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The Lebesgue density theorem in separable metric spaces

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Proposition Let X be a separable metrizable space and μ a locally finite quasi-Radon measure on X. Then there is a metric ρ on X, compatible with its topology, such that for every measurable $E \subseteq X$ there is a negligible set F such that

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for every $x \in E \setminus F$, where $B(x,\delta) = \{y : \rho(x,y) \le \delta\}$.

proof (a) Since X is Lindelöf and μ is locally finite, there is a sequence of open sets of finite measure covering X. It follows at once that μ is σ-finite; and since μ is inner regular with respect to the closed sets, it is also outer regular with respect to the open sets (Fremlin 03, 412Wb).

Let $\mathfrak{A}$ be the family of subsets of X with negligible boundaries; then $\mathfrak{A}$ is an algebra of subsets of X (the 'Jordan algebra' of (X,μ)). Because μ is complete and measures every open set, it measures every member of $\mathfrak{A}$. Let $\mathbb{A}$ be the family of finite partitions of X into sets belonging to $\mathfrak{A}$. For $\mathcal{A}$, $\mathcal{A}' \in \mathbb{A}$ say that $\mathcal{A} \preccurlyeq \mathcal{A}'$ if $\mathcal{A}$ refines $\mathcal{A}'$, that is, every member of $\mathcal{A}$ is included in some member of $\mathcal{A}'$. Then $\preccurlyeq$ is a partial order on $\mathbb{A}$ under which $\mathbb{A}$ is downwards-directed (because if $\mathcal{A}$, $\mathcal{A}' \in \mathbb{A}$ then $\{A \cap A' : A \in \mathcal{A},\ A' \in \mathcal{A}'\}$ belongs to $\mathbb{A}$). If $f : X \to \mathbb{R}$ is a bounded continuous function, then $\{\gamma : \mu f^{-1}[\{\gamma\}] > 0\}$ must be countable, and for all but countably many γ the sets $\{x : f(x) < \gamma\}$ and $\{x : f(x) \le \gamma\}$ belong to $\mathfrak{A}$. So for any $\epsilon > 0$ there is an $\mathcal{A} \in \mathbb{A}$ such that $f[A]$ has diameter at most ϵ for every $A \in \mathcal{A}$.

Because X is second-countable and completely regular, there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of continuous functions from X to $[0,1]$ such that whenever $G \subseteq X$ is open and $x \in G$ there is an $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f_n(y) = 0$ for $y \in X \setminus G$.

(b) Choose families $\langle \mathcal{A}_n\rangle_{n\in\mathbb{N}}$, $\langle g_{nA}\rangle_{n\in\mathbb{N},A\in\mathcal{A}_n}$, $\langle H_n\rangle_{n\in\mathbb{N}}$ and $\langle F_n\rangle_{n\in\mathbb{N}}$ as follows. Start with $\mathcal{A}_0 = \{X\}$ and $F_0 = \emptyset$. Given that F_n is a finite family of continuous functions from X to $[0,1]$ and that $\mathcal{A}_n \in \mathbb{A}$, let H_n be an open set, including $\bigcup_{A\in\mathcal{A}_n} \partial A$, of measure at most $2^{-n}\min\{\mu A : A \in \mathcal{A}_n,\ \mu A > 0\}$. Then $A \setminus H_n$ is closed and $A \cup H_n$ is open for every $A \in \mathcal{A}_n$. Because X is normal, we can choose $\langle g_{nA}\rangle_{A\in\mathcal{A}_n}$ such that each g_{nA} is a continuous function from X to $[0,1]$, $g_{nA}(x) = 1$ for $x \in A\setminus H_n$ and $g_{nA}(x) = 0$ for $x \notin A\cup H_n$. Set $F_{n+1} = F_n \cup \{f_n\} \cup \{g_{nA} : A \in \mathcal{A}_n\}$. Because $\mathbb{A}$ is downwards-directed there is a $\mathcal{A}_{n+1} \in \mathbb{A}$, refining $\mathcal{A}_n$, such that $\operatorname{diam} f[A] \le 2^{-n}$ for every $f \in F_{n+1} \cup \{f_{n+1}\}$ and $A \in \mathcal{A}_{n+1}$. Continue.

