Version of 28.6.16

## Covering with transversals and extensions of Lebesgue measure

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**1 Theorem** Let $X$ be a set, $\kappa$ an infinite cardinal and $\langle R_\xi\rangle_{\xi<\kappa}$ a family of equivalence relations on $X$. Suppose that

whenever $x\in X$ and $J\subseteq\kappa$ is infinite there is a finite set $I\subseteq J$ such that $\#(\{y:(x,y)\in R_\xi\ \forall\ \xi\in I\})\le\kappa$.

Then we have a family $\langle X_\xi\rangle_{\xi<\kappa}$, covering $X$, such that $X_\xi$ is a transversal for $R_\xi$ for every $\xi<\kappa$.

**proof (a)** For an equivalence relation $R$ on $X$ and a subset $A$ of $X$, I will say that $A$ is **$R$-free** if $(x,y)\notin R$ for all distinct points $x$, $y$ of $A$. So a transversal for $R$ is just a maximal $R$-free set.

**(b)(i)** I will say that a subset $A$ of $X$ is **well-filled** if

whenever $I\subseteq\kappa$ is finite, $\langle x_\xi\rangle_{\xi\in I}$ is a family of points of $A$, and $\#(\{y:(x_\xi,y)\in R_\xi\ \forall\ \xi\in I\})\le\kappa$, then $\{y:(x_\xi,y)\in R_\xi\ \forall\ \xi\in I\}\subseteq A$.

Observe that

if $\mathcal{A}$ is an upwards-directed family of well-filled subsets of $X$, then $\bigcup\mathcal{A}$ is well-filled;

the intersection of any non-empty family of well-filled subsets of $X$ is well-filled;

if $B\subseteq X$ there is a well-filled subset $A$ of $X$, including $B$, with $\#(A)\le\max(\kappa,\#(B))$.

**(ii)** If $C\subseteq X$ is well-filled, there is a non-decreasing family $\langle B_\alpha\rangle_{\alpha\le\#(C)}$ of well-filled subsets of $C$, covering $C$, such that $\#(B_\alpha)\le\max(\kappa,\#(\alpha))$ for every $\alpha\le\#(C)$ and $B_\alpha=\bigcup_{\beta<\alpha}B_\beta$ for every limit ordinal $\alpha\le\lambda$. **P** Enumerate $C$ as $\langle x_\alpha\rangle_{\alpha<\#(C)}$ and take $B_\alpha$ to be the smallest well-filled set including $\{x_\beta:\beta<\alpha\}$. **Q**

**(iii)** If $C\subseteq X$ is well-filled and $x\in X\setminus C$, then $J=\{\xi:\xi<\kappa,\ \exists\ y\in C,\ (x,y)\in R_\xi\}$ is finite. **P?** Otherwise, there is a finite $I\subseteq J$ such that $D=\{y:(x,y)\in R_\xi\ \forall\ \xi\in I\}$ has cardinal at most $\kappa$. For each $\xi\in I$ choose $z_\xi\in C$ such that $(x,z_\xi)\in R_\xi$. Then $D=\{y:(z_\xi,y)\in R_\xi\ \forall\ \xi\in I\}$. As $C$ is supposed to be well-filled, $D\subseteq C$. But $x\in D\setminus C$. **XQ**

**(d)** If $C\subseteq X$ and $g:C\to\mathcal{P}\kappa$ is a function, I will say that a **$g$-splitting** of $C$ is a function $f:C\to\kappa$ such that $f(x)\notin g(x)$ for every $x\in C$ and $f^{-1}[\{\xi\}]$ is $R_\xi$-free for every $\xi<\kappa$.

**(e)** (The key.) Suppose that $\lambda$ is a cardinal, $C\subseteq X$ is a well-filled set, $\#(C)=\lambda$ and $g:C\to[\kappa]^{<\omega}$ is a function. Then there is a $g$-splitting function $f:C\to\kappa$. **P** Induce on $\lambda$.

