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The density algebra

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This note extends remarks in Fremlin 03, §491.

1 Order-continuity properties of density

1A The context (For general definitions, see Fremlin 02, Fremlin 03 and Fremlin 08?.) For $A \subseteq \mathbb{N}$ let $d^*(A) = \limsup_{n\to\infty} \frac{1}{n}\#(A \cap n)$ be its upper asymptotic density. Write $\mathcal{Z}$ for the density ideal $\{A : d^*(A) = 0\}$, $\mathfrak{Z}$ for the density algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. We have a strictly positive submeasure $\bar{d}^*$ on $\mathfrak{Z}$ defined by setting $\bar{d}^*(A^\bullet) = d^*(A)$ for $A \subseteq \mathbb{N}$ (Fremlin 03, 491I).

1B More definitions (a) A Boolean algebra $\mathfrak{A}$ is **weakly** (λ, κ)**-distributive** if whenever $\langle A_\xi\rangle_{\xi<\lambda}$ is a family of partitions of unity in $\mathfrak{A}$, all of size at most κ, then there is a partition C of unity in $\mathfrak{A}$ such that $\{a : a \in A_\xi,\ a \cap c \ne 0\}$ is finite for every $c \in C$ and $\xi < \lambda$ (Koppelberg 89, 14.23).

1C Theorem (a) Suppose that $A \subseteq \mathfrak{Z}$ is non-empty and downwards-directed, and $\#(A) < \mathfrak{p}$. Then A has a lower bound c such that $\bar{d}^*(c) = \inf_{a\in A} \bar{d}^*(a)$.

(b) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{Z}$, there is an $a \in \mathfrak{Z}$ such that $a_n \subseteq a$ for every $n \in \mathbb{N}$ and $\bar{d}^*(a) = \sup_{n\in\mathbb{N}} \bar{d}^*(a_n)$.

(c) $\bar{d}^* : \mathfrak{Z} \to [0, 1]$ is order-continuous on the left, in the sense that if $A \subseteq \mathfrak{Z}$ is non-empty and upwards-directed and has supremum b, then $\bar{d}^*(b) = \sup_{a\in A} \bar{d}^*(a)$.

proof (a) Let $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$ be a downwards-directed set, of cardinal less than $\mathfrak{p}$, such that $A = \{I^\bullet : I \in \mathcal{A}\}$. Set $\gamma = \inf_{a\in A} \bar{d}^*(a) = \inf_{I\in\mathcal{A}} d^*(I)$. Let P be the family of triples (K, n, I) where $K \subseteq n \in \mathbb{N}$ and $I \in \mathcal{A}$; say that $(K, n, I) \le (K', n', I')$ if $n \le n'$, $K = K' \cap n$, $I' \subseteq I$ and $K' \setminus I \subseteq n$. Then $\le$ is a partial order on P. If $(K, n, I) \in P$ and $J \in \mathcal{A}$ is included in I, then $(K, n, I) \le (K, n, J)$; so P is σ-centered upwards. If $I \in \mathcal{A}$ then $Q_I = \{(K, n, I') : I' \subseteq I\}$ is cofinal with P. If $m \in \mathbb{N}$ then $Q'_m = \{(K, n, I) : n > m,\ \frac{1}{n}\#(K \cap n) \ge \gamma - 2^{-m}\}$ is cofinal with P. So there is an upwards-directed $R \subseteq P$ meeting every Q_I and every Q'_m. Setting $J = \bigcup\{K : (K, n, I) \in R\}$, $c = J^\bullet$, $d^*J \ge \gamma$ and $J \setminus I$ is finite for every $I \in \mathcal{A}$, so $\bar{d}^*c = \gamma$ and $c \subseteq a$ for every $a \in A$.

(b) Let $\langle I_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{P}\mathbb{N}$ such that $a_n = I_n^\bullet$ for every n, and set $\gamma = \sup_{n\in\mathbb{N}} \bar{d}^*(a_n) = \sup_{n\in\mathbb{N}} d^*(I_n)$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\#(I_n \cap m) \le (\gamma + 2^{-n})m$ whenever $m \ge k_n$, and set $I = \bigcup_{n\in\mathbb{N}} I_n \cap k_{n+1} \setminus k_n$, $c = I^\bullet$. Then $I_n \setminus I$ is finite so $a_n \subseteq c$ for every n. If $k_n \le m < k_{n+1}$, $\#(I \cap m) \le \#(I_n \cap m) \le (\gamma + 2^{-n})m$, so $d^*(I) \le \gamma$ and $\bar{d}^*(c) \le \gamma$.

(c) ? Suppose, if possible, otherwise.

(i) Set $\gamma = \bar{d}^*(b)$, $\gamma' = \sup_{a\in A} \bar{d}^*(a)$ and $\epsilon = \frac{1}{4}(\gamma - \gamma') > 0$. Let $J \subseteq \mathbb{N}$ be such that $J^\bullet = b$, and set $\mathcal{A} = \{I : I \subseteq \mathbb{N},\ I^\bullet \in A\}$, so that $\mathcal{A}$ is upwards-directed. Let $\langle n_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $n_{k+1} \ge kn_k$ and $\#(J \cap n_k) > (\gamma - \epsilon)n_k$ for every k. Set $n'_k = \lfloor \epsilon n_k\rfloor$, so that $\#(J \cap n_k \setminus n'_k) \ge (\gamma - 2\epsilon)n_k$ for every k, $\lim_{k\to\infty} \frac{n'_k}{n_k} = \epsilon$ and $\lim_{k\to\infty} \frac{n_k}{n_{k+1}} = 0$. For $I \subseteq \mathbb{N}$, $K \in [\mathbb{N}]^\omega$ set

$$\beta(K, I) = \limsup_{k\to K} \frac{1}{n_k}\#(I \cap n_k \setminus n'_k) = \lim_{n\to\infty} \sup_{k\in K\setminus n} \frac{1}{n_k}\#(I \cap n_k \setminus n'_k);$$

for $K \in [\mathbb{N}]^\omega$, set $\alpha(K) = \sup_{I\in\mathcal{A}} \beta(K, I)$.

(ii) Choose $\langle K_r\rangle_{r\in\mathbb{N}}$, $\langle I_r\rangle_{r\in\mathbb{N}}$ inductively, as follows. $K_0 = \mathbb{N}$. Given K_r, let $I_r \in \mathcal{A}$ be such that $\beta(K_r, I_r) > \alpha(K_r) - 2^{-r}$; as $\mathcal{A}$ is upwards-directed, we can arrange that $I_r \supseteq I_{r-1}$ if $r > 0$. Given K_r and I_r, set

$$K_{r+1} = \{k : k \in K_r,\ \#(I_r \cap n_k \setminus n'_k) \ge \alpha(K_r) - 2^{-r}\},$$

so that $K_{r+1}\subseteq K_r$ is infinite and the induction continues.

(iii) Looking back at the proof of (b), we see that there is an $L\subseteq\mathbb{N}$ such that $I_r\setminus L$ is finite for every r and $d^*(L)\le\sup_{r\in\mathbb{N}}d^*(I_r)\le\gamma'$. Now we can find a strictly increasing sequence $\langle k(r)\rangle_{r\in\mathbb{N}}$ such that

$$k(r)\in K_{r+1},\quad \#(L\cap n_{k(r)})\le(\gamma'+\epsilon)n_{k(r)}$$

for every $r\in\mathbb{N}$. Set $C=(J\setminus L)\cap\bigcup_{r\in\mathbb{N}}(n_{k(r)}\setminus n'_{k(r)})$. Then, for each r,

$$\#(C\cap n_{k(r)})\ge(\gamma-2\epsilon)n_{k(r)}-(\gamma'+\epsilon)n_{k(r)}\ge\epsilon n_{k(r)},$$

and $d^*(C)>0$. As $C\subseteq J$, we have $0\ne C^\bullet\subseteq b$. There must therefore be an $a\in A$ such that $a\cap C^\bullet\ne 0$, and an $I\in\mathcal{A}$ such that $d^*(C\cap I)>0$; set $D=C\cap I$ and $\eta=\frac{1}{4}d^*(D)>0$.

(iv) For every $r_0\in\mathbb{N}$ there is an $r\ge r_0$ such that $\#(D\cap n_{k(r+1)}\setminus n'_{k(r+1)})\ge 2\eta\epsilon n_{k(r+1)}$. **P** We may suppose that r_0 is so large that $n'_{k+1}\ge n_k$ and $3\eta n'_{k+1}-n_k\ge 2\eta\epsilon n_{k+1}$ for every $k\ge k(r_0)$. Then there is a least $n\ge n_{k(r_0)+1}$ such that $\#(D\cap n)\ge 3\eta n$. Let $r\ge r_0$ be such that $n_{k(r)}<n\le n_{k(r+1)}$. As $D\subseteq C$ does not meet $n'_{k(r+1)}\setminus n_{k(r)}$, $n\ge n'_{k(r+1)}$. Now

$$\begin{aligned}\#(D\cap n_{k(r+1)}\setminus n'_{k(r+1)})&\ge\#(D\cap n)-n_{k(r)}\\&\ge 3\eta n'_{k(r+1)}-n_{k(r+1)-1}\ge 2\eta\epsilon n_{k(r+1)}.\end{aligned}$$ **Q**

(v) Let $s\in\mathbb{N}$ be such that $2^{-s}\le\eta\epsilon$. Then $\beta(K_s,D\cup I_s)>\alpha(K_s)$. **P** Given $r_0\in\mathbb{N}$, let $r_1\ge\max(s,r_0)$ be such that $I_s\setminus L\subseteq n'_{k(r_1)}$. Then there is an $r\ge r_1$ such that $\#(D\cap n_{k(r)}\setminus n'_{k(r)})\ge 2\eta\epsilon n_{k(r)}$. On the other hand, $k(r)\in K_{r+1}\subseteq K_{s+1}$ so $\#(I_s\cap n_{k(r)}\setminus n'_{k(r)})\ge\alpha(K_s)-2^{-s}$; and as $D\cap L=\emptyset$, $D\cap I_s\setminus n'_{k(r)}$ is empty. We therefore have $k(r)\in K_s$ and

$$\begin{aligned}\#((D\cup I_s)\cap n_{k(r)}\setminus n'_{k(r)})&=\#(D\cap n_{k(r)}\setminus n'_{k(r)})+\#(I_s\cap n_{k(r)}\setminus n'_{k(r)})\\&\ge 2\eta\epsilon n_{k(r)}+(\alpha(K_s)-2^{-s})n_{k(r)}\ge(\alpha(K_s)+\eta\epsilon)n_{k(r)}.\end{aligned}$$

Since this happens for infinitely many r, $\beta(K_s,D\cup I_s)\ge\alpha(K_s)+\eta\epsilon$. **Q**

However, there must be an $I'\in\mathcal{A}$ including $I_s\cup I$, so that $\beta(K_s,I')\ge\beta(K_s,D\cup I_s)>\alpha(K_s)$; contradicting the definition of $\alpha(K_s)$. **X**

This contradiction proves the result.

