeta) = \omega$. Then $f_\zeta \restriction A$ is order-reversing so does not attain its minimum. **XQ**

Thus $\#(A) < \kappa^+$. As A is arbitrary, S witnesses that $P_\psi(\kappa^+, \kappa^+)$ is false.

4F MC-dense families: Definition (see AVILÉS PLEBANEK & RODRÍGUEZ P09) Let (X, Σ, μ) be a measure space. A family S is **MC-dense over** (X, Σ, μ) if it is hereditary and whenever $F \in \Sigma$, $\gamma < \mu F$ and $\langle A_n \rangle_{n\in\mathbb{N}}$ is a sequence of sets covering X, then there is an $I \in S$ such that $\mu^*(F \cap \bigcup\{A_n : n \in \mathbb{N}, I \cap A_n \ne \emptyset\}) \ge \gamma$.

4G Theorem (AVILÉS PLEBANEK & RODRÍGUEZ P09, 3.4) Let (X, Σ, μ) be a measure space such that there is an uncountable disjoint family of subsets of X of full outer measure. Then there is a compact hereditary family $S \subseteq [X]^{<\omega}$ which is MC-dense over (X, Σ, μ).

proof (a) By 4B, there is a hereditary compact $S_0 \subseteq [\mathfrak{c}]^{<\omega}$ such that for every finite $I \subseteq \mathfrak{c}$ there is a $J \in S_0$ such that $J \subseteq I$ and $\#(J) \ge \sqrt{\#(I)}$. Let $\langle D_\xi \rangle_{\xi<\omega_1}$ be an uncountable disjoint family of subsets of X with full outer measure. Set

$$S = \{I : I \in [X]^{<\omega},\ \#(I \cap D_\xi) \le 1 \text{ for every } \xi < \omega_1,\ \{\xi : \xi < \omega_1,\ I \cap D_\xi \ne \emptyset\} \in S_0\}.$$

Then S is a compact hereditary family of finite subsets of X.

(b) S is MC-dense. **P** Suppose that $F \in \Sigma$, $\gamma < \mu F < \infty$ and that $\langle A_n \rangle_{n\in\mathbb{N}}$ covers X. For each $\xi < \omega_1$, $\mu^*(D_\xi \cap F) = \mu F > \gamma$, so there is a finite $K_\xi \subseteq \mathbb{N}$ such that $\mu^*(D_\xi \cap F \cap \bigcup_{n\in K_\xi} A_n) \ge \gamma$; of course we may suppose that $D_\xi \cap A_n \ne \emptyset$ for every $n \in K_\xi$. Let $K \in [\mathbb{N}]^{<\omega}$ be such that $P = \{\xi : K_\xi = K\}$ is infinite. Set $m = \#(K)$; then P has a subset of cardinal m^2, so there is an $I \subseteq P$ such that $\#(I) = m$ and $I \in S_0$. Enumerate I as $\langle \xi_i \rangle_{i<m}$ and K as $\langle k_i \rangle_{i<m}$; for each $i < m$ take $x_i \in D_{\xi_i} \cap A_{k_i}$; consider $J = \{x_i : i < m\}$. Then $J \in S$ and

$$\mu^*(F \cap \bigcup\{A_n : n \in \mathbb{N},\ I \cap A_n \ne \emptyset\}) \ge \mu^*(F \cap \bigcup_{n\in K} A_n) \ge \gamma.$$

As F, γ and $\langle A_n \rangle_{n\in\mathbb{N}}$ are arbitrary, S is MC-dense. **Q**

Remark Recall that if (X, Σ, μ) is a probability space in which singletons are negligible and there is no quasi-measurable cardinal less than or equal to $\#(X)$, then there is an uncountable disjoint family of subsets of X with full outer measure (FREMLIN 08, 547E).

4H Proposition Let κ be an atomlessly-measurable cardinal, and μ an atomless κ-additive probability with domain $\mathcal{P}\kappa$. Then there is a compact hereditary family $S \subseteq [\kappa]^{<\omega}$ which is MC-dense over $(\kappa, \mathcal{P}\kappa, \mu)$.

proof (a) There is a stochastically independent family $\langle E_\xi \rangle_{\xi<\kappa}$ of subsets of κ of measure $\frac{1}{2}$. **P** By the Gitik-Shelah theorem (GITIK & SHELAH 89, or FREMLIN 08, 543E), the Maharam type of μ is at least κ. Since this applies to any normalized measure of the form $\frac{1}{\mu E}\mu \llcorner E$, where $\mu E > 0$, we see that every non-zero principal ideal of the measure algebra $\mathfrak{A}$ of μ has Maharam type at least κ. So the measure algebra $\mathfrak{B}_\kappa$ of the usual measure on $\{0,1\}^\kappa$ can be embedded into $\mathfrak{A}$ (Fremlin 02, 322P), and there is a stochastically independent family $\langle e_\xi\rangle_{\xi<\kappa}$ of elements of $\mathfrak{A}$ of measure $\frac{1}{2}$. Take E_ξ such that $E_\xi^\bullet = e_\xi$ for each ξ. **Q**

(b) Re-coding a family of the type in (a), we can get an independent family $\langle E_{\xi n}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ of sets of measure $\frac{1}{2}$. In this case, setting $F_\xi = \bigcap_{n\in\mathbb{N}} E_{\xi n}$, every F_ξ is negligible. Now let S be the family of finite subsets I of κ such that

$$\eta \in E_{\xi k} \setminus F_\xi \text{ whenever } \eta \in I,\ \xi \in I \cap \eta \text{ and } k < \#(I).$$

Clearly S is hereditary. To see that S is compact, observe that if ξ, $\eta \in I \in S$ and $\xi < \eta$, there is an $n \in \mathbb{N}$ such that $\eta \notin E_{\xi n} \setminus F_\xi$, and in this case $\#(I) \le n$.

(c) S is MC-dense over $(\kappa, \mathcal{P}\kappa, \mu)$. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets covering κ, and $\gamma < 1$. Then there is a finite family $\langle B_i\rangle_{i<m}$ of sets, all of non-zero measure, such that for every $i < m$ there is an $n \in \mathbb{N}$ such that $B_i \subseteq A_n$, and $\mu(\bigcup_{i<m} B_i) \ge \gamma$. Because the $E_{\xi n}$ are independent, there is a countable $M \subseteq \kappa$ such that $\langle E_{\xi n}\rangle_{\xi\in\kappa\setminus M,n\in\mathbb{N}}$ are independent of each other and also of the algebra generated by $\{B_i : i < m\}$ (Fremlin 01, 272Q[1]).

