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Illanes's ω-resolvability theorem

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I write out a proof of the principal theorem of Illanes 96, there credited to E.K.van Douwen.

1 Density algebras

1A Definitions (a) A **density structure** is a pair $(\mathfrak{A},\mathfrak{G})$ where

$\mathfrak{A}$ is a Boolean algebra,

$\mathfrak{G}$ is a Dedekind complete order-closed subalgebra of $\mathfrak{A}$,

$\sup A$ is defined in $\mathfrak{A}$ whenever $A\subseteq\mathfrak{A}$ is such that $\operatorname{upr}(a,\mathfrak{G})\cap\operatorname{upr}(b,\mathfrak{G})=0$ for all distinct a, $b\in A$.

[Here $\operatorname{upr}(a,\mathfrak{G})=\min\{g:a\subseteq g\in\mathfrak{G}\}$, as in Fremlin 12, 313S.]

(b) Suppose that $(\mathfrak{A},\mathfrak{G})$ is a density structure, $a\in\mathfrak{A}$ and κ is a cardinal.

(i) a is **κ-resolvable** if there is a disjoint family $\langle a_\xi\rangle_{\xi<\kappa}$ such that $a_\xi\subseteq a$ and $\operatorname{upr}(a_\xi,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$ for every $\xi<\kappa$.

(ii) a is **resolvable** if it is 2-resolvable.

1B Lemma Let $(\mathfrak{A},\mathfrak{G})$ be a density structure and κ a cardinal.

(a) If $a\in\mathfrak{A}$ is κ-resolvable then $a\cap g$ is κ-resolvable for every $g\in\mathfrak{G}$.

(b) If $a\in\mathfrak{A}$ and $C\subseteq\mathfrak{G}$, then $a\cap\sup C$ is κ-resolvable iff $a\cap g$ is κ-resolvable for every $g\in C$.

(c) If $a\in\mathfrak{A}$ is not resolvable, there is a $g_0\in\mathfrak{G}$ such that $a\cap g_0$ is a relative atom over $\mathfrak{G}$ (definition: Fremlin 12, 331A).

proof (a) There is a disjoint $\langle a_\xi\rangle_{\xi<\kappa}$ such that $a_\xi\subseteq a$ and $\operatorname{upr}(a_\xi,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$ for every $\xi<\kappa$; now $\langle a_\xi\cap g\rangle_{\xi<\kappa}$ is disjoint, $a_\xi\cap g\subseteq a\cap g$ and

$$\operatorname{upr}(a_\xi\cap g,\mathfrak{G})=\operatorname{upr}(a_\xi,\mathfrak{G})\cap g=\operatorname{upr}(a,\mathfrak{G})\cap g=\operatorname{upr}(a\cap g,\mathfrak{G})$$

for every ξ, by Fremlin 12, 313Sc.

(b) If $a\cap\sup C$ is κ-resolvable, then $a\cap g=(a\cap\sup C)\cap g$ is κ-resolvable for every $g\in C$, by (a). If $a\cap g$ is κ-resolvable for every $g\in C$, let D be a maximal disjoint subset of $\{g:g\in\mathfrak{G},\ g\subseteq g'$ for some $g'\in C\}$. For $g\in D$, $a\cap g$ is κ-resolvable, by (a) again; let $\langle a_{g\xi}\rangle_{\xi<\kappa}$ be a disjoint family such that $a_{g\xi}\subseteq a\cap g$ and $\operatorname{upr}(a_{g\xi},\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$ for every $\xi<\kappa$. By the last clause in the definition 1Aa, $a_\xi=\sup_{g\in D}a_{g\xi}$ is defined for every $\xi<\kappa$. Now $\langle a_\xi\rangle_{\xi<\kappa}$ is disjoint and if $\xi<\kappa$ then $a_\xi\subseteq a\cap\sup C$ and

$$\operatorname{upr}(a_\xi,\mathfrak{G})=\sup_{g\in D}\operatorname{upr}(a_{g\xi},\mathfrak{G})$$

(Fremlin 12, 313Sb)

$$=\sup_{g\in D}\operatorname{upr}(a\cap g,\mathfrak{G})=\sup_{g\in D}\operatorname{upr}(a,\mathfrak{G})\cap g$$
$$=\operatorname{upr}(a,\mathfrak{G})\cap\sup D=\operatorname{upr}(a,\mathfrak{G})\cap\sup C$$

(because D is maximal, so $\sup D=\sup C$)

$$=\operatorname{upr}(a\cap\sup C,\mathfrak{G}).$$

Thus a is κ-resolvable.