1D Proposition $\mathfrak{Z}$ is weakly (σ,∞)-distributive.

proof Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{Z}$. For each $n\in\mathbb{N}$ let A^*_n be

$$\{b:b\in\mathfrak{Z},\ \{a:a\in A_n,\ a\cap b\ne 0\}\text{ is finite}\},$$

so that A^*_n is an order-dense ideal of $\mathfrak{Z}$. Then $\bigcap_{n\in\mathbb{N}}A^*_n$ is order-dense. **P** Suppose that $b\in\mathfrak{Z}$ is non-zero. Choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $b_0=b$. Given that $\bar d^*(b_n)>\frac{1}{2}\bar d^*(b)$, $A^*_n\cap[0,b_n]$ is upwards-directed and has supremum b_n; by 1Cc, there is a $b_{n+1}\in A^*_n\cap[0,b_n]$ such that $\bar d^*(b_{n+1})>\frac{1}{2}\bar d^*(b)$. Continue.

At the end of the induction, 1Ca tells us that there is an $a\in\mathfrak{Z}$ such that $a\subseteq b_n$ for every n and $\bar d^*(a)>0$. Now $0\ne a\subseteq b$ and $a\in\bigcap_{n\in\mathbb{N}}A^*_n$. **Q**

There is therefore a partition C of unity in $\mathfrak{Z}$ included in $\bigcap_{n\in\mathbb{N}}A^*_n$, that is to say, if $c\in C$ and $n\in\mathbb{N}$, then c meets only finitely many members of A_n. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{Z}$ is weakly (σ,∞)-distributive (Fremlin 02, 316H).

1E Corollary The regular open algebra of $\mathbb{R}$ cannot be regularly embedded in $\mathfrak{Z}$.

proof $\mathrm{RO}(\mathbb{R})$ is not weakly (σ,∞)-distributive (Fremlin 02, 316J).

1F Examples (a)(i) There is a downwards-directed set $A\subseteq\mathfrak{Z}$ such that $\inf A=0$ and $\bar d^*(a)=1$ for every $a\in A$.

(ii) There are families $\langle a_\xi\rangle_{\xi<\omega_1}$, $\langle c_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{Z}$ such that $a_\eta\subseteq c_\xi$ whenever $\eta<\xi<\omega_1$, $\bar d^*(c_\xi)\le\frac{1}{2}$ for every $\xi<\omega_1$, but $\bar d^*(c)=1$ whenever $c\in\mathfrak{Z}$ and $\{\xi:a_\xi\subseteq c\}$ is uncountable.

(b) Suppose that $\mathfrak{c}=\omega_1$.

(i) There is a non-increasing family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{Z}$, with infimum 0, such that $\bar d^*(c_\xi)=1$ for every ξ.

(ii) There is a non-decreasing family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{Z}$ such that $\bar d^*(c_\xi)\le\frac12$ for every ξ, but $\bar d^*(a)=1$ for every upper bound a of $\{c_\xi:\xi<\omega_1\}$.

proof (a)(i) Set $K_n=\{2^n(2m+1):m\in\mathbb{N}\}$, $I_n=\bigcup_{k\in K_n}[k!,(k+1)![$, $c_n=I_n^\bullet$; then $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint and $\bar d^*(c_n)=1$ for every n. Set $A=\{a:a\in\mathfrak{Z},\, c_n\subseteq a$ for all but finitely many $n\}$; then A is downwards-directed, $\inf A=0$ and $\bar d^*(a)=0$ for every $a\in A$.

(ii) $(\boldsymbol{\alpha})$For $\xi<\omega_1$, choose $f_\xi\in\mathbb{N}^{\mathbb{N}}$, $I_\xi\subseteq\mathbb{N}$, $I'_\xi\subseteq I_\xi$ as follows. The inductive hypothesis will be that I_η is infinite and $\#(I_\eta\cap n^2)\le n$ for every $n\in\mathbb{N}$ and $\eta<\xi$, and that $I_\eta\cap I_\zeta$ is finite whenever $\zeta<\eta<\xi$. For the inductive step to ξ, enumerate ξ as $\langle\theta(\xi,i)\rangle_{i<\#(\xi)}$. Choose $f_\xi(i)$ inductively such that

$f_\xi(i)\ge i^2$,
$f_\xi(i)\notin I_{\theta(\xi,j)}$ whenever $j<\min(i,\#(\xi))$,
if $i<\#(\xi)$ then $f_\xi(i)\in I_{\theta(\xi,i)}$.

Set $I_\xi=f_\xi[\mathbb{N}]$,

$$I'_\xi=\{f_\xi(i):i\in\mathbb{N},\, i<\#(\xi),\, f_\xi(i)\notin I'_{\theta(\xi,i)}\}.$$

$(\boldsymbol{\beta})$ For $n\in\mathbb{N}$, set

$$L(n)=\{i:n!\le i<(n+1)!,\ i\text{ is even}\},\quad L'(n)=\{i:n!\le i<(n+1)!,\ i\text{ is odd}\}.$$

For $\xi<\omega_1$ set

$$A_\xi=\bigcup\{L(n):n\in I'_\xi\}\cup\bigcup\{L'_n:n\in I_\xi\setminus I'_\xi\},\quad a_\xi=A_\xi^\bullet.$$

$(\boldsymbol{\gamma})$ If $K\subseteq\omega_1$ is finite, then $\bar d^*(\sup_{\xi\in K}a_\xi)\le\frac12$. **P** Set $A=\bigcup_{\xi\in K}A_\xi$. There is a $k\in\mathbb{N}$ such that $I_\xi\cap I_\eta\subseteq k$ for all distinct ξ, $\eta\in K$. For $n\ge k$, $A\cap[n!,(n+1)![$ is either L_n or L'_n or empty, so $\bar d^*(\sup_{\xi\in K}a_\xi)=d^*(A)$ is at most $\frac12$. **Q**

By (b), it follows that for each $\xi<\omega_1$ there is a $c_\xi\in\mathfrak{Z}$ such that $a_\eta\subseteq c$ for every $\eta<\xi$ and $\bar d^*(c)\le\frac12$.

$(\boldsymbol{\delta})$ Now suppose that $c\in\mathfrak{Z}$ is such that $D=\{\xi:a_\xi\subseteq c\}$ is uncountable. Let $C\subseteq\mathbb{N}$ be such that $c=C^\bullet$. Take any $\epsilon>0$. Then there is a $k\in\mathbb{N}$ such that

$$D'=\{\xi:\#(A_\xi\cap m\setminus C)\le\epsilon m\text{ for every }m\ge k!\}$$

is uncountable. Let $\xi\in D'$ be such that $D'\cap\xi$ is infinite. Then $M=\{i:i\in\mathbb{N},\,\theta(\xi,i)\in D',\, f_\xi(i)\ge k\}$ is infinite. But for every $i\in M$, setting $l_i=f_\xi(i)!$, $l'_i=(f_\xi(i)+1)!$,

$$A_\xi\cup A_{\theta(\xi,i)}\supseteq l'_i\setminus l_i.$$

So $\#((l'_i\setminus l_i)\setminus C)\le 2\epsilon l'_i$ and $\#(C\cap l'_i)\ge l'_i(1-2\epsilon)-l_i$. As this is true for infinitely many i, $\bar d^*(c)=d^*(C)=\ge 1-2\epsilon$. As ϵ is arbitrary, $\bar d^*(c)=1$.

Thus $\langle a_\xi\rangle_{\xi<\omega_1}$ and $\langle c_\xi\rangle_{\xi<\omega_1}$ have the required properties.

(b)(i) Enumerate $\mathfrak{Z}^+=\mathfrak{Z}\setminus\{0\}$ as $\langle a_\xi\rangle_{\xi<\omega_1}$. Choose $\langle c_\xi\rangle_{\xi<\omega_1}$ inductively. $c_0=1$. Given that $d^*(c_\xi)=1$, we can partition it into c, c' with $\bar d^*(c)=\bar d^*(c')=1$; take $c_{\xi+1}$ to be one of these not including a_ξ. For non-zero countable limit ordinals ξ, use (a) to see that there is a c_ξ such that $\bar d^*(c_\xi)=1$ and $c_\xi\subseteq c_\eta$ for every $\eta<\xi$. Now no a_ξ can be a lower bound for $\{c_\eta:\eta<\omega_1\}$.

