\[ p = t, \text{ following Malliaris-Shelah and Steprāns} \]

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I attempt a proof, based on that sketched in Steprāns N13, of the theorem in Malliaris & Shelah 16 that \( p = t \).

1 Gaps, interpolation and chain-additivity

1A Definitions Let \( P \) be a partially ordered set and \( \lambda, \kappa \) non-zero cardinals.

(a) A \((\lambda, \kappa^*)\)-gap in \( P \) is a pair \((\langle x_\xi \rangle_{\xi < \lambda}, \langle y_\eta \rangle_{\eta < \kappa})\) of families in \( P \) such that
\[ x_\xi < x_{\xi'} \leq y_{\eta'} < y_\eta \text{ for every } \xi < \xi' < \lambda \text{ and } \eta < \eta' < \kappa, \]
 whenever \( z \in P \) is such that \( x_\xi < x_{\xi'} \leq z \leq y_{\eta'} < y_\eta \text{ whenever } \xi < \lambda \text{ and } \eta < \kappa. \]

(b) A peculiar \((\lambda, \kappa^*)\)-gap in \( P \) is a pair \((\langle x_\xi \rangle_{\xi < \lambda}, \langle y_\eta \rangle_{\eta < \kappa})\) of families in \( P \) such that
\[ x_\xi < x_{\xi'} \leq y_{\eta'} < y_\eta \text{ for every } \xi < \xi' < \lambda \text{ and } \eta < \eta' < \kappa, \]
 whenever \( z \in P \) is such that \( z < y_\eta \text{ for every } \eta < \kappa \), there is a \( \xi < \lambda \) such that \( z < x_\xi \),
 whenever \( z \in P \) is such that \( x_\xi < z < \xi < \lambda \), there is an \( \eta < \kappa \) such that \( y_\eta < z. \)

1B Definitions Let \( (P, \leq) \) be a partially ordered set.

(a) The chain-additivity of \( P \), \( \text{chadd} \ P \), is the least cardinal of any totally ordered subset of \( P \) with no upper bound in \( P \); or \( \infty \) if there is no such set.

Note that \( \text{chadd} \ P \) is either 0 (if \( P \) is empty) or \( \infty \) (if every maximal chain in \( P \) has a greatest member) or a regular infinite cardinal \( \kappa \), and in the last case there is a strictly increasing family \( \langle y_\eta \rangle_{\eta < \kappa} \) in \( P \) with no upper bound in \( P \).

If \( P \) is upwards-directed then \( \text{chadd} \ P = \text{add} \ P \) as defined in Fremlin 08, 511Bb.

(b)(i) If \( \lambda \) is a cardinal, say that \( \lambda \) has the \( < \kappa \)-interpolation property if whenever \( A, B \subseteq P \) are non-empty, \( a \leq b \) for every \( a \in A \) and \( b \in B \), and \( \max(\#(A), \#(B)) \prec \kappa \), then there is a \( c \in P \) such that \( a \leq c \leq b \) whenever \( a \in A \) and \( b \in B \).

(ii) The interpolation number of \( P \), \( \text{interp} \ P \), is the greatest cardinal \( \kappa \) such that \( \lambda \) has the \( < \kappa \)-interpolation property, or \( \infty \) if there is no such \( \kappa \). (For this use of ‘\( \infty \)’, see Fremlin 08, 511C.)

Note that \( \text{interp} \ P = \infty \) iff \( P \) is Dedekind complete, and that \( \text{interp} \ P \geq \omega \) if \( P \) is a lattice.

1C Lemma Suppose that \( P \) is a lattice. Write \( \text{chgap} \ P \) for the least cardinal \( \kappa \) such that there is a \((\lambda_0, \lambda_1^*)\)-gap in \( P \) with cardinals \( \lambda_0, \lambda_1 \leq \kappa \), or \( \infty \) if there is no such \( \kappa \). Then \( \text{interp} \ P = \text{chgap} \ P \).

\text{proof} (a) Suppose that \((\langle x_\xi \rangle_{\xi < \lambda_0}, \langle y_\eta \rangle_{\eta < \lambda_1})\) is a \((\lambda_0, \lambda_1^*)\)-gap. Then \( \{x_\xi : \xi < \lambda_0\}, \{y_\eta : \eta < \lambda_1\} \) witness that \( \text{interp} \ P \leq \text{max}(\lambda_0, \lambda_1) \). As \((\langle x_\xi \rangle_{\xi < \lambda_0}, \langle y_\eta \rangle_{\eta < \lambda_1})\) is arbitrary, \( \text{interp} \ P \leq \text{chgap} \ P \).

(b) Suppose that \( A, B \subseteq P \) are non-empty sets with cardinal less than \( \text{chgap} \ P \) and \( a \leq b \) for every \( a \in A \) and \( b \in B \).

(i) If \( A \) is well-ordered and \( B \) is downwards well-ordered (that is, \( (B, \geq) \) is well-ordered), then there is \( a \in P \) such that \( a \leq c \leq b \) for every \( a \in A \) and \( b \in B \). \( \textbf{P} \) Set \( \lambda_0 = \text{cf} A, \lambda_1 = \text{cf} B \), and let \( \langle x_\xi \rangle_{\xi < \lambda_0} \) be the increasing enumeration of a cofinal subset of \( A \) with cardinal \( \lambda_0 \), and \( \langle y_\eta \rangle_{\eta < \lambda_1} \) the decreasing enumeration of a coinitial subset of \( B \) with cardinal \( \lambda_1 \); then \( \lambda_0 \leq \#(A) < \text{chgap} \ P \) and \( \lambda_1 \leq \#(B) < \text{chgap} \ P \), so \((\langle x_\xi \rangle_{\xi < \lambda_0}, \langle y_\eta \rangle_{\eta < \lambda_1})\) cannot be a \((\lambda_0, \lambda_1^*)\)-gap and there must be a \( c \in P \) such that \( x_\xi \leq c \leq y_\eta \) for all \( \xi \) and \( \eta \), so that \( a \leq c \leq b \) whenever \( a \in A \) and \( b \in B \). \( \textbf{Q} \)

(ii) If \( A \) is well-ordered then there is a \( c \in P \) such that \( a \leq c \leq b \) for every \( a \in A \) and \( b \in B \). \( \textbf{P} \) If \( B \) is finite, this is trivial, as \( \text{inf} B \) is defined in \( P \). So suppose that \( B \) is infinite. Set \( \lambda_0 = \text{cf} A \) and let \( \langle x_\xi \rangle_{\xi < \lambda_0} \) be the increasing enumeration of a cofinal subset of \( A \) with cardinal \( \lambda_0 \); of course \( \lambda_0 \prec \text{chgap} \ P \). Set
A' = \{ y : a \leq y \text{ for every } a \in A \} = \{ y : x_\xi \leq y \text{ for every } \xi < \lambda_0 \} \supseteq B.

Enumerate B as \( (b_\eta)_{\eta < \lambda_1} \), where \( \lambda_1 < \text{chgap } P \), and choose a non-increasing family \( (y_\eta)_{\eta \leq \lambda_1} \) in A' inductively, as follows. Start with \( y_0 = b_0 \). Given \( y_\eta \), where \( \eta < \lambda_1 \), set \( y_{\eta+1} = y_\eta \wedge b_\eta \). Given a non-zero limit ordinal \( \beta \leq \lambda_1 \) and a non-increasing family \( (y_\eta)_{\eta < \beta} \) in A', the set \( \{ y_\eta : \eta < \beta \} \) is downwards well-ordered and has cardinal at most \#(\beta) \leq \lambda_1 < \text{chgap } P \), so by (ii) there is a \( y_\beta \in P \) such that \( a_\xi \leq y_\beta \leq y_\eta \) for every \( \xi < \lambda_0 \) and \( \eta < \beta \), and the induction continues.

At the end of the induction, consider \( c = y_{\lambda_1} \). Then \( c \in A'; \) since also \( c \leq y_{\eta+1} \leq b_\eta \) for every \( \eta < \lambda_1 \), c serves. Q

(iii) In any case, there is a \( c \in P \) such that \( a \leq c \leq b \) for every \( a \in A \) and \( b \in B \). If A is finite, take \( c = \sup A \). Otherwise, enumerate A as \( (a_\xi)_{\xi < \lambda_0} \) and choose \( (x_\xi)_{\xi < \lambda_0} \) inductively, as follows. Start with \( x_0 = a_0 \). If \( \xi < \lambda_0 \), set \( x_{\xi+1} = x_\xi \lor a_\xi \). If \( \alpha \leq \lambda_0 \) is a non-zero limit ordinal, (ii) tells us that there is an \( x_\alpha \in P \) such that \( x_\xi \leq x_\alpha \leq b \) whenever \( \xi < \alpha \) and \( b \in B \). At the end of the induction, take \( c = x_{\lambda_0} \).

