# SPACES OF FINITE LENGTH

D. H. FREMLIN

[Received 18 December 1990]

ABSTRACT

I discuss the topological properties of metric spaces of finite one-dimensional Hausdorff measure.

## *Introduction*

Let $(X, \rho)$ be a metric space. Define $\mu_\rho^* = \mu_{1,\rho}^*$: $\mathcal{P}X \to [0, \infty]$ by writing

$$\mu_\rho^*(A) = \sup_{\delta>0} \inf \left\{ \sum_{i\in\mathbb{N}} \operatorname{diam}_\rho(A_i) : A \subseteq \bigcup_{i\in\mathbb{N}} A_i \subseteq X,\ \operatorname{diam}_\rho(A_i) \leqslant \delta\ \forall i \in \mathbb{N} \right\}$$

for each $A \subseteq X$, the one-dimensional Hausdorff outer measure on $X$. If $\mu_\rho^*(X) < \infty$, I will say that $X$ is *of finite length*. In this paper I seek to investigate the topological properties of metric spaces of finite length, concentrating on connected spaces. The basis of the work is the fact that a space $X$ of finite length is a 'finite cut space', that is, its topology has a base consisting of sets with finite boundary (corresponding to the 'regular curves' of **[22]**). (See 1C below.) This is already enough to prove some remarkable properties; for instance, if $X$ is connected then it is locally connected (2A–2B), and if it is connected and a Borel set in its metric completion then it is path-connected (3G). I then set out to develop a structure theory for spaces of finite length, showing how the measure theory associated with the outer measure $\mu_\rho^*$ connects with the topology (§ 4). A striking result from this part of the theory is in 4I: if $C \subseteq X$ is connected and $\mu_\rho^*(C)$ is finite, then $C$ is actually $\mu_\rho$-measurable and $\mu_\rho(\bar{C} \setminus C) = 0$. In § 5, I show that metric spaces of finite length can be embedded homeomorphically in 'good' spaces of finite length, indeed, as subspaces of compact connected spaces of finite length in $\mathbb{R}^3$ (5H). I end the paper with topological characterizations of finite length in general separable metric spaces which extend some of those given for continua in **[6]**.

## 1. *Basic definitions and results*

I list the fundamental known facts on which I shall rely.

1A. *Length.* Let $(X, \rho)$ be a metric space, and define $\mu_\rho^*$ by the formula in the Introduction, interpreting $\inf \emptyset$ as $\infty$, so that $\mu_\rho^*(A) = \infty$ if $A$ is non-separable.

(a) $\mu_\rho^*$ is a metric outer measure **[7**, p. 7**]**. The associated measure $\mu_\rho$ defined by Carathéodory's method **[7**, p. 3**]** is defined on a $\sigma$-algebra including the $\sigma$-algebra of Borel sets **[7**, p. 6**]**. For any $A \subseteq X$ there is a $G_\delta$ set $E \supseteq A$ such that $\mu_\rho(E) = \mu_\rho^*(A)$ **[7**, p. 8**]**, so that $\mu_\rho$ is outer regular for the $G_\delta$ sets. If $E \subseteq X$ is $\mu_\rho$-measurable and $\mu_\rho(E) < \infty$ then $\mu_\rho(E) = \sup\{\mu_\rho(F) : F \subseteq E,\ F \text{ is closed in } X\}$ **[7**, p. 8**]**.

---

1991 *Mathematics Subject Classification*: 28A75, 54F50.

*Proc. London Math. Soc.* (3) 64 (1992) 449–486.

(b) If $Y \subseteq X$ is any set and $\sigma = \rho \restriction Y \times Y$ is the induced metric on $Y$, then $\mu_\sigma^*$ is the restriction of $\mu_\rho^*$ to $\mathcal{P}Y$.

(c) If $X = \mathbb{R}$ and $\rho$ is the usual metric of $\mathbb{R}$ then $\mu_\rho^*$ is Lebesgue outer measure on $\mathbb{R}$ and $\mu_\rho$ is Lebesgue measure [**7**, p. 12].

(d) If $\Gamma \subseteq X$ is an arc (i.e., a homeomorphic image of $[0, 1]$), then $\mu_\rho(\Gamma)$ is just the length of $\Gamma$ [**7**, p. 29].

(e) If $(Y, \sigma)$ is another metric space and $f: X \to Y$ is Lipschitz-1 (that is, $\sigma(f(x), f(x')) \leq \rho(x, x')$ for all $x, x' \in X$), then $\mu_\sigma^*(f[A]) \leq \mu_\rho^*(A)$ for every $A \subseteq X$ [**7**, p. 10].

(f) If $(Y, \sigma)$ is another metric space and $f: X \to Y$ is Lipschitz-1, then

$$\mu_\rho(X) \geq \int^* \#^*(f^{-1}[\{y\}])\mu_\sigma(dy).$$

(See [**6**, Theorem 1; **3**, Theorem; **8**, p. 176]. In this formula, interpret '$\#^*(I)$' as $\#(I) \in \mathbb{N}$ if $I$ is finite, and as $\infty$ if $I$ is infinite, and '$\int^* h(y)\mu_\sigma(dy)$' as the infimum of the integrals $\int g(y)\mu_\sigma(dy)$ as $g$ runs over the $\mu_\sigma$-measurable functions from $Y$ to $[0, \infty]$ such that $g(y) \geq h(y)$ for every $y \in Y$.)

1B. *Finite cut spaces.* Let $(X, \mathfrak{T})$ be a topological space. For $A \subseteq X$ write $\partial A$ for $\bar{A}\setminus\operatorname{int} A$, the boundary of $A$. I will say that $(X, \mathfrak{T})$ is a *finite cut space* if

$$\mathcal{G} = \{G: G \in \mathfrak{T}, \partial G \text{ is finite}\}$$

is a base for $\mathfrak{T}$. (In this paper I will reserve the symbol $\mathcal{G}$ for this context.) Note that if $G$ and $H$ belong to $\mathcal{G}$ so do $G \cap H$, $G \cup H$ and $X\setminus\bar{G}$ (because the sets with finite boundary form a subalgebra of $\mathcal{P}X$ containing the closures and interiors of its members).

Compact metrizable connected finite cut spaces are treated in [**22**], where they are called 'regular continua'.

1C. The essential link between 1A and 1B is the following long-known result.

PROPOSITION. *A metric space of finite length is a finite cut space.*

*Proof.* [**6**, § 3, Corollary].

## 2. *Finite cut spaces*

In this section I give an account of the elementary properties of finite cut spaces. I am concerned primarily with separable metric spaces; but where an argument applies more generally I allow weaker hypotheses. From the point of view of this paper, the most important fact is 2A: connected Hausdorff finite cut spaces are locally connected. Most of the other results amount to saying that finite cut spaces are well-behaved in various ways; thus 2E–2G show that components are regularly arranged, while 2H–2J do the same for path-components. The technical result 2F(a) will be very useful later.

2A. THEOREM. *A connected Hausdorff finite cut space is locally connected.*

*Proof.* Let $X$ be a connected Hausdorff finite cut space, $x$ a point of $X$, $U$ a neighbourhood of $x$, $C$ the component of $U$ containing $x$; I have to show that $C$ is a neighbourhood of $x$. If $U=X$ then $C=X$ and the result is trivial. Otherwise, let $G_0$ be an open set with finite boundary containing $x$ and included in $U$. Because $X$ is Hausdorff, there is a neighbourhood $V$ of $x$, included in $G_0$, with $\bar{V}\cap\partial G_0=\emptyset$. Let $G_1$ be an open set with finite boundary containing $x$ and included in $V$; then $\bar{G}_1\subseteq U$.

Let $\mathscr{E}$ be the algebra of relatively open-and-closed subsets of $\bar{G}_1$. If $E, E'\in\mathscr{E}$ and $E\cap\partial G_1=E'\cap\partial G_1$ then $E\Delta E'\subseteq G_1$ so $E\Delta E'$ is both open and closed in $X$; because $X$ is connected and $G_1\subseteq U\neq X$, $E\Delta E'=\emptyset$ and $E=E'$.

Because $\partial G_1$ is finite, $\mathscr{E}$ must also be finite. Its atoms therefore constitute a finite partition of $\bar{G}_1$ into closed connected sets. Let $C_0$ be the atom of $\mathscr{E}$ containing $x$; then $C_0\cap G_1$ is open, so $C_0$ must be a neighbourhood of $x$; also $C_0\subseteq C$, so $C$ is also a neighbourhood of $x$, as required.

Remark. For metric spaces, this result is given in [**14b**, § 51.IV, Theorem 1]

2B. Corollary. *A connected metric space of finite length is locally connected.*

2C. It is convenient to collect here some very elementary facts about finite cut spaces.

Lemma. *Let $X$ be a finite cut space.*

(a) *Any subspace of $X$, with the subspace topology, is a finite cut space.*

(b) *If $X$ is Hausdorff, it is regular.*

(c) *If $X$ is separable and metrizable then it has at most countably many non-singleton components.*

*Proof.* (a) If $Y\subseteq X$ and $G\subseteq X$, then $\partial_Y(G\cap Y)$, the boundary of $G\cap Y$ taken in $Y$, is a subset of $\partial_X G$.

(b) Take $x\in X$ and a neighbourhood $U$ of $x$. Then there is an open set $G$ with finite boundary such that $x\in G\subseteq U$. Now, because $X$ is Hausdorff, there is a neighbourhood $V$ of $x$ such that $V\subseteq G$ and $\bar{V}\cap\partial G=\emptyset$. So $\bar{V}\subseteq U$.

(c) Let $\mathscr{H}$ be a countable base for the topology consisting of open sets with finite boundary, and let $D$ be the countable set $\bigcup_{H\in\mathscr{H}}\partial H$; then every non-singleton component of $X$ must meet $D$.

2D. Corollary. *If $X$ is a connected Hausdorff finite cut space then*

$$\mathscr{G}'=\{G\colon\ G\subseteq X \text{ is open and connected, } \partial G \text{ is finite}\}$$

*is a base for the topology of $X$.*

*Proof.* If $x\in X$ and $U$ is a neighbourhood of $x$, take an open set $H$ with finite boundary such that $x\in H\subseteq\bar{H}\subseteq U$ (using 2C(b)); let $G$ be the component of $H$ containing $x$. By 2A, $G$ is open. Moreover, $G$ is relatively closed in $H$, so $\partial G\subseteq\partial H$ is finite.

2E. Theorem. *Let $X$ be a Lindelöf Hausdorff finite cut space. If $C$ is a component of $X$ and $F$ is a closed subset of $X$ disjoint from $C$, there is an open-and-closed subset of $X$ separating $C$ from $F$.*

*Proof.* Let us write $C^*$ for the intersection of all the open-and-closed subsets of $X$ including $C$. Let $\mathcal{G}$ be the family of open sets with finite boundaries.

(a) If $G \in \mathcal{G}$ and $\bar{G} \cap C^* = \emptyset$, there is an open-and-closed subset $W$ of $X$ separating $C^*$ from $\bar{G}$. For there is surely an open-and-closed set $W_0$ such that $C^* \subseteq W_0$ and $\partial G \cap W_0 = \emptyset$; now try $W = W_0 \backslash G = W_0 \backslash \bar{G}$.

(b) If $E$ is a closed subset of $X$ disjoint from $C^*$, then there is an open-and-closed subset $W$ of $X$ separating $C^*$ from $E$. For let $\mathcal{H}$ be

$$\{G:\ G \in \mathcal{G}, \text{ either } \bar{G} \cap C^* = \emptyset \text{ or } \bar{G} \cap E = \emptyset\}.$$

Because $X$ is regular (2C), $\mathcal{H}$ is an open cover of $X$; because $X$ is Lindelöf, there is a sequence $\langle H_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{H}$ covering $X$. Now we can find increasing sequences $\langle U_n \rangle_{n \in \mathbb{N}}$, $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that

$$(\bar{U}_n \cup C^*) \cap (\bar{V}_n \cup E) = \emptyset, \quad \partial U_n \subseteq C^*, \quad H_n \subseteq U_{n+1} \cup V_{n+1}$$

for every $n \in \mathbb{N}$. To see this, start with $U_0 = V_0 = \emptyset$. Given $U_n$ and $V_n$, use (a) of this proof to find an open-and-closed set $W_n \supseteq C^*$ such that $W_n \cap \bar{V}_n = \emptyset$ and $W_n \cap (\partial H_n \backslash C^*) = \emptyset$ and, if $\bar{H}_n \cap C^* = \emptyset$, then $W_n \cap \bar{H}_n = \emptyset$. Try

$$U_{n+1} = U_n \cup (H_n \cap W_n), \quad V_{n+1} = V_n \cup (H_n \backslash \bar{U}_{n+1}).$$

A straightforward calculation checks that this works.

On completing the induction, set $W = \bigcup_{n \in \mathbb{N}} U_n$; $W$ is open-and-closed because its complement is $\bigcup_{n \in \mathbb{N}} V_n$, and of course $C^* \subseteq W \subseteq X \backslash E$.

(c) It follows that $C^*$ is connected. For if $G$ and $H$ are open sets in $X$ with $C^* \subseteq G \cup H$, $C^* \cap G \cap H = \emptyset$, then $C^* \cap G = C^* \backslash H$ and $C^* \cap H$ are disjoint closed sets. Being regular and Lindelöf, $X$ is normal [**14**, § 14.I, Theorem 1], so there are disjoint open sets $G_1$, $H_1$ with $C^* \cap G \subseteq G_1$, $C^* \cap H \subseteq H_1$; so that $C^* \subseteq G_1 \cup H_1$. Applying (b) with $E = X \backslash (G_1 \cup H_1)$, we can find an open-and-closed $W$ with $C^* \subseteq W \subseteq G_1 \cup H_1$. In this case $W \cap G_1$ and $W \cap H_1$ are open-and-closed, so one includes $C$ and the other is disjoint from $C^*$. Thus one of $C^* \cap G$, $C^* \cap H$ is empty. As $G$, $H$ are arbitrary, $C^*$ is connected.

(d) Accordingly $C = C^*$ and (b) gives the result.

2F. Corollary. *Let $(X, \mathfrak{T})$ be a Lindelöf Hausdorff finite cut space. Write $\mathcal{G}$ for the family of open sets with finite boundary.*

(a) *The family*

$$\mathcal{G}^* = \{G:\ G \in \mathcal{G}, \partial G \text{ lies within one component of } \bar{G}\}$$

*is a base for $\mathfrak{T}$; in fact, every member of $\mathcal{G}$ has a finite partition into members of $\mathcal{G}^*$.*

(b) *If $E$, $F$ are closed subsets of $X$ such that no component of $X$ meets both $E$ and $F$, then $E$ and $F$ can be separated by an open-and-closed set.*

*Proof.* (a) Take any $G \in \mathcal{G}$. If $G = \emptyset$ then $G \in \mathcal{G}^*$ and we have finished. Otherwise, let $C_1, \ldots, C_k$ be the components of $\bar{G}$ meeting $\partial G$. Applying Theorem 2E to $\bar{G}$, we have a partition $W_1, \ldots, W_k$ of $\bar{G}$ into relatively open-and-closed sets such that $C_i \subseteq W_i$ for each $i$. Set $G_i = G \cap W_i$ for each $i$; then $G_1, \ldots, G_k$ is a partition of $G$ into open sets, and $\bar{G}_i = W_i$, so $\partial G_i = W_i \backslash G \subseteq C_i \subseteq \bar{G}_i$ for each $i$, and each $G_i$ belongs to $\mathcal{G}^*$.

(b) By the theorem, each component of $X$ can be separated by an open-and-closed set from at least one of $E$, $F$. Consequently,

$$\mathcal{W} = \{W\colon\ W \text{ is open and closed, } W \cap E = \emptyset \text{ or } W \cap F = \emptyset\}$$

is an open cover of $X$. Let $\langle W_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{W}$ covering $X$. Set $W'_n = W_n \setminus \bigcup_{i<n} W_i$ for each $n \in \mathbb{N}$; then $\langle W'_n \rangle_{n \in \mathbb{N}}$ is a disjoint cover of $X$ by members of $\mathcal{W}$. Set $W = \bigcup \{W'_n\colon\ n \in \mathbb{N},\ E \cap W'_n \neq \emptyset\}$; then $W$ is open-and-closed and $E \subseteq W \subseteq X \setminus F$.

Remark. Note that if $X$ is connected then $\bar{G}$ is connected for every $G \in \mathcal{G}^*$.

2G. Proposition. *Let $X$ be a Hausdorff finite cut space. Then*

$$R = \{(x, y)\colon\ x,\ y \text{ belong to the same component of } X\}$$

*is closed in $X \times X$.*

*Proof.* Take $(x, y) \in \bar{R}$. If $x = y$ then of course $(x, y) \in R$. If $x \neq y$, take an open set $G$ with finite boundary such that $x \in G$ and $y \notin \bar{G}$. Then $(x, y)$ belongs to the closure of $R \cap (G \times (X \setminus \bar{G}))$; but this is included in

$$\bigcup \{C \times C\colon\ C \text{ is a component of } X \text{ meeting } \partial G\},$$

which is a closed subset of $R$, so again $(x, y) \in R$.

2H. Proposition. *Let $X$ be a Hausdorff first-countable finite cut space. Then*

$$Q = \{(x, y)\colon\ x,\ y \text{ belong to the same path-component of } X\}$$

*is closed in $X \times X$.*

*Proof.* Let $x$, $y$ be distinct points of $X$ such that $(x, y) \in \bar{Q}$. Choose decreasing sequences $\langle G_n \rangle_{n \in \mathbb{N}}$, $\langle H_n \rangle_{n \in \mathbb{N}}$ of open sets with finite boundaries such that $G_0 \cap H_0 = \emptyset$ and $\{G_n\colon\ n \in \mathbb{N}\}$, $\{H_n\colon\ n \in \mathbb{N}\}$ are bases of neighbourhoods of $x$, $y$ respectively. For each $n \in \mathbb{N}$ there is a continuous function $\gamma_n\colon [0, 1] \to X$ such that $\gamma_n(0) \in G_n$ and $\gamma_n(1) \in H_n$, because $Q$ meets $G_n \times H_n$. For $k \leq n$ set

$$t_{nk} = \inf\{t\colon\ \gamma_n(t) \notin G_k\}, \qquad a_{nk} = \gamma_n(t_{nk}) \in \partial G_k,$$
$$u_{nk} = \sup\{u\colon\ \gamma_n(u) \notin H_k\}, \qquad b_{nk} = \gamma_n(u_{nk}) \in \partial H_k.$$

Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$; because $\partial G_k$, $\partial H_k$ are finite,

$$a_k = \lim_{n \to \mathcal{F}} a_{nk}, \qquad b_k = \lim_{n \to \mathcal{F}} b_{nk}$$

are defined in $\partial G_k$, $\partial H_k$ for each $k$. Moreover, there are paths $\sigma_k$, $\gamma$, $\tau_k$ in $X$ such that

$\sigma_k$ runs from $a_{k+1}$ to $a_k$ within $\bar{G}_k$,

$\gamma$ runs from $a_0$ to $b_0$,

$\tau_k$ runs from $b_k$ to $b_{k+1}$ within $\bar{H}_k$

for every $k \in \mathbb{N}$, obtained by taking appropriate pieces of appropriate $\gamma_n$. Putting these together, one sees that

$$\gamma^* = (x)^\frown \ldots {}^\frown \sigma_1{}^\frown \sigma_0{}^\frown \gamma^\frown \tau_0{}^\frown \tau_1{}^\frown \ldots {}^\frown (y)$$

is a path from $x$ to $y$, so $(x, y) \in Q$.

2I. Corollary. *If $X$ is a Hausdorff first-countable finite cut space then*
(a) *path-components of $X$ are closed;*
(b) *if $X$ is path-connected it is locally path-connected.*

*Proof.* (a) This is immediate from 2H.

(b) If $G$ is an open set with finite boundary, then every path-component of $\bar{G}$ must meet $\partial G$, so $\bar{G}$ has only finitely many path-components, which are all closed, by (a). Now if $x \in G$, one of the path-components of $\bar{G}$ is a path-connected neighbourhood of $x$ included in $\bar{G}$. As $X$ is regular, this shows that $X$ is locally path-connected.

2J. Proposition. *Let $X$ be a metrizable finite cut space and $\alpha \geqslant 0$. Let $Q_\alpha$ be the set of pairs $(x, y) \in X \times X$ such that for every $\beta > \alpha$ there is a path of length at most $\beta$ from $x$ to $y$ in $X$. Then $Q_\alpha$ is a closed set in $X \times X$.*

*Proof.* The argument follows that of 2H. As before, take distinct $x$, $y$ such that $(x, y) \in \bar{Q}_\alpha$. Fix $\varepsilon > 0$. Take $\langle G_n \rangle_{n \in \mathbb{N}}$, $\langle H_n \rangle_{n \in \mathbb{N}}$ and $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ as before, but this time require also that the length of each path $\gamma_n$ is at most $\alpha + \varepsilon$. Writing $\mathrm{lh}(\gamma \restriction [t, u])$ for the length of the path defined by $\gamma \restriction [t, u]$, take the paths $\sigma_k$, $\gamma$, $\tau_k$ such that

$$\mathrm{lh}(\sigma_k) \leqslant 2^{-k}\varepsilon + \lim_{n \to \mathscr{F}} \mathrm{lh}(\gamma_n \restriction [t_{n,k+1}, t_{nk}]),$$

$$\mathrm{lh}(\gamma) \leqslant \varepsilon + \lim_{n \to \mathscr{F}} \mathrm{lh}(\gamma_n \restriction [t_{n0}, u_{n0}]),$$

$$\mathrm{lh}(\tau_k) \leqslant 2^{-k}\varepsilon + \lim_{n \to \mathscr{F}} \mathrm{lh}(\gamma_n \restriction [u_{nk}, u_{n,k+1}]).$$

In this case we shall have

$$\mathrm{lh}(\gamma^*) = \sum_{k \in \mathbb{N}} \mathrm{lh}(\sigma_k) + \mathrm{lh}(\gamma) + \sum_{k \in \mathbb{N}} \mathrm{lh}(\tau_k) \leqslant 5\varepsilon + \lim_{n \to \mathscr{F}} \mathrm{lh}(\gamma_n) \leqslant \alpha + 6\varepsilon.$$

As $\varepsilon$ is arbitrary, $(x, y) \in Q_\alpha$.