(ii) Enumerate $\{A:A\subseteq\mathbb{N},\, d^*(A)<1\}$ as $\langle A_\xi\rangle_{\xi<\omega_1}$. For $n\in\mathbb{N}$, set $L_n=\{i:n!\le i<(n+1)!\}$. Let $\langle I_\xi\rangle_{\xi<\omega_1}$ be a family in $\mathcal{P}\mathbb{N}$ such that $I_\eta\setminus I_\xi$ is finite and $I_{\xi+1}\setminus I_\xi$ is infinite for $\eta\le\xi<\omega_1$. (Cf. Fremlin 03, 419A.) Choose $\langle C_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. The inductive hypothesis will be that whenever $\zeta\le\eta<\xi$ then $C_\eta\subseteq\bigcup_{n\in I_\eta}L_n$ and $\#(C_\eta\cap L_n\cap i)\le\frac12(i-n!)$ for every $n\in\mathbb{N}$ and $i\in L_n$, and $C_\zeta\setminus C_\eta$ is finite.

Start with $C_0=\emptyset$. Given C_ξ, then set

$$D_{\xi n}=\{i:i\in L_n\setminus A_\xi,\ \#(i\cap L_n\cap D_{\xi n})\le\frac12(i-n!)\}\text{ for }n\in\mathbb{N},$$

$$C_{\xi+1}=C_\xi\cup\bigcup\nolimits_{n\in I_{\xi+1}\setminus I_\xi}D_{\xi n}.$$

Observe that if $\delta=\frac{1}{2}(1-d^*(A_\xi))>0$, then for all n large enough we shall have $\#(L_n\setminus A_\xi)\ge\delta\#(L_n)$, so that $\#(D_{\xi n})\ge\frac{1}{2}\delta\#(L_n)$; consequently $d^*(C_{\xi+1}\setminus A_\xi)\ge\frac{1}{2}\delta>0$.

For the inductive step to a non-zero countable limit ordinal ξ, let $\langle\eta_k\rangle_{k\in\mathbb{N}}$ be a non-decreasing cofinal sequence in ξ, and $\langle n_k\rangle_{k\in\mathbb{N}}$ a strictly increasing sequence such that $I_{\eta_k}\setminus I_\xi\subseteq n_k$ and $C_{\eta_k}\setminus C_{\eta_{k+1}}\subseteq n_k!$ for every k. Set

$$C_\xi=\bigcup_{k\in\mathbb{N}}C_{\eta_k}\setminus n_k!.$$

Then for $n_k\le n<n_{k+1}$, $C_\xi\cap L_n=C_{\eta_k}\cap L_n$ is appropriately thin, and is empty unless $n\in I_\xi$.

Set $c_\xi=C_\xi^\bullet$ for each ξ. Then $\langle c_\xi\rangle_{\xi<\omega_1}$ is non-decreasing, and $\bar{d}^*(c_\xi)\le\frac{1}{2}$ for every ξ. **?** If $a\in\mathfrak{Z}$ is an upper bound for $\{c_\xi:\xi<\omega_1\}$ and $\bar{d}^*(a)<1$, ther is a $\xi<\omega_1$ such that $a=A_\xi^\bullet$, and now $c_{\xi+1}\setminus a\ne0$. **X**

2 Bits & pieces

2A Lemma If $a\in\mathfrak{Z}$ and $\langle\gamma_\xi\rangle_{\xi<\mathfrak{c}}$ is any family in $[0,\bar{d}^*(a)]$, there is a disjoint family $\langle b_\xi\rangle_{\xi<\mathfrak{c}}$ such that $b_\xi\subseteq a$ and $\bar{d}^*(b_\xi)=\gamma_\xi$ for every $\xi<\mathfrak{c}$.

proof Set $\gamma=\bar{d}^*(a)$. Let $A\subseteq\mathbb{N}$ be such that $A^\bullet=a$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\#(A\cap k_{n+1}\setminus k_n)\ge(\gamma-2^{-n})k_{n+1}$ for every n. Let $\langle I_\xi\rangle_{\xi<\mathfrak{c}}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ (Fremlin 08?, 5A1Fa). Set $A_\xi=\bigcup_{n\in I_\xi}A\cap k_{n+1}\setminus k_n$. Then $A_\xi\subseteq A$ and $d^*(A_\xi)=\gamma$ for every ξ, and $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ is almost disjoint.

Now, for $\xi<\mathfrak{c}$, define $B_\xi\subseteq A_\xi$ by saying that

$$B_\xi=\{i:i\in A_\xi,\ \#(i\cap B_\xi)\le\gamma_\xi i\}.$$

Then $\#(i\cap B_\xi)\le1+\gamma_\xi(i-1)$ for every $i\in\mathbb{N}$, so $d^*(B_\xi)\le\gamma_\xi$. **?** If $d^*(B_\xi)<\gamma_\xi$, let n be such that $\#(B_\xi\cap i)\le\gamma_\xi i$ whenever $i\ge n$; then $B_\xi\supseteq A_\xi\setminus n$ and $d^*(B_\xi)\ge\gamma$. **X**

So we can set $b_\xi=B_\xi^\bullet$ for every ξ.

2B Proposition For any $b\in\mathfrak{Z}$ there is a positive additive functional μ on $\mathfrak{Z}$ such that $\mu b=\bar{d}^*b$ and $\mu a\le\bar{d}^*a$ for every $a\in\mathfrak{Z}$. **P** Take $B\subseteq\mathbb{N}$ representing b. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $d^*(B)=\lim_{n\to\infty}\frac{1}{k_n}\#(B\cap k_n)$. Take a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and set $\mu A^\bullet=\lim_{n\to\mathcal{F}}\frac{1}{k_n}\#(A\cap k_n)$ for every $A\subseteq\mathbb{N}$. **Q**

Note that μ is countably additive (in the sense of Fremlin 02, 326E), because $\bar{d}^*$ is sequentially order-continuous.

2C Proposition $c(\mathfrak{Z}_a)=\mathfrak{c}$ for every non-zero $a\in\mathfrak{Z}$.

proof Represent a as $A^\bullet$. Take a strictly increasing sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ such that $\#(A\cap k_{n+1}\setminus k_n)\ge(d^*(A)-2^{-n})k_{n+1}$ for every n. Let $\langle K_\xi\rangle_{\xi<\mathfrak{c}}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$. Set $a_\xi=(A\cap\bigcup_{n\in K_\xi}k_{n+1}\setminus k_n)^\bullet$. Then $\langle a_\xi\rangle_{\xi<\mathfrak{c}}$ is disjoint and $\bar{d}^*(a_\xi)=\bar{d}^*(a)$ for every ξ. **Q**

2D Proposition $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ can be regularly embedded in $\mathfrak{Z}$.

proof Define $\pi:\mathcal{P}\mathbb{N}\to\mathcal{P}\mathbb{N}$ by setting $\pi A=\bigcup_{n\in A}2^{n+1}\setminus2^n$. Then $\pi A\in\mathcal{Z}$ iff A is finite, so π descends to an injective Boolean homomorphism $\bar{\pi}:\mathcal{P}\mathbb{N}/[\mathbb{N}]^\omega\to\mathfrak{Z}$. **?** If $\bar{\pi}$ is not order-continuous, there is a non-empty downwards-directed set $P\subseteq\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ such that $\inf P=0$ in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ but there is a non-zero $c\in\mathfrak{Z}$ which is a lower bound for $\bar{\pi}[P]$. Set $\mathcal{A}=\{A:A^\bullet\in P\}$ and let C represent c; then $C\setminus\pi A\in\mathcal{Z}$ for every $A\in\mathcal{A}$. Consider $K=\{n:\#(C\cap2^{n+1}\setminus2^n)\ge\frac{1}{3}d^*(C)\}$. This is infinite. If $A\in\mathcal{A}$, then $\{n:\#((C\setminus\pi A)\cap2^{n+1}\setminus2^n)\ge\frac{1}{6}d^*(C)\}$ must be finite, so $\{n:n\in K,\ \pi A\cap2^{n+1}\setminus2^n=\emptyset\}$ must be finite and $K\setminus A$ is finite, so $K^\bullet$ is a non-zero lower bound for P. **X** So $\bar{\pi}$ is a regular embedding of $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ in $\mathfrak{Z}$.

2E Corollary $\mathfrak{Z}$ has an (ω_1,ω_1^*)-gap, that is, families $\langle a_\xi\rangle_{\xi<\omega_1}$, $\langle b_\xi\rangle_{\xi<\omega_1}$ such that $a_\eta\subset a_\xi\subseteq b_\xi\subset b_\eta$ whenever $\eta<\xi<\omega_1$ but there is no $c\in\mathfrak{Z}$ such that $a_\xi\subseteq c\subseteq b_\xi$ for every $\xi<\omega_1$.

proof Let $\langle a_\xi\rangle_{\xi<\omega_1}$, $\langle b_\xi\rangle_{\xi<\omega_1}$ be an (ω_1,ω_1^*)-gap in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ (Fremlin 84, 21L), and consider $\langle\bar\pi a_\xi\rangle_{\xi<\omega_1}$, $\langle\bar\pi b_\xi\rangle_{\xi<\omega_1}$. **?** If $c\in\mathfrak{Z}$ is such that $\bar\pi a_\xi\subseteq c\subseteq\bar\pi b_\xi$ for every ξ, let $C\subseteq\mathbb{N}$ be such that $C^\bullet=C$, and consider $D=\{n:\#(C\cap 2^{n+1}\setminus 2^n)\ge 2^{n-1}\}$; then $a_\xi\subseteq D^\bullet\subseteq b_\xi$ in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ for every $\xi<\omega_1$. **X**

2F Proposition (a) $\mathfrak{Z}$ is isomorphic to the simple product $\mathfrak{Z}^{\mathbb{N}}$.
(b) $\mathfrak{Z}$ has the σ-interpolation property.

proof (a) Set $A_n=\{2^{n+1}(2i+1):i\in\mathbb{N}\}$, $a_n=A_n^\bullet\in\mathfrak{Z}$. Then each principal ideal $\mathfrak{Z}_{a_n}$ is isomorphic to $\mathfrak{Z}$ (see Fremlin 03, 491Xo), and the map $A\mapsto\langle A\cap A_n\rangle_{n\in\mathbb{N}}:\mathcal{P}\mathbb{N}\to\prod_{n\in\mathbb{N}}\mathcal{P}A_n$ descends to an isomorphism from $\mathfrak{Z}$ to $\mathfrak{Z}^{\mathbb{N}}$.

