Version of 19.3.16

**Symmetry and measurability**

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**1 Proposition** Let $X$ be a set, $\Sigma$ a $\sigma$-algebra of subsets of $X$ and $W$ a member of $\Sigma\widehat{\otimes}\Sigma$. Then $W=W^{-1}$ iff $W$ is in the $\sigma$-algebra of subsets of $X^2$ generated by $\{E^2:E\in\Sigma\}$.

**Remark** Following Fremlin 00 and Fremlin 01 we write $\Sigma\widehat{\otimes}\Sigma$ for the $\sigma$-algebra of subsets of $X^2$ generated by $\{E\times F:E,\,F\in\Sigma\}$ and $W^{-1}$ for $\{(y,x):(x,y)\in W\}$.

**proof (a)** Write $\mathcal{W}$ for the $\sigma$-algebra of subsets of $X^2$ generated by $\{E^2:E\in\Sigma\}$. Because $\{W:W^{-1}=W\}$ is a $\sigma$-algebra of subsets of $X^2$ containing $A^2$ for every $A\subseteq X$, $W^{-1}=W$ for every $W\in\mathcal{W}$.

**(b)** Suppose that $E,\,F\in\Sigma$ and $E\cap F=\emptyset$.

**(i)** For every $W\in\Sigma\widehat{\otimes}\Sigma$ there is a $W_1\in\mathcal{W}$ such that $W_1\cap(E\times F)=W\cap(E\times F)$. **P** Consider the set

$$\mathcal{V}=\{V:V\subseteq X^2\text{ and there is a }W_1\in\mathcal{W}\text{ such that }V\cap(E\times F)=W_1\cap(E\times F)\}.$$

Then $\mathcal{V}$ is a $\sigma$-algebra of subsets of $X$. If $G,\,H\in\Sigma$ set

$$W_1=((G\cap E)\cup(H\cap F))^2\setminus((G\cap E)^2\cup(H\cap F)^2)\in\mathcal{W}$$

and observe that

$$W_1\cap(E\times F)=(G\cap E)\times(H\cap F)=(G\times H)\cap(E\times F).$$

So $G\times H\in\mathcal{V}$. As $G$ and $H$ are arbitrary, $\mathcal{V}\supseteq\Sigma\widehat{\otimes}\Sigma$. **Q**

**(ii)** If $W\in\Sigma\widehat{\otimes}\Sigma$ and $W=W^{-1}$ then $W\cap((E\times F)\cup(F\times E))\in\mathcal{W}$. **P** Observe first that

$$(E\times F)\cup(F\times E)=(E\cup F)^2\setminus(E^2\cup F^2)\in\mathcal{W}.$$

Now (i) tells us that there is a $W_1\in\mathcal{W}$ such that $W_1\cap(E\times F)=W\cap(E\times F)$. So

$$\begin{aligned}W\cap(F\times E)&=W^{-1}\cap(F\times E)=(W\cap(E\times F))^{-1}\\&=(W_1\cap(E\times F))^{-1}=W_1^{-1}\cap(F\times E)=W_1\cap(F\times E)\end{aligned}$$

and

$$W\cap((E\times F)\cup(F\times E))=W_1\cap((E\times F)\cup(F\times E))\in\mathcal{W}.\ \mathbf{Q}$$

**(c)** Now take any $W\in\Sigma\widehat{\otimes}\Sigma$ such that $W=W^{-1}$. Then there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ in $\Sigma$ such that $W\in\mathrm{T}\widehat{\otimes}\mathrm{T}$ where $\mathrm{T}$ is the $\sigma$-subalgebra of $\Sigma$ generated by $\langle H_n\rangle_{n\in\mathbb{N}}$.

**(i)** For each $n\in\mathbb{N}$ let $\mathrm{T}_n$ be the finite subalgebra of $\mathrm{T}$ generated by $\{H_i:i<n\}$. Let $\mathcal{A}_n$ be the set of atoms of $\mathrm{T}_n$, and set $V_n=\bigcup_{A\in\mathcal{A}_n}A^2$. By (b-ii), $W\cap((A\times A')\cup(A'\times A))\in\mathcal{W}$ whenever $A,\,A'\in\mathcal{A}_n$ are distinct. So

$$W\setminus V_n=\bigcup_{A,A'\in\mathcal{A}_n,A\neq A'}W\cap((A\times A')\cup(A'\times A))$$

belongs to $\mathcal{W}$. Setting $V=\bigcap_{n\in\mathbb{N}}V_n$. $W\setminus V=\bigcup_{n\in\mathbb{N}}W\setminus V_n$ belongs to $\mathcal{W}$.

