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$\mathfrak{p} = \mathfrak{t}$, following Malliaris-Shelah and Steprāns

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I attempt a proof, based on that sketched in Steprāns n13, of the theorem in Malliaris & Shelah 16 that $\mathfrak{p} = \mathfrak{t}$.

1 Gaps, interpolation and chain-additivity

1A Definitions Let P be a partially ordered set and λ, κ non-zero cardinals.

(a) A **(λ, κ^*)-gap** in P is a pair $(\langle x_\xi \rangle_{\xi<\lambda}, \langle y_\xi \rangle_{\eta<\kappa})$ of families in P such that
$x_\xi < x_{\xi'} \le y_{\eta'} < y_\eta$ whenever $\xi < \xi' < \lambda$ and $\eta < \eta' < \kappa$,
there is no $z \in P$ such that $x_\xi \le z \le y_\eta$ whenever $\xi < \lambda$ and $\eta < \kappa$.

(a) A **peculiar** (λ, κ^*)**-gap** in P is a pair $(\langle x_\xi \rangle_{\xi<\lambda}, \langle y_\xi \rangle_{\eta<\kappa})$ of families in P such that
$x_\xi < x_{\xi'} \le y_{\eta'} < y_\eta$ whenever $\xi < \xi' < \lambda$ and $\eta < \eta' < \kappa$,
whenever $z \in P$ is such that $z \le y_\eta$ for every $\eta < \kappa$, there is a $\xi < \lambda$ such that $z \le x_\xi$,
whenever $z \in P$ is such that $x_\xi \le z$ for every $\xi < \lambda$, there is an $\eta < \kappa$ such that $y_\eta \le z$.

1B Definitions Let $(P, \le)$ be a partially ordered set.

(a) The **chain-additivity** of P, $\operatorname{chadd} P$, is the least cardinal of any totally ordered subset of P with no upper bound in P; or ∞ if there is no such set.

Note that $\operatorname{chadd} P$ is either 0 (if P is empty) or ∞ (if every maximal chain in P has a greatest member) or a regular infinite cardinal κ, and in the last case there is a strictly increasing family $\langle p_\xi \rangle_{\xi<\kappa}$ in P with no upper bound in P.

If P is upwards-directed then $\operatorname{chadd} P = \operatorname{add} P$ as defined in Fremlin 08, 511Bb.

(b)(i) If κ is a cardinal, say that P has the $< \kappa$**-interpolation property** if whenever A, $B \subseteq P$ are non-empty, $a \le b$ for every $a \in A$ and $b \in B$, and $\max(\#(A), \#(B)) < \kappa$, then there is a $c \in P$ such that $a \le c \le b$ whenever $a \in A$ and $b \in B$.

(ii) The **interpolation number** of P, $\operatorname{interp} P$, is the greatest cardinal κ such that P has the $< \kappa$-interpolation property, or ∞ if there is no such κ. (For this use of '∞', see Fremlin 08, 511C.)

Note that $\operatorname{interp} P = \infty$ iff P is Dedekind complete, and that $\operatorname{interp} P \ge \omega$ if P is a lattice.

1C Lemma Suppose that P is a lattice. Write $\operatorname{chgap} P$ for the least cardinal κ such that there is a (λ_0, λ_1^*)-gap in P with cardinals λ_0, $\lambda_1 \le \kappa$, or ∞ if there is no such κ. Then $\operatorname{interp} P = \operatorname{chgap} P$.

proof (a) Suppose that $(\langle x_\xi \rangle_{\xi<\lambda_0}, \langle y_\xi \rangle_{\eta<\lambda_1})$ is a (λ_0, λ_1^*)-gap. Then $\{x_\xi : \xi < \lambda_0\}$, $\{y_\eta : \eta < \lambda_1\}$ witness that $\operatorname{interp} P \le \max(\lambda_0, \lambda_1)$. As $(\langle x_\xi \rangle_{\xi<\lambda_0}, \langle y_\xi \rangle_{\eta<\lambda_1})$ is arbitrary, $\operatorname{interp} P \le \operatorname{chgap} P$.

(b) Suppose that A, $B \subseteq P$ are non-empty sets with cardinal less than $\operatorname{chgap} P$ and $a \le b$ for every $a \in A$ and $b \in B$.

(i) If A is well-ordered and B is downwards well-ordered (that is, $(B, \ge)$ is well-ordered), then there is a $c \in P$ such that $a \le c \le b$ for every $a \in A$ and $b \in B$. **P** Set $\lambda_0 = \operatorname{cf} A$, $\lambda_1 = \operatorname{ci} B$, and let $\langle x_\xi \rangle_{\xi<\lambda_0}$ be the increasing enumeration of a cofinal subset of A with cardinal λ_0, and $\langle y_\eta \rangle_{\eta<\lambda_1}$ the decreasing enumeration of a coinitial subset of B with cardinal λ_1; then $\lambda_0 \le \#(A) < \operatorname{chgap} P$ and $\lambda_1 \le \#(B) < \operatorname{chgap} P$, so $(\langle x_\xi \rangle_{\xi<\lambda_0}, \langle y_\eta \rangle_{\eta<\lambda_1})$ cannot be a (λ_0, λ_1^*)-gap and there must be a $c \in P$ such that $x_\xi \le c \le y_\eta$ for all ξ and η, so that $a \le c \le b$ whenever $a \in A$ and $b \in B$. **Q**

(ii) If A is well-ordered then there is a $c \in P$ such that $a \le c \le b$ for every $a \in A$ and $b \in B$. **P** If B is finite, this is trivial, as $\inf B$ is defined in P. So suppose that B is infinite. Set $\lambda_0 = \operatorname{cf} A$ and let $\langle x_\xi \rangle_{\xi<\lambda_0}$ be the increasing enumeration of a cofinal subset of A with cardinal λ_0; of course $\lambda_0 < \operatorname{chgap} P$. Set

$$A' = \{y : a \le y \text{ for every } a \in A\} = \{y : x_\xi \le y \text{ for every } \xi < \lambda_0\} \supseteq B.$$

Enumerate B as $\langle b_\eta\rangle_{\eta<\lambda_1}$, where $\lambda_1 < \operatorname{chgap} P$, and choose a non-increasing family $\langle y_\eta\rangle_{\eta\le\lambda_1}$ in A' inductively, as follows. Start with $y_0 = b_0$. Given y_η, where $\eta < \lambda_1$, set $y_{\eta+1} = y_\eta \wedge b_\eta$. Given a non-zero limit ordinal $\beta \le \lambda_1$ and a non-increasing family $\langle y_\eta\rangle_{\eta<\beta}$ in A', the set $\{y_\eta : \eta < \beta\}$ is downwards well-ordered and has cardinal at most $\#(\beta) \le \lambda_1 < \operatorname{chgap} P$, so by (ii) there is a $y_\beta \in P$ such that $a_\xi \le y_\beta \le y_\eta$ for every $\xi < \lambda_0$ and $\eta < \beta$, and the induction continues.

At the end of the induction, consider $c = y_{\lambda_1}$. Then $c \in A'$; since also $c \le y_{\eta+1} \le b_\eta$ for every $\eta < \lambda_1$, c serves. **Q**

(iii) In any case, there is a $c \in P$ such that $a \le c \le b$ for every $a \in A$ and $b \in B$. **P** If A is finite, take $c = \sup A$. Otherwise, enumerate A as $\langle a_\xi\rangle_{\xi<\lambda_0}$ and choose $\langle x_\xi\rangle_{\xi\le\lambda_0}$ inductively, as follows. Start with $x_0 = a_0$. If $\xi < \lambda_0$, set $x_{\xi+1} = x_\xi \vee a_\xi$. If $\alpha \le \lambda_0$ is a non-zero limit ordinal, (ii) tells us that there is an $x_\alpha \in P$ such that $x_\xi \le x_\alpha \le b$ whenever $\xi < \alpha$ and $b \in B$. At the end of the induction, take $c = x_{\lambda_0}$.

(iv) As A and B are arbitrary, $\operatorname{chgap} P \le \operatorname{interp} P$ and the two are equal.

1D Lemma Let P be a lattice with the $<\omega_1$-interpolation property which is not Dedekind σ-complete. Then $\operatorname{interp} P \le \#(P)$.

proof (a) Suppose that there is a countable subset A of P with an upper bound but no least upper bound. Then A must be infinite; let $\langle p_n\rangle_{n\in\mathbb{N}}$ be an enumeration of A, and set $p'_n = \sup_{i\le n} p_i$, so that $\langle p'_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence which is not eventually constant, and has a strictly increasing subsequence $\langle p''_n\rangle_{n\in\mathbb{N}}$. Let B the set of upper bounds of A. Because B has no infimum, it is surely infinite; set $\kappa = \#(B)$ and enumerate B as $\langle q_\eta\rangle_{\eta<\kappa}$. Choose $\langle q'_\eta\rangle_{\eta<\beta}$ in B as follows. Start with $q'_0 = q_0$. Given $q'_\eta \in B$, q'_η is not the least member of B, so there is a first $\zeta_\eta < \kappa$ such that $q'_\eta \not\le q_{\zeta_\eta}$; set $q'_{\eta+1} = q'_\eta \wedge q_{\zeta_\eta}$. Given $\langle q'_{\eta'}\rangle_{\eta'<\eta}$ where $\eta < \kappa$ is a non-zero limit ordinal, then if there is a member of B less than or equal to $q'_{\eta'}$ for every $\eta' < \eta$, take such a member for q'_η; otherwise set $\beta = \eta$ and stop. If the induction continues to the end, then there cannot be a member of B less than or equal to $q'_{\eta+1} \le q_\eta$ for every $\eta < \kappa$, so set $\beta = \kappa$.

Thus we have a strictly decreasing family $\{q'_\eta : \eta < \beta\}$ in B with no lower bound in B, where $\omega \le \beta \le \kappa \le \#(P)$. Set $\lambda = \operatorname{cf}\beta$ and let $\langle\eta_\theta\rangle_{\lambda<\theta}$ be the increasing enumeration of a cofinal subset of β. Then $(\langle p''_n\rangle_{n\in\mathbb{N}}, \langle q'_{\eta_\theta}\rangle_{\theta<\lambda})$ is an (ω, λ^*)-gap in P. As P has the $<\omega_1$-interpolation property, $\lambda > \omega$, so

$$\operatorname{interp} P \le \lambda \le \beta \le \kappa \le \#(P).$$

(b) If there is a countable subset of P with a lower bound but no greatest lower bound, argue similarly, or apply (a) to $(P, \ge)$.

1E Definitions (a) Write $\mathfrak{p}$ for the least cardinal of any downwards-directed set $A \subseteq [\mathbb{N}]^\omega$ for which there is no $b \in [\mathbb{N}]^\omega$ such that $b \setminus a$ is finite for every $a \in A$.

(b) Write $\mathfrak{t}$ for the least cardinal κ for which there is a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $[\mathbb{N}]^\omega$ such that $a_\eta \setminus a_\xi$ is finite whenever $\xi < \eta < \kappa$, but there is no $a \in [\mathbb{N}]^\omega$ such that $a \setminus a_\xi$ is finite for every $\xi < \kappa$.

2 Reduced products and internal sets

Most of the rest of the arguments in this note will be based on a fragment of the model theory of ultrapowers. For the next few sections, fix an ultrafilter $\mathcal{F}$ on a set I.

