Version of 19.9.15

On Prokhorov spaces

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For notation see Fremlin 03.

1. Theorem $\mathbb{R}$, with the right-facing Sorgenfrey topology, is not a Prokhorov space.

proof The proof follows the argument of Fremlin 03, 439S, itself based on Preiss 73.

(a) Write $\mathfrak{T}$ for the usual topology on $[0,1]$ and $\mathfrak{S}$ for the subspace topology on $[0,1]$ when $\mathbb{R}$ is given the right-facing Sorgenfrey topology.

Note first that a subset K of $[0,1]$ is $\mathfrak{S}$-compact iff it is $\mathfrak{T}$-closed and well-capped, that is, every non-empty subset of K has a greatest member, that is, there is no strictly increasing sequence in K. **P** (i) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in K, set $x=\sup_{n\in\mathbb{N}}x_n$; then $\{[x,1]\}\cup\{[0,x_n[\,:n\in\mathbb{N}\}$ is a cover of K by members of $\mathfrak{S}$ with no finite subcover. (ii) If K is not $\mathfrak{T}$-closed then it cannot be $\mathfrak{S}$-compact because $\mathfrak{S}$ is finer than $\mathfrak{T}$. (iii) If K is $\mathfrak{T}$-closed and well-capped and $\mathcal{G}\subseteq\mathfrak{S}$ covers K, set $A=\{x:x\in[0,1]$, $K\cap[0,x]$ is covered by finitely many members of $\mathcal{G}\}$. Then $0\in A$ so $c=\sup A$ is defined in $[0,1]$. Because K is well-capped, there must be a $c'<c$ such that $K\cap\,]c',c[\,=\emptyset$; now there is an $x\in A\cap\,]c',c]$. (α) If $c\in K$ then there is a $G\in\mathcal{G}$ containing c. If y is such that $[c,y]\subseteq G$, then $K\cap[0,y]\subseteq(K\cap[0,x])\cup G$ is covered by finitely many members of $\mathcal{G}$ so $y\in A$ and $y\le c$; but this means, first, that $c\in A$, and, second, that $c=1$. So in this case $1\in A$ and K is covered by finitely many members of $\mathcal{G}$. (β) If $c\notin K$ then $K\cap[0,c]=K\cap[0,x]$ so $c\in A$. **?** If $c<1$ then there is a $y\in\,]c,1]$ such that $[c,y]\cap K=\emptyset$, in which case $K\cap[0,y]=K\cap[0,x]$ and $y\in A$, which is impossible. **X** So in this case also $1=c\in A$ and $\mathcal{G}$ has a finite subcover. **Q**

It follows that all $\mathfrak{S}$-compact sets are countable.

(b) There is a non-decreasing sequence $\langle X_k\rangle_{k\in\mathbb{N}}$ of non-empty $\mathfrak{S}$-compact subsets of $[0,1[$ such that

(i) whenever $k\in\mathbb{N}$, $x\in X_k$ and $\delta>0$, then $X_{k+1}\cap[x,x+\delta]$ is infinite,

(ii) setting $X=\bigcup_{k\in\mathbb{N}}X_k$, there is no strictly increasing sequence in X with supremum in X,

(iii) $\mathfrak{S}$ and $\mathfrak{T}$ agree on X.