(c) Define $\rho : X \times X \to [0,1]$ by setting

$$\rho(x,y) = \sup_{n\in\mathbb{N}} \max_{f\in F_{n+1}} \min(2^{-n}, |f(x) - f(y)|)$$

for x, $y \in X$. Then ρ is a metric on X, defining its topology. **P** Directly from the form of its definition we see that ρ is a pseudometric. If $G \subseteq X$ is open and $x \in G$, there is an $n \in \mathbb{N}$ such that $f_n(x) = 1$ and $f[X \setminus G] = \{0\}$; now $f_n \in F_{n+1}$ so $\rho(x,y) \ge \min(2^{-n}, |f_n(x) - f_n(y)|) \ge 2^{-n}$ for every $y \in X \setminus G$. So every open set is ρ-open; it follows at once that ρ is a metric. If $G \subseteq X$ is ρ-open and $x \in G$, there is an $n \ge 1$ such that $B(x, 2^{-n}) \subseteq G$. But

$$\begin{aligned} B(x,2^{-n}) &= \{y : \max{}_{f\in F_n} \min(2^{-n+1}, |f(x) - f(y)|) \le 2^{-n}\} \\ &\supseteq \{y : |f(x) - f(y) < 2^{-n} \text{ for every } f \in F_n\} \end{aligned}$$

is a neighbourhood of x, so G is a neighbourhood of x. Thus every ρ-open set is open and ρ is compatible with the given topology on X. **Q**

(d) Suppose that $n \ge 1$, $A \in \mathcal{A}_n$ and $x \in A \setminus H_n$.

(i) If $y \in A \setminus H_n$, then $g_{nA}(x) = g_{nA}(y) = 1$ and $g_{nA'}(x) = g_{nA'}(y) = 0$ for every $A' \in \mathcal{A}_n \setminus \{A\}$; also $|f(x) - f(y)| \le 2^{-n-1}$ for every $f \in F_n \cup \{f_n\}$. But this means that $|f(x) - f(y)| \le 2^{-n-1}$ for every $f \in F_{n+1}$, so that $\rho(x,y) \le 2^{-n-1}$.

(ii) If $y \in A' \setminus H_n$ where $A' \in \mathcal{A}_n \setminus \{A\}$, then $|g_{nA}(x) - g_{nA}(y)| = 1$ while $g_{nA} \in F_{n+1}$ so $\rho(x,y) \ge 2^{-n}$.

(iii) So if $2^{-n-1} \le \delta < 2^{-n}$ then $A \setminus H_n \subseteq B(x,\delta) \subseteq A \cup H_n$. Consequently $\mu(B(x,\delta) \triangle A) \le 2^{-n}\mu A$ if $\mu A > 0$.

(e) Set $\mathcal{A}^* = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. Because $\langle \mathcal{A}_n \rangle_{n\in\mathbb{N}}$ is a sequence of partitions, each refining the previous one, $\mathcal{A}^*$ is well-capped in the sense that every non-empty subset of $\mathcal{A}^*$ has a maximal element. Consequently, if $\mathcal{I} \subseteq \mathcal{A}^*$ and $\mathcal{J}$ is the set of maximal elements of $\mathcal{I}$, $\bigcup \mathcal{I} = \bigcup \mathcal{J}$; moreover, since for any two members of $\mathcal{A}^*$ either they are disjoint or one is included in another, $\mathcal{J}$ is a disjoint family.

(f) Set

$$H = \bigcap_{n\in\mathbb{N}} \bigcup_{m\ge n} H_m \cup \bigcup\{A : A \in \mathcal{A}^*,\, \mu A = 0\}.$$

Then H is negligible. If $x \in X \setminus H$ and $\delta > 0$, there is an $n \ge 1$ such that $2^{-n} \le \delta$ and $x \notin H_n$; now there is an $A \in \mathcal{A}_n$ containing x and $B(x,\delta) \supseteq A_n \setminus H_n$ has measure at least $(1-2^{-n})\mu A > 0$. On the other hand, there is an open set of finite measure containing x, so $B(x,\delta)$ must have finite measure for all sufficiently small δ. Accordingly we can speak of

$$\liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)}$$

whenever $x \in X \setminus H$ and $E \in \operatorname{dom}\mu$.