2A Suppose that X_i is a non-empty set for each $i \in I$,

(a) We have an equivalence relation on $\prod_{i\in I} X_i$ given by saying that $\langle x_i\rangle_{i\in I} \sim \langle y_i\rangle_{i\in I}$ if $\{i : x_i = y_i\}$ belongs to $\mathcal{F}$. I will write $\langle x_i\rangle^\bullet_{i\in I}$ for the equivalence class of $\langle x_i\rangle_{i\in I}$. The set of equivalence classes is the **reduced product** of $\langle X_i\rangle_{i\in I}$ mod $\mathcal{F}$, which I will denote $\prod_{i\in I} X_i|\mathcal{F}$. (See Fremlin 08, 5A2A.)

(b) A subset $\boldsymbol{Z}$ of $\boldsymbol{X}$ is **internal** if it corresponds to a member of $\prod_{i\in I} \mathcal{P}X_i|\mathcal{F}$, that is, if there is a family $\langle Z_i\rangle_{i\in I}$ such that $Z_i \subseteq X_i$ for every $i \in I$ and $\boldsymbol{Z} = \{\langle x_i\rangle^\bullet_{i\in I} : \{i : x_i \in Z_i\} \in \mathcal{F}$; note that if every Z_i is non-empty this is in a natural one-to-one correspondence with $\prod_{i\in I} Z_i|\mathcal{F}$.

(c) Because $\mathcal{F}$ is an ultrafilter, the family of internal subsets of $\boldsymbol{X}$ is an algebra of sets containing all singleton sets, therefore every finite subset of $\boldsymbol{X}$.

(d) If $\boldsymbol{Z}$ is a non-empty internal subset of $\boldsymbol{X}$, then a subset of $\boldsymbol{Z}$ is internal in $\boldsymbol{Z}$ iff it is internal in $\boldsymbol{X}$. **P Q**

(e) Generally, when I use an italic bold upper-case letter like $\boldsymbol{X}$ or $\boldsymbol{P}$, you should take it that I am thinking of a set together with an associated structure of internal sets.

2B Let $\boldsymbol{X} = \prod_{i\in I} X_i|\mathcal{F}$ and $\boldsymbol{Y} = \prod_{i\in I} Y_i|\mathcal{F}$ be two reduced products mod $\mathcal{F}$. Then we have a natural bijection between $\boldsymbol{X} \times \boldsymbol{Y}$ and $\prod_{i\in I} X_i \times Y_i|\mathcal{F}$, identifying $(\langle x_i\rangle^\bullet_{i\in I}, \langle y_i\rangle^\bullet_{i\in I})$ with $\langle (x_i, y_i)\rangle^\bullet_{i\in I}$. This gives us an associated notion of 'internal' subset of $\boldsymbol{X} \times \boldsymbol{Y}$, being one corresponding to an internal subset of $\prod_{i\in I} X_i \times Y_i|\mathcal{F}$.

The same idea applies to products of any finite number of reduced products mod $\mathcal{F}$.

2C Again suppose that $\boldsymbol{X} = \prod_{i\in I} X_i|\mathcal{F}$ and $\boldsymbol{Y} = \prod_{i\in I} Y_i|\mathcal{F}$ are two reduced products mod $\mathcal{F}$.

(a) If $\boldsymbol{Z} \subseteq \boldsymbol{X}$ and $\boldsymbol{W} \subseteq \boldsymbol{Y}$ are internal, then $\boldsymbol{Z} \times \boldsymbol{W}$ is an internal subset of $\boldsymbol{X} \times \boldsymbol{Y}$.

(b) If $\boldsymbol{W}$ is an internal subset of $\boldsymbol{X} \times \boldsymbol{Y}$ and $\boldsymbol{Z}$ is an internal subset of $\boldsymbol{X}$, then $\boldsymbol{W}[\boldsymbol{Z}]$ is an internal subset of $\boldsymbol{Y}$. (For if $\boldsymbol{W}$ corresponds to $\langle W_i\rangle_{i\in I}$ and $\boldsymbol{Z}$ to $\langle Z_i\rangle_{i\in I}$, then $\boldsymbol{W}[\boldsymbol{Z}]$ corresponds to $\langle W_i[Z_i]\rangle_{i\in I}$.) In particular, any section $\boldsymbol{W}[\{\boldsymbol{x}\}]$, where $\boldsymbol{x} \in \boldsymbol{X}$, is an internal subset of $\boldsymbol{Y}$.

(c) If $W_i \subseteq X_i \times Y_i$ is the graph of a function for each i, then the corresponding internal relation $\boldsymbol{W} \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ will be the graph of a function, its domain being the internal subset of $\boldsymbol{X}$ corresponding to $\langle \operatorname{dom} W_i\rangle_{i\in I}$.

(d) If $X_i = Y_i$ and W_i is a partial order on X_i for each i, then $\boldsymbol{W}$ will be a partial order on $\boldsymbol{X}$. If $X_i = Y_i$ and W_i is a total order on X_i for each i, then $\boldsymbol{W}$ will be a total order on $\boldsymbol{X}$. If $X_i = Y_i$ and W_i is a well-ordering of X_i for each i, then every non-empty *internal* subset of $\boldsymbol{X}$ will have a $\boldsymbol{W}$-least member. (For if $\boldsymbol{Z} \subseteq \boldsymbol{X}$ corresponds to $\langle Z_i\rangle_{i\in I}$ and $\boldsymbol{x} = \langle x_i\rangle^\bullet_{i\in I} \in \boldsymbol{Z}$, define $\langle z_i\rangle_{i\in I}$ by saying that

$$\begin{aligned} z_i &\text{ is the } W_i\text{-least member of } Z_i \text{ if } Z_i \ne \emptyset, \\ &= x_i \text{ otherwise;} \end{aligned}$$

then $\langle z_i\rangle^\bullet_{i\in I}$ is the $\boldsymbol{W}$-least member of $\boldsymbol{Z}$.)

(e) Conversely, if $\boldsymbol{W}$ is an internal subset of $\boldsymbol{X} \times \boldsymbol{X}$ and is a partial order, then there is a family $\langle W_i\rangle_{i\in I}$ such that W_i is a partial order on X_i for each i and $\boldsymbol{W}$ corresponds to $\langle W_i\rangle_{i\in I}$. **P** By the definition of 'internal subset of $\boldsymbol{X} \times \boldsymbol{X}$' there is a family $\langle W'_i\rangle_{i\in I}$ such that $\boldsymbol{W}$ corresponds to $\langle W'_i\rangle_{i\in I}$. Set $\Delta_i = \{(x, x) : x \in X_i\}$ for $i \in I$. Now consider

$$J = \{i : W'_i \not\supseteq \Delta_i\}, \quad K = \{i : W'_i \circ W'_i \not\subseteq W'_i\} \quad L = \{i : W_i \cap W_i^{-1} \not\subseteq \Delta_i\}.$$

? If $J \in \mathcal{F}$, take $x_i \in X_i$ such that $(x_i, x_i) \notin W'_i$ for $i \in J$ and set $\boldsymbol{x} = \langle x_i\rangle^\bullet_{i\in I}$; then $(\boldsymbol{x}, \boldsymbol{x}) \notin \boldsymbol{W}$. **X**

? If $K \in \mathcal{F}$, take x_i, y_i, $z_i \in X_i$ such that, for $i \in K$, $(x_i, y_i) \in W'_i$, $(y_i, z_i) \in W'_i$ but $(x_i, z_i) \notin W'_i$; setting $\boldsymbol{x} = \langle x_i\rangle^\bullet_{i\in I}$, $\boldsymbol{y} = \langle y_i\rangle^\bullet_{i\in I}$ and $\boldsymbol{z} = \langle z_i\rangle^\bullet_{i\in I}$, $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W}$ and $(\boldsymbol{y}, \boldsymbol{z}) \in \boldsymbol{W}$ but $(\boldsymbol{x}, \boldsymbol{z}) \notin \boldsymbol{W}$. **X**

? If $L \in \mathcal{F}$, take x_i, $y_i \in X_i$ such that, for $i \in L$, $(x_i, y_i) \in W'_i$ and $(y_i, x_i) \in W'_i$ but $x_i \ne y_i$. Setting $\boldsymbol{x} = \langle x_i\rangle^\bullet_{i\in I}$ and $\boldsymbol{y} = \langle y_i\rangle^\bullet_{i\in I}$, $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W}$ and $(\boldsymbol{y}, \boldsymbol{x}) \in \boldsymbol{W}$ but $\boldsymbol{x} \ne \boldsymbol{y}$. **X**

Consequently, $M = I \setminus (J \cup K \cup L)$ belongs to $\mathcal{F}$, while W'_i is a partial order on X_i for every $i \in M$. Setting $W_i = W'_i$ for $i \in M$, $W_i = \Delta_i$ for $i \in J \cup K \cup L$, we have a suitable family. **Q**

(f) If $\boldsymbol{W}$ is an internal subset of $\boldsymbol{X} \times \boldsymbol{Y}$, then its projection $\{\boldsymbol{x} : \exists\, \boldsymbol{y}, (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W}\}$ is an internal subset of $\boldsymbol{X}$. **P** If $\boldsymbol{W}$ corresponds to $\langle W_i\rangle_{i\in I}$, consider $A_i = \{x : \exists\, y, (x, y) \in W_i\}$ for each $i \in I$. **Q** Hence, or otherwise, $\{\boldsymbol{x} : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{W} \text{ for every } \boldsymbol{y} \in \boldsymbol{Y}\}$ is an internal subset of $\boldsymbol{X}$.

2D Power sets (a) Once more, suppose that we have a family $\langle X_i\rangle_{i\in I}$ of non-empty sets and the reduced product $\boldsymbol{X} = \prod_{i\in I} X_i|\mathcal{F}$. Then we can form the reduced product $\prod_{i\in I} \mathcal{P}X_i|\mathcal{F}$.

(b) If $\langle Z_i\rangle_{i\in I}$ and $\langle Z'_i\rangle_{i\in I}$ belong to $\prod_{i\in I} \mathcal{P}X_i$, and we look at the corresponding internal sets $\boldsymbol{Z}$, $\boldsymbol{Z}'$ as defined in 2Ab, we find that $\boldsymbol{Z} = \boldsymbol{Z}'$ iff $\{i : Z_i = Z'_i\} \in \mathcal{F}$. **P** If $J = \{i : Z_i = Z'_i\}$ belongs to $\mathcal{F}$, then for any $\langle x_i\rangle_{i\in I} \in \prod_{i\in I} X_i$

$$\langle x_i\rangle^\bullet_{i\in I}\in\boldsymbol{Z}\iff\{i:i\in I,\,x_i\in Z_i\}\in\mathcal{F}\iff\{i:i\in J,\,x_i\in Z_i\}\in\mathcal{F}$$
$$\iff\{i:i\in J,\,x_i\in Z'_i\}\in\mathcal{F}\iff\langle x_i\rangle^\bullet_{i\in I}\in\boldsymbol{Z}'$$

and $\boldsymbol{Z}=\boldsymbol{Z}'$. If $K=\{i:Z_i\not\subseteq Z'_i\}$ belongs to $\mathcal{F}$, choose $z_i\in Z_i\setminus Z'_i$ for $i\in K$, and take $\langle z_i\rangle^\bullet_{i\in I}\in\boldsymbol{Z}\setminus\boldsymbol{Z}'$ and $\boldsymbol{Z}\ne\boldsymbol{Z}'$. Similarly, $\boldsymbol{Z}\ne\boldsymbol{Z}'$ if $\{i:Z'_i\not\subseteq Z_i\}\in\mathcal{F}$. So if $J\notin\mathcal{F}$ then $\boldsymbol{z}\ne\boldsymbol{Z}'$. **Q**

(c) Thus we have a natural bijection between the reduced product $\prod_{i\in I}\mathcal{P}X_i|\mathcal{F}$ and the algebra $\boldsymbol{\mathcal{P}X}$ of internal subsets of $\boldsymbol{X}$. Accordingly we have a notion of internal subset of $\boldsymbol{\mathcal{P}X}$, being one corresponding to a family $\langle\mathcal{A}_i\rangle_{i\in I}$ where $\mathcal{A}_i\subseteq X_i$ for each i, so that $\boldsymbol{\mathcal{P}\mathcal{P}X}$ can be identified with $\prod_{i\in I}\mathcal{P}\mathcal{P}X_i|\mathcal{F}$.