Choose $\langle \xi_i\rangle_{i<m}$ inductively, as follows. Given that $\langle \xi_i\rangle_{i<j}$ is a strictly increasing family in $\kappa \setminus M$, where $j < m$, we have

$$\mu(B_j \cap \textstyle\bigcap_{i<j,k<m} E_{\xi_i k} \setminus F_{\xi_i}) = \mu(B_j \cap \bigcap_{i<j,k<m} E_{\xi_i k}) = \mu B_j \cdot \prod_{i<j,k<m} \mu E_{\xi_i k} > 0,$$

so we can take $\xi_j \in B_j \cap \bigcap_{i<j,k<m} E_{\xi_i k} \setminus F_{\xi_i}$ such that $\xi_j > \xi_i$ for every $i < j$, and continue. At the end of the construction, set $I = \{\xi_i : i < m\}$, and see that $I \in S$, while

$$\mu(\textstyle\bigcup\{A_n : I \cap A_n \ne \emptyset\}) \ge \mu(\bigcup_{i<m} B_i) \ge \gamma.$$

As $\langle A_n\rangle_{n\in\mathbb{N}}$ and γ are arbitrary, S is MC-dense. **Q**

4I Proposition Suppose that there are infinitely many measurable cardinals. Then there is a probability space (X, Σ, μ) in which all singletons are negligible but there is no MC-dense compact hereditary subset of $[X]^{<\omega}$.

proof (a) Let $\langle \kappa_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence of measurable cardinals, and set $\kappa = \sup_{n\in\mathbb{N}} \kappa_n$. For each $n \in \mathbb{N}$ let $\mathcal{F}_n$ be a non-principal κ_n-complete ultrafilter on κ_n, and define $\mu : \mathcal{P}\kappa \to [0,1]$ by setting

$$\mu A = \textstyle\sum_{n\in\mathbb{N},A\cap\kappa_n\in\mathcal{F}_n} 2^{-n-1}.$$

Then $(\kappa, \mathcal{P}\kappa, \mu)$ is a probability space in which every singleton is negligible. Let $S \subseteq [\kappa]^{<\omega}$ be a hereditary MC-compact set. The rest of the argument will be devoted to showing that S is not compact.

(b) For each $n \in \mathbb{N}$ there are families $\langle D_i\rangle_{i\le n}$, $\langle T_i\rangle_{i\le n+1}$ such that

(α) $D_i \in \mathcal{F}_i$, $D_i \cap \kappa_{i-1} = \emptyset$ for every $i \le n$,

(counting κ_{-1} as 0),

(β) if $m \le n+1$ and $A_i \in \mathcal{F}_i$ for every $i < m$, then there is an $I \in T_m$ such that $I \cap A_i \ne \emptyset$ for every $i \le m$,

(γ) $T_i \subseteq [\kappa]^{<\omega}$ is hereditary for every $i \le n+1$,

(δ) if $m \le n+1$, $J \in T_m$ and $\xi_i \in D_i$ for $m \le i \le n$, then $J \cup \{\xi_i : m \le i \le n\} \in S$.

P Choose the D_i, T_i by downwards induction on i, as follows. Start with $T_{n+1} = S$. Of course (γ) and (δ) are satisfied, while (α) is so far vacuous. If $A_i \in \mathcal{F}_i$ for $i \le n$, set

$$A'_i = A_i \setminus \bigcup_{j<i} A'_j \text{ for } i \le n,$$

$$= \kappa \setminus \bigcup_{j<i} A'_j \text{ for } i > n.$$

[1] Later editions only.

Note that $A'_i \in \mathcal{F}_i$ for $i \le n$. Since $\langle A'_i \rangle_{i\in\mathbb{N}}$ covers κ, there is a $J \in S$ such that $\mu^*(\bigcup\{A'_i : J \cap A'_i \ne \emptyset\}) > 1 - 2^{-n-1}$. As $\mu A'_i = 2^{-i-1}$ for $i \le n$, J meets $A'_i \subseteq A_i$ for every $i \le n$, while $J \in T_{n+1}$. As $\langle A_i \rangle_{i\le n}$ is arbitrary, (β) is satisfied.

For the downwards step to $m \le n$, set

$$T_{m\xi} = \{J : J \subseteq \kappa_{m-1},\ J \cup \{\xi\} \in T_{m+1}\}$$

for each $\xi < \kappa_m$. Because κ_m is strongly inaccessible, $\#(\mathcal{P}([\kappa_{m-1}]^{<\omega})) < \kappa_m$, and there is a $T_m \subseteq [\kappa_{m-1}]^{<\omega}$ such that $D_m = \{\xi : \kappa_{m-1} \le \xi < \kappa_m,\ T_{m\xi} = T_m\}$ belongs to $\mathcal{F}_m$. Now (α) and (γ) are well in hand, because every $T_{m\xi}$ is hereditary. If $A_i \in \mathcal{F}_i$ for every $i < m$, there is an $I \in T_{m+1}$ such that $I \cap A_i \ne \emptyset$ for $i < m$ and $I \cap D_m \ne \emptyset$; take $\xi \in I \cap D_m$; then $J = I \cap \kappa_m$ belongs to $T_{m\xi} = T_m$, and $J \cap A_i \ne \emptyset$ for $i < m$. Thus (β) is satisfied at the new level. If $J \in T_m$ and $\xi_i \in D_i$ for $m \le i \le n$, then $J \cup \{\xi_m\} \in T_{m+1}$, so $J \cup \{\xi_i : m \le i \le n\} \in S$. So (δ) is satisfied, and the induction proceeds. **Q**

At the end of the induction, observe that $T_0 \ne \emptyset$ (by (β)), so that we have a family $\langle D_i \rangle_{i\le n}$ such that $D_i \in \mathcal{F}_i$ for every $i \le n$ and $\{\xi_0, \dots, \xi_n\} \in S$ whenever $\xi_i \in D_i$ for every $i \le n$.

(c) For each $n \in \mathbb{N}$, take $\langle D_{ni} \rangle_{i\le n}$ as in (β). Set $D^*_i = \bigcap_{n\ge i} D_{ni}$; then $D^*_i \in \mathcal{F}_i$ for $i \in \mathbb{N}$. Take any sequence $\langle \xi_i \rangle_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} D^*_i$; then $\{\xi_i : i \in \mathbb{N}\}$ is infinite and $\{\xi_i : i \le n\} \in S$ for every n, so S is not compact, as claimed.

5 Connexions with precalibers of measure algebras

5A Measure-precaliber pairs: Definition (Fremlin 08, 511E) Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra. Say that a **measure-precaliber pair** of $(\mathfrak{A}, \bar\mu)$ is pair (κ, λ) of cardinals such that whenever $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ with $\inf_{\xi<\kappa} \bar\mu a_\xi > 0$, there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is centered. A **measure-precaliber** of $(\mathfrak{A}, \bar\mu)$ is a cardinal κ such that (κ, κ) is a measure-precaliber pair. Observe that if κ has uncountable cofinality, it is a measure-precaliber of $(\mathfrak{A}, \bar\mu)$ iff it is a precaliber of $\mathfrak{A}$, and that ω is a measure-precaliber of any probability algebra. I will say that (κ, λ) is a **measure-precaliber pair of probability algebras** if it is a measure-precaliber pair of every probability algebra; in particular, every cardinal less than $\mathfrak{m}_{\mathrm{K}}$ is a measure-precaliber of probability algebras (Fremlin 08, 525Ud[1]).

