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Vector-valued gauge integrals

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1A The Riemann integral One of the ways of defining the Riemann integral of a real-valued function over an interval $[a,b]\subseteq\mathbb{R}$ is to say that

$$\int_a^b f \simeq \sum_{i=0}^n f(x_i)(a_{i+1}-a_i)$$

where $a=a_0\le x_0\le a_1\le x_1\le a_2\le\ldots\le a_n\le x_n\le a_{n+1}=b$. Of course in order to get a good approximation we have to pick suitable strings a_i.

What are we going to keep, and what are we going to change?

We are going to keep the *finite* sums, and we are going to keep the idea of 'multiplication'. So some of the algebra is going to stay the same.

The first thing I want to change is the formula '$a_{i+1}-a_i$'. This is to be replaced by 'the measure of the interval C_i between a_i and a_{i+1}', so we get a formula

$$\int_a^b f \simeq \sum_{i=0}^n f(x_i)\nu C_i$$

where $C_0,\ldots,C_n$ are sets and ν is a measure of size (in the present case, of length).

I didn't say what kind of interval the C_i should be (open, closed, half-open). This is because it is going to be very convenient to insist on the 'intervals', or whatever kind of set we are going to allow, being disjoint. Open intervals would do this. But we shall also want to have neat and tidy unions $C_0\cup\ldots\cup C_n$. And to get $C_0\cup\ldots\cup C_n=[a,b]$ we are likely to have to use some half-open intervals. The reason we can be casual about this when looking at the ordinary Riemann integral is that $]a_i,a_{i+1}[$, $]a_i,a_{i+1}]$, $[a_i,a_{i+1}[$ and $[a_i,a_{i+1}]$ all have the same measure. But already with a Riemann-Stieltjes integral we have to be more careful.

Next, expressing sums in the form $\sum_{i=0}^n\ldots_i$ clutters the formulae and is a waste of a letter. So I am going to ask you to join me in using a formula

$$S_{\boldsymbol{t}}(f,\nu)=\sum_{(x,C)\in\boldsymbol{t}} f(x)\,\nu C$$

where $\boldsymbol{t}$ is a finite set of pairs (x,C). (Because I'll always be looking at commutative additions, it won't matter if we throw away any indication of the order in which one is supposed to add things up. Also we'll never want to count any term (x,C) more than once. The formula I wrote down above for the Riemann integral allows us to start with $a_0=x_0=a_1=x_1=a_2=a$, so that the sum begins with $f(a)(a-a)+f(a)(a-a)+\ldots$. This is just wasting time and we shan't miss it.)

Now for another change. The $f(x)$ and νC don't have to be real numbers. We just need things which we can multiply together and then add up. To make the algebra come right, we shall want the values $f(x)$ to belong to a linear space U, the values νC to belong to a linear space V (of course allowing $V=U$), and the values $f(x)\nu C$ to belong to a third linear space W, the 'multiplication' corresponding to a bilinear operator from $U\times V$ to W. Because I might want to play with this a bit, I will give it a name $\langle\,|\,\rangle$, and demand that

$$\langle u_1+u_2|v\rangle=\langle u_1|v\rangle+\langle u_2|v\rangle,\quad \langle u|v_1+v_2\rangle=\langle u|v_1\rangle+\langle u|v_2\rangle,$$

$$\langle\alpha u|v\rangle=\langle u|\alpha v\rangle=\alpha\langle u|v\rangle$$

for all u, u_1, $u_2\in U$, v, v_1, $v_2\in V$ and scalars α. So our guiding formula is

$$\int f\,d\nu\simeq S_{\boldsymbol{t}}(f,\nu)=\sum_{(x,C)\in\boldsymbol{t}}\langle f(x)|\nu C\rangle.$$

(What scalars? Easiest to take our linear spaces to be real. But the algebra is the same if you take complex scalars.)

⊕One of the things gauge integrals are good for is giving us definitions of vector-valued integrals of vector-valued functions over vector-valued measures. Making sense of these definitions will be another matter, of course.

1B Which pairs (x, C)? ⊕/⊗ Another thing about gauge integrals is that at every step we have an enormous amount of choice. There is a handful of standard examples, beginning with the Riemann integral, in which we have constructions which are clearly going to be a permanent part of mathematics. Elsewhere the formulations must be regarded as experimental.

However, the following pattern seems to cover enough cases to be a useful limitation to impose. For the rest of the week, we shall have a set X, a family $\mathcal{C}$ of subsets of X, and a set $Q \subseteq X \times \mathcal{C}$ of allowable pairs; and we shall look at the set T_Q of **tagged partitions** $\boldsymbol{t}$ which are finite subsets of Q and such that $C \cap C' = \emptyset$ whenever (x, C) and (x', C') are distinct members of $\boldsymbol{t}$.

> **Examples (i)** For the Riemann integral $\mathrm{R}\!\!\int_a^b$, X will be $[a, b]$, $\mathcal{C}$ will be the family of subintervals (open, closed or half-open) of $[a, b]$, and Q will be $\{(x, C) : x \in [a, b],\ C \in \mathcal{C},\ x \in \overline{C}\}$. So a Riemann sum $\sum_{i=0}^{n} f(x_i)(a_{i+1} - a_i)$ can be represented as $S_{\boldsymbol{t}}(f, \nu)$ where $\boldsymbol{t} = \{(x_i, [a_i, a_{i+1}[) : i \le n\} \cup \{(b, \{b\})\}$. (I put the last term in because one of the rules later, for the Riemann integral, will be that we shall have a preference for tagged partitions $\boldsymbol{t}$ such that $\bigcup_{(x,C)\in\boldsymbol{t}} C = [a, b]$.)
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> **(ii)** For the Birkhoff integral on a σ-finite measure space (X, Σ, μ), $\mathcal{C}$ will be the family of measurable sets of finite measure, and Q will be $\{(x, C) : x \in X,\ C \in \mathcal{C}\}$.
>
> **(iii)** For the Henstock integral on $\mathbb{R}$ (never mind if you haven't heard of this), X will be $\mathbb{R}$, $\mathcal{C}$ will be the family of bounded subintervals of $\mathbb{R}$, and Q will be $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, just as for the Riemann integral.