(d) In 2Ae I spoke of a 'structure of internal sets'; the vagueness was deliberate, as I intended to include not only the subalgebra $\boldsymbol{\mathcal{P}X}$ of $\mathcal{P}\boldsymbol{X}$ but the repeated sets-of-internal-sets algebras $\boldsymbol{\mathcal{P}\mathcal{P}X}$, $\boldsymbol{\mathcal{P}\mathcal{P}\mathcal{P}X}$ and so on. (For this note, happily, we do not have to go far along this road.)

2E Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of non-empty sets, and $\boldsymbol{X}=\prod_{i\in I}X_i|\mathcal{F}$ their reduced product. Then the relation $\subseteq$ on $\boldsymbol{\mathcal{P}X}$ is internal.

proof As in 2D, we can identify $\boldsymbol{\mathcal{P}X}$ with $\prod_{i\in I}\mathcal{P}X_i|\mathcal{F}$, so that we think of internal subsets of $\boldsymbol{X}$ as equivalence classes $\langle Z_i\rangle^\bullet_{i\in I}$ where $Z_i\subseteq X_i$ for $i\in I$. Now if we have two families $\langle W_i\rangle_{i\in I}$, $\langle Z_i\rangle_{i\in I}\in\prod_{i\in I}\mathcal{P}X_i$ representing internal sets $\boldsymbol{W}$, $\boldsymbol{Z}\subseteq\boldsymbol{X}$, we have $\boldsymbol{W}\subseteq\boldsymbol{Z}$ iff $J=\{i:W_i\subseteq Z_i\}$ belongs to $\mathcal{F}$. **P** This is a trifling refinement of 2Db. If $J\in\mathcal{F}$ and $\boldsymbol{x}=\langle x_i\rangle^\bullet_{i\in I}\in\boldsymbol{W}$, then $\{i:x_i\in Z_i\}\supseteq J\cap\{i:x_i\in W_i\}$ belongs to $\mathcal{F}$ and $\boldsymbol{x}\in\boldsymbol{Z}$. If $I\setminus J\in\mathcal{F}$, choose $x_i\in W_i\setminus Z_i$ for $i\in I\setminus J$, $x_i\in X_i$ for $i\in J$; then $\boldsymbol{x}=\langle x_i\rangle^\bullet_{i\in I}$ belongs to $\boldsymbol{W}\setminus\boldsymbol{Z}$ and $\boldsymbol{W}\not\subseteq\boldsymbol{Z}$. **Q**

Now this means that if we look at the internal subset of $\boldsymbol{\mathcal{P}X}\times\boldsymbol{\mathcal{P}X}$ corresponding to the family $\langle\subseteq_i\rangle_{i\in I}\in\prod_{i\in I}\mathcal{P}X_i\times\mathcal{P}X_i$, where $\subseteq_i=\{(W,Z):W\subseteq Z\subseteq X_i\}$ for each i, we find that it is precisely the relation $\subseteq$.

2F Definitions (a) Let $\mathrm{Ufm}_{<\omega}(\mathcal{F})$ be the class of structures isomorphic to structures $\prod_{i\in I}X_i|\mathcal{F}$, together with the corresponding algebras of internal sets, where every X_i is finite and not empty.

(b) Let $\mathrm{Po}_{<\omega}(\mathcal{F})$ be the class of non-empty partially ordered sets $(\boldsymbol{P},\boldsymbol{\le})$ where $\boldsymbol{P}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$ and $\boldsymbol{\le}$ is an internal relation on $\boldsymbol{P}$ which is a partial order. As noted in 2Ce, we must then be able to identify $(\boldsymbol{P},\boldsymbol{\le})$ with a structure $\prod_{i\in I}(P_i,\le_i)|\mathcal{F}$ where P_i is finite and $\le_i$ is a partial order on P_i for every i.

(c) Let $\mathrm{Lo}_{<\omega}(\mathcal{F})\subseteq\mathrm{Po}_{<\omega}(\mathcal{F})$ be the class of non-empty totally ordered sets belonging to $\mathrm{Po}_{<\omega}(\mathcal{F})$. If $(\boldsymbol{X},\boldsymbol{\le})\in\mathrm{Lo}_{<\omega}(\mathcal{F})$, we can identify it with a structure $\prod_{i\in I}(X_i,\le_i)|\mathcal{F}$ where X_i is finite and $\le_i$ is a total order on X_i for every i.

2G Proposition (a) If $\boldsymbol{X}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$ and $\boldsymbol{Z}$ is a non-empty internal subset of $\boldsymbol{X}$, then $\boldsymbol{Z}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$.

(b) If $\boldsymbol{X}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$ then $\boldsymbol{\mathcal{P}X}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$.

(c) If $\boldsymbol{X}$, $\boldsymbol{Y}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$ then $\boldsymbol{X}\times\boldsymbol{Y}\in\mathrm{Ufm}_{<\omega}(\mathcal{F})$.

proof (a) If $\boldsymbol{X}\cong\prod_{i\in I}X_i|\mathcal{F}$ where X_i is finite for every $i\in I$, then $\boldsymbol{Z}\cong\prod_{i\in I}Z_i|\mathcal{F}$ where $Z_i\subseteq X_i$ is finite for every $i\in I$.

(b) If $\boldsymbol{X}\cong\prod_{i\in I}X_i|\mathcal{F}$ where X_i is finite for every $i\in I$, then $\boldsymbol{\mathcal{P}X}\cong\prod_{i\in I}\mathcal{P}X_i|\mathcal{F}$ and $\mathcal{P}X_i$ is finite for every $i\in I$.

(c) If $\boldsymbol{X}\cong\prod_{i\in I}X_i|\mathcal{F}$ and $\boldsymbol{Y}\cong\prod_{i\in I}Y_i|\mathcal{F}$ where X_i and Y_i are finite for every $i\in I$, then $\boldsymbol{X}\times\boldsymbol{Y}\cong\prod_{i\in I}X_i\times Y_i|\mathcal{F}$ and $X_i\times Y_i$ is finite for every $i\in I$.

2H Lemma (a) Suppose that $(\boldsymbol{P},\boldsymbol{\le})\in\mathrm{Po}_{<\omega}(\mathcal{F})$. Then every non-empty internal subset of $\boldsymbol{P}$ has a maximal element.

(b) Suppose that $(\boldsymbol{P},\boldsymbol{\le})\in\mathrm{Lo}_{<\omega}(\mathcal{F})$.

(i) $(\boldsymbol{P},\boldsymbol{\le})$ is isomorphic to $(\boldsymbol{P},\boldsymbol{\ge})$.

(ii) Every non-empty internal subset of $\boldsymbol{P}$ has greatest and least members.

proof (a) The point is just that this is true for all finite partially ordered sets, and it is a first-order property. More explicitly, if $(\boldsymbol{P},\boldsymbol{\le})\cong\prod_{i\in I}(P_i,\le_i)|\mathcal{F}$, and $\boldsymbol{Z}$ is a non-empty internal subset of $\boldsymbol{P}$, then $\boldsymbol{Z}$ corresponds to $\prod_{i\in I} Z_i$ where $Z_i \subseteq P_i$ is non-empty for every $i \in I$. Now if $z_i \in Z_i$ is $\le_i$-maximal for every i, $\boldsymbol{z} = \langle z_i\rangle^\bullet_{i\in I}$ is $\boldsymbol{\le}$-maximal in $\boldsymbol{Z}$.

(b)(i) In this case,

$$(\boldsymbol{P}, \boldsymbol{\ge}) \cong \prod_{i\in I}(P_i, \ge)|\mathcal{F} \cong \prod_{i\in I}(P_i, \le)|\mathcal{F} \cong (\boldsymbol{P}, \boldsymbol{\le}).\ \mathbf{Q}$$

(ii) This is a special case of (a).

2I Lemma If $\boldsymbol{P} \in \mathrm{Lo}_{<\omega}(\mathcal{F})$ is infinite, then $\omega_1 \le \mathrm{interp}\,\boldsymbol{P} \le \omega^{\#(I)}$.

proof We can suppose that $\boldsymbol{P}$ is a reduced product $\prod_{i\in I}(P_i, \le_i)$ where every P_i is finite and every $\le_i$ is a total order.

(a) $\boldsymbol{P}$ has the $<\omega_1$-interpolation property. **P** If $\langle \boldsymbol{p}_k\rangle_{k\in\mathbb{N}}$ and $\langle \boldsymbol{q}_k\rangle_{k\in\mathbb{N}}$ are sequences in $\boldsymbol{P}$ with $\boldsymbol{p}_j \le \boldsymbol{q}_k$ for all j, $k \in \mathbb{N}$, express $\boldsymbol{p}_k$ as $\langle p_{ki}\rangle^\bullet_{i\in I}$ and $\boldsymbol{q}_k$ as $\langle q_{ki}\rangle^\bullet_{i\in I}$, where p_{ki}, $q_{ki} \in P_i$ for $i \in I$ and $k \in \mathbb{N}$. Set

$$A_l = \{i : i \in I,\ p_{ji} \le q_{ki} \text{ whenever } j,\ k < l\},$$

so that $\langle A_l\rangle_{l\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{F}$ starting with $A_0 = I$. Set

$$\begin{aligned} p_i^* &= \max(\{p_{0i}\} \cup \{p_{ji} : j < l\}) \text{ if } i \in A_l \setminus A_{l+1}, \\ &= \max\{p_{ji} : j \in \mathbb{N}\} \text{ if } i \in \bigcap_{l\in\mathbb{N}} A_l. \end{aligned}$$

Then

$$\{i : p_{ki} \le p_i^* \le q_{ki}\} \supseteq A_{k+1} \in \mathcal{F}$$

for every k, so

$$\boldsymbol{p}_k \le \langle p_i^*\rangle^\bullet_{i\in I} \le \boldsymbol{q}_k$$

for every $k \in \mathbb{N}$. **Q**

(b) There is a sequence $\langle \boldsymbol{p}_k\rangle_{k\in\mathbb{N}}$ in $\boldsymbol{P}$ with no supremum in $\boldsymbol{P}$. **P** Let $\langle p'_{ki}\rangle_{k<\#(P_i)}$ be the increasing enumeration of P_i for each i. As $\boldsymbol{P}$ is infinite, $A_k = \{i : \#(P_i) \ge k\} \in \mathcal{F}$ for each k. So if we set

$$\begin{aligned} p_{ki} &= p'_{ki} \text{ if } k < \#(X_i), \\ &= \max P_i \text{ if } k \ge \#(X_i), \\ \boldsymbol{p}_k &= \langle p_{ki}\rangle^\bullet_{i\in I} \end{aligned}$$

for $k \in \mathbb{N}$, $\langle \boldsymbol{p}_k\rangle_{k\in\mathbb{N}}$ will be strictly increasing. If $\boldsymbol{q} = \langle q_i\rangle^\bullet_{i\in I}$ is an upper bound for $\{\boldsymbol{p}_k : k \in \mathbb{N}\}$, and we take $\boldsymbol{q}' = \langle q'_i\rangle_{i\in I} \in \prod_{i\in I} X_i$ such that q'_i is the predecessor of q_i in P_i whenever $q_i \ne \min P_i$, then $\boldsymbol{q}' < \boldsymbol{q}$ and $\boldsymbol{q}'$ is still an upper bound of $\{\boldsymbol{p}_k : k \in \mathbb{N}\}$, so $\{\boldsymbol{p}_k : k \in \mathbb{N}\}$ has no least upper bound. **Q**

(c) Since $\boldsymbol{P}$ has a greatest member $\langle \max P_i\rangle^\bullet_{i\in I}$, 1D tells us that

$$\mathrm{interp}\,\boldsymbol{P} \le \#(\boldsymbol{P}) \le \#(\prod_{i\in I} P_i) \le \omega^{\#(I)}.$$

3 Interp$_{<\omega}$ and Chadd$_{<\omega}$

As in §2, take a fixed ultrafilter $\mathcal{F}$ on a fixed set I.

