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1 Definition Let E be a Riesz space and $\| \|$ a Riesz norm on E. Construct inductively a family $\langle \mathcal{G}_\xi \rangle_{\xi \in \mathrm{On}}$ as follows. The inductive hypothesis will be that every $A \in \mathcal{G}_\xi$ is a non-empty subset of E^+ with a supremum in E. Start with $\mathcal{G}_0 = \{\{x\} : x \in E^+\}$. For each ordinal ξ,

$$\mathcal{G}_{\xi+1} = \{\bigcup \mathcal{A} : \emptyset \ne \mathcal{A} \subseteq \mathcal{G}_\xi, \quad \{\sup A : A \in \mathcal{A}\} \text{ is upwards-directed and has a supremum in } E\}.$$

For non-zero limit ordinals ξ, $\mathcal{G}_\xi = \bigcup_{\eta<\xi} \mathcal{G}_\eta$. At the end of the induction, set $\mathcal{G} = \bigcup_{\xi \in \mathrm{On}} \mathcal{G}_\xi$. Of course $\mathcal{G} = \mathcal{G}_\zeta$ for some ζ. Observe that $\bigcup \mathcal{A} \in \mathcal{G}$ whenever $\mathcal{A}$ is a non-empty subset of $\mathcal{G}$, $\{\sup A : A \in \mathcal{A}\}$ is upwards-directed and $\bigcup \mathcal{A}$ has a least upper bound in E.

2 Lemma For every $\xi \le \zeta$ and $\alpha \ge 0$,
(a) $\alpha A = \{\alpha x : x \in A\}$ belongs to $\mathcal{G}_\xi$ whenever $A \in \mathcal{G}_\xi$,
(b) $A + B = \{x + y : x \in A,\ y \in B\}$ belongs to $\mathcal{G}_\xi$ whenever $A,\ B \in \mathcal{G}_\xi$,
(c) $A \wedge B = \{x \wedge y : x \in A,\ y \in B\}$ belongs to $\mathcal{G}_\xi$ whenever $A,\ B \in \mathcal{G}_\xi$.

proof Induce on ξ. The point is that if $A,\ B \subseteq E$ have suprema, then $\sup(\alpha A) = \alpha \sup A$, $\sup(A + B) = \sup A + \sup B$ and $\sup(A \wedge B) = \sup A \wedge \sup B$ are defined (Fremlin 02, 351D and 352Ea).

The induction starts with the elementary case in which A, B, $A + B$ and $A \wedge B$ are singleton sets for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. For the inductive step to $\xi + 1$, if A^*, $B^* \in \mathcal{G}_{\xi+1}$ let $\mathcal{A}$, $\mathcal{B} \subseteq \mathcal{G}_\xi$ be such that $A^* = \bigcup \mathcal{A}$, $B^* = \bigcup \mathcal{B}$, $\{\sup A : A \in \mathcal{A}\}$ is upwards-directed and $\{\sup B : B \in \mathcal{B}\}$ is upwards-directed. Then $\{\alpha A : A \in \mathcal{A}\}$ is a subset of $\mathcal{G}_\xi$ with union αA^*,

$$\{\sup(\alpha A) : A \in \mathcal{A}\} = \{\alpha \sup A : A \in \mathcal{A}\}$$

is upwards-directed, and $\sup(\alpha A^*) = \alpha \sup A^*$. Next, $\{A + B : A \in \mathcal{A},\ B \in \mathcal{B}\}$ is a subset of $\mathcal{G}_\xi$ with union $A^* + B^*$,

$$\{\sup(A + B) : A \in \mathcal{A},\ B \in \mathcal{B}\} = \{\sup A : A \in \mathcal{A}\} + \{\sup B : B \in \mathcal{B}\}$$

is upwards-directed, and $\sup(A^* + B^*) = \sup A^* + \sup B^*$. Similarly, $\{A \wedge B : A \in \mathcal{A},\ B \in \mathcal{B}\}$ is a subset of $\mathcal{G}_\xi$ with union $A^* \wedge B^*$,

$$\{\sup(A \wedge B) : A \in \mathcal{A},\ B \in \mathcal{B}\} = \{\sup A : A \in \mathcal{A}\} \wedge \{\sup B : B \in \mathcal{B}\}$$

is upwards-directed, and $\sup(A^* \wedge B^*) = \sup A^* \wedge \sup B^*$.