(c) Set $C=\{g:g\in\mathfrak{G},\ a\cap g$ is resolvable$\}$ and $g_0=1\setminus\sup C$. By (b), $a\cap\sup C$ is resolvable, so $a\not\subseteq\sup C$ and $a\cap g_0\ne 0$. If $b\subseteq a\cap g_0$, consider $g=\operatorname{upr}(b,\mathfrak{G})\cap\operatorname{upr}(a\setminus b,\mathfrak{G})$. Then $\operatorname{upr}(a\cap g\cap b,\mathfrak{G})=$

$\operatorname{upr}(a,\mathfrak{G})\cap g$ and $\operatorname{upr}(a\cap g\setminus b,\mathfrak{G}) = \operatorname{upr}(a\setminus b,\mathfrak{G})\cap g$ are both equal to g, so their union $\operatorname{upr}(a\cap g,\mathfrak{G})$ is also equal to g. and $a\cap g$ is resolvable. Thus $g\in C$; but $g\subseteq \operatorname{upr}(b,\mathfrak{G})\subseteq g_0$, so $g=0$. Accordingly $(a\setminus b)\cap \operatorname{upr}(b,\mathfrak{G})=0$. But this means that $b=a\cap g_0\cap \operatorname{upr}(b,\mathfrak{G})$ belongs to $\{a\cap g_0\cap g' : g'\in\mathfrak{G}\}$. As b is arbitrary, $a\cap g_0$ is a relative atom over $\mathfrak{G}$.

1C Lemma Let $(\mathfrak{A},\mathfrak{G})$ be a density structure, $a\in\mathfrak{A}$ and $d\subseteq a$ a relative atom over $\mathfrak{G}$. If $n\in\mathbb{N}$ and a is $(n+1)$-resolvable then $a\setminus d$ is n-resolvable. If a is resolvable, $\operatorname{upr}(a\setminus d,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$.

proof (a) Let $a_0,\dots,a_n$ be disjoint and such that $a_i\subseteq a$ and $\operatorname{upr}(a_i,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$ for every $i\le n$. For each $i\le n$, let $g_i\in\mathfrak{G}$ be such that $d\cap a_i=d\cap g_i$; we can suppose that $g_i\subseteq\operatorname{upr}(a,\mathfrak{G})$. If $i\ne j$ then $a_j\cap g_i\subseteq a\cap g_i\setminus d$ and

$$\begin{aligned}\operatorname{upr}(a_j\cap g_i,\mathfrak{G})&=\operatorname{upr}(a_j,\mathfrak{G})\cap g_i=\operatorname{upr}(a,\mathfrak{G})\cap g_i\\&\supseteq\operatorname{upr}(a\cap g_i\setminus d,\mathfrak{G})\supseteq\operatorname{upr}(a_j\cap g_i,\mathfrak{G}),\end{aligned}$$

so $\langle a_j\cap g_i\rangle_{j\le n,j\ne i}$ witnesses that $a\cap g_i\setminus d$ is n-resolvable. And setting $g=1\setminus\sup_{i\le n}g_i$, $a_i\cap g\cap d=0$ for every i, so $\langle a_i\cap g\rangle_{i<n}$ witnesses that $a\cap g\setminus d$ is n-resolvable. By 1Bb, $a\setminus d$ is n-resolvable.