3A Definitions (a) Write $\mathrm{Interp}_{<\omega}(\mathcal{F})$ for $\min\{\mathrm{interp}\,\boldsymbol{P} : \boldsymbol{P} \in \mathrm{Lo}_{<\omega}(\mathcal{F})\}$.

(b) Write $\mathrm{Chadd}_{<\omega}(\mathcal{F})$ for $\min\{\mathrm{chadd}\,\boldsymbol{P} : \boldsymbol{P} \in \mathrm{Po}_{<\omega}(\mathcal{F})\}$.

3B Lemma Suppose that $\boldsymbol{X} \in \mathrm{Lo}_{<\omega}(\mathcal{F})$ and that we have sets $A \subseteq \boldsymbol{X}$, $\mathcal{Z} \subseteq \boldsymbol{PX})$ such that $\#(A)$, $\#(\mathcal{Z})$ are both less than $\min(\mathrm{Chadd}_{<\omega}(\mathcal{F}), \mathrm{interp}\,\boldsymbol{X})$ and every member of $\mathcal{Z}$ is an internal set including A. Then there is an internal set $\boldsymbol{Z}^* \subseteq \boldsymbol{X}$ such that $A \subseteq \boldsymbol{Z}^* \subseteq \bigcap \mathcal{Z}$.

Remark Note that there is no suggestion that A or $\mathcal{Z}$ should be an internal set.

proof (a) If either A or $\mathcal{Z}$ is finite, the result is trivial. Otherwise, set $\kappa = \max(\#(A), \#(\mathcal{Z}))$ and let $\langle \boldsymbol{x}_\xi \rangle_{\xi<\kappa}$, $\langle \boldsymbol{Z}_\xi \rangle_{\xi<\kappa}$ run over A, $\mathcal{Z}$ respectively.

Because $\operatorname{interp} \boldsymbol{X} < \infty$ (2I), we have a (λ_0, λ_1^*)-gap in $\boldsymbol{X}$ with $\max(\lambda_0, \lambda_1^*) = \operatorname{interp} \boldsymbol{X}$; as $(\boldsymbol{X}, \boldsymbol{\le}) \cong (\boldsymbol{X}, \boldsymbol{\ge})$, we can suppose that $\lambda_1 \le \lambda_0$ and we have a strictly increasing family $\langle \boldsymbol{y}_\eta \rangle_{\eta<\operatorname{interp}\boldsymbol{X}}$ in $\boldsymbol{X}$.

Let $\boldsymbol{P} = \boldsymbol{P}(\boldsymbol{X} \times \boldsymbol{X})$ be the set of internal subsets of $\boldsymbol{X} \times \boldsymbol{X}$. For $\boldsymbol{p} \in \boldsymbol{P}$ and $\boldsymbol{e} \in \boldsymbol{X}$, write $\boldsymbol{p} \lceil \boldsymbol{e}$ for $\{(\min(\boldsymbol{z}, \boldsymbol{e}), \boldsymbol{x}) : (\boldsymbol{z}, \boldsymbol{x}) \in \boldsymbol{p}\}$. Observe that $(\boldsymbol{p}\lceil\boldsymbol{e})\lceil\boldsymbol{e}' = \boldsymbol{p}\lceil\min(\boldsymbol{e}, \boldsymbol{e}')$ for all $\boldsymbol{p}$, $\boldsymbol{e}$ and $\boldsymbol{e}'$, so we have a partial order $\boldsymbol{\le}$ on $\boldsymbol{P}$ defined by saying that $\boldsymbol{p}' \boldsymbol{\le} \boldsymbol{p}$ if there is an $\boldsymbol{e} \in \boldsymbol{X}$ such that $\boldsymbol{p}' = \boldsymbol{p}\lceil\boldsymbol{e}$. Now $(\boldsymbol{P}, \boldsymbol{\le}) \in \mathrm{Po}_{<\omega}(\mathcal{F})$. **P** We have just to repeat the formula in each coordinate. Suppose that $(\boldsymbol{X}, \boldsymbol{\le})$ is isomorphic to the reduced product $\prod_{i\in I}(X_i, \le_i)|\mathcal{F}$ where $(X_i, \le_i)$ is a finite totally ordered set for each i. If $i \in I$, $p \subseteq X_i^2$ and $e \in X_i$, set $p\lceil e = (\min(z, e), x) : (z, x) \in p\}$; for p', $p \subseteq X_i^2$ say that $p' \le_i p$ if there is an $e \in X_i$ such that $p' = p\lceil e$. If now $\boldsymbol{p}'$, $\boldsymbol{p} \in \boldsymbol{P}$, we can identify them with $\langle p_i' \rangle_{i\in I}^\bullet$, $\langle p_i \rangle_{i\in I}^\bullet$ respectively, where p_i, $p_i' \subseteq X_i^2$ for each i (2B). If $\boldsymbol{e}$ corresponds to $\langle e_i \rangle_{i\in I}^\bullet \in \prod_{i\in I} X_i|\mathcal{F}$, $\boldsymbol{p}\lceil\boldsymbol{e}$ corresponds to $\langle p_i\lceil e_i \rangle_{i\in I}^\bullet$. So if $\boldsymbol{p}' \le \boldsymbol{p}$, $\{i : p_i' \le_i p_i\}$ belongs to $\mathcal{F}$; and, conversely, if $J = \{i : p_i' \le_i p_i\}$ belongs to $\mathcal{F}$, we can find a family $\langle e_i \rangle_{i\in I} \in \prod_{i\in I} X_i$ such that $J \supseteq \{i : p_i' = p_i\lceil e_i\}$, in which case $\boldsymbol{p}' \boldsymbol{\le} \boldsymbol{p}$. Thus $\boldsymbol{P}$ is isomorphic to $\langle(\mathcal{P}(X_i^2), \le_i)\rangle_{i\in I}^\bullet$ and belongs to $\mathrm{Po}_{<\omega}(\mathcal{F})$. **Q**

(b) Choose a non-decreasing family $\langle \boldsymbol{p}_\eta \rangle_{\eta\le\kappa}$ in $\boldsymbol{P}$ inductively, as follows. The inductive hypothesis will be that $\boldsymbol{p}_\eta \in \boldsymbol{P}$, $\boldsymbol{p}_{\eta'} = \boldsymbol{p}_\eta\lceil\boldsymbol{y}_{\eta'}$ whenever $\eta' \le \eta$, and $(\boldsymbol{y}_\eta, \boldsymbol{x}_\xi) \in \boldsymbol{p}_\eta$ whenever $\xi < \kappa$.

Start with $\boldsymbol{p}_0 = \{\boldsymbol{y}_0\} \times \boldsymbol{X}$. Given $\boldsymbol{p}_\eta$ where $\eta < \kappa$, set

$$\boldsymbol{p}_{\eta+1} = \boldsymbol{p}_\eta \cup \{(\boldsymbol{y}_{\eta+1}, \boldsymbol{x}) : (\boldsymbol{y}_\eta, \boldsymbol{x}) \in \boldsymbol{p}_\eta,\ \boldsymbol{x} \in \boldsymbol{Z}_\eta\}.$$

Then $\boldsymbol{p}_\eta = \boldsymbol{p}_{\eta+1}\lceil\boldsymbol{y}_\eta \le \boldsymbol{p}_{\eta+1}$, and $(\boldsymbol{y}_{\eta+1}, \boldsymbol{x}_\xi) \in \boldsymbol{p}_{\eta+1}$ whenever $\xi < \kappa$, because $\boldsymbol{x}_\xi \in A \subseteq \boldsymbol{Z}_\eta \in \mathcal{Z}$.

For the inductive step to a non-zero limit ordinal $\eta \le \kappa$, we have

$$\operatorname{cf}\eta \le \kappa < \mathrm{Chadd}_{<\omega}(\mathcal{F}) \le \operatorname{chadd}\boldsymbol{P},$$

so there is an upper bound $\boldsymbol{p}'$ of $\{\boldsymbol{p}_{\eta'} : \eta' < \eta\}$ in $\boldsymbol{P}$. For each $\xi < \kappa$, set

$$\boldsymbol{e}_\xi = \max\{\boldsymbol{z} : (\boldsymbol{z}, \boldsymbol{x}_\xi) \in \boldsymbol{p}'\}$$

which is defined because $\boldsymbol{p}'$ is an internal subset of $\boldsymbol{X}^2$, so $\{\boldsymbol{z} : (\boldsymbol{z}, \boldsymbol{x}_\xi) \in \boldsymbol{p}'\}$ is an internal subset of $\boldsymbol{X}$, and is non-empty because $(\boldsymbol{y}_0, \boldsymbol{x}_\xi) \in \boldsymbol{p}_0 = \boldsymbol{p}'\lceil\boldsymbol{y}_0$. If $\eta' < \eta$, then $(\boldsymbol{y}_{\eta'}, \boldsymbol{x}_\xi) \in \boldsymbol{p}_{\eta'} = \boldsymbol{p}'\lceil\boldsymbol{y}_{\eta'}$, so $\boldsymbol{y}_{\eta'} \le \boldsymbol{e}_\xi$ and $\boldsymbol{y}_{\eta'} \le \min(\boldsymbol{y}_\eta, \boldsymbol{e}_\xi)$. Because $\kappa < \operatorname{interp}\boldsymbol{X}$, there must be an $\boldsymbol{e} \in \boldsymbol{X}$ such that $\boldsymbol{y}_{\eta'} \le \boldsymbol{e} \le \min(\boldsymbol{y}_\eta, \boldsymbol{e}_\xi)$ whenever $\eta' < \eta$ and $\xi < \kappa$. Set

$$\boldsymbol{p}'' = \boldsymbol{p}'\lceil\boldsymbol{e}, \quad \boldsymbol{p}_\eta = \boldsymbol{p}'' \cup \{(\boldsymbol{y}_\eta, \boldsymbol{x}) : (\boldsymbol{e}, \boldsymbol{x}) \in \boldsymbol{p}''\}.$$

For $\eta' < \eta$, we have

$$\boldsymbol{p}_{\eta'} = \boldsymbol{p}'\lceil\boldsymbol{y}_{\eta'} = \boldsymbol{p}''\lceil\boldsymbol{y}_{\eta'} = \boldsymbol{p}_\eta\lceil\boldsymbol{y}_{\eta'} \le \boldsymbol{p}_\eta,$$

while if $\xi < \kappa$ then $(\boldsymbol{e}_\xi, \boldsymbol{x}_\xi) \in \boldsymbol{p}'$, $(\boldsymbol{e}, \boldsymbol{x}_\xi) \in \boldsymbol{p}''$ and $(\boldsymbol{y}_\eta, \boldsymbol{x}_\xi) \in \boldsymbol{p}_\eta$. Of course $\boldsymbol{z} \le \boldsymbol{y}_\eta$ whenever $(\boldsymbol{z}, \boldsymbol{x}) \in \boldsymbol{p}_\eta$, so the induction continues.