**(i)** If $\lambda\le\kappa$, enumerate $C$ as $\langle x_\eta\rangle_{\eta<\lambda}$ and choose $\langle f(\eta)\rangle_{\eta<\lambda}$ inductively such that $f(\eta)\in\kappa\setminus(g(x_\eta)\cup\{f(\zeta):\zeta<\eta\})$ for each $\eta$; now set $C_\xi=\{x_\eta\}$ if $f(\eta)=\xi$ and $C_\xi=\emptyset$ if there is no such $\eta$.

**(ii)** For the inductive step to $\lambda>\kappa$, (b-ii) tells us that there will be a non-decreasing family $\langle B_\alpha\rangle_{\alpha\le\lambda}$ of well-filled subsets of $C$, covering $C$, such that $\#(B_\alpha)\le\max(\kappa,\#(\alpha))$ for every $\alpha\le\lambda$ and $B_\alpha=\bigcup_{\beta<\alpha}B_\beta$ for every limit ordinal $\alpha\le\lambda$. For each $\alpha<\lambda$ and $x\in B_{\alpha+1}$ set

$$\begin{aligned}g_\alpha(x)&=g(x)\text{ if }x\in B_\alpha,\\&=g(x)\cup\{\xi:\exists\ y\in B_\alpha,\ (x,y)\in R_\xi\}\text{ otherwise.}\end{aligned}$$

By (b-iii), $g_\alpha(x)$ is finite for every $x\in B_\alpha$. Also $\#(B_{\alpha+1})<\lambda$. By the inductive hypothesis, there is a $g_\alpha$-splitting $f_\alpha:B_{\alpha+1}\to\kappa$.

Define $f:C\to\kappa$ by setting $f(x)=f_\alpha(x)$ whenever $\alpha<\lambda$ and $x\in B_{\alpha+1}\setminus B_\alpha$. Then $f(x)$ never belongs to $g(x)$ because $f_\alpha(x)$ never belongs to $g(x)$. Next, if $x$, $y\in C$ are distinct and $f(x)=f(y)=\xi$ then there are $\alpha$, $\beta<\lambda$ such that $x\in B_{\alpha+1}\setminus B_\alpha$ and $y\in B_{\beta+1}\setminus B_\beta$. Now

— if $\alpha=\beta$ we have $f_\alpha(x)=f_\alpha(y)=\xi$ so $(x,y)\notin R_\xi$ because $f_\alpha$ is splitting;

— if $\beta < \alpha$ then $y \in B_\alpha$ while $\xi \notin g_\alpha(x)$ so $(x, y) \notin R_\xi$;

— and similarly $(x, y) \notin R_\xi$ if $\alpha < \beta$.

As $x$, $y$ are arbitrary $f$ is $g$-splitting and the induction continues. **Q**

**(f)** Applying (e) with $C = X$ and $g(x) = \emptyset$ for every $x \in X$, we see that there is a splitting $f : X \to \kappa$. Now take $X_\xi$ to be a maximal $R_\xi$-free set including $f^{-1}[\{\xi\}]$ for each $\xi$ to get the required covering of $X$ by transversals.

**2 Corollary** Let $r \geq 1$ be an integer, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a sequence of linear subspaces of $\mathbb{R}^r$ such that $\{n : x \in V_n\}$ is finite for every non-zero $x \in \mathbb{R}^r$. Then $\mathbb{R}^r$ can be covered by a sequence $\langle X_n \rangle_{n \in \mathbb{N}}$ of sets such that $\#(X_n \cap (z + V_n)) = 1$ for every $n \in \mathbb{N}$ and $z \in \mathbb{R}^r$.

**proof** Set $R_n = \{(x, y) : x - y \in V_n\}$ for $n \in \mathbb{N}$. If $x \in \mathbb{R}^r$ and $J \subseteq \mathbb{N}$ is infinite, then $\bigcap_{n \in J} V_n = \{0\}$ so there is a finite set $I \subseteq J$ such that $\bigcap_{n \in I} V_n = \{0\}$ and $\#(\{y : (x, y) \in R_n \ \forall\ n \in I\}) = 1$. So Theorem 1 gives the result.