The definition here is written out for probability algebras. But suppose that $\mathfrak{A}$ is any Boolean algebra. If (κ, λ) is a measure-precaliber pair of probability algebras, and $A \subseteq \mathfrak{A} \setminus \{0\}$ has positive intersection number (Fremlin 02, 391H), and $\langle a_\xi \rangle_{\xi<\kappa}$ is a family in A, then there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a_\xi : \xi \in \Gamma\}$ is centered. **P** By Kelley's theorem (Fremlin 02, 391I), there is an additive functional $\nu : \mathfrak{A} \to [0,1]$ such that $\nu 1 = 1$ and $\inf_{a\in A} \nu a > 0$. (I am passing over the trivial case $\mathfrak{A} = \{0\}$.) Let I be the ideal $\{a : \nu a = 0\}$ and $\mathfrak{B}$ the quotient algebra $\mathfrak{A}/I$; then we have a functional $\bar\nu : \mathfrak{B} \to [0,1]$ defined by setting $\bar\nu a^\bullet = \nu a$ for every $a \in \mathfrak{A}$. Next, we have a probability algebra $(\widehat{\mathfrak{B}}, \bar\mu)$ defined by taking a metric completion of $\mathfrak{B}$ and extending $\bar\nu$ (Fremlin 02, 393B). We are supposing that κ is a measure-precaliber pair of $(\mathfrak{B}, \bar\mu)$. So there is a $\Gamma \in [\kappa]^\lambda$ such that $\{a^\bullet_\xi : \xi \in \Gamma\}$ is centered in $\widehat{\mathfrak{B}}$. It follows at once that $\{a_\xi : \xi \in \Gamma\}$ is centered in $\mathfrak{A}$. **Q**

5B Note on Example 1Cb The point of 1Cb is that if the continuum hypothesis is true then ω_1 is not a measure-precaliber of the measure algebra of Lebesgue measure on $[0,1]$. Generally, if (κ, λ) is not a measure-precaliber pair of probability algebras, then $P(\kappa, \lambda)$ is false. **P** Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\langle a_\xi \rangle_{\xi<\kappa}$ a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa} \bar\mu A_\xi = \epsilon > 0$. Set $S = \{I : I \in [\kappa]^{<\omega},\ \inf_{\xi\in I} a_\xi \ne 0\}$. Then S is ψ-filling over κ, where $\psi(n) = \epsilon n$ for every n. Also there is no $A \in [\kappa]^\lambda$ such that $[A]^{<\omega} \subseteq S$. So S witnesses that $P(\kappa, \lambda)$ is false. **Q**

5C Lemma Let S be a set and $\langle E_t \rangle_{t\in T}$ a non-empty family of subsets of S. Then the following are equiveridical:

(i) there is a non-negative finitely additive functional $\nu : \mathcal{P}S \to [0,1]$ such that $\nu E_t > 0$ for every $t \in T$;

(ii) there are a probability space (X, Σ, μ) and a family $\langle F_t \rangle_{t\in T}$ in Σ such that $\mu F_t > 0$ for every $t \in T$ and, for $I \in [T]^{<\omega}$, $\bigcap_{t\in I} F_t = \emptyset$ whenever $\bigcap_{t\in I} E_t = \emptyset$.

proof (i)⇒(ii) Re-normalizing ν if necessary, we may suppose that $\nu S = 1$. Let $\mathcal{N}$ be the ideal $\{N : N \subseteq S,\ \nu N = 0\}$ and $\mathfrak{B}$ the quotient Boolean algebra $\mathcal{P}S/\mathcal{N}$; then we have a strictly positive finitely additive functional $\bar\nu : \mathfrak{B} \to [0,1]$ defined by setting $\bar\nu E^\bullet = \nu E$ for every $E \subseteq S$. We can therefore complete $(\mathfrak{B}, \bar\nu)$ to form a probability algebra $(\mathfrak{A}, \bar\mu)$ (see Fremlin 02, 393B). Now let (X, Σ, μ) be the Stone space of $(\mathfrak{A}, \bar\mu)$

(Fremlin 02, 321J-321K). For $t \in T$ let $F_t \subseteq X$ be the open-and-closed set corresponding to the image $E_t^\bullet$ of E_t in $\mathfrak{B} \subseteq \mathfrak{A}$. Then

$$\mu F_t = \bar{\mu} E_t^\bullet = \bar{\nu} E_t^\bullet = \nu E_t > 0$$

for every $t \in T$. Also, if $\bigcap_{t\in I} E_t = \emptyset$, where $I \subseteq T$ is finite, then

$$\mu(\bigcap_{t\in I} F_t) = \bar{\mu}(\inf_{t\in I} E_t^\bullet) = \bar{\nu}(\inf_{t\in I} E_t^\bullet)$$
$$= \bar{\nu}(\bigcap_{t\in I} E_t)^\bullet = 0;$$

because $\bigcap_{t\in I} F_t$ is an open subset of X, it must be empty. So we have a suitable structure $(X, \mu, \langle F_t\rangle_{t\in T})$.

(ii)⇒(i) Now suppose that we have (X, Σ, μ) and $\langle F_t\rangle_{t\in T}$ as in (ii). Then there is a finitely additive functional $\tilde{\mu} : \mathcal{P}X \to [0, 1]$ extending μ (Fremlin 02, 391F). Let Z be the Stone space of $\mathcal{P}S$ (Fremlin 02, 311E-311F). For any $x \in X$, $\{E_t : t \in T, x \in F_t\}$ has the finite intersection property, so there is an $f(x) \in Z$ such that $f(x) \in \widehat{E_t}$ whenever $x \in F_t$, writing $\widehat{E_t}$ for the open-and-closed subset of Z corresponding to $E_t \in \mathcal{P}S$. Observe that $F_t \subseteq f^{-1}[\widehat{E_t}]$ for every $t \in T$. Set $\nu C = \tilde{\mu} f^{-1}[\widehat{C}]$ for every $C \subseteq S$; because $C \mapsto \widehat{C} : \mathcal{P}S \to \mathcal{P}Z$ is a Boolean homomorphism, ν is a finitely additive functional. For any $t \in T$,

$$\nu E_t = \tilde{\mu} f^{-1}[\widehat{E_t}] \geq \tilde{\mu} F_t = \mu F_t > 0,$$

so ν witnesses that (i) is true.