I have just mentioned the **spread** $\bigcup_{(x,C)\in\boldsymbol{t}} C$ of a tagged partition $\boldsymbol{t}$; I will denote this $H_{\boldsymbol{t}}$.

1C What do we mean by $\doteqdot$? Remember that in our formula

$$\int f\,d\nu \doteqdot S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} \langle f(x) | \nu C \rangle,$$

the sums $S_{\boldsymbol{t}}(f, \nu)$ belong to a linear space W. To make sense of approximation in W, we need a topology, and this had better be a Hausdorff topology so that the integrals are uniquely defined as limiting values of these sums. To make sense of the ordinary algebra of integration, the topology should be a linear space topology, so that $(w, w') \mapsto w + w'$ and $(\alpha, w) \mapsto \alpha w$ are continuous. In fact I think that in all the examples I'll be looking at, U and V also will be linear topological spaces, and the bilinear map $\langle\,|\,\rangle$ will be continuous.

1D When do we get a good approximation? Now we come to the fun bit. To make $S_{\boldsymbol{t}}(f, \nu)$ a good approximation to $\int f\,d\nu$, we are going to have to ask quite a lot of the tagged partition $\boldsymbol{t}$. Again, I am going to specialize to criteria of two particular forms. (I ought to remark that some very interesting gauge integrals have been introduced which cannot be expressed in this way.)

(a) We may require $\boldsymbol{t}$ to be δ-fine where $\delta \subseteq X \times \mathcal{P}X$ and $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. Of course the requirement in 1B that $\boldsymbol{t} \subseteq Q$ is just like this. The difference is that we are going to have a whole family Δ of **gauges** $\delta \subseteq X \times \mathcal{P}X$, and we are going to pick one of them after we have been told how good an approximation to the integral is required.

> **Examples (i)** For the Riemann integral on $[a, b]$, Δ will be the family of **uniform metric gauges** δ_η of the form
>
> $$\{(x, A) : x \in [a, b],\ A \subseteq [a, b],\ \operatorname{diam} A \le \eta\}$$
>
> for some $\eta > 0$. (So a Riemann sum $\sum_{i=0}^{n} f(x_i)(a_{i+1} - a_i)$ will correspond to a δ_η-fine tagged partition if $a_{i+1} - a_i \le \eta$ for every i.)
>
> **(ii)** For the Birkhoff integral on a σ-finite measure space (X, Σ, μ), Δ will be the family of gauges of the form
>
> $$\delta_{\mathcal{E}} = \{(x, A) : \text{there is an } E \in \mathcal{E} \text{ including } A \cup \{x\}\}$$
>
> where $\mathcal{E}$ is a countable partition of X into measurable sets.
>
> **(iii)** For the Henstock integral on $\mathbb{R}$, Δ will be the family of **neighbourhood gauges** of the form
>
> $$\delta_{\boldsymbol{G}} = \{(x, A) : x \in \mathbb{R},\ A \subseteq G_x\}$$

where $\boldsymbol{G} = \langle G_x \rangle_{x\in\mathbb{R}}$ and G_x is an open set containing x for every $x \in \mathbb{R}$.

Note that in all these three examples, Δ is downwards-directed in the sense that whenever δ_1, $\delta_2 \in \Delta$ there is a $\delta \in \Delta$ such that $\delta \subseteq \delta_1 \cap \delta_2$.

(i) $\delta_\eta \cap \delta_{\eta'} = \delta_{\min(\eta,\eta')}$.

(ii) $\delta_{\mathcal{E}} \cap \delta_{\mathcal{E}'} = \delta_{\{E\cap E' : E\in\mathcal{E}, E'\in\mathcal{E}'\}}$.

(iii) $\delta_{\langle G_x\rangle_{x\in\mathbb{R}}} \cap \delta_{\langle G'_x\rangle_{x\in\mathbb{R}}} = \delta_{\langle G_x\cap G'_x\rangle_{x\in\mathbb{R}}}$.

This is something we can work round if we have to, but I am going to try to ensure that all my families Δ of gauges have this property.

(b) You will see that the empty set will always be a δ-fine tagged partition, so we are certainly going to have to ask more. The second type of criterion I will use will concern the spreads $H_{\boldsymbol{t}}$ rather than the individual members of $\boldsymbol{t}$. The idea is that we are going to want the residues $X \setminus H_{\boldsymbol{t}}$ to be small (empty if possible, but often it's not), and also, in many cases, of particular shapes. To express this, I will suppose that we have a family $\mathfrak{R}$ of collections $\mathcal{R}$ of 'residual' subsets of X, and that for $\mathcal{R} \subseteq \mathcal{P}X$ a tagged partition $\boldsymbol{t}$ will be '$\mathcal{R}$-filling' if $X \setminus H_{\boldsymbol{t}} \in \mathcal{R}$. It will be convenient later to insist from the beginning that $\emptyset \in \mathcal{R}$ for every $\mathcal{R} \in \mathfrak{R}$.

Examples (i) For the Riemann integral on $[a, b]$, we are going to demand $H_{\boldsymbol{t}} = [a, b]$ (this corresponds to the requirements $a = a_0$, $a_{n+1} = b$), so that $\mathfrak{R}$ will have only one member, $\mathcal{R} = \{\emptyset\}$.