(c) At the end of the induction, set $\boldsymbol{Z}^* = \{\boldsymbol{x} : (\boldsymbol{y}_\kappa, \boldsymbol{x}) \in \boldsymbol{p}_\kappa\}$. Then $\boldsymbol{Z}^*$ is an internal set because $\boldsymbol{p}_\kappa$ is, and contains every $\boldsymbol{x}_\xi$ by the construction of $\boldsymbol{p}_\kappa$. If $\eta < \kappa$ and $\boldsymbol{x} \in \boldsymbol{Z}^*$, then

$$(\boldsymbol{y}_{\eta+1}, \boldsymbol{x}) = (\min(\boldsymbol{y}_\kappa, \boldsymbol{y}_{\eta+1}), \boldsymbol{x}) \in \boldsymbol{p}_\kappa\lceil\boldsymbol{y}_{\eta+1} = \boldsymbol{p}_{\eta+1},$$

so $\boldsymbol{x} \in \boldsymbol{Z}_\eta$. Thus $A \subseteq \boldsymbol{Z}^* \subseteq \bigcap\mathcal{Z}$, as required.

3C Corollary Suppose that $\boldsymbol{X} \in \mathrm{Lo}_{<\omega}(\mathcal{F})$ and $\boldsymbol{h} : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ is an internal function. Let $A \subseteq \boldsymbol{X}$ and $\boldsymbol{w} \in \boldsymbol{X}$ be such that $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}') \le \boldsymbol{w}$ for all $\boldsymbol{x}$, $\boldsymbol{x}' \in A$ and $\#(A) < \min(\mathrm{Chadd}_{<\omega}(\mathcal{F}), \operatorname{interp}\boldsymbol{X})$. Then there is an internal set $\boldsymbol{D} \subseteq \boldsymbol{X}$ such that $A \subseteq \boldsymbol{D}$ and $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}') \le \boldsymbol{w}$ for all $\boldsymbol{x}$, $\boldsymbol{x}' \in \boldsymbol{D}$.

proof For $\boldsymbol{x} \in A$, set $\boldsymbol{Z}_{\boldsymbol{x}} = \{\boldsymbol{x}' : \boldsymbol{x}' \in \boldsymbol{X},\ \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}') \le \boldsymbol{w}\}$. Then $\boldsymbol{Z}_{\boldsymbol{x}}$ is an internal subset of $\boldsymbol{X}$ including A. Applying 3B to A and $\mathcal{Z} = \{\boldsymbol{Z}_{\boldsymbol{x}} : \boldsymbol{x} \in A\}$, we see that there is an internal set $\boldsymbol{Z} \subseteq \boldsymbol{X}$ such that $A \subseteq \boldsymbol{Z} \subseteq \boldsymbol{Z}_{\boldsymbol{x}}$ for every $\boldsymbol{x} \in A$. Now

$$\boldsymbol{D} = \{\boldsymbol{x} : \boldsymbol{x} \in \boldsymbol{Z},\ \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}') \le \boldsymbol{w} \text{ for every } \boldsymbol{x}' \in \boldsymbol{Z}\}$$

is an internal set including A, and $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{x}') \le \boldsymbol{w}$ for all $\boldsymbol{x}$, $\boldsymbol{x}' \in \boldsymbol{D}$.

3D Lemma Suppose that $\boldsymbol{X} \in \mathrm{Po}_{<\omega}(\mathcal{F})$ and that $\boldsymbol{Y} \in \mathrm{Ufm}_{<\omega}(\mathcal{F})$. Let $D \subseteq \boldsymbol{X}$ be a well-ordered set with order type less than $\mathrm{Chadd}_{<\omega}(\mathcal{F})$, and $F : D \to \boldsymbol{Y}$ a function. Then there is an internal function $\boldsymbol{h} : \boldsymbol{X} \to \boldsymbol{Y}$ extending F.

proof (a) Write α for $\operatorname{otp} D$. Let $\boldsymbol{P}$ be the set of internal partial functions from subsets of $\boldsymbol{X}$ to $\boldsymbol{Y}$, that is, the set of internal subsets $\boldsymbol{p}$ of $\boldsymbol{X} \times \boldsymbol{Y}$ such that $\boldsymbol{y} = \boldsymbol{y}'$ whenever $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{x}, \boldsymbol{y}') \in \boldsymbol{p}$. Then $(\boldsymbol{P}, \subseteq) \in \mathrm{Po}_{<\omega}(\mathcal{F})$, being isomorphic to $\prod_{i \in I}(P_i, \subseteq)|\mathcal{F}$ where each P_i is the set of partial functions from subsets of X_i to Y_i. (See 2Ce.) So $\alpha < \operatorname{chadd} \boldsymbol{P}$.

Let $\langle \boldsymbol{d}_\beta \rangle_{\beta<\alpha}$ be the increasing enumeration of D.

(b) Choose a non-decreasing family $\langle \boldsymbol{p}_\beta \rangle_{\beta<\alpha}$ inductively, as follows. The inductive hypothesis will be that $\boldsymbol{p}_\beta \in \boldsymbol{P}$ and $\boldsymbol{d}_\beta$ is the greatest element of $\operatorname{dom} \boldsymbol{p}_\beta$. Start with $\boldsymbol{p}_0 = \{(\boldsymbol{d}_0, F(\boldsymbol{d}_0))\}$. Given $\langle \boldsymbol{p}_\gamma \rangle_{\gamma<\beta}$, where $\beta < \alpha$, this is a totally ordered subset of $\boldsymbol{P}$ of cofinality less than $\operatorname{chadd} \boldsymbol{P}$, so has an upper bound $\boldsymbol{q} \in \boldsymbol{P}$; set

$$\boldsymbol{p}_\beta = \{(\boldsymbol{x}, \boldsymbol{y}) : (\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{q},\ \boldsymbol{x} < \boldsymbol{d}_\beta\} \cup \{(\boldsymbol{d}_\beta, F(\boldsymbol{d}_\beta))\}.$$

(c) At the end of the induction, $\langle \boldsymbol{p}_\beta \rangle_{\beta<\alpha}$ is still a totally ordered subset of $\boldsymbol{P}$ with cofinality less than $\operatorname{chadd} \boldsymbol{P}$, so has an upper bound $\boldsymbol{q}^* \in \boldsymbol{P}$; let $\boldsymbol{h}$ be any internal function extending $\boldsymbol{q}^*$ to a function from $\boldsymbol{X}$ to $\boldsymbol{Y}$. Now

$$\boldsymbol{h}(\boldsymbol{d}_\beta) = \boldsymbol{q}^*(\boldsymbol{d}_\beta) = \boldsymbol{p}_\beta(\boldsymbol{d}_\beta) = F(\boldsymbol{d}_\beta)$$

for every $\beta < \alpha$, so $\boldsymbol{h} \supseteq F$.

3E Lemma Suppose that $\boldsymbol{X} \in \mathrm{Po}_{<\omega}(\mathcal{F})$ and that $\boldsymbol{Y} \in \mathrm{Ufm}_{<\omega}(\mathcal{F})$. Suppose that $D \subseteq \boldsymbol{X}$ is a well-ordered set with order type less than $\mathrm{Chadd}_{<\omega}(\mathcal{F})$, and $F : D^2 \to \boldsymbol{Y}$ a function. Then there is an internal function from $\boldsymbol{X}^2$ to $\boldsymbol{Y}$ extending F.

proof Set $\boldsymbol{Z} = \prod_{i \in I} Z_i|\mathcal{F}$, where Z_i is the set of functions from X_i to Y_i for each $i \in I$; note that each Z_i is finite. For $\boldsymbol{d} \in D$, define $F_{\boldsymbol{d}} : D \to \boldsymbol{Y}$ by setting $F_{\boldsymbol{d}}(\boldsymbol{d}') = F(\boldsymbol{d}, \boldsymbol{d}')$ for $\boldsymbol{d}' \in D$. By 3D, we have an internal function $\boldsymbol{h}_{\boldsymbol{d}} : \boldsymbol{X} \to \boldsymbol{Y}$ extending $F_{\boldsymbol{d}}$, and $\boldsymbol{h}_{\boldsymbol{d}}$ can be represented by a member $\boldsymbol{z}_d$ of $\boldsymbol{Z}$.

By 3D again, there is an internal function $\boldsymbol{h}' : \boldsymbol{X} \to \boldsymbol{Z}$ such that $\boldsymbol{h}'(\boldsymbol{d}) = \boldsymbol{z}_{\boldsymbol{d}}$ for every $\boldsymbol{d} \in D$. Suppose that $\boldsymbol{h}'$ corresponds to $\langle h'_i \rangle_{i \in I}$ where $h'_i : X_i \to Z_i$ is a function for each i. If we set $h_i(x, x') = h'_i(x)(x')$ for x, $x' \in X_i$, then $\langle h_i \rangle_{i \in I}$ corresponds to an internal function $\boldsymbol{h} : \boldsymbol{X}^2 \to \boldsymbol{Y}$. If $\boldsymbol{d}$, $\boldsymbol{d}' \in D$ correspond to $\langle d_i \rangle^\bullet_{i \in I}$ and $\langle d'_i \rangle^\bullet_{i \in I}$ respectively, then $\boldsymbol{h}(\boldsymbol{d}, \boldsymbol{d}')$ corresponds to

$$\langle h_i(d_i, d'_i) \rangle^\bullet_{i \in I} = \langle h'_i(d_i)(d'_i) \rangle^\bullet_{i \in I} = \langle h'_i(d_i) \rangle^\bullet_{i \in I}(\langle d'_i \rangle^\bullet_{i \in I})$$

and

$$\boldsymbol{h}(\boldsymbol{d}, \boldsymbol{d}') = \boldsymbol{h}'(\boldsymbol{d})(\boldsymbol{d}') = \boldsymbol{h}_{\boldsymbol{d}}(\boldsymbol{d}') = F_{\boldsymbol{d}}(\boldsymbol{d}') = F(\boldsymbol{d}, \boldsymbol{d}').$$

So $\boldsymbol{h}$ extends F, as required.