## 3. *Connected metrizable finite cut spaces*

This section is devoted to the properties of connected metric finite cut spaces. By 2A they are locally connected, and I begin with a survey of simple facts about connected, locally connected metric spaces (3A–3C). Then I give some straightforward results about finite cut spaces (3D–3F) before ending the section with one of the principal theorems of this paper (3G), a striking sufficient condition for a finite cut space to be path-connected.

3A. Lemma. *Let $Z$ be a complete metric space and $X$ a connected, locally connected $G_\delta$ set in $Z$. Then $X$ is path-connected.*

*Proof.* This is the Mazurkiewicz–Moore–Menger theorem [**2**, 10.3.10].

3B. Lemma. *Let $(X, \rho)$ be a connected, locally connected metric space with* $\mathrm{diam}_\rho(X) < \infty$.

(a) *There is a metric $\sigma$ on $X$, equivalent to $\rho$, given by the formula*

$$\sigma(x, y) = \inf\{\mathrm{diam}_\rho(C)\colon\ C \subseteq X \text{ is connected and } x, y \in C\}.$$

(b) *Write $\hat{X}_\sigma$ for the completion of $X$ with respect to $\sigma$, and $\hat{\sigma}$ for the metric of $\hat{X}_\sigma$; then the sets*

$$U_X(x, \alpha) = \{y\colon\ y \in X,\ \sigma(y, x) < \alpha\},$$
$$U_X(w, \alpha) = \{y\colon\ y \in X,\ \hat{\sigma}(y, w) < \alpha\},$$
$$U(w, \alpha) = \{v\colon\ v \in \hat{X}_\sigma,\ \hat{\sigma}(v, w) < \alpha\}$$

*are connected whenever $x \in X$, $w \in \hat{X}_\sigma$, $\alpha > 0$.*

(c) *$\hat{X}_\sigma$ is connected, locally connected and path-connected.*

(d) *If $G$ is a regular open subset of $\hat{X}_\sigma$ (that is, the interior of a closed subset of $\hat{X}_\sigma$), then the boundary $\partial G$ of $G$ in $\hat{X}_\sigma$ is a subset of $\overline{\partial_X(G \cap X)}$, the closure in $\hat{X}_\sigma$ of the boundary $\partial_X(G \cap X)$ of $G \cap X$ in $X$.*

(e) *If $G$ is a connected open subset of $\hat{X}_\sigma$, then $G \cap X$ is connected.*

*Proof.* Most of this must be standard, but I sketch the arguments.

(a) $\sigma$ is a metric because $\rho$ is a metric and $X$ is connected and $\mathrm{diam}_\rho(X) < \infty$; $\sigma$ is equivalent to $\rho$ because $X$ is locally connected.

(b) First,

$$U_X(x, \alpha) = \bigcup\{C\colon\ C \subseteq X \text{ is connected, } x \in C,\ \mathrm{diam}_\rho(C) < \alpha\}$$

is connected. Now

$$U_X(w, \alpha) = \bigcup\{U_X(z, \alpha - \hat{\sigma}(z, w))\colon\ z \in X,\ \hat{\sigma}(z, w) < \tfrac{1}{2}\alpha\}$$

is connected. Because it is dense in $U(w, \alpha)$, so is the latter.

(c) By (b), $\hat{X}_\sigma$ is connected and locally connected. So $\hat{X}_\sigma$ is path-connected by 3A.

(d) Suppose, if possible, otherwise; take $w \in \partial G \setminus \overline{\partial_X(G \cap X)}$. Set $\alpha = \hat{\sigma}(w, \partial_X(G \cap X)) > 0$. Then $U_X(w, \alpha)$ is a connected subset of $X$ meeting both $G$ and $X \setminus G$ (because $w \in \partial G = \bar{G} \cap \overline{\hat{X}_\sigma \setminus G}$ and $X$ is dense in $\hat{X}_\sigma$) but not meeting $\partial_X(G \cap X)$; which is impossible.

(e) Suppose, if possible, otherwise. Then $G \cap X$ can be partitioned into two non-empty relatively open sets $H_0$, $H_1 \subseteq X$. Set $V = \mathrm{int}\, \bar{H}_0$, taken in $\hat{X}_\sigma$, so that $V$ is a regular open subset of $\hat{X}_\sigma$ and $H_0 \subseteq V \cap X \subseteq \bar{H}_0 \cap X$. Because $G$ is connected and meets both $V$ and $\hat{X}_\sigma \setminus V$, it meets $\partial V$; by (d), $G$ meets $\partial_X(V \cap X) \subseteq \partial_X H_0$; but of course this is not so.

REMARK. Of course the hypothesis '$\mathrm{diam}_\rho(X) < \infty$' is nearly irrelevant, being used only to ensure that $\sigma(x, y)$ is always finite.

3C. LEMMA. *Let $(X, \rho)$ be a connected, locally connected, separable metrizable space. Write*

$\mathrm{Br}(X) = \{x$: *there is an open connected $G \subseteq X$ such that $G \setminus \{x\}$ is not connected*$\}$.

*Then* $\mathrm{Br}(X)$ *is a $K_\sigma$ set in $X$ (that is, a countable union of compact sets).*

*Proof.* (a) Let $\rho$ be a metric, defining the topology of $X$, such that $\operatorname{diam}_\rho(X) < \infty$. Construct $\sigma$ as in 3B. Consider

$$\mathrm{Br}_0(X) = \{x\colon\ X\setminus\{x\} \text{ is not connected}\}.$$

Let $D \subseteq X$ be a countable dense set. For each pair $y$, $z$ of points of $D$ let $K_{yz}$ be the intersection of all the closed connected subsets of $\hat{X}_\sigma$ containing both $y$ and $z$. Because $\hat{X}_\sigma$ is path-connected, one of these closed connected sets will be compact, and $K_{yz}$ is also compact. Set

$$B_0 = \bigcup \{K_{yz}\setminus\{y, z\}\colon\ y, z \in D\};$$

then $B_0$ is a $K_\sigma$ subset of $\hat{X}_\sigma$.

If $x \in \mathrm{Br}_0(X)$, then $X\setminus\{x\}$ is not connected, so $\hat{X}_\sigma\setminus\{x\}$ is not connected (by (e) above), and has at least two components, which must meet $D$; take $y$, $z \in D$ belonging to different components of $\hat{X}_\sigma\setminus\{x\}$. Of course every connected subset of $\hat{X}_\sigma$ containing both $y$ and $z$ must also contain $x$, so $x \in K_{yz}$. As $x$ is arbitrary, this shows that $\mathrm{Br}_0(X) \subseteq B_0$.

If $w \in B_0$, take $y$, $z \in D$ such that $w \in K_{yz}\setminus\{y, z\}$. Then $\hat{X}_\sigma\setminus\{w\}$ includes no arc from $y$ to $z$, so cannot be path-connected; but it is surely a locally connected $G_\delta$ subset of $\hat{X}_\sigma$, so cannot be connected. Accordingly its dense subset $X\setminus\{w\}$ cannot be connected. Thus $w \in \mathrm{Br}_0(X)$; as $w$ is arbitrary, $\mathrm{Br}_0(X)$ is equal to $B_0$ and is a $K_\sigma$ set.

(b) Now let $\mathcal{U}$ be any countable base for the topology of $X$ consisting of connected open sets. By (a),

$$B = \{x\colon\ \exists U \in \mathcal{U} \text{ such that } U\setminus\{x\} \text{ is not connected}\}$$

is a $K_\sigma$ set in $X$. Of course $B \subseteq \mathrm{Br}(X)$. But also, if $x \in \mathrm{Br}(X)$, let $G$ be a connected open subset of $X$ such that $G\setminus\{x\}$ is not connected. Let $G'$, $G''$ be disjoint non-empty open sets with union $G\setminus\{x\}$, and let $U \in \mathcal{U}$ be such that $x \in U \subseteq G$. Because $G$ is connected, $x \in \overline{G'} \cap \overline{G''}$, so that $U \cap G'$, $U \cap G''$ form a partition of $U\setminus\{x\}$ into non-empty open sets, and $x \in B$.

Thus $\mathrm{Br}(X) = B$ is a $K_\sigma$ set.

Remark. Compare [**22**, Theorem III.5.3 and elsewhere]. $\mathrm{Br}(X)$ is the set of 'local separating points' in $X$ [**21**].

3D. Lemma. *Let $(X, \rho)$ be a connected metric finite cut space with* $\operatorname{diam}_\rho(X) < \infty$. *Let $\sigma$ be the metric on $X$ defined as in* 3B. *Then there is a $G_\delta$ subset $W$ of $\hat{X}_\sigma$, including $X$, which is a finite cut space.*

*Proof.* Let $H \subseteq X$ be an open subset of $X$ with finite boundary $\partial_X H$ in $X$. Set $V = \operatorname{int} \bar{H}$, taken in $\hat{X}_\sigma$. Then $\partial V \subseteq \overline{\partial_X H} = \partial_X H$, by 3B(d), so $\partial V$ is finite.

So if we write

$$W_n = \bigcup \{V\colon\ V \subseteq \hat{X}_\sigma \text{ is open, } \operatorname{diam}_{\hat{\sigma}}(V) \leq 2^{-n},\ \partial V \text{ is finite}\},$$

we see that $X \subseteq W_n$ for each $n \in \mathbb{N}$, and the $G_\delta$ set $W = \bigcap_{n \in \mathbb{N}} W_n$ is a finite cut space, as required.

3E. Theorem. *A connected metrizable finite cut space is separable.*

*Proof.* Let $(X, \rho)$ be a connected metric finite cut space.

(a) Consider first the case in which $X$ is path-connected. Suppose, if possible, that $X$ is not separable. Let $\varepsilon > 0$ be such that there is an uncountable set $A \subseteq X$ with $\rho(a, a') > \varepsilon$ for all distinct $a, a' \in A$. Fix $x_0 \in X$ and for each $a \in A$ let $\Gamma_a$ be a path from $x_0$ to $a$. Set

$$\mathcal{G}_\varepsilon = \{G\colon\ G \subseteq X \text{ is open, } \partial G \text{ is finite, } \operatorname{diam}(G) \leq \varepsilon\}.$$

For $n \in \mathbb{N}$ set

$$W_n = \{x\colon\ x \in X, \exists G \in \mathcal{G}_\varepsilon \text{ such that } \rho(x, X \setminus G) > 2^{-n}\}.$$

Then $\langle W_n \rangle_{n \in \mathbb{N}}$ is an increasing sequence of open sets covering $X$, so there is an $n \in \mathbb{N}$ such that $A' = \{a\colon\ a \in A,\ \Gamma_a \subseteq W_n\}$ is uncountable.

For each $x \in W_n$ choose $G_x \in \mathcal{G}_\varepsilon$ such that $\rho(x, X \setminus G_x) > 2^{-n}$. Choose a sequence $\langle \mathcal{H}_k \rangle_{k \in \mathbb{N}}$ of finite subsets of $\mathcal{G}_\varepsilon$ by the rule

$$\mathcal{H}_0 = \{G_{x_0}\}, \quad \mathcal{H}_{k+1} = \{G_x\colon\ x \in W_n \cap \bigcup \{\partial H\colon\ H \in \mathcal{H}_k\}\}$$

for each $k$. Set $\mathcal{H} = \bigcup_{k \in \mathbb{N}} \mathcal{H}_k$, $U = \bigcup \mathcal{H}$. Then $\mathcal{H}$ is a countable family of sets of diameter less than $\varepsilon$, so there must be some $a \in A' \setminus U$.

Of course $x_0 \in U$. Take $z$ to be the first point of $\Gamma_a$ not belonging to $U$. Let $w$ be a point on $\Gamma_a$, strictly preceding $z$ in $\Gamma_a$, such that $\rho(x, z) \leq 2^{-n}$ for every $x$ lying between $w$ and $z$ in $\Gamma_a$. Then $w \in U$ so there are $k \in \mathbb{N}$, $H \in \mathcal{H}_k$ such that $w \in H$. Now $z \notin H$ so there is a point $x$ of $\partial H$ lying between $w$ and $z$ in $\Gamma_a$. Observe that $x \in \Gamma_a \subseteq W_n$. But now $\rho(z, x) \leq 2^{-n}$ so $z \in G_x \in \mathcal{H}_{k+1}$; which is impossible.

Thus $X$ must be separable if it is path-connected.

(b) For the general case, we may replace $\rho$ by an equivalent bounded metric $\rho'$. Because $X$ is locally connected (2A) we may now form a metric $\sigma$ from $\rho'$ by the method of 3B. Take the $G_\delta$ set $W \subseteq \hat{X}_\sigma$ as in 3D. Then $W$ is connected (because $X$ is dense in $W$) and locally connected (by 2A again) and therefore path-connected (by 3A). Applying (a), we find that $W$ is separable so that $X$ is also separable.

3F. Lemma. *Let $(X, \mathfrak{T})$ be a separable metrizable finite cut space. Write*

$$\mathcal{G} = \{G\colon\ G \in \mathfrak{T},\ \partial G \text{ is finite}\},$$

$$\mathcal{G}^* = \{G\colon\ G \in \mathcal{G},\ \partial G \text{ lies within one component of } \bar{G}\}.$$

*Then there is a sequence $\langle \mathcal{U}_k \rangle_{k \in \mathbb{N}}$ of finite subsets of $\mathcal{G}^*$ such that*

(i) $\mathcal{U}_0 = \{X\}$;

(ii) *for each $k \in \mathbb{N}$, $\mathcal{U}_k$ is disjoint and $\bigcup \mathcal{U}_k$ is dense in $X$;*

(iii) *if $j \leq k$, $U \in \mathcal{U}_j$ and $V \in \mathcal{U}_k$ then either $V \subseteq U$ or $V \cap U = \emptyset$;*

(iv) *if $x \in H \in \mathfrak{T}$ then there is a $k \in \mathbb{N}$ such that*

$$F_k(x) = \bigcup \{\bar{U}\colon\ U \in \mathcal{U}_k,\ x \in \bar{U}\} \subseteq H;$$

(v) *$F_k(x)$ is a neighbourhood of $x$ for every $x \in X$, $k \in \mathbb{N}$;*

(vi) *if $\sigma$ is a pseudometric on $X$ and $\sigma(\bar{U}, \bar{V}) > 0$ whenever $k \in \mathbb{N}$, $U, V \in \mathcal{U}_k$ and $\bar{U} \cap \bar{V} = \emptyset$, then the topology $\mathfrak{T}_\sigma$ defined by $\sigma$ includes $\mathfrak{T}$.*

*Proof.* By definition, $\mathscr{G}$ is a base for the topology of $X$; let $\langle G_i\rangle_{i\in\mathbb{N}}$ run over a subset of $\mathscr{G}$ which is still a base for the topology. Choose $\langle \mathscr{U}_k\rangle_{k\in\mathbb{N}}$ inductively as follows: $\mathscr{U}_0=\{X\}$; given $\mathscr{U}_k$, set

$$\mathscr{U}'_k=\{U\cap G_k:\ U\in\mathscr{U}_k\}\cup\{U\setminus\bar{G}_k:\ U\in\mathscr{U}_k\}.$$

Then $\mathscr{U}'_k$ is disjoint and $\mathscr{U}'_k\subseteq\mathscr{G}$. By 2E(a) we can find a finite disjoint family $\mathscr{U}_{k+1}\subseteq\mathscr{G}^*$ such that every member of $\mathscr{U}'_k$ is a union of members of $\mathscr{U}_{k+1}$. It is easy to see that this process produces a sequence satisfying (i)–(v).

As for (vi), given $x\in H\in\mathfrak{T}$, take $j\leqslant k\in\mathbb{N}$ such that

$$F_k(x)\subseteq\operatorname{int}F_j(x)\subseteq F_j(x)\subseteq H.$$

Set $\delta=\min\{\sigma(\bar{U},\bar{V}):\ U,\ V\in\mathscr{U}_k,\ \bar{U}\cap\bar{V}=\varnothing\}>0$. If $\sigma(y,x)<\delta$, take $U,V\in\mathscr{U}_k$ such that $x\in\bar{U}$, $y\in\bar{V}$; then $\sigma(\bar{U}\cap\bar{V})<\delta$ so $\bar{U}\cap\bar{V}\neq\varnothing$. Now $\bar{U}\subseteq F_k(x)$ so $\bar{V}\cap\operatorname{int}F_j(x)\neq\varnothing$ and $V\cap F_j(x)\neq\varnothing$. But we have $V\subseteq V'$ for some $V'\in\mathscr{U}_j$, in which case $V'\cap F_j(x)\neq\varnothing$ and $\overline{V'}\subseteq F_j(x)$; thus $y\in F_j(x)\subseteq H$.

This shows that $H\supseteq\{y:\ \sigma(y,x)<\delta\}$ is a neighbourhood of $x$ in $\mathfrak{T}_\sigma$. As $x$ and $H$ are arbitrary, $\mathfrak{T}\subseteq\mathfrak{T}_\sigma$.

3G. THEOREM. *Let $X$ be a connected metrizable finite cut space which is (homeomorphic to) a Borel subset of a complete metric space. Then $X$ is path-connected.*

*Proof.* The argument will depend on Martin's theorem that Borel games are determined [**15, 16**]. As often happens with arguments of this kind, there seems to be a choice between letting the details swamp the ideas, and leaving rather a lot of work to the reader. What I aim to do is to set up an infinite game with Borel payoff set in which a winning strategy for Player I leads to a path in $X$ and a winning strategy for Player II would lead to a decomposition of $X$ into open-and-closed sets.

Of course we know already that $X$ is separable, by 3E. It is convenient to note at once that, for a metric space, the property of being a Borel set in its completion is a topological one [**14**, § 35.IV]. So we may take any metric $\rho$ on $X$ defining its topology and $X$ will be a Borel set in the corresponding completion $\hat{X}_\rho$. I choose to take for $\rho$ a totally bounded metric [**14**, § 22.II, Corollary 1a], so that $\hat{X}_\rho$ will be compact. In the discussion below, topological notions such as closure and boundary are to be taken in $X$ unless otherwise indicated. For $A\subseteq X$, I will write $A^\sim$ for the closure of $A$ in $\hat{X}_\rho$, so that $\bar{A}$ (the closure of $A$ in $X$) is just $X\cap A^\sim$.

Now take any two distinct points $x$, $y$ of $X$ and a sequence $\langle\mathscr{U}_k\rangle_{k\in\mathbb{N}}$ as in 3F above, with the associated family $\langle F_k(z)\rangle_{k\in\mathbb{N},z\in X}$. Note that $\bar{G}$ must be connected for every $G\in\mathscr{G}^*$ (see the remark following 2E). Now set

$$D_k=\{x,y\}\cup\bigcup\{\partial U:\ U\in\mathscr{U}_k\}=\{x,y\}\cup(X\setminus\bigcup\mathscr{U}_k),$$

so that each $D_k$ is a finite set and $D_k\subseteq D_{k+1}$ for each $k\in\mathbb{N}$.

By a (I, $k$)-*chain* from $d$ to $d'$ I shall mean a finite chain of the form

$$(d_0,U_1,d_1,U_2,\dots,U_m,d_m)$$

where $d_0=d$, $d_1,\dots,d_m=d'$ are distinct points of $D_k$, $U_1,\dots,U_m$ are (not necessarily distinct) members of $\mathscr{U}_k$, and $d_{i-1}$, $d_i$ both belong to $\bar{U}_i$ for $1\leqslant i\leqslant m$. I will say that such a chain is *covered* by a set $F$ if $\bar{U}_i\subseteq F$ for $1\leqslant i\leqslant k$.

Now to describe the game. Player I must begin with the move

$$(x, X, y),$$

the unique (I, 0)-chain from $x$ to $y$. Given that Player I has played, for his $k$th move, a (I, $k-1$)-chain $(d_0, U_1, \ldots, U_m, d_m)$, Player II must reply by choosing one of the links $(d_{i-1}, U_i, d_i)$ from the chain. Player I must now, for his $(k+1)$st move, choose a (I, $k$)-chain from $d_{i-1}$ to $d_i$ which is covered by $\bar{U}_i$. (Such a chain always exists because $\bar{U}_i$ is connected and $\{U\colon U \in \mathcal{U}_k, U \subseteq U_i\}$ is a finite family of open sets with union dense in $U_i$.)

For any particular play $p$ of the game, write $V_k(p)$ for the open set, belonging to $\mathcal{U}_k$, in the link chosen by Player II for his $(k+1)$st move in that play. Observe that $V_0(p) = X$ and that $V_{k+1}(p) \subseteq V_k(p)$ for every $k$. Now Player I wins the play $p$ if $\bigcap_{k \in \mathbb{N}} \overline{V_k(p)} \neq \emptyset$; otherwise Player II wins.