5D Definition Suppose that T is a set and $S \subseteq [T]^{<\omega}$ is hereditary. I will say that S **accommodates a measure** if there is a finitely additive functional ν on $\mathcal{P}S$ such that $\nu\{I : t \in I \in S\} > 0$ for every $t \in T$; that is, if $\langle E_t\rangle_{t\in T}$ satisfies the conditions of Lemma 5C, where $E_t = \{I : t \in I \in S\}$ for $t \in T$.

In this case, setting $T_n = \{t : \nu E_t \geq 2^{-n}\}$ and $\psi_n(i) = 2^{-n}i$, we have a sequence $\langle T_n\rangle_{n\in\mathbb{N}}$ of subsets of T covering T such that S is ψ_n-filling over T_n for every $n \in \mathbb{N}$.

The importance of this is the following. If κ is a precaliber of probability algebras, then whenever (X, Σ, μ) is a probability space and $\langle E_\xi\rangle_{\xi<\kappa}$ is a family in Σ such that $\mu E_\xi > 0$ for every $\xi < \kappa$, there is a set $A \subseteq \kappa$, of cardinal κ, such that $\bigcap_{\xi\in I} E_\xi \neq \emptyset$ for every finite $I \subseteq A$. So if a hereditary set $S \subseteq [\kappa]^{<\omega}$ accommodates a measure, there is an $A \in [\kappa]^\kappa$ such that $[A]^{<\omega} \subseteq S$.

5E We can use the ideas of §§5C-5D from another angle, as follows.

Proposition (G.Plebanek) Suppose that $S \subseteq [T]^{<\omega}$ is ψ-filling over T, where $\inf_{n\geq 1} \dfrac{\psi(n)}{n} > 0$. Let (κ, λ) be a measure-precaliber pair of probability algebras and $\langle A_\xi\rangle_{\xi<\kappa}$ a family of infinite subsets of T. Then there is a $\Gamma \in [\kappa]^\lambda$ such that for every $I \in [\Gamma]^{<\omega}$ there is a $J \in S$ such that $I \subseteq \{\xi : \xi < \kappa, J \cap A_\xi \neq \emptyset\}$.

proof Write $\epsilon = \inf_{n\geq 1} \dfrac{\psi(n)}{n}$,

$$\tilde{S} = \{I : I \in [\kappa]^{<\omega}, \exists\, J \in S, J \cap A_\xi \neq \emptyset \text{ for every } \xi \in I\}.$$

For $\xi < \kappa$, set $B_\xi = \{I : \xi \in I \in \tilde{S}\}$. Then $\mathcal{C} = \{B_\xi : \xi < \kappa\}$ has intersection number at least ϵ. **P** Given a finite family $\langle C_k\rangle_{k\in K}$ in $\mathcal{C}$ (*not* supposed distinct), express each C_k as B_{ξ_k}. Because every A_{ξ_k} is infinite, we can find a family $\langle t_k\rangle_{k\in K}$ in T, *these* being all distinct, such that $t_k \in A_{\xi_k}$ for each k. Because S is ψ-filling over T, there is a set $L \subseteq k$ with $\#(L) \geq \epsilon k$ such that $J = \{t_k : k \in L\}$ belongs to S. Set $I = \{\xi_k : k \in L\}$; then J witnesses that $I \in \tilde{S}$, while also $I \in \bigcap_{k\in L} C_k$. **Q**

By the remarks in §5A, there is a $\Gamma \in [\kappa]^\lambda$ such that $\{B_\xi : \xi \in \Gamma\}$ is centered, which is what we need to know.

6 Large cardinals

6A For singular cardinals κ we have the following fragment of information concerning $P(\kappa, \lambda)$.

Proposition (Apter & Džamonja 01) Suppose that κ and λ are infinite cardinals such that $\mathrm{cf}(\lambda) > \mathrm{cf}(\kappa)$. If $P(\kappa, \lambda)$ is true, then there is a $\kappa_0 < \kappa$ such that $P(\kappa', \lambda)$ is true whenever $\kappa_0 \leq \kappa' \leq \kappa$.

proof ? Suppose, if possible, otherwise. Then there is an increasing family $\langle\kappa_\xi\rangle_{\xi<\mathrm{cf}(\kappa)}$ of cardinals, with supremum κ, such that $P(\kappa_\xi,\lambda)$ is false for every $\xi<\mathrm{cf}(\kappa)$. So for each $\xi<\mathrm{cf}(\kappa)$ we can find a $\frac{1}{2}$-filling family $\mathcal{S}_\xi\subseteq[\kappa_\xi]^{<\omega}$ such that $[A]^{<\omega}\not\subseteq\mathcal{S}_\xi$ for any $A\subseteq[\kappa_\xi]^\lambda$. Now set $\kappa'_\xi=\sup_{\eta<\xi}\kappa_\eta$ for $\xi<\mathrm{cf}(\kappa)$, and consider

$$\mathcal{S}=\{I:I\in[\kappa]^{<\omega},\ I\cap\kappa_\xi\setminus\kappa'_\xi\in\mathcal{S}_\xi\text{ for every }\xi<\mathrm{cf}(\kappa)\}.$$

It is easy to check that $\mathcal{S}$ is $\frac{1}{2}$-filling over κ, so there ought to be a set $A\in[\kappa]^\lambda$ such that $[A]^{<\omega}\subseteq\mathcal{S}$. But now, setting $A_\xi=A\cap\kappa_\xi\setminus\kappa'_\xi$, $[A_\xi]^{<\omega}\subseteq\mathcal{S}_\xi$ for every $\xi<\mathrm{cf}(\kappa)$, so $\#(A_\xi)<\lambda$ for every $\xi<\mathrm{cf}(\kappa)$; since $\mathrm{cf}(\kappa)<\mathrm{cf}(\lambda)$, $\#(A)<\lambda$. **X**

6B Proposition (Apter & Džamonja 01) If κ is a Ramsey cardinal (Jech 78, §29, p. 328), then $P(\kappa,\kappa)$ is true.

proof If $S\subseteq[\kappa]^{<\omega}$ is $\frac{1}{2}$-filling, then then there is an $A\in[\kappa]^\kappa$ such that for every $n\in\mathbb{N}$ either $[A]^n\subseteq S$ or $[A]^n\cap S=\emptyset$. But the latter is impossible. So $[A]^{<\omega}\subseteq S$.

6C Lemma Let κ be a quasi-measurable cardinal (Fremlin 08, §542), and $\mathcal{I}$ an ω_1-saturated normal ideal in $\mathcal{P}\kappa$. Let $S\subseteq[\kappa]^{<\omega}$ be a compact hereditary family.