**Remark** This is a fractional extension of the main result in Davies 63, proved by the same method.

**3 Theorem** If $n \in \mathbb{N}$ and $\mathfrak{c} > \omega_n$, then whenever $V_0, \dots, V_n \subseteq \mathbb{R}^2$ are lines and $A_0, \dots, A_n$ cover $\mathbb{R}^2$ there must be an $i \leq n$ and an $x \in \mathbb{R}^2$ such that $A_i \cap (x + V_i)$ is uncountable.

**proof** In fact I seek to show, by induction on $m$, that whenever $\omega_m < \mathfrak{c}$ and $V_0, \dots, V_m \subseteq \mathbb{R}^2$ are lines there is a set $A \subseteq \mathbb{R}^2$ of cardinal $\omega_{m+1}$ such that whenever $\mathcal{D}$ is a countable cover of $A$ there is a $D \in \mathcal{D}$ such that for every $i \leq m$ there is an $x \in \mathbb{R}^2$ such that $D \cap (x + V_i)$ is uncountable.

To start the induction, given $m = 0$ and a line $V_0$, just take $A \in [V_0]^{\omega_1}$. For the inductive step to $m > 1$, take lines $V_0, \dots, V_m$. By the inductive hypothesis, there is a set $B \in [\mathbb{R}^2]^{\omega_m}$ such that whenever $\mathcal{D}$ is a countable cover of $B$ there is a $D \in \mathcal{D}$ such that for every $i < m$ there is an $x \in \mathbb{R}^2$ such that $D \cap (x + V_i)$ is uncountable. Observe that the same is true for $y + B$ for every $y \in \mathbb{R}^2$. Because $\mathfrak{c} > \omega_m$, we can find a family $\langle y_\xi \rangle_{\xi < \omega_{m+1}}$ in $V_m$ such that $\langle y_\xi + B \rangle_{\xi < \omega_{m+1}}$ is disjoint. Set $A = \bigcup_{\xi < \omega_{m+1}} y_\xi + B$. Then $\#(A) = \omega_{m+1}$.

Let $\mathcal{D}$ be a countable cover of $A$. Set

$$\mathcal{D}' = \{D : D \in \mathcal{D},\ D \cap (x + V_m) \text{ is countable for every } x \in \mathbb{R}^2\}$$

If $D \in \mathcal{D}'$ and $x \in B$, then $\{\xi : x + y_\xi \in D\}$ must be countable. There is therefore a $\xi < \omega_{m+1}$ such that $x + y_\xi \notin \bigcup \mathcal{D}'$ for any $x \in B$, that is, $(y_\xi + B) \cap \bigcup \mathcal{D}' = \emptyset$. So $y_\xi + B \subseteq \bigcup \mathcal{D}''$ where $\mathcal{D}'' = \mathcal{D} \setminus \mathcal{D}'$. Now there must be a $D \in \mathcal{D}''$ such that for every $i < m$ there is an $x \in \mathbb{R}^2$ such that $D \cap (x + V_i)$ is uncountable. But now we see that there is also an $x \in \mathbb{R}^2$ such that $D \cap (x + V_m)$ is uncountable. As $\mathcal{D}$ is arbitrary, the induction proceeds.

**4 Theorem** Suppose that $n \in \mathbb{N}$ and there is a Sierpiński subset of $\mathbb{R}$ (Fremlin 08, 537A) of cardinal $\omega_{n+1}$. Then whenever $V_0, \dots, V_n \subseteq \mathbb{R}^2$ are lines, $\mathcal{D}$ is a countable family of subsets of $\mathbb{R}^2$ and $\bigcup \mathcal{D}$ has non-zero inner Lebesgue measure, there must be a $D \in \mathcal{D}$ such that for every $i \leq n$ there is an $x \in \mathbb{R}^2$ such that $D \cap (x + V_i)$ has non-zero one-dimensional Hausdorff outer measure.