(ii) For the classical integral on a σ-finite (X, Σ, μ), we are going to have

$$\mathfrak{R} = \{\mathcal{R}_{F\epsilon} : F \in \Sigma,\ \mu F < \infty,\ \epsilon > 0\},$$

where $\mathcal{R}_{F\epsilon} = \{E : E \in \Sigma,\ \mu(E \cap F) \le \epsilon\}$. Thus $\boldsymbol{t}$ will be $\mathcal{R}_{F\epsilon}$-filling if $H_{\boldsymbol{t}}$ covers F, up to a set of measure at most ϵ.

(iii) For the Henstock integral on $\mathbb{R}$, we shall have

$$\mathfrak{R} = \{\mathcal{R}_C : C \in \mathcal{C}\},$$

where

$$\mathcal{R}_C = \{\mathbb{R} \setminus C' : C \subseteq C' \in \mathcal{C}\} \cup \{\emptyset\}$$

for $C \in \mathcal{C}$. Thus $\boldsymbol{t}$ will be $\mathcal{R}_C$-filling if $H_{\boldsymbol{t}}$ is an interval including C.

Once again, in these examples, and I will do my best to keep it so, $\mathfrak{R}$ is downwards-directed.

(i) The singleton set $\{\{\emptyset\}\}$ is surely downwards-directed.

(ii) $\mathcal{R}_{F\epsilon} \cap \mathcal{R}_{F'\epsilon'} \supseteq \mathcal{R}_{F\cup F',\min(\epsilon,\epsilon')}$.

(iii) $\mathcal{R}_{C_1} \cap \mathcal{R}_{C_2} \supseteq \mathcal{R}_C$ if $C \supseteq C_1 \cup C_2$.

1E Compatibility Let me run through the structure again as it stands. We have a set X, a family $\mathcal{C}$ of subsets of X, a set $Q \subseteq X \times \mathcal{C}$, the corresponding set T_Q of tagged partitions $\boldsymbol{t}$, a downwards-directed family Δ of gauges, and a downwards-directed family $\mathfrak{R}$ of residual classes. Next, we have linear spaces U, V and W, a bilinear function $\langle\,|\,\rangle : U \times V \to W$, and a Hausdorff linear space topology on W; and finally we have functions $f : X \to U$ and $\nu : \mathcal{C} \to V$ and sums $S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} \langle f(x)|\nu C\rangle$. I am going to want to say

(*) $\int f\,d\nu = w$ if for every open set $G \subseteq W$ containing w there are a $\delta \in \Delta$ and an $\mathcal{R} \in \mathfrak{R}$ such that $S_{\boldsymbol{t}}(f, \nu) \in G$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling.

But of course this will be out of the question unless, for every $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$, there *is* a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}$; because otherwise (*) will be vacuously satisfied for every w. So we need to be sure that we have compatible families Δ and $\mathfrak{R}$ in this sense.

Examples (i) For the Riemann integral, we just have to observe that if $\eta > 0$ then $n = \lceil \frac{b-a}{\eta} \rceil$ is finite, so we can take $a_i = \min(b, a + i\eta)$ for $i \le n + 1$.

(ii) For the classical integral, given $\delta = \delta_{\mathcal{E}}$ and $\mathcal{R} = \mathcal{R}_{F\epsilon}$, there is a finite subset $\mathcal{E}_0$ of $\mathcal{E} \setminus \{\emptyset\}$ such that $\mu(F \setminus \bigcup \mathcal{E}_0) \le \epsilon$; for $E \in \mathcal{E}_0$ choose $x_E \in E$, and set $\boldsymbol{t} = \{(x_E, F \cap E) : E \in \mathcal{E}_0\}$.

(iii) For the Henstock integral, given a neighbourhood gauge $\delta = \delta_{\langle G_x\rangle_{x\in\mathbb{R}}}$ and a residual class $\mathcal{R}_C$, take a, b such that $C \subseteq [a,b]$ and set

$$A = \{c : c \in [a,b], \text{ there is a } \delta\text{-fine } \boldsymbol{t} \text{ such that } H_{\boldsymbol{t}} = [a,c]\}.$$

Start with $\boldsymbol{t} = \{(a,\{a\})\}$ to see that $a \in A$, so A is a non-empty subset of $[a,b]$ and has a supremum d say. There is an $\eta > 0$ such that $[d-\eta, d+\eta] \subseteq G_d$. There is a $c \in [d-\eta, d] \cap C$; let $\boldsymbol{t}$ be a δ-fine partition such that $H_{\boldsymbol{t}} = [a,c]$. If $c = b$ then $\boldsymbol{t}$ is already δ-fine and $\mathcal{R}_C$-filling. Otherwise, set $\boldsymbol{t}' = \boldsymbol{t} \cup \{(d,]c, \min(d+\eta, b)]\}$. Then $\boldsymbol{t}'$ is a δ-fine partition and $H_{\boldsymbol{t}'} = [a, \min(d+\eta, b)]$. So $\min(d+\eta, b) \in C$; as $\min(d+\eta, b) \le d \le b$, we have $d = b = \min(d+\eta, b)$ and $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}_C$-filling.

$\bigoplus$**Remark** Note that, at least in the definition, we don't need to say anything about 'measurability' of the function f, and the function ν doesn't have to be defined on any more sets than those in $\mathcal{C}$. So, for instance, with the Henstock-Stieltjes integral, we can take any real-valued function ν defined on bounded intervals of $\mathbb{R}$ with the property that if C and C' are adjoining disjoint intervals then $\nu(C \cup C') = \nu C + \nu C'$. For the definition, we don't even have to say that, though if it isn't true we are in danger of having few ν-integrable functions. But we certainly don't have to trouble ourselves with countable additivity for the time being.