Let $P$ be the set of all plays in the game, and give it its natural metrizable topology. To see that the set $P_{\mathrm{I}}$ of plays won by Player I is a Borel set in $P$, note first that if $z \in \bigcap_{k \in \mathbb{N}} \overline{V_k(p)}$ and $H$ is any neighbourhood of $z$ in $X$, then there is a $k \in \mathbb{N}$ such that $F_k(z) \subseteq H$; in particular, $\overline{V_k(p)} \subseteq H$. This means that if $p \in P_{\mathrm{I}}$ then $\inf_{k \in \mathbb{N}} \operatorname{diam}_\rho(V_k(p)) = 0$. Of course,

$$P' = \left\{p\colon\ p \in P,\ \inf_{k \in \mathbb{N}} \operatorname{diam}_\sigma(V_k(p)) = 0\right\}$$

is a $G_\delta$ set in $P$. Next, for any $p \in P'$, we have a $w_p \in \hat{X}_\sigma$ given by

$$\{w_p\} = \bigcap_{k \in \mathbb{N}} V_k(p)^{\sim},$$

and the map $p \mapsto w_p\colon P' \to \hat{X}_\sigma$ is continuous. Because $X$ is a Borel set in $\hat{X}_\sigma$, $P_{\mathrm{I}} = \{p\colon\ p \in P',\ w_p \in X\}$ is a Borel set in $P$.

It follows by Martin's theorem that either Player I or Player II has a winning strategy. Before tracing the consequences of this dichotomy, I make the following observation. If $Q \subseteq P$ is any closed set, write

$$W(Q) = \bigcap_{k \in \mathbb{N}} \bigcup \{V_k(q)^{\sim}\colon\ q \in Q\} \subseteq \hat{X}_\sigma.$$

Then we have also

$$W(Q) = \bigcup \left\{\bigcap_{k \in \mathbb{N}} V_k(q)^{\sim}\colon\ q \in Q\right\}.$$

For suppose that $w \in W(Q)$. Then we can find $q_k \in Q$ such that $w \in V_k(q_k)^{\sim}$ for each $k \in \mathbb{N}$. Let $q \in Q$ be any cluster point of $\langle q_k \rangle_{k \in \mathbb{N}}$ (here we need to note that there are only finitely many moves available to a player at any particular point in the game, so that $P$ is compact); then for any $k \in \mathbb{N}$ there is an $i \geqslant k$ such that the plays $q_i$ and $q$ agree down to Player II's $(k+1)$st move, so that $w \in V_i(q_i)^{\sim} \subseteq V_k(q_i)^{\sim} = V_k(q)^{\sim}$. Thus $w \in \bigcap_{k \in \mathbb{N}} V_k(q)^{\sim}$.

Now let us look at what it means for one of the players to have a winning strategy. The idea of the game is that when Player I offers a chain $(d_0, U_1, \ldots, U_m, d_m)$ he is claiming that there is a path from $d_0$ to $d_m$ through the intervening sets and points $\bar{U}_i$, $d_i$ in that order; while when Player II responds with the link $(d_{i-1}, U_i, d_i)$ he is claiming that such a path will be defective in that link. I have to show that a strategy for either player will in some sense be sufficiently continuous to ensure that his claims can be assembled to form a path or a disconnection of $X$.

*Case* 1. Suppose that Player I has a winning strategy. Let $P_{\mathrm{I}}^* \subseteq P_{\mathrm{I}}$ be the set of all plays in which Player I follows his strategy; then $P_{\mathrm{I}}^*$ is a closed set in $P$. For each $k \in \mathbb{N}$ consider

$$C_k = \bigcup \{V_k(p)^{\sim}: p \in P_{\mathrm{I}}^*\}.$$

This is a connected subset of $X$ containing both $x$ and $y$. For if $(d_0, U_1, \ldots, U_m, d_m)$ is the $(k+1)$st move by Player I in any play $p \in P_{\mathrm{I}}^*$, then every link $(d_{i-1}, U_i, d_i)$ is a possible response by Player II, so $U_i^{\sim} \subseteq C_k$ for every $i$. Also, every $\bar{U}_i$ is connected so every $U_i^{\sim}$ is connected. This means that if $(d, U, d')$ is the $k$th move by Player II in $p$, then $d$ and $d'$ belong to the same component of $C_k$. An easy induction on $k$ now shows that every $C_k$ is connected.

Consequently,

$$W(P_{\mathrm{I}}^*) = \bigcap_{k \in \mathbb{N}} C_k^{\sim}$$

is a compact connected subset of $\hat{X}_\sigma$, because $\langle C_k^{\sim} \rangle_{k \in \mathbb{N}}$ is a decreasing sequence of compact connected sets. But also

$$W(P_{\mathrm{I}}^*) = \bigcup \left\{ \bigcap_{k \in \mathbb{N}} V_k(p)^{\sim}: p \in P_{\mathrm{I}}^* \right\} = \{w_p: p \in P_{\mathrm{I}}^*\} \subseteq X.$$

So $x$ and $y$ both belong to a connected compact subset $C = W(P_{\mathrm{I}}^*)$ of $X$. By 2A, $C$ is locally connected; by 3A, it is path-connected; so $x$ and $y$ are joined by a path in $C$ which is also a path in $X$.

*Case* 2. Now suppose, if possible, that Player II has a winning strategy. Let $P_{\mathrm{II}}^* \subseteq P \backslash P_{\mathrm{I}}$ be the set of plays in which Player II follows his strategy; again, $P_{\mathrm{II}}^*$ is closed in $P$. Define an equivalence relation on $D = \bigcup_{k \in \mathbb{N}} D_k$ by writing $d \sim d'$ if either $d = d'$ or there is a finite chain

$$d = d_0, U_1, d_1, \ldots, U_m, d_m = d'$$

where for $1 \leq i \leq m$ we have $U_i \in \mathcal{U} = \bigcup_{k \in \mathbb{N}} \mathcal{U}_k$, $d_{i-1} \in \bar{U}_i$, $d_i \in \bar{U}_i$, and no play in $P_{\mathrm{II}}^*$ has $(d_{i-1}, U_i, d_i)$ for any of Player II's moves.

Let us investigate the relation $\sim$. First, note that if $p \in P_{\mathrm{II}}^*$ then $\bigcap_{k \in \mathbb{N}} \overline{V_k(p)} = \emptyset$, that is, $X \cap \bigcap_{k \in \mathbb{N}} V_k(p)^{\sim} = \emptyset$, so $X \cap W(P_{\mathrm{II}}^*) = \emptyset$. Let $z$ be any point of $X$. Then there is a $k \in \mathbb{N}$ such that $z \notin \bigcup \{V_k(p)^{\sim}: p \in P_{\mathrm{II}}^*\}$; because $\mathcal{U}_k$ is finite, there is a connected open neighbourhood $H$ of $z$ which does not meet $V_k(p)$ for any $p \in P_{\mathrm{II}}^*$; in which case any two members of $D \cap H$ must be equivalent for $\sim$. It follows at once that

$$\{\operatorname{int} \bar{C}: C \text{ is an equivalence class in } D \text{ for } \sim\}$$

is a disjoint cover of $X$ by open sets. As $X$ is supposed to be connected, we find that $D$ is itself the sole equivalence class for $\sim$.

In particular, $x \sim y$. Take a finite chain

$$x = d_0, U_1, d_1, \ldots, U_m, d_m = y$$

witnessing this, with all the $d_i$ distinct. Player I can use this chain to mark out a play, compatible with Player II's strategy, as follows. He must of course start with $(x, X, y)$, to which Player II must respond $(x, X, y)$. Now, given that Player II's

$k$th move was of the form $(d_r, V, d_s)$ where $r<s$, $U_{r+1} \cup \ldots \cup U_s \subseteq V$, and $d_i \notin D_{k-1}$ if $r<i<s$, then Player I examines those $d_i$, for $r \leqslant i \leqslant s$, which belong to $D_k$. He will be able to use all of these to form the unique (I, $k$)-chain with links of the form $(d_i, U, d_j)$ where $U_{i+1} \cup \ldots \cup U_j \subseteq U \subseteq V$. He takes such a chain for his $(k+1)$st move. Now sooner or later this will lead to Player II being confronted with a (I, $k$)-chain in which all the links are of the form $(d_{i-1}, U, d_i)$. But the $d_i$, $U_i$ were chosen among those links which he has renounced for any move; and his strategy breaks down.

So Player II does not have a winning strategy, Player I does have a winning strategy, and there is a path from $x$ to $y$ in $X$. As $x$ and $y$ are arbitrary, $X$ is path-connected.

3H. Remarks. In 4M below I give an example of a connected finite cut subspace of $\mathbb{R}^2$ which is not path-connected. Under special axioms the hypothesis '$X$ is Borel in its completion' can be materially relaxed, since $X$ and $P_{\mathrm{I}}$ are of virtually the same type by the criteria of descriptive set theory, and it appears consistent to suppose that there are many more determined games than the Borel games (see [**18**, Chapter 6]). Note that R. L. Moore, building on a construction of B. Knaster, gave an example of a connected locally connected $K_\sigma$ subset of the plane which is not pathwise connected ([**17**, **13**]; see also [**11**; § 3-8]); thus in 3G we really need to know that $X$ is a finite cut space.

## 4. *Connected spaces of finite length*

I turn now to the special properties of spaces of finite length. By 1C, we can use all the results of §§ 2–3. But we have in addition some remarkable interactions between the length measure and the topology which lead us to an effective structure theory for these spaces.

The starting point is M. Bognár's theorem that a connected set must have the same length as its closure (4A(c)). Next, the length measure on a connected set defines an intrinsic distance (4B) which has a variety of useful properties (4C–4F). The 'structure theorem' is 4G–4H; it gives an effective description of connected spaces of finite length enabling us to draw 4I–4L as straightforward corollaries. I end with an example (4M) of a connected space of finite length which is not path-connected; the example seems to demand the ideas of 4G.

4A. Proposition. *Let $(X, \rho)$ be a metric space and $C \subseteq X$ a connected set.*

(a) $\mu_\rho^*(C) \geqslant \operatorname{diam}_\rho(C)$.

(b) *If $\mu_\rho^*(C) < \infty$ then $C$ is totally bounded.*

(c) *In any case, $\mu_\rho(\bar{C}) = \mu_\rho^*(C)$.*

*Proof.* (a) If $x \in C$, then $y \mapsto \rho(y, x)$: $X \to \mathbb{R}$ is a Lipschitz-1 map onto an interval $I$ of $\mathbb{R}$; writing $\lambda$ for Lebesgue measure, we have

$$\sup_{y \in C} \rho(y, x) = \lambda I \leqslant \mu_\rho^*(C)$$

by 1A(e); as $x$ is arbitrary, $\operatorname{diam}_\rho(C) \leqslant \mu_\rho^*(C)$.

(b) If $x \in C$ and $0 < \varepsilon < \sup_{y \in C} \rho(y, x)$, then

$$y \mapsto \rho(y, x)\colon\ C \cap U(x, \varepsilon) \to [0, \varepsilon[$$

is surjective, where $U(x, \varepsilon)=\{y: \rho(y, x)<\varepsilon\}$. Consequently (as in (a))

$$\mu_\rho^*(C\cap U(x, \varepsilon))\geq\varepsilon.$$

But note also that the open sets $U(x, \varepsilon)$ are $\mu_\rho$-measurable (see 1A(a)). Consequently, if $x_0, \ldots, x_n$ are points of $C$ which are at least a distance $2\varepsilon$ apart, so that the balls $U(x_i, \varepsilon)$ are disjoint,

$$\mu_\rho^*(C)\geq\sum_{i\leq n}\mu_\rho^*(C\cap U(x_i, \varepsilon)),$$

and $n+1\leq\mu_\rho^*(C)/\varepsilon$. As $\varepsilon$ is arbitrary, $C$ must be totally bounded.

(c) (This is the main theorem of [**1**]; but I give a shorter proof.) We may suppose that $\mu_\rho^*(C)<\infty$ and that $C\neq\emptyset$. Fix $\varepsilon>0$. Let $E\supseteq C$ be a $G_\delta$ set such that $\mu_\rho(E)=\mu_\rho^*(C)$ and $F\subseteq E$ a closed set such that $\mu_\rho(F)\geq\mu_\rho(E)-\varepsilon$ (see 1A(a)), so that $\mu_\rho^*(C\backslash F)\leq\varepsilon$.

Take $n\in\mathbb{N}$ and set $G_n=\{y: \rho(y, F)<2^{-n}\}$. Take any $\delta$ belonging to $]0, \min(\varepsilon, 2^{-n})]$. Let $\langle x_i\rangle_{i\in I}$ be a maximal family in $\bar{C}\backslash G_n$ subject to the condition that $\rho(x_i, x_j)\geq 2\delta$ if $i\neq j$. I claim that $\#(I)\leq\varepsilon/\delta$. To see this, we may of course suppose that $\#(I)\geq 2$. For each $i\in I$, set $C_i=\{x: x\in C, \ \rho(x, x_i)<\delta\}$; then $C_i\neq C$, so the Lipschitz-1 function $x\mapsto\rho(x, x_i)$ takes all values in $]0, \delta[$ on $C_i$, and $\mu_\rho^*(C_i)\geq\delta$. As in the proof of (b), we get

$$\delta\#(I)\leq\sum_{i\in I}\mu_\rho^*(C_i)=\mu_\rho^*\Big(\bigcup_{i\in I}C_i\Big)\leq\mu_\rho^*(C\backslash F)\leq\varepsilon,$$

which is what I said.

Now set $B_i=\{x: \rho(x, x_i)<2\delta\}$ for each $i\in I$; we have $\operatorname{diam}_\rho(B_i)\leq 4\delta$ for each $i$, and $\bar{C}\backslash G_n\subseteq\bigcup_{i\in I}B_i$, $\sum_{i\in I}\operatorname{diam}_\rho(B_i)\leq 4\delta\#(I)\leq 4\varepsilon$. As $\delta$ is arbitrary, $\mu_\rho(\bar{C}\backslash G_n)\leq 4\varepsilon$. As $n$ is arbitrary, $\mu_\rho(\bar{C}\backslash F)\leq 4\varepsilon$ and

$$\mu_\rho(\bar{C})\leq\mu_\rho(F)+4\varepsilon\leq\mu_\rho(E)+4\varepsilon=\mu_\rho^*(C)+4\varepsilon.$$

As $\varepsilon$ is arbitrary, $\mu_\rho(\bar{C})=\mu_\rho^*(C)$.

REMARK. For a stronger form of (c), see Corollary 4I below.

4B. We now have a result paralleling 3B above.

THEOREM. *Let $(X, \rho)$ be a connected metric space of finite length. Define $\sigma: X\times X\to\mathbb{R}$ by setting*

$$\sigma(x, y)=\inf\{\mu_\rho^*(C): C\subseteq X \text{ is connected, } x, y\in C\}.$$

*Then $\sigma$ is a metric on $X$, equivalent to $\rho$, and $\mu_\sigma^*=\mu_\rho^*$.*

*Proof.* (a) Of course $\sigma$ is a pseudometric, and $\sigma(x, y)\geq\rho(x, y)$ for all $x$, $y$, so $\sigma$ is a metric. If $x\in X$ and $\varepsilon>0$, let $U$ be an open $\rho$-neighbourhood of $x$ such that $\mu_\rho(U)\leq\varepsilon$. Let $V$ be the component of $U$ containing $x$; by 2A above, $V$ is a neighbourhood of $x$, and $\sigma(x, y)\leq\mu_\rho(V)\leq\varepsilon$ for every $y\in V$. This shows that $\sigma$ is $\rho$-continuous and defines the same topology on $X$.

(b) Because $\rho\leq\sigma$, $\mu_\rho^*(A)\leq\mu_\sigma^*(A)$ for every $A\subseteq X$. On the other hand, given $A\subseteq X$ and $\alpha>\mu_\rho^*(A)$ and $\delta>0$, set

$$\mathscr{G}_\delta=\{G: G\subseteq X \text{ is open, } \partial G \text{ is finite, } \mu_\rho(G)\leq\delta\}.$$

Then $\mathscr{G}_\delta$ is a base for the topology of $X$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathscr{G}_\delta$ covering $X$, and set $H_n=G_n\backslash\bigcup_{i<n}G_i$ for each $i<n$; then every $H_n$ belongs to $\mathscr{G}_\delta$

and $D = X \backslash \bigcup_{n\in\mathbb{N}} H_n \subseteq \bigcup_{n\in\mathbb{N}} \partial G_n$ is countable. Because every $H_n$ is $\mu_\rho$-measurable, $\sum_{n\in\mathbb{N}} \mu_\rho^*(A \cap H_n) = \mu_\rho^*(A\backslash D) = \mu_\rho^*(A) < \alpha$. Choose open sets $H_n'$ such that $A \cap H_n \subseteq H_n' \subseteq H_n$ for each $n$ and $\sum_{n\in\mathbb{N}} \mu_\rho(H_n') \leq \alpha$. Let $\mathscr{H}$ be the set of components of $\bigcup_{n\in\mathbb{N}} H_n'$; then each member of $\mathscr{H}$ is open, so $\mathscr{H}$ is countable, and also every member $H$ of $\mathscr{H}$ must be included in some $H_n$, so $\operatorname{diam}_\sigma(H) \leq \mu_\rho(H) \leq \delta$. Now $\mathscr{H}' = \mathscr{H} \cup \{\{x\}\colon x \in D\}$ is a countable cover of $A$ by sets of $\sigma$-diameter at most $\delta$, and

$$\sum_{H\in\mathscr{H}'} \operatorname{diam}_\sigma(H) = \sum_{H\in\mathscr{H}} \operatorname{diam}_\sigma(H) \leq \sum_{H\in\mathscr{H}} \mu_\rho(H) = \sum_{n\in\mathbb{N}} \mu_\rho(H_n') \leq \alpha.$$

As $\delta$ is arbitrary, $\mu_\sigma^*(A) \leq \alpha$; as $A$ and $\alpha$ are arbitrary, $\mu_\sigma^* \leq \mu_\rho^*$ and $\mu_\sigma^* = \mu_\rho^*$.

REMARK. The move from $\rho$ to $\sigma$ evidently corresponds to re-parametrizing an arc by its arc-length distance. For compact $X$ this is due to **[5]**.

4C. DEFINITIONS. Let $(X, \rho)$ be a metric space.

(a) I say that $(X, \rho)$ has the *almost geodesic property* if

$$\rho(x, y) = \inf\{\mu_\rho^*(C)\colon\ C \subseteq X \text{ is connected, } x, y \in C\}$$

for all $x, y \in X$. Observe that if $\sigma$ is constructed by the process of 4B then $(X, \sigma)$ necessarily has the almost geodesic property.

(b) A *geodesic* in $X$ is an arc $\Gamma$ such that the length of $\Gamma$ (necessarily equal to $\mu_\rho(\Gamma)$, see 1A(d)) is precisely the distance between the two endpoints of $\Gamma$. Note that any subarc of a geodesic is again a geodesic.

(c) I say that $(X, \rho)$ has the *geodesic property* if for any distinct $x, y \in X$ there is a geodesic with endpoints $x, y$.

REMARK. For compact $X$, the geodesic and almost geodesic properties (which by 4D below coincide) correspond to $\rho$ being 'convex' in the sense of **[5]**.

4D. PROPOSITION. *If $(X, \rho)$ is a compact metric space with a dense subset $Y$ which has the almost geodesic property then $X$ has the geodesic property.*

*Proof.* Let $x, y$ be distinct points of $X$. Let $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ be sequences in $Y$ converging to $x$, $y$ respectively. For each $n \in \mathbb{N}$ let $C_n$ be a connected subset of $Y$, containing $x_n$ and $y_n$, with $\mu_\rho^*(C_n) \leq \rho(x_n, y_n) + 2^{-n}$. Then $\bar{C}_n$ is a compact connected subset of $X$ with $\mu_\rho(\bar{C}_n) = \mu_\rho^*(C_n) \leq \rho(x_n, y_n) + 2^{-n}$, by 4A(c).

By 3.16–3.19 of **[7]**, there is a compact connected set $C \subseteq X$, a cluster point of $\langle C_n\rangle_{n\in\mathbb{N}}$ for the Hausdorff metric on the space of closed subsets of $X$, containing both $x$ and $y$ and with $\mu_\rho(C) \leq \liminf_{n\to\infty} \mu_\rho(C_n) = \rho(x, y)$. (The arguments of **[7]** are cast for subsets of $\mathbb{R}^n$; but of course they apply equally well in any compact metric space. See also 5C below.) Now by 2B and 3A, $C$ is path-connected, so there is an arc $\Gamma \subseteq C$ joining $x$ and $y$; the length of $\Gamma$ is $\mu_\rho(\Gamma)$ (by 1A(d)) and must be exactly $\rho(x, y)$.

4E. COROLLARY. *If $(X, \rho)$ is a connected metric space of finite length and $\sigma$ is constructed as in* 4B, *then the completion of $(X, \sigma)$ has the geodesic property.*

*Proof.* Because $\mu_\sigma(X) = \mu_\rho(X) < \infty$, $(X, \sigma)$ is totally bounded (4A(b)) and its completion is compact; now 4D gives the result.