(g) Let $E \subseteq X$ be a measurable set of finite measure, $\gamma < 1$ and $\epsilon > 0$; set $\gamma' = \frac{1}{2}(1+\gamma)$ and

$$E' = \{x : x \in E \setminus H,\, \liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} < \gamma\}.$$

Now we have an open set $G \supseteq E'$ such that $\mu G \le \epsilon + \mu^* E'$. Consider $\mathcal{I} = \{A : A \in \mathcal{A}^*,\, A \subseteq G,\, \mu^*(E \cap A) \le \gamma' \mu A\}$. Then $E' \subseteq \bigcup \mathcal{I}$. **P** Take $x \in E'$. There are i, $n \in \mathbb{N}$ such that

$$f_i(x) = 1, \quad f_i(y) = 0 \text{ for every } y \in X \setminus G,$$

$$n > i, \quad \gamma(1+2^{-n}) \le \gamma', \quad x \notin \bigcup_{m\ge n} H_m.$$

Now there are an $m > n$ and a $\delta \in [2^{-m-1}, 2^{-m}[$ such that $\mu(E \cap B(x,\delta)) \le \gamma\mu B(x,\delta)$. In this case, if A is the member of $\mathcal{A}_m$ including x, $\mu A > 0$ (because $x \notin H$) so $\mu(A \triangle B(x,\delta)) \le 2^{-m}\mu A$, by (d-iii). We also know that $f_i \in F_{m-1}$ so $\operatorname{diam} f_i[A] \le 2^{-m} < 1$ and $A \subseteq G$. Now

$$\begin{aligned}\mu(E\cap A) &\le \gamma\mu B(x,\delta) + 2^{-m}\mu A \le \gamma(\mu A + 2^{-m}\mu A) + 2^{-m}\mu A \\ &\le (\gamma + 2^{-n})\mu A \le \gamma'\mu A.\end{aligned}$$

So $A \in \mathcal{I}$ and $x \in \bigcup \mathcal{I}$. **Q**

Let $\mathcal{J}$ be the set of maximal members of $\mathcal{I}$; then $\mathcal{J}$ is disjoint and $E' \subseteq \bigcup \mathcal{J}$, by (e) above. So

$$\begin{aligned}\mu^* E' &\le \mu(E \cap \bigcup \mathcal{J}) = \sum_{A\in\mathcal{J}} \mu(E\cap A) \\ &\le \gamma' \sum_{A\in\mathcal{A}} \mu A \le \gamma'\mu G \le \gamma'(\mu^* E' + \epsilon).\end{aligned}$$

At this point, recall that $\epsilon > 0$ was arbitrary, so $\mu^* E' \le \gamma' \mu^* E'$ and $\mu^* E' = 0$.

Thus we see that

$$\{x : x \in E \setminus H,\, \liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} < \gamma\}$$

is negligible. And $\gamma < 1$ was arbitrary, so

$$\{x : x \in E \setminus H,\, \liminf_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} < 1\}$$

is negligible. As H is negligible,

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for almost every $x \in E$.

(h) Thus ρ has the declared property for measurable sets E of finite measure. But suppose now that $E \subseteq X$ is any measurable set. As noted in (a), there is a sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of open sets of finite measure covering X. Now (g) tells us that, for each n,

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap G_n\cap B(x,\delta))}{\mu B(x,\delta)} = 1 \text{ for almost every } x \in E \cap G_n,$$

that is,

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1 \text{ for almost every } x \in E \cap G_n.$$

So in fact we have

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1 \text{ for almost every } x \in E \cap \bigcup_{n\in\mathbb{N}} G_n,$$

that is,

$$\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1 \text{ for almost every } x \in E,$$

which is what we wanted to know.

Question Suppose that X is Čech-complete, that is, there is some metric on X, defining its topology, with respect to which X is complete. Can we find ρ, as above, also making X complete? Note that the metric above is for practical purposes totally bounded. What if $X = [0,1] \setminus \mathbb{Q} \cong \mathbb{N}^{\mathbb{N}}$?

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Reference

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003 (`https://www.essex.ac.`