(a) Defining $\partial^\gamma_{\mathcal{I}}$, for ordinals γ, as in 2Bb, there is a countable ordinal γ such that $\partial^\gamma_{\mathcal{I}}S=\emptyset$.

(b) Defining $\tilde\partial^\gamma$ as in 2Be, there is a $D\in\mathcal{I}$ such that $\tilde\partial^\gamma(S\cap[\kappa\setminus D]^{<\omega})=\emptyset$.

proof (a)(i) For $n\ge1$ define an ideal $\mathcal{I}_n$ of subsets of κ^n inductively by setting $\mathcal{I}_1=\mathcal{I}$, and for $n\ge1$ taking $\mathcal{I}_{n+1}$ to be the family of those subsets W of κ^{n+1} such that

$$\{x:x\in\kappa^n,\ W[\{x\}]\notin\mathcal{I}\}\in\mathcal{I}_n,$$

where I write $W[\{x\}]=\{\xi:(x,\xi)\in W\}$, identifying κ^{n+1} with $\kappa^n\times\kappa$. Then it is easy to check that every $\mathcal{I}_n$ is a κ-additive ideal of subsets of κ^n containing all singletons. Also it is ω_1-saturated. **P** Induce on n. If $\langle W_\alpha\rangle_{\alpha<\omega_1}$ is a family in $\mathcal{P}\kappa^{n+1}\setminus\mathcal{I}_{n+1}$, set $V_\alpha=\{x:x\in\kappa^n,\ W_\alpha[\{x\}]\notin\mathcal{I}\}$, so that $V_\alpha\notin\mathcal{I}_n$ for every n. Because $\mathcal{I}_n$ is ω_1-saturated, therefore κ-saturated, while $\kappa>\omega_1$ (Fremlin 08, 542B), there is an $x\in\kappa^n$ such that $A=\{\alpha:x\in V_\alpha\}$ is uncountable (Fremlin 08, 541Cb); now $\langle W_\alpha[\{x\}]\rangle_{\alpha\in A}$ cannot be disjoint, so $\langle W_\alpha\rangle_{\alpha<\omega_1}$ is not disjoint. **Q**

(ii) For $A\subseteq\kappa$, $n\ge1$ write $A^{\uparrow n}$ for $\{(\xi_1,\dots,\xi_n):\xi_1,\dots,\xi_n\in A,\ \xi_1<\xi_1<\dots<\xi_n\}$. If $n\ge1$ and $W\in\mathcal{I}_n$, then there is an $A\in\mathcal{I}$ such that $W\cap(\kappa\setminus A)^{\uparrow n}=\emptyset$. **P** Induce on n. For $n=1$ we need only take $A=W$. For the inductive step to $n+1$, set $V=\{x:x\in\kappa^n,\ W[\{x\}]\notin\mathcal{I}\}$, so that $V\in\mathcal{I}_n$. By the inductive hypothesis we have a $B\in\mathcal{I}$ such that $V\cap(\kappa\setminus B)^{\uparrow n}=\emptyset$. Now, for $\xi<\kappa$, set $E_\xi=\bigcup\{W[\{x\}]:x\in((\xi+1)\setminus B)^{\uparrow n}\}$. Then E_ξ is the union of fewer than κ members of $\mathcal{I}$ and belongs to $\mathcal{I}$. Because $\mathcal{I}$ is normal,

$$A=B\cup\bigcup_{\xi<\kappa}E_\xi\setminus(\xi+1)$$

belongs to $\mathcal{I}$. If $(x,\xi)\in(\kappa\setminus A)^{\uparrow n+1}$, then $x\in((\zeta+1)\setminus B)^{\uparrow n}$ where ζ is the last coordinate of x, so $W[\{x\}]\setminus(\zeta+1)\subseteq E_\zeta\setminus(\zeta+1)$ does not contain ξ and $(x,\xi)\notin W$. Thus we have an appropriate set A. **Q**

(iii) Now suppose that $S\subseteq[\kappa]^{<\omega}$ is compact and hereditary. Set $W^\alpha_n=\{(\xi_1,\dots,\xi_n):\{\xi_1,\dots,\xi_n\}\in\partial^\alpha_{\mathcal{I}}S\}$ for each α. Because every $\mathcal{I}_n$ is ω_1-saturated, $W^\alpha_n\setminus W^{\alpha+1}_n\in\mathcal{I}_n$ for all but countably many α, and there is an $\alpha<\omega_1$ such that $W^\alpha_n\setminus W^{\alpha+1}_n\in\mathcal{I}_n$ for every $n\in\mathbb{N}$. Now, by (ii), there is a $C\in\mathcal{I}$ such that $(\kappa\setminus C)^{\uparrow n}\cap W^\alpha_n\setminus W^{\alpha+1}_n$ is empty for every n (we can manage every n simultaneously because $\mathcal{I}$ is a σ-ideal); and of course we can suppose also that C contains 0 and every successor ordinal less than κ, because $\mathcal{I}$ is normal.

What this means is that if $I\cap C=\emptyset$ and $I\in\partial^\alpha_{\mathcal{I}}S\setminus\{\emptyset\}$ then there is a $\xi\in\kappa\setminus C$ such that $I\subseteq\xi$ and $I\cup\{\xi\}\in\partial^\alpha_{\mathcal{I}}S$. **P** Express I as $\{\xi_1,\dots,\xi_n\}$ where $\xi_1<\dots<\xi_n$. Then

$$(\xi_1,\dots,\xi_n)\in W^\alpha_n\cap(\kappa\setminus C)^{\uparrow n}\subseteq W^{\alpha+1}_n,$$

so $I\in\partial^{\alpha+1}_{\mathcal{I}}S$ and $\{\xi:I\cup\{\xi\}\in\partial^\alpha_{\mathcal{I}}S\}$ does not belong to $\mathcal{I}$, so contains something not in $C\cup I$. **Q**

Accordingly $\partial^\alpha_{\mathcal{I}}S\cap[\kappa\setminus C]^{<\omega}$ has no maximal element other than possibly $\emptyset$, and must be empty or $\{\emptyset\}$. Since $\partial^\alpha_{\mathcal{I}}S$ is hereditary, it must itself be included in $\{\emptyset\}$, and $\partial^{\alpha+1}_{\mathcal{I}}S$ is empty. So we can take $\gamma=\alpha+1$.