(b) Let $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathfrak{Z}$ such that $a_m\subseteq b_n$ for all m, $n\in\mathbb{N}$. Let $\langle I_n\rangle_{n\in\mathbb{N}}$, $\langle J_n\rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{P}\mathbb{N}$ such that $I_n^\bullet=\sup_{i\le n}a_i$ and $J_n^\bullet=\inf_{i\le n}b_i$ for every $n\in\mathbb{N}$, $\langle I_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and $\langle J_n\rangle_{n\in\mathbb{N}}$ is non-increasing. Then $d^*(I_n\setminus J_n)=0$ for each n. Let $\langle r_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $\#((I_n\setminus J_n)\cap m)\le 2^{-n}m$ whenever $m\ge r_n$; then $\#(((I_n\setminus J_n)\setminus r_n)\cap m)\le 2^{-n}m$ for every m. Set $I=\bigcap_{n\in\mathbb{N}}(r_n\cup J_n)$. Then $I\setminus J_n\subseteq r_n$ is finite for every n. Next, for any $n\in\mathbb{N}$,

$$I_n\setminus I=\bigcup_{k\in\mathbb{N}}((I_n\setminus J_k)\setminus r_k.$$

Set $J'=\bigcup_{k<n}I_n\setminus J_k$; then $d^*J'=0$. Set

$$J''=\bigcup_{k\ge n}((I_n\setminus J_k)\setminus r_k)=\subseteq\bigcup_{k\ge n}((I_k\setminus J_k)\setminus r_k)$$

and

$$\#(J''\cap m)\le\sum_{k=n}^\infty 2^{-k}m=2^{-n+1}m$$

for every m, so $d^*(J'')\le 2^{-n+1}$ and $d^*(I_n\setminus I)\le 2^{-n+1}$. As $\langle I_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, $d^*(I_n\setminus I)=0$ for every n. So, setting $c=I^\bullet$, we have $a_n\subseteq c\subseteq b_n$ for every n.

2G Proposition For a, $b\in\mathfrak{Z}$, set $\rho(a,b)=\bar d^*(a\triangle b)$, so that ρ is a metric on $\mathfrak{Z}$ (Fremlin 02, 392H[1]). If $\mathfrak{C}\subseteq\mathfrak{Z}$ is a subalgebra which is closed and has weight less than $\mathfrak{p}$ for the metric topology of $\mathfrak{Z}$, then $\mathfrak{C}$ is order-closed.

proof Let $\kappa<\mathfrak{p}$ be the weight of $\mathfrak{C}$. Suppose that $A\subseteq\mathfrak{C}$ is non-empty and upwards-directed and has a supremum b in $\mathfrak{Z}$. Then there is a dense subset D of A of cardinal at most κ; let D' be the set of suprema of finite subset of D. Then b is an upper bound of D'; moreover, if c is any upper bound of D', then $\{a:a\subseteq c\}$ is topologically closed, so includes A, and $c\supseteq b$. Thus $b=\sup D'$.

The set $\{b\setminus a:a\in D'\}$ is downwards-directed and has cardinal less than $\mathfrak{p}$; by 1Ca, it has a lower bound c such that $\bar d^*(c)=\inf_{a\in D'}\bar d^*(b\setminus a)$; but c must be 0, so

$$0=\inf_{a\in D'}\bar d^*(b\setminus a)\ge\inf_{a\in A}\bar d^*(b\setminus a)$$

and $b\in\overline{A}\subseteq\mathfrak{C}$. As A is arbitrary, $\mathfrak{C}$ is order-closed (Fremlin 02, 313E(a-i)).

2H Proposition $\operatorname{Aut}\mathfrak{Z}$ has many involutions.

proof If $a\in\mathfrak{Z}^+$, let $I\subseteq\mathbb{N}$ be such that $I^\bullet=a$. Let $f:\mathbb{N}\to I$ be the increasing enumeration of I. Define a bijection $h:\mathbb{N}\to\mathbb{N}$ by saying that $h(n)=n$ for $n\in\mathbb{N}\setminus I$ and $h(f(2i))=f(2i+1)$, $h(f(2i+1))=f(2i)$ for $i\in\mathbb{N}$. Then $d^*(h[J])=d^*(J)$ for every $J\subseteq\mathbb{N}$, so we have a Boolean automorphism $\pi:\mathfrak{Z}\to\mathfrak{Z}$ defined by saying that $\pi(J^\bullet)=(h^{-1}[J])^\bullet$ for every $J\subseteq\mathbb{N}$; now π is an involution with support a.

3 Cardinal functions

3A As the cellularity $c(\mathfrak{Z})$ of $\mathfrak{Z}$ is $\mathfrak{c}=\#(\mathfrak{Z})$ (2C), we have $\operatorname{link}(\mathfrak{Z})=d(\mathfrak{Z})=\pi(\mathfrak{Z})=\mathfrak{c}$ (Fremlin 08?, 511J).

3B Proposition The Maharam type $\tau(\mathfrak{Z})$ of $\mathfrak{Z}$ is at least $\mathfrak{p}$.
[Strengthened in 3H.]

[1]Formerly 393B.

proof If $D \subseteq \mathfrak{Z}$ and $\#(D) < \mathfrak{p}$, let $\mathfrak{D}$ be the subalgebra of $\mathfrak{Z}$ generated by D, and $\mathfrak{C}$ the topological closure of $\mathfrak{D}$ (see 2G). Then $\mathfrak{C}$ is a subalgebra of $\mathfrak{Z}$, because the Boolean operations are topologically continuous (Fremlin 02, 392H), and $w(\mathfrak{C}) \le \#(\mathfrak{D}) < \mathfrak{p}$. So 2G tells us that $\mathfrak{C}$ is order-closed. On the other hand, $\mathfrak{C}$ is certainly not equal to $\mathfrak{Z}$, because the topological density of $\mathfrak{Z}$ is $\mathfrak{c}$, by 2A. So $\mathfrak{Z}$ is not the order-closed subalgebra of itself generated by D. As D is arbitrary, $\tau(\mathfrak{Z}) \ge \mathfrak{p}$.

3C Proposition The weak distributivity $\mathrm{wdistr}(\mathfrak{Z})$ of $\mathfrak{Z}$ is ω_1.

proof By 1D, $\mathrm{wdistr}(\mathfrak{Z}) \ge \omega_1$. As the measure algebra $\mathfrak{B}_\mathfrak{c}$ of the usual measure on $\{0,1\}^\mathfrak{c}$ is regularly embedded in $\mathfrak{Z}$ (Fremlin 03, 491P), $\omega_1 = \mathrm{wdistr}(\mathfrak{B}_c) \ge \mathrm{wdistr}(\mathfrak{Z})$ (Fremlin 08?, 524Mb and 514Eb).

3D Proposition The Martin number $\mathfrak{m}(\mathfrak{Z})$ of $\mathfrak{Z}$ is at least $\mathfrak{m}_{\sigma\text{-linked}}$.

proof Take $\kappa < \mathfrak{m}_{\sigma\text{-linked}}$, a family $\langle D_\xi\rangle_{\xi<\kappa}$ of order-dense subsets of $\mathfrak{Z}$, and $\tilde{d} \in \mathfrak{Z}^+$.

(a)(i) Let $\tilde{A} \subseteq \mathbb{N}$ be such that $\tilde{A}^\bullet = \tilde{d}$, and set $\epsilon = \frac{1}{3}\bar{d}^*(\tilde{A}) > 0$. Let $\langle m_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\#(\tilde{A}\cap m_k) \ge 2\epsilon m_k$ and and $m_{k+1} \ge \max(k, \frac{1}{\epsilon})m_k$ for every n; set $m'_k = \lfloor \epsilon m_k \rfloor$ and $L_k = m_k \setminus m'_k$ for each k, so that $\#(\tilde{A} \cap L_k) \ge \epsilon\#(L_k)$ for every k, $\langle L_k\rangle_{k\in\mathbb{N}}$ is disjoint, $\lim_{k\to\infty} \frac{\#(L_k)}{m_k} = 1-\epsilon$ and $\lim_{k\to\infty} \frac{m_{k-1}}{\#(L_k)} = 0$. For $I \subseteq \mathbb{N}$ set $C_I = \bigcup_{k\in I} L_k$; for $I \in [\mathbb{N}]^\omega$ and $A \subseteq \mathbb{N}$, set

$$\delta^*(I,A) = \limsup_{k\in I, k\to\infty} \frac{\#(A\cap L_k)}{\#(L_k)},$$

$$\delta(I,A) = \lim_{k\in I, k\to\infty} \frac{\#(A\cap L_k)}{\#(L_k)}$$

if the limit is defined. Note that whenever A, $B \subseteq \mathbb{N}$ and $I \in [\mathbb{N}]^\omega$,

$\delta^*(I,A) = \delta^*(I, A\cap C_I)$,
$\delta^*(I,C_I) = 1$,
$\delta^*(I,\tilde{A}) \ge \epsilon$,
$\delta^*(I,B) \le \delta^*(I,A)$ if $B \setminus A$ is finite,
$\delta^*(I,A) \ge \delta^*(J,A)$ whenever $J \in [\mathbb{N}]^\omega$ and $J \setminus I$ is finite,
there is a $J \in [I]^\omega$ such that $\delta(J,A)$ is defined and equal to $\delta^*(I,A)$,
if $\delta(I,A)$ is defined then $\delta(J,A)$ is defined and equal to $\delta(I,A)$ whenever $J \in [\mathbb{N}]^\omega$ and $J \setminus I$ is finite,
if $\delta(I,A)$ is defined and $\delta^*(I, B\cap A) = 0$, then $\delta^*(I, A\cup B) = \delta(I,A) + \delta^*(I,B)$.