P I give an inductive construction of the sets X_k, together with functions $g_k:X_k\to\,]0,\infty[$, as follows. Set $X_0=\{0\}$ and $g_0(0)=1$. Given that $X_k\subseteq[0,1[$ is $\mathfrak{S}$-compact and contains 0 and that $g_k:X_k\to\,]0,\infty[$ is such that $x<y-g_k(y)$ whenever $x<y$ in X_k, of course X_k is $\mathfrak{T}$-closed. Let $\mathcal{I}_k$ be the set of $\mathfrak{T}$-components of $[0,1[\,\setminus X_k$; then each member of $\mathcal{I}_k$ is an open interval with endpoints in $X_k\cup\{1\}$. For each $J\in\mathcal{I}_k$ choose a strictly decreasing sequence $\langle x_{Jj}\rangle_{j\in\mathbb{N}}$ in J with infimum $\inf J$ and such that if $\sup J<1$ then $x_{j0}<\sup J-g(\sup J)$. Set $X_{k+1}=X_k\cup\{x_{Jj}:J\in\mathcal{I}_k,\ j\in\mathbb{N}\}$. If $A\subseteq X_{k+1}\setminus X_k$ is non-empty, consider $\mathcal{J}=\{J:J\in\mathcal{I}_k,\ A\cap J\ne\emptyset\}$; since $\min J\in X_k$ for every $J\in\mathcal{J}$, there is a $J\in\mathcal{J}$ with greatest minumum, and if now $j\in\mathbb{N}$ is minimal subject to $x_{Jj}\in A$, we have $x_{Jj}=\max A$. It follows that every non-empty subset of X_{k+1} has a greatest element. On the other hand, X_{k+1} is $\mathfrak{T}$-closed because every strictly decreasing sequence in X_{k+1} has infimum in X_k. So X_{k+1} is $\mathfrak{S}$-compact. Now set $g_{k+1}(x)=g_k(x)$ for every $x\in X_k$ and for $J\in\mathcal{I}_k$, $i\in\mathbb{N}$ set

$$g_{k+1}(x_{J_i})=\frac{1}{2}(x_{J_i}-x_{J,i+1}).$$

Finally, if $x<y$ in X_{k+1}, then

— if x, $y\in X_k$ we have $x<y-g_k(y)=y-g_{k+1}(y)$;

— if $y\in X_k$ and $x\notin X_k$ then $x=x_{Ji}$ for some $J\in\mathcal{I}_k$ and $i\in\mathbb{N}$; if $\sup J=y$ then $x\le x_{j0}<y-g_k(y)=y-g_{k+1}(y)$; otherwise, $\sup J\in X_k$ and $x<\sup J<y-g_{k+1}(y)$;

— if $y\notin X_k$ then $y=x_{Ji}$ for some $J\in\mathcal{I}_k$ and $i\in\mathbb{N}$, and $x\le x_{J,i+1}<y-g_{k+1}(y)$.

Continue.

(i) follows directly from the construction. As for (ii), $g=\bigcup_{k\in\mathbb{N}}g_k$ is a strictly positive real-valued function on X and $x<g(y)$ whenever $x<y$ in X, so no strictly increasing sequence in X can have supremum in X.

Finally, both $\mathfrak{S}$ and $\mathfrak{T}$ are first-countable, any sequence in $\mathbb{R}$ has a subsequence which is either non-increasing or non-decreasing, a non-increasing sequence in $[0,1]$ converges to its infimum for both $\mathfrak{S}$ and $\mathfrak{T}$, and there is no strictly increasing sequence in X with a supremum in X; so a sequence in X with a $\mathfrak{T}$-limit in X has a $\mathfrak{S}$-limit and the two topologies agree on X. **Q**

(c) For $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$ set

$$f(x,A) = \inf_{y \in A \cap]-\infty,x]} x - y, \qquad \rho(x,A) = \inf_{y \in A} |x-y|$$

counting $\inf \emptyset$ as ∞. If $\langle \epsilon_k \rangle_{k \in \mathbb{N}}$ is any sequence in $]0,\infty[$, and $F \subseteq [0,1]$ is a countable $\mathfrak{T}$-closed set, then there is an $x^* \in X \setminus F$ such that $f(x^*, X_k) < \epsilon_k$ for every $k \in \mathbb{N}$. **P** We can suppose that $\lim_{k\to\infty} \epsilon_k = 0$. Define $\langle H_k \rangle_{k \in \mathbb{N}}$ inductively, as follows. $H_0 = \mathbb{R}$. Given H_k, set $H_{k+1} = H_k \cap \{x : f(x, X_k \cap H_k) < \epsilon_k\}$. Observe that $X_k \cap H_k \subseteq H_{k+1} \subseteq H_k$ and that H_k is $\mathfrak{S}$-open, for every k. At the same time,

$$H_{k+1} = (X_k \cap H_k) \cup ((H_k \setminus X_k) \cap \bigcup_{y \in X_k \cap H_k}]y, y+\epsilon_k[);$$

because every X_k is $\mathfrak{T}$-closed and therefore $\mathfrak{T}$-G_δ, we see that every H_k will be $\mathfrak{T}$-G_δ.