The inductive step to a limit ordinal is elementary.

3 Theorem For any $\alpha \ge 0$, the following are equiveridical:
(i) there is a Fatou norm $\| \|'$ on E such that $\|x\|' \le \|x\| \le \alpha \|x\|'$ for every $x \in E$;
(ii) $\|\sup A\| \le \alpha \sup_{x \in A} \|x\|$ for every $A \in \mathcal{G}$.

proof (i)⇒(ii) If (i) is true, then $\|\sup A\|' = \sup_{x \in A} \|x\|'$ whenever $\xi \le \zeta$ and $A \in \mathcal{G}_\zeta$. **P** Induce on ξ. The case of limit ordinals is elementary. For the inductive step to $\xi + 1$, if $A^* \in \mathcal{G}_{\xi+1}$ express it as $\bigcup \mathcal{A}$ where $\mathcal{A} \subseteq \mathcal{G}_\xi$ and $\{\sup A : A \in \mathcal{A}\}$ is upwards-directed. Then $\sup A^* = \sup_{A \in \mathcal{A}} \sup A$, so

$$\|\sup A^*\|' = \sup_{A \in \mathcal{A}} \|\sup A\|' = \sup_{A \in \mathcal{A}} \sup_{x \in A} \|x\| = \sup_{x \in A^*} \|x\|. \quad \mathbf{Q}$$

Consequently, for any $A \in \mathcal{G} = \mathcal{G}_\zeta$,

$$\|\sup A\| \le \alpha \|\sup A\|' = \alpha \sup_{x \in A} \|x\|' \le \alpha \sup_{x \in A} \|x\|.$$

(ii)⇒(i) If (ii) is true, then for every $x \in E$ set

$$\|x\|' = \inf\{\sup_{y\in A}\|y\| : A \in \mathcal{G},\ |x| \le \sup A\}.$$

Since $\{|x|\} \in \mathcal{G}$, $\|x\|' \le \|x\|$. Of course $\|x\|' \le \|y\|'$ whenever $|x| \le |y|$. Using Lemma 2(a-b), with $\xi = \zeta$, we see that $\|\alpha x\|' \le \alpha\|x\|'$ and $\|x + y\|' \le \|x\|' + \|y\|'$ whenever $\alpha \ge 0$ and x, $y \in E$. And condition (ii) tells us that $\|x\| \le \alpha \sup_{y\in A}\|y\|$ whenever $A \in \mathcal{G}$ and $\sup A = x$, so $\|x\| \le \alpha\|x\|'$ for every x. In particular, $\|x\|' \ne 0$ for every non-zero x, and $\|\ \|'$ is a Riesz norm.

Finally, if $A \subseteq E^+$ is a non-empty upwards-directed set with supremum z, $\|z\|' \le \sup_{x\in A}\|x\|'$. **P** Let $\epsilon > 0$. For each $x \in A$, let $A_x \in \mathcal{G}$ be such that $x \le \sup A_x$ and $\sup_{y\in A_x}\|y\| \le \|x\|' + \epsilon$. Set $B_x = \{y \wedge x : y \in A_x\}$. By (c) of Lemma 2, applied to A_x and $\{x\}$, $B_x \in \mathcal{G}$, while of course $\sup B_x = x \wedge \sup A_x = x$. Accordingly $B = \bigcup_{x\in A} B_x$ belongs to $\mathcal{G}$, $\sup B = z$ and

$$\|z\|' \le \sup_{y\in B}\|y\| \le \sup_{x\in A}\sup_{y\in A_x}\|y\| \le \epsilon + \sup_{x\in A}\|x\|'.$$

As ϵ is arbitrary, we have the result. **Q**

So $\|\ \|'$ is a Fatou norm.