Also, of course, $A \mapsto \delta^*(I,A)$ is a submeasure, for every $I \in [\mathbb{N}]^\omega$.

(ii) If $A \subseteq \mathbb{N}$ and $I \in [\mathbb{N}]^\omega$, then

$$\epsilon d^*(A\cap C_I) \le \delta^*(I,A) \le \frac{1}{1-\epsilon} d^*(A\cap C_I).$$

P $(\boldsymbol{\alpha})$Let $\eta > 0$. Let $k_0 \in I$ be such that $\#(A\cap L_k) \le (\delta^*(I,A)+\eta)\#(L_k)$ and $m_{k-1} \le \eta m_k$ for every $k \ge k_0$. Take any $n > m_{k_0}$; let k, l be successive members of I such that $m_k < n \le m_l$. If $n \le m'_l$ then

$$\begin{aligned}\#(A\cap C_I\cap n) = \#(A\cap C_I \cap m_k) &\le \#(A\cap L_k) + m_{k-1}\\ &\le (\delta^*(I,A)+\eta)\#(L_k) + m_{k-1}\\ &\le (\delta^*(I,A)+\eta)m_k + \eta m_k \le (\delta^*(I,A)+2\eta)n.\end{aligned}$$

If $m'_l < n \le m_l$ then

$$\#(A\cap C_I\cap n) \le \#(A\cap L_l) + m_{l-1} \le (\delta^*(I,A)+2\eta)m_l \le \frac{\delta^*(I,A)+2\eta}{\epsilon} n.$$

So in both cases $\frac{1}{n}\#(A\cap C_I\cap n) \le \frac{\delta^*(I,A)+2\eta}{\epsilon}$; as η is arbitrary, $\epsilon d^*(A\cap C_I) \le \delta^*(I,A)$.

(β) Given $\eta > 0$, let $n_0 \in \mathbb{N}$ be such that $\#(A \cap C_I \cap n) \le (d^*(A \cap C_I) + \eta)n$ for every $n \ge n_0$. If $k \in I$ is such that $m'_k \ge n_0$, then

$$\frac{\#(A \cap L_k)}{\#(L_k)} \le \frac{\#(A \cap m_k)}{\#(L_k)} \le \frac{(d^*(A \cap C_I)+\eta)m_k}{\#(L_k)} \le \frac{d^*(A \cap C_I)+\eta}{1-\epsilon}.$$

As η is arbitrary, $\delta^*(I, A) \le \frac{d^*(A \cap C_I)}{1-\epsilon}$. **Q**

(iii) If $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of $\mathbb{N}$ and $I \in [\mathbb{N}]^\omega$, there is a $B \subseteq \mathbb{N}$ such that $B \setminus A_n$ is finite for every $n \in \mathbb{N}$ and $\delta^*(I, B) = \sup_{n\in\mathbb{N}} \delta^*(I, A_n)$. **P** Let $\langle k_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence such that $\#(A_n \cap L_k) \le (\delta^*(I, A_n) + 2^{-n})\#(L_k)$ whenever $k \ge k_n$. Set $B = \bigcup_{n\in\mathbb{N}} A_n \setminus m_{k_n}$. **Q**

(b) Suppose that $I \in [\mathbb{N}]^\omega$, $\eta > 0$ and that $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$ is an upwards-directed family such that $\{A^\bullet : A \in \mathcal{A}\}$ is order-dense in $\mathfrak{Z}$. Then there are a $J \in [I]^\omega$ and an $A \in \mathcal{A}$ such that $\delta^*(J, A) \ge 1 - \eta$. **P?** Otherwise, choose $\langle J_n \rangle_{n\in\mathbb{N}}$, $\langle \beta_n \rangle_{n\in\mathbb{N}}$ and $\langle A_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $J_0 = I$. Given J_n, set $\beta_n = \sup\{\delta^*(J, A) : A \in \mathcal{A},\ J \in [J_n]^\omega\}$, and choose $J \in [J_n]^\omega$, $A_n \in \mathcal{A}$ such that $\delta^*(J, A_n) \ge \beta_n - 2^{-n}$; as $\mathcal{A}$ is upwards-directed, we may suppose that $A_n \supseteq A_{n-1}$ if $n \ge 1$. Let $J_{n+1} \in [J]^\omega$ be such that $\delta(J_{n+1}, A_n)$ is defined and equal to $\delta^*(J, A_n)$, and continue.

At the end of the induction, let $J \in [\mathbb{N}]^\omega$ be such that $J \setminus J_n$ is finite for every $n \in \mathbb{N}$. Observe that $\delta(J, A_n)$ is defined for every n. By (a-iii), there is a $B \subseteq \mathbb{N}$ such that $A_n \setminus B$ is finite for every n and

$$\delta^*(J, B) = \sup_{n\in\mathbb{N}} \delta^*(J, A_n) \le 1 - \eta.$$

So $\delta^*(J, C_J \setminus B) \ge \eta$ and $d^*(C_J \setminus B) > 0$. There is therefore an $A \in \mathcal{A}$ such that $0 \ne A^\bullet \subseteq (C_J \setminus B)^\bullet$. In this case, $d^*(A \cap C_J) > 0$ and $\delta^*(J, A) > 0$. Let n be such that $2^{-n} < \delta^*(J, A)$. Let $A' \in \mathcal{A}_n$ be such that $A' \supseteq A \cup A_n$. Since $d^*(A \cap A_n) \le d^*(A \cap B) = 0$, $\delta^*(J, A \cap A_n) = 0$ and

$$\begin{aligned}\delta^*(J_n, A') &\ge \delta^*(J_n, A \cup A_n) \ge \delta^*(J, A \cup A_n)\\ &= \delta(J, A_n) + \delta^*(J, A) > \delta(J_n, A_n) + 2^{-n} \ge \beta_n;\end{aligned}$$

contradicting the definition of β_n. **XQ**

(c) Suppose that $I \in [\mathbb{N}]^\omega$ and that $\mathcal{D} \subseteq \mathcal{P}\mathbb{N}$ is such that $\{A^\bullet : A \in \mathcal{D}\}$ is order-dense in $\mathfrak{Z}$ and $B \in \mathcal{D}$ whenever $B \subseteq A \in \mathcal{D}$. Then there are a $K \in [I]^\omega$ and a disjoint sequence $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{D}$ such that $\delta(K, A_n)$ is defined for every $n \in \mathbb{N}$ and $\sum_{n=0}^\infty \delta(K, A_n) = 1$. **P** Choose $\langle J_n \rangle_{n\in\mathbb{N}}$, $\langle \mathcal{I}_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. $J_0 = I$. Given J_n, use (b) to find $J \in [J_n]^\omega$ and $\mathcal{I}_n \in [\mathcal{D}]^{<\omega}$ such that $\delta^*(J, \bigcup \mathcal{I}_n) \ge 1 - 2^{-n}$; let $J_n \in [J]^\omega$ be such that $\delta(J_n, \bigcup \mathcal{I}_n)$ is defined and equal to $\delta^*(J, \bigcup \mathcal{I}_n)$. At the end of the induction, let $J \in [I]^\omega$ be such that $J \setminus J_n$ is finite for every n. Then $\delta(J, \bigcup \mathcal{I}_n)$ is defined and greater than or equal to $1 - 2^{-n}$ for every n.

Let $\langle A'_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ running over $\bigcup_{n\in\mathbb{N}} \mathcal{I}_n$. Set $A_n = A'_n \setminus \bigcup_{i<n} A'_i$ for each n. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$ such that $K_0 = J$ and $K_{n+1} \in [K_n]^\omega$ and $\delta(K_{n+1}, A_n)$ is defined for every n; let $K \in [J]^\omega$ be such that $K \setminus K_n$ is finite for every n, so that $\delta(K, A_n)$ is defined for every n. For $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $\{A'_i : i \le n\} \supseteq \mathcal{I}_k$. In this case,

$$\begin{aligned}\sum_{i=0}^n \delta(K, A_i) &= \delta(K, \bigcup_{i\le n} A_i) = \delta(K, \bigcup_{i\le n} A'_i)\\ &\ge \delta(K, \bigcup \mathcal{I}_k) = \delta(J, \bigcup \mathcal{I}_k) \ge 1 - 2^{-k}.\end{aligned}$$

As k is arbitrary, $\sum_{n=0}^\infty \delta(K, A_m) = 1$. **Q**

(d) For $\xi < \kappa$, set

$$\mathcal{D}_\xi = \{A : A \subseteq \mathbb{N}, \text{ there is some } d \in D_\xi \text{ such that } A^\bullet \subseteq d\}.$$

Choose $\langle I_\xi \rangle_{\xi\le\kappa}$, $\langle A_{\xi n} \rangle_{\xi<\kappa, n\in\mathbb{N}}$ as follows. $I_0 = \mathbb{N}$. Given $I_\xi \in [\mathbb{N}]^\omega$, where $\xi < \kappa$, let $I_{\xi+1}$ and $\langle A_{\xi n} \rangle_{n\in\mathbb{N}}$ be such that $I_{\xi+1}$ is an infinite subset of I_ξ, $\langle A_{\xi n} \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{D}_\xi$, $\delta(I_{\xi+1}, A_{\xi n})$ is defined for every n and $\sum_{n=0}^\infty \delta(I_{\xi+1}, A_{\xi n}) = 1$; this is possible by (c). Given that $\xi \le \kappa$ is a non-zero limit ordinal and $\langle I_\eta \rangle_{\eta<\xi}$ is a family of infinite sets such that $I_\eta \setminus I_\zeta$ is finite whenever $\zeta \le \eta < \xi$, then $\#(\xi) \le \kappa < \mathfrak{m}_{\sigma\text{-linked}} \le \mathfrak{p}$, so there is an infinite I_ξ such that $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$; continue.