**(ii)** $V=\bigcap_{E\in\mathrm{T}}E^2\cup(X\setminus E)^2$. **P** $(\alpha)$ Take any $(x,y)\in V$. If $i\in\mathbb{N}$ then $(x,y)\in V_{i+1}$ and $x\in H_i$ iff $y\in H_i$, since any atom of $\mathrm{T}_{i+1}$ is either included in $H_i$ or disjoint from $H_i$. Now

$$\{E:E\in\mathrm{T},\,(x,y)\in E^2\cup(X\setminus E)^2\}$$

is a $\sigma$-subalgebra of T containing $H_i$ for every $i$, so is the whole of T, and $(x,y) \in \bigcap_{E\in\mathrm{T}} E^2 \cup (X \setminus E)^2$. $(\beta)$ Conversely, if $(x,y) \in X^2 \setminus V$, there are $A$, $A' \in \mathrm{T}$ such that $A \cap A' = \emptyset$ and $(x,y) \in A \times A'$, so that $(x,y) \notin A^2 \cup (X \setminus A)^2$. **Q**

**(iii)** The map $x \mapsto (x,x) : X \to X^2$ is $(\mathrm{T}, \mathrm{T}\widehat{\otimes}\mathrm{T})$-measurable, so $E = \{x : (x,x) \in W\}$ belongs to T. Now $W \cap V = (E \times E) \cap V$. **P** For $(x,y) \in V$,

$$\begin{aligned}(x,y) \in E \times E &\iff x \in E \text{ and } y \in E\\ &\iff x \in E\\ &\iff (x,x) \in W\\ &\iff x \in W[\{x\}]\\ &\iff y \in W[\{x\}]\end{aligned}$$

(because the map $z \mapsto (x,z) : X \to X^2$ is $(\mathrm{T}, \mathrm{T}\widehat{\otimes}\mathrm{T})$-measurable, so $W[\{x\}] = \{z : (x,z) \in W$ belongs to T)

$$\iff (x,y) \in W. \textbf{ Q}$$

So $W \cap V \in \mathcal{W}$. Putting this together with (i) just above, $W \in \mathcal{W}$.

**(iv)** Thus we see that $W \in \mathcal{W}$ whenever $W \in \Sigma\widehat{\otimes}\Sigma$ and $W = W^{-1}$, and the proof is complete.

**2 Remark** The natural question arises: if $W \subseteq X^3$ belongs to $\Sigma\widehat{\otimes}\Sigma\widehat{\otimes}\Sigma$ and is fully symmetric in the sense that if $(x,y,z) \in W$ then $(y,z,x)$, $(z,x,y)$, $(x,z,y)$, $(y,x,z)$, $(z,y,x)$ all belong to $W$, does $W$ belong to the $\sigma$-algebra generated by $\{E^3 : E \in \Sigma\}$? This is certainly not the case, even if $X = \{0,1\}$ and $\Sigma = \mathcal{P}X$; consider $W = \{(1,0,0), (0,1,0), (0,0,1)\}$. We do not know if there is any other interesting way of generating the $\sigma$-algebra of fully symmetric measurable sets.

**3 Theorem** Let $X$ be a set, $G$ a finite group, and $\bullet$ an action of $G$ on $X$. Let $\mathcal{C}$ be an algebra of subsets of $X$ such that $g\bullet C \in \mathcal{C}$ whenever $g \in G$, $C \in \mathcal{C}$. Write $\mathcal{D}$ for $\{C : C \in \mathcal{C},\ g\bullet C = C$ for every $g \in G\}$. If $E$ belongs to the $\sigma$-algebra of sets generated by $\mathcal{C}$ and $g\bullet E = E$ for every $g \in G$, then $E$ belongs to the $\sigma$-algebra generated by $\mathcal{D}$.