(iv) As A and B are arbitrary, chgap P \leq \text{interp } P \) and the two are equal.

1D Lemma Let P be a lattice with the \( < \omega_1 \)-interpolation property which is not Dedekind \( \sigma \)-complete. Then interp P \leq \#(P).

proof (a) Suppose that there is a countable subset A of P with an upper bound but no least upper bound. Then A must be infinite; let \( (p_\eta)_{\eta \in \mathbb{N}} \) be an enumeration of A, and set \( p'_\eta = \sup_{\eta \leq \eta} p_\eta \), so that \( (p'_\eta)_{\eta \in \mathbb{N}} \) is a non-decreasing sequence which is not eventually constant, and has a strictly increasing subsequence \( (p''_\eta)_{\eta \in \mathbb{N}} \).

Let B be the set of upper bounds of A. Because B has no infimum, it is surely infinite; set \( \kappa = \#(B) \) and enumerate B as \( (q_\eta)_{\eta < \kappa} \). Choose \( (q''_\eta)_{\eta < \beta} \in B \) as follows. Start with \( q_0'' = q_0 \). Given \( q_\eta'' \in B, q_\eta'' \) is not the least member of B, so there is a first \( \zeta_\eta < \kappa \) such that \( q_\eta'' \not\leq q_{\zeta_\eta} \); set \( q_{\eta+1}'' = q_\eta'' \wedge q_{\zeta_\eta} \). Given \( (q''_\eta)_{\eta < \eta} \) where \( \eta < \kappa \) is a non-zero limit ordinal, then if there is a member of B less than or equal to \( q_{\eta}'' \) for every \( \eta < \eta \), take such a member for \( q''_\eta \); otherwise set \( \beta = \eta \) and stop. If the induction continues to the end, then there cannot be a member of B less than or equal to \( q_{\eta}'' \) for every \( \eta < \kappa \), so set \( \beta = \kappa \).

Thus we have a strictly decreasing family \( \{ q''_\eta : \eta < \beta \} \) in B with no lower bound in B, where \( \omega \leq \beta \leq \kappa \leq \#(P) \). Set \( \lambda = \text{cf } \beta \) and let \( (q_\eta)_{\lambda < \beta} \) be the increasing enumeration of a cofinal subset of \( \beta \). Then \( (q''_\eta)_{\eta \in \mathbb{N}, \eta < \lambda} \) is an \( (\omega, \lambda^*) \)-gap in P. As P has the \( < \omega_1 \)-interpolation property, \( \lambda > \omega \); so

\[ \text{interp } P \leq \lambda \leq \beta \leq \kappa \leq \#(P). \]

(b) If there is a countable subset of P with a lower bound but no greatest lower bound, argue similarly, or apply (a) to (P, \geq).

1E Definitions (a) Write p for the least cardinal of any downwards-directed set A \( \subseteq [\mathbb{N}]^\omega \) for which there is no b \in [\mathbb{N}]^\omega \) such that b \( \setminus a \) is finite for every a \in A.

(b) Write t for the least cardinal \( \kappa \) for which there is a family \( \{ a_\xi : \xi < \kappa \} \subseteq [\mathbb{N}]^\omega \) such that \( a_\eta \setminus a_\xi \) is finite whenever \( \xi < \eta < \kappa \), but there is no a \in [\mathbb{N}]^\omega \) such that \( a \setminus a_\xi \) is finite for every \( \xi < \kappa \).

2 Reduced products and internal sets

Most of the rest of the arguments in this note will be based on a fragment of the model theory of ultrafilters. For the next few sections, fix an ultrafilter \( \mathcal{F} \) on a set I.

2A Suppose that \( X_i \) is a non-empty set for each i \( \in I \),

(a) We have an equivalence relation on \( \prod_{i \in I} X_i \), given by saying that \( \langle x_i \rangle_{i \in I} \sim \langle y_i \rangle_{i \in I} \) if \( \{ i : x_i = y_i \} \) belongs to \( \mathcal{F} \). I will write \( \langle x_i \rangle_{i \in I}^* \) for the equivalence class of \( \langle x_i \rangle_{i \in I} \). The set of equivalence classes is the reduced product of \( (X_i)_{i \in I} \) mod \( \mathcal{F} \), which I will denote \( \prod_{i \in I} X_i / \mathcal{F} \). (See FRELMIN 08, 5A2A.)

(b) A subset \( Z \) of \( X \) is internal if it corresponds to a member of \( \prod_{i \in I} \mathcal{P} X_i / \mathcal{F} \), that is, if there is a family \( \{ Z_i \}_{i \in I} \) such that \( Z_i \subseteq X_i \) for every \( i \in I \) and \( Z = \{ \langle x_i \rangle_{i \in I} : \{ i : x_i \in Z_i \} \in \mathcal{F} \} \); note that if every \( Z_i \) is non-empty this is in a natural one-to-one correspondence with \( \prod_{i \in I} Z_i / \mathcal{F} \).

(c) Because \( \mathcal{F} \) is an ultrafilter, the family of internal subsets of \( X \) is an algebra of sets containing all singleton sets, therefore every finite subset of \( X \).

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(d) If $Z$ is a non-empty internal subset of $X$, then a subset of $Z$ is internal in $Z$ iff it is internal in $X$. P Q

(e) Generally, when I use an italic bold upper-case letter like $X$ or $P$, you should take it that I am thinking of a set together with an associated structure of internal sets.

2B Let $X = \prod_{i \in J} X_i | F$ and $Y = \prod_{i \in J} Y_i | F$ be two reduced products mod $F$. Then we have a natural bijection between $X \times Y$ and $\prod_{i \in J} X_i \times Y_i | F$, identifying $\langle (x_i)_{i \in J}, (y_i)_{i \in J} \rangle$ with $\langle (x_i, y_i) \rangle_{i \in J}$. This gives us an associated notion of ‘internal’ subset of $X \times Y$, being one corresponding to an internal subset of $\prod_{i \in J} X_i \times Y_i | F$.

The same idea applies to products of any finite number of reduced products mod $F$.

2C Again suppose that $X = \prod_{i \in J} X_i | F$ and $Y = \prod_{i \in J} Y_i | F$ are two reduced products mod $F$.

(a) If $Z \subseteq X$ and $W \subseteq Y$ are internal, then $Z \times W$ is an internal subset of $X \times Y$.

(b) If $W$ is an internal subset of $X \times Y$ and $Z$ is an internal subset of $X$, then $W[Z]$ is an internal subset of $Y$. (For if $W$ corresponds to $(W_i)_{i \in I}$ and $Z$ to $(Z_i)_{i \in I}$, then $W[Z]$ corresponds to $(W_i[Z_i])_{i \in I}$.) In particular, any section $W([x])$, where $x \in X$, is an internal subset of $Y$.

(c) If $W_i \subseteq X_i \times Y_i$ is the graph of a function for each $i$, then the corresponding internal relation $W \subseteq X \times Y$ will be the graph of a function, its domain being the internal subset of $X$ corresponding to $\langle \text{dom } W_i \rangle_{i \in I}$.

(d) If $X_i = Y_i$ and $W_i$ is a partial order on $X_i$ for each $i$, then $W$ will be a partial order on $X$. If $X_i = Y_i$ and $W_i$ is a total order on $X_i$ for each $i$, then $W$ will be a total order on $X$. If $X_i = Y_i$ and $W_i$ is a well-ordering of $X_i$ for each $i$, then every non-empty internal subset of $X$ will have a $W$-least member. (For if $Z \subseteq X$ corresponds to $(Z_i)_{i \in I}$ and $x = (x_i)_{i \in I} \in Z$, define $(z_i)_{i \in I}$ by saying that $z_i$ is the $W_i$-least member of $Z_i$ if $Z_i \neq \emptyset$,

\[ z_i = x_i \text{ otherwise; } \]

then $(z_i)_{i \in I}$ is the $W$-least member of $Z$.)