4 Theorem Let E be a weakly (σ,∞)-distributive Riesz space with the countable sup property. If $\|\ \|$ is a weakly Fatou norm on E, there is an equivalent Fatou norm on E.

proof (a) For A, $B \subseteq E^+$ say that $B \preccurlyeq A$ if for every $y \in B$ there is an $x \in A$ such that $y \le x$.

(b) Let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be constructed as in Definition 1. Then whenever $\xi \le \zeta$ and $A \in \mathcal{G}_\xi$, there is an upwards-directed $B \preccurlyeq A$ with supremum $\sup A$. **P** Induce on ξ. The step to limit ξ is elementary. For the inductive step to $\xi + 1$, take $A^* \in \mathcal{G}_{\xi+1}$, and express it as $\bigcup\mathcal{A}$ where $\mathcal{A} \subseteq \mathcal{G}_\xi$ and $C = \{\sup A : A \in \mathcal{A}\}$ is upwards-directed. We can suppose that $\{0\} \in \mathcal{A}$. Set $z = \sup A^* = \sup C$. Because E has the countable sup property, there is a sequence $\langle z_n\rangle_{n\in\mathbb{N}}$ in C with supremum z; because C is upwards-directed, we can take $\langle z_n\rangle_{n\in\mathbb{N}}$ to be non-decreasing, and also $z_0 = 0$.

For each $n \in \mathbb{N}$, take $A_n \in \mathcal{A}$ such that $z_n = \sup A_n$. By the inductive hypothesis, there is an upwards-directed set $B_n \preccurlyeq A_n$ with supremum z_n. Choose $\langle A'_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $A'_0 = B_0 = A_0 = \{0\}$. Given that $n > 0$, A'_{n-1} is upwards-directed and $A'_{n-1} \preccurlyeq B_m$ for every $m \ge n-1$, set $B_{nm} = \{z_n \wedge y : y \in B_m\}$ for each $m \ge n$. Then B_{nm} is upwards-directed and has supremum $z_n \wedge z_m = z_n$. Because E is weakly (σ,∞)-distributive, there is a set $\tilde{A}_n$, with supremum z_n, such that $\tilde{A}_n \preccurlyeq B_{nm}$ for every $m \ge n$ (see Fremlin 02, 368N). Since of course $B_{nm} \preccurlyeq B_m$, $\tilde{A}_n \preccurlyeq B_m$ for $m \ge n$. Now set

$$A'_n = A'_{n-1} \cup \{y \vee \tilde{y} : y \in A'_{n-1},\ \tilde{y} \in \tilde{A}_n\}.$$

Then A'_n is upwards-directed. Since $A'_{n-1} \cup \tilde{A}_n \preccurlyeq B_m$ and B_m is upwards-directed, $A'_n \preccurlyeq B_m$ for $m \ge n$, and the induction can continue.

At the end of this induction, set $B = \bigcup_{n\in\mathbb{N}} A'_n$. Because $\langle A'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of upwards-directed sets, B is upwards-directed. Because

$$A'_n \preccurlyeq B_n \preccurlyeq A_n \subseteq A^*$$

for each n, $B \preccurlyeq A^*$. And because $\tilde{A}_n \preccurlyeq A'_n \subseteq B$ and $\sup\tilde{A}_n = z_n$ for every n, any upper bound of B must be an upper bound of $\{z_n : n \in \mathbb{N}\}$, and $z = \sup B$.

Thus the inductive step to $\mathcal{G}_{\xi+1}$ is successful, and we have the result. **Q**

(c) It follows that if $\|\ \|$ is a weakly Fatou norm, and $\alpha \ge 0$ is such that $\|\sup A\| \le \alpha \sup_{x\in A}\|x\|$ whenever $A \subseteq E^+$ is non-empty and has a supremum, then condition (ii) of Theorem 3 is satisfied, and there is an equivalent Fatou norm on E.

References

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