At the end of the induction, setting $I=I_\kappa$, we have $\sum_{n=0}^\infty\delta(I,A_{\xi n})=1$ for every n.

(e) For $k\in I$ let ν_k be the uniform probability measure on L_k. Let $\mathcal{F}$ be any non-principal ultrafilter on I, and consider the probability algebra reduced product $(\mathfrak{A},\bar\mu)=\prod_{k\in I}(\mathcal{P}L_k,\nu_k)|\mathcal{F})$ (Fremlin 02, §328[2]). For $A\subseteq\mathbb{N}$, set $\theta(A)=\langle A\cap L_k\rangle^\bullet_{k\in I}\in\mathfrak{A}$ (see the construction in Fremlin 02, 328A). Then θ is a surjective Boolean homomorphism, and

$$\bar\mu\theta(A)=\lim_{k\to\mathcal{F}}\nu_k(A\cap L_k)=\delta(I,A)$$

whenever $\delta(I,A)$ is defined. For $\xi<\kappa$ and $n\in\mathbb{N}$, set $a_{n\xi}=\theta(A_{\xi n})$. Then $\langle a_{\xi n}\rangle_{n\in\mathbb{N}}$ is disjoint and $\sum_{n=0}^\infty\bar\mu a_{\xi n}=1$. Set $\tilde a=\theta(\tilde A)$, so that $\bar\mu\tilde a\ge\epsilon$ and $\tilde a\ne 0$.

(f)(i) There is a family $\langle n_\xi\rangle_{\xi<\kappa}$ in $\mathbb{N}$ such that $\{\tilde a\}\cup\{a_{\xi n_\xi}:\xi<\kappa\}$ is centered in $\mathfrak{A}$. **P** For each $\xi<\kappa$, the set $E_\xi=\{e:e\in\mathfrak{A}^+,\ e\subseteq a_{\xi n}$ for some $n\}$ is coinitial with $\mathfrak{A}^+$. Now the downwards Martin number of $\mathfrak{A}^+$ is $\mathfrak{m}(\mathfrak{A})\ge\mathfrak{m}_{\sigma\text{-linked}}$ (Fremlin 08?, 524N), so there must be a downwards-directed set $R\subseteq\mathfrak{A}^+$ containing $\tilde a$ and meeting every E_ξ (Fremlin 08?, 517B, inverted). In this case, there is for each $\xi<\kappa$ a unique n_ξ such that $a_{\xi n_\xi}$ includes some member of R, and $\{\tilde a\}\cup\{a_{\xi n_\xi}:\xi<\kappa\}$ is centered. **Q**

(ii) For each $\xi<\kappa$, let $d_\xi\in D_\xi$ be such that $A^\bullet_{\xi n_\xi}\subseteq d_\xi$. Then $\{\tilde d\}\cup\{d_\xi:\xi<\kappa\}$ is centered in $\mathfrak{Z}$. **P** If K is a finite subset of κ, set $A=\tilde A\cap\bigcap_{\xi\in K}A_{\xi n_\xi}$. Then $\theta(A)=\tilde a\cap\inf_{\xi\in K}a_{\xi n_\xi}$ is non-zero, so

$$0<\bar\mu\theta(A)=\lim_{k\to\mathcal{F}}\frac{\#(A\cap L_k)}{\#(L_k)}\le\delta^*(I,A)$$

and $d^*(A)>0$, that is,

$$0\ne A^\bullet=\tilde d\cap\inf_{\xi\in K}A^\bullet_{\xi n_\xi}\subseteq\tilde d\cap\inf_{\xi\in K}d_\xi.\ \mathbf{Q}$$

(g) Thus we have a linked set $\{\tilde d\}\cup\{d_\xi:\xi<\kappa\}$ in $\mathfrak{Z}$ containing $\tilde d$ and meeting every D_ξ. As $\tilde d$ and $\langle D_\xi\rangle_{\xi<\kappa}$ are arbitrary, $\mathfrak{m}(\mathfrak{Z})\ge\mathfrak{m}_{\sigma\text{-linked}}$.

3E Proposition The Freese-Nation number $\mathrm{FN}(\mathfrak{Z})$ of $\mathfrak{Z}$ is at least $\mathrm{FN}(\mathcal{P}\mathbb{N})$ and at most $\max(\mathrm{FN}^*(\mathcal{P}\mathbb{N}),(\mathrm{cf}\mathcal{N})^+)$, where $\mathrm{FN}^*(\mathcal{P}\mathbb{N})$ is the regular Freese-Nation number of $\mathcal{P}\mathbb{N}$, and $\mathcal{N}$ is the Lebesgue null ideal.

proof (a) As $\mathfrak{B}_\mathfrak{c}$ is regularly embedded in $\mathfrak{Z}$, $\mathrm{FN}(\mathfrak{Z})\ge\mathrm{FN}(\mathfrak{B}_\mathfrak{c})=\mathrm{FN}(\mathcal{P}\mathbb{N})$ (Fremlin 08?, 518C and 524N).

(b) Set $\kappa=\max(\mathrm{FN}^*(\mathcal{P}\mathbb{N}),(\mathrm{cf}\mathcal{N})^+)$. Recall that $\mathrm{cf}\mathcal{Z}=\mathrm{cf}\mathcal{N}$ (Fremlin 08?, 526Ha); let $\mathcal{A}\subseteq\mathcal{Z}$ be a cofinal family of cardinal $\mathrm{cf}\mathcal{N}<\kappa$, containing $\emptyset$. Let $f:\mathcal{P}\mathbb{N}\to[\mathcal{P}\mathbb{N}]^{<\kappa}$ be a Freese-Nation function. For each $a\in\mathfrak{Z}$, let $I_a\subseteq\mathbb{N}$ be such that $I_a^\bullet=a$, and set

$$g(a)=\bigcup_{A\in\mathcal{A}}\{I^\bullet:I\in f(I_a\cup A)\}$$

Because κ is regular, $\#(g(a))<\kappa$. If $a,b\in\mathfrak{Z}$ and $a\subseteq b$, there is an $A\in\mathcal{A}$ such that $I_a\subseteq I_b\cup A$. Now there is an $I\in f(I_a)\cap f(I_b\cup A)$ such that $I_a\subseteq I\subseteq I_b\cup A$, and $I^\bullet\in g(a)\cap g(b)$, $a\subseteq I^\bullet\subseteq b$. Thus $g:\mathfrak{Z}\to[\mathfrak{Z}]^{<\kappa}$ is a Freese-Nation function, and $\mathrm{FN}(\mathfrak{Z})\le\kappa$.

3F Theorem The Dedekind completion $\widehat{\mathfrak{Z}}$ of $\mathfrak{Z}$ and the Dedekind completion $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})\widehat\otimes\mathfrak{B}_\mathfrak{c}$ of the free product $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})\otimes\mathfrak{B}_\mathfrak{c}$ are isomorphic.

proof Farah 06, Theorem 1.3, or Fremlin 08?, 556S.

3G Lemma Every member of $(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})\widehat\otimes\mathfrak{B}_\mathfrak{c}$ is expressible as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ and $b_i\in\mathfrak{B}_\mathfrak{c}$ for each $i\in I$.

proof

3H Proposition $\tau(\widehat{\mathfrak{Z}})\ge\mathrm{wdistr}(\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega})$.

proof

[2] Later editions only; see `http://www.essex.ac.uk/maths/people/fremlin/cont32.htm`.

4 Homogeneity

4A Corollary $\widehat{\mathfrak{Z}}$ is homogeneous and $\operatorname{Aut}(\widehat{\mathfrak{Z}})$ is simple.

proof $\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ and $\mathfrak{B}_{\mathfrak{c}}$ are homogeneous, so their free product also is (Fremlin 02, 316Q). The Dedekind completion of a homogeneous Boolean algebra is homogeneous (Fremlin 02, 316P). So $\widehat{\mathfrak{Z}}$ is homogeneous, by Theorem 3F. Now Fremlin 02, 382S tells us that $\operatorname{Aut}\mathfrak{Z}$ is simple.

4B Theorem [CH] $\mathfrak{Z}$ is homogeneous and $\operatorname{Aut}\mathfrak{Z}$ is simple.

proof If the continuum hypothesis is true, $\mathfrak{Z}$ is homogeneous (Just & Krawczyk 84, Farah 03, 8.2). Once again Fremlin 02, 382S tells us that $\operatorname{Aut}\mathfrak{Z}$ is simple.