(e) Conversely, if $W$ is an internal subset of $X \times X$ and is a partial order, then there is a family $(W_i)_{i \in I}$ such that $W_i$ is a partial order on $X_i$ for each $i$ and $W$ corresponds to $(W_i)_{i \in I}$. P By the definition of ‘internal subset of $X \times X$’ there is a family $(W'_i)_{i \in I}$ such that $W$ corresponds to $(W'_i)_{i \in I}$. Set $\Delta_i = \{(x, x) : x \in X_i\}$ for $i \in I$. Now consider

\[ J = \{i : W'_i \not\supseteq \Delta_i\}, \quad K = \{i : W_i \circ W'_i \not\subseteq W'_i\} \quad L = \{i : W_i \cap W'_i \not\subseteq \Delta_i\}. \]

? If $J \in F$, take $x_i \in X_i$ such that $(x_i, x_i) \notin W'_i$ for $i \in J$ and set $x = (x_i)_{i \in J}$; then $(x, x) \notin W$. X

? If $K \in F$, take $x_i, y_i, z_i \in X_i$ such that, for $i \in K$, $(x_i, y_i) \in W'_i, (y_i, z_i) \in W'_i$ but $(x_i, z_i) \notin W'_i$; setting $x = (x_i)_{i \in K}, y = (y_i)_{i \in K}$ and $z = (z_i)_{i \in K}$, $(x, y) \in W$ and $(y, z) \in W$ but $(x, z) \notin W$. X

? If $L \in F$, take $x_i, y_i \in X_i$ such that, for $i \in L$, $(x_i, y_i) \in W'_i$ and $(y_i, x_i) \in W'_i$ but $x_i \neq y_i$. Setting $x = (x_i)_{i \in L} \text{ and } y = (y_i)_{i \in L}, (x, y) \in W$ and $(y, x) \in W$ but $x \neq y$. X

Consequently, $M = I \setminus (J \cup K \cup L)$ belongs to $F$, while $W'_i$ is a partial order on $X_i$ for every $i \in M$. Setting $W_i = W'_i$ for $i \in M, W_i = \Delta_i$ for $i \in J \cup K \cup L$, we have a suitable family. Q

(f) If $W$ is an internal subset of $X \times Y$, then its projection $\{x : \exists y, (x, y) \in W\}$ is an internal subset of $X$. P If $W$ corresponds to $(W_i)_{i \in I}$, consider $A_i = \{x : \exists y, (x, y) \in W_i\}$ for each $i \in I$. Q Hence, or otherwise, $\{x : (x, y) \in W \text{ for every } y \in Y\}$ is an internal subset of $X$.

2D Power sets (a) Once more, suppose that we have a family $(X_i)_{i \in I}$ of non-empty sets and the reduced product $X = \prod_{i \in I} X_i | F$. Then we can form the reduced product $\prod_{i \in I} P X_i | F$.

(b) If $(Z_i)_{i \in I}$ and $(Z'_i)_{i \in I}$ belong to $\prod_{i \in I} P X_i$, and we look at the corresponding internal sets $Z, Z'$ as defined in 2A(b), we find that $Z = Z'$ iff $\{i : Z_i = Z'_i\} \in F$. P If $J = \{i : Z_i = Z'_i\}$ belongs to $F$, then for any $(x_i)_{i \in I} \in \prod_{i \in I} X_i$ D.H.FREMLIN
\[(x_i)_{i \in I} \in Z \iff \{i : i \in I, x_i \in Z_i\} \in F \iff \{i : i \in J, x_i \in Z_i\} \in F \iff \langle x_i \rangle_{i \in I} \in Z' \]

and \(Z = Z'\). If \(K = \{i : Z_i \subseteq Z_i'\}\) belongs to \(F\), choose \(z_i \in Z_i \setminus Z_i'\) for \(i \in K\), and take \(\langle z_i \rangle_{i \in I} \in Z \setminus Z'\) and \(Z \neq Z'\). Similarly, \(Z \neq Z'\) if \(\{i : Z_i' \not\subseteq Z_i\}\) \(\in F\). So if \(J \notin F\) then \(z \neq Z'\). \(Q\)

(c) Thus we have a natural bijection between the reduced product \(\prod_{i \in I} PX_i | F\) and the algebra \(PX\) of internal subsets of \(X\). Accordingly we have a notion of internal subset of \(PX\), being one corresponding to a family \(\langle A_i \rangle_{i \in I}\) where \(A_i \subseteq X_i\) for each \(i\), so that \(PPX\) can be identified with \(\prod_{i \in I} PPX_i | F\).

(d) In 2A I spoke of a ‘structure of internal sets’; the vagueness was deliberate, as I intended to include only the subalgebra \(PX\) of \(PX\) but the repeated sets-of-internal-sets algebras \(PPX, PPPX\) and so on. (For this note, happily, we do not have to go far along this road.)

2E Proposition Let \(\langle X_i \rangle_{i \in I}\) be a family of non-empty sets, and \(X = \prod_{i \in I} X_i | F\) their reduced product. Then the relation \(\subseteq\) on \(PX\) is internal.

proof As in 2D, we can identify \(PX\) with \(\prod_{i \in I} PX_i | F\), so that we think of internal subsets of \(X\) as equivalence classes \(\langle Z_i \rangle_{i \in I}\) where \(Z_i \subseteq X_i\) for \(i \in I\). Now if we have two families \(\langle W_i \rangle_{i \in I}, \langle Z_i \rangle_{i \in I}\) \(\in \prod_{i \in I} PX_i\) representing internal sets \(W, Z \subseteq X\), we have \(W \subseteq Z\) if \(J = \{i : W_i \subseteq Z_i\}\) belongs to \(F\). \(P\) This is a trivial refinement of 2Db. If \(J \in F\) and \(x = \langle x_i \rangle_{i \in I} \in W\), then \(\{i : x_i \in Z_i\} \supseteq J \cap \{i : x_i \in W_i\}\) belongs to \(F\) and \(x \in Z\). If \(I \setminus J \in F\), choose \(x_i \in W_i \setminus Z_i\) for \(i \in I \setminus J, x_i \in X_i\) for \(i \in J\); then \(x = \langle x_i \rangle_{i \in I}\) belongs to \(W \setminus Z\) and \(W \not\subseteq Z\).

Now this means that if we look at the internal subset of \(PX \times PX\) corresponding to the family \(\langle \subseteq_i \rangle_{i \in I} \in \prod_{i \in I} PX_i \times PX_i\), where \(\subseteq_i = \{\langle W, Z \rangle : W \subseteq Z \subseteq X_i\}\) for each \(i\), we find that it is precisely the relation \(\subseteq\).

2F Definitions (a) Let \(Umf_{\subseteq}(F)\) be the class of structures isomorphic to structures \(\prod_{i \in I} X_i | F\), together with the corresponding algebras of internal sets, where every \(X_i\) is finite and not empty.

(b) Let \(Po_{\subseteq}(F)\) be the class of non-empty partially ordered sets \((P, \leq)\) where \(P \in Umf_{\subseteq}(F)\) and \(\leq\) is an internal relation on \(P\) which is a partial order. As noted in 2Ce, we must then be able to identify \((P, \leq)\) with a structure \(\prod_{i \in I} (P_i, \leq_i) | F\) where \(P_i\) is finite and \(\leq_i\) is a partial order on \(P_i\) for every \(i\).

(c) Let \(Lo_{\subseteq}(F) \subseteq Po_{\subseteq}(F)\) be the class of non-empty totally ordered sets belonging to \(Po_{\subseteq}(F)\). If \((X, \leq) \in Lo_{\subseteq}(F)\), we can identify it with a structure \(\prod_{i \in I} (X_i, \leq_i) | F\) where \(X_i\) is finite and \(\leq_i\) is a total order on \(X_i\) for every \(i\).

2G Proposition (a) If \(X \in Umf_{\subseteq}(F)\) and \(Z\) is a non-empty internal subset of \(X\), then \(Z \in Umf_{\subseteq}(F)\).

(b) If \(X \in Umf_{\subseteq}(F)\) then \(PX \in Umf_{\subseteq}(F)\).

(c) If \(X, Y \in Umf_{\subseteq}(F)\) then \(X \times Y \in Umf_{\subseteq}(F)\).

proof (a) If \(X \cong \prod_{i \in I} X_i | F\) where \(X_i\) is finite for every \(i \in I\), then \(Z \cong \prod_{i \in I} Z_i | F\) where \(Z_i \subseteq X_i\) is finite for every \(i \in I\).

(b) If \(X \cong \prod_{i \in I} X_i | F\) where \(X_i\) is finite for every \(i \in I\), then \(PX \cong \prod_{i \in I} PX_i | F\) and \(PX_i\) is finite for every \(i \in I\).

(c) If \(X \cong \prod_{i \in I} X_i | F\) and \(Y \cong \prod_{i \in I} Y_i | F\) where \(X_i\) and \(Y_i\) are finite for every \(i \in I\), then \(X \times Y \cong \prod_{i \in I} X_i \times Y_i | F\) and \(X_i \times Y_i\) is finite for every \(i \in I\).