Consequently, $E = \bigcap_{k \in \mathbb{N}} H_k$ is a $\mathfrak{T}$-G_δ subset of $\mathbb{R}$, while $X_k \cap H_k \subseteq E$ for every k. In particular, $E \cap X$ includes X_0 and is not empty. Next, for each k, $\rho(x, E \cap X_k) \le f(x, E \cap X_k) < \epsilon_k$ for every $x \in H_{k+1}$ and therefore for every $x \in E$; accordingly $E \cap X$ is $\mathfrak{T}$-dense in E.

Moreover, if $x \in E \cap X$, there is a $k \in \mathbb{N}$ such that $x \in X_k$; we must have $x \in H_{k+1}$. By the construction in (b), there is a strictly decreasing sequence in X_{k+1} with infimum x, and this sequence will eventually lie in H_{k+1} because H_{k+1} is $\mathfrak{S}$-open.

So every $\mathfrak{T}$-neighbourhood of x contains infinitely many points of $H_{k+1} \cap X_{k+1} \subseteq E \cap X$. Thus $E \cap X$ has no $\mathfrak{T}$-isolated points; it follows that E has no $\mathfrak{T}$-isolated points. By 4A2Mc and 4A2Me of Fremlin 03, E is uncountable.

There is therefore a point $z \in E \setminus F$. Let $m \in \mathbb{N}$ be such that $\rho(z, F) \ge \epsilon_m$ for every $y \in F$. As $z \in H_{m+1}$, there is an $x^* \in H_m \cap X_m$ such that $x^* \le z < x^* + \epsilon_m$, so $x^* \notin F$. Let $k \in \mathbb{N}$. If $k \ge m$ then certainly $f(x^*, X_k) = 0 < \epsilon_k$. If $k < m$ then $x^* \in H_{k+1}$ so $f(x^*, X_k) \le f(x^*, H_k \cap X_k) < \epsilon_k$. Thus we have a suitable x^*. **Q**

(d) For n, $k \in \mathbb{N}$ set

$$G_{kn} = \{x : x \in [0,1] \setminus X_k,\ \rho(x, X_n) > 2^{-k}\}.$$

Then G_{kn} is a $\mathfrak{T}$-open subset of $[0,1]$.

(e)(i) Write A_1 for the set of $\mathfrak{T}$-Radon probability measures μ on $[0,1]$ such that $\mu G_{kn} \le 2^{-n}$ for all k, $n \in \mathbb{N}$. Then A_1 is a narrowly closed subset of the set $P_{\mathrm{R}}([0,1], \mathfrak{T})$ of $\mathfrak{T}$-Radon probability measures on $[0,1]$, which is itself narrowly compact (Fremlin 03, 437R(f-ii)).

(ii) $\mu([0,1] \setminus X) = 0$ for every $\mu \in A_1$. **P** Let $K \subseteq [0,1] \setminus X$ be $\mathfrak{T}$-compact, and $n \in \mathbb{N}$. Then K and X_n are disjoint $\mathfrak{T}$-compact sets, so there is some $k \in \mathbb{N}$ such that $|x - y| > 2^{-k}$ for every $x \in X_n$ and $y \in K$. In this case $K \subseteq G_{kn}$ so $\mu K \le 2^{-n}$. As n is arbitrary, $\mu K = 0$; as K is arbitrary, $\mu([0,1] \setminus X) = 0$. **Q**

(iii) Write A_2 for the set of $\mathfrak{T}$-Radon probability measures μ on X such that $\mu(G_{kn} \cap X) \le 2^{-n}$ for all k, $n \in \mathbb{N}$. By Fremlin 03, 437Nb, the set $P_{\mathrm{R}}(X, \mathfrak{T})$ of $\mathfrak{T}$-Radon probability measures on X, with its narrow topology, is homeomorphic to the subset D of $P_{\mathrm{R}}([0,1], \mathfrak{T})$ consisting of $\mathfrak{T}$-Radon measures μ on $[0,1]$ such that $\mu([0,1] \setminus X) = 0$; and a homeomorphism from D to $P_{\mathrm{R}}(X, \mathfrak{T})$ is given by taking $\mu \in D$ to the subspace measure μ_X on X. Now $A_2 = \{\mu_X : \mu \in A_1\}$, so A_2 is compact in $P_{\mathrm{R}}(X, \mathfrak{T})$ for the narrow topology.