(b) For $\xi<\kappa$, set

$$F_\xi=\bigcup\{\{\eta:\eta<\kappa,\,I\cup\{\eta\}\in\partial^\beta_{\mathcal{I}}S\}:\beta\le\alpha,\,I\in[\xi+1]^{<\omega}\setminus\partial^{\beta+1}_{\mathcal{I}}S\}\in\mathcal{I}.$$

Then

$$D=\{0\}\cup\{\xi+1:\xi<\kappa\}\cup\bigcup\nolimits_{\xi<\kappa}F_\xi\setminus(\xi+1)$$

belongs to $\mathcal{I}$. Setting $S'=\kappa\setminus D$, we find that $\tilde\partial^\beta S'\subseteq\partial^\beta_{\mathcal{I}}S$ for every $\beta\le\gamma$. **P** Induce on β. Start with $\beta=0$, $S'\subseteq S$. For the inductive step to a successor ordinal $\beta+1$ where $\beta\le\alpha$, take $I\in\tilde\partial^{\beta+1}S'$. Then there is an η such that $I\subseteq\eta<\kappa$ and $I\cup\{\eta\}\in\tilde\partial^\beta S'$. Set $\xi=\sup I$; as $\eta\notin D$, $\eta>\xi+1$ and $\eta\notin F_\xi$. On the other hand, $I\cup\{\eta\}\in\partial^\beta S$, by the inductive hypothesis; so $I\in\partial^{\beta+1}_{\mathcal{I}}S$. As I is arbitrary, the induction proceeds. The step to a non-zero limit ordinal is trivial, so we have the result. **Q** In particular,

$$\tilde\partial^\gamma S'\subseteq\partial^\gamma_{\mathcal{I}}S=\emptyset,$$

as required.

6D Proposition If κ is a quasi-measurable cardinal, then $P(\kappa,\omega)$ is true.

proof Putting 2D and 6Ca together, we see that there can be no $S\subseteq[\kappa]^{<\omega}$ which is compact, hereditary and $\frac12$-filling over κ.

6E Corollary At least if it is consistent to suppose that there are two-valued-measurable cardinals, it is consistent to suppose that $P(\mathfrak{c},\omega)$ is true.

6F Remark Dodos & Kanellopoulos p05 have shown that if we impose describability conditions on our filling families, we get a similar result in ZFC. Specifically, their Theorem 5 shows a little more than the following: if $S\subseteq[\,[0,1]\,]^{<\omega}$ has the Baire property in the restricted sense (see Kuratowski 66), and is ψ-filling where $\inf_{n\ge1}\dfrac{\psi(n)}{n}>0$, then there is an uncountable compact set $K\subseteq[0,1]$ such that $[K]^{<\omega}\subseteq S$.

7 Constructions of $\frac12$-filling sets

7A(a) If $I,\,J\subseteq\mathbb{N}$ write $I\Cup J$ for $\{2n:n\in I\}\cup\{2n+1:n\in J\}$. For $R,\,S\subseteq\mathcal{P}\mathbb{N}$ write $R\boxplus S$ for $\{I\Cup J:I\in R,\,J\in S\}\cup\{J\Cup I:I\in R,\,J\in S\}$. Observe that if R and S are hereditary so is $R\boxplus S$. Write $\mathcal{K}$ for the ideal of finite sets.

I will say that a set S is **quasi-$\frac12$-filling** over T if it is hereditary and whenever $J\subseteq T$ is a finite set with an *even* number of elements, there is an $I\in S\cap\mathcal{P}J$ such that $\#(I)=\frac12\#(J)$. As in 2E(a), any quasi-$\frac12$-filling family includes a minimal quasi-$\frac12$-filling family.

(b) If $S\subseteq[T]^{<\omega}$ is quasi-$\frac12$-filling over T, and $\mathcal{I}$ is a proper ideal of subsets of T containing singletons, with additivity κ, then $\operatorname{rank}_{\mathcal{I}}S>\omega$. **P** Setting $\psi(n)=\lfloor\frac12 n\rfloor$, S is ψ-filling over T, so we can use Proposition 4C. **Q**

(c) If $R,\,S\subseteq[\mathbb{N}]^{<\omega}$ are hereditary so is $R\boxplus S$. If $R,\,S\subseteq[\mathbb{N}]^{<\omega}$ are compact so is $R\boxplus S$ (because the map $(I,J)\mapsto I\Cup J$ is continuous).

(d) If $R\subseteq[\mathbb{N}]^{<\omega}$ is $\frac12$-filling over $\mathbb{N}$ and $S\subseteq[\mathbb{N}]^{<\omega}$ is quasi-$\frac12$-filling over $\mathbb{N}$ then $R\boxplus S$ is $\frac12$-filling over $\mathbb{N}$. **P** Let $K\in[\mathbb{N}]^{<\omega}$. Set $K_1=\{n:2n\in K\}$, $K_2=\{n:2n+1\in K\}$, $m_i=\#(K_i)$ for both i. (i) If m_1 is even, there are $I_1\in S\cap\mathcal{P}K_1$ such that $\#(I_1)\ge\frac{m_1}{2}$ and $I_2\in R\cap\mathcal{P}K_2$ such that $\#(I_2)\ge\frac{m_2}{2}$; now $I_1\Cup I_2\in(R\boxplus S)\cap\mathcal{P}K$ and $\#(I_1\Cup I_2)\ge\frac12\#(K)$. (ii) If m_2 is even, similarly. (iii) If m_1 and m_2 are both odd, set $K_1'=K_1\setminus\{\max K_1\}$. Then there are $I_1\in S\cap\mathcal{P}K_1'$ such that $\#(I_1)\ge\frac{m_1-1}{2}$ and $I_2\in R\cap\mathcal{P}K_2$ such that $\#(I_2)\ge\frac{m_2+1}{2}$; now $I_1\Cup I_2\in(R\boxplus S)\cap\mathcal{P}K$ and $\#(I_1\Cup I_2)\ge\frac12\#(K)$. So all cases are covered. **Q**

7B Notation Write $\ddagger(T,F,V,S)$ to mean

(α) $F \subseteq T$, $T \setminus F$ is infinite;
(β) $S \subseteq [T]^{<\omega}$ is compact and hereditary;
(γ) $\sup_{K \in S} \#(K \setminus F)$ is finite;
(δ) whenever $J \in [T]^{<\omega}$ and $\#(J \cap V) \geq \#(J \setminus (F \cup V))$ there is a $K \in S \cap \mathcal{P}J$ such that $\#(K) \geq \frac{1}{2}\#(J)$.