2H Lemma (a) Suppose that \((P, \leq) \in Po_{\subseteq}(F)\). Then every non-empty internal subset of \(P\) has a maximal element.

(b) Suppose that \((P, \leq) \in Lo_{\subseteq}(F)\).

(i) \((P, \leq)\) is isomorphic to \((P, \geq)\).

(ii) Every non-empty internal subset of \(P\) has greatest and least members.

proof (a) The point is just that this is true for all finite partially ordered sets, and it is a first-order property. More explicitly, if \((P, \leq) \cong \prod_{i \in I} (P_i, \leq_i) | F\), and \(Z\) is a non-empty internal subset of \(P\), then \(Z\) corresponds
to $\prod_{i \in I} Z_i$ where $Z_i \subseteq P_i$ is non-empty for every $i \in I$. Now if $z_i \in Z_i$ is $\leq_i$-maximal for every $i$, $z = \langle z_i \rangle_{i \in I}$ is $\leq$-maximal in $Z$.

(b)(i) In this case,
\[(P, \geq) \cong \prod_{i \in I} \langle \langle P_i, \geq \rangle \rangle = \prod_{i \in I} \langle P_i, \leq \rangle = (P, \leq). \quad \text{Q} \]

(ii) This is a special case of (a).

21 Lemma If $P \in \text{Lo}_{<\omega}(\mathcal{F})$ is infinite, then $\omega_1 \leq \text{interp} P \leq \omega^{\#(I)}$.

proof We can suppose that $P$ is a reduced product $\prod_{i \in I} (P_i, \leq_i)$ where every $P_i$ is finite and every $\leq_i$ is a total order.

(a) $P$ has the $< \omega_1$-interpolation property. If $(p_k)_{k \in \mathbb{N}}$ and $(q_k)_{k \in \mathbb{N}}$ are sequences in $P$ with $p_j \leq q_k$ for all $j, k \in \mathbb{N}$, express $p_k$ as $\langle p_{k_1} \rangle_{i \in I}$ and $q_k$ as $\langle q_{k_1} \rangle_{i \in I}$, where $p_{k_i}, q_{k_i} \in P_i$ for $i \in I$ and $k \in \mathbb{N}$. Set
\[A_i = \{i : i \in I, p_{ji} \leq q_{ki} \text{ whenever } j, k < l\},\]
so that $(A_i)_{i \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{F}$ starting with $A_0 = I$. Set
\[p_i^* = \max(\{p_{0i}\} \cup \{p_{ji} : j < l\}) \text{ if } i \in A_i \setminus A_{i+1},\]
\[= \max\{p_{ji} : j \in \mathbb{N}\} \text{ if } i \in \bigcap_{l \in \mathbb{N}} A_l.\]

Then
\[\{i : p_{ki} \leq p_i^* \leq q_{ki}\} \supseteq A_{k+1} \in \mathcal{F}\]
for every $k$, so
\[p_k \leq \langle p_i^* \rangle_{i \in I} \leq q_k\]
for every $k \in \mathbb{N}$. \quad \text{Q}

(b) There is a sequence $(p_k)_{k \in \mathbb{N}}$ in $P$ with no supremum in $P$. Let $\langle p_k' \rangle_{k < \#(P)}$ be the increasing enumeration of $P$, for each $i$. As $P$ is infinite, $A_k = \{i : \#(P_i) \geq k\} \in \mathcal{F}$ for each $k$. So if we set
\[p_{ki} = p_{k_i}' \text{ if } k < \#(X_i),\]
\[= \max P_i \text{ if } k \geq \#(X_i),\]
\[p_k = \langle p_{ki} \rangle_{i \in I}\]
for $k \in \mathbb{N}$, $(p_k)_{k \in \mathbb{N}}$ will be strictly increasing. If $q = \langle q_{ki} \rangle_{i \in I}$ is an upper bound for $\{p_k : k \in \mathbb{N}\}$, and we take $q' = \langle q_{ki}' \rangle_{i \in I} \in \prod_{i \in I} X_i$ such that $q_i'$ is the predecessor of $q_i$ in $P_i$ whenever $q_i \neq \min P_i$, then $q' < q$ and $q'$ is still an upper bound of $\{p_k : k \in \mathbb{N}\}$, so $\{p_k : k \in \mathbb{N}\}$ has no least upper bound. \quad \text{Q}

(c) Since $P$ has a greatest member $\langle \max P_i \rangle_{i \in I}$, 1D tells us that
\[\text{interp} P \leq \#(P) \leq \#(\prod_{i \in I} P_i) \leq \omega^{\#(I)}.\]

3 Interp$_{<\omega}$ and Chadd$_{<\omega}$

As in §2, take a fixed ultrafilter $\mathcal{F}$ on a fixed set $I$.

3A Definitions (a) Write $\text{Interp}_{<\omega}(\mathcal{F})$ for $\min\{\text{interp} P : P \in \text{Lo}_{<\omega}(\mathcal{F})\}$.
(b) Write $\text{Chadd}_{<\omega}(\mathcal{F})$ for $\min\{\text{chadd} P : P \in \text{Po}_{<\omega}(\mathcal{F})\}$.

3B Lemma Suppose that $X \in \text{Lo}_{<\omega}(\mathcal{F})$ and that we have sets $A \subseteq X, \ Z \subseteq PX$ such that $(A), \ #(Z)$ are both less than $\min(\text{Chadd}_{<\omega}(\mathcal{F}), \ \text{interp} X)$ and every member of $Z$ is an internal set including $A$. Then there is an internal set $Z^* \subseteq X$ such that $A \subseteq Z^* \subseteq \bigcap Z$.

Remark Note that there is no suggestion that $A$ or $Z$ should be an internal set.
proof (a) If either $A$ or $Z$ is finite, the result is trivial. Otherwise, set $\kappa = \max(\#(A), \#(Z))$ and let $(x_\ell)_{\ell \in \kappa}, (z_\ell)_{\ell \in \kappa}$ run over $A, Z$ respectively.

Because \( \text{interp} X < \infty \) \( (2I) \), we have a \((\lambda_0, \lambda_1)\)-gap in $X$ with \( \max(\lambda_0, \lambda_1^\ast) = \text{interp} X \); as \((X, \leq) \cong (X, \geq)\), we can suppose that $\lambda_0 \leq \lambda_1$ and we have a strictly increasing family $(y_\eta)_{\eta < \text{interp} X}$ in $X$.

Let $P = \mathcal{P}(X \times X)$ be the set of internal subsets of $X \times X$. For $p \in P$ and $e \in X$, write $p/e$ for $\{(\min(\{z, e\}, x) : (z, x) \in p)\}$. Observe that $(p/e') e' = p/\min(e, e')$ for all $p, e$ and $e'$, so we have a partial order $\leq$ on $P$ defined by saying that $p' \leq p$ if there is an $e \in X$ such that $p' = p/e$. Now $(P, \leq) \in \mathcal{P}(\omega_\omega(F))$.

We have just to repeat the formula in each coordinate. Suppose that $(X, \leq)$ is isomorphic to the reduced product $\prod_{i \in I}(X_i, \leq_i)|\mathcal{F}$ where $(X_i, \leq_i)$ is a finite totally ordered set for each $i$. If \( i \in I \), \( p \subseteq X_i^2 \) and \( e \in X_i \), set $p/e = \{(\min(\{z, e\}, x) : (z, x) \in p)\}$; for $p', p \subseteq X_i^2$ say that $p' \leq_i P$ if there is an $e \in X_i$ such that $p' = p/e$. If now $p', p \in P$, we can identify them with $(p_i')_{i \in I}, (p_i')_{i \in I}$ respectively, where $p_i, p_i' \subseteq X_i^2$ for each $i$ (2B). If $e$ corresponds to $(e_i')_{i \in I} \subseteq \prod_{i \in I} X_i|\mathcal{F}$, $p/e$ corresponds to $(p_i|e_i')_{i \in I}$. So if $p' \leq p$, \( \{i : p_i' \leq_i p_i\} \) belongs to $\mathcal{F}$; and, conversely, if $J = \{i : p_i' \leq_i p_i\}$ belongs to $\mathcal{F}$, we can find a family $(e_i')_{i \in I} \subseteq \prod_{i \in I} X_i$ such that $J \subseteq \{i : p_i' = p_i|e_i\}$, in which case $p' \leq p$. Thus $P$ is isomorphic to $(\mathcal{P}(X^2), \leq_{\omega\omega}(F))$ and belongs to $\mathcal{P}(\omega_\omega(F))$. Q.E.D.