(iv) Because $\mathfrak{S}$ and $\mathfrak{T}$ agree on X, we can think of A_2 as the set of $\mathfrak{S}$-Radon probability measures μ on X such that $\mu(G_{kn} \cap X) \le 2^{-n}$ for all k, $n \in \mathbb{N}$, and it is compact in $P_{\mathrm{R}}(X, \mathfrak{S})$ for the narrow topology.

(v) Repeating the argument of (ii)-(iii) with $\mathfrak{S}$ instead of $\mathfrak{T}$, we now see that $P_{\mathrm{R}}(X, \mathfrak{S})$ is homeomorphic to the set of $\mathfrak{S}$-Radon measures μ on $[0,1]$ such that $\mu([0,1] \setminus X) = 0$, and that A_2 is homeomorphic to the set A of $\mathfrak{S}$-Radon measures μ on $[0,1]$ such that $\mu G_{kn} \le 2^{-n}$ for all k, $n \in \mathbb{N}$. So again we have a narrowly compact set of measures.

(f) A, regarded as a subset of $P_{\mathrm{R}}([0,1], \mathfrak{S})$, is not uniformly tight. **P** Let $K \subseteq [0,1]$ be $\mathfrak{S}$-compact. Consider the set C of those $w \in [0,1]^{[0,1]}$ such that $w(x) = 0$ for every $x \in K$, $\sum_{x \in [0,1]} w(x) \le 1$ and

$\sum_{x\in G_{kn}} w(x) \le 2^{-n}$ for all k, $n \in \mathbb{N}$. Then C is a compact subset of $[0,1]^{[0,1]}$. If $D \subseteq C$ is any non-empty upwards-directed set, then $\sup D$, taken in $[0,1]^{[0,1]}$, belongs to C. By Zorn's Lemma, C has a maximal member w say. **?** Suppose, if possible, that $\sum_{x\in X} w(x) = \gamma < 1$. For each $n \in \mathbb{N}$, let $L_n \subseteq X$ be a finite set such that $\sum_{x\in L_n} w(x) \ge \gamma - 2^{-n-1}$, and $m_n \in \mathbb{N}$ such that $L_n \subseteq X_{m_n}$. Because K is countable and $\mathfrak{T}$-closed, (c) tells us that there is an $x^* \in X \setminus K$ such that $f(x^*, X_n) < 2^{-m_n}$ for every $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $x^* \in X_r$ and $\gamma + 2^{-r} \le 1$, and set $w'(x^*) = w(x^*) + 2^{-r}$, $w'(x) = w(x)$ for every $x \in [0,1] \setminus \{x^*\}$. Then certainly $w' \in [0,1]^{[0,1]}$ and $\sum_{x\in[0,1]} w'(x) \le 1$. If k, $n \in \mathbb{N}$ and $x^* \notin G_{kn}$, then

$$\sum_{x\in G_{kn}} w'(x) = \sum_{x\in G_{kn}} w(x) \le 2^{-n}.$$

If $x^* \in G_{kn}$, then $n < r$ and

$$2^{-k} < \rho(x^*, X_n) \le f(x^*, X_n) < 2^{-m_n},$$

so $m_n < k$ and $L_n \subseteq X_k$ and

$$\sum_{x\in G_{kn}} w(x) \le \sum_{x\in[0,1]\setminus X_k} w(x) \le \sum_{x\in[0,1]\setminus L_n} w(x) \le 2^{-n-1},$$

$$\sum_{x\in G_{kn}} w'(x) \le 2^{-n-1} + 2^{-r} \le 2^{-n}.$$

Thus $w' \in C$ and w was not maximal. **X**

Accordingly $\sum_{x\in[0,1]} w(x) = 1$ and the point-supported measure μ defined by w is a probability measure on $[0,1]$. By the definition of C, $\mu \in A$ and $\mu([0,1] \setminus K) = 1$. As K is arbitrary, A cannot be uniformly tight. **Q**

(g) Thus A witnesses that $[0,1]$, with the topology $\mathfrak{S}$, is not a Prokhorov space. Since $[0,1]$ is a closed subset of $\mathbb{R}$ with the right-facing Sorgenfrey topology, the latter is not a Prokhorov space (Fremlin 03, 437Vb).