3F Lemma If $\boldsymbol{X} \in \mathrm{Lo}_{<\omega}(\mathcal{F})$, κ is a cardinal and there is a (κ, κ^*)-gap in $\boldsymbol{X}$, then $\mathrm{Chadd}_{<\omega}(\mathcal{F}) \le \kappa$.

proof (a) Of course $\boldsymbol{X}$ must be infinite. Consider the partial ordering $\preccurlyeq$ on $[\boldsymbol{X}]^2$ defined by saying that $I \preccurlyeq J$ if $\min I \le \min J$ and $\max J \le \max I$. Then $([\boldsymbol{X}]^2, \preccurlyeq)$ is isomorphic to a member of $\mathrm{Po}_{<\omega}(\mathcal{F})$. **P** Suppose that $\boldsymbol{X} \cong \prod_{i \in I}(X_i, \le_i)|\mathcal{F}$ where $(X_i, \le_i)$ is a finite non-empty totally ordered set for each i. Since $\#(\boldsymbol{X}) > 1$, $K = \{i : \#(X_i) \ge 2\} \in \mathcal{F}$; set

$$(X'_i, \le'_i) = (X_i, \le_i) \text{ for } i \in K,$$
$$= (\{0, 1\}, \le) \text{ for } i \in I \setminus K.$$

On $[X'_i]^2$ define $\preccurlyeq_i$ by saying that $I \preccurlyeq_i J$ if $\min I \le'_i \min J$ and $\max J \le'_i \max I$. Then

$$([\boldsymbol{X}]^2, \preccurlyeq) \cong \prod_{i \in I}([X'_i]^2, \le'_i)|\mathcal{F} \in \mathrm{Po}_{<\omega}(\mathcal{F}). \textbf{ Q}$$

(b) Let $(\langle \boldsymbol{x}_\xi \rangle_{\xi<\kappa}, \langle \boldsymbol{y}_\xi \rangle_{\xi<\kappa})$ be a (κ, κ^*)-gap in $\boldsymbol{X}$. Then $\langle \{\boldsymbol{x}_\xi, \boldsymbol{y}_\xi\} \rangle_{\xi<\kappa}$ is a strictly increasing family in $[\boldsymbol{X}]^2$. **?** If it has an upper bound $I \in [\boldsymbol{X}]^2$, then $\boldsymbol{x}_\xi \le \min I \le \max I \le \boldsymbol{y}_\eta$ for all ξ, $\eta < \kappa$, which is supposed to be impossible. **X** So $\kappa \ge \operatorname{chadd}[\boldsymbol{X}]^2 \ge \mathrm{Chadd}_{<\omega}(\mathcal{F})$.

3G Theorem $\mathrm{Chadd}_{<\omega}(\mathcal{F}) \le \mathrm{Interp}_{<\omega}(\mathcal{F})$.

proof ? Suppose otherwise.

(a) Of course $\mathrm{Interp}_{<\omega}(\mathcal{F})$ cannot be ∞. Set $\kappa = \mathrm{Interp}_{<\omega}(\mathcal{F})$ and let $\boldsymbol{X} \in \mathrm{Lo}_{<\omega}(\mathcal{F})$ be such that $\mathrm{interp}\,\boldsymbol{X} = \kappa$. By 1C, there is a (λ, λ_1^*)-gap in $\boldsymbol{X}$ with $\max(\lambda, \lambda_1) = \kappa$; since $(\boldsymbol{X}, \le)$ is isomorphic to $(\boldsymbol{X}, \ge)$ (see 2Fa), we can take it that $\lambda \le \lambda_1 = \kappa$. We are supposing that $\kappa < \mathrm{Chadd}_{<\omega}(\mathcal{F})$. By 3F, there is no (κ, κ^*)-gap in $\boldsymbol{X}$, so $\lambda < \kappa$. Let $(\langle \boldsymbol{x}_\eta \rangle_{\eta<\lambda}, \langle \boldsymbol{x}'_\xi \rangle_{\xi<\kappa})$ be a (λ, κ^*)-gap in $\boldsymbol{X}$.

(b) Because $(\boldsymbol{X}, \ge)$ and $(\boldsymbol{X}, \le)$ are isomorphic, and $\boldsymbol{X}$ has a strictly decreasing family $\langle \boldsymbol{x}'_\xi \rangle_{\xi<\kappa}$, there is also a strictly increasing family $\langle \boldsymbol{d}_\xi \rangle_{\xi<\kappa}$ in $\boldsymbol{X}$. Let $G : \lambda^+ \times \lambda^+ \to \lambda$ be such that $\beta \mapsto G(\alpha, \beta) : \alpha \to \lambda$ is injective for every $\alpha < \lambda^+$. Because $\lambda^+ \le \kappa < \mathrm{Chadd}_{<\omega}(\mathcal{F})$, 3E tells us that there is an internal function $\boldsymbol{h} : \boldsymbol{X}^2 \to \boldsymbol{X}$ such that $\boldsymbol{h}(\boldsymbol{d}_\alpha, \boldsymbol{d}_\beta) = \boldsymbol{x}_{G(\alpha,\beta)}$ for all α, $\beta < \lambda^+$. Now 3C tells us that for every $\xi < \kappa$ there is an internal set $\boldsymbol{D}_\xi \supseteq \{\boldsymbol{d}_\alpha : \alpha < \min(\lambda^+, \xi + 1)\}$ such that $\boldsymbol{h}(\boldsymbol{d}, \boldsymbol{d}') \le \boldsymbol{x}'_\xi$ for all $\boldsymbol{d}$, $\boldsymbol{d}' \in \boldsymbol{D}_\xi$.

(c) Let $\boldsymbol{Q}$ be the family of internal subsets $\boldsymbol{q}$ of $\boldsymbol{X}^3$ such that

$$\boldsymbol{h}(\boldsymbol{d}', \boldsymbol{d}'') \le \boldsymbol{y} \text{ whenever } (\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}),\ (\boldsymbol{z}', \boldsymbol{y}', \boldsymbol{d}'),\ (\boldsymbol{z}', \boldsymbol{y}'', \boldsymbol{d}'') \in \boldsymbol{q} \text{ and } \boldsymbol{z} \le \boldsymbol{z}'\}.$$

Then $\boldsymbol{Q}$, partially ordered by inclusion, belongs to $\mathrm{Po}_{<\omega}(\mathcal{F})$. **P** We can suppose that $(\boldsymbol{X}, \le) = \prod_{i\in I}(X_i, \le_i)|\mathcal{F}$ where $(X_i, \le_i)$ is a finite totally ordered set for every $i \in I$. Because $\boldsymbol{h}$ is an internal function, we have a family $\langle h_i \rangle_{i\in I}$ such that $h_i : X_i^2 \to X_i$ is a function for each $i \in I$ and $\boldsymbol{h}$ can be regarded as $\langle h_i \rangle^\bullet_{i\in I}$. If we set

$$Q_i = \{q : q \subseteq X_i^3,\ h_i(d', d'') \le {}_i y \text{ whenever } (z, y, d),\ (z', y', d'),\ (z', y'', d'') \in q \text{ and } z \le {}_i z'\},$$

then we can identify $(\boldsymbol{Q}, \subseteq)$ with $\prod_{i\in I}(Q_i, \subseteq)|\mathcal{F}$. **Q**[1]

Accordingly $\mathrm{chadd}\,\boldsymbol{Q} > \kappa$.

(d) There is a non-decreasing family $\langle \boldsymbol{q}_\xi \rangle_{\xi<\kappa}$ in $\boldsymbol{Q}$ such that, for each $\xi < \kappa$,

if $\beta < \min(\lambda^+, \xi + 1)$ and $\boldsymbol{d}_\beta \le \boldsymbol{z} \le \boldsymbol{d}_\xi$ then there is a $\boldsymbol{y}$ such that $(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}_\beta) \in \boldsymbol{q}_\xi$,

if $(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}) \in \boldsymbol{q}_\xi$ then $\boldsymbol{z} \le \boldsymbol{d}_\xi$ and $\boldsymbol{x}'_\xi \le \boldsymbol{y}$,

$(\boldsymbol{d}_\xi, \boldsymbol{x}'_\xi, \boldsymbol{d}_0) \in \boldsymbol{q}$.

P Start the induction with $\boldsymbol{q}_0 = \{(\boldsymbol{d}_0, \boldsymbol{x}'_0, \boldsymbol{d}_0)\}$. Given $\langle \boldsymbol{q}_\eta \rangle_{\eta<\xi}$ where $0 < \xi < \kappa$, take an upper bound $\boldsymbol{q}$ of $\{\boldsymbol{q}_\eta : \eta < \xi\}$ in $\boldsymbol{Q}$. For $\alpha < \min(\lambda^+, \xi)$, the set

$$\{(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{e}) : \boldsymbol{e},\ \boldsymbol{y},\ \boldsymbol{z} \in \boldsymbol{X},\ \boldsymbol{z} < \boldsymbol{d}_\alpha \text{ or } \boldsymbol{e} < \boldsymbol{z} \text{ or } (\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}_\alpha) \in \boldsymbol{Q}\}$$

is an internal subset of $\boldsymbol{X}^3$, so

$$\boldsymbol{E}_\alpha = \{\boldsymbol{e} : \boldsymbol{e} \in \boldsymbol{X}, \text{ for every } \boldsymbol{z} \in [\boldsymbol{d}_\alpha, \boldsymbol{e}] \text{ there is a } \boldsymbol{y} \text{ such that } (\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}_\alpha) \in \boldsymbol{q}\}$$

is an internal subset of $\boldsymbol{X}$ (use 2Cf); since there is a $\boldsymbol{y}$ such that $(\boldsymbol{d}_\alpha, \boldsymbol{y}, \boldsymbol{d}_\alpha) \in \boldsymbol{q}_\alpha \subseteq \boldsymbol{q}$, $\boldsymbol{d}_\alpha \in \boldsymbol{E}_\alpha$; by 2H(b-ii), $\boldsymbol{E}_\alpha$ has a greatest element $\boldsymbol{e}_\alpha$ say.

Because $\boldsymbol{q}_\eta \in \boldsymbol{Q}$ and $\boldsymbol{q}_\eta \subseteq \boldsymbol{q}$ for $\alpha \le \eta < \xi$, $\boldsymbol{d}_\eta \le \boldsymbol{e}_\alpha$ whenever $\eta < \xi$ and $\alpha < \min(\lambda^+, \xi)$. Now $\xi < \kappa = \mathrm{interp}\,\boldsymbol{X}$ so there is a $\boldsymbol{e} \in \boldsymbol{X}$ such that $\boldsymbol{d}_\eta \le \boldsymbol{e} \le \boldsymbol{e}_\alpha$ for every $\eta < \xi$ and $\alpha < \min(\lambda^+, \xi)$; replacing $\boldsymbol{e}$ by $\min(\boldsymbol{e}, \boldsymbol{d}_\xi)$ if necessary, we can suppose that $\boldsymbol{e} \le \boldsymbol{d}_\xi$. Set

$$\boldsymbol{q}_\xi = \{(\boldsymbol{z}, \max(\boldsymbol{y}, \boldsymbol{x}'_\xi), \boldsymbol{d}) : (\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}) \in \boldsymbol{q},\ \boldsymbol{z} < \boldsymbol{e}\} \cup \{(\boldsymbol{z}, \boldsymbol{x}'_\xi, \boldsymbol{d}) : \boldsymbol{e} \le \boldsymbol{z} \le \boldsymbol{d}_\xi,\ \boldsymbol{d} \in \boldsymbol{D}_\xi).$$

This continues the induction.