(b) Choose a non-decreasing family $(p_\eta)_{\eta \leq \kappa}$ in $P$ inductively, as follows. The inductive hypothesis will be that $p_\eta \in P$, $p_{\eta'} = p_\eta|y_{\eta'}$ whenever $\eta' < \eta$, and $(y_{\eta}, x_\ell) \in p_\eta$ whenever $\xi < \kappa$.

Start with $p_0 = \{y_0\} \times X$. Given $p_\eta$, where $\eta < \kappa$, set

$p_{\eta+1} = p_\eta \cup \{(y_{\eta+1}, x) : (y_{\eta}, x) \in p_\eta, x \in Z_\eta\}$.

Then $p_\eta = p_{\eta+1}[y_\eta \leq \eta+1]$ and $(y_{\eta+1}, x_\ell) \in p_{\eta+1}$ whenever $\xi \leq \kappa$, because $x_\xi \in A \subseteq Z_\eta \subseteq Z$.

For the inductive step to a non-zero limit ordinal $\eta \leq \kappa$, we have

$e\eta \leq \kappa < \text{Chadd}_{\omega\omega}(F) \leq \text{chadd} P$,

so there is an upper bound $p'$ of $(p_\eta : \eta' < \eta)$ in $P$. For each $\xi < \kappa$, set

$e_\xi = \max\{z : (z, x_\xi) \in p'\}$

which is defined because $p'$ is an internal subset of $X^2$, so $\{z : (z, x_\xi) \in p'\}$ is an internal subset of $X$, and is non-empty because $(y_\eta, x_\xi) \in p_\eta = p'|y_\eta$. If $\eta' < \eta$, then $(y_{\eta'}, x_\xi) \in p_{\eta'} = p'|y_{\eta'}$, so $y_{\eta'} \leq e_{\xi}$ and $y_{\eta} \leq \min(y_{\eta}, e_{\xi})$. Because $\kappa < \text{interp} X$, there must be an $e \in X$ such that $y_{\eta'} \leq e \leq \min(y_{\eta}, e_{\xi})$ whenever $\eta' < \eta$ and $\xi < \kappa$. Set

$p'' = p'|e, \ p_\eta = p'' \cup \{(y_{\eta}, x) : (e, x) \in p''\}$.

For $\eta' < \eta$, we have

$p_{\eta'} = p'|y_{\eta'} = p''[y_{\eta'} = p_\eta|y_{\eta'} \leq p_\eta]$,

while if $\xi < \kappa$ then $(e_{\xi}, x_\xi) \in p', (e, x_\xi) \in p''$ and $(y_{\eta}, x_\xi) \in p_\eta$. Of course $x \leq y_\eta$ whenever $(z, x) \in p_\eta$, so the induction continues.

(c) At the end of the induction, set $Z^* = \{x : (y_\kappa, x) \in p_\kappa\}$. Then $Z^*$ is an internal set because $p_\kappa$ is, and contains every $x_\xi$ by the construction of $p_\kappa$. If $\eta < \kappa$ and $x \in Z^*$, then

$(y_{\eta+1}, x) = (\min(y_\kappa, y_{\eta+1}), x) \in p_\kappa[y_{\eta+1} = p_{\eta+1},$ so $x \in Z_\eta$. Thus $A \subseteq Z^* \subseteq \cap Z_\eta$ as required.

3C Corollary Suppose that $X \in L_{\text{chadd}_{\omega\omega}(F)}$ and $h : X \times X \to X$ is an internal function. Let $A \subseteq X$ and $w \in X$ be such that $h(x, x') \leq w$ for all $x, x' \in A$ and $\#(A) < \min(\text{Chadd}_{\omega\omega}(F), \text{interp} X)$. Then there is an internal set $D \subseteq X$ such that $A \subseteq D$ and $h(x, x') \leq w$ for all $x, x' \in D$.

proof For $x \in A$, set $Z_x = \{x' : x' \in X, h(x, x') \leq w\}$. Then $Z_x$ is an internal subset of $X$ including $A$. Applying 3B to $A$ and $Z = \{Z_x : x \in A\}$, we see that there is an internal set $T \subseteq X$ such that $A \subseteq Z \subseteq T_x$ for every $x \in A$. Now

$D = \{x : x \in Z, h(x, x') \leq w \text{ for every } x' \in Z\}$

is an internal set including $A$, and $h(x, x') \leq w$ for all $x, x' \in D$.
3D Lemma Suppose that $X \in \text{Po}_{<\omega}(F)$ and that $Y \in \text{Ufm}_{<\omega}(F)$. Let $D \subseteq X$ be a well-ordered set with order type less than \text{Chadd}_{<\omega}(F)$, and $F : D \to Y$ a function. Then there is an internal function $h : X \to Y$ extending $F$.

proof (a) Write $\alpha$ for otp $D$. Let $P$ be the set of internal partial functions from subsets of $X$ to $Y$, that is, the set of internal subsets $p$ of $X \times Y$ such that $y = y'$ whenever $(x, y)$ and $(x, y') \in p$. Then $(P, \subseteq) \in \text{Po}_{<\omega}(F)$, being isomorphic to $\prod_{i \in I} (P_i, \subseteq)|F$ where each $P_i$ is the set of partial functions from subsets of $X_i$ to $Y_i$.

(See 2Ce.) So $\alpha < \text{chad} P$.

Let $(\beta_\beta)_{\beta \alpha}$ be the increasing enumeration of $D$.

(b) Choose a non-decreasing family $(p_\beta)_{\beta < \alpha}$ inductively, as follows. The inductive hypothesis will be that $p_\beta \in P$ and $d_\beta$ is the greatest element of dom $p_\beta$. Start with $p_0 = \{(d_0, F(d_0))\}$. Given $d_{\beta + 1} < \beta$, where $\beta < \alpha$, this is a totally ordered subset of $P$ of cofinality less than chad $P$, so has an upper bound $q \in P$; set

$$p_\beta = \{(x, y) : (x, y) \in q, x < d_\beta\} \cup \{(d_\beta, F(d_\beta))\}$$

for every $\beta < \alpha$, so $h \supseteq F$.

3E Lemma Suppose that $X \in \text{Po}_{<\omega}(F)$ and that $Y \in \text{Ufm}_{<\omega}(F)$. Suppose that $D \subseteq X$ is a well-ordered set with order type less than \text{Chadd}_{<\omega}(F)$, and $F : D^2 \to Y$ a function. Then there is an internal function from $X^2$ to $Y$ extending $F$.

proof Set $Z = \prod_{i \in I} Z_i|F$, where $Z_i$ is the set of functions from $X_i$ to $Y_i$ for each $i \in I$; note that each $Z_i$ is finite. For $d \in D$, define $F_d : D \to Y$ by setting $F_d(d') = F(d, d')$ for $d' \in D$. By 3D, we have an internal function $h_d : X \to Y$ extending $F_d$, and $h_d$ can be represented by a member $z_d$ of $Z$.

By 3D again, there is an internal function $h' : X \to Z$ such that $h'(d) = z_d$ for every $d \in D$. Suppose that $h'$ corresponds to $(h'_i)_{i \in I}$ where $h'_i : X_i \to Z_i$ is a function for each $i$. If we set $h_i(x, x') = h'_i(x)(x')$ for $x, x' \in X_i$, then $(h_i)_{i \in I}$ corresponds to an internal function $h : X^2 \to Y$. If $d, d' \in D$ correspond to $(d_i)_{i \in I}$ and $(d'_i)_{i \in I}$ respectively, then $h(d, d')$ corresponds to

$$h_i(d_i, d'_i)^{\ast_{i \in I}} = (h'_i(d_i))^{\ast_{i \in I}} = (h'_i(d_i))^{\ast_{i \in I}}(d'_i)^{\ast_{i \in I}}$$

and

$$h(d, d') = h'(d, d') = h_d(d') = F(d, d').$$

So $h$ extends $F$, as required.

3F Lemma If $X \in \text{Lo}_{<\omega}(F)$, $\kappa$ is a cardinal and there is a $(\kappa, \kappa^*)$-gap in $X$, then $\text{Chadd}_{<\omega}(F) \leq \kappa$.

proof (a) Of course $X$ must be infinite. Consider the partial ordering $\leq$ on $|X|^2$ defined by saying that $I \leq J$ if min $I \leq$ min $J$ and max $I \leq$ max $J$. Then $|X|^2, \leq$ is isomorphic to a member of $\text{Po}_{<\omega}(F)$. Suppose that $X \cong \prod_{i \in I} (X_i, \leq_i)|F$ where $(X, \leq_i)$ is a finite non-empty totally ordered set for each $i$. Since $\#(X) > 1$, $K = \{i : \#(X_i) \geq 2\} \in F$; set

$$(X_i, \leq_i) = (X_i, \leq_i) \text{ for } i \in K,$$

$$= \{(0, 1), \leq_i\} \text{ for } i \in I \setminus K.$$

On $|X|^2$ define $\leq_i$ by saying that $I \leq_i J$ if min $I \leq_i$ min $J$ and max $I \leq_i$ max $J$. Then $|X|^2, \leq$ is isomorphic to $\prod_{i \in I} (|X_i|^2, \leq)|F \in \text{Po}_{<\omega}(F)$.