(b) Now suppose that a is resolvable. Then we can repeat the argument above with $n=1$, observing that $d\cap g_0\cap a_1=d\cap g_1\cap a_0=0$. In this case, $\operatorname{upr}(a\setminus d,\mathfrak{G})$ includes both $\operatorname{upr}(a_1\cap g_0,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})\cap g_0$ and $\operatorname{upr}(a_0\cap g_1,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})\cap g_1$ and $\operatorname{upr}(a_0\cap g,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})\cap g$. So $\operatorname{upr}(a\setminus d,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$.

1D Lemma Let $(\mathfrak{A},\mathfrak{G})$ be a density structure. Suppose that $a\in\mathfrak{A}$ is n-resolvable for every $n\in\mathbb{N}$. Then there is a $d^*\subseteq a$ such that $\operatorname{upr}(d^*,\mathfrak{G})=\operatorname{upr}(a\setminus d^*,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$ and $a\setminus d^*$ is n-resolvable for every $n\in\mathbb{N}$.

proof (a) Suppose that $d\subseteq a$ is a relative atom over $\mathfrak{G}$. By 1C, $\operatorname{upr}(a\setminus d,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$ and $a\setminus d$ is n-resolvable for every $n\in\mathbb{N}$.

(b) Suppose that $g\in\mathfrak{G}$ is such that d is resolvable whenever $d\subseteq a\cap g$ and $\operatorname{upr}(d,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$. Then there is a $d\subseteq a\cap g$ such that $\operatorname{upr}(d,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$ and $a\setminus d$ is n-resolvable for every $n\in\mathbb{N}$. **P** Choose a non-increasing $\langle d_n\rangle_{n\in\mathbb{N}}$ such that $d_0=a\cap g$ and

$$\operatorname{upr}(d_{n+1},\mathfrak{G})=\operatorname{upr}(d_n\setminus d_{n+1},\mathfrak{G})=\operatorname{upr}(d_n,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$$

for every n. Take $d=d_0\setminus d_1$. **Q**

(c) Let C be the set of those $g\in\mathfrak{G}$ such that there is a $d\subseteq a\cap g$ with $\operatorname{upr}(d,\mathfrak{G})=\operatorname{upr}(a\cap g\setminus d,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$ and $a\setminus d$ is n-resolvable for every $n\in\mathbb{N}$. Then C is a π-base for $\mathfrak{G}$. **P** Take $g_0\in\mathfrak{G}\setminus\{0\}$. If d is resolvable whenever $d\subseteq a\cap g_0$ and $\operatorname{upr}(d,\mathfrak{G})=\operatorname{upr}(a\cap g_0,\mathfrak{G})$, then (b) tells us that $g_0\in C$. Otherwise, let $d\subseteq a\cap g_0$ be such that $\operatorname{upr}(d,\mathfrak{G})=\operatorname{upr}(a\cap g_0,\mathfrak{G})$ and d is not resolvable. By 1Bc, there is a $g\in\mathfrak{G}$ such that $d\cap g$ is a relative atom over $\mathfrak{G}$; as $d\subseteq g_0$ we can take it that $g\subseteq g_0$. Now $\operatorname{upr}(d\cap g,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$, while $\operatorname{upr}(a\cap g\setminus d,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})$ and $a\cap g\setminus d$ is n-reducible for every $n\in\mathbb{N}$, by 1C. So $g\in C$. As g_0 is arbitrary, C is a π-base. **Q**

(d) Let $D\subseteq C$ be a maximal disjoint set. Then $\sup D=1$. For $g\in D$, let $d_g\subseteq a\cap g$ be such that $\operatorname{upr}(d_g,\mathfrak{G})=\operatorname{upr}(a\cap g,\mathfrak{G})=\operatorname{upr}(a\cap g\setminus d_g,\mathfrak{G})$ and $a\cap g\setminus d_g$ is n-resolvable for every $n\in\mathbb{N}$. Set $d^*=\sup_{g\in D}d_g$. Then

$$\operatorname{upr}(d^*,\mathfrak{G})=\sup_{g\in D}\operatorname{upr}(a,\mathfrak{G})\cap g=\operatorname{upr}(a,\mathfrak{G})$$

and

$$\operatorname{upr}(a\setminus d^*,\mathfrak{G})=\sup_{g\in D}\operatorname{upr}(a\setminus d_g,\mathfrak{G})\cap g=\operatorname{upr}(a,\mathfrak{G}).$$

Next, $a\cap g\setminus d^*=a\cap g\setminus d_g$ is n-resolvable whenever $n\in\mathbb{N}$ and $g\in D$; by 1Bb, $a\setminus d^*$ is n-resolvable whenever $n\in\mathbb{N}$, as required.