2. Remark This gives an answer to Problem 12.15 in Wheeler 83.[1]

Because the argument above so closely follows Preiss' proof that $\mathbb{Q}$ is not a Prokhorov space, and noting that it uses a set X which is homeomorphic to $\mathbb{Q}$ (being countable and without isolated points), it's natural to ask whether the result here can be derived directly from Preiss'. However, at least the simplest approach fails.

3. Proposition Give $\mathbb{R}$ its right-facing Sorgenfrey topology. Then $\mathbb{Q}$ is not homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$.

proof (a) Let $f : \mathbb{Q} \to \mathbb{R}^{\mathbb{N}}$ be a continuous function; write f_n for its nth coordinate, so that $f(q) = \langle f_n(q)\rangle_{n\in\mathbb{N}}$ for $q \in \mathbb{Q}$. Let $\langle r_n\rangle_{n\in\mathbb{N}}$ be an enumeration of $\mathbb{Q}$. Note that if $g : \mathbb{Q} \to \mathbb{R}$ is continuous and $q \in \mathbb{Q}$, then $q \in \operatorname{int}\{q' : g(q') \ge q\}$, because $g^{-1}[\,[q,\infty[\,]$ is open.

(b) Choose open sets U_n, V_n, W_n, $G_n \subseteq \mathbb{Q}$ and points q'_n, $q_n \in \mathbb{Q}$ inductively, as follows. $U_0 = \mathbb{Q}$. Given U_n, let $V_n \subseteq U_n$ be a non-empty open set such that $r_n \notin \overline{V}_n$ and f_n is bounded below on V_n. Given V_n, then if there is a non-empty open subset of V_n on which f_n is constant, take such a set for W_n; otherwise, set $W_n = V_n$. Let q'_n be any point of W_n. Let $G_n \subseteq W_n$ be an open neighbourhood of q'_n such that $f_n(q) \ge f_n(q'_n)$ whenever $q \in G_n$. Now take $q_n \in G_n \setminus \{q'_n\}$ such that $f_j(q_n) \ne f_j(q'_n)$ for any $j \le n$ such that $\{q : q \in G_n,\ f_j(q) = f_j(q'_n)\}$ has empty interior. Set $U_{n+1} = \{q : q \in G_n,\ f_j(q) < f_j(q_n)$ whenever $j \le n$ and $f_j(q'_n) < f_j(q_n)\}$, and continue.

(c) At the end of the induction, $\langle q_n\rangle_{n\in\mathbb{N}}$ can have no limit in $\mathbb{Q}$ because $q_n \in V_j$ whenever $j \le n$ and $r_j \notin \overline{V}_j$. On the other hand, if $j \in \mathbb{N}$ then $\langle f_j(q_n)\rangle_{n\ge j}$ is non-increasing. **P** If f_j is constant on W_j, this is immediate, because $q_n \in W_j$ for $n \ge j$. Otherwise, for any $n \ge j$, $\{q : q \in G_n,\ f_j(q) = f_j(q'_n)\}$ has empty interior, so $f_j(q'_n) < f_j(q_n)$, $f_j(q) < f_j(q_n)$ for every $q \in U_{n+1}$ and $f_j(q_{n+1}) < f_j(q_n)$. **Q**

At the same time we know that $\langle f_j(q_n)\rangle_{n\in\mathbb{N}}$ is bounded below in $\mathbb{R}$ because f_j is bounded below on V_j. So $\lim_{n\to\infty} f_j(q_n) = \inf_{n\ge j} f_j(q_n)$ is defined in $\mathbb{R}$. Accordingly $\lim_{n\to\infty} f(q_n)$ is defined in $\mathbb{R}^{\mathbb{N}}$. But this means either that $f[\mathbb{Q}]$ is not closed in $\mathbb{R}^{\mathbb{N}}$ or that f is not a homeomorphism between $\mathbb{Q}$ and $f[\mathbb{Q}]$.