(b) Let $(x_\xi)_{\xi < \kappa}$ be a $(\kappa, \kappa^*)$-gap in $X$. Then $\{(x_\xi, y_\xi)_{\xi < \kappa} \text{ is a strictly increasing family in } |X|^2$. If it has an upper bound $I \in |X|^2$, then $x_\xi \leq \min I \leq \max I \leq y_\eta$ for all $\xi, \eta < \kappa$, which is supposed to be impossible. So $\kappa \geq \text{chadd}|X|^2 \geq \text{Chadd}_{<\omega}(F)$.

D.H.Fremlin
3G Theorem  \( \text{Chadd}_{<\omega}(F) \leq \text{Interp}_{<\omega}(F) \).  

**Proof** Suppose otherwise.

(a) Of course \( \text{Interp}_{<\omega}(F) \) cannot be \( \infty \). Set \( \kappa = \text{Interp}_{<\omega}(F) \) and let \( X \in \text{Lo}_{<\omega}(F) \) be such that \( \text{interp} \, X = \kappa \). By 1C, there is a \((\lambda, \lambda^+)\)-gap in \( X \) with \( \text{max}(\lambda, \lambda^+) = \kappa \); since \( (X, \leq) \) is isomorphic to \( (X, \geq) \) (see 2Fa), we can take it that \( \lambda \leq \lambda^+ = \kappa \). We are supposing that \( \kappa < \text{Chadd}_{<\omega}(F) \). By 3F, there is no \((\kappa, \kappa^+)\)-gap in \( X \), so \( \lambda < \kappa \). Let \( (\langle x_\eta \rangle_{\eta < \lambda}, \langle x_\xi \rangle_{\xi < \kappa}) \) be a \((\lambda, \kappa^+)\)-gap in \( X \).

(b) Because \( (X, \geq) \) and \( (X, \leq) \) are isomorphic, and \( X \) has a strictly decreasing family \( \langle x_\xi \rangle_{\xi < \kappa} \), there is also a strictly increasing family \( \langle d_\xi \rangle_{\xi < \kappa} \) in \( X \). Let \( G : \lambda^+ \times \lambda^+ \rightarrow \lambda \) be such that \( \beta \rightarrow G(\alpha, \beta) : \alpha \rightarrow \lambda \) is injective for every \( \alpha < \lambda^+ \). Because \( \lambda^+ \leq \kappa < \text{Chadd}_{<\omega}(F) \), 3E tells us that there is an internal function \( h : X^2 \rightarrow X \) such that \( h(d_\alpha, d_\beta) = x_{G(\alpha, \beta)} \) for all \( \alpha, \beta < \lambda^+ \). Now 3C tells us that for every \( \xi < \kappa \) there is an internal set \( \Delta_\xi \supseteq \{d_\alpha : \alpha < \min(\lambda^+, \xi + 1)\} \) such that \( h(d, d') \leq x_\xi \) for all \( d, d' \in \Delta_\xi \).

(c) Let \( Q \) be the family of internal subsets \( q \) of \( X^3 \) such that 
\[ h(d(d'), d'(d'')) \leq y \text{ whenever } (z, y, d), (z', y', d'), (z'', y'', d'') \in q \text{ and } z < z' \].

Then \( Q \), partially ordered by inclusion, belongs to \( \text{Po}_{<\omega}(F) \). We can suppose that \( (X, \leq) = \prod_{\eta \in I}(X_\eta, \leq_\eta) \) if \( (X_\eta, \leq_\eta) \) is a finite totally ordered set for every \( \eta \in I \). Because \( h \) is an internal function, we have a family \( \langle h_\eta \rangle_{\eta \in I} \) such that \( h_\eta : X_\eta^2 \rightarrow X_\xi \) is a function for each \( \eta \in I \) and \( h \) can be regarded as \( \langle h_\eta \rangle_{\eta \in I} \). If we set
\[ Q_\eta = \{q : q \subseteq X_\eta^2, h(d(d'), d'(d'')) \leq y \text{ whenever } (z, y, d), (z', y', d'), (z'', y'', d'') \in q \text{ and } z < z' \}, \]
then we can identify \( \langle Q, \subseteq \rangle \) with \( \prod_{\eta \in I}(Q_\eta, \subseteq) \) \( F \).

Accordingly chad \( Q > \kappa \).

(d) There is a non-decreasing family \( \langle q_\xi \rangle_{\xi < \kappa} \) in \( Q \) such that, for each \( \xi < \kappa \),
\[ \begin{align*}
&\text{if } \beta < \min(\lambda^+, \xi + 1) \text{ and } d_\beta \leq z \leq d_\xi \text{ then there is a } y \text{ such that } (z, y, d_\beta) \in q_\xi, \\
&\text{if } (z, y, d) \in q_\xi \text{ then } z \leq d_\xi \text{ and } x_\xi \leq y.
\end{align*} \]

Let \( \langle d_\xi, x_\xi, d_0 \rangle \in q_\eta \).

**Proof.** Start the induction with \( q_0 = \{\langle d_0, x_0', d_0 \rangle\} \). Given \( \langle q_\eta \rangle_{0 < \xi} \) where \( 0 < \kappa < \xi \), take an upper bound \( q \) of \( \{q_\eta : \eta < \xi \} \) in \( Q \). For \( \alpha < \min(\lambda^+, \xi) \), the set
\[ \{ (z, y, e) : e, y, z \in X, z < d_\alpha \text{ or } e < z \text{ or } (z, y, d_\alpha) \in Q \} \]
is an internal subset of \( X^3 \), so
\[ E_\alpha = \{e : e \in X \text{ for every } z \in [d_\alpha, e] \text{ there is a } y \text{ such that } (z, y, d_\alpha) \in q\} \]
is an internal subset of \( X \) (use 2Fb); since there is a \( y \) such that \( (d_\alpha, y, d_\alpha) \in q_\eta \subseteq q_\eta \subseteq E_\alpha \); by 2Hb(2), \( E_\alpha \) has a greatest element \( e_\alpha \), say.

Because \( q_\eta \subseteq Q \) and \( q_\eta \subseteq q \) for \( \alpha < \eta < \xi, d_\eta \leq e_\alpha \) whenever \( \eta < \xi \) and \( \alpha < \min(\lambda^+, \xi) \). Now \( \xi < \kappa = \text{interp} \, X \) so there is a \( e \in X \) such that \( d_\eta \leq e < e_\alpha \) for every \( \eta < \xi \) and \( \alpha < \min(\lambda^+, \xi) \); replacing \( e \) by \( \min(e, d_\xi) \) if necessary, we can suppose that \( e \leq d_\xi \). Set
\[ q_\xi = \{(z, \max(y, x_\xi), d) : (z, y, d) \in q, z < e \} \cup \{(z, x_\xi, d) : e \leq z \leq d_\xi, d \in d_\xi \}. \]

This continues the induction.

(e) At the end of the induction take an upper bound \( q \) of \( \{q_\xi : \xi < \kappa \} \) in \( Q \). For \( \alpha < \lambda^+ \) take \( e_\alpha \) maximal subject to
\[ \text{for every } z \in [d_\alpha, e_\alpha] \text{ there is a } y \text{ such that } (z, y, d_\alpha) \in q. \]

\( ^1 \) Alternatively, check that
\[ R = \{(z, y, d, z', y', d') : z \leq z' = z'' = y < h(d', d'') \} \]
is an internal subset of \( X^3 \); note that \( Q = \{q : q \cap R = \emptyset\} \) and that \( q \rightarrow q^3 \cap R \) is an internal function.