4F. LEMMA. *Let $(X, \sigma)$ be a metric space with the almost geodesic property, and $(\hat{X}_\sigma, \hat{\sigma})$ its completion. Then $(\hat{X}_\sigma, \hat{\sigma})$ has all the properties of* 3B(b)–(c) *above.*

*Proof.* These properties were all deduced from the first, that $U_X(x, \alpha)$ is connected for every $x \in X$, $\alpha > 0$. But in the present context

$$U_X(x, \alpha) = \bigcup \{C\colon\ C \subseteq X \text{ is connected, } x \in C,\ \mu_\sigma(C) < \alpha\},$$

so it is surely connected.

4G. THEOREM. *Let $(X, \rho)$ be a connected metric space of finite length. Set*

$$\mathrm{Br}(X) = \{x\colon \textit{ there is a connected open set } G \subseteq X \textit{ such that } G \backslash \{x\} \textit{ is not connected}\}.$$

*Then* $\mathrm{Br}(X)$ *is a $K_\sigma$ subset of $X$ and* $\mu_\rho(X \backslash \mathrm{Br}(X)) = 0$.

*Proof.* By 2B and 3C, $\mathrm{Br}(X)$ is a $K_\sigma$ set, so is $\mu_\rho$-measurable. To find $\mu_\rho(\mathrm{Br}(x))$, first take $\sigma$ to be the metric on $X$ defined by the formula in 4B, so that $\mu_\rho^* = \mu_\sigma^*$ and $\mu_\rho = \mu_\sigma$. Let $(\hat{X}_\sigma, \hat{\sigma})$ be the completion of $(X, \sigma)$, so that $(\hat{X}_\sigma, \hat{\sigma})$ is a compact connected metric space of finite length with the geodesic property (4E).

Now let $D \subseteq X$ be a countable dense set and for each pair $y$, $z$ of distinct points in $D$ choose a geodesic $\Gamma_{yz}$ from $y$ to $z$ in $\hat{X}_\sigma$. Set

$$Y = \bigcup \{\Gamma_{yz}\colon\ y, z \in D, y \neq z\};$$

then (ignoring the trivial case in which $D$ is a singleton) $Y$ is a dense connected subset of $\hat{X}_\sigma$, so we have

$$\mu_{\hat{\sigma}}(Y) = \mu_{\hat{\sigma}}(\hat{X}_\sigma) = \mu_{\hat{\sigma}}^*(X) = \mu_\sigma(X) = \mu_\rho(X),$$

using 4A(c) twice.

The point is that if $\Gamma$ is any geodesic in $\hat{X}_\sigma$, then $\mu_{\hat{\sigma}}(\Gamma \backslash \mathrm{Br}(X)) = 0$. To see this, let $u$ and $v$ be the endpoints of $\Gamma$, and take $\varepsilon > 0$. Let $V \supseteq \Gamma$ be an open set in $\hat{X}_\sigma$ such that $\mu_{\hat{\sigma}}(V \backslash \Gamma) \leq \varepsilon$. Define $h\colon X \to \Gamma$ by saying that $h(x)$ is that point of $\Gamma$ for which $\sigma(u, h(x)) = \min(\sigma(u, x), \sigma(u, v))$, for each $x \in X$. Then $h$ is Lipschitz-1, so $\mu_\sigma^*(h[V \backslash \Gamma]) \leq \varepsilon$, by 1A(e). Set $A = h[V \backslash \Gamma] \cup \{u, v\}$. Take any $w \in \Gamma \backslash A$. Let $\delta > 0$ be such that $V_1 = \{w'\colon\ \hat{\sigma}(w', w) < \delta\} \subseteq V$. Then

$$\{w'\colon\ w' \in V_1, h(w') = w\} = \{w\},$$

so $V_1 \backslash \{w\}$ is not connected and its dense subset $(X \cap V_1) \backslash \{w\}$ is not connected. But (as remarked in 4F) $X \cap V_1$ is a connected open set in $X$, so $w \in \mathrm{Br}(X)$. Thus $\Gamma \backslash A \subseteq \mathrm{Br}(X)$, and $\mu_{\hat{\sigma}}^*(\Gamma \backslash \mathrm{Br}(X)) \leq \mu_{\hat{\sigma}}^*(A) \leq \varepsilon$. As $\varepsilon$ is arbitrary, $\mu_{\hat{\sigma}}(\Gamma \backslash \mathrm{Br}(X)) = 0$.

Because $Y$ is a countable union of geodesics, $\mu_{\hat{\sigma}}(Y \backslash \mathrm{Br}(X)) = 0$ and

$$\begin{aligned}\mu_\rho(X \backslash \mathrm{Br}(X)) &= \mu_\sigma(X \backslash \mathrm{Br}(X)) = \mu_{\hat{\sigma}}(X \backslash \mathrm{Br}(X)) \\ &\leq \mu_{\hat{\sigma}}(\hat{X}_\sigma \backslash \mathrm{Br}(X)) = \mu_{\hat{\sigma}}(Y \backslash \mathrm{Br}(X)) = 0,\end{aligned}$$

as claimed.

4H. REMARKS. The theorem 4G, as stated, is sufficient for the corollaries below. But to gain a mental picture of these spaces, it seems useful to look at the set $Y \subseteq \hat{X}_\sigma$ of the proof. This is a countable union of arcs, each of which has almost all its points (as measured by $\mu_\sigma$, which along the arc is the natural copy of Lebesgue measure) in $X$, and indeed almost all these points are 'local separating points' in $X$, disconnecting some open set. The proof of 3C gives us just a little more: if $\mathcal{U}$ is any base for the topology of $X$, then each local separating point of $X$ disconnects some (component of some) member of $\mathcal{U}$. If we think of $Y$ as a countable union of arcs of which any two intersect at most in an endpoint of one of them (see [**7**, 3.13–3.14]), we get the kind of picture arising in Example 4M.

4I. Corollary. *Let $(X, \rho)$ be a metric space and $C \subseteq X$ a connected set with $\mu_\rho^*(C) < \infty$. Then $C$ is $\mu_\rho$-measurable and $\mu_\rho(\bar{C} \setminus C) = 0$.*

*Proof.* Applying 4G to $(C, \rho \restriction C^2)$ we see that there is a $K_\sigma$ set $B \subseteq C$ with $\mu_\rho(C \setminus B) = 0$; now $B$ is $\mu_\rho$-measurable so $C$ also is. Accordingly 4A(c) tells us that $\mu_\rho(\bar{C} \setminus C) = 0$.

4J. Corollary. *Let $(X, \rho)$ be a metric space and $C \subseteq X$ a connected set with $\mu_\rho^*(C) < \infty$. Then $C$ is a 'regular 1-set', that is, it is $\mu_\rho$-measurable and, writing $U(x, \alpha) = \{y: \rho(y, x) < \alpha\}$, we have*

$$\lim_{\alpha \downarrow 0} \frac{\mu_\rho(C \cap U(x, \alpha))}{2\alpha} = \lim_{\alpha \downarrow 0} \frac{\mu_\rho(C \cap U(x, \alpha))}{\operatorname{diam}_\rho(C \cap U(x, \alpha))} = 1$$

*for $\mu_\rho$-almost all $x \in C$.*

*Proof.* We may suppose that $X$ is complete. In this case $\bar{C}$ is compact (4A(b)), so the arguments of [**7**, § 3.2] tell us that $\bar{C}$ is a regular 1-set. But as $\mu_\rho(\bar{C} \setminus C) = 0$, it follows at once that $C$ is a regular 1-set.

4K. Corollary. *Let $(X, \rho)$ be a connected metric space of finite length. Then $\mu_\rho$ is inner regular for the compact sets, that is, is a Radon measure in the sense of* [**9**].

*Proof.* Because $\mu_\rho$ is certainly inner regular for the closed sets (1A(a)), 4G shows that it will be inner regular for the $K_\sigma$ sets, and therefore for the compact sets.

4L. Corollary. *Let $(X, \rho)$ be a metric space of finite length, $(Y, \sigma)$ any metric space, and $f: X \to Y$ a Lipschitz-1 function. If $E \subseteq X$ is $\mu_\rho$-measurable and lies entirely within one component of $X$, then*

$$y \mapsto \#^*(E \cap f^{-1}[\{y\}]): Y \to \mathbb{N} \cup \{\infty\}$$

*is $\mu_\sigma$-measurable.*

*Proof.* We may of course suppose that $X$ is itself connected. By 4K there is an increasing sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ of compact subsets of $E$ with $\lim_{n \to \infty} \mu_\rho(E \setminus K_n) = 0$. Let $\mathcal{U}$ be a countable base for the topology of $X$. Set

$$g_n(y) = \#^*(K_n \cap f^{-1}[\{y\}]), \quad g(y) = \#^*(E \cap f^{-1}[\{y\}])$$

for $y \in Y$, $n \in \mathbb{N}$. Then $\{y\colon g_n(y) \geq k\}$ is just

$$\{y\colon \exists U_1, \ldots, U_k \in \mathcal{U} \text{ such that } \bar{U}_i \cap \bar{U}_j = \emptyset \ \forall i \neq j,\ y \in f[K_n \cap \bar{U}_j]\ \forall i\},$$

so is $K_\sigma$ and $\mu_\sigma$-measurable, for every $n$, $k \in \mathbb{N}$. Thus each $g_n$ is $\mu_\sigma$-measurable. Now

$$\left\{y\colon g(y) \neq \sup_{n \in \mathbb{N}} g_n(y)\right\} \subseteq f\left[E \Big\backslash \bigcup_{n \in \mathbb{N}} K_n\right]$$

is $\mu_\sigma$-negligible because $f$ is Lipschitz-1 and $E \backslash \bigcup_{n\in\mathbb{N}} K_n$ is $\mu_\rho$-negligible (1A(e)). So $g$ is also $\mu_\sigma$-measurable.

4M. Example. There is a subspace $X$ of $\mathbb{R}^2$, with $\mu_\rho^*(X) < \infty$, where $\rho$ is the usual metric of $\mathbb{R}^2$, such that $X$ is connected but not path-connected.

*Proof.* (a) Construct sequences $\langle s_i \rangle_{i\in\mathbb{N}}$, $\langle S_i \rangle_{i\in\mathbb{N}}$ such that
$s_0 \in \mathbb{R}^2$,
$S_i$ is the circle centre $s_i$ radius $2^{-i}$,
$s_i \in \bigcup_{j<i} S_j$ if $i > 0$,
$S_i \cap \{s_j\colon j \in \mathbb{N}\}$ is dense in $S_i$,
for every $i \in \mathbb{N}$. Set $Y = \bigcup_{i\in\mathbb{N}} S_i$; then $Y$ is a connected subset of $\mathbb{R}^2$ and $\mu_\rho(Y) = 4\pi < \infty$. Set $Z = \bar{Y}$; then $Z$ is a compact connected subset of $\mathbb{R}^2$ and $\mu_\rho(Z) = \mu_\rho(Y) < \infty$, by 4A(c).

Consider $B = \mathrm{Br}(Z)$, in the notation of 3C and 4G. Then $\mu_\rho(B) = \mu_\rho(Z)$. Let $A \subseteq Z \backslash B$ be a Bernstein set in $Z \backslash B$ (that is, such that $A$ and $(Z\backslash B)\backslash A$ meet every uncountable Borel subset of $Z\backslash B$). Try $X = B \cup A$. Then $B \subseteq X \subseteq Z$ so $\mu_\rho(X)$ exists and is finite.

(b) Suppose, if possible, that $X$ is not connected. Then there are disjoint relatively open sets $G$, $H \subseteq Z$ such that $X \subseteq G \cup H$ and $X \cap G$, $X \cap H$ are not empty and $G = Z \backslash \bar{H}$. Examine $\partial G = \bar{G} \backslash G \subseteq Z$. As this is disjoint from $X$ it meets neither $B$ nor $A$. But it is compact so it must be countable. Also it is not empty (because $Z$ is connected) so it must have an isolated point $z$ say. Because $Z$ is locally connected (2B), $z$ has a connected neighbourhood $U$ such that $U \cap \partial G = \{z\}$. Then $U$ meets both $G$ and $H$ and $U \backslash \{z\} \subseteq G \cup H$ so $U\backslash\{z\}$ is not connected and $z \in \mathrm{Br}(Z) = B$, which is absurd.

(c) Suppose, if possible, that $X$ includes some arc $\Gamma$. Then $\Gamma \backslash B$ is a Borel subset of $Z$ included in $A$, so must be countable; because $B$ is $K_\sigma$, there must be an arc $\Gamma_1 \subseteq \Gamma \cap B$. Recall that by the argument of 3C, $B = \mathrm{Br}(z)$ may be constructed as $\bigcup_{U\in\mathcal{U}} \mathrm{Br}_0(U)$ where $\mathcal{U}$ is a countable base for the topology of $Z$ consisting of sets with finite boundary; again because every $\mathrm{Br}_0(U)$ is $K_\sigma$, there must be a $U \in \mathcal{U}$ and an arc $\Gamma_2 \subseteq \Gamma_1 \cap \mathrm{Br}_0(U)$.

Next observe that there is an arc $\Gamma_3 \subseteq \Gamma_2$ which is a subarc of a closed Jordan curve $S \subseteq U$. For there is surely a $j \in \mathbb{N}$ such that $\mu_\rho(\Gamma_2 \cap S_j) > 0$.

(i) If $S_j$ and $\Gamma_2$ share any arc, then this arc will contain infinitely many points $s_i$, and there will be circles $S_i \subseteq U$ which meet $\Gamma_2$ twice or more, so that consecutive points of $\Gamma_2$ along $S_i$ will serve as endpoints of $\Gamma_3$.

(ii) If $S_j$ and $\Gamma_2$ do not share any arc, then $S_j \backslash \Gamma_2$ must have infinitely many components; all but finitely many of these must lie within $U$, so that the endpoints of one of them will serve as endpoints of $\Gamma_3$.

The set $U\backslash\Gamma_3$ is an open set in the locally connected space $Z$, so has only countably many components. For each component $C$ of $U\backslash\Gamma_3$, $C$ is relatively open in $Z$ and relatively closed in $U\backslash\Gamma_3$, but cannot be relatively closed in $U$, because $U$ is connected; so $\bar{C}\cap\Gamma_3\neq\emptyset$. There is therefore a countable set $D\subseteq\Gamma_3$ such that $\bar{C}\cap D\neq\emptyset$ for every component $C$ of $U\backslash\Gamma_3$. Let $z$ be any point of $\Gamma_3\backslash D$, and examine the components of $U\backslash\{z\}$. If $C$ is a component of $U\backslash\{z\}$, then either $S\backslash\{z\}\subseteq C$ or $(S\backslash\{z\})\cap\bar{C}=\emptyset$; but the latter is impossible, because it would force $C$ to be a component of $U\backslash\Gamma_3$, in which case $\bar{C}\cap D$ would have to be non-empty, so that $\bar{C}$ would meet $S\backslash\{z\}$. So in fact $U\backslash\{z\}$ has only one component, and $z\notin \mathrm{Br}_0(U)$; but $U$ was chosen with $\Gamma_2$ so that $\Gamma_2\subseteq \mathrm{Br}_0(U)$.

Thus we have a contradiction, resolvable only by abandoning the idea that there is an arc $\Gamma\subseteq X$.

(d) Assembling (a)–(c) we see that $X$ has the required properties.

## 5. *Embedding theorems*

I give some results to show that spaces which are topologically of finite length can be expressed as subsets of 'good' spaces.

5A. Lemma. *Let $(X,\rho)$ be a non-empty metric space of finite length. Then there are a metric $\sigma$ on $X$ and a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of components of $X$ such that*

*$\sigma$ is equivalent to $\rho$;*

*$\bigcup_{n\in\mathbb{N}} C_n$ is dense in $X$;*

*$\sum_{n\geq 1}\sigma(C_n,\bigcup_{i<n}C_i)<\infty$;*

*$\mu_\sigma(C_n)=\mu_\rho C_n$ for each $n\in\mathbb{N}$.*

*Proof.* If $X$ has only finitely many components, this is trivial; so suppose otherwise.

(a) Take $\mathcal{G}$, $\mathcal{G}^*$ and $\langle\mathcal{U}_k\rangle_{k\in\mathbb{N}}$ from 3F. Set $\mathcal{U}=\bigcup_{k\in\mathbb{N}}\mathcal{U}_k$, $D=\bigcup_{U\in\mathcal{U}}\partial U$. Then there is a family $\langle t_U\rangle_{U\in\mathcal{U}}$ such that

$t_U\in\bar{U}$ for every $U\in\mathcal{U}$;

if $U,V\in\mathcal{U}$ and $U\subseteq V$ and $t_V\in\bar{U}$ then $t_U=t_V$;

if $U\in\mathcal{U}$ and $\partial U\neq\emptyset$ then $t_U\in\partial U$.

For we may choose $\langle t_U\rangle_{U\in\mathcal{U}_k}$ for each $k$ in turn, taking care that

if $U\in\mathcal{U}_k$, $V\in\mathcal{U}_{k-1}$, $U\subseteq V$ and $t_V\in\bar{U}$ then $t_U=t_V$;

if $U\in\mathcal{U}_k$ and either $k=0$ or $t_V\notin\bar{U}$, where $V$ is the member of $\mathcal{U}_{k-1}$ including $U$, and if $\bar{U}\cap D\neq\emptyset$, then $t_U\in\bar{U}\cap D$ is taken to minimise

$$\min\{j:\ \exists W\in\mathcal{U}_j,\ t_U\in\partial W\};$$

if $t_U$ cannot be chosen by either of these rules (so that, in particular, $\partial U=\emptyset$), any $t_U\in U$ is taken.

It is straightforward to check that this construction works.

(b) Now let $\mathcal{C}$ be the set of components of $X$ containing some $t_U$. Then $Y=\bigcup\mathcal{C}\supseteq\{t_U:\ U\in\mathcal{U}\}$ is dense, so $\mathcal{C}$ must be infinite. We may enumerate $\mathcal{C}$ as $\langle C_n\rangle_{n\in\mathbb{N}}$ in such a way that for each $k\in\mathbb{N}$ the set

$$\{i:\ \exists U\in\mathcal{U}_k,\ t_U\in C_i\}$$

is of the form $\{i:\ i\leq n\}$ for some $n$. (The point is that

$$\{t_U:\ U\in\mathcal{U}_k\}\subseteq\{t_U:\ U\in\mathcal{U}_{k+1}\}$$

for each $k$.) In this case we find that, for each $U \in \mathcal{U}_k$, $n \in \mathbb{N}$,

$$t_U \in C_n \quad \Leftrightarrow \quad n = \min\{i\colon\ C_i \cap \bar{U} \neq \emptyset\},$$

since if $C_i \cap \bar{U} \neq \emptyset$ either $C_i \cap \partial U \neq \emptyset$, so that $t_U \in C_i$, or $C_i \subseteq U$, so that if $t_V \in C_i$ either $t_U = t_V$, and $i = n$, or $V \notin \bigcup_{j \leq k} \mathcal{U}_j$, and $i > n$.

Observe in particular that $t_X \in C_0$.

We need also to know that if $U \in \mathcal{U}$ and $t_U \in C_n$ then $\bar{U} \cap C_n$ is connected. For if $\partial U \neq \emptyset$ then $\partial U$ lies within a single component $C$ of $\bar{U}$ and $t_U \in \partial U$, so $\partial U \subseteq C \subseteq C_n$ and $C_n \cap \bar{U} = C$ is connected; while if $\partial U = \emptyset$ then $C_n \subseteq U$ and $C_n \cap \bar{U} = C_n$ is connected.

(c) For each $n \geq 1$, choose $U_n$, $U_n'$, $s_n$ and $s_n'$ as follows. Take the first $k$ such that $C_n \cap \{t_U\colon\ U \in \mathcal{U}_k\} \neq \emptyset$, and take $U_n \in \mathcal{U}_k$ such that $t_{U_n} \in C_n$; set $s_n = t_{U_n}$; now observe that $k > 0$ because $n > 0$, so that we may take $U_n'$ to be the member of $\mathcal{U}_{k-1}$ including $U_n$, and set $s_n' = t_{U_n'}$. I seek a metric $\sigma$, equivalent to $\rho$, such that $\sigma(s_n, s_n') \leq 2^{-n}$ for each $n \geq 1$.

(d) Set

$$R = \bigcup_{n \in \mathbb{N}} (C_n \times C_n) \cup \{(s_n, s_n')\colon\ n \geq 1\} \cup \{(s_n', s_n)\colon\ n \geq 1\},$$

so that $R$ is a symmetric subset of $Y \times Y$. Observe that if $U \in \mathcal{U}$ and $C_n$ is the component of $X$ containing $t_U$ and $x \in \bar{U} \backslash C_n$, $(x, y) \in R$ then $y \in \bar{U}$. For we know that $\partial U \subseteq C_n$. If $x$ and $y$ belong to the same $C_m$, then $C_m \cap \partial U = \emptyset$ so $y \in C_m \subseteq U$. Otherwise, we have $\{x, y\} = \{s_m, s_m'\}$ for some $m$. Now if $V$ is any member of $\mathcal{U}$ such that $t_V \in \bar{U} \backslash C_n$, then $t_V \in \bar{V} \cap U$, so $V \cap U \neq \emptyset$ and either $V \subseteq U$ or $V \supseteq U$; but the latter is impossible, because $t_V \neq t_U$, so $V \subset U$. Accordingly, if $x = s_m = t_{U_m} \in \bar{U} \backslash C_n$, then $U_m \subset U$ so $U_m' \subseteq U$ and $y = s_m' = t_{U_m'} \in \bar{U}$; while if $x = s_m' = t_{U_m'} \in \bar{U} \backslash C_n$ then $U_m \subseteq U_m' \subseteq U$ and $y = s_m \in \bar{U}$.

