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Peres' 33 Directions

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I discuss the configuration used by Conway & Kochen p06 and Conway & Kochen 09 in the so-called Free Will Theorem.

1. The problem For a given set of lines through the origin in $\mathbb{R}^3$ I will say that a colouring in two colours, white and black, is **acceptable** if

— whenever two of the lines are orthogonal, at least one is black,

— whenever three of the lines are mutually orthogonal, at least one is white.

The question is, whether there is an acceptable colouring of the set of all lines through 0, and the answer is 'no'. This is because there is a finite set of lines for which there is no acceptable colouring. (Subject to the axiom of choice, this has to be so, but since this is supposed to have something to do with quantum mechanics, and by extension something to do with the real world, it's better if we can find a Euclidean proof.) I do not know what are the smallest sets of lines with this property, but a very pretty configuration is described in Peres 91.

2. The configuration There are much more attractive and meaningful ways of putting this (see §5 below), but I will describe it in terms of coordinates, as follows. Set $\gamma = \frac{1}{\sqrt{2}}$. Consider

— the six points of the forms $(\pm1,0,0)$, $(0,\pm1,0)$, $(0,0,\pm1)$,

— the twelve points of the forms $(0,\pm1,\pm1)$, $(\pm1,0,\pm1)$, $(\pm1,\pm1,0)$,

— the twenty-four points of the forms $(\pm1,\pm\gamma,0)$, $(\pm1,0,\pm\gamma)$, $(\pm\gamma,\pm1,0)$, $(\pm\gamma,0,\pm1)$, $(0,\pm1,\pm\gamma)$, $(0,\pm\gamma,\pm1)$,

— the twenty-four points of the forms $(\pm1,\pm\gamma,\pm\gamma)$, $(\pm\gamma,\pm1\pm\gamma)$, $(\pm\gamma,\pm\gamma,\pm1)$.

These sixty-six points lie on the faces of a $2\times2\times2$ cube centred at $(0,0,0)$ and define thirty-three lines through $(0,0,0)$.

Note that on each face of the cube we have thirteen points, being the the midpoint of each edge and nine points on a square grid; the midpoints of the edges and the corners of the grid being equally spaced around a circle of radius 1.

The really important thing is that the configuration is invariant under any rotation or reflection which leaves the cube fixed.

3. Theorem There is no acceptable colouring of the thirty-three lines of Peres' configuration.

proof ? Suppose there were. Transfer the colouring to the sixty-six points at which the lines intersect the surface of the cube, as listed in §2, so that each of these points is now white or black.

(a) $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ lie on mutually orthogonal lines, so one of them has to be white. Rotating the cube, we can suppose that it has been turned so that $X=(1,0,0)$ is white. (I will give the points the labels they have in Conway & Kochen p06, Endnote 4.) In this case, the sixteen points on lines orthogonal to X,

$$(0,\pm1,0),\quad (0,0,\pm1),\quad (0,\pm1,\pm1),\quad (0,\pm\gamma,\pm1),\quad (0,\pm1,\pm\gamma)$$

must all be black.

(b) Next, consider the four points $(1,\pm\gamma,\pm\gamma)$. We have two orthogonal triples

$$(1,\gamma,-\gamma),\quad (1,-\gamma,\gamma),\quad (0,1,1),$$

$$(1,\gamma,\gamma),\quad (1,-\gamma,-\gamma),\quad (0,1,-1)$$

in which the final point is black, so that one (and only one) of the other two is white. So we have two white points, and the possible pairs of white points (since no pair of white points can lie on orthogonal lines) are

$$(1,\gamma,-\gamma),\quad (1,\gamma,\gamma),$$

$$(1,\gamma,\gamma),\quad (1,-\gamma,\gamma),$$

$$(1,-\gamma,\gamma),\quad (1,-\gamma,-\gamma),$$

$$(1,-\gamma,-\gamma),\quad (1,\gamma,-\gamma).$$

Rotating the cube around the axis through X, we see that any of the four pairs can be moved to the pair C', C, where $C=(1,-\gamma,\gamma)$ and $C'=(1,\gamma,\gamma)$. So again I suppose the cube turned so that these are white.

(c)(i) C and $D=(\gamma,1,0)$ lie on orthogonal lines, so D is black; C' and $D'=(-\gamma,1,0)$ lie on orthogonal lines, so D' is black.