7C Lemma Suppose that $\ddagger(T, V, F, S)$. Let $I \in S \cap \mathcal{P}F$, $t \in T \setminus F$. Then there are S', V', I^* and $\tilde{I}$ such that

$\ddagger(T, F, V', S')$;
$S' \supseteq S$, $S' \cap \mathcal{P}F = S \cap \mathcal{P}F$;
$V' \supseteq V$, $t \in V'$, $K \cap V = \emptyset$ for every $K \in S' \setminus S$;
$I \subseteq I^* \in S'$;
$I^* \subseteq \tilde{I}$ and I^* is the only member of $S' \cap \mathcal{P}\tilde{I}$ such that $\#(I^*) \geq \frac{1}{2}\#(\tilde{I})$;
$\tilde{I} \subseteq F \cup V'$ and $\#(F \cap \tilde{I}) < \frac{1}{2}\#(\tilde{I})$.

proof (a) Set $m = \sup_{K \in S} \#(K \setminus F)$. Applying clause ($\delta$) of 7B to $J \in [F]^{<\omega}$, we see that $S \cap \mathcal{P}F$ is $\frac{1}{2}$-filling over F; by 2F, there is an $I' \in [F]^{<\omega}$ such that for every non-empty $J \in [F \setminus I']^{<\omega}$ there is a $K \in S \cap \mathcal{P}J$ with $\#(K) > \frac{1}{2}\#(J)$; of course we may suppose that $I' \supseteq I$. Set $k = m + 1 + 3\#(I') + \#(V)$. Let $I^* \in [T \setminus V]^k$ be such that $I^* \cap F = I'$ and $t \in I^* \cup V$. Let $\tilde{I} \in [T]^{2k-1}$ be such that $\tilde{I} \cap F = I'$ and $I^* \cup V \subseteq \tilde{I}$. Set $V' = \tilde{I} \setminus F$, $m' = 2\#(V') = 2(2m + 1 + 5\#(I') + 2\#(V))$,

$$S' = S \cup \{K : K \in [T]^{<\omega}, K \cap F \in S,\ \#(K \setminus F) \leq m',\ K \cap V = \emptyset,$$
$$\text{either } K \cap \tilde{I} = I^* \text{ or } \#(K \cap \tilde{I}) < k\}.$$

Then all the requirements on S', V', I^* and $\tilde{I}$ are easily verified, with the exception of clause (δ) of 7B.

(b) So let $J \in [T]^{<\omega}$ be such that $\#(J \cap V') < \#(J \setminus (F \cup V'))$. Note first that

$$\#(J \setminus F) < 2\#(J \cap V') \leq m'.$$

case 1 Suppose that $\#(J \cap V) \geq \#(J \cap (F \setminus V))$. Then there is a $K \in S \cap \mathcal{P}J$ such that $\#(K) \geq \frac{1}{2}\#(J)$, and $K \in S'$.

case 2 Suppose that $\#(J \cap V) < \#(J \cap (F \setminus V))$ and $\#(\tilde{I} \setminus J) \leq 2\#(I')$. Then

$$\#(J \cap \tilde{I} \setminus (F \cup V)) \geq \#(\tilde{I} \setminus F) - 2\#(I') - \#(V) = \#(\tilde{I}) - 3\#(I') - \#(V) = k + m.$$

Take $L \subseteq J \cap \tilde{I} \setminus (F \cup V)$ such that $\#(L) = k - 1 \geq \frac{1}{2}(\#(J \cap \tilde{I}) - 1)$.

case 2a If $J \cap F \setminus \tilde{I} \neq \emptyset$ then (because $F \cap \tilde{I} = I'$) there is a $K_1 \in S \cap \mathcal{P}(J \cap F \setminus \tilde{I})$ such that $\#(K_1) \geq \frac{1}{2}(1 + \#(J \cap F \setminus \tilde{I}))$. Set $K = K_1 \cup L \cup (J \setminus (F \cup \tilde{I})) \subseteq J$. Then $K \cap F = K_1 \in S$ and $K \cap V = \emptyset$ and $\#(K \setminus F) \leq \#(J \setminus F) \leq m'$. Also $\#(K \cap \tilde{I}) = k - 1 < \#(\tilde{I})$, so $K \in S'$. Finally

$$\#(K) \geq \tfrac{1}{2}(\#(J \cap F \setminus \tilde{I}) - 1) + \tfrac{1}{2}(\#(J \cap \tilde{I}) - 1) + \#(J \setminus (F \cup \tilde{I})) \geq \tfrac{1}{2}\#(J).$$

case 2b If $J \cap F \subseteq \tilde{I}$ and $J \not\supseteq \tilde{I}$ set $K = L \cup (J \setminus \tilde{I}) \subseteq J$. Then $K \cap F = \emptyset \in S$, $K \cap V = \emptyset$, $\#(K \setminus F) \leq m'$ and $\#(K \cap \tilde{I}) < \frac{1}{2}\#(\tilde{I})$, so $K \in S'$. Now

$$\#(K) = k - 1 + \#(J \setminus \tilde{I}) \geq \tfrac{1}{2}\#(J \cap \tilde{I}) + \#(J \setminus \tilde{I}) \geq \tfrac{1}{2}\#(J).$$

case 2c If $J \cap F \subseteq \tilde{I}$ and $J \supseteq \tilde{I}$ set $K = I^* \cup (J \setminus \tilde{I}) \subseteq J$. Then $K \cap F = I' \in S$, $K \cap V = \emptyset$, $\#(K \setminus F) \leq m'$ and $K \cap \tilde{I} = I^*$, so $K \in S'$. This time

$$\#(K) = k + \#(J \setminus \tilde{I}) \geq \tfrac{1}{2}\#(\tilde{I}) + \#(J \setminus \tilde{I}) \geq \tfrac{1}{2}\#(J).$$

case 3 Suppose that $\#(J \cap V) < \#(J \cap (F \setminus V))$ and $\#(\tilde{I} \setminus J) > 2\#(I')$. Let $K_0 \in S \cap \mathcal{P}(J \cap F)$ be such that $\#(K_0) \geq \frac{1}{2}\#(J \cap F)$. Then

$$\#(K_0) + \#(J \cap (F \setminus V)) > \tfrac{1}{2}(\#(J \cap F) + \#(J \cap V) + \#(J \cap (F \setminus V)) = \tfrac{1}{2}\#(J),$$

so there is an $M \subseteq J \cap (F \setminus V)$ such that $\#(K_0) + \#(M) = \lceil \frac{1}{2}\#(J) \rceil$. We can suppose that the points of M are taken, as far as possible, from $J \setminus (F \cup \tilde{I})$, so that

$$\#(M \cap \tilde{I}) \le \lceil \frac{1}{2}\#(J \cap \tilde{I} \setminus F) \rceil \le \frac{1}{2}(\#(J \cap \tilde{I} \setminus F) + 1)$$
$$\le \frac{1}{2}(\#(J \cap \tilde{I}) + 1) \le \frac{1}{2}(\#(\tilde{I}) - 2\#(I')) = k - \frac{1}{2} - \#(I'),$$

and $\#(M \cap \tilde{I}) \le k - 1 - \#(I')$. Set $K = K_0 \cup M \subseteq J$. Then $K \cap F = K_0 \in S$, $K \cap V = \emptyset$, $\#(K \setminus F) \le m'$ and

$$\#(K \cap \tilde{I}) = \#(K_0 \cap \tilde{I}) + \#(M \cap \tilde{I}) \le \#(I') + k - 1 - \#(I') < \tfrac{1}{2}\#(\tilde{I}),$$

so $K \in S'$. And $\#(K) \ge \frac{1}{2}\#(J)$ by the choice of M.