(e) Define $\theta\colon\ Y \times Y \to [0, \infty]$ by setting

$$\theta(x, y) = \inf\{\mu_\rho(C)\colon\ C \subseteq X \text{ is connected, } x, y \in C\}$$

if $x$ and $y$ belong to the same $C_n$,

$$\theta(s_n, s_n') = \theta(s_n', s_n) = 2^{-n}$$

if $n \geq 1$, and

$$\theta(x, y) = \infty$$

if $(x, y) \in (Y \times Y) \backslash R$. Note that $\theta \restriction C_n \times C_n$ is a metric on each $C_n$, as in 4B. Now define $\sigma_0\colon\ Y \times Y \to [0, \infty]$ by setting

$$\sigma_0(x, y) = \inf\left\{\sum_{i<n} \theta(x_i, x_{i+1})\colon\ x_0, \ldots, x_n \in Y, x_0 = x, x_n = y\right\}.$$

Of course $\sigma_0(x, z) \leq \sigma_0(x, y) + \sigma_0(y, z)$ for all $x$, $y$, $z \in Y$.

(f) If $U \in \mathcal{U}$ and $y \in \bar{U} \cap C_m$ then

$$\sigma_0(t_U, y) \leq \sum_{i \leq m} \mu_\rho(\bar{U} \cap C_i) + \sum \{2^{-i}\colon\ 1 \leq i \leq m, s_i, s_i' \in \bar{U}\};$$

the proof is a simple induction on $m$. For we know that if $t_U \in C_n$ then we must have $m \geq n$, and either $m = n$ and $\sigma_0(t_U, y) \leq \theta(t_U, y) \leq \mu_\rho(\bar{U} \cap C_n)$, because $\bar{U} \cap C_n$ is connected, as remarked in (b), or $m > n$ and $C_m \subseteq U$, so that $s_m \in U$,

$s'_m \in \bar{U} \cap \bigcup_{i<m} C_i$ (see (d)) and, using the inductive hypothesis to bound $\sigma_0(t_U, s'_m)$, we have

$$\begin{aligned}\sigma_0(t_U, y) &\leq \sigma_0(t_U, s'_m) + \theta(s'_m, s_m) + \theta(s_m, y)\\ &\leq \sum_{i<m} \mu_\rho(\bar{U} \cap C_i) + \sum \{2^{-i}\colon\ 1 \leq i < m,\ s_i, s'_i \in \bar{U}\} + 2^{-m} + \mu_\rho C_m\\ &= \sum_{i\leq m} \mu_\rho(\bar{U} \cap C_i) + \sum \{2^{-i}\colon\ 1 \leq i \leq m,\ s_i, s'_i \in \bar{U}\}.\end{aligned}$$

Consequently

$$\operatorname{diam}_{\sigma_0}(\bar{U} \cap Y) \leq 2\Big(\mu_\rho(\bar{U}) + \sum \{2^{-i}\colon\ i \geq 1,\ s_i, s'_i \in \bar{U}\}\Big).$$

(g) In particular, $\sigma_0(t_X, y) < \infty$ for every $y \in Y$, and $\sigma_0$ is finite-valued, and therefore a pseudometric. Also every point of $X$ has neighbourhoods intersecting $Y$ in sets of arbitrarily small $\sigma_0$-diameter. For given $x \in X$, $\varepsilon > 0$ there is a neighbourhood $G$ of $x$ such that

$$\mu^*_\rho(G) + \sum \{2^{-i}\colon\ s_i, s'_i \in G\} \leq \varepsilon;$$

now there is a $k \in \mathbb{N}$ such that $F_k(x) \subseteq G$. If $x \in \partial U$ for some $U \in \mathcal{U}_k$ then $x \in Y$ and $\sigma_0(x, y) \leq 2\varepsilon$ for every $y \in F_k(x) \cap Y$, so $\operatorname{diam}_{\sigma_0}(F_k(x) \cap Y) \leq 4\varepsilon$. Otherwise, $F_k(x) = \bar{U}$ for that $U \in \mathcal{U}_k$ containing $x$, so $\operatorname{diam}_{\sigma_0}(F_k(x) \cap Y) \leq 2\varepsilon$.

(h) If $x$, $y \in C_n$ for some $n \in \mathbb{N}$, then $\sigma_0(x, y) = \theta(x, y)$. For of course $\sigma_0(x, y) \leq \theta(x, y)$. On the other hand, if $x = x_0, \ldots, x_m = y$ is a chain from $x$ to $y$ in $Y$, and if $\sum_{i<m} \theta(x_i, x_{i+1}) < \infty$, so that $(x_{i-1}, x_i) \in R$ for each $i$, then either every $x_i \in C_n$ so that $\sum_{i<m} \theta(x_i, x_{i+1}) \geq \theta(x, y)$ (because $\theta \restriction C_n^2$ is a metric), or there are first and last $j$, $k$ such that $x_j \notin C_n$, $x_k \notin C_n$. But in this case $x_{j-1} = x_{k+1} \in C_n$, because the linkages between different components of $Y$ formed by the pairs $(s_m, s'_m)$ yield no cycles. So

$$\sum_{i<m} \theta(x_i, x_{i+1}) \geq \theta(x, x_{j-1}) + \theta(x_{k+1}, y) \geq \theta(x, y).$$

As $x_0, \ldots, x_m$ are arbitrary, $\sigma_0(x, y) \geq \theta(x, y)$ and the two are equal.

(i) Let $U$, $V \in \mathcal{U}$ be such that $\bar{U} \cap \bar{V} = \varnothing$. Then $\sigma_0(\bar{U} \cap Y, \bar{V} \cap Y) > 0$. To see this, let $C_l$ be the component of $X$ containing $t_U$ and $C_m$ the component containing $t_V$. Set

$$\delta = \min(\{2^{-l}, 2^{-m}\} \cup \{\rho(x, y)\colon\ x \in \partial U, y \in \partial V\}) > 0.$$

Now let $x_0, \ldots, x_n$ be points of $Y$ such that $x_0 \in \bar{U}$, $x_n \in \bar{V}$ and $\sum_{i<n} \theta(x_i, x_{i+1}) < \infty$. Of course $(x_{i-1}, x_i) \in R$ for $1 \leq i \leq n$. Take $j$, $k$ to be first and last such that $x_{j+1} \notin \bar{U}$ and $x_{k-1} \notin \bar{V}$. Then by (d) we must have $x_j \in C_l$, $x_k \in C_m$.

If $l = m$, then

$$\sum_{i<n} \theta(x_i, x_{i+1}) \geq \sigma_0(x_j, x_k) = \theta(x_j, x_k),$$

by (h) above. But also any connected set containing both $x_j$ and $x_k$ must meet both $\partial U$ and $\partial V$ so must have $\rho$-diameter at least $\delta$, and $\theta(x_j, x_k) \geq \delta$, so that $\sum_{i<n} \theta(x_i, x_{i+1}) \geq \delta$.

If $l \neq m$, we must have at least one of the jumps $(s_l, s_l')$, $(s_m', s_m)$ appearing among the pairs $(x_{i-1}, x_i)$ for $j < i \leq k$; so in this case

$$\sum_{i<n} \theta(x_i, x_{i+1}) \geq \min(\theta(s_l, s_l'), \theta(s_m', s_m)) \geq \delta.$$

Because $x_0, \ldots, x_n$ are arbitrary, we have $\sigma_0(\bar{U} \cap Y, \bar{V} \cap Y) \geq \delta > 0$, as required.

(j) From (f) we see that there is a unique extension of $\sigma_0$ to a pseudometric $\sigma$ on $X$ which is continuous with respect to $\rho$; that is, the topology $\mathfrak{T}_\sigma$ defined by $\sigma$ is included in the original topology $\mathfrak{T}_\rho$. But now observe that if $U \in \mathcal{U}$ then $U \cap Y$ is dense in $\bar{U}$. Accordingly, $\sigma(\bar{U}, \bar{V}) = \sigma_0(\bar{U} \cap Y, \bar{V} \cap Y) > 0$ whenever $U, V \in \mathcal{U}$ and $\bar{U} \cap \bar{V} = \emptyset$, by (i). By (vi) of 3F, $\mathfrak{T}_\sigma \supseteq \mathfrak{T}_\rho$.

(k) Thus $\sigma$ is equivalent to $\rho$. We already know that $\bigcup_{n \in \mathbb{N}} C_n$ is dense in $X$. Next,

$$\sum_{n \geq 1} \sigma\Big(C_n, \bigcup_{i<n} C_i\Big) \leq \sum_{n \geq 1} \sigma(s_n, s_n') \leq \sum_{n \geq 1} 2^{-n} < \infty.$$

Finally, $\sigma$ agrees with $\theta$ on each $C_n$; but by 4B it follows that $\mu_\sigma^*$ agrees with $\mu_\rho^*$ on each $C_n$, so that $\mu_\sigma(C_n) = \mu_\rho(C_n)$.

5B. Theorem. *Let $(X, \rho)$ be a metric space of finite length. Then $X$ is homeomorphic to a subset of a compact connected metric space of finite length.*

*Proof.* We may of course suppose that $X \neq \emptyset$. By 5A, we may find a metric $\sigma$ on $X$, equivalent to $\rho$, and a sequence $\langle C_n \rangle_{n \in \mathbb{N}}$ of components in $X$ such that $\bigcup_{n \in \mathbb{N}} C_n$ is dense, $\sum_{n \geq 1} \sigma(C_n, \bigcup_{i<n} C_i) < \infty$ and $\mu_\sigma(C_n) = \mu_\rho(C_n)$ for each $n$. Choose $\langle s_n \rangle_{n \geq 1}$, $\langle s_n' \rangle_{n \geq 1}$ such that $s_n \in C_n$ and $s_n' \in \bigcup_{i<n} C_i$ for each $n \geq 1$ and $\sum_{n \geq 1} \sigma(s_n, s_n') < \infty$. Take $W$ to be $X \cup \bigcup_{n \geq 1} \Gamma_n$ where each $\Gamma_n$ is a new arc drawn from $s_n$ to $s_n'$. Let $\tau$ be a metric on $W$, extending $\sigma$, and giving length $\sigma(s_n, s_n')$ to each $\Gamma_n$. If we set $Y' = \bigcup_{n \in \mathbb{N}} C_n \cup \bigcup_{n \geq 1} \Gamma_n$, then $Y'$ is a dense connected subset of $W$ and

$$\mu_\tau(Y') = \sum_{n \in \mathbb{N}} \mu_\sigma(C_n) + \sum_{n \geq 1} \sigma(s_n, s_n')$$
$$= \sum_{n \in \mathbb{N}} \mu_\rho(C_n) + \sum_{n \geq 1} \sigma(s_n, s_n') < \infty.$$

By 4A(c), $\mu_\tau(W) < \infty$. So $(W, \tau)$ is totally bounded (4A(b)) and its completion $(\hat{W}_\tau, \hat{\tau})$ is compact; moreover, $\mu_{\hat{\tau}}(\hat{W}_\tau) < \infty$ by 4A(c) again, and $\hat{W}_\tau$ is connected because it has a dense connected subspace $Y'$. Thus $\hat{W}_\tau$ is a compact connected metric space of finite length in which $X$ is homeomorphically embedded.

5C. The next lemma seems interesting enough to be worth giving in greater generality and precision than is quite needed here. It complements Theorem 3.18 of [7].

Lemma. *Let $(X, \rho)$ be a metric space, $Y \subseteq X$. Suppose that $\alpha \geq 0$ is such that whenever $\mathcal{H}$ is a finite family of open sets in $X$, all meeting $Y$, there is a connected set $C \subseteq X$, meeting every member of $\mathcal{H}$, such that $\mu_\rho(C) \leq \alpha$. Then $\mu_\rho^*(Y) \leq \alpha$.*

*Proof.* We may suppose that $X$ is complete and that $Y \neq \varnothing$. Now $Y$ is totally bounded. For given $\varepsilon > 0$, let $\langle y_i \rangle_{i \in I}$ be a maximal family in $Y$ such that $\rho(y_i, y_j) \geq 2\varepsilon$ whenever $i \neq j$. Then for any $\delta \in ]0, \varepsilon[$, finite $J \subseteq I$, we may consider $\mathscr{H} = \{U(y_i, \delta): i \in J\}$, writing $U(y, \gamma) = \{x: x \in X, \rho(x, y) < \gamma\}$. If $C$ is any connected subset of $X$ meeting every member of $\mathscr{H}$, then (unless $J$ is a singleton) $\mu_\rho^*(C \cap U(y_i, \varepsilon)) \geq \varepsilon - \delta$ for each $i \in J$; so $\mu_\rho^*(C) \geq \#(J)(\varepsilon - \delta)$. But we are supposing that such a $C$ can always be found with $\mu_\rho(C) \leq \alpha$, so we get $\#(J) \leq \max(1, \alpha/(\varepsilon - \delta))$. Because $J$ and $\delta$ are arbitrary, $\#(I) \leq \max(1, \alpha/\varepsilon)$. Because $\varepsilon$ is arbitrary, $Y$ is totally bounded.

Now let $\delta > 0$. Set $\eta = \delta/(1 + 3\alpha\delta^{-1}) > 0$. Let $\mathscr{H}$ be a finite open cover of $Y$ by sets of diameter at most $\eta$, all meeting $Y$. Then there is a connected set $C \subseteq X$, meeting every member of $\mathscr{H}$, with $\mu_\rho(C) \leq \alpha$. We know that $\mu_\rho(\bar{C}) \leq \alpha$ (4A(c)) so that $\bar{C}$ is a finite cut space (1C) and locally connected (2A) and pathwise connected (3A). Accordingly there is a tree $F \subseteq \bar{C}$ meeting every member of $\mathscr{H}$ (as in the proof of 3.18 in [7]). Now we may express $F$ as $\bigcup_{i<k} F_i$ where

$$\operatorname{diam}_\rho(F_i) \leq \delta \quad \text{for each } i < k,$$

$$\sum_{i<k} \operatorname{diam}_\rho(F_i) \leq \mu_\rho(F) \leq \mu_\rho(\bar{C}) \leq \alpha,$$

$$k \leq 1 + 3\delta^{-1}\mu_\rho(F) \leq 1 + 3\alpha\delta^{-1}$$

[7, Lemma 3.17]. Set

$$E_i = \{y: \rho(y, F_i) \leq 2\eta\}$$

for each $i < k$. Then

$$\bigcup_{i<k} E_i = \{y: \rho(y, F) \leq 2\eta\} \supseteq \bigcup \mathscr{H} \supseteq Y,$$

$$\operatorname{diam}_\rho(E_i) \leq \operatorname{diam}_\rho(F_i) + 4\eta \leq 5\delta \quad \text{for all } i < k,$$

$$\sum_{i<k} \operatorname{diam}_\rho(E_i) \leq 4\eta k + \sum_{i<k} \operatorname{diam}_\rho(F_i) \leq 4\eta(1 + 3\alpha\delta^{-1}) + \alpha = \alpha + 4\delta.$$

As $\delta$ is arbitrary, $\mu_\rho^*(Y) \leq \alpha$, as claimed.

5D. LEMMA. *Let $(X, \sigma)$ be a compact metric space of finite length with the geodesic property and $A \subseteq X$ a non-empty finite set. Then we may find a connected compact $K \subseteq X$ and a Lipschitz-1 function $g: X \to K$ such that*

(i) $A \subseteq K$,

(ii) *$g(x) = x$ for every $x \in K$,*

(iii) *$g^{-1}[\{a\}] = \{a\}$ for every $a \in A$,*

(iv) *$\mu_\sigma$-almost every point of $K$ has an open neighbourhood in $K$ homeomorphic to the open interval $]0, 1[$.*

*Proof.* If $\#(X) \leq 1$, this is trivial; so let us suppose that $\operatorname{diam}_\sigma(X) > 0$.

(a) First note that there is a countable compact set $F \subseteq X$ such that whenever $a \in A$, $x \in X \setminus \{a\}$ there is a $y \in F$ with $\sigma(x, y) < \sigma(a, y)$. To see this, set $\gamma_n = \operatorname{diam}_\sigma(X)/(\sqrt{2})^n$ for each $n \in \mathbb{N}$, and choose for each $a \in A$, $n \in \mathbb{N}$ an $\alpha_{an} \in ]\gamma_{n+1}, \gamma_n]$ such that $F_{an} = \{x: \sigma(a, x) = \alpha_{an}\}$ is finite; such an $\alpha_{an}$ will always exist by 1A(e). Set $F = A \cup \bigcup_{a \in A, n \in \mathbb{N}} F_{an}$; then $F$ is countable and compact because $A$ is finite. If $a \in A$, $x \in X \setminus \{a\}$, let $n$ be such that $\gamma_{n+1} < \sigma(x, a) \leq \gamma_n$. Let

$\Gamma$ be a geodesic from $a$ to $x$, and let $y$ be the point of $\Gamma$ such that $\sigma(a, y) = \alpha_{a,n+1}$. Then $y \in F$ and $\sigma(a, x) \leq \gamma_n = 2\gamma_{n+2} < 2\sigma(a, y)$, while also (because $\Gamma$ is a geodesic) $\sigma(a, y) + \sigma(y, x) = \sigma(a, x)$; so $\sigma(x, y) < \sigma(a, y)$.

(b) Consider the family $\Phi$ of all Lipschitz-1 functions $f$: $X \to X$ such that $f(y) = y$ for every $y \in F$. Then $\Phi$ is a compact subset of the space of all continuous functions from $X$ to itself with the topology of uniform convergence, and composition is continuous on $\Phi$; moreover, the map $f \mapsto f[X]$ is a continuous function from $\Phi$ to the space $\mathcal{K}$ of compact subsets of $X$ with the Hausdorff metric. Because composition is continuous on $\Phi$, the set $\Phi_0$ of idempotent functions (i.e., functions such that $f \circ f = f$) is closed in $\Phi$. Accordingly, $\{f[X]: f \in \Phi_0\}$ is a compact set in $\mathcal{K}$ and must have a minimal member $K$; take $g \in \Phi_0$ with $g[X] = K$. Observe that if $a \in A$ and $x \in X \setminus \{a\}$, there is a $y \in F$ which is nearer to $x$ than to $a$; but $g(y) = y$, so $y$ is also nearer to $g(x)$ than to $a$, and $g(x) \neq a$. Thus $g^{-1}[\{a\}] = \{a\}$ for every $a \in A$.

(c) I have still to prove (iv). If $x$ and $y$ are distinct points of $K$, there is a geodesic $\Gamma$ from $x$ to $y$ in $X$. But because $g$ is Lipschitz-1 and $g(x) = x$, $g(y) = y$, $g[\Gamma]$ must be a geodesic from $x$ to $y$ in $K$. Thus $K$ has the geodesic property. Being separable, it has a dense connected subset $Y$ which is a countable union of geodesics (as in the proof of 4G), and $\mu_\sigma(Y) = \mu_\sigma(K)$ by 4A(c). Thus $\mu_\sigma$-almost every member of $K$ lies on a geodesic in $K$.

(d) Now let $\Gamma$ be any geodesic in $K$, and $\varepsilon > 0$. Let $V$ be an open neighbourhood of $\Gamma$ in $X$ such that $\mu_\sigma(\bar{V} \setminus \Gamma) < \varepsilon$ and $\partial V$ is finite. (The first condition is satisfied by any neighbourhood small enough, and the second is achievable because $X$ is a finite cut space and $\Gamma$ is compact.) Take $x_0$, $x_1$ to be the endpoints of $\Gamma$ and let $h$: $X \to \Gamma$ be the Lipschitz-1 function defined by saying that

$$h(x) \in \Gamma, \quad \sigma(h(x), x_0) = \min(\sigma(x, x_0), \sigma(x_1, x_0)) \quad \text{for every } x \in X.$$

Then $\mu_\sigma(h[\bar{V} \setminus \Gamma]) < \varepsilon$ and $h[F]$ is countable so there is an open set $H \supseteq h[(\bar{V} \setminus \Gamma) \cup F] \cup \{x_0, x_1\}$ with $\mu_\sigma(H) \leq \varepsilon$. The components of $H \cap \Gamma$ are all intervals in $\Gamma$, with lengths totalling at most $\varepsilon$. Let $E$ be the union of all those components of $H \cap \Gamma$ which meet $h[F \cup \partial V] \cup \{x_0, x_1\}$; because $F \cup \partial V$ is compact, $E$ is a finite union of connected sets, and $\mu_\sigma(\bar{E}) = \mu_\sigma(E) \leq \mu_\sigma(H) \leq \varepsilon$.

(e) Let $y$ be any point of $\Gamma \setminus \bar{E}$. The component $J$ of $\Gamma \setminus \bar{E}$ containing $y$ is an interval in $\Gamma$; let $y_0, y_1 \in \bar{E}$ be its endpoints. Note that neither belongs to $H$, so that $\bar{V} \cap h^{-1}[\{y_i\}] = \{y_i\}$ for both $i$; also $\partial V \cap h^{-1}[\bar{J}] = \emptyset$, so

$$\bar{V} \cap h^{-1}[\bar{J}] = V \cap h^{-1}[J \cup \{y_0, y_1\}] = (V \cap h^{-1}[J]) \cup \{y_0, y_1\}.$$

Examine $W = V \cap h^{-1}[J]$. Because $J$ is relatively open in $\Gamma$, $W$ is open in $X$; and $\bar{W} \subseteq \bar{V} \cap h^{-1}[\bar{J}] = W \cup \{y_0, y_1\}$. Thus $\partial W = \{y_0, y_1\}$.