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[1] I am indebted to J.Pachl for the reference.

4. Proposition Let X be a compact metrizable space and $\mathcal{K}$ a family of compact subsets of X such that $\#(\mathcal{K})$ is less than $\operatorname{cov}\mathcal{M} = \mathfrak{m}_{\text{countable}}$, the least cardinal of any cover of $\mathbb{R}$ by meager sets (Fremlin 08, 522S). Then $X \setminus \bigcup\mathcal{K}$ is Prokhorov.

proof (a) Let $\mathcal{U}$ be a countable base for the topology of X which is closed under finite unions. Write Y for $X \setminus \bigcup\mathcal{K}$. Let $A \subseteq P_{\mathrm{R}}(Y)$ be a narrowly compact set. Let $\epsilon > 0$.

(b) For an open set $G \subseteq X$, set

$$\theta(G) = \sup_{\mu \in A} \mu(G \cap Y).$$

Then θ is a submeasure, order-continuous on the left (Fremlin 02, 392A and 386Yb), because $G \mapsto \mu(G \cap Y)$ is for every $\mu \in A$. Set $\mathcal{V} = \{U : U \in \mathcal{U},\ \theta(U) < \epsilon\}$, ordered by $\subseteq$. For $K \in \mathcal{K}$, set $\mathcal{V}_K = \{V : V \in \mathcal{V},\ K \subseteq V\}$. Then $\mathcal{V}_K$ is cofinal with $\mathcal{V}$. **P** Take any $V \in \mathcal{V}$. As $X \setminus K$ is open in X, it is a Prokhorov space (Fremlin 03, 437Vc), and it includes Y. Let $A' \subseteq P_{\mathrm{R}}(X \setminus K)$ be the set of extensions of members of A to Radon probability measures on $X \setminus K$, as in Fremlin 03, 437Nb, so that A' is narrowly compact. There is therefore a compact set $L \subseteq X \setminus K$ such that $\nu((X \setminus K) \setminus L) \le \frac{1}{2}(\epsilon - \theta(V))$ for every $\nu \in A'$, that is, $\mu(Y \setminus L) \le \frac{1}{2}(\epsilon - \theta(V))$ for every $\mu \in A$, that is, $\theta(X \setminus L) \le \frac{1}{2}(\epsilon - \theta(V))$. Next, there is a $U \in \mathcal{U}$ such that $K \subseteq U \subseteq X \setminus L$ because K is compact, L is closed and $\mathcal{U}$ is a base for the topology of X; and now $V \cup U \in \mathcal{U}$, $K \subseteq V \cup U$ and $\theta(V \cup U) \le \theta(V) + \theta(U) < \epsilon$, so we have $V \subseteq V \cup U \in \mathcal{V}_K$. **Q**

(c) Because $\#(\mathcal{K}) < \mathfrak{m}_{\text{countable}}$ and $\mathcal{V}$ is countable, there is an upwards-directed subset $\mathcal{W}$ of $\mathcal{V}$ meeting every $\mathcal{V}_K$ (Fremlin 08, 517B). Set $H = \bigcup\mathcal{W}$; then $H \supseteq \bigcup\mathcal{K}$. So $L = X \setminus H$ is a compact set included in Y, while

$$\mu(Y \setminus L) \le \theta(H) = \sup_{G \in \mathcal{W}} \theta G \le \epsilon$$

for every $\mu \in A$. As A and ϵ are arbitrary, Y is a Prokhorov space.

5. Proposition For a cardinal κ, $\mathbb{Q}$ is embeddable in $\mathbb{R}^\kappa$ as a closed subset iff κ is at least $\mathfrak{d}$, the cofinality of $\mathbb{N}^{\mathbb{N}}$.

proof (a) Suppose there is a function $f : \mathbb{Q} \to \mathbb{R}^\kappa$ such that $f[\mathbb{Q}]$ is closed in $\mathbb{R}^\kappa$ and f is a homeomorphism between $\mathbb{Q}$ and its image. For $\xi < \kappa$, $q \in \mathbb{Q}$ set $f_\xi(q) = f(q)(\xi)$, so that $f_\xi : \mathbb{Q} \to \mathbb{R}$ is continuous. Set

$$G_\xi = \bigcup\{U : U \subseteq \mathbb{R} \text{ is open, } f_\xi[U \cap \mathbb{Q}] \text{ is bounded in } \mathbb{R}\}.$$