(e) At the end of the induction take an upper bound $\boldsymbol{q}$ of $\{\boldsymbol{q}_\xi : \xi < \kappa\}$ in $\boldsymbol{Q}$, For $\alpha < \lambda^+$ take $\boldsymbol{e}_\alpha$ maximal subject to

$$\text{for every } \boldsymbol{z} \in [\boldsymbol{d}_\alpha, \boldsymbol{e}_\alpha] \text{ there is a } \boldsymbol{y} \text{ such that } (\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}_\alpha) \in \boldsymbol{q}.$$

[1] Alternatively, check that

$$\boldsymbol{R} = \{(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{d}, \boldsymbol{z}', \boldsymbol{y}', \boldsymbol{d}', \boldsymbol{z}'', \boldsymbol{y}'', \boldsymbol{d}'') : \boldsymbol{z} \le \boldsymbol{z}' = \boldsymbol{z}'',\ \boldsymbol{y} < \boldsymbol{h}(\boldsymbol{d}', \boldsymbol{d}'')\}$$

is an internal subset of $\boldsymbol{X}^9$; note that $\boldsymbol{Q} = \{\boldsymbol{q} : \boldsymbol{q}^3 \cap \boldsymbol{R} = \emptyset\}$ and that $\boldsymbol{q} \mapsto \boldsymbol{q}^3 \cap \boldsymbol{R}$ is an internal function.

As in the inductive step in (d) above, $\boldsymbol{e}_\alpha \ge \boldsymbol{d}_\xi$ for every $\xi < \kappa$. So if $\boldsymbol{y}_\alpha = \min\{\boldsymbol{y} : (\boldsymbol{z},\boldsymbol{y},\boldsymbol{d}) \in \boldsymbol{q},\ \boldsymbol{z} \le \boldsymbol{e}_\alpha\}$, $\boldsymbol{y}_\alpha \le \boldsymbol{x}'_\xi$ for every $\xi < \kappa$ and there is a $\theta(\alpha) < \lambda$ such that $\boldsymbol{y}_\alpha \le \boldsymbol{x}_{\theta(\alpha)}$. Let $\eta < \lambda$ be such that $A = \{\alpha : \theta(\alpha) \le \eta\}$ has cardinal λ^+; let $\alpha \in A$ be such that $\#(A \cap \alpha) = \lambda$; then there must be a $\beta \in A \cap \alpha$ such that $G(\alpha,\beta) > \eta$. Set $\boldsymbol{e} = \min(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta)$; then there are $\boldsymbol{y}'$, $\boldsymbol{y}''$ such that $(\boldsymbol{e},\boldsymbol{y}',\boldsymbol{d}_\alpha)$ and $(\boldsymbol{e},\boldsymbol{y}'',\boldsymbol{d}_\beta)$ belong to $\boldsymbol{q}$. We therefore have

$$\begin{aligned}\boldsymbol{h}(\boldsymbol{d}_\alpha,\boldsymbol{d}_\beta) &\le \min\{\boldsymbol{y} : (\boldsymbol{z},\boldsymbol{y},\boldsymbol{d}) \in \boldsymbol{q},\ \boldsymbol{z} \le \boldsymbol{e}\} = \max(\boldsymbol{y}_\alpha,\boldsymbol{y}_\beta)\\ &\le \max(\boldsymbol{x}_{\theta(\alpha)},\boldsymbol{x}_{\theta(\beta)}) \le \boldsymbol{x}_\eta < \boldsymbol{x}_{G(\alpha,\beta)} = \boldsymbol{h}(\boldsymbol{d}_\alpha,\boldsymbol{d}_\beta)\end{aligned}$$

which is impossible. **X**

(f) This contradiction shows that $\mathrm{Chadd}_{<\omega}(\mathcal{F})$ is indeed less than or equal to $\mathrm{Interp}_{<\omega}(\mathcal{F})$.

4 A forcing notion

Let $\mathbb{P}$ be the forcing notion $([\mathbb{N}]^\omega, \subseteq^*, \mathbb{N}, \downarrow)$, where $A \subseteq^* B$ if $A \setminus B$ is finite.

4A Proposition $\mathfrak{t}$ is the largest cardinal such that $\mathbb{P}$ is $\mathfrak{t}$-closed in the sense of Kunen 80, 6.12.

proof Immediate from the definition.

4B Proposition (a) $\mathbb{P}$ preserves cofinalities and cardinals up to and including $\mathfrak{t}$.

(b) $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} = (\mathcal{P}\mathbb{N})\check{}$.

(c)(i) $\Vdash_{\mathbb{P}} \mathfrak{t} = \check{\mathfrak{t}}$.

(ii) $\Vdash_{\mathbb{P}} \mathfrak{p} = \check{\mathfrak{p}}$.

proof (a) Kunen 80, 6.15.

(b) We just need to know that $\mathbb{P}$ is countably closed.

(c)(i)(α) Let $\langle a_\xi\rangle_{\xi<\mathfrak{t}}$ be a $\subseteq^*$-decreasing family in $[\mathbb{N}]^\omega$ with no $\subseteq^*$-lower bound in $[\mathbb{N}]^\omega$. Then

$\Vdash_{\mathbb{P}} \langle\check{a}_\xi\rangle_{\xi<\check{\mathfrak{t}}}$ is a $\subseteq^*$-decreasing family in $[\mathbb{N}]^\omega$

and as $\Vdash_{\mathbb{P}} \mathcal{P}\mathbb{N} = (\mathcal{P}\mathbb{N})\check{}$,

$\Vdash_{\mathbb{P}} \{\check{a}_\xi : \xi < \check{\mathfrak{t}}\}$ has no $\subseteq^*$-lower bound in $[\mathbb{N}]^{<\omega}$, so $\mathfrak{t} \le \check{\mathfrak{t}}$.

(β) Suppose that $\kappa < \mathfrak{t}$, $p \in \mathbb{P}$ and $\langle\dot{a}_\xi\rangle_{\xi<\kappa}$ is a family of $\mathbb{P}$-names such that

$p \Vdash \langle\dot{a}_\xi\rangle_{\xi<\check{\kappa}}$ is a $\subseteq^*$-decreasing family in $[\mathbb{N}]^\omega$.

Because $\mathbb{P}$ is $\mathfrak{t}$-closed, there are a q stronger than p and a family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathcal{P}\mathbb{N}$ such that $q \Vdash_{\mathbb{P}} \dot{a}_\xi = \check{a}_\xi$ for every $\xi < \kappa$. Now

$q \Vdash \langle\check{a}_\xi\rangle_{\xi<\check{\kappa}}$ is a $\subseteq^*$-decreasing family in $[\mathbb{N}]^\omega$,

so in fact $\langle a_\xi\rangle_{\xi<\kappa}$ is a $\subseteq^*$-decreasing family in $[\mathbb{N}]^\omega$; as $\kappa < \mathfrak{t}$, there is a $\subseteq^*$-lower bound a of $\{a_\xi : \xi < \kappa\}$ in $[\mathbb{N}]^\omega$, and now

$\Vdash_{\mathbb{P}} \check{a}$ is a $\subseteq^*$-lower bound of $\{\check{a}_\xi : \xi < \check{\kappa}\}$ in $[\mathbb{N}]^\omega$,

so

$q \Vdash_{\mathbb{P}} \check{a}$ is a $\subseteq^*$-lower bound of $\{\dot{a}_\xi : \xi < \check{\kappa}\}$ in $[\mathbb{N}]^\omega$.

As $\langle\dot{a}_\xi\rangle_{\xi<\kappa}$ is arbitrary,

$\Vdash_{\mathbb{P}} \check{\kappa} < \mathfrak{t}$.

As κ is arbitrary,

$\Vdash_{\mathbb{P}} \check{\mathfrak{t}} \le \mathfrak{t}$ so $\check{\mathfrak{t}} = \mathfrak{t}$.

(ii) Argue similarly, using the fact that $\mathfrak{p} \le \mathfrak{t}$ so $\mathbb{P}$ is $\mathfrak{p}$-closed.

4C Proposition Let $\dot{\mathcal{G}}$ be the $\mathbb{P}$-name $\{(\check{A}, A) : A \in [\mathbb{N}]^\omega\}$. Then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{G}} \text{ is a non-principal ultrafilter on } \mathbb{N}.$$

proof It is easy to see that

$$\Vdash_{\mathbb{P}} \dot{\mathcal{G}} \text{ is a filter on } \mathbb{N}.$$

Now if $A \in [\mathbb{N}]^\omega$ and $\dot{C}$ is a $\mathbb{P}$-name such that $A \Vdash_{\mathbb{P}} \dot{C} \in [\mathbb{N}]^\omega$, there are a $C \subseteq \mathbb{N}$ and an infinite $A' \subseteq^* A$ such that $A' \Vdash \dot{C} = \check{C}$ (4Bb); now if $A' \cap C$ is infinite, $A' \cap C \Vdash \dot{C} \in \dot{\mathcal{G}}$; otherwise, $A' \setminus C$ is infinite and $A' \setminus C \Vdash \mathbb{N} \setminus \dot{C} \in \dot{\mathcal{G}}$. So $\Vdash_{\mathbb{P}} \dot{\mathcal{G}}$ is an ultrafilter. Finally, if $n \in \mathbb{N}$,

$$\mathbb{N} \subseteq^* \mathbb{N} \setminus \{n\} \Vdash \mathbb{N} \setminus \{n\} \in \dot{\mathcal{G}}$$

and $\Vdash_{\mathbb{P}} \dot{\mathcal{G}}$ is non-principal.

4D Proposition $\Vdash_{\mathbb{P}} \mathfrak{t} \le \mathrm{Chadd}_{<\omega}(\dot{\mathcal{G}})$.

proof Let $\dot{\boldsymbol{P}}$, $\dot{R}$ be $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \dot{R} \subseteq \dot{\boldsymbol{P}} \in \mathrm{Po}_{<\omega}(\dot{\mathcal{G}}),\ \dot{R} \text{ is well-ordered, } \mathrm{otp}(\dot{R}) < \mathfrak{t}.$$

We can suppose that

$\Vdash_{\mathbb{P}}$ there is a sequence $\langle (P_n, \le_n)\rangle_{n\in\mathbb{N}}$ of non-empty finite partially ordered sets such that $\dot{\boldsymbol{P}} = \prod_{n\in\mathbb{N}} P_n|\dot{\mathcal{G}}$.

Take any $A \in [\mathbb{N}]^\omega$. By 4B(c-i), $\Vdash_{\mathbb{P}} \mathrm{otp}(\dot{R}) < \check{\mathfrak{t}}$ and there are a $B \in [A]^\omega$ and an ordinal $\alpha < \mathfrak{t}$ such that $B \Vdash \mathrm{otp}(\dot{R}) = \check{\alpha}$. Let $\langle \dot{\boldsymbol{p}}_\xi\rangle_{\xi<\alpha}$ be a family of $\mathbb{P}$-names such that

$$B \Vdash \langle \dot{\boldsymbol{p}}_\xi\rangle_{\xi<\check{\alpha}} \text{ is the increasing enumeration of } \dot{R}.$$

Next, we have families $\langle (\dot{P}_n, \dot{\le}_n)\rangle_{n\in\mathbb{N}}$ and $\langle \dot{p}_{\xi n}\rangle_{\xi<\alpha, n\in\mathbb{N}}$ of $\mathbb{P}$-names such that

$B \Vdash (\dot{P}_n, \dot{\le}_n)$ is a non-empty finite partially ordered set, $\dot{\boldsymbol{P}} = \prod_{n\in\mathbb{N}} \dot{P}_n|\dot{\mathcal{G}}$, and $\dot{\boldsymbol{p}}_\xi = \langle \dot{p}_{\xi n}\rangle^\bullet_{n\in\mathbb{N}}$

for every $\xi < \alpha$. Because $\mathbb{P}$ is $\mathfrak{t}$-closed, there are an infinite $C \subseteq B$ and families $\langle (P_n, \le_n)\rangle_{n\in\mathbb{N}}$ and $\langle p_{\xi n}\rangle_{\xi<\alpha, n\in\mathbb{N}}$ such that

$$(P_n, \le_n) \text{ is a non-empty finite partially ordered set and } p_{\xi n} \in P_n,$$

$$C \Vdash (\dot{P}_n, \dot{\le}_n) = (\check{P}_n, \check{\le}_n) \text{ and } \dot{p}_{\xi n} = \check{p}_{\xi n}$$

for every $n \in \mathbb{N}$ and $\xi < \alpha$.