**Measure Theory**
As in the inductive step in (d) above, \( e_\alpha \geq d_\xi \) for every \( \xi < \kappa \). So if \( y_\alpha = \min\{y : (z, y, d) \in q, \ z \leq e_\alpha\} \), \( y_\alpha \leq x_\xi \) for every \( \xi < \kappa \) and there is a \( \theta(\alpha) < \lambda \) such that \( y_\alpha \leq x_{\theta(\alpha)} \). Let \( \eta < \lambda \) be such that \( A = \{\alpha : \theta(\alpha) \leq \eta\} \) has cardinal \( \lambda^+ \); let \( \alpha \in A \) be such that \( \#(A \cap \alpha) = \lambda \); then there must be a \( \beta \in A \cap \alpha \) such that \( G(\alpha, \beta) > \eta \). Set \( e = \min(e_\alpha, e_\beta) \); then there are \( y', y'' \) such that \( (e, y', d_\alpha) \) and \( (e, y'', d_\beta) \) belong to \( q \). We therefore have

\[
h(d_\alpha, d_\beta) \leq \min\{y : (z, y, d) \in q, \ z \leq e\} = \max(y_\alpha, y_\beta) \\
\leq \max(x_{\theta(\alpha)}, x_{\theta(\beta)}) \leq x_\eta < x_{G(\alpha, \beta)} = h(d_\alpha, d_\beta)
\]

which is impossible. \( \Box \)

(f) This contradiction shows that \( \text{Chad}_{<\omega}(F) \) is indeed less than or equal to \( \text{Interp}_{<\omega}(F) \).

4 A forcing notion

Let \( \mathbb{P} \) be the forcing notion \( ([\mathbb{N}]^\omega, \subseteq^*, \mathbb{N}, \downarrow) \), where \( A \subseteq B \) if \( A \setminus B \) is finite.

4A Proposition \( t \) is the largest cardinal such that \( \mathbb{P} \) is \( t \)-closed in the sense of Kunen 80, 6.12.

proof Immediate from the definition.

4B Proposition (a) \( \mathbb{P} \) preserves cofinalities and cardinals up to and including \( t \).

(b) \( \Vert \mathbb{P} \mathbb{N} = (\mathbb{P}\mathbb{N})^* \).

(c)(i) \( \Vert \mathbb{P} t = t \).

(ii) \( \Vert \mathbb{P} p = p \).

proof (a) Kunen 80, 6.15.

(b) We just need to know that \( \mathbb{P} \) is countably closed.

(c)(i)(a) Let \( \langle a_\xi \rangle_{\xi < \kappa} \) be a \( \subseteq^*- \)decreasing family in \( [\mathbb{N}]^\omega \) with no \( \subseteq^* \)-lower bound in \( [\mathbb{N}]^\omega \). Then \( \Vert \mathbb{P} \langle a_\xi \rangle_{\xi < 1} \) is a \( \subseteq^* \)-decreasing family in \( [\mathbb{N}]^\omega \) and as \( \Vert \mathbb{P} \mathbb{P}\mathbb{N} = (\mathbb{P}\mathbb{N})^* \), \( \Vert \mathbb{P} \{a_\xi : \xi < t\} \) has no \( \subseteq^* \)-lower bound in \( [\mathbb{N}]^{<\omega} \), so \( t \leq t \).

(\( \beta \)) Suppose that \( \kappa < t \), \( p \in \mathbb{P} \) and \( \langle a_\xi \rangle_{\xi < \kappa} \) is a family of \( \mathbb{P} \)-names such that \( p \Vert \mathbb{P} \langle a_\xi \rangle_{\xi < \kappa} \) is a \( \subseteq^* \)-decreasing family in \( [\mathbb{N}]^\omega \). Because \( \mathbb{P} \) is \( t \)-closed, there are a \( q \) stronger than \( p \) and a family \( \langle a_\xi \rangle_{\xi < \kappa} \) in \( \mathbb{P}\mathbb{N} \) such that \( q \Vert \mathbb{P} a_\xi = a_\xi \) for every \( \xi < \kappa \). Now \( q \Vert \mathbb{P} \langle a_\xi \rangle_{\xi < \kappa} \) is a \( \subseteq^* \)-decreasing family in \( [\mathbb{N}]^\omega \); so in fact \( \langle a_\xi \rangle_{\xi < \kappa} \) is a \( \subseteq^* \)-decreasing family in \( [\mathbb{N}]^\omega \); as \( \kappa < t \), there is a \( \subseteq^* \)-lower bound \( a \) of \( \{a_\xi : \xi < \kappa\} \) in \( [\mathbb{N}]^\omega \), and now \( \Vert \mathbb{P} a \) is a \( \subseteq^* \)-lower bound of \( \{a_\xi : \xi < \kappa\} \) in \( [\mathbb{N}]^\omega \), so \( \langle a_\xi \rangle_{\xi < \kappa} \) is arbitrary, \( \Vert \mathbb{P} \kappa < t \).

As \( \kappa \) is arbitrary, \( \Vert \mathbb{P} t \leq t \) so \( \mathbb{P} \) is \( p \)-closed.

4C Proposition Let \( \hat{G} \) be the \( \mathbb{P} \)-name \( \{\hat{A}, A : A \in [\mathbb{N}]^\omega\} \). Then \( \Vert \mathbb{P} \hat{G} \) is a non-principal ultrafilter on \( \mathbb{N} \).

proof It is easy to see that

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Now if $A \in [\mathbb{N}]^\omega$ and $\dot{C}$ is a $\mathbb{P}$-name such that $\mathbf{P}\models \dot{C} \in [\mathbb{N}]^\omega$, there are a $C \subseteq \mathbb{N}$ and an infinite $A' \subseteq^* A$ such that $A' \Vdash \dot{C} = \bar{C}$ (4Bb); now if $A' \cap C$ is infinite, $A' \cap C \Vdash \dot{C} \in \dot{G}$; otherwise, $A' \setminus C$ is infinite and $A' \setminus C \Vdash \mathbb{N} \setminus \bar{C} \in \dot{G}$. So $\mathbf{P}\models \dot{G}$ is an ultrafilter. Finally, if $n \in \mathbb{N}$,

$$\mathbf{P}\models \bar{N} \subseteq^* \mathbb{N} \setminus \{n\} \models \mathbb{N} \setminus \{n\} \in \dot{G}$$

and $\mathbf{P}\models \dot{G}$ is non-principal.

4D Proposition $\mathbf{P}\models t \leq \text{Chadd}_{\omega}(\dot{G})$.

**proof** Let $\dot{P}$, $\dot{R}$ be $\mathbb{P}$-names such that

$$\mathbf{P}\models \dot{R} \subseteq \dot{P} \in \text{Po}_{\omega}(\dot{G})$$

$\dot{R}$ is well-ordered, $\text{otp}(\dot{R}) < t$.

We can suppose that

$$\mathbf{P}\models \text{there is a sequence } \langle (P_n, \leq_n) \rangle_{n \in \mathbb{N}} \text{ of non-empty finite partially ordered sets such that } \dot{P} = \prod_{n \in \mathbb{N}} P_n[\dot{G}]$$

Take any $A \in [\mathbb{N}]^\omega$. By 4B(c-i), $\mathbf{P}\models \text{otp}(\dot{R}) < t$ and there are a $B \in [A]^\omega$ and an ordinal $\alpha < t$ such that $B \Vdash \text{otp}(\dot{R}) = \alpha$. Let $\langle \dot{p}_\alpha \rangle_{\alpha < \omega}$ be a family of $\mathbb{P}$-names such that

$$B \Vdash \langle \dot{p}_\alpha \rangle_{\alpha < \omega} \text{ is the increasing enumeration of } \dot{R}.$$

Next, we have families $\langle (\dot{P}_n, \dot{\leq}_n) \rangle_{n \in \mathbb{N}}$ and $\langle \dot{p}_\xi \rangle_{\xi < \alpha, n \in \mathbb{N}}$ of $\mathbb{P}$-names such that

$$B \Vdash (\dot{P}_n, \dot{\leq}_n) \text{ is a non-empty finite partially ordered set, } \dot{P} = \prod_{n \in \mathbb{N}} P_n[\dot{G}] \text{, and } \dot{p}_\xi = \langle \dot{p}_\xi \rangle_{n \in \mathbb{N}}$$

for every $\xi < \alpha$. Because $\mathbb{P}$ is t-closed, there are an infinite $C \subseteq B$ and families $\langle (P_n, \leq_n) \rangle_{n \in \mathbb{N}}$ and $\langle \dot{p}_\xi \rangle_{\xi < \alpha, n \in \mathbb{N}}$ such that

$$(P_n, \leq_n) \text{ is a non-empty finite partially ordered set and } p_\xi \in P_n,$$

for every $n \in \mathbb{N}$ and $\xi < \alpha$.