1E Theorem Let $(\mathfrak{A},\mathfrak{G})$ be a density structure. If $a\in\mathfrak{A}$ is n-resolvable for every $n\in\mathbb{N}$, it is ω-resolvable.

proof Choose $\langle a_m\rangle_{m\in\mathbb{N}}$ inductively, as follows. Given that $\sup_{i<m}a_i\subseteq a$ and $a\setminus\sup_{i<m}a_i$ is n-resolvable for every $n\in\mathbb{N}$, use 1D to find $a_m\subseteq a\setminus\sup_{i<m}a_i$ such that

$$\operatorname{upr}(a_m,\mathfrak{G})=\operatorname{upr}(a\setminus\sup_{i\le m}a_i,\mathfrak{G})=\operatorname{upr}(a\setminus\sup_{i<m}a_i,\mathfrak{G})$$

and $a\setminus\sup_{i\le m}a_i$ is n-resolvable for every n. Then $\langle a_m\rangle_{m\in\mathbb{N}}$ is disjoint and

$$\operatorname{upr}(a_m,\mathfrak{G})=\operatorname{upr}(a\setminus\sup_{i<m}a_i,\mathfrak{G})=\operatorname{upr}(a,\mathfrak{G})$$

for every m, so a is ω-reducible.

2 Topological spaces

2A Proposition Let X be a topological space. Write $\mathcal{N}\mathrm{wd}$ for the ideal of nowhere dense subsets of X and $\mathfrak{A}$ for the quotient Boolean algebra $\mathcal{P}X/\mathcal{N}\mathrm{wd}$. Set $\mathfrak{G}=\{G^\bullet:G\subseteq X\text{ is open}\}\subseteq\mathfrak{A}$.

(a) $(\mathfrak{A},\mathfrak{G})$ is a density structure.

(b) A subset A of X is dense iff $\operatorname{upr}(A^\bullet,\mathfrak{G})=1$.

proof (a) Since the union of two open sets is again open, $\mathfrak{G}$ is closed under $\cup$. If $G\subseteq X$ is open, then $\overline{G}\setminus G$ is nowhere dense, so

$$1\setminus G^\bullet=1\setminus\overline{G}^\bullet=(X\setminus\overline{G})^\bullet\in\mathfrak{G}.$$

Thus $\mathfrak{G}$ is closed under complementation. Also $0=\emptyset^\bullet\in\mathfrak{G}$, so $\mathfrak{G}$ is a subalgebra of $\mathfrak{A}$ (Fremlin 12, 312B).

If $C\subseteq\mathfrak{G}$ then $\sup C$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{G}$. **P** Set $\mathcal{G}=\{G:G\subseteq X\text{ is open, }G^\bullet\in C\}$ and $g=(\bigcup\mathcal{G})^\bullet$. Then g is an upper bound of $\{G^\bullet:G\in\mathcal{G}\}=C$. If $a\in\mathfrak{A}$ is an upper bound of C, express it as $A^\bullet$ where $A\subseteq X$. Then $(G\setminus A)^\bullet=G^\bullet\setminus a=0$, that is, $G\setminus A$ is nowhere dense, for every $G\in\mathcal{G}$. But this implies that $(\bigcup\mathcal{G})\setminus A$ is nowhere dense and $g\subseteq a$. Thus $g=\sup C$ in $\mathfrak{A}$, while $g\in\mathfrak{G}$. **Q** It follows that $\mathfrak{G}$ is order-closed in $\mathfrak{A}$ (Fremlin 12, 313Ea) and also that $\mathfrak{G}$ is Dedekind complete.

