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Flat GOPS

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A The game Each of two players starts with cards numbered $1, \ldots, n$. At each trick, the players select cards from those remaining to them and compare; higher card wins the trick (in case of a tie, neither wins); winner of the game is the player with the most tricks.

This is GOPS or Goofspiel, as described in `http://en.wikipedia.org/wiki/GOPS`, but with every trick given the same value.

B Proposition There is no strategy which defeats the random strategy in which cards are chosen in random order.

proof (a) I will say that a **configuration** is a quadruple (I, J, a, b) where I, J are finite subsets of $\{1, 2, \ldots\}$, both of the same size, represent the players' hands at some stage in the game, and a, $b \in \{0, 1, 2, \ldots\}$ represent the numbers of tricks they have won. Its **value** $V(I, J, a, b)$ is the expected winnings of the first player (the one whose holding is I and who has won a tricks so far) if he plays correctly, the second player plays at random, and the payoff is $+1$ for a win, 0 for a draw and -1 for a loss; so that if $I = J = \emptyset$ then the value is 1 if $a > b$, 0 if $a = b$ and -1 if $a < b$.

If (I, J, a, b) is a configuration and $i \in I$, then $V_i(I, J, a, b)$ is the expected winnings of the first player if he starts by playing card i; so that

$$V_i(I, J, a, b) = \frac{1}{\#(J)} \textstyle\sum_{j \in J} V(I \setminus \{i\}, J \setminus \{j\}, a + h(i - j), b + h(j - i))$$

where $h(\alpha) = 1$ if $\alpha > 0$, 0 otherwise, and

$$V(I, J, a, b) = \max_{i \in I} V_i(I, J, a, b).$$

(b) I set out to prove by induction on $\#(I)$ that $V_i(I, J, a, b) = V_{i'}(I, J, a, b)$ whenever (I, J, a, b) is a configuration and i, $i' \in I$. If $\#(I) = 1$ this is trivial. For the inductive step to $\#(I) = n > 1$, take distinct i, $i' \in I$ and observe that

$$\begin{aligned} V_i(I, J, a, b) &= \frac{1}{n} \sum_{j \in J} V(I \setminus \{i\}, J \setminus \{j\}, a + h(i - j), b + h(j - i)) \\ &= \frac{1}{n} \sum_{j \in J} V_{i'}(I \setminus \{i\}, J \setminus \{j\}, a + h(i - j), b + h(j - i)) \end{aligned}$$

(by the inductive hypothesis)

$$\begin{aligned} &= \frac{1}{n(n-1)} \sum_{\substack{j \in J \\ j' \in J \setminus \{j\}}} V(I \setminus \{i, i'\}, J \setminus \{j, j'\}, \\ &\qquad\qquad a + h(i - j) + h(i' - j'), b + h(j - i) + h(j' - i')) \\ &= \frac{1}{n(n-1)} \sum_{\substack{j' \in J \\ j \in J \setminus \{j'\}}} V(I \setminus \{i', i\}, J \setminus \{j', j\}, \\ &\qquad\qquad a + h(i' - j') + h(i - j), b + h(j' - i') + h(j - i)) \\ &= V_{i'}(I, J, a, b) \end{aligned}$$

and the induction continues.

(c) So the first player's expected payoff doesn't depend on what he does, and he might as well play at random; in which case the expected payoff is precisely zero, by symmetry.