4C Theorem [OCA + $\mathfrak{m} > \omega_1$] (a) $\operatorname{Aut}\mathfrak{Z}$, regarded as a subgroup of $\operatorname{Aut}\widehat{\mathfrak{Z}}$, is not ergodic.
(b) $\mathfrak{Z}$ is not homogeneous.
(c) $\operatorname{Aut}\mathfrak{Z}$ is not simple.

proof (a)(i) In the terminology of Farah 00 or Fremlin n05, $\mathcal{Z} = \operatorname{Exh}(\nu)$ where ν is the entirely non-pathological lower semi-continuous submeasure on $\mathbb{N}$ defined by setting $\nu a = \sup_{n\ge 1}\frac{1}{n}\#(a\cap n)$ for $a \subseteq \mathbb{N}$. So by Farah 00, 3.4.1-3.4.2 or Fremlin n05, 5H, every Boolean automorphism $\pi : \mathfrak{Z} \to \mathfrak{Z}$ is representable by a bijective function $h : A \to B$, where A, $B \subseteq \mathbb{N}$ are cofinite, in the sense that $\pi(I^\bullet) = (h^{-1}[I])^\bullet$ for every $I \subseteq \mathbb{N}$.

(ii) For $n \in \mathbb{N}$ set $M_n = \{i : 2^{n^2} \le i < 2^{n^2+1}\}$; note that $\lim_{n\to\infty}\#(M_n)/\sum_{m<n}\#(M_m) = 0$. Set $I = \bigcup_{n\in\mathbb{N}} M_{2n}$, $J = \bigcup_{n\in\mathbb{N}} M_{2n+1}$, $a = I^\bullet \in \mathfrak{Z}$, $b = J^\bullet$. Then $d^*(I)$, $d^*(J)$ are both $\frac{1}{2}$ so a and b are non-zero. If $\pi : \mathfrak{Z} \to \mathfrak{Z}$ is a Boolean automorphism such that $b \cap \pi a \ne 0$, let $h : A \to B$ represent π in the sense of (a) above. Then $b \cap \pi a = (J \cap h^{-1}[I])^\bullet$.

Set

$$I_0 = \{i : i \in I \cap h[J],\ h^{-1}(i) < i\}, \qquad J_0 = \{i : i \in J \cap h^{-1}[I],\ h(i) < i\}.$$

If $n \in \mathbb{N}$ and $i \in I_0 \cap M_{2n}$, then $h^{-1}(i) \in \bigcup_{m<n} M_{2m+1}$ so $\#(I_0 \cap M_{2n}) \le \sum_{m<2n}\#(M_m)$. Accordingly

$$d^*(I_0) \le \limsup_{n\to\infty}\frac{1}{\#(M_{2n})}\#(B \cap h[J] \cap M_{2n}) = 0.$$

Similarly, $d^*(J_0) = 0$. But now observe that

$$(J \setminus J_0) \cap h^{-1}[I \setminus I_0] = \emptyset$$

so

$$b \cap \pi a = (J \setminus J_0)^\bullet \cap (h^{-1}[I \setminus I_0])^\bullet = 0.$$

(iii) So if we take d to be the supremum in $\widehat{\mathfrak{Z}}$ of $\{\pi a : \pi \in \operatorname{Aut}\mathfrak{Z}\}$, we shall have $\hat\pi d = d$ for every $\pi \in \operatorname{Aut}\mathfrak{Z}$, writing $\hat\pi \in \operatorname{Aut}\widehat{\mathfrak{Z}}$ for the automorphism of $\widehat{\mathfrak{Z}}$ extending π; while d is neither 0 nor 1, since $a \subseteq d$ and $d \cap b = 0$. Accordingly $\operatorname{Aut}\mathfrak{Z}$ does not act ergodically on $\widehat{\mathfrak{Z}}$.

(b) Taking a and b from (a), at least one of the principal ideals $\mathfrak{Z}_a$, $\mathfrak{Z}_b$, $\mathfrak{Z}_{1\setminus a}$ and $\mathfrak{Z}_{1\setminus b}$ is not isomorphic to $\mathfrak{Z}$.

(c) Taking a from (a), let $I \lhd \mathfrak{Z}$ be the ideal generated by $\{\pi a : \pi \in \operatorname{Aut}\mathfrak{Z}\}$. Then I is a proper ideal. Let H be the set of those $\pi \in \operatorname{Aut}\mathfrak{Z}$ supported by members of I (definition: Fremlin 02, 381B). Then $H \lhd \operatorname{Aut}\mathfrak{Z}$, by Fremlin 02, 381Eb, 381Eh and 381Ej; and H is non-trivial by Proposition 2H.

4D Proposition Let D be the set of those $d \in \mathfrak{Z}^+$ such that there are regular embeddings both from $\mathfrak{Z}$ to the principal ideal $\mathfrak{Z}_d$ and from $\mathfrak{Z}_d$ to $\mathfrak{Z}$. Then D is order-dense in $\mathfrak{Z}$.

proof (a) Let $a \in \mathfrak{Z}^+$; express a as $A^\bullet$ where $A \in \mathcal{P}\mathbb{N}\setminus\mathcal{Z}$. Set $I_n = \{i : 2^n \le i < 2^{n+1}\}$ for $n \in \mathbb{N}$. Then $\limsup_{n\to\infty} 2^{-n}\#(A \cap I_n) > 0$. Let $\epsilon > 0$ be such that $\{n : \#(A \cap I_n) \ge 2^n\epsilon\}$ is infinite; let $\langle k(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that

$$\#(A\cap I_{k(n)})\ge 2^{k(n)}\epsilon\ge 2^{n+1}$$

for every n. For each $n\in\mathbb{N}$, let $\langle A_{ni}\rangle_{i<2^n}$ be a disjoint family of subsets of $A\cap I_{k(n)}$ such that $\#(A_{ni})=\lfloor 2^{k(n)-n}\epsilon\rfloor\ge 2^{k(n)-n-1}\epsilon$ for each $i<2^n$. Set $E=\bigcup_{n\in\mathbb{N}}\bigcup_{i<2^n}A_{ni}$ and $e=E^\bullet\subseteq a$. Then $e\ne 0$ in $\mathfrak{Z}$.

(b) Define $\phi:\mathcal{P}\mathbb{N}\to\mathcal{P}E$ by setting

$$\phi I=\bigcup_{n\in\mathbb{N}}\bigcup\{A_{ni}: i<2^n,\ 2^n+i\in I\}$$

for $I\subseteq\mathbb{N}$. Then ϕ is a Boolean homomorphism, and $\phi I\in\mathcal{Z}$ whenever $I\in\mathcal{Z}$. **P**

$$\begin{aligned}2^{-k(n)}\#(I_{k(n)}\cap\phi I)&=2^{-k(n)}\sum_{i\in I\cap I_n}\#(A_{n,i-2^n})\\&\le 2^{-k(n)}\#(I\cap I_n)\cdot 2^{k(n)-n}\epsilon=2^{-n}\epsilon\#(I\cap I_n)\to 0\end{aligned}$$

as $n\to\infty$; and of course $\#(I_j\cap\phi I)=0$ if $j\ne k(n)$ for every n. **Q** So we have a Boolean homomorphism $\pi:\mathfrak{Z}\to\mathfrak{Z}_e$ defined by setting $\pi I^\bullet=(\phi I)^\bullet$ for every $I\subseteq\mathbb{N}$.

π is injective. **P** If $I\in\mathcal{P}\mathbb{N}\setminus\mathcal{Z}$, then the same formulae give us

$$\begin{aligned}\limsup_{n\to\infty}2^{-k(n)}\#(I_{k(n)}\cap\phi I)&=\limsup_{n\to\infty}2^{-k(n)}\sum_{i\in I\cap I_n}\#(A_{n,i-2^n})\\&\ge\limsup_{n\to\infty}2^{-k(n)}\#(I\cap I_n)\cdot 2^{k(n)-n-1}\epsilon\\&=2\epsilon\limsup_{n\to\infty}2^{-n}\#(I\cap I_n)>0\end{aligned}$$

and $\phi I\notin\mathcal{Z}$. **Q**

π is a regular embedding. **P?** Otherwise, there is a partition C of unity in $\mathfrak{Z}$ such that d is not the supremum of $\pi[C]$ in $\mathfrak{Z}_e$. Let $B\subseteq E$ be such that $B\notin\mathcal{Z}$ and $B^\bullet\cap\pi c=0$ for every $c\in C$; let $\delta>0$ be such that $L=\{n:\#(B\cap I_{k(n)})\ge 2^{k(n)}\delta\}$ is infinite. For each $n\in L$, set $K_n=\{i: i<2^n,\ \#(B\cap A_{ni})\ge 2^{k(n)-n-1}\delta\}$; then $\#(K_n)\ge 2^n\cdot\dfrac{\delta}{4\epsilon}$. So $J=\bigcup_{n\in\mathbb{N}}(2^n+K_n)\notin\mathcal{Z}$, and there is a $c\in C$ such that $J^\bullet\cap c\ne 0$. Let $\tilde J\subseteq J$ be such that $\tilde J^\bullet=J^\bullet\cap c$. Then there is an $\eta>0$ such that $\tilde L=\{n:\#(\tilde J\cap I_n)\ge 2^n\eta\}$ is infinite. If $n\in L$, then

$$\#(I_{k(n)}\cap B\cap\phi\tilde J)\ge 2^{k(n)-n-1}\delta\cdot\#(\tilde J\cap I_n)\ge 2^{k(n)-1}\delta\eta.$$

But this means that $B\cap\phi\tilde J\notin\mathcal{Z}$ and $B^\bullet\cap c\ne 0$. **XQ**

(c)(i) Set $m(n)=\#(E\cap I_{k(n)})\ge 2^{k(n)-1}\epsilon$ for each n, and let $\langle n_j\rangle_{j\in\mathbb{N}}$ be an unbounded monotonic slowly increasing sequence in $\mathbb{N}$ such that $n_0=0$, $n_{j+1}\le n_j+1$ for every j and

$$\lim_{j\to\infty}\frac{m(n_j)}{\sum_{i<j}m(n_i)}=\lim_{n\to\infty}\frac{\sum_{n_i<n}n_i}{\sum_{n_i=n}n_i}=0.$$

Let $\langle M_j\rangle_{j\in\mathbb{N}}$ be the partition of $\mathbb{N}$ such that $\#(M_j)=m(n_j)$ and $\max M_j<\min M_{j+1}$ for each j. For each j, let $f_j:M_j\to E\cap I_{k(n_j)}$ be a bijection, and set $f=\bigcup_{j\in\mathbb{N}}f_j$, so that $f:\mathbb{N}\to E$ is a surjection.