Now suppose that $A\subseteq\mathfrak{A}$ is such that $\langle\operatorname{upr}(a,\mathfrak{G})\rangle_{a\in A}$ is disjoint. For each $a\in A$, take an open set $G_a\subseteq X$ such that $\operatorname{upr}(a,\mathfrak{G})=G_a^\bullet$; then $G_a\cap G_b$ is nowhere dense, therefore empty, whenever a and b are distinct. For each $a\in A$ let $B_a\subseteq G_a$ be such that $B_a^\bullet=a$. Set $B=\bigcup_{a\in A}B_a$. Then $B^\bullet$ is an upper bound of A in $\mathfrak{A}$. If c is any upper bound of A, and $c=C^\bullet$ where $C\subseteq X$, then $G_a\cap B\setminus C=B_a\setminus C$ is nowhere dense for every $a\in A$, so $B\setminus C=(\bigcup_{a\in A}G_a)\cap B\setminus C$ is nowhere dense, and $B^\bullet\subseteq c$; accordingly $\sup A=B^\bullet$ is defined in $\mathfrak{A}$. Thus all the conditions of 1Aa are satisfied.

(b)

$$\begin{aligned}A\text{ is dense}&\iff A\cap G\notin\mathcal{N}\mathrm{wd}\text{ for every non-empty open }G\subseteq X\\&\iff A^\bullet\cap g\ne0\text{ for every non-zero }g\in\mathfrak{G}\\&\iff\operatorname{upr}(A^\bullet,\mathfrak{G})=1.\end{aligned}$$

2B Theorem (Illanes 96, Theorem 5) Let X be a topological space such that for every $n\in\mathbb{N}$ there is a disjoint family $\langle D_i\rangle_{i<n}$ of dense subsets of X. Then there is a disjoint sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of dense subsets.

proof Take $\mathfrak{A}$ and $\mathfrak{G}$ as in 2A. For any $n\in\mathbb{N}$, we have a disjoint family $\langle D_i\rangle_{i<n}$ of dense sets; now $\langle D_i^\bullet\rangle_{i<n}$ is a disjoint family in $\mathfrak{A}$ such that $\operatorname{upr}(D_i^\bullet,\mathfrak{G})=1$ for every $i<n$, by 2Ab. Thus $1\in\mathfrak{A}$ is n-resolvable for every $n\in\mathbb{N}$. By Theorem 1E, 1 is ω-resolvable and we have a disjoint sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\operatorname{upr}(d_n,\mathfrak{G})=1$ for every n. Choose $B_n\subseteq X$ such that $B_n^\bullet=d_n$ for $n\in\mathbb{N}$, and set $D_n=B_n\setminus\bigcup_{i<n}B_i$; then $\operatorname{upr}(D_n^\bullet,\mathfrak{G})=\operatorname{upr}(d_n,\mathfrak{G})=1$ for each n, while $\langle D_n\rangle_{n\in\mathbb{N}}$ is disjoint. By 2Ab in the other direction, every D_n is dense in X, so we have a suitable sequence.

In §1 I gave a formula for 'κ-resolvable' element of $\mathfrak{A}$ for arbitrary cardinals κ. In Illanes 96 there is a corresponding definition for κ-resolvable topological space. For $\kappa\le\omega$ the construction in 2A makes these definitions match comfortably, so that Illanes's theorem can be deduced from 1E above. For uncountable κ we can expect to meet cases in which $1\in\mathfrak{A}$ is κ-resolvable but X does not have a disjoint family of κ dense sets.

3 Proto-decomposable spaces

I adapt an idea from Fremlin 87.