1F Theorem (a) Under the conditions of 1E, $\int$ is bilinear, in the sense that

$$\int (f_1 + f_2) d\nu \text{ exists and is equal to } \int f_1\, d\nu + \int f_2\, d\nu$$

whenever f_1, $f_2 : X \to U$ and $\nu : \mathcal{C} \to V$ are such that $\int f\, d\nu_1$ and $\int f\, d\nu_2$ are defined,

$$\int f\, d(\nu_1 + \nu_2) \text{ exists and is equal to } \int f\, d\nu_1 + \int f\, d\nu_2$$

whenever $f : X \to U$ and ν_1, $\nu_2 : \mathcal{C} \to V$ are such that $\int f\, d\nu_1$ and $\int f\, d\nu_2$ are defined,

$$\int (\alpha f) d\nu \text{ and } \int f\, d(\alpha\nu) \text{ exist and are equal to } \alpha \int f\, d\nu$$

whenever $f : X \to U$ and $\nu : \mathcal{C} \to V$ are such that $\int f\, d\nu$ is defined, and $\alpha \in {\mathbb{R} \atop \mathbb{C}}$.

(b) Suppose now that we have linear spaces U_1 and V_1, a topological linear space W_1, a bilinear map $\langle\,|\,\rangle_1 : U_1 \times V_1 \to W_1$, a continuous linear operator $T : W \to W_1$ and linear operators $\hat{T} : U \to U_1$ and $\tilde{T} : V \to V_1$ such that $T(\langle u|v\rangle) = \langle \hat{T}u | \tilde{T}v\rangle_1$ for all $u \in U$ and $v \in V$. Then $\int (\hat{T}f) d(\tilde{T}\nu) = T(\int f d\nu)$ whenever $f : X \to U$ and $\nu : \mathcal{C} \to V$ are such that $\int f d\nu$ is defined in W.

proof (a) This is because all the functions $S_{\boldsymbol{t}}$ are bilinear, and addition and scalar multiplication in W are continuous.

(b) Similarly, $T(S_{\boldsymbol{t}}(f,\nu)) = S_{\boldsymbol{t}}(\hat{T}f, \tilde{T}\nu)$ for every $\boldsymbol{t}$, and T is continuous.

Remark The statement of (b) looks rather involved. The easy examples are when one of U, V is one-dimensional and we can identify W with the other. More generally, there are further examples when W is a tensor product of U and V.

2 Saks-Henstock indefinite integrals

2A Resumé We are working with a set X, a family $\mathcal{C}$ of subsets of X, a set $Q \subseteq X \times \mathcal{C}$, the set

$$T_Q = \{\boldsymbol{t} : \boldsymbol{t} \text{ is a finite subset of } Q,\ C \cap C' = \emptyset \text{ for all distinct } (x,C),\ (x',C') \in \boldsymbol{t}\},$$

a downwards-directed family Δ of gauges $\delta \subseteq X \times \mathcal{P}X$, and a downwards-directed family $\mathfrak{R}$ of residual classes $\mathcal{R} \subseteq \mathcal{P}X$, all containing $\emptyset$. Off-stage, we have linear spaces U, V and W, a bilinear operator $\langle\,|\,\rangle : U \times V \to W$, and a Hausdorff linear space topology on W.

2B Subdivisions I am going to add some further conditions to the list of properties of X, $\mathcal{C}$, Q, Δ and $\mathfrak{R}$.

(a) First, $\mathfrak{R}$ should not only be downwards-directed, but should also have the following property:

for every $\mathcal{R} \in \mathfrak{R}$ there is a $\mathcal{R}' \in \mathfrak{R}$ such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint.

(Since we are still going to insist that $\emptyset$ should belong to every member of $\mathfrak{R}$, this will imply that $\mathcal{R}' \subseteq \mathcal{R}$. It's important that we look only at the unions of disjoint sets A, B here. In cases like the Henstock integral, where we have sharp restrictions on the shapes of members of residual sets – residual sets are always complements of bounded intervals – the condition here will be trivially satisfied.)

(b) Let $\mathcal{E}_0$ be the family of subsets of X expressible in the form $E = \bigcup \mathcal{C}_0$ where $\mathcal{C}_0 \subseteq \mathcal{C}$ is a finite disjoint set. (So $H_{\boldsymbol{t}} \in \mathcal{E}_0$ whenever $\boldsymbol{t} \in T_Q$.) Then we want

(i) whenever C, $C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C' \in \mathcal{E}_0$,

(ii) whenever $\mathcal{R} \in \mathfrak{R}$ and $E \in \mathcal{E}_0$, there is an $E' \in \mathcal{E}_0$ such that $E \subseteq E'$ and $X \setminus E' \in \mathcal{R}$.

(c) Finally, we shall assume that Δ is downwards-directed and we have a kind of super-compatibility, as follows:

whenever $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$, there is a δ-fine $\boldsymbol{t} \in T$ such that $H_{\boldsymbol{t}} \subseteq C$ and $C \setminus H_{\boldsymbol{t}} \in \mathcal{R}$.

(d) It is easy to check that all of these conditions are satisfied by the three leading examples I've been examining. (Of course the arguments for (c) are based on the arguments for simple compatibility in 1E.) It is also not hard to confirm that (a)-(c) here (together, of course, with the list of conditions in 2A) imply that Δ and $\mathfrak{R}$ are compatible (see FREMLIN 03, 481Hf), so that we have a well-defined notion of integral.

***2C** For completeness, I mention two technical points.

Lemma Suppose that X, $\mathcal{C}$, Q, T_Q, Δ and $\mathfrak{R}$ satisfy the conditions of 2A-2B.

(a) For any $\mathcal{R} \in \mathfrak{R}$ there is a sequence $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$ in $\mathfrak{R}$ such that whenever $J \subseteq \mathbb{N}$ is finite, $A_i \in \mathcal{R}_i$ for $i \in J$ and $\langle A_i \rangle_{i \in J}$ is disjoint then $\bigcup_{i \in J} A_i \in \mathcal{R}$.

(b) Let $\mathcal{E}$ be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$. Then for any $E \in \mathcal{E}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ there is a δ-fine $\boldsymbol{t} \in T_Q$ such that $H_{\boldsymbol{t}} \subseteq E$ and $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}$.

proof (a) FREMLIN 03, 481He

(b) FREMLIN 03, 482Aa.