For $\xi < \alpha$, set

$$E_\xi = \{(n, p) : n \in C,\ p \in P_n,\ p_{\xi n} \le_n p\}.$$

If $\xi \le \eta < \alpha$, then $E_\eta \setminus E_\xi$ is finite. **P?** Otherwise, set $D = \{n : \exists\, p,\ (n, p) \in E_\eta \setminus E_\xi\}$; because every P_n is finite, D is an infinite subset of C. If $n \in D$ there is a $p \in P_n$ such that $p_{\eta n} \le p$ but $p_{\xi n} \not\le_n p$, so that $p_{\xi n} \not\le_n p_{\eta n}$; now

$$D \Vdash \check{p}_{\xi n} \check{\not\le}_n \check{p}_{\eta n} \text{ for every } n \in \check{D},$$

so we have

$$D \Vdash \check{D} \in \dot{\mathcal{G}} \text{ and } \dot{p}_{\xi n} = \check{p}_{\xi n} \dot{\not\le}_n \check{p}_{\eta n} = \dot{p}_{\eta n} \text{ for every } n \in \check{D}$$

and

$$D \Vdash \langle \dot{p}_{\xi n}\rangle^\bullet_{n\in\mathbb{N}} \not\le \langle \dot{p}_{\eta n}\rangle^\bullet_{n\in\mathbb{N}},$$

contrary to the choice of $\langle \dot{p}_{\xi n}\rangle_{\xi<\alpha, n\in\mathbb{N}}$. **X**

Since every E_ξ is a subset of the countable set $\{(n, p) : n \in C,\ p \in P_n\}$, and $\mathrm{cf}\,\alpha < \mathfrak{t}$, there is an infinite $E \subseteq \{(n, p) : n \in C,\ p \in P_n\}$ such that $E \setminus E_\xi$ is finite for every $\xi < \alpha$. Now set $D = \{n : \exists\, p,\ (n, p) \in E\}$, so that D is infinite, and for each $n \in D$ take $q_n \in P_n$ such that $(n, q_n) \in E$; for other $n \in \mathbb{N}$ take any $q_n \in P_n$. In this case, for any $\xi < \alpha$,

$$\{n : n \in D,\ p_{\xi n} \not\le q_n\} \subseteq \{n : n \in D,\ (n, q_n) \notin E_\xi\}$$

is finite, so

$$D \subseteq^* \{n : n \in D,\ (n, q_n) \in E_\xi\} \subseteq \{n : n \in \mathbb{N},\ p_{\xi n} \le q_n\}.$$

But now observe that, writing D_ξ for $\{n : n \in D,\, p_{\xi n} \le q_n\}$,

$$D \subseteq^* D_\xi \Vdash \check D_\xi \in \dot{\mathcal{G}} \text{ and } \dot p_{\xi n} \le \check q_n \text{ for every } n \in \check D_\xi, \text{ so that } \langle \dot p_{\xi n}\rangle^\bullet_{n\in\mathbb{N}} \le \langle \check q_n\rangle^\bullet_{n\in\mathbb{N}} \text{ in } \dot{\boldsymbol{P}}.$$

As ξ is arbitrary,

$$D \Vdash \langle \check q_n\rangle^\bullet_{n\in\mathbb{N}} \text{ is an upper bound for } \dot R, \text{ and } \dot R \text{ is bounded above in } \dot{\boldsymbol{P}}.$$

As A is arbitrary,

$$\Vdash_{\mathbb{P}} \dot R \text{ is bounded above.}$$

As $\dot R$ is arbitrary,

$$\Vdash_{\mathbb{P}} \mathfrak{t} \le \operatorname{chadd} \dot{\boldsymbol{P}}.$$

As $\dot{\boldsymbol{P}}$ is arbitrary,

$$\Vdash_{\mathbb{P}} \mathfrak{t} \le \operatorname{Chadd}_{<\omega}(\dot{\mathcal{G}}),$$

as claimed.

4E Lemma Let $\preceq$ be the partial ordering on $\mathbb{N}^{\mathbb{N}}$ defined by saying that $f \preceq g$ if either $f = g$ or $\{n : g(n) \le f(n)\}$ is finite. If $\kappa \le \mathfrak{p}$ is an infinite cardinal and there is a peculiar $(\kappa, \mathfrak{p}^*)$-gap in $(\mathbb{N}^{\mathbb{N}}, \preceq)$, then $\mathfrak{p} = \mathfrak{t}$.

proof (a) Let $(\langle f_\xi\rangle_{\xi<\kappa}, \langle g_\eta\rangle_{\eta<\mathfrak{p}})$ be such a gap; we can suppose that f_ξ, $g_\eta \le g_0$ for every $\xi < \kappa$ and $\eta < \mathfrak{p}$. Let $\dot{\boldsymbol{P}}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\boldsymbol{P}} = \textstyle\prod_{n\in\mathbb{N}}(\check g_0(n)+1)|\dot{\mathcal{G}} \in \mathrm{Lo}_{<\omega}(\dot{\mathcal{G}}).$$

Then for each $\xi < \kappa$, $\eta < \mathfrak{p}$ we have $\mathbb{P}$-names $\dot{\boldsymbol{p}}_\xi$, $\dot{\boldsymbol{q}}_\eta$ such that

$$\Vdash_{\mathbb{P}} \dot{\boldsymbol{p}}_\xi = \check f_\xi^\bullet \in \dot{\boldsymbol{P}},\ \dot{\boldsymbol{q}}_\eta = \check g_\eta^\bullet \in \dot{\boldsymbol{P}}.$$

Now from 4Ba and 4B(c-ii) we have

$$\Vdash_{\mathbb{P}} \check\kappa \text{ is a cardinal less than } \mathfrak{p} \text{ and } \check f_\xi \prec \check f_{\xi'} \prec \check g_{\eta'} \prec \check g_\eta \text{ whenever } \xi < \xi' < \check\kappa \text{ and } \eta < \eta' < \mathfrak{p};$$

since we also know that $\Vdash_{\mathbb{P}} \dot{\mathcal{G}}$ is a free filter, we have

$$\Vdash_{\mathbb{P}} \dot{\boldsymbol{p}}_\xi < \dot{\boldsymbol{p}}_{\xi'} < \dot{\boldsymbol{q}}_{\eta'} < \dot{\boldsymbol{q}}_\eta \text{ whenever } \xi < \xi' < \check\kappa \text{ and } \eta < \eta' < \mathfrak{p}.$$

(b) ? If

$$\not\Vdash_{\mathbb{P}} (\langle \dot{\boldsymbol{p}}_\xi\rangle_{\xi<\check\kappa}, \langle \dot{\boldsymbol{q}}_\eta\rangle_{\eta<\mathfrak{p}}) \text{ is a } (\check\kappa, \mathfrak{p}^*)\text{-gap in } \dot P,$$

there are an $A \in [\mathbb{N}]^\omega$ and a $\mathbb{P}$-name $\dot h$ such that

$$A \Vdash \dot h \in \textstyle\prod_{n\in\mathbb{N}}(\check g_0(n)+1) \text{ and } \dot{\boldsymbol{p}}_\xi \le \dot h^\bullet \le \dot{\boldsymbol{q}}_\eta \text{ for every } \xi < \check\kappa \text{ and } \eta < \mathfrak{p}.$$

Because $\mathbb{P}$ is countably closed, there are an infinite $B \subseteq A$ and an $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$B \Vdash \dot h = \check h.$$

Next, for each $\xi < \kappa$, we have

$$B \Vdash \check f_\xi^\bullet < \check h^\bullet,$$

that is,

$$B \Vdash \{n : \check f_\xi(n) \le \check h(n)\} \in \dot{\mathcal{G}},$$

that is,

$$B \Vdash \{n : f_\xi(n) < h(n)\}\check{} \in \dot{\mathcal{G}},$$

that is,

$$B \subseteq^* \{n : f_\xi(n) < h(n)\}.$$

But this means that if we set

$$h'(n) = h(n) \text{ for } n \in B,$$
$$= g_0(n) \text{ for } n \in \mathbb{N} \setminus B,$$

we shall have $f_\xi \prec h'$; and this is true for every $\xi < \kappa$. Because $(\langle f_\xi \rangle_{\xi<\kappa}, \langle g_\eta \rangle_{\eta<\mathfrak{p}})$ is a peculiar gap, there is an $\eta < \mathfrak{p}$ such that $g_\eta \preceq h'$, in which case $B \subseteq^* \{n : g_{\eta+1}(n) < h(n)\}$; running the argument above backwards, we see that

$$B \Vdash \dot{\boldsymbol{q}}_{\eta+1} < \dot{h}^\bullet,$$

contrary to the choice of A and $\dot{h}$. **X**

(c) We conclude that

$$\Vdash_{\mathbb{P}} (\langle \dot{\boldsymbol{p}}_\xi \rangle_{\xi<\check{\kappa}}, \langle \dot{\boldsymbol{q}}_\eta \rangle_{\eta<\mathfrak{p}}) \text{ is a } (\check{\kappa}, \mathfrak{p}^*)\text{-gap in } \dot{\boldsymbol{P}}, \text{ so that } \mathfrak{p} \geq \operatorname{interp} \dot{\boldsymbol{P}}.$$

But now 3F and 4D, together with the Forcing Theorem (Kunen 80, VII.4.2), tell us that

$$\Vdash_{\mathbb{P}} \mathfrak{t} \leq \operatorname{Chadd}_{<\omega}(\dot{\mathcal{G}}) \leq \operatorname{Interp}_{<\omega}(\dot{\mathcal{G}}) \leq \operatorname{interp} \dot{\boldsymbol{P}} \leq \mathfrak{p}.$$

Accordingly, by 4Bc,

$$\Vdash_{\mathbb{P}} \check{\mathfrak{t}} \leq \check{\mathfrak{p}}$$

and $\mathfrak{t} \leq \mathfrak{p}$, so in fact $\mathfrak{t} = \mathfrak{p}$.

4F Theorem (Malliaris & Shelah 16) $\mathfrak{p} = \mathfrak{t}$.

proof ? Otherwise, there are an uncountable regular $\kappa < \mathfrak{p}$ and a $(\kappa, \mathfrak{p}^*)$-gap in $(\mathbb{N}^{\mathbb{N}}, \preceq)$, by Shelah 09, 1.12 or Fremlin n14, 2H (see parts (c)-(g) of the proof). And 4E tells us that this can happen only if $\mathfrak{p} = \mathfrak{t}$. **X**

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