(f) We may therefore define $g_1$: $X \to K$ by setting

$$g_1(x) = \begin{cases} g(x) & \text{if } g(x) \in X \setminus W, \\ h(g(x)) & \text{if } g(x) \in \bar{W}, \end{cases}$$

and $g_1$ will be continuous. In fact, $g_1$ will be Lipschitz-1, because $g$ and $h$ and $h \circ g$ are Lipschitz-1 and $X$ has the geodesic property, so that if $z \in g^{-1}[X \setminus W]$,

$z' \in g^{-1}[\bar{W}]$ then

$$\begin{aligned}\sigma(z, z') &\geq \sigma(g(z), g(z')) \\ &= \min(\sigma(g(z), y_0) + \sigma(y_0, g(z')), \sigma(g(z), y_1) + \sigma(y_1, g(z'))) \\ &\geq \min(\sigma(g_1(z), y_0) + \sigma(y_0, g_1(z')), \sigma(g_1(z), y_1) + \sigma(y_1, g_1(z'))) \\ &= \sigma(g_1(z), g_1(z')).\end{aligned}$$

Furthermore, $g_1$ is idempotent. For if $x \in g^{-1}[X\backslash W]$, then $g_1(x) = g(x) \in X\backslash W$, so $g(g_1(x)) = g(x) \in X\backslash W$ and $g_1^2(x) = g(x) = g_1(x)$. While if $x \in g^{-1}[W]$, then $g_1(x) = h(g(x)) \in \Gamma \subseteq K$, so $g_1(x) = g(g_1(x)) = h(g(g_1(x)))$ and $g_1^2(x) = g_1(x)$.

Finally, if $x \in F$, then $g(x) = x \notin W$, because $\bar{J} \cap h[F] = \emptyset$, so $F \cap W = \emptyset$. Putting these together, we have $g_1 \in \Phi_0$. Also, $g_1[X] \subseteq K$. By the choice of $K$, we must have $g_1[X] = K$. Now consider $K \cap W$. If $x \in K \cap W$ then $g(x) = x \in W$ so $g_1(x) = h(x) \in \Gamma$; but also $g_1(x) = x$, because $g_1$ is idempotent. Thus $K \cap W \subseteq \Gamma$ and $J = \Gamma \cap W = K \cap W$ is a relatively open set in $K$, containing $y$, and homeomorphic to $]0, 1[$.

(g) Thus if we write $M$ for the set of points in $K$ which have relatively open neighbourhoods in $K$ homeomorphic to $]0, 1[$, we see that $\Gamma\backslash\bar{E} \subseteq M$ and $\mu_\sigma^*(\Gamma\backslash M) \leq \mu_\sigma(\bar{E}) \leq \varepsilon$. As $\varepsilon$ is arbitrary, $\mu_\sigma(\Gamma\backslash M) = 0$; as $\Gamma$ is arbitrary, $\mu_\sigma(K\backslash M) = 0$.

This completes the proof.

5E. Lemma. *Let $(X, \rho)$ be a compact connected metric space of finite length and $B \subseteq \mathbb{R}^3$ an open ball. Suppose that $I \subseteq X$ is a finite set and $h$: $I \to \partial B$ an injection. Let $\varepsilon > 0$. Then we can find $f$, $\langle B_i \rangle_{i\in\mathbb{N}}$ such that*

(i) *$f$: $X \to \bar{B}$ is a continuous function;*

(ii) *$f^{-1}[\partial B] = I$ and $f$ extends $h$;*

(iii) *each $B_i$ is an open ball included in $B$;*

(iv) *$\bar{B}_i \cap \bar{B}_j = \emptyset$ for $i \neq j$;*

(v) *$\sum_{i\in\mathbb{N}} \operatorname{diam}(B_i) \leq \varepsilon$;*

(vi) *for every $i \in \mathbb{N}$, $f^{-1}[\bar{B}_i]$ is connected, $\operatorname{diam}_\rho(f^{-1}[\bar{B}_i]) \leq \varepsilon$ and $f^{-1}[\partial B_i]$ is finite;*

(vii) *$f \restriction X\backslash f^{-1}[\bigcup_{i\in\mathbb{N}} B_i]$ is injective;*

(viii) *$f[X]\backslash\bigcup_{i\in\mathbb{N}} B_i$ can be covered by finitely many straight-line segments, none perpendicular to any axis of $\mathbb{R}^3$.*

*Proof.* (a) The result is trivial if $\#(X) \leq 1$; so let us suppose henceforth that $X$ has more than one point.

Let $\sigma$ be the metric on $X$ constructed by the method of 4B. Then $(X, \sigma)$ has the geodesic property (4E). Note that $\rho \leq \sigma$. Let $\delta$ be such that $0 < \delta \leq \frac{1}{2}\varepsilon$, $\delta < \operatorname{diam}_\sigma(X)$ and $\delta < \sigma(x, x')$ for all distinct $x, x' \in I$.

(b) Let $A \subseteq X$ be a finite set, with $\operatorname{diam}_\sigma(A) > \delta$, such that $I \subseteq A$, $\sigma(x, A) \leq \delta$ for every $x \in X$, and any connected set $C$ including $A$ has $\mu_\sigma(X\backslash C) < \delta$ (using 5C). Let $K$, $g$ be obtained from $X$, $\sigma$, $A$ as in Lemma 5D. Then $K = g[X]$ is connected, so $\mu_\sigma(X\backslash K) < \delta$. Because $g$ is Lipschitz-1, $\mu_\sigma(g[X\backslash K]) < \delta$ (1A(e)). Write

$$M = \{x\colon x \in K, x \text{ has an open neighbourhood in } K \text{ homeomorphic to } ]0, 1[\},$$

so that $\mu_\sigma(K\backslash M) = 0$. Fix on a countable dense set $D \subseteq K$ and a point $d_0 \in D$.

(c) Let $G$ be a relatively open subset of $K$, including

$$(K\backslash M)\cup g[X\backslash K]\cup D\cup I,$$

and with $\mu_\sigma(G)\leqslant\delta$. Set

$$H=\bigcup\{V:\ V\subseteq K \text{ is relatively open in } K,\ \mu_\sigma(V\backslash G)=0\}.$$

Then $H$ is relatively open in $K$, $\mu_\sigma(H)=\mu_\sigma(G)\leqslant\delta$, and $V\subseteq H$ whenever $V\subseteq K$ is relatively open and $\mu_\sigma(V\backslash H)=0$.

(d) Because $K$ is locally connected (2A), the components of $H$ are relatively open in $K$. If $x$, $x'$ are distinct points of $I$, then $\sigma(x,x')>\delta\geqslant\mu_\sigma(H)\geqslant\operatorname{diam}_\sigma(C)$ for each component $C$ of $H$, so $x$ and $x'$ must belong to distinct components of $H$. Enumerate $I$ as $\langle x_i\rangle_{i<n'}$. Because $K\backslash M$ is compact, it is covered by finitely many components of $H$; enumerate the components of $H$ meeting $(K\backslash M)\cup I\cup\{d_0\}$ as $\langle H_i\rangle_{i<n}$ where $n\geqslant n'$ and $x_i\in H_i$ for $i<n'$.

Choose open balls $B_i\subset B$, for $i<n$, such that their closures are disjoint, $\operatorname{diam}(B_i)\leqslant 2^{-i-1}\varepsilon$ for each $i$, and for $i<n'$ the ball $B_i$ is internally tangent to $B$ at $h(x_i)\in\partial B$.

(e) Now consider $K'=K\backslash\bigcup_{i<n}H_i$. This is a compact subset of $M$, so each of its points lies in a relatively open subset of $K$ homeomorphic to $]0,1[$, and there is a finite cover $\langle E_j\rangle_{j<m}$ of $K'$ by relatively open subsets of $K\backslash\{d_0\}$ homeomorphic to $]0,1[$. Observe that the union of two such sets cannot be homeomorphic to the unit circle $S^1$, because it would then be a proper open-and-closed subset of $K$. Consequently, if two of the $E_j$ meet, their union is also homeomorphic to $]0,1[$, so we may take it that the $E_j$ are disjoint. If $C$ is any connected relatively open subset of $K$, and $j<m$, then $E_j\cap C$ is either an open interval in $E_j$ or the union of two open intervals, one at each end of $E_j$. In any case, $\partial_K(E_j\cap C)$, the boundary of $E_j\cap C$ in $K$, is finite. So

$$\partial_K C\subseteq\bigcup_{j<m}\partial_K(E_j\cap C)$$

is finite for every component $C$ of $H$.

(f) If $C_0$, $C_1$ are distinct components of $H$ then their closures are disjoint. For otherwise there is a point $x\in\partial_K C_0\cap\partial_K C_1$. Now $x\in E_j$ for some $j<m$ and $C_0\cap E_j$, $C_1\cap E_j$ must include open intervals in $E_j$ abutting at $x$. Thus $x$ belongs to an open interval $V\subseteq E_j$ such that $V\backslash(C_0\cup C_1)=\{x\}$. But now $V$ is relatively open in $K$ and $\mu_\sigma(V\backslash H)=0$ so that $x\in V\subseteq H$, which is absurd.

(g) Set $J=\bigcup_{i<n}\partial_K H_i$. Then $J$ is finite, and each point of $J$ belongs to $\partial_K H_i$ for exactly one $i<n$. We may therefore choose a function $h_1\colon J\cup I\to\bar{B}$ such that $h_1$ extends $h$, $h_1$ is injective and $h_1(x)\in B\cap\partial B_i$ whenever $x\in\partial_K H_i$. (Of course $I\cap J=\emptyset$.)

Examine $K'=\bigcup_{j<m}E_j\backslash\bigcup_{i<n}H_i$ again. Each $E_j\backslash\bigcup_{i<n}H_i$ is a finite disjoint union of arcs with endpoints in $J$, so we may express $K'$ as $\bigcup_{k<l}\Gamma_k$ where the $\Gamma_k$ are disjoint arcs. For each $k<l$ let $u_k$, $u'_k\in J$ be the endpoints of $\Gamma_k$. Choose a disjoint family $\langle\Delta_k\rangle_{k<l}$ of polygonal arcs in $B\backslash\bigcup_{i<n}B_i$ such that, for each $k$,

(i) the endpoints of $\Delta_k$ are $h_1(u_k)$, $h_1(u'_k)$,

(ii) no line segment of $\Delta_k$ is perpendicular to any axis in $\mathbb{R}^3$,

(iii) $\Delta_k\cap\bigcup_{i<n}\bar{B}_i=\{h_1(u_k),h_1(u'_k)\}$ precisely;

there is room for these because all the $\bar{B}_i$ are disjoint (and we have three dimensions to move in). Note that as $\mu_\sigma(H) \leqslant \delta < \operatorname{diam}_\sigma(A) \leqslant \operatorname{diam}_\sigma(K) \leqslant \mu_\sigma(K)$, we surely have $l > 0$.

(h) Now enumerate the components of $H \setminus \bigcup_{i<n} H_i$ as $\langle H_i \rangle_{i \geqslant n}$. (They must be infinite in number because $D \subseteq H$ is dense in $K$ and $\mu_\sigma(H) < \mu_\sigma(K) = \mu_\sigma(\bar{H})$, while $\mu_\sigma(\bar{C}) = \mu_\sigma(C)$ for every component $C$ of $H$, by 4A(c).) For each $i \geqslant n$, $H_i$ is an open interval in exactly one $\Gamma_k$; let us define $\hat{k}(i)$ by taking $H_i \subseteq \Gamma_{\hat{k}(i)}$. Let $y_i$, $y_i'$ be the endpoints of $H_i$ in $\Gamma_{\hat{k}(i)}$; note that the $y_i$, $y_i'$ are all distinct from each other and from the $u_k$, $u_k'$, by (f).

Assign in turn, for each $i \geqslant n$, an open ball $B_i \subseteq B$ and points $z_i$, $z_i' \in B$ in such a way that

- ($\alpha$) $z_i$, $z_i'$ are distinct points of $\Delta_{\hat{k}(i)}$ belonging to the interior of one of the straight-line segments constituting $\Delta_{\hat{k}(i)}$, and $B_i$ is the open ball with these points as a diameter;
- ($\beta$) $\bar{B}_i \cap \bar{B}_j = \bar{B}_i \cap \Delta_k = \varnothing$ for $j < i$, $k < l$, $k \neq \hat{k}(i)$;
- ($\gamma$) $\bar{B}_i \cap \Delta_{\hat{k}(i)}$ is precisely the closed line segment with endpoints $z_i$, $z_i'$;
- ($\delta$) if $n \leqslant j < i$ and $k = \hat{k}(i) = \hat{k}(j)$, then $h(u_k)$, $z_i$, $z_i'$, $z_j$, $z_j'$, $h(u_k')$ are distinct and appear in the same order along $\Delta_k$ as $u_k$, $y_i$, $y_i'$, $y_j$, $y_j'$, $u_k'$ appear along $\Gamma_k$;
- ($\varepsilon$) $\operatorname{diam}(B_i) \leqslant 2^{-i-1}\varepsilon$.

Moreover, take care to do this in such a way that $\Delta_k \cap \bigcup_{i \geqslant n} B_i$ is dense in $\Delta_k$ for each $k < l$; this will be possible because $\Gamma_k \cap \bigcup_{i \geqslant n} H_i$ is dense in $\Gamma_k$ for each $k$.

(i) Set $K'' = K \setminus H$; then $\{y_i : i \geqslant n\} \cup \{y_i' : i \geqslant n\}$ is dense in $K''$, and there is a unique continuous injection $h_2$: $K'' \cup I \to \bar{B}$ such that $h_2(y_i) = z_i$ and $h_2(y_i') = z_i'$ for every $i \geqslant n$, $h_2(u_k) = h_1(u_k)$ and $h_2(u_k') = h_1(u_k')$ for every $k < l$, and $h_2(x_i) = h(x_i)$ for every $i < n'$. Observe that $h_2[\Gamma_k \setminus H] = \Delta_k \setminus \bigcup_{i \in \mathbb{N}} B_i$ for every $k < l$, so that $h_2^{-1}[\partial B] = I$.

(j) By Tietze's theorem, there is for each $i \in \mathbb{N}$ a continuous $\phi_i$: $\bar{H}_i \to \bar{B}_i$ extending $h_2 \restriction \bar{H}_i$ and such that $\phi_i^{-1}[\partial B_i] = h_2^{-1}[\partial B_i] = \partial H_i \cup (H_i \cap I)$. Now we have a common extension $\phi = h_2 \cup \bigcup_{i \in \mathbb{N}} \phi_i$: $K \to \bar{B}$, which is continuous because $h_2$ and all the $\phi_i$ are continuous and $\lim_{i \to \infty} \operatorname{diam}(B_i) = 0$.

Examine the function $\phi$. We see that

$\phi^{-1}[\partial B] = I$ and $\phi$ extends $h$;
$\phi^{-1}[\bar{B}_i] = \bar{H}_i$ for each $i \in \mathbb{N}$;
$\phi^{-1}[\partial B_i] = \partial H_i \cup (H_i \cap I)$ for each $i \in \mathbb{N}$;
$\phi^{-1}[B_i] = H_i \setminus I$ for each $i \in \mathbb{N}$;
$\phi \restriction K'' \cup I = h_2$ is injective;
$\phi[K] \setminus \bigcup_{i<n} B_i \subseteq \bigcup_{k<l} \Delta_k \cup h[I]$.

(k) We need to know that $g^{-1}[\bar{H}_i] \subseteq X$ is connected for every $i \in \mathbb{N}$; this is because $\partial_K H_i \cap g[X \setminus K] = \varnothing$, so that $g^{-1}[\bar{H}_i] = \bar{H}_i \cup g^{-1}[H_i]$. But of course any component $C$ of $g^{-1}[H_i]$ must be open in $X$ (because $X$ is of finite length, so is locally connected, and $g^{-1}[H_i]$ is open), so $\bar{C} \cap \bar{H}_i \supseteq \partial C \neq \varnothing$. Thus every component of $g^{-1}[\bar{H}_i]$ meets $\bar{H}_i$; because $\bar{H}_i$ is connected, so is $g^{-1}[\bar{H}_i]$.

Note also that $g[X \setminus K] \subseteq H \setminus I$, because $I \subseteq A$, so $g^{-1}[\{x\}] = \{x\}$ for each $x \in I$.

(l) Now set $f = \phi g$: $X \to \bar{B}$. We see that

(i) $f$ is continuous;
(ii) $f^{-1}[\partial B] = g^{-1}[I] = I$ and $f$ extends $h$, because $g(x) = x$ for $x \in I$;
(iii) each $B_i$ is an open ball included in $B$;
(iv) $\bar{B}_i \cap \bar{B}_j$ for $i \neq j$;

(v) $\sum_{i\in\mathbb{N}} \operatorname{diam}(B_i) \leq \sum_{i\in\mathbb{N}} 2^{-i-1}\varepsilon = \varepsilon$;

(vi) (α) for each $i \in \mathbb{N}$, $f^{-1}[\bar{B}_i] = g^{-1}[\bar{H}_i]$ is connected;

(β) for each $i \in \mathbb{N}$,

$$\begin{aligned}\operatorname{diam}_\rho(f^{-1}[\bar{B}_i]) &\leq \operatorname{diam}_\sigma(g^{-1}[\bar{H}_i])\\ &\leq \mu_\sigma(g^{-1}[\bar{H}_i]) \leq \mu_\sigma(H_i) + \mu_\sigma(X\backslash K) \leq 2\delta \leq \varepsilon;\end{aligned}$$

(γ) for each $i \in \mathbb{N}$, $f^{-1}[\partial B_i] = g^{-1}[(H_i \cap I) \cup \partial H_i] = (H_i \cap I) \cup \partial H_i$ is finite, because $g[X\backslash K]$ does not meet $\partial H \cup I$;

(vii) $f \restriction X\backslash f^{-1}[\bigcup_{i\in\mathbb{N}} B_i] = f \restriction (X\backslash g^{-1}[H]) \cup I = f \restriction K'' \cup I = h_2$ is injective;

(viii) $f[X]\backslash\bigcup_{i\in\mathbb{N}} B_i \subseteq \bigcup_{k<l} \Delta_k \cup h[I]$ can be covered by finitely many line segments, none perpendicular to any axis.

Thus the lemma is proved.

5F. As in 5C, I give a lemma in a form which goes a little further than is absolutely necessary.

LEMMA. *Let $Y \subseteq \mathbb{R}^n$ be a connected set, where $n \geq 1$. Write $\rho$ for the Euclidean metric of $\mathbb{R}^n$. Then*

$$\mu_\rho(Y) \leq \sum_{i<n} \int_* \#^*(\{y\colon\ y(i) = \alpha\})\, d\alpha,$$

*where $\int_* h(\alpha)\, d\alpha$ is the supremum of the integrals $\int g(\alpha)\, d\alpha$ as $g$ runs over the Borel measurable functions with $g(\alpha) \leq h(\alpha)$ for every $\alpha \in \mathbb{R}$.*

*Proof.* (a) For any $A \subseteq \mathbb{R}^n$, set

$$\psi(A) = \sum_{i<n} \int_* \#^*(\{a\colon\ a \in A,\ a(i) = \alpha\})\, d\alpha.$$

Observe that $\psi(A \cup A') \geq \psi(A) + \psi(A')$ if $A \cap A' = \emptyset$. I wish to prove that $\mu_\rho(Y) \leq \psi(Y)$.

(b) Suppose first that $A \subseteq \mathbb{R}^n$ is connected, $a \in A$, $\delta > 0$ and that $A \nsubseteq U(a, \delta) = \{x\colon\ \rho(x, a) < \delta\}$. Then $\psi(A \cap U(a, \delta)) \geq \delta$. For let $\eta > 0$, and consider the sets $E_i = \pi_i[A \cap U(a, \delta)]$ for each $i < n$, where $\pi_i\colon\ \mathbb{R}^n \to \mathbb{R}$ is the $i$th coordinate map. Let $C_i$ be the component of $E_i$ containing $\pi_i(a)$. Then there is an interval $J_i \supseteq C_i$ such that neither endpoint of $J_i$ belongs to $E_i$ and $\lambda(J_i) \leq \lambda(C_i) + \eta$, where we write $\lambda$ for Lebesgue linear measure. Set $V = \bigcap_{i<n} \pi^{-1}[J_i]$. Then $V \cap A \cap U(a, \delta)$ is relatively open-and-closed in $A \cap U(a, \delta)$; because $A$ is connected and not included in $U(a, \delta)$, there must be a point

$$x \in \overline{V \cap A \cap U(a, \delta)} \cap \partial U(a, \delta).$$

But now $\pi_i(x) \in \bar{J}_i$ for each $i$, so

$$\begin{aligned}\delta = \rho(a, x) &\leq \sum_{i<n} |\pi_i(a) - \pi_i(x)| \leq \sum_{i<n} \lambda(J_i)\\ &\leq n\eta + \sum_{i<n} \lambda_*(E_i) \leq n\eta + \psi(A \cap U(a, \delta)).\end{aligned}$$

As $\eta$ is arbitrary, we have the result.