3A Definition I will say that a **weakly proto-decomposable space** is a quadruple $(X,\mathcal{E},\mathcal{I},\mathcal{U})$ where

X is a set,

$\mathcal{E}$ is an algebra of subsets of X,

$\mathcal{I}$ is an ideal of subsets of X,

$\mathcal{E}\cap\mathcal{I}$ is cofinal with $\mathcal{I}$,

$\mathcal{U}\subseteq\mathcal{E}$

and whenever $\mathcal{V}\subseteq\mathcal{U}$ is disjoint then

$\bigcup\mathcal{V}\in\Sigma$,

if $\langle A_V\rangle_{V\in\mathcal{V}}$ is a family in $\mathcal{I}$ such that $A_V\subseteq V$ for every $V\in\mathcal{V}$, then $\bigcup_{V\in\mathcal{V}}A_V\in\mathcal{I}$,

if $E\in\mathcal{E}$ and $E\setminus\bigcup\mathcal{V}\notin\mathcal{I}$, then there is a non-empty $U\in\mathcal{U}$ such that $U\cap\bigcup\mathcal{V}=\emptyset$ and $U\setminus E\in\mathcal{I}$.

Remark I am following the definition of 'proto-decomposable measurable space with negligibles' in Fremlin 87, 1Bh, except that I no longer assume that $\mathcal{E}$ is a σ-algebra or that $\mathcal{I}$ is a σ-ideal. In particular, we have the examples 1G, 6Bf and 7C in Fremlin 87, in which, respectively,

— $\mathcal{E}$ is a σ-algebra of subsets of X, $\mathcal{I}$ is a σ-ideal of subsets of X such that $\mathcal{E}\cap\mathcal{I}$ is cofinal with $\mathcal{I}$ and is ω_1-saturated in $\mathcal{E}$, and $\mathcal{U}=\mathcal{E}\setminus\mathcal{I}$;

— $(X,\mathcal{E},\mu)$ is a complete strictly localizable measure space, $\mathcal{I}$ is the ideal of negligible sets, and $\mathcal{U}$ is the family of measurable sets of non-zero finite measure;

— $(X,\mathcal{U})$ is a topological space, $\mathcal{E}$ is the algebra of sets with the Baire property and $\mathcal{I}$ is the ideal of meager sets.

3B Theorem Let $(X,\mathcal{E},\mathcal{I},\mathcal{U})$ be a weakly proto-decomposable space. Set $\mathfrak{A}=\mathcal{P}X/\mathcal{I}$ and $\mathfrak{G}=\{E^\bullet:E\in\mathcal{E}\}\subseteq\mathfrak{A}$. Then $(\mathfrak{A},\mathfrak{G})$ is a density structure.

proof (a) Because $\mathcal{E}$ is a subalgebra of $\mathcal{P}X$, $\mathfrak{G}$ is a subalgebra of $\mathfrak{A}$. If $C\subseteq\mathfrak{G}$, then $\sup C$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{G}$. **P** Let $\mathcal{V}\subseteq\mathcal{U}$ be a maximal disjoint set such that for every $V\in\mathcal{V}$ there is a $g\in C$ such that $V^\bullet\subseteq g$, and consider $G=\bigcup\mathcal{V}$. Then $G\in\mathcal{E}$ so $G^\bullet\in\mathfrak{G}$. If $g\in C$ there is an $E\in\mathcal{E}$ such that $E^\bullet=g$. By the maximality of $\mathcal{V}$ there is no non-empty $U\in\mathcal{U}$ such that $U\cap G=\emptyset$ and $U\setminus E\in\mathcal{I}$, so $E\setminus G\in\mathcal{I}$ and $g\subseteq G^\bullet$. Thus $G^\bullet$ is an upper bound of C. If $a\in\mathfrak{A}$ is an upper bound of C, express a as $A^\bullet$ where $A\subseteq X$. If $V\in\mathcal{V}$, there is a $g\in C$ such that $V^\bullet=g\subseteq a$, so $V\setminus A\in\mathcal{I}$; consequently $(\bigcup\mathcal{V})\setminus A=\bigcup_{V\in\mathcal{V}}V\setminus A$ belongs to $\mathcal{I}$ and $G^\bullet\subseteq a$. Accordingly $\sup C=G^\bullet$ is defined and belongs to $\mathfrak{G}$. **Q**