2D Saks-Henstock Lemma The point about the conditions in 2B is that they are satisfied by a decent proportion of the current crop of leading examples, and they are sufficient to lead to a useful kind of indefinite integral.

Theorem Suppose that X, $\mathcal{C}$, Q, T_Q, Δ, $\mathfrak{R}$, U, V, W and $\langle\,|\,\rangle$ satisfy the conditions of 2A-2B, and moreover W is complete in its given linear space topology. Let $\mathcal{E}$ be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$. If $f : X \to U$ and $\nu : \mathcal{C} \to V$ are functions, then $\int f\,d\nu$ is defined in the sense of 1E iff there is a function $F : \mathcal{E} \to W$ such that

(α) F is additive[1]

and for every neighbourhood G of 0 in W there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that

(β) $S_{\boldsymbol{t}}(f,\nu) - F(H_{\boldsymbol{t}}) \in G$ for every δ-fine $\boldsymbol{t} \in T$,

(γ) $F(E) \in G$ whenever $E \in \mathcal{E} \cap \mathcal{R}$.

In this case, F is uniquely determined by the conditions (α)-(γ), and $F(X) = \int f d\nu$.

proof (a) Suppose that $\int f\,d\nu$ is defined.

(i) I had better begin by showing that while the hypotheses allow $\emptyset \in \mathcal{C}$ and $\nu\emptyset \neq 0$, this will not upset the result. In fact for any neighbourhood G of 0 in W there is a $\delta \in \Delta$ such that $S_{\boldsymbol{t}}(f,\nu) \in G$ whenever $\boldsymbol{t} \in T$ is δ-fine and $H_{\boldsymbol{t}} = \emptyset$. **P** Take a neighbourhood G_1 of 0 in W, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that $G_1 - G_1 \subseteq G$ and $S_{\boldsymbol{s}}(f,\nu) - \int f\,d\nu \in G_1$ whenever $\boldsymbol{s} \in T_Q$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t} \in T_Q$ is δ-fine and $H_{\boldsymbol{t}} = \emptyset$, take any δ-fine $\mathcal{R}$-filling $\boldsymbol{s} \in T$, and consider $\boldsymbol{s}' = \boldsymbol{s} \setminus \boldsymbol{t}$, $\boldsymbol{s}'' = \boldsymbol{s} \cup \boldsymbol{t}$. Because $H_{\boldsymbol{s}} \cap H_{\boldsymbol{t}} = \emptyset$, both $\boldsymbol{s}'$ and $\boldsymbol{s}''$ belong to T_Q; both are δ-fine; and because $H_{\boldsymbol{s}'} = H_{\boldsymbol{s}''} = H_{\boldsymbol{s}}$, both are $\mathcal{R}$-filling. So

$$\begin{aligned} S_{\boldsymbol{t}}(f,\nu) &= S_{\boldsymbol{s}''}(f,\nu) - S_{\boldsymbol{s}'}(f,\nu) = (S_{\boldsymbol{s}''}(f,\nu) - \int f d\nu) - (S_{\boldsymbol{s}'}(f,\nu) - \int f d\nu) \\ &\in G_1 - G_1 \subseteq G, \end{aligned}$$

[1]that is, $F(E \cup E') = F(E) \cup F(E')$ whenever E, $E' \in \mathcal{E}$ are disjoint.

as required. **Q**

(ii) For $E \in \mathcal{E}$, write T'_E for the set of those $\boldsymbol{t} \in T_Q$ such that, for every $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. For any $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and finite $\mathcal{D} \subseteq \mathcal{E}$ there is a δ-fine $\boldsymbol{t} \in \bigcap_{E \in \mathcal{D}} T'_E$ such that $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}$ for every $E \in \mathcal{D}$. **P** Let $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that whenever $J \subseteq \mathbb{N}$ is finite, $A_i \in \mathcal{R}_i$ for $i \in J$ and $\langle A_i \rangle_{i \in J}$ is disjoint then $\bigcup_{i \leq n} A_i \in \mathcal{R}$ (2Ca). Let $\mathcal{E}^*$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{D}$, and enumerate the atoms of $\mathcal{E}^*$ as $\langle E_i \rangle_{i<n}$. By 2Cb, there is for each $i < n$ a δ-fine $\boldsymbol{s}_i \in T_Q$ such that $H_{\boldsymbol{s}_i} \subseteq E_i$ and $E_i \setminus H_{\boldsymbol{s}_i} \in \mathcal{R}_i$. Set $\boldsymbol{t} = \bigcup_{i<n} \boldsymbol{s}_i$. If $E \in \mathcal{D}$ then $E = \bigcup_{i \in J} E_i$ for some $J \subseteq n$. For any $(x, C) \in \boldsymbol{t}$, there is some $i < n$ such that $(x, C) \in \boldsymbol{s}_i$ and $C \subseteq E_i$, so that $C \subseteq E$ if $i \in J$, $C \cap E = \emptyset$ otherwise; thus $\boldsymbol{t} \in T'_E$. Moreover, $E \setminus H_{\boldsymbol{t}} = \bigcup_{i \in J}(E_i \setminus H_{\boldsymbol{s}_i})$ belongs to $\mathcal{R}$. **Q**

(iii) We therefore have a filter $\mathcal{F}^*$ on T generated by sets of the form

$$T_{E\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T'_E \text{ is } \delta\text{-fine, } E \setminus H_{\boldsymbol{t}} \in \mathcal{R}\}$$

as δ runs over Δ, $\mathcal{R}$ runs over $\mathfrak{R}$ and E runs over $\mathcal{E}$. For $\boldsymbol{t} \in T_Q$ and $E \subseteq X$ set $\boldsymbol{t}_E = \{(x, C) : (x, C) \in \boldsymbol{t},\, C \subseteq E\}$. Now $F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ is defined in W for every $E \in \mathcal{E}$. **P** For any neighbourhood G of 0 in W, there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that $\int f\,d\nu - S_{\boldsymbol{t}}(f, \nu) \in G$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T_Q$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}'$. If $\boldsymbol{t}$, $\boldsymbol{t}'$ belong to $T_{E,\delta,\mathcal{R}'} = T_{X \setminus E,\delta,\mathcal{R}'}$, then set

$$\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t}',\, C \subseteq E\} \cup \{(x, C) : (x, C) \in \boldsymbol{t},\, C \cap E = \emptyset\}.$$