**Remark** Of course 'one-dimensional Hausdorff measure' on a line $V \subseteq \mathbb{R}^2$ is just the copy of one-dimensional Lebesgue measure under any isometry between $\mathbb{R}$ and $V$ (Fremlin 01, §264).

**proof** Write $\mu_L$ for two-dimensional Lebesgue measure and $\mu_{H1}$ for one-dimensional Hausdorff measure on $\mathbb{R}^2$.

**(a)** If $V \subseteq \mathbb{R}^2$ is a line, there is a set $A \subseteq \mathbb{R}^2$ of cardinal $\omega_1$ such that whenever $\mathcal{D}$ is a countable family of subsets of $\mathbb{R}^2$ and $\mu_L(A \setminus \bigcup \mathcal{D}) = 0$, there are a $D \in \mathcal{D}$ and an $x \in \mathbb{R}^2$ such that $\mu^*_{H1}(V \cap (x + D)) > 0$. **P** Let $W \subseteq \mathbb{R}^2$ be a line orthogonal to $V$. For $D \subseteq \mathbb{R}^2$, write

$$C(D) = \{x : x \in W,\ \mu^*_{H1}(V \cap (x + D)) > 0\}.$$

Note that $- : V \times W \to \mathbb{R}^2$ is an isomorphism between the product measure $\mu_{H1} \times \mu_{H_1}$ on $V \times W$ and $\mu_L$ (see Fremlin 01, 251N). Let $B \subseteq V$, $B' \subseteq W$ be sets of cardinal $\omega_1$ which are not $\mu_{H1}$-negligible, and set $A = B - B'$. Then $B' \subseteq C(A)$, so $C(A)$ is not $\mu_{H1}$-negligible. On the other hand, if $E \subseteq \mathbb{R}^2$ is

$\mu_L$-negligible, $\mu_{H1}C(E) = 0$, by Fubini's theorem. Now if $\mathcal{D}$ is countable and $A' = A \cap \bigcup \mathcal{D}$ is $\mu_L$-negligible, $\bigcup_{D\in\mathcal{D}} C(D) \supseteq C(A) \setminus C(A')$ so there is a $D \in \mathcal{D}$ such that $C(D)$ is not empty. **Q**

**(b)** If $m \le n$ and $V_0, \dots, V_m \subseteq \mathbb{R}^2$ are lines, there is a set $A \subseteq \mathbb{R}^2$ of cardinal $\omega_{m+1}$ such that whenever $\mathcal{D}$ is a countable family of subsets of $\mathbb{R}^2$ and $\mu_L(A \setminus \bigcup \mathcal{D}) = 0$, there is a $D \in \mathcal{D}$ such that for every $i \le n$ there is an $x \in \mathbb{R}^2$ such that $\mu^*_{H1}(D \cap (x + V_i)) > 0$. **P** For $m = 0$ this is (a) above. For the inductive step to $m$ when $1 \le m \le n$, take lines $V_0, \dots, V_m$. By the inductive hypothesis, there is a set $B \in [\mathbb{R}^2]^{\omega_m}$ such that whenever $\mathcal{D}$ is countable and $\mu_L(B \setminus \bigcup \mathcal{D}) = 0$ there is a $D \in \mathcal{D}$ such that for every $i < m$ there is an $x \in \mathbb{R}^2$ such that $\mu^*_{H1}(D \cap (x + V_i)) > 0$. Observe that $y + B$ will have the same property for every $y \in \mathbb{R}^2$. Now we have a $\mu_{H1}$-Sierpiński subset $C$ of $V_m$ with cardinal $\omega_{m+1}$. Choose a family $\langle y_\xi \rangle_{\xi<\omega_{m+1}}$ in $C$ such that $\langle y_\xi + B \rangle_{\xi<\omega_{m+1}}$ is disjoint. Set $A = \bigcup_{\xi<\omega_{m+1}} y_\xi + B$. Then $\#(A) = \omega_{m+1}$.