Thus in all cases we can find a $K \in S' \cap \mathcal{P}J$ such that $\#(K) \ge \frac{1}{2}\#(J)$, and (δ) of 7B is satisfied.

7D Proposition Suppose that T, F, S are such that $F \subseteq T$, $\#(F) \le \omega$, $\#(T \setminus F) = \omega$ and $S \subseteq [F]^{<\omega}$ is compact and $\frac{1}{2}$-filling. Then there is a compact minimal $\frac{1}{2}$-filling $\tilde{S} \subseteq [T]^{<\omega}$ such that $S = \tilde{S} \cap [F]^{<\omega}$.

proof Let $\langle I_n \rangle_{n \in \mathbb{N}}$ run over S, and let $\langle t_n \rangle_{n \in \mathbb{N}}$ enumerate $T \setminus F$. Choose $\langle S_n \rangle_{n \in \mathbb{N}}$, $\langle V_n \rangle_{n \in \mathbb{N}}$, $\langle I_n^* \rangle_{n \in \mathbb{N}}$ and $\langle \tilde{I}_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $S_0 = S$ and $V_0 = \emptyset$. Given that $\ddagger(T, F, V_n, S_n)$ is true, use 7C to choose I_n^*, $\tilde{I}_n$, V_{n+1} and S_{n+1} so that

$\ddagger(T, F, V_{n+1}, S_{n+1})$;
$S_{n+1} \supseteq S_n$, $S_{n+1} \cap \mathcal{P}F = S_n \cap \mathcal{P}F$;
$V_{n+1} \supseteq V_n$, $t_n \in V_{n+1}$, $K \cap V_n = \emptyset$ for every $K \in S_{n+1} \setminus S_n$;
$I_n \subseteq I_n^* \in S_{n+1}$;
$I_n^* \subseteq \tilde{I}_n$ and I_n^* is the only member of $S_{n+1} \cap \mathcal{P}\tilde{I}_n$ such that $\#(I_n^*) \ge \frac{1}{2}\#(\tilde{I}_n)$;
$\tilde{I}_n \subseteq F \cup V_{n+1}$ and $\#(F \cap \tilde{I}_n) < \frac{1}{2}\#(\tilde{I}_n)$.

At the end of the induction set $S_\infty = \bigcup_{n \in \mathbb{N}} S_n$. Then $S_\infty \subseteq [T]^{<\omega}$ is hereditary, and $S_\infty \cap [F]^{<\omega} = S$. Next, S_∞ is $\frac{1}{2}$-filling over T. **P** If $J \in [T]^{<\omega}$, there is an $n \in \mathbb{N}$ such that $J \setminus F \subseteq \{t_i : i \le n\} \subseteq V_{n+1}$. In this case $\#(J \setminus (F \cup V_{n+1})) \le \#(J \cap V_{n+1})$ so there is a $K \in S_{n+1} \cap \mathcal{P}J$ such that $\#(K) \ge \frac{1}{2}\#(J)$, and of course $K \in S_\infty$. **Q**

Also S_∞ is compact. **P** Suppose that $[A]^{<\omega} \subseteq S_\infty$. Then $[A \cap F]^{<\omega} \subseteq S$ so $A \cap F$ is finite. If $A \setminus F$ is not empty, let n be such that $t_n \in A$. If $K \in [A]^{<\omega}$ there is an $m \in \mathbb{N}$ such that $K \cup \{t_n\} \in S_{m+1} \setminus S_m$. In this case, $K \cup \{t_n\}$ cannot meet V_m, so $m \le n$. Thus $[A]^{<\omega} \subseteq S_{n+1}$; as S_{n+1} is compact, A is finite; as A is arbitrary, S_∞ is compact. **Q**

Now let $\tilde{S} \subseteq S_\infty$ be a minimal $\frac{1}{2}$-filling family. Then $\tilde{S} \cap [F]^{<\omega} \subseteq S$. **?** If $\tilde{S} \cap [F]^{<\omega} \ne S$, let n be such that $I_n \notin \tilde{S}$. There must be some $K \in \tilde{S} \cap \mathcal{P}\tilde{I}_n$ such that $\#(K) \ge \frac{1}{2}\#(\tilde{I}_n)$. Now $K \not\supseteq I_n$ so $K \ne I_n^*$ and $K \notin S_{n+1}$. Because $\#(F \cap \tilde{I}_n) < \frac{1}{2}\#(\tilde{I}_n)$, $K \cap V_{n+1} \ne \emptyset$ so $K \notin S_{m+1} \setminus S_m$ for any $m > n$ and $K \notin S_\infty$. **X**

Thus $S = \tilde{S} \cap [F]^{<\omega}$, as required.

7E Corollary Set $\mathcal{I} = [\mathbb{N}]^{<\omega}$. Then for any $\alpha < \omega_1$ there is a compact minimal $\frac{1}{2}$-filling $S \subseteq [\mathbb{N}]^{<\omega}$ such that $\mathrm{rank}_{\mathcal{I}}(S) \ge \alpha$.

proof We have only to construct a compact $\frac{1}{2}$-filling R with $\mathrm{rank}_{\mathcal{I}}(R) \ge \alpha$, and then use 7D to express it as the trace of a compact minimal $\frac{1}{2}$-filling family.

8 Problems

8A The problems DU(a) and DU(b) from my problem list can be expressed as

DU(a): is $P(\omega_1, \omega)$ true?

DU(b): does $\mathfrak{m} > \omega_1$ imply $P(\omega_1, \omega_1)$?

Of course we can also ask, more generally,

is $P(\omega_1, \omega_1)$ consistent?

and, in the light of Corollary 6E,

is $P(\mathfrak{c},\mathfrak{c})$ consistent? if κ is two-valued-measurable and we add κ random reals, does $P(\mathfrak{c},\mathfrak{c})$ become true?

is it consistent to suppose that $P(\kappa,\omega)$ is false for every κ? what about $P(\mathfrak{c}^+,\omega)$?

From §§5C-5D above, we see that the situation for families accommodating measures is much more familiar, and corresponds to questions about calibers of measure algebras, as considered in Comfort & Negrepontis 82 and Fremlin 08, §525. So we have to ask

if $S \subseteq [T]^{<\omega}$ is $\frac{1}{2}$-filling over T, does it necessarily accommodate a measure, in the sense of §5D?

This question is raised, in effect, in Galvin & Prikry 00. A positive answer seems wildly improbable but no counter-example is known.

In Propostion 4B, $\mathfrak{c}$ is the natural limit of the argument given; but is it really best possible? What about 2^{ω_1}? 4F

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