Thus $\mathfrak{G}$ is a Dedekind complete order-closed subalgebra of $\mathfrak{A}$. Now suppose that $C\subseteq\mathfrak{A}$ and $\operatorname{upr}(a,\mathfrak{G})\cap\operatorname{upr}(b,\mathfrak{G})=0$ whenever a, $b\in C$ are distinct. Let $\mathcal{V}\subseteq\mathcal{U}$ be a maximal disjoint family such that for every $V\in\mathcal{V}$ there is an $a\in C$ such that $V^\bullet\subseteq\operatorname{upr}(a,\mathfrak{G})$. For each $V\in\mathcal{V}$ choose $a_V\in C$ such that $V^\bullet\subseteq\operatorname{upr}(a_V,\mathfrak{G})$ and $A_V\subseteq V$ such that $A_V^\bullet=a_V\cap V^\bullet$. Set $A=\bigcup_{V\in\mathcal{V}}A_V$.

? If $A^\bullet$ is not an upper bound of C, there is a $b\in C$ such that $b'=b\setminus A^\bullet$ is not 0. Let $E\in\mathcal{E}$ be such that $E^\bullet=\operatorname{upr}(b',\mathfrak{G})\subseteq\operatorname{upr}(b,\mathfrak{G})$. By the maximality of $\mathcal{V}$, $E\setminus\bigcup\mathcal{V}\in\mathcal{I}$. So there is a $B\subseteq E\cap\bigcup\mathcal{V}$ such that $B^\bullet=b'$. As $B\notin\mathcal{I}$, there must be a $V\in\mathcal{V}$ such that $B\cap V\notin\mathcal{I}$, that is, $b'\cap\operatorname{upr}(a_V,\mathfrak{G})\ne0$, in which case $b\cap\operatorname{upr}(a_V,\mathfrak{G})\ne0$; but this means that $b=a_V$ and

$$A_V^\bullet=a_V\cap V^\bullet\supseteq b'\cap V^\bullet=(B\cap V)^\bullet,$$

which is non-zero and disjoint from $A^\bullet$, even though $A_V\subseteq A$. **X** So $A^\bullet$ is an upper bound of C.

On the other hand, if d is any upper bound of C, $A^\bullet\subseteq d$. **P** Let $D\subseteq X$ be such that $D^\bullet=d$. For $V\in\mathcal{V}$,

$$(V\cap A\setminus D)^\bullet=(A_V\setminus D)^\bullet\subseteq a_V\setminus d=0,$$

so $V\cap A\setminus D\in\mathcal{I}$; because $\mathcal{V}\subseteq\mathcal{U}$ is disjoint and $A\subseteq\bigcup\mathcal{V}$, $A\setminus D\in\mathcal{I}$ and $A^\bullet\subseteq d$. **Q**

Thus $A^\bullet$ is the least upper bound of C. As C is arbitary, the third clause of 1Aa is satisfied and $(\mathfrak{A},\mathfrak{G})$ is a density structure.

3C Corollary Let (X,Σ,μ) be a strictly localizable measure space. If for every $n\in\mathbb{N}$ there is a disjoint family $\langle D_i\rangle_{i<n}$ of subsets of full outer measure, then there is a disjoint sequence of sets of full outer measure.

proof Completing the measure leaves it strictly localizable (Fremlin 16, 212G), and does not change the family of sets of full outer measure (Fremlin 16, 212Eb), so we may suppose that μ is complete. In this case, we have a proto-decomposable space $(X,\Sigma,\mathcal{I},\mathcal{U})$, as noted in 3A, and the associated density structure $(\mathfrak{A},\mathfrak{G})$ as in 3B. If $D\subseteq X$ then, just as in 2Ab,

$$D \text{ has full outer measure } \iff D \cap E \notin \mathcal{I} \text{ for every } E \in \Sigma \setminus \mathcal{I}$$
$$\iff \operatorname{upr}(D^\bullet, \mathfrak{G}) = 1,$$

so we can read the conclusion off from Theorem 1E, just as in 2B.

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