Let $\mathcal{D}$ be a countable family of sets such that $\mu_L(A \setminus \bigcup \mathcal{D}) = 0$. Set

$$\mathcal{D}' = \{D : D \in \mathcal{D},\ \mu_{H1}(D \cap (x + V_m)) = 0 \text{ for every } x \in \mathbb{R}^2\}$$

If $D \in \mathcal{D}'$ and $x \in B$, then $\mu_{H1}(V_m \cap (D - x)) = 0$ so $\{\xi : x + y_\xi \in D\} = \{\xi : y_\xi \in D - x\}$ must be countable. So there is a $\xi < \omega_{m+1}$ such that $x + y_\xi \notin \bigcup \mathcal{D}'$ for any $x \in B$, that is, $(y_\xi + B) \cap \bigcup \mathcal{D}' = \emptyset$. So $(y_\xi + B) \setminus \bigcup \mathcal{D}''$ is $\mu_L$-negligible, where $\mathcal{D}'' = \mathcal{D} \setminus \mathcal{D}'$. Now there must be a $D \in \mathcal{D}''$ such that for every $i < m$ there is an $x \in \mathbb{R}^2$ such that $\mu_{H1}(D \cap (x + V_i)) > 0$. But now we see that there is also an $x \in \mathbb{R}^2$ such that $\mu_{H1}(D \cap (x + V_m)) > 0$. As $\mathcal{D}$ is arbitrary, the induction proceeds. **Q**

**(c)** Now suppose that $\mathcal{D}$ is a countable family of subsets of $\mathbb{R}^2$ and $\bigcup \mathcal{D}$ has non-zero inner Lebesgue measure. Let $E \subseteq \bigcup \mathcal{D}$ be such that $\mu_L E > 0$. Set $Q = \mathbb{Q} \times \mathbb{Q}$. Then $Q$ is a topologically dense subset of $\mathbb{R}^2$ so $E - F$ meets $Q$ whenever $F \subseteq \mathbb{R}^2$ and $\mu^*_L F > 0$ (Fremlin 03, 443D) and $E + Q$ is $\mu_L$-conegligible. Set $\mathcal{D}' = \{D + q : q \in Q\}$; then $\mathcal{D}'$ is countable and $\bigcup \mathcal{D}'$ is $\mu_L$-conegligible. By (b), with $m = n$, there are a $D' \in \mathcal{D}'$ such that for every $i \le n$ there is an $x_i \in \mathbb{R}^2$ such that $\mu^*_{H1}(D' \cap (x_i + V_i)) > 0$. Let $q \in Q$ be such that $D = D' - q$ belongs to $\mathcal{D}$; then $\mu^*_{H1}(D \cap ((x_i - q) + V_i)) > 0$ for every $i$. So we have an appropriate $D$.

**5 Corollary** Suppose that $n \in \mathbb{N}$ and there is a Sierpiński subset of $\mathbb{R}$ with cardinal $\omega_{n+1}$. Then whenever $V_0, \dots, V_n$ are lines in $\mathbb{R}^2$, there is an extension of Lebesgue measure $\mu_L$ on $\mathbb{R}^2$ to a measure $\lambda$ such that $\lambda D = 0$ whenever $D \subseteq \mathbb{R}^2$ and there is an $i \le n$ such that $\mu_{H1}(D \cap (x + V_i)) = 0$ for $\mu_L$-almost every $x \in \mathbb{R}^2$.

**proof** For $i \le n$, set

$$\mathcal{D}_i = \{D : D \subseteq \mathbb{R}^2,\ \mu_{H1}(D \cap (x + V_i)) = 0 \text{ for every } x \in \mathbb{R}^2\}.$$

Now there is a measure $\lambda$ on $\mathbb{R}^2$, extending $\mu_L$, such that $\lambda D = 0$ for every $D \in \bigcup_{i\le n} \mathcal{D}_i$. **P?** Otherwise, there is a countable set $\mathcal{D} \subseteq \bigcup_{i\le n} \mathcal{D}_i$ such that $\mu^*_L(\bigcup \mathcal{D}) > 0$ (Fremlin 03, 417A). But this is impossible, by Theorem 4. **XQ**