(ii) $Z=(0,0,1)$, D and $E=(1,-\gamma,0)$ form an orthogonal triple in which Z and D are black, so E is white; Z, D' and $E'=(1,\gamma,0)$ form an orthogonal triple in which Z and D' are black, so E' is white.

(iii) $F=(\gamma,1,-\gamma)$ and $G=(\gamma,1,\gamma)$ lie on lines orthogonal to E, so are black; $F'=(-\gamma,1,\gamma)$ and $G'=(-\gamma,1,-\gamma)$ lie on lines orthogonal to E', so are black.

(iv) F, F' and $U=(1,0,1)$ form an orthogonal triple in which the first two are black, so U is white; G, G' and $V=(1,0,-1)$ form an orthogonal triple in which the first two are black, so V is white.

(v) But U and V lie on orthogonal lines, so this is impossible. **X**

4. Remarks Once we have identified our first white point X, the rest of the argument in §3 is just a matter of identifying so many black points on other faces that there must be impossibly many white points on the face of the cube containing X.

The proof in §3 does not trouble to give labels to thirty-three lines, but all the others are potentially present because we used simplification-by-symmetry arguments in parts (a) and (b), relying on the configuration being invariant under all rotations of the cube in §2, and all the four types of line identified in §2 have roles in the chase around the diagram. In fact, as Gould & Aravind 09 point out, Peres' configuration is minimal in that omitting any line from it gives a set with an acceptable colouring. Specifically, I believe that we have the following four types of colouring, corresponding to the four types of line described in §2:

— excluding the line through $(1,-\gamma,-\gamma)$, there is an acceptable colouring in which the lines through

$(1,0,0)$, $(1,-1,0)$, $(1,0,-1)$, $(1,-\gamma,0)$, $(1,0,\gamma)$, $(-\gamma,1,0)$, $(\gamma,0,1)$, $(1,-\gamma,\gamma)$, $(-\gamma,1,\gamma)$, $(\gamma,-\gamma,1)$

are white, and the rest black;

— excluding the line through $(1,\gamma,0)$, there is an acceptable colouring in which the lines through

$(1,0,0)$, $(1,-1,0)$, $(1,0,-1)$, $(1,-\gamma,0)$, $(1,0,-\gamma)$, $(-\gamma,0,1)$, $(1,-\gamma,-\gamma)$, $(1,\gamma,-\gamma)$, $(-\gamma,1,\gamma)$, $(-\gamma,\gamma,1)$

are white, and the rest black;

— excluding the line through $(1,1,0)$, there is an acceptable colouring in which the lines through

$(1,0,0)$, $(1,-1,0)$, $(1,0,-1)$, $(1,-\gamma,0)$, $(1,0,-\gamma)$, $(1,0,\gamma)$, $(-\gamma,1,0)$, $(1,-\gamma,-\gamma)$, $(1,-\gamma,\gamma)$, $(-\gamma,1,\gamma)$

are white, and the rest black;

— excluding the line through $(0,1,0)$, there is an acceptable colouring in which the lines through

$(1,0,0)$, $(1,-1,0)$, $(1,0,-1)$, $(1,-\gamma,0)$, $(1,0,-\gamma)$, $(-\gamma,1,0)$, $(1,-\gamma,-\gamma)$, $(1,-\gamma,\gamma)$, $(-\gamma,1,\gamma)$, $(-\gamma,\gamma,1)$

are white, and the rest black.

5. Alternative description of Peres' configuration A more geometric way of representing the 33 directions is as follows. Start with a cube $\mathcal{C}$. Take the three cubes $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$ obtained from $\mathcal{C}$ by rotating it through 45° about one of its axes. Each of these has thirteen axes of symmetry (three through centres of opposite faces, four long diagonals and six through midpoints of opposite edges). Some of these axes coincide; together (if we use the most natural coordinate frame in this context) they amount to the 33 directions of §2.

As I find that the confusion has spread to the internet, it is perhaps worth pointing out that the deservedly lauded biography Roberts 15 makes an error on this point (p. 278). Peres' directions are *not* the axes of symmetry of the union $\mathcal{C} \cup \mathcal{C}' \cup \mathcal{C}'' \cup \mathcal{C}'''$ sketched there.