Then G_ξ is open and $\mathbb{Q} \subseteq G_\xi$. Now $\mathbb{Q} = \bigcap_{\xi<\kappa} G_\xi$. **P?** Otherwise, take $x \in \bigcap_{\xi<\kappa} G_\xi \setminus \mathbb{Q}$. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{Q}$ containing $U \cap \mathbb{Q}$ for every neighbourhood U of x. For $\xi < \kappa$, $f_\xi[[\mathcal{F}]]$ is an ultrafilter on $\mathbb{R}$; because $x \in G_\xi$, $f_\xi[[\mathcal{F}]]$ contains a bounded set and is convergent. Accordingly $f[[\mathcal{F}]]$ converges in $\mathbb{R}^\kappa$, and the limit must belong to $\overline{f[\mathbb{Q}]}$. It is therefore of the form $f(q)$ for some $q \in \mathbb{Q}$. But as $x \ne q$ there is a neighbourhood V of q such that $x \notin \overline{V}$, $\mathbb{Q} \setminus \overline{V} \in \mathcal{F}$ and $f(q) \in \overline{f[\mathbb{Q} \setminus V]}$; which is impossible because f is supposed to be a homeomorphism between $\mathbb{Q}$ and $f[\mathbb{Q}]$. **XQ**

Consequently $\{[-n, n] \setminus G_\xi : n \in \mathbb{N},\ \xi < \kappa\}$ is a cover of $\mathbb{R} \setminus \mathbb{Q}$ by at most $\max(\omega, \kappa)$ compact sets. But $\mathbb{R} \setminus \mathbb{Q}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and every compact subset of $\mathbb{N}^{\mathbb{N}}$ has an upper bound in $\mathbb{N}^{\mathbb{N}}$. So $\mathfrak{d} \le \max(\omega, \kappa)$; as $\mathfrak{d}$ is uncountable, $\mathfrak{d} \le \kappa$.

(b) Now suppose that $\kappa \ge \mathfrak{d}$. Using the same ideas as in the last part of (a) above, we have a family $\langle K_\xi \rangle_{\xi<\kappa}$ of compact sets with union $\mathbb{R} \setminus \mathbb{Q}$. Set $G_\xi = \mathbb{R} \setminus K_\xi$ for each ξ, so that $\mathbb{Q} = \bigcap_{\xi<\kappa} G_\xi$. This means that $\mathbb{Q}$ will be homeomorphic to

$$Q = \{x : x \in \prod_{\xi<\kappa} G_\xi,\ x(\xi) = x(\eta) \text{ for all } \xi, \eta < \kappa\},$$

which is a closed subset of $\prod_{\xi<\kappa} G_\xi$.

Next note that, for any $\xi < \kappa$, G_ξ is homeomorphic to a closed subset of $\mathbb{R}^2$. **P** G_ξ has a partition into countably many non-empty open intervals; topologically it is the direct sum of these intervals, and each is homeomorphic to $\mathbb{R}$; consequently G_ξ is homeomorphic to $\mathbb{R} \times I$ for some countable set I, and is homeomorphic to a closed subset of $\mathbb{R}^2$. **Q** Consequently $\prod_{\xi<\kappa} G_\xi$ is homeomorphic to a closed subset of $(\mathbb{R}^2)^\kappa$. But this means that $\mathbb{Q} \cong Q$ is homeomorphic to a closed subset of $(\mathbb{R}^2)^\kappa \cong \mathbb{R}^\kappa$.

6. Corollary $\mathbb{R}^{\mathfrak{d}}$ is not a Prokhorov space.

proof A closed subset of a Prokhorov space is Prokhorov (Fremlin 03, 437Vb) and $\mathbb{Q}$ is not Prokhorov, by Preiss' theorem.

7. Problem Is it relatively consistent with ZFC to suppose that $\mathbb{R}^{\omega_1}$ is a Prokhorov space?

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