Then $\boldsymbol{s} \in T_E$ is δ-fine, and also $E \setminus H_{\boldsymbol{s}} = E \setminus H_{\boldsymbol{t}'}$, $(X \setminus E) \setminus H_{\boldsymbol{s}} = (X \setminus E) \setminus H_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$; so their union $X \setminus H_{\boldsymbol{s}}$ belongs to $\mathcal{R}$, and $\boldsymbol{s}$ is $\mathcal{R}$-filling. Accordingly

$$S_{\boldsymbol{t}_E}(f, \nu) - S_{\boldsymbol{t}'_E}(f, \nu) = S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{s}}(f, \nu) \in G - G.$$

As G is arbitrary and W is complete, this is enough to show that $\lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ is defined. **Q**

(iv) $F(\emptyset) = 0$. **P** Let G be a neighbourhood of 0 in W. By (i), there is a $\delta \in \Delta$ such that $S_{\boldsymbol{t}}(f, \nu) \in G$ whenever $\boldsymbol{t} \in T$ is δ-fine and $H_{\boldsymbol{t}} = \emptyset$. Since $\{\boldsymbol{t} : \boldsymbol{t} \text{ is } \delta\text{-fine}\}$ belongs to $\mathcal{F}^*$,

$$F(\emptyset) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_\emptyset}(f, \nu) \in \overline{G};$$

as G is arbitrary, $F(\emptyset) = 0$. **Q**

If E, $E' \in \mathcal{E}$, then

$$S_{\boldsymbol{t}_{E \cup E'}}(f, \nu) + S_{\boldsymbol{t}_{E \cap E'}}(f, \nu) = S_{\boldsymbol{t}_E}(f, \nu) + S_{\boldsymbol{t}_{E'}}(f, \nu)$$

for every $\boldsymbol{t} \in T'_E \cap T'_{E'}$; as $T'_E \cap T'_{E'}$ belongs to $\mathcal{F}^*$,

$$F(E \cup E') + F(E \cap E') = F(E) + F(E').$$

Since $F(\emptyset) = 0$, $F(E \cup E') = F(E) + F(E')$ whenever $E \cap E' = \emptyset$, and F is additive.

(v) Now suppose that G is a neighbourhood of 0 in W. Let G_1 be a neighbourhood of 0 in W such that $G_1 - G_1 + G_1 - G_1 \subseteq G$. Let $\delta \in \Delta$ and $\mathcal{R}^* \in \mathfrak{R}$ be such that $S_{\boldsymbol{t}}(f, \nu) - \int f\,d\nu \in G_1$ for every δ-fine, $\mathcal{R}^*$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}^*$ for all disjoint A, $B \in \mathcal{R}$.

($\boldsymbol{\alpha}$) If $\boldsymbol{t} \in T$ is δ-fine, then $S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}}) \in G$. **P** There is a δ-fine $\boldsymbol{s} \in T$ such that
$S_{\boldsymbol{s}}(f, \nu) - \int f\,d\nu \in G_1$,
for every $(x, C) \in \boldsymbol{s}$, either $C \subseteq H_{\boldsymbol{t}}$ or $C \cap H_{\boldsymbol{t}} = \emptyset$,
$(X \setminus H_{\boldsymbol{t}}) \setminus H_{\boldsymbol{s}} \in \mathcal{R}$, $H_{\boldsymbol{t}} \setminus H_{\boldsymbol{s}} \in \mathcal{R}$,
$\sum_{(x,C) \in \boldsymbol{s}, C \subseteq H_{\boldsymbol{t}}} \langle f(x) | \nu C \rangle - F(H_{\boldsymbol{t}}) \in G_1$
because the set of $\boldsymbol{s}$ with these properties belongs to $\mathcal{F}^*$. Now, setting $\boldsymbol{s}_1 = \{(x, C) : (x, C) \in \boldsymbol{s},\, C \subseteq H_{\boldsymbol{t}}\}$ and $\boldsymbol{t}' = \boldsymbol{t} \cup (\boldsymbol{s} \setminus \boldsymbol{s}_1)$, $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}^*$-filling, like $\boldsymbol{s}$, so

$$\begin{aligned} S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}}) &= S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{s}_1}(f, \nu) + S_{\boldsymbol{s}_1}(f, \nu) - F(H_{\boldsymbol{t}}) \\ &= S_{\boldsymbol{t}'}(f, \nu) - S_{\boldsymbol{s}}(f, \nu) + S_{\boldsymbol{s}_1}(f, \nu) - F(H_{\boldsymbol{t}}) \\ &= (S_{\boldsymbol{t}'}(f, \nu) - \int f d\nu) - (S_{\boldsymbol{s}}(f, \nu) - \int f d\nu) + (S_{\boldsymbol{s}_1}(f, \nu) - F(H_{\boldsymbol{t}})) \\ &\in G_1 - G_1 + G_1 \subseteq G. \end{aligned}$$ **Q**