For $n\in\mathbb{N}$, set $\tilde M_n=\bigcup_{n_j=n}M_j$, $r_n=\min\tilde M_n$; then $\#(\tilde M_n)=r_{n+1}-r_n$ and $\lim_{n\to\infty}\dfrac{r_n}{r_{n+1}}=0$. Note also that $2^{k(n)}\le\dfrac{2m(n)}{\epsilon}$ for every n, while $\lim_{n\to\infty}\dfrac{m(n)}{r_n}=0$, so $\lim_{n\to\infty}\dfrac{2^{k(n)}}{r_n}=0$.

(ii) If $I\subseteq E$ and $I\in\mathcal{Z}$, then $f^{-1}[I]\in\mathcal{Z}$. **P**

$$\begin{aligned}\limsup_{j\to\infty}\frac{\#(f^{-1}[I])}{\#(M_j)}&=\limsup_{j\to\infty}\frac{\#(I\cap I_{k(n_j)})}{m(n_j)}\\&\le\limsup_{j\to\infty}2^{-k(n_j)}\#(I\cap I_{k(n_j)})\cdot\limsup_{j\to\infty}\frac{2^{k(n_j)}}{m(n_j)}\le 0\cdot\frac{2}{\epsilon}=0.\end{aligned}$$

As $\lim_{j\to\infty}\dfrac{\#(M_j)}{\min M_j}=0$, $f^{-1}[I]\in\mathcal{Z}$. **Q** So we have a Boolean homomorphism $\theta:\mathfrak{Z}_e\to\mathfrak{Z}$ defined by saying that $\theta(I^\bullet)=(f^{-1}[I])^\bullet$ for every $I\subseteq E$.

(iii) If $I \subseteq E$ and $f^{-1}[I] \in \mathcal{Z}$, then $I \in \mathcal{Z}$. **P**

$$\begin{aligned}\limsup_{n\to\infty} 2^{-k(n)}\#(I \cap I_{k(n)}) &= \limsup_{j\to\infty} 2^{-k(n_j)}\#(I \cap I_{k(n_j)}) \\ &\le \limsup_{j\to\infty} \frac{\#(I\cap I_{k(n_j)})}{m(n_j)} = \limsup_{n\to\infty} \frac{\#(f^{-1}[I]\cap\tilde M_n)}{\#(\tilde M_n)} \\ &\le \limsup_{n\to\infty} \frac{\#(f^{-1}[I]\cap r_{n+1})}{r_{n+1}} \cdot \frac{r_{n+1}}{r_{n+1}-r_n} \\ &\le d^*(f^{-1}[I]) \cdot 1 = 0. \ \mathbf{Q}\end{aligned}$$

So θ embeds $\mathfrak{Z}_e$ in $\mathfrak{Z}$.

(iv) Let $\mathcal{J}$ be the family of non-empty sets J of the form $\bigcup_{i\in K} M_i$ where $n_i = n_j$ for all i, $j \in K$. We need to know that if $B \subseteq \mathbb{N}$, then there are infinitely many $J \in \mathcal{J}$ such that $\#(B\cap J) \ge \frac{1}{5}d^*(B)\cdot(1+\max J)$. **P** Set $\delta = \frac{1}{5}d^*(B)$. We can suppose that $\delta > 0$. Take any $n^* \in \mathbb{N}$ such that $r_n \le \delta r_{n+1}$, $2^{k(n)} \le \delta r_n$ for every $n \ge n^*$. Then there is an $m \ge r_{n^*+1}$ such that $\#(B \cap m) \ge 4\delta m$. Let $n > n^*$ be such that $m \in \tilde M_n$.

case 1 If $\#(B \cap r_n) \ge \frac{1}{2}\#(B \cap m)$, set $J = \tilde M_{n-1}$. Then

$$\#(B \cap J) \ge 2\delta m - r_{n-1} \ge 2\delta r_n - r_{n-1} \ge \delta r_n = \delta(1 + \max J).$$

case 2 Otherwise, set $J = \bigcup\{M_j : n_j = n,\ \max M_j < m\}$. Then

$$\#(B \cap J) \ge \#(B \cap m) - \#(B \cap r_n) - 2^{k(n)} \ge 2\delta m - 2^{k(n)} \ge \delta m \ge \delta(1 + \max J).$$

As n^* is arbitrary, at least one of these happens infinitely often. **Q**

(v) θ is a regular embedding. **P?** Otherwise, there are a partition C of unity in $\mathfrak{Z}_e$ and a non-zero $b \in \mathfrak{Z}$ such that $b \cap \theta c = 0$ for every $c \in C$. Express b as $B^\bullet$ where $B \in \mathcal{P}\mathbb{N} \setminus \mathcal{Z}$. By (iv), there are a $\delta > 0$ and an infinite sequence $\langle J_l\rangle_{l\in\mathbb{N}}$ of distinct members of $\mathcal{J}$ such that $\#(B \cap J_l) \ge \delta(1 + \max J_l)$. For each l there is an $p(l)$ such that $J_l \subseteq \tilde M_{p(l)}$; taking a subsequence if necessary, we may suppose that $\langle p(l)\rangle_{l\in\mathbb{N}}$ is strictly increasing. For each l, let K_l be such that $J_l = \bigcup_{j\in K_l} M_j$; we have $n_j = p(l)$ for each $j \in K_l$; set $s_l = \#(K_l)$, so that $\#(J_l) = 2^{k(p(l))}s_l$ and $f[J_l] \subseteq E \cap I_{k(p(l))}$.

For $l \in \mathbb{N}$, set

$$V_l = \{i : \#(\{j : j \in B \cap J_l,\ f(j) = i\}) \ge \frac{1}{2}\delta s_l\} \subseteq E \cap I_{k(p(l))}.$$

Since f is injective on M_j for each $j \in K_l$,

$$2^{k(p(l))}s_l\delta = \delta\#(J_l) \le \delta(1 + \max J_l) \le \#(B \cap J_l) \le s_l\#(V_l) + 2^{k(p(l))-1}\delta s_l$$

and $\#(V_l) \ge 2^{k(p(l))}\delta$. Accordingly $V = \bigcup_{l\in\mathbb{N}} V_l$ does not belong to $\mathcal{Z}$ and there is a $c \in C$ such that $V^\bullet \cap c \ne 0$. Let $\tilde V \subseteq V$ be such that $\tilde V^\bullet = V^\bullet \cap c$. Then there is an $\eta > 0$ such that $\tilde L = \{l : \#(\tilde V \cap I_{k(p(l))}) \ge 2^{k(p(l))}\eta\}$ is infinite. For $l \in \tilde L$,

$$\begin{aligned}\#(B \cap f^{-1}[\tilde V] \cap (1 + \max J_l)) &\ge \#(B \cap f^{-1}[\tilde V] \cap J_l) \ = \#(B \cap f^{-1}[\tilde V \cap I_{k(p(l))}]) \\ &\ge \frac{1}{2}\delta s_l\#(\tilde V \cap I_{k(p(l))}) \ge \frac{1}{2}\delta s_l \cdot 2^{k(p(l))}\eta \\ &\ge \frac{1}{2}\delta\eta\#(J_l) \ge \frac{1}{2}\delta\eta\#(B \cap J_l) \ \ge \frac{1}{2}\delta^2\eta(1 + \max J_l).\end{aligned}$$

But this means that $B \cap f^{-1}[\tilde V] \notin \mathcal{Z}$ and $b \cap \theta c \ne 0$; which is impossible. **XQ**

(vi) Thus we have our regular embeddings in both directions, and $e \in D$, while $0 \ne e \subseteq a$. As a is arbitrary, D is order-dense, as claimed.

5 Problems

5A In ZFC, can we find a non-decreasing family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{Z}$ such that $\sup_{\xi<\kappa} \bar d^*(a_\xi) < \inf_{b\in B} \bar d^*(b)$, where B is the set of upper bounds of $\{a_\xi : \xi < \kappa\}$?

Can it be done with $\kappa < \mathfrak{m}$?

[By 1Cb, this cannot be done with $\kappa = \omega$. Subject to CH, 1F(b-ii) gives such an example with $\kappa = \mathfrak{c} = \omega_1$.]

5B In ZFC, can we find a non-increasing family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathfrak{Z}$ such that $\inf_{\xi<\kappa} \bar{d}^*(a_\xi) > \sup_{b\in B} \bar{d}^*(b)$, where B is the set of lower bounds of $\{a_\xi : \xi < \kappa\}$?

[By 1Ca, this cannot be done with $\kappa < \mathfrak{p}$. Subject to CH, 1F(b-i) gives such an example with $\kappa = \mathfrak{c} = \omega_1$.]

5C Can $\tau(\mathfrak{Z})$ be less than $\mathfrak{c}$?

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