For $\xi < \alpha$, set

$$E_\xi = \{ (n, p) : n \in C, p \in P_n, p_\xi \leq p \}.$$ 

If $\xi \leq \eta < \alpha$, then $E_\eta \setminus E_\xi$ is finite. **P?** Otherwise, set $D = \{ n : \exists p \ (n, p) \in E_\eta \setminus E_\xi \}$; because every $P_n$ is finite, $D$ is an infinite subset of $C$. If $n \in D$ there is a $p \in P_n$ such that $p_\eta \leq p$ but $p_\xi \not\leq p$, so that $p_\xi \not\leq p_\eta$; now

$$D \Vdash \dot{p}_\xi \dot{\not\leq} \dot{p}_n \text{ for every } n \in D,$$

so we have

$$D \Vdash \dot{D} \notin \dot{G} \text{ and } \dot{p}_\xi \dot{\not\leq} \dot{p}_n \text{ for every } n \in D$$

and

$$D \Vdash \langle \dot{p}_\xi \rangle_{n \in \mathbb{N}} \not\leq \langle \dot{p}_n \rangle_{n \in \mathbb{N}}.$$

contrary to the choice of $\langle \dot{p}_\xi \rangle_{\xi < \alpha, n \in \mathbb{N}}$.

Since every $E_\xi$ is a subset of the countable set $\{ (n, p) : n \in C, p \in P_n \}$, and if $\alpha < t$, there is an infinite $E \subseteq \{ (n, p) : n \in C, p \in P_n \}$ such that $E \setminus E_\xi$ is finite for every $\xi < \alpha$. Now set $D = \{ n : \exists p \ (n, p) \in E \}$, so that $D$ is infinite, and for each $n \in D$ take $q_n \in P_n$ such that $(n, q_n) \in E$; for other $n \in \mathbb{N}$ take any $q_n \in P_n$. In this case, for any $\xi < \alpha$,

$$\{ n : n \in D, p_\xi \not\leq q_n \} \subseteq \{ n : n \in D, (n, q_n) \not\in E_\xi \}$$

is finite, so

$$D \supseteq^* \{ n : n \in D, (n, q_n) \in E_\xi \} \subseteq \{ n : n \in \mathbb{N}, p_\xi \leq q_n \}.$$

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But now observe that, writing \(D_\xi\) for \(\{n : n \in D, p_{\xi n} \leq q_n\}\),
\[
D \subseteq^* D_\xi \models \hat{D}_\xi \in \mathcal{G}
\] and \(p_{\xi n} \leq \hat{q}_n\) for every \(n \in D_\xi\), so that \(\langle p_{\xi n}\rangle_{n \in \mathbb{N}} \leq \langle \hat{q}_n\rangle_{n \in \mathbb{N}}\) in \(\mathcal{P}\).

As \(\xi\) is arbitrary,
\[
D \models \langle \hat{q}_n \rangle_{n \in \mathbb{N}} \text{ is an upper bound for } \hat{R}, \text{ and } \hat{R} \text{ is bounded above in } \mathcal{P}.
\]
As \(A\) is arbitrary,
\[
\models \bar{\varphi} \hat{R} \text{ is bounded above.}
\]
As \(\hat{R}\) is arbitrary,
\[
\models \bar{\varphi} \varphi \leq \mathrm{chadd} \mathcal{P}.
\]
As \(\mathcal{P}\) is arbitrary,
\[
\models \bar{\varphi} \varphi \leq \mathrm{Chadd}_{<\omega} (\mathcal{G}),
\]
as claimed.

**4E Lemma** Let \(\preceq\) be the partial ordering on \(\mathbb{N}^\omega\) defined by saying that \(f \preceq g\) if either \(f = g\) or \(\{n : g(n) \leq f(n)\}\) is finite. If \(\kappa \leq p\) is an infinite cardinal and there is a peculiar \((\kappa, p^* )\)-gap in \((\mathbb{N}^\omega, \preceq )\), then \(p \equiv t\).

**Proof (a)** Let \(\langle (f_\xi)_{\xi < \kappa}, (g_\eta)_{\eta < p}\rangle\) be such a gap; we can suppose that \(f_\xi, g_\eta \leq g_0\) for every \(\xi < \kappa\) and \(\eta < p\). Let \(\mathcal{P}\) be a \(\mathcal{P}\)-name such that
\[
\models \bar{\varphi} \mathcal{P} = \prod_{n \in \mathbb{N}} (g_0(n) + 1) \mathcal{G} \in \mathrm{Lo}_{<\omega} (\mathcal{G})
\]
Then for each \(\xi < \kappa\), \(\eta < p\) we have \(\mathcal{P}\)-names \(\hat{p}_\xi, \hat{q}_\eta\) such that
\[
\models \bar{\varphi} \hat{p}_\xi \in \mathcal{P}, \hat{q}_\eta = \hat{q}_p^* \in \mathcal{P}.
\]
Now from 4Ba and 4B(c-i) we have
\[
\models \bar{\varphi} \hat{p}_\xi \prec \hat{p}_{\xi'} \prec \hat{q}_{\eta'} \prec \hat{q}_\eta \text{ whenever } \xi < \xi' < \kappa \text{ and } \eta < \eta' < p;
\]
since we also know that \(\models \bar{\varphi} \mathcal{G}\) is a free filter, we have
\[
\models \bar{\varphi} \hat{p}_\xi \prec \hat{p}_{\xi'} \prec \hat{q}_{\eta'} \prec \hat{q}_\eta \text{ whenever } \xi < \xi' < \kappa \text{ and } \eta < \eta' < p.
\]

(b) If
\[
\models \bar{\varphi} \langle (\hat{p}_\xi)_{\xi < \kappa}, (\hat{q}_\eta)_{\eta < p} \rangle\text{ is a } (\kappa, p^*)\text{-gap in }\mathcal{P},
\]
there are an \(A \in [\mathbb{N}]^\omega\) and a \(\mathcal{P}\)-name \(\hat{h}\) such that
\[
A \models \hat{h} \in \prod_{n \in \mathbb{N}} (g_0(n) + 1) \text{ and } \hat{p}_\xi \leq \hat{h}^* \leq \hat{q}_\eta \text{ for every } \xi < \kappa \text{ and } \eta < p.
\]
Because \(\mathcal{P}\) is countably closed, there are an infinite \(B \subseteq A\) and an \(h \in \mathbb{N}^\omega\) such that
\[
B \models \hat{h} = \hat{h}.
\]
Next, for each \(\xi < \kappa\), we have
\[
B \models \hat{f}^*_\xi \prec \hat{h}^*,
\]
that is,
\[
B \models \{n : f_\xi(n) \leq h(n)\} \in \mathcal{G},
\]
that is,
\[
B \models \{n : f_\xi(n) < h(n)\}^+ \in \mathcal{G},
\]
that is,
\[
B \subseteq^* \{n : f_\xi(n) < h(n)\}.
\]
But this means that if we set
\[ h'(n) = h(n) \text{ for } n \in B, \]
\[ = g_0(n) \text{ for } n \in \mathbb{N} \setminus B, \]

we shall have \( f_\xi \prec h' \); and this is true for every \( \xi < \kappa \). Because \((\langle f_\xi \rangle_{\xi < \kappa}, \langle g_\eta \rangle_{\eta < p})\) is a peculiar gap, there is an \( \eta < p \) such that \( g_\eta \preceq h' \), in which case \( B \subseteq \{ n : g_{\eta+1}(n) < h(n) \} \); running the argument above backwards, we see that
\[ B \models \dot{q}_{\eta+1} < \dot{h}', \]
contrary to the choice of \( A \) and \( \dot{h} \).

\textbf{(c) We conclude that}
\[ \| P (\langle \dot{p}_\xi \rangle_{\xi < \kappa}, \langle \dot{q}_\eta \rangle_{\eta < p}) \| \text{ is a } (\check{\kappa}, \check{p}^*)\text{-gap in } \dot{P}, \]
so that \( p \geq \text{interp } \dot{P} \).

But now 3F and 4D, together with the Forcing Theorem (KUNEN 80, VII.4.2), tell us that
\[ \| P t \leq \text{Chadd}_{<\omega}(\dot{G}) \leq \text{Interp}_{<\omega}(\dot{G}) \leq \text{interp } \dot{P} \leq p. \]

Accordingly, by 4Bc,
\[ \| P \check{t} \leq \check{p} \]
and \( t \leq p \), so in fact \( t = p \).

\textbf{4F Theorem} (MALLIARIS & SHELAH 16) \( p = t \).

\textbf{proof} Otherwise, there are an uncountable regular \( \kappa < p \) and a \((\kappa, p^*)\)-gap in \((\mathbb{N}, \preceq)\), by SHELAH 09, 1.12 or FREMLIN 14, 2H (see parts (c)-(g) of the proof). And 4E tells us that this can happen only if \( p = t \).

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\textbf{References}