(β) If $E \in \mathcal{E} \cap \mathcal{R}$ then $F(E) \in G$. **P** There is a $\boldsymbol{t}$ such that
$\boldsymbol{t} \in T'_E$ is δ-fine,
$E \setminus H_{\boldsymbol{t}}$ and $(X \setminus E) \setminus H_{\boldsymbol{t}}$ both belong to $\mathcal{R}$,
$S_{\boldsymbol{t}_E}(f, \nu) - F(E) \in G_1$
(once again, the set of candidates belongs to $\mathcal{F}^*$, so is not empty). In this case $\boldsymbol{t}$ and $\boldsymbol{t}_{X \setminus E}$ are both $\mathcal{R}^*$-filling and δ-fine, so

$$\begin{aligned} F(E) &= S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}_{X\setminus E}}(f, \nu) - S_{\boldsymbol{t}_E}(f, \nu) + F(E) \\ &= (S_{\boldsymbol{t}}(f, \nu) - \int f d\nu) - (S_{\boldsymbol{t}_{X\setminus E}}(f, \nu) - \int f d\nu) - (S_{\boldsymbol{t}_E}(f, \nu) - F(E)) \\ &\in G_1 - G_1 - G_1 \subseteq G. \ \mathbf{Q} \end{aligned}$$

Thus F has all the required properties.

(vi) To see that F is unique, suppose that $F' : \mathcal{E} \to \mathbb{R}$ is another function with the same properties, and take $E \in \mathcal{E}$ and a neighbourhood G of 0 in W. Then there are δ, $\delta' \in \Delta$ and $\mathcal{R}$, $\mathcal{R}' \in \mathfrak{R}$ such that
$S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}}) \in G$ for every δ-fine $\boldsymbol{t} \in T$,
$S_{\boldsymbol{t}}(f, \nu) - F'(H_{\boldsymbol{t}}) \in G$ for every δ-fine $\boldsymbol{t} \in T$,
$F(E) \in G$ whenever $E \in \mathcal{E} \cap \mathcal{R}$,
$F'(E) \in G$ whenever $E \in \mathcal{E} \cap \mathcal{R}'$.
Now taking $\delta'' \in \Delta$ such that $\delta'' \subseteq \delta \cap \delta'$, and $\mathcal{R}'' \in \mathfrak{R}$ such that $\mathcal{R}'' \subseteq \mathcal{R} \cap \mathcal{R}'$, there is a δ''-fine $\boldsymbol{t} \in T$ such that $H_{\boldsymbol{t}} \subseteq E$ and $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}''$. In this case

$$S_{\boldsymbol{t}}(f, \nu) - F(E) = S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}}) - F(E \setminus H_{\boldsymbol{t}})$$

(because F is additive)

$$\in G - G$$

because $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}'' \subseteq \mathcal{R}$ and $\boldsymbol{t}$ is δ''-fine, therefore δ-fine. Similarly, $S_{\boldsymbol{t}}(f, \nu) - F'(E) \in G - G$ so $F'(E) - F(E) \in G - G - G + G$. As G and E are arbitrary, $F = F'$.

(b) In the other direction, suppose that $F : \mathcal{E} \to W$ has the properties (α)-(γ). Take a neighbourhood G of 0 in W. Let G_1 be a neighbourhood of 0 in W such that $G_1 - G_1 \subseteq G$. Then there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}}) \in G_1, \quad F(E) \in G_1$$

whenever $\boldsymbol{t} \in T_Q$ is δ-fine and $E \in \mathcal{E} \cap \mathcal{R}$. Now suppose that $\boldsymbol{t} \in T_Q$ is δ-fine and $\mathcal{R}$-filling. Then

$$S_{\boldsymbol{t}}(f, \nu) - F(X) = (S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}})) - F(X \setminus H_{\boldsymbol{t}}) \in G_1 - G_1 \subseteq G.$$

As G is arbitrary, $\int f d\nu$ is defined and equal to $F(X)$.

Remark In this context, I will call F the **Saks-Henstock indefinite integral** of f with respect to ν. (**Warning!** Do not suppose it is like any other kind of indefinite integral you have seen.)

3 Why bother?

⊕3A What are gauge integrals good at? (a) With practice, you will I hope find that you can quite often devise gauge integrals which will at least agree with your favourite classical integrals (whether scalar- or vector-valued) on the functions the classical integrals will integrate. Thus the Henstock integral $\not{\!\!\!\int}$ on $\mathbb{R}$, by which I mean the gauge integral got from the configuration of Example (iii) and taking νC to be the length of C for every bounded interval $C \subseteq \mathbb{R}$, extends the Lebesgue integral and various kinds of improper Riemann integral. (This takes a bit of proving, I have to admit. See Fremlin 03, §483.) Typically, gauge integrals go farther. But of course one rather likes the idea of integrating more functions.

(b) While the limiting process described in (*) of 1E is a complex one, and a good deal more difficult than a simple limit of sequences, we can get a definition of an integral from just this one process. Ordinary approaches to either Riemann or Lebesgue integration ask for repeated limit stages which have to be done in just the right order. In the Riemann integral, we quickly find ourselves looking at 'improper' Riemann integrals, which are limits of Riemann integrals; while in the Lebesgue integral, we use one limit process to measure sets, and a second to integrate functions.

(c) Gauge integrals are good at integrating derivatives. For instance:

Theorem If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $\gamma = \lim_{x\to\infty} f(x) - \lim_{x\to-\infty} f(x)$ is defined in $\mathbb{R}$, then $\not\int f'$ is defined and equal to γ.

proof In 2D, take $F(\bigcup \mathcal{C}_0) = \sum_{C\in\mathcal{C}_0} f(\sup C) - f(\inf C)$, $F(\mathbb{R} \setminus \bigcup \mathcal{C}_0) = \gamma - F(\bigcup \mathcal{C}_0)$ whenever $\mathcal{C}_0 \subseteq \mathcal{C} \setminus \{\emptyset\}$ is finite and disjoint. [We need to check that this is well-defined, that is, we get the same sum when $\bigcup \mathcal{C}_0 = \bigcup \mathcal{C}_1$.] To get $|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f, \nu)| \le \epsilon$ for $\delta_{\boldsymbol{G}}$-fine $\boldsymbol{t}$, take G_x small enough so that

$$|f(b) - f(a) - (b - a)f'(x)| \le \tfrac{\epsilon|b-a|}{\pi(1+a^2+b^2)}$$

whenever $a \le x \le b$ and a, b both belong to G_x. To get $|F(\mathbb{R} \setminus C')| \le \epsilon$ whenever C' is an interval including C, we just have to take C big enough.