**proof (a)** Consider first the case in which $\mathcal{C}$ is countable. Write $\mathcal{W}_0$ for the $\sigma$-algebra generated by $\mathcal{D}$.

**(i)** For $x$, $y \in X$ say that $x \sim y$ if $\{C : x \in C \in \mathcal{C}\} = \{C : y \in C \in \mathcal{C}\}$. Then $\sim$ is an equivalence relation on $X$. If $x \sim y$ and $g \in G$, then

$$\{C : g\bullet x \in C \in \mathcal{C}\} = \{g^{-1}\bullet C : x \in C \in \mathcal{C}\} = \{g^{-1}\bullet C : y \in C \in \mathcal{C}\}$$

(because $g^{-1}\bullet C \in \mathcal{C}$ for every $C \in \mathcal{C}$)

$$= \{C : g\bullet y \in C \in \mathcal{C}\},$$

so $g\bullet x \sim g\bullet y$. It follows that if $x$, $y \in X$ and $g \in G$, then $g\bullet x \sim g\bullet y$ iff $x \sim y$.

Note that if $x$, $y \in X$ and $x \not\sim y$, there is a $C \in \mathcal{C}$ such that $x \in C$ and $y \notin C$, because $X \setminus C \in \mathcal{C}$ for every $C \in \mathcal{C}$.

**(ii)** For $x \in X$, set $H_x = \{h : h \in G,\ h\bullet x \sim x\}$, so that $H_x$ is a subgroup of $G$. Then there is a $C \in \mathcal{C}$ such that $x \in C$, $H_y \subseteq H_x$ for every $y \in C$ and $h\bullet C = C$ for every $h \in H_x$. **P** For $g \in G \setminus H_x$, $g^{-1} \notin H_x$ so we can choose $C_g \in \mathcal{C}$ be such that $x \in C_g$ and $g^{-1}\bullet x \notin C_g$. Set $C' = X \cap \bigcap_{g\in G\setminus H_x}(C_g \setminus g\bullet C_g)$. Then $x \in C' \in \mathcal{C}$. If $y \in C'$ and $g \in G \setminus H_x$, then $g\bullet y \in g\bullet C_g$, $g\bullet y \notin C_g$, $g\bullet y \not\sim y$ and $g \notin H_y$. So $H_y \subseteq H_x$. Now set $C = \bigcap_{h\in H_x} h\bullet C'$; then $C \in \mathcal{C}$, $h\bullet C' = C'$ for every $h \in H_x$, $H_y \subseteq H_x$ for every $y \in C'$, and $x \in C$ because $x \in C'$ and $x \sim h\bullet x$ for every $h \in H_x$. **Q**

**(iii)** For $n \in \mathbb{N}$, set $V_n = \{x : \#(H_x) = n\}$. Then $g\bullet V_n = V_n$ for every $g \in G$. **P** If $x \in V_n$, then

$$\begin{aligned} H_{g\bullet x} &= \{h : h\bullet g\bullet x \sim g\bullet x\} = \{h : g^{-1}\bullet h\bullet g\bullet x \sim x\} \\ &= \{h : (g^{-1}hg)\bullet x \sim x\} = \{h : g^{-1}hg \in H_x\} = gH_xg^{-1}, \end{aligned}$$

so $\#(H_{g\bullet x}) = \#(H_x) = n$ and $g\bullet x \in V_n$. **Q**

**(iv)** If $n \in \mathbb{N}$ and $x \in \bigcup_{i\le n} V_i$, there is a $D \in \mathcal{D}$ such that $x \in D \subseteq \bigcup_{i\le n} V_i$. **P** By (ii), there is a $C \in \mathcal{C}$ such that $x \in C$ and $\#(H_y) \le \#(H_x) \le n$ for every $y \in C$. By (iii), $\#(H_{g\bullet y}) \le n\}$ whenever $y \in C$ and $g \in G$. So if we set $D = \bigcup_{g\in G} g\bullet C$, we have $D \subseteq \bigcup_{i\le n} V_i$ and $x \in D \in \mathcal{D}$. **Q** Because $\mathcal{C}$ is countable, $\bigcup_{i\le n} V_i \in \mathcal{W}_0$. As this is true for every $n$, $V_n \in \mathcal{W}_0$ for every $n$.