(c) We may suppose that $\psi(Y)<\infty$. In this case $\mu_\rho(Y)<\infty$. For let $\delta>0$; let $\langle y_i\rangle_{i\in I}$ be a maximal family in $Y$ subject to the requirement that $\rho(y_i, y_j)\geq 2\delta$ for $i\neq j$, and consider $\psi(Y\cap U(y_i, \delta))$ for each $i$. We have

$$\psi(Y)\geq\sum_{i\in I}\psi(Y\cap U(y_i, \delta))\geq\delta\#(I),$$

at least if $\#(I)>1$, and $\#(I)\leq\max(1, \delta^{-1}\psi(Y))$. Now $Y\subseteq\bigcup_{i<n}U(y_i, 2\delta)$ and $\sum_{i\in I}\operatorname{diam}_\rho U(y_i, 2\delta)\leq 4\delta\#(I)\leq 4\max(\delta, \psi(Y))$, while $\operatorname{diam}(U(y_i, 2\delta))\leq 4\delta$ for each $i$. As $\delta$ is arbitrary, $\mu_\rho(Y)\leq 4\psi(Y)$.

(d) To remove the constant 4, we may argue as follows. Consider the compact connected set $\bar{Y}$. We have $\mu_\rho(\bar{Y}\backslash Y)=0$ (4I), so that

$$\lambda(\{\alpha\colon \#^*(\{y\colon y\in Y, y(i)=\alpha\})\neq\#^*(\{y\colon y\in\bar{Y}, y(i)=\alpha\})\})\leq\lambda(\pi_i[\bar{Y}\backslash Y])=0$$

for each $i$, and $\psi(Y)=\psi(\bar{Y})$. Now consider any arc $\Gamma\subseteq\mathbb{R}^n$, with endpoints $a$, $b$. Then surely $\psi(\Gamma)\geq\sum_{i<n}|a(i)-b(i)|\geq\rho(a, b)$. Breaking $\Gamma$ up into subarcs, we see at once that $\psi(\Gamma)$ is greater than or equal to the length of $\Gamma$, which is just $\mu_\rho(\Gamma)$ (1A(d)). But now, given $\delta>0$, we can find a tree $F\subseteq\bar{Y}$, a finite union of arcs, with $\mu_\rho(F)\geq\mu_\rho(Y)-\delta$; so that

$$\psi(Y)=\psi(\bar{Y})\geq\psi(F)\geq\mu_\rho(F)\geq\mu_\rho(Y)-\delta.$$

As $\delta$ is arbitrary, $\mu_\rho(Y)\leq\psi(Y)$.

5G. Lemma. *Let $Y\subseteq\mathbb{R}^n$ be a compact connected set. Suppose that for each $i<n$ the set $\{y\colon y\in Y, y(i)=\alpha\}$ is finite for almost all $\alpha\in\mathbb{R}$. Then $Y$ is homeomorphic to a subset of $\mathbb{R}^n$ of finite length.*

*Proof.* We may suppose that $Y\subseteq[0, 1]^n$. For each $i<n$, set $g_i(\alpha)=\#^*(\{y\colon y\in Y, y(i)=\alpha\})$. Then $g_i$ is measurable (because $Y$ is compact), and finite almost everywhere, by hypothesis. Define $\phi_i\colon [0, 1]\to[0, 1]$ by setting

$$\phi_i(\alpha)=\int_0^\alpha(1+g_i(\beta))^{-1}\,d\beta$$

for each $\alpha\in[0, 1]$. Define $\phi\colon [0, 1]^n\to[0, 1]^n$ by setting $\phi(y)=\langle\phi_i(y(i))\rangle_{i<n}$ for each $y\in[0, 1]^n$. Then $\phi[Y]$ is homeomorphic to $Y$, and

$$\int\#^*(\{z\colon z\in\phi[Y], z(i)=\alpha\})\,d\alpha\leq 1$$

for each $i$. By 5F, $\phi[Y]$ has finite length.

Remark. The hypotheses of this lemma can be significantly relaxed.

5H. Theorem. *Let $(X, \rho)$ be a metric space of finite length. Then it is homeomorphic to a subspace of some compact connected subspace of $\mathbb{R}^3$ of finite length.*

*Proof.* In view of 5B, it will be enough to consider the case in which $X$ is itself compact and connected.

(a) The first step is to construct sequences $\langle h_n\rangle_{n\in\mathbb{N}}$, $\langle\mathcal{B}_n\rangle_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$,

(i) $h_n$ is a continuous function from $X$ to $\mathbb{R}^3$,

(ii) $\mathcal{B}_n$ is a family of open balls in $\mathbb{R}^3$ with disjoint closures,

(iii) $h_n^{-1}[\bar{B}]$ is connected and $h_n^{-1}[\partial B]$ is finite, for every $B \in \mathcal{B}_n$,

(iv) $h_n \restriction X \backslash h_n^{-1}[\bigcup \mathcal{B}_n]$ is injective,

(v) $h_n[X] \backslash \bigcup \mathcal{B}_n$ can be covered by finitely many line segments, none perpendicular to any axis of $\mathbb{R}^3$,

(vi) $\bigcup \mathcal{B}_{n+1} \subseteq \bigcup \mathcal{B}_n$,

(vii) $h_{n+1}$ agrees with $h_n$ on $X \backslash h_n^{-1}[\bigcup \mathcal{B}_n]$,

(viii) $h_{n+1}^{-1}[B] = h_n^{-1}[B]$ for every $B \in \mathcal{B}_n$,

(ix) $\lim_{n\to\infty} \sum \{\operatorname{diam}(B)\colon B \in \bigcup_{r \geqslant n} \mathcal{B}_r\} = 0$,

(x) $\lim_{n\to\infty} \sup_{B \in \mathcal{B}_n} \operatorname{diam}_\rho(h_n^{-1}[\bar{B}]) = 0$.

*Construction.* Start with $B_0$ an open ball of diameter 1; set $\mathcal{B}_0 = \{B_0\}$ and let $h_0\colon X \to B_0$ be any continuous function.

Given $h_n$ and $\mathcal{B}_n$, let $B_n$ be a member of $\mathcal{B}_n$ with maximal diameter. Set $X_n = h_n^{-1}[\bar{B}_n]$, $I_n = h_n^{-1}[\partial B_n]$. By Lemma 5E, we can find $\langle B_{ni} \rangle_{i \in \mathbb{N}}$, $f_n$ such that

($\alpha$) $f_n\colon X_n \to \bar{B}_n$ is a continuous function;

($\beta$) $f_n^{-1}[\partial B_n] = I_n$ and $f_n$ extends $h_n \restriction I_n$;

($\gamma$) each $B_{ni}$ is an open ball included in $B_n$;

($\delta$) $\bar{B}_{ni} \cap \bar{B}_{nj} = \varnothing$ if $i \neq j$;

($\varepsilon$) $\sum_{i \in \mathbb{N}} \operatorname{diam}(B_{ni}) \leqslant 2^{-n-1}$;

($\zeta$) for each $i \in \mathbb{N}$, $f_n^{-1}[\bar{B}_{ni}]$ is connected, $\operatorname{diam}_\rho(f_n^{-1}[\bar{B}_{ni}]) \leqslant 2^{-n}$ and $f_n^{-1}[\partial B_{ni}]$ is finite;

($\eta$) $f_n \restriction X_n \backslash f_n^{-1}[\bigcup_{i \in \mathbb{N}} B_{ni}]$ is injective;

($\theta$) $f_n[X_n] \backslash \bigcup_{i \in \mathbb{N}} B_{ni}$ can be covered by finitely many line segments, none perpendicular to any axis of $\mathbb{R}^3$.

Now set

$$\mathcal{B}_{n+1} = (\mathcal{B}_n \backslash \{B_n\}) \cup \{B_{ni}\colon i \in \mathbb{N}\},$$

$$h_{n+1}(x) = f_n(x) \text{ for } x \in X_n, \quad = h_n(x) \text{ if } x \in X \backslash X_n.$$

It is easy to check that this construction of $\langle h_n \rangle_{n \in \mathbb{N}}$ and $\langle B_n \rangle_{n \in \mathbb{N}}$ achieves (i)–(viii). For (ix), observe that if we set $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n = \{B_0\} \cup \{B_{ni}\colon n, i \in \mathbb{N}\}$, then $\sum_{B \in \mathcal{B}} \operatorname{diam}(B) \leqslant 2$; consequently $\{B_n\colon n \in \mathbb{N}\}$ must be the whole of $\mathcal{B}$, so that

$$\sum \left\{\operatorname{diam}(B)\colon B \in \bigcup_{r \geqslant n} \mathcal{B}_r\right\} = \sum_{r \geqslant n} \operatorname{diam}(B_r) \to 0$$

as $n \to \infty$. As for (x), given $\varepsilon > 0$, let $n \in \mathbb{N}$ be such that $2^{-n} \leqslant \varepsilon$. Then

$$\sum_{B \in \mathcal{B}_n} \operatorname{diam}_\rho(h_n^{-1}[\bar{B}]) \leqslant \sum_{B \in \mathcal{B}_n} \mu_\rho(h_n^{-1}[\bar{B}]) \leqslant \mu_\rho(X) < \infty,$$

so $\mathcal{A} = \{B\colon B \in \mathcal{B}_n,\ \operatorname{diam}_\rho(h_n^{-1}[\bar{B}]) > \varepsilon\}$ is finite. Let $m \geqslant n$ be such that

$\mathscr{A} \subseteq \{B_r\colon r < m\}$. Then

$$\bigcup_{r \geq m} \mathscr{B}_r \subseteq (\mathscr{B}_n \setminus \mathscr{A}) \cup \{B_{ri}\colon r \geq n, i \in \mathbb{N}\},$$

so $\operatorname{diam}_\rho(h_r^{-1}[\bar{B}]) \leq \varepsilon$ whenever $r \geq m$ and $B \in \mathscr{B}_r$. (Note that it is a consequence of (i)–(viii) that $h_r^{-1}[\bar{B}] = h_n^{-1}[\bar{B}]$ whenever $r \geq n$ and $B \in \mathscr{B}_n$.)

(b) Now (vii)–(ix) show that $\langle h_n \rangle_{n \in \mathbb{N}}$ is a uniformly convergent sequence of functions; let $h$ be its limit, so that $h\colon X \to \mathbb{R}^3$ is continuous. If $n \in \mathbb{N}$ and $x \in X \setminus h_n^{-1}[\bigcup \mathscr{B}_n]$, then $h_r(x) = h_n(x)$ for every $r \geq n$, so $h(x) = h_n(x)$; while if $B \in \mathscr{B}_n$ and $x \in h_n^{-1}[B]$, then $h_r(x) \in B$ for every $r \geq n$, so $h(x) \in \bar{B}$. Accordingly we have $h[X] \setminus \bigcup \{\bar{B}\colon B \in \mathscr{B}_n\} \subseteq h_n[X] \setminus \bigcup \mathscr{B}_n$ covered by a finite number of line segments, none perpendicular to any axis. Now take any $i < 3$. Let $\pi_i\colon \mathbb{R}^3 \to \mathbb{R}$ be the $i$th coordinate map, and write $\lambda$ for Lebesgue measure. Because

$$\sum \left\{\lambda(\pi_i[\bar{B}])\colon B \in \bigcup_{r \geq n} \mathscr{B}_r\right\} \leq \sum \left\{\operatorname{diam}(B)\colon B \in \bigcup_{r \geq n} \mathscr{B}_r\right\} \to 0$$

as $n \to \infty$, we see that for almost every $\alpha \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $\alpha \notin \bigcup \{\pi_i[\bar{B}]\colon B \in \mathscr{B}_n\}$. But for any such $\alpha$, $h[X] \cap \pi_i^{-1}[\{\alpha\}]$ must be finite, being covered by finitely many line segments not parallel to the $i$th axis, so each meeting $\pi_i^{-1}[\{\alpha\}]$ in at most one point.

Thus Lemma 5G tells us that $h[X]$ is homeomorphic to a subspace of $\mathbb{R}^3$ of finite length.

(c) Finally, $h$ is injective. To see this, take any distinct points $x, y \in X$. Let $n \in \mathbb{N}$ be such that $\operatorname{diam}_\rho(h_n^{-1}[\bar{B}]) < \rho(x, y)$ for every $B \in \mathscr{B}_n$. Let us examine four possible cases.

(α) If neither $h_n(x)$ nor $h_n(y)$ belongs to $\bigcup \mathscr{B}_n$ then we have $h(x) = h_n(x) \neq h_n(y) = h(y)$, using Condition (a)(iv) above.

(β) If $h_n(x) \notin \bigcup \mathscr{B}_n$ and $h_n(y) \in B \in \mathscr{B}_n$ then $y \in h_n^{-1}[\bar{B}]$, so $x \notin h_n^{-1}[\bar{B}]$ and $h(x) = h_n(x) \notin \bar{B}$, $h(y) \in \bar{B}$; so $h(x) \neq h(y)$.

(γ) Similarly, if $h_n(x) \in \bigcup \mathscr{B}_n$ and $h_n(y) \notin \bigcup \mathscr{B}_n$ then $h(x) \neq h(y)$.

(δ) Finally, if $h_n(x) \in B \in \mathscr{B}_n$ and $h_n(y) \in B' \in \mathscr{B}_n$ then $B \neq B'$, so $\bar{B} \cap \bar{B}' = \varnothing$ and $h(x) \in \bar{B}$ must be different from $h(y) \in \bar{B}'$.

Thus $h(x) \neq h(y)$ in all cases; as $x$ and $y$ are arbitrary, $h$ is injective. Because $X$ is supposed to be compact, it follows that $X$ is homeomorphic to $h[X]$, and therefore to some subspace of $\mathbb{R}^3$ of finite length.

5I. Remarks. Note that any separable metrizable finite cut space must be homeomorphic to some subspace of $\mathbb{R}^3$, by Hurewicz's theorem [**12**, Theorem V.3]. The extra work above, answering a question of [**6**, p. 140], seems necessary to show that if we start with a space of finite length, we can finish with a space of finite length for the usual metric of $\mathbb{R}^3$.

In both 5B and 5H, I offer constructions for new equivalent metrics still giving finite length to a space. It is natural to pause for a moment to consider the relationship between $\mu_\sigma$ and $\mu_\rho$ if $\sigma$ and $\rho$ are equivalent metrics on the same space $X$ both giving it finite length. Even if $X = [0, 1]$, it is possible for $\mu_\rho$ and $\mu_\sigma$ to be mutually orthogonal as measures. So we may ask: given a metric space $(X, \rho)$ of finite length, can it be embedded in a space $(Y, \sigma)$ of finite length, in such a way that $\mu_\rho$ and $\mu_\sigma \restriction X$ are mutually absolutely continuous (or even more closely related), and $Y$ is compact, or connected, or $\mathbb{R}^3$?

## 6. *Topological characterizations of finite length*

In **[21, 4, 10, 6]**, some remarkable characterizations of compact connected spaces of finite length are given. Here I show that many of their formulations are sufficient to describe metric spaces of finite length even without assuming connectedness or compactness.

6A. DEFINITION. A topological space $(X, \mathfrak{T})$ is *topologically of finite length* if there is a metric $\rho$ on $X$, defining the topology $\mathfrak{T}$, for which $\mu_\rho^*(X) < \infty$.

REMARKS. Of course a space which is topologically of finite length has to be metrizable, and moreover has to be separable, as remarked in 1A. We know also that it must be a finite cut space (1C). So generally in this section I shall be dealing with separable metrizable finite cut spaces.

6B. THEOREM. *Let $(X, \mathfrak{T})$ be a separable metrizable finite cut space. Then the following are equivalent:*

(a) *$X$ is topologically of finite length;*

(b) *for each pair $x$, $y$ of distinct points of $X$ there is a finite family $\mathcal{K}_{xy}$ of perfect non-empty subsets of $X$ such that every connected subset of $X$ containing both $x$ and $y$ includes some member of $\mathcal{K}_{xy}$;*

(c) *for each pair $x$, $y$ of distinct points of $X$ there is a finite family $\mathcal{A}_{xy}$ of subsets of $X$ such that no countable compact subset of $X$ includes any member of $\mathcal{A}_{xy}$ and every closed connected subset of $X$ containing both $x$ and $y$ does include some member of $\mathcal{A}_{xy}$;*

(d) *for each pair $x$, $y$ of distinct points of $X$ there is a continuous function $f\colon X \to \mathbb{R}$ such that*

$$\{\alpha\colon f(x) < \alpha < f(y), f^{-1}[\{\alpha\}] \text{ is finite}\}$$

*is uncountable.*

*Scheme of proof.* It is obvious that (b) implies (c); the proof will therefore be given in three parts 6C, 6E and 6F below, showing respectively that (a)$\Rightarrow$(b), that (c)$\Rightarrow$(a) and that (a)$\Rightarrow$(d)$\Rightarrow$(c). In between is a lemma (6D) which is supposed to clarify the difference between (b) and (c), which is actually very small.

6C. *Proof of* 6B (a)$\Rightarrow$(b). Let $\rho$ be a metric on $X$ defining $\mathfrak{T}$ and with $\mu_\rho^*(X) < \infty$. Let $x$ and $y$ be distinct points of $X$. If they belong to different components of $X$ take $\mathcal{K}_{xy} = \emptyset$ and stop. Otherwise let $C$ be the component of $X$ containing them. Let $\mathcal{H}$ be the set of connected relatively open subsets of $C$ with finite boundaries in $C$; then $\mathcal{H}$ is a base for the topology of $C$ (see 2C). Define $f\colon C \to \mathbb{R}$ by setting $f(z) = \rho(z, x)$ for each $z \in C$; then $f$ is Lipschitz-1. For $H \in \mathcal{H}$, define $g_H\colon \mathbb{R} \to \mathbb{N} \cup \{\infty\}$ by setting $g_H(\alpha) = \#^*(H \cap f^{-1}[\{\alpha\}])$ for each $\alpha \in \mathbb{R}$; then $g_H$ is measurable for Lebesgue measure $\lambda$ (4L) and finite almost everywhere (1A(f)). Let $\mathcal{U} \subseteq \mathcal{H}$ be a countable base for the topology of $C$. Then there is a perfect set $K \subseteq ]0, \rho(x, y)[$ such that $\lambda(K) > 0$, $g_U \restriction K$ is continuous for every $U \in \mathcal{U}$, and $g_C \restriction K$ is constant and finite. Suppose that $g_C(\alpha) = n$ for every $\alpha \in K$. Of course $n \geq 1$ because $f[C] \supseteq [0, \rho(x, y)]$.

Fix $\alpha_0 \in K$ such that $\lambda(K \cap [\alpha_0 - \delta, \alpha_0 + \delta]) > 0$ for every $\delta > 0$. Enumerate $f^{-1}[\{\alpha_0\}]$ as $\langle x_i \rangle_{i<n}$. Take a disjoint family $\langle U_i \rangle_{i<n}$ in $\mathcal{U}$ such that $x_i \in U_i$ for each $i$. Then $g_{U_i}(\alpha_0) = 1$ for each $i$; because $g_{U_i} \restriction K$ is continuous, there is a $\delta > 0$ such that $g_{U_i}(\alpha) = 1$ for every $i < n$, $\alpha \in K' = K \cap [\alpha_0 - \delta, \alpha_0 + \delta]$. We may suppose also that $\delta$ is so small that $|\alpha_0 - f(z)| > \delta$ for every $z \in \bigcup_{i<n} \partial U_i \cup \{x, y\}$ (the boundaries being taken in $C$).

Set $E_i = U_i \cap f^{-1}[K']$ for each $i < n$. Then each $E_i$ is a Borel subset of $X$ and $f[E_i] = K'$ so $\mu_\rho(E_i) \geq \lambda(K') > 0$. Recall that $\mu_\rho \restriction \mathcal{P}C$ is inner regular for the compact sets (4K), so there is for each $i$ a perfect set $K_i \subseteq E_i$ with $\mu_\rho(K_i) > 0$.

Suppose, if possible, that there is a connected set $C' \subseteq X$, containing both $x$ and $y$, but not including any $K_i$. Take $z_i \in K_i \setminus C'$ for each $i < n$, and examine

$$V = \left\{z\colon\ z \in C \setminus \bigcup_{i<n} U_i, f(z) < \alpha_0\right\} \cup \bigcup_{i<n} \{z\colon\ z \in U_i, f(z) < f(z_i)\},$$

$$W = \left\{z\colon\ z \in C \setminus \bigcup_{i<n} U_i, f(z) > \alpha_0\right\} \cup \bigcup_{i<n} \{z\colon\ z \in U_i, f(z) > f(z_i)\}.$$

Then $V$ and $W$ are relatively open in $C$, because if $z \in \partial U_i$ then either $f(z) < \min(\alpha_0, f(z_i))$ or $f(z) > \max(\alpha_0, f(z_i))$. Also $V \cup W = C \setminus \{z_i\colon\ i < n\} \supseteq C'$, $V \cap W = \emptyset$, $x \in V \cap C'$ and $y \in W \cap C'$. Of course this is impossible, because $C'$ is supposed to be connected.

Thus any connected subset of $X$ containing both $x$ and $y$ must include some $K_i$, and we can set $\mathcal{K}_{xy} = \{K_i\colon\ i < n\}$ to witness (b).