If now we have $i \le n$ and a set $D \subseteq \mathbb{R}^2$ such that $\mu_{H1}(D \cap (x + V_i)) = 0$ for $\mu_L$-almost every $x \in \mathbb{R}^2$, consider $C = \{z : z \in \mathbb{R}^2,\ \mu^*_{H1}(D \cap (z + V_i)) > 0\}$. Then $\lambda C = \mu_L C = 0$. But if we set $D' = D \setminus C$ then $\mu_{H1}(D' \cap (x + V_i)) = 0$ for every $x \in \mathbb{R}^2$ so $\lambda D' = 0$ and $\lambda D = 0$. Thus we have a suitable $\lambda$.

**6 Corollary** Suppose that there is a Sierpiński subset of $\mathbb{R}$ of cardinal $\omega_\omega$. Then there is a finitely additive extension $\lambda$ of Lebesgue measure on $\mathbb{R}^2$ such that $\lambda D = 0$ whenever $V \subseteq \mathbb{R}^2$ is a line and $D \subseteq \mathbb{R}^2$ is such that $\mu_{H1}(D \cap (x + V)) = 0$ for almost every $x \in \mathbb{R}^2$.

**Remark** Here $\lambda$ will be a functional from an algebra $\mathrm{T}$ of subsets of $\mathbb{R}^2$ to $[0, \infty]$ such that $\lambda(A \cup B) = \lambda A + \lambda B$ whenever $A$, $B \in \mathrm{T}$ are disjoint.

For circumstances in which there are large Sierpiński sets, see Fremlin 08, 544G and 552E.

**proof** Let $\mathcal{V}$ be the set of lines in $\mathbb{R}^2$. For $V \in \mathcal{V}$, set

$$\mathcal{D}_V = \{D : D \subseteq \mathbb{R}^2,\ \mu_{H1}(D \cap (x + V)) = 0 \text{ for almost every } x \in \mathbb{R}^2\}.$$

Corollary 5 tells us that for each finite $\mathcal{I} \subseteq \mathcal{V}$ we have a (countably) additive $\lambda_{\mathcal{I}}$ extending $\mu_L$ such that $\lambda_{\mathcal{I}} D = 0$ for every $D \in \bigcup_{V\in\mathcal{I}} D_V$. Write $\Sigma_{\mathcal{I}}$ for $\operatorname{dom} \lambda_{\mathcal{I}}$, and set

$$\lambda'_{\mathcal{I}} D = \lambda_{\mathcal{I}} D \text{ for } D \in \Sigma_{\mathcal{I}},$$
$$= 0 \text{ for other } D \subseteq \mathbb{R}^2.$$

Let $\mathcal{F}$ be an ultrafilter on $[\mathcal{V}]^{<\omega}$ containing $\{\mathcal{I} : V \in \mathcal{I} \in [\mathcal{V}]^{<\omega}\}$ for every $V \in \mathcal{V}$, and set

$$\mathrm{T} = \bigcup_{\mathcal{K}\in\mathcal{F}} \bigcap_{\mathcal{I}\in\mathcal{K}} \Sigma_{\mathcal{I}},$$

$$\lambda D = \lim_{\mathcal{I}\to\mathcal{F}} \lambda'_{\mathcal{I}} D \text{ in } [0,\infty] \text{ for every } D \in \mathrm{T}.$$

Then T is an algebra of subsets of $\mathbb{R}^2$ including $\operatorname{dom}\mu_L \cup \mathcal{D}_V$ for every $V \in \mathcal{V}$, and because $+ : [0,\infty] \times [0,\infty] \to [0,\infty]$ is continuous, $\lambda$ is additive. Since every $\lambda'_{\mathcal{I}}$ extends $\mu_L$, so does $\lambda$; and since $\lambda'_{\mathcal{I}} D = 0$ whenever $V \in \mathcal{I}$ and $D \in \mathcal{D}_V$, $\lambda$ is zero on $\bigcup_{V\in\mathcal{V}} \mathcal{D}_V$, as required.

**Acknowledgement** Correspondence with Alan Sokal.

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