**(v)** Now suppose that $n \in \mathbb{N}$ and $x \in V_n$. Let $C$ be a set as in (ii) above. If $B \in \mathcal{C}$ there is a $D \in \mathcal{D}$ such that $C \cap V_n \cap B = C \cap V_n \cap D$. **P** If $y \in C \cap V_n$, then $H_y \subseteq H_x$, but $\#(H_y) = \#(H_x)$, so $H_y = H_x$ and $g\bullet y \sim y$ for every $g \in H_x$. Now $D = \bigcup_{g\in G} g\bullet(B \cap C)$ belongs to $\mathcal{D}$ and $C \cap V_n \cap D = C \cap V'_n \cap B$, because if $y \in C \cap V_n \cap D$ there is a $g \in G$ such that $g^{-1}\bullet y \in B \cap C$; in this case, $y \in C \cap (g^{-1}\bullet C)$, so $g \in H_x = H_y$, $y \sim g^{-1}\bullet y \in B$ and $y \in B$. **Q**

Of course the set $\bigcup_{W\in\mathcal{W}_0}\{E : C \cap V_n \cap W = C \cap V_n \cap E\}$ is a $\sigma$-algebra of sets, because $\mathcal{W}_0$ is; and we have just seen that it includes $\mathcal{C}$.

**(vi)** Now suppose that $E$ is in the $\sigma$-algebra generated by $\mathcal{C}$ and that $g\bullet E = E$ for every $g \in G$. Take any $x \in E$. Then there are an $n \in \mathbb{N}$ and $D \in \mathcal{D}$ such that $x \in D \cap V_n$ and $D \cap V_n \cap E \in \mathcal{W}_0$. **P** Set $n = \#(H_x)$ so that $x \in V_n$. By (v), there are a $C \in \mathcal{C}$ and $W \in \mathcal{W}_0$ such that $C \cap V_n \cap W = C \cap V_n \cap E$. Set $D = \bigcup_{g\in G} g\bullet C$. Since $V_n$, $W$ and $E$ are all invariant under the action of $G$,

$$\begin{aligned} D \cap V_n \cap E &= \bigcup_{g\in G}(g\bullet C) \cap V_n \cap E = \bigcup_{g\in G}(g\bullet C) \cap (g\bullet V_n) \cap (g\bullet E) \\ &= \bigcup_{g\in G} g\bullet(C \cap V_n \cap E) = \bigcup_{g\in G} g\bullet(C \cap V_n \cap W) = D \cap V_n \cap W \in \mathcal{W}_0. \ \mathbf{Q} \end{aligned}$$

As $x$ is arbitrary, we see that

$$E = \bigcup\{D \cap V_n \cap W : D \in \mathcal{D},\, n \in \mathbb{N},\, D \cap V_n \cap W \in \mathcal{W}_0\}.$$

As $\mathcal{D}$ is countable, $E$ is a countable union of members of $\mathcal{W}_0$ and itself belongs to $\mathcal{W}_0$. As $E$ is arbitrary, we have the result, at least if $\mathcal{C}$ is countable.

**(b)** For the general case, take $E$ in the $\sigma$-algebra generated by $\mathcal{C}$ such that $g\bullet E = E$ for every $g \in G$. Then there is a countable set $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $E$ is in the $\sigma$-algebra generated by $\mathcal{C}_0$. Let $\mathcal{C}'$ be the smallest family of sets such that

$$\mathcal{C}_0 \subseteq \mathcal{C}',$$

$$C \cap C',\ X \setminus C,\ g\bullet C \in \mathcal{C}' \text{ whenever } C,\ C' \in \mathcal{C}' \text{ and } g \in G.$$

Then $\mathcal{C}'$ is a countable algebra of subsets of $X$. Set $\mathcal{D}' = \{D : D \in \mathcal{C}',\ g\bullet D = D \text{ for every } g \in G\}$. By (a), $E$ belongs to the $\sigma$-algebra generated by $\mathcal{D}'$, so certainly belongs to the $\sigma$-algebra generated by $\mathcal{D}$. Thus we have the result in the general case.

**References**

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