6. Changing the rules Of course the result here depends on the precise sense of 'acceptable' colouring as defined in §1, and this is curiously asymmetric. If we make the requirement more stringent, saying that

— whenever two of the lines are orthogonal, at least one must be black,

— whenever two of the lines are orthogonal, at least one must be white,

then no orthogonal triple of lines can be coloured. If we weaken it to

— whenever three of the lines are mutually orthogonal, at least one is white and at least one is black,

then we have a colouring of the set of all lines through the origin, as follows.

Example For a point $x = (\xi_1, \xi_2, \xi_3)$ of the unit sphere in $\mathbb{R}^3$ colour it black if either $\xi_1^2 > \frac{1}{3}$ or $\xi_1^2 = \frac{1}{3}$ and $\xi_2^2 \ge \frac{1}{3}$; otherwise colour it white.

If we have an orthonormal pair $x = (\xi_1, \xi_2, \xi_3)$, $y = (\eta_1, \eta_2, \eta_3)$ then we cannot have $\xi_2^2 = \xi_2^2 = \eta_1^2 = \eta_2^2 = \frac{1}{3}$. **P?** If this were the case, then we must also have $\xi_3^2 = \eta_3^2 = \frac{1}{3}$. But this would mean that

$$0 = (x|y) = \pm\tfrac{1}{3} \pm \tfrac{1}{3} \pm \tfrac{1}{3},$$

which is impossible. **XQ**

If we now have an orthonormal triple $x = (\xi_1, \xi_2, \xi_3)$, $y = (\eta_1, \eta_2, \eta_3)$, $z = (\zeta_1, \zeta_2, \zeta_3)$ then $\xi_1^2 + \eta_1^2 + \zeta_1^2 = \xi_2^2 + \eta_2^2 + \zeta_2^2 = 1$ so either

— one of ξ_1^2, η_1^2, ζ_1^2 is greater than $\frac{1}{3}$ and another is less than $\frac{1}{3}$, in which case one of x, y, z is black and another is white,

or $\xi_1^2 = \eta_1^2 = \zeta_1^2 = \frac{1}{3}$; and in this latter case, ξ_2^2 and η_2^2 cannot both be equal to $\frac{1}{3}$, so one of ξ_2^2, η_2^2, ζ_2^2 is greater than $\frac{1}{3}$ and another is less than $\frac{1}{3}$, in which case again one of x, y, z is black and another is white.

Now since in the above colouring of S_2 antipodeal points are always of the same colour, we can transfer it to the lines through the origin, giving each line the colour of the points where it crosses the sphere; and we shall now have a colouring of the set of all lines through $\mathbf{0}$ in which no orthogonal triple is monochromatic.

7. Two questions (a) What is the smallest size of any set of lines through the origin in $\mathbb{R}^3$ which has no acceptable colouring (in the sense of §1)? The examples in §4 show that there is no proper subset of Peres' configuration without an acceptable colouring, but tell us nothing about other sets of lines.

(b) Let us say that a set L of lines through the origin in $\mathbb{R}^3$ is **ortho-closed** if whenever ℓ_0, $\ell_1 \in L$ are orthogonal then the line orthogonal to both belongs to L. Is there a finite ortho-closed set of lines with no acceptable colouring?

8. Concluding remarks I have a definite view on the relevance of this construction to what we know as 'free will', but do not feel it necessary to discuss the matter here.

Jack Grahl has posted pdfs of a design for a cardboard cube with the 33 directions and their complementary planes marked on it; see `https://github.com/jwg4/33_directions/tree/master/output`.

References

Conway J. & Kochen S. [p06] 'The free will theorem', arXiv:quant-ph/0604079v1, 11.4.06.

Conway J.H. & Kochen S. [09] 'The strong free will theorem', Notices of the A.M.S. 56 (2009) 226-232.

Gould E. & Aravind P.K. [09] ‘Isomorphism between the Peres and Penrose proofs of the BKS theorem in three dimensions’, arxiv.org/pdf/0909.4502.

Peres A. [91] ‘Two simple proofs of the Kochen-Specker theorem’, J. Phys. A Math. Gen. 24 (1991) L175.

Roberts S. [15] *Genius at Play.* Bloomsbury, 2015.