6D. Lemma. *Let $X$ be a connected, locally connected metric space. Suppose that $A_0, \ldots, A_n$ are subsets of $X$ which are not relatively compact in $X$. Then for any distinct $x$, $y \in X$ there is a closed connected $C \subseteq X$ such that $x$, $y \in C$ but $A_i \nsubseteq C$ for every $i \leq n$.*

*Proof.* We may suppose that $X$ is a dense subset of a locally connected complete metric space $(\hat{X}_\sigma, \hat{\sigma})$ constructed as in 3B. For each $i \leq n$ let $\langle a_{ij} \rangle_{j \in \mathbb{N}}$ be a sequence in $A_i$ with no cluster point in $X$. Let $F_i$ be the set of cluster points of $\langle a_{ij} \rangle_{j \in \mathbb{N}}$ in $\hat{X}_\sigma$, and set $F = \bigcup_{i \leq n} F_i \subseteq \hat{X}_\sigma \setminus X$. Then $\hat{X}_\sigma \setminus F$ is connected and locally connected, so by the Mazurkiewicz–Moore–Menger theorem (3A) it is path-connected and there is a path $\Gamma \subseteq \hat{X}_\sigma \setminus F$ from $x$ to $y$. For each $i \leq n$ there must be a $j(i) \in \mathbb{N}$ such that $a_{i,j(i)} \notin \Gamma$, because $F_i \cap \Gamma = \emptyset$ and $\Gamma$ is compact. Because $\hat{X}_\sigma$ is locally connected, there is a connected open set $W \subseteq \hat{X}_\sigma$ such that $\Gamma \subseteq W$ and $a_{i,j(i)} \neq \bar{W}$ for each $i \leq n$. By 3B(e), $W \cap X$ is connected, so $C = X \cap \overline{W \cap X}$ is a relatively closed connected subset of $X$ not containing any $a_{i,j(i)}$, so not including any $A_i$, while of course it contains $x$ and $y$.

6E. *Proof of* 6B (c)$\Rightarrow$(a). Now let us assume that $(X, \mathfrak{T})$ satisfies Condition (c), and seek a metric $\rho$ defining $\mathfrak{T}$ for which $\mu_\rho^*(X) < \infty$.

(a) Write $\mathcal{K}$ for the set of uncountable compact subsets of $X$. For distinct $x$, $y \in X$ write

$$\mathcal{K}_{xy} = \mathcal{K} \cap \{\bar{A}\colon\ A \in \mathcal{A}_{xy}\}.$$

If $C \subseteq X$ is a closed connected set containing both $x$ and $y$, then $C$ is locally connected (2A), so we can apply 6D to $C$ and

$$\mathscr{A} = \{A: A \in \mathscr{A}_{xy}, A \subseteq C, \bar{A} \notin \mathscr{K}\}$$

to find a closed connected $C' \subseteq C$ not including any member of $\mathscr{A}$, but containing both $x$ and $y$; now $C'$ must include some $A \in \mathscr{A}_{xy}$, in which case $\bar{A} \in \mathscr{K}_{xy}$ and $\bar{A} \subseteq C$. Thus every closed connected set containing both $x$ and $y$ must include some member of $\mathscr{K}_{xy}$.

(b) Take $\mathscr{G}$, $\mathscr{G}^*$, $\langle \mathscr{U}_k \rangle_{k \in \mathbb{N}}$ and $\langle F_k(x) \rangle_{x \in X, k \in \mathbb{N}}$ as in Lemma 3F above. Write

$$\mathscr{U} = \bigcup_{k \in \mathbb{N}} \mathscr{U}_k, \quad D_k = \bigcup \{\partial U: U \in \mathscr{U}_k\},$$

$$\mathscr{E}_k = \bigcup \{\mathscr{K}_{xy}: x, y \in D_k, x \neq y\}, \quad \mathscr{E} = \bigcup_{k \in \mathbb{N}} \mathscr{E}_k.$$

Then $\mathscr{E}$ is countable, so there is a family $\langle \eta_K \rangle_{K \in \mathscr{E}}$ of strictly positive real numbers such that $\sum_{K \in \mathscr{E}} \eta_K \leq 1$. For each $K \in \mathscr{E}$, there is a subset of $K$ homeomorphic to $\{0, 1\}^{\mathbb{N}}$, so there is an atomless Radon probability $\mu_K$ on $X$ with $\mu_K(K) = 1$. Set $\mu = \sum_{K \in \mathscr{E}} \eta_K \mu_K$. Then $\mu$ is an atomless Radon measure on $X$, $\mu X \leq 1$, and $\mu K \geq \eta_K > 0$ for every $K \in \mathscr{E}$.

(c) For $U \in \mathscr{U}$ set

$$\nu(U) = \mu(\bar{U}) + \inf\{\varepsilon_k: k \in \mathbb{N}, U \in \mathscr{U}_k\},$$

where $\varepsilon_k = 1/(k + \#(\mathscr{U}_k))$ for each $k \in \mathbb{N}$. For $x, y \in X$ set

$$\rho(x, y) = \inf\left\{\sum_{i \leq n} \nu(U_i): n \in \mathbb{N}, U_0, \dots, U_n \in \mathscr{U}, \right.$$
$$\left. x \in \bar{U}_0, y \in \bar{U}_n, \bar{U}_i \cap \bar{U}_{i+1} \neq \varnothing \; \forall i < n\right\}.$$

Then

$$0 \leq \rho(x, y) \leq \nu(X) \leq 2,$$
$$\rho(x, z) \leq \rho(x, y) + \rho(y, z) \quad \text{for all } x, y, z \in X.$$

(d) If $x \in X$ and $\varepsilon > 0$, let $G$ be an open neighbourhood of $x$ such that $\mu(G) \leq \varepsilon$. Let $k \geq \varepsilon^{-1}$ be such that $F_k(x) \subseteq G$. Then $F_k(x)$ is a neighbourhood of $x$ and if $y \in F_k(x)$ there is a $U \in \mathscr{U}_k$ such that $x$, $y$ both belong to $\bar{U}$, so that

$$\rho(x, y) \leq \nu(U) \leq \mu(\bar{U}) + \varepsilon_k \leq \mu(G) + \frac{1}{k} \leq 2\varepsilon.$$

As $x$ and $\varepsilon$ are arbitrary, $\rho$ is a continuous pseudometric, and the topology $\mathfrak{T}_\rho$ defined by $\rho$ is no larger than $\mathfrak{T}$.

(e) Let $k \in \mathbb{N}$. Set

$$\delta_k = \min(\{\varepsilon_i: i \leq k\} \cup \{\eta_K: K \in \mathscr{E}_k\}) > 0.$$

Then $\rho(\bar{V}_0, \bar{V}_1) \geq \delta_k$ whenever $V_0$, $V_1$ are members of $\mathscr{U}_k$ with $\bar{V}_0 \cap \bar{V}_1 = \varnothing$. For suppose, if possible, otherwise. Then there must be a finite chain $U_0, \dots, U_n \in \mathscr{U}$ such that $\bar{U}_0 \cap \bar{V}_0 \neq \varnothing$, $\bar{U}_n \cap \bar{V}_1 \neq \varnothing$, $\bar{U}_i \cap \bar{U}_{i+1} \neq \varnothing$ for $i < n$ and $\sum_{i \leq n} \nu(U_i) < \delta_k$. Take such a chain of minimal length. Observe that for $i \leq n$, $j \leq k$ we have $\nu(U_i) < \delta_k \leq \varepsilon_j$ so $U_i$ must belong to some $\mathscr{U}_r$ for $r > k$. So $V_0 \not\subseteq U_0$ and $V_1 \not\subseteq U_1$. On the other hand, we cannot have $U_0 \subseteq V_0$ or $U_n \subseteq V_n$, because $n$ is minimal. So in fact $U_0 \cap V_0 = \varnothing$ and $\partial U_0 \cap \partial V_0 \neq \varnothing$; similarly, $\partial U_n \cap \partial V_1 \neq \varnothing$. The same argument shows that, because $U_i \not\subseteq U_{i+1}$ and $U_{i+1} \not\subseteq U_i$, $\partial U_i \cap \partial U_{i+1} \neq \emptyset$ for each $i<n$, if $n>0$. However, all the $U_i$ are supposed to belong to $\mathcal{G}^*$, so each $\bar{U}_i$ has a component $C_i$ including $\partial U_i$. The union $C=\bigcup_{i<n} C_i$ is now a connected subset of $\bigcup_{i\leq n} \bar{U}_i$ meeting both $\partial V_0$ and $\partial V_1$; take $x \in C \cap \partial V_0$, $y \in C \cap \partial V_1$. Then $C$ is a closed connected set, so by (a) above includes some member $K$ of $\mathcal{K}_{xy}$. In this case $K \in \mathcal{E}_k$, so

$$\delta_k \leq \eta_K \leq \mu(K) \leq \mu(C) \leq \sum_{i\leq n} \mu(\bar{U}_i) \leq \sum_{i\leq n} \nu(U_i) < \delta_k,$$

which is impossible.

This shows that $\rho(\bar{V}_0, \bar{V}_1) \geq \delta_k$, as claimed. By (vi) of 3F, $\mathfrak{T} \subseteq \mathfrak{T}_\sigma$, so $\mathfrak{T}=\mathfrak{T}_\sigma$.

(f) Finally, take any $\delta>0$. Because every $\mu_K$ is atomless with compact support,

$$\lim_{k\to\infty} \sup\{\mu_K(\bar{U}):\ U \in \mathcal{U}_k\}=0$$

for every $K \in \mathcal{E}$. It follows that

$$\lim_{k\to\infty} \sup\{\mu(\bar{U}):\ U \in \mathcal{U}_k\}=0.$$

Let $k$ be such that $k \geq 1/\delta$ and $\mu(\bar{U}) \leq \delta$ for every $U \in \mathcal{U}_k$. Then

$$\mathrm{diam}_\rho(\bar{U}) \leq \nu(U) \leq \mu(\bar{U})+\varepsilon_k \leq 2\delta$$

for every $U \in \mathcal{U}_k$. Moreover,

$$\sum_{U\in\mathcal{U}_k} \mathrm{diam}_\rho(\bar{U}) \leq \sum_{U\in\mathcal{U}_k} \mu(\bar{U})+\varepsilon_k=\mu(X)+\#(\mathcal{U}_k)\varepsilon_k \leq 2,$$

because $\mu$ is atomless, so $\mu(\bar{U})=\mu(U)$ for every $U \in \mathcal{U}_k$. As $\delta$ is arbitrary, $\mu_\rho^*(X) \leq 2<\infty$. So $\rho$ is a metric witnessing (a).

6F. *Proof of* 6B(a)$\Rightarrow$(d)$\Rightarrow$(c). For (a)$\Rightarrow$(d), take a metric $\rho$ witnessing (a), and set $f(z)=\rho(z,x)$ for $z \in X$; by 1A(f), $f^{-1}[\{\alpha\}]$ is finite for almost all $\alpha \in ]0, \rho(y,x)[=]f(x), f(y)[$.

Now assume (d); I have to show that (c) is true. The argument for this is mostly in 6C. If $x$ and $y$ do not belong to the same component $C$ of $X$, take $\mathcal{A}_{xy}=\emptyset$ and stop. Otherwise, take $\mathcal{H}$ and $\mathcal{U}$ as in 6C, and a continuous function $f$: $X\to\mathbb{R}$ such that

$$B_0=\{\alpha:\ f(x)<\alpha<f(y), f^{-1}[\{\alpha\}] \text{ is finite}\}$$

is uncountable. Let $n \in \mathbb{N}$ be such that

$$B_1=\{\alpha:\ \alpha \in B_0,\ \#(f^{-1}[\{\alpha\}])=n\}$$

is uncountable; of course $n \geq 1$ because $f[C] \supseteq B_0$. For each $\alpha \in B_1$, there is a disjoint family $\langle U_i\rangle_{i<n}$ in $\mathcal{U}$ such that $U_i$ meets $f^{-1}[\{\alpha\}]$ for each $i$; because $B_1$ is uncountable and $\mathcal{U}$ is countable, there is a fixed family $\langle U_i\rangle_{i<n}$ in $\mathcal{U}$ such that

$$B_2=\{\alpha:\ \alpha \in B_1,\ U_i \cap f^{-1}[\{\alpha\}] \neq \emptyset\ \forall i<n\}$$

is uncountable. Take $\alpha_0 \in B_2$ such that $B_2 \cap [\alpha_0-\delta, \alpha_0+\delta]$ is uncountable for every $\delta>0$. Fix $\delta>0$ such that $|\alpha_0-f(z)|>\delta$ for every $z \in \{x, y\} \cup \bigcup_{i<n} \partial_C U_i$, taking the boundaries in $C$, as in 6C. Write $B_3=B_2 \cap [\alpha_0-\delta, \alpha_0+\delta]$.

Set $E_i=U_i \cap f^{-1}[B_3]$ for each $i<n$. Then, just as in 6C, every connected set $C'$ containing both $x$ and $y$ must include some $E_i$. So we may take $\mathcal{A}_{xy}=\{E_i:\ i<n\}$.

6G. Corollary. *Let $X$ be a separable metrizable finite cut space.*

(a) *If every component of $X$ is topologically of finite length, so is $X$.*

(b) *If there is a finite-to-one continuous function from $X$ to a space which is topologically of finite length, then $X$ is topologically of finite length.*

*Proof.* (a) This is immediate from the criteria 6B(b) or 6B(c).

(b) We may suppose that there is a finite-to-one continuous function $\phi: X \to Y$ where $(Y, \sigma)$ is a metric space of finite length. If $x$ and $y$ are distinct points of $X$, take a neighbourhood $U$ of $x$, with finite boundary, such that $\bar{U}$ does not contain either $y$ or any point of $\phi^{-1}[\{\phi(x)\}]$ other than $x$. Then $\delta = \min\{\sigma(\phi(x), \phi(z)): z \in \partial U\} > 0$. Define $f: X \to \mathbb{R}$ by setting $f(z) = \min(\delta, \sigma(\phi(x), \phi(z)))$ for $z \in U$, $f(z) = \delta$ if $z \in X \backslash U$. Then by 1A(f) there are uncountably many $\alpha \in ]f(x), f(y)[ = ]0, \delta[$ such that $\{w: w \in Y, \sigma(w, \phi(x)) = \alpha\}$ is finite, and for any such $\alpha$ we shall have $f^{-1}[\{\alpha\}]$ finite. Thus $f$ witnesses 6B(d) for $x$ and $y$. As $x$ and $y$ are arbitrary, $X$ is topologically of finite length.

6H. Remarks. It is now easy to see that the conditions (A), (B), (C) of [**6**, Theorem 3], are equivalent for all separable metric spaces. To bring their conditions (D) and (E) into play we should add the phrase 'of which $X_0$ is compact' to the phrase 'given any two disjoint closed subsets $X_0$, $X_1$ in $X$'.

6I. Using Theorem 5B we can add another, elementary, characterization of finite length.

Proposition. *A topological space $X$ is topologically of finite length if and only if there are a metric $\sigma$ on $X$ defining its topology, a bounded set $A \subseteq \mathbb{R}$ and a surjection from $A$ onto $X$ which is Lipschitz-1 for $\sigma$ and the usual metric of $A$.*

*Proof.* (a) Suppose that $X$ is topologically of finite length. We may express $X$ as a subset of a compact connected metric space $(Z, \rho)$ of finite length, by 5B. Now there is a Lipschitz-1 surjection $f: [0, 2\mu_\rho(Z)] \to Z$. Set $A = f^{-1}[X]$, $\sigma = \rho \restriction X^2$; then $\sigma$, $A$ and $f \restriction A$ witness the condition.

(b) If $X$ satisfies the condition then $\mu_\sigma^*(X) \leq \lambda^*(A) < \infty$, by 1A(c) and 1A(e); so $X$ is topologically of finite length.

Concluding remarks. The obvious challenge left open here is to find some appropriate extension of these ideas to two- and higher-dimensional Hausdorff measure. The difficulties are likely to be immense; see for instance [**20**]. By and large, the results of § 1 generalize (see [**6, 7**]); if $X$ has finite $d$-dimensional Hausdorff measure, its topology has a base consisting of sets whose boundaries have finite $(d-1)$-dimensional measure. When we come to connectedness, however, it is not clear what we should aim to do. The standard examples of connected spaces which are not locally connected, etc., can be embedded as bounded sets of $\mathbb{R}^2$ of zero two-dimensional measure; and we can have a dense connected subset of $\mathbb{R}^n$ of zero two-dimensional measure, so that 4A dies. It may be that some $m$-dimensional analogue of connectedness (see, for example, [**11**, §§ 4–9]) is relevant here.

When we come to §§ 5–6, a number of direct questions present themselves. For instance, following 6B(d), we can ask the following:

*Let $(X, \rho)$ be a separable metric space, and suppose that for each $x \in X$, open $G$ containing $x$, there is a continuous function $f$: $X \to \mathbb{R}$ such that $f(x) = 0$, $f(y) = 1$ for every $y \in X \setminus G$, and $\{\alpha$: $\alpha \in ]0, 1[$, $\mu_{k,\rho}(f^{-1}[\{\alpha\}]) < \infty\}$ is uncountable. Does it follow that there is an equivalent metric $\sigma$ on $X$ such that $\mu_{k+1,\sigma}(X) < \infty$?*

(Here I write $\mu_{k,\rho}$ for $k$-dimensional Hausdorff measure.) Or, following 5B and 5H, we can ask:

*If $(X, \rho)$ is a metric space of finite k-dimensional Hausdorff measure, is it necessarily homeomorphic to a subspace of a compact metric space, or of $\mathbb{R}^{2k+1}$, of finite k-dimensional measure?*

Finally, I gave 6I above as a foundation for asking:

*If $(X, \rho)$ is a metric space of finite k-dimensional Hausdorff measure, can we find an equivalent metric $\sigma$ on $X$, a bounded set $A \subseteq \mathbb{R}^k$ and a surjection from $A$ onto $X$ which is Lipschitz-1 for $\sigma$ and the usual metric of $A$?*

*Added in proof* (*November 1991*). I have shown that if $(X, \rho)$ is any metric space of finite length, there is an $f$: $X \to \mathbb{R}^3$ such that $f$ is a homeomorphism between $X$ and $f[X]$ and $\mu_\tau^* f[A] = \mu_\rho^* A$ for every $A \subseteq X$, where $\tau$ is the Euclidean metric of $\mathbb{R}^3$. The proof is in University of Essex Research Reports 91–22 and 91–25, titled respectively 'Embedding spaces of finite length in continua' and 'Embedding spaces of finite length in $\mathbb{R}^3$'.

*Acknowledgements*

I should like to acknowledge the help of a number of friends and colleagues. M. Burke introduced me to the problems here, and carefully read an early draft. C. A. Rogers and R. O. Davies made encouraging and helpful comments. M. Talagrand suggested the use of Borel determinacy in the context of 3F. R. D. Mauldin referred me to **[6]**, and M. Laczkovich referred me to **[1]**. The phrase 'finite cut space' was proposed by the referee, who also suggested the possibility of a result along the lines of 5H.

*References*

**1.** M. BOGNÁR, 'On the exterior linear measure', *Acta. Math. Hungar.* 53 (1989) 159–168.
**2.** A. CSÁSZÁR, *General topology* (Adam Hilger, 1978).
**3.** S. EILENBERG, 'On $\Phi$ measures', *Ann. Soc. Polon. Math.* 17 (1938) 252–253.
**4.** S. EILENBERG, 'On continua of finite length', *Ann. Soc. Polon. Math.* 17 (1938) 253–254.
**5.** S. EILENBERG, 'Continua of finite linear measure II', *Amer. J. Math.* 66 (1944) 425–427.
**6.** S. EILENBERG and O. G. HARROLD JR, 'Continua of finite linear measure I', *Amer. J. Math.* 65 (1943) 137–146.
**7.** K. J. FALCONER, *The geometry of fractal sets* (Cambridge University Press, 1985).
**8.** H. FEDERER, *Geometric measure theory* (Springer, Berlin, 1969).
**9.** D. H. FREMLIN, *Topological Riesz spaces and measure theory* (Cambridge University Press, 1974).
**10.** O. G. HARROLD, 'Continua of finite length and certain product sets', *Bull. Amer. Math. Soc.* 46 (1940) 951–953.
**11.** J. G. HOCKING and G. S. YOUNG, *Topology* (Addison-Wesley, Reading, Mass., 1961).
**12.** W. HUREWICZ and H. WALLMAN, *Dimension theory* (Princeton University Press, 1948).

**13.** B. KNASTER, 'Un continu dont tout sous-continu est indécomposable', *Fund. Math.* 3 (1922) 247–286.
**14.** K. KURATOWSKI, *Topology,* vol. I (Academic Press, New York, 1966).
**14b.** K. KURATOWSKI, *Topology,* vol. II (Academic Press, New York, 1968).
**15.** D. A. MARTIN, 'Borel determinacy', *Ann. of Math.* 102 (1975) 363–371.
**16.** D. A. MARTIN, 'A purely inductive proof of Borel determinacy', [**19**, pp. 303–308].
**17.** R. L. MOORE, 'A connected and regular point set which contains no arc', *Bull. Amer. Math. Soc.* 32 (1926) 331–332.
**18.** Y. N. MOSCHOVAKIS, *Descriptive set theory* (North-Holland, Amsterdam, 1982).
**19.** A. NERODE and R. A. SHORE (eds), *Recursion theory,* Proceedings of Symposia in Pure Mathematics 42 (American Mathematical Society, Providence, R.I., 1985).
**20.** D. PREISS, 'Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability and densities', *Ann. of Math.* 125 (1987) 537–643.
**21.** G. T. WHYBURN, 'Concerning continua of finite degree and local separating points', *Amer. J. Math.* 57 (1935) 11–18.
**22.** G. T. WHYBURN, *Analytic topology* (American Mathematical Society, Providence, R.I., 1942).

*Department of Mathematics*
*University of Essex*
*Wivenhoe Park*
*Colchester CO4 3SQ*
*Essex*