⊗3B What are gauge integrals bad at? Rather a lot of things. Subspaces don't work well. (You should *not* think of the function F of 2D as an ordinary indefinite integral; $F(E)$ there does not have to have much to do with $\int f \times \chi E \, d\nu$.) Nor do products. There are convergence theorems (see 3J below), but it takes hard work to make them give the full value of the classical theorems, and often the easiest way to prove them is to find a classical integral associated with your gauge integral, and use the convergence theorems for that. (Thus for a non-negative function $f : \mathbb{R} \to [0, \infty[$, $\not\int f$ is equal to the ordinary Lebesgue integral of f if either is defined.)

⊗3C Riemann-Stieltjes integrals If we take the tagged-partition structure of the Riemann integral, as defined in Example (i), but use a functional ν with an atom, then we can have problems. Taking $\mathcal{C}$ to be the family of subintervals of $[0, 1]$, set $\nu C = 1$ if $\frac{1}{2} \in C$, 0 otherwise. Then, for any $f : [0, 1] \to \mathbb{R}$ and $\boldsymbol{t} \in T_Q$,

$$S_{\boldsymbol{t}}(f, \nu) = f(x) \text{ if } (x, C) \in \boldsymbol{t} \text{ and } \frac{1}{2} \in C,$$

$$= 0 \text{ if } \frac{1}{2} \notin H_{\boldsymbol{t}}.$$

So if $\boldsymbol{t}$ is δ_η-fine, where δ_η is the uniform metric gauge $\{(x, A) : x \in [0, 1],\ A \subseteq [0, 1],\ \operatorname{diam} A \le \eta\}$, and $\{\emptyset\}$-filling, we shall have $S_{\boldsymbol{t}}(f, \nu) = f(x)$ for some x belonging to the closure of an interval C, of diameter at most η, containing $\frac{1}{2}$. This means that the gauge integral $\int f d\nu$ will be defined, and equal to α, iff

for every $\epsilon > 0$ there is an $\eta \in \left]0, \frac{1}{2}\right[$ such that $|f(x) - \alpha| \le \epsilon$ whenever there is a δ_η-fine tagged partition $\boldsymbol{t}$ with $H_{\boldsymbol{t}} = [0, 1]$ and an interval C such that $(x, C) \in \boldsymbol{t}$ and $\frac{1}{2} \in C$.

But of course for any $x \in [\frac{1}{2} - \eta, \frac{1}{2} + \eta]$ we can find such $\boldsymbol{t}$ and C; just take C to be $[\frac{1}{2} - \eta, \frac{1}{2} + \eta]$ and fill $\left[0, \frac{1}{2} - \eta\right[$ and $\left]\frac{1}{2} + \eta, 1\right]$ by the method in 1E. So we find that

$\int f d\nu = \alpha$ iff for every $\epsilon > 0$ there is an $\eta \in \left]0, \frac{1}{2}\right[$ such that $|f(x) - \alpha| \le \epsilon$ whenever $x \in [\frac{1}{2} - \eta, \frac{1}{2} + \eta]$;

that is, iff f is continous at $\frac{1}{2}$ and $f(\frac{1}{2}) = \alpha$.

Next, given that f is continous at $\frac{1}{2}$, its Saks-Henstock indefinite integral F is defined by saying that

$$F(E) = f(\frac{1}{2}) \text{ if } \frac{1}{2} \in E,$$

$$= 0 \text{ otherwise.}$$

(For this function satisfies (α)-(γ) of 2D.) But if we look at $\int f \times \chi E \, d\nu$, this will be defined only if $f \times \chi E$ is continuous at $\frac{1}{2}$; that is, either $f(\frac{1}{2}) = 0$ or $\frac{1}{2}$ does not lie on the boundary of E. In particular, if $f = \chi X$ is the constant function with value 1, and $C = \left[0, \frac{1}{2}\right[$ or $[0, \frac{1}{2}]$, then $\int f \times \chi C \, d\nu$ is undefined, although $F(C)$ is defined and is 0 or 1 respectively.

In the language of Riemann-Stieltjes integration, the functional ν here corresponds to a jump function $g = \chi[0, \frac{1}{2}]$ or $g = \chi\left[0, \frac{1}{2}\right[$; and Riemann-Stieltjes integrals $\int f dg$ are problematic when f and g share a point of discontinuity.

3D Multipliers The phenomenon in 3C is related to a general question. For any notion of integration $\int \ldots d\nu$, we can ask: for which functions g will it be true that $\int g \times f\, d\nu$ is defined whenever $\int f d\nu$ is defined? We say that such a function g is a **multiplier** for the integral. In the context of this note, where we have functions $f : X \to U$ where U is a vector space, the natural multipliers to look for are real-valued functions g defined on X so that we can interpret $g \times f$ with the formula $(g \times f)(x) = g(x)f(x)$. In 3N, for Riemann integration with respect to the point mass ν, the multipliers will be just the functions continuous at $\frac{1}{2}$. For ordinary Riemann integration with respect to length, the multipliers turn out to be the Riemann integrable functions. For the Lebesgue integral on $\mathbb{R}^n$, the multipliers are the essentially bounded Lebesgue measurable functions. For gauge integrals they can be difficult to characterize. But for the Henstock integral on $\mathbb{R}$, we have an elegant result: $g : \mathbb{R} \to \mathbb{R}$ is a multiplier iff there is a function g_0 of bounded variation such that $g = g_0$ almost everywhere in Lebesgue's sense. (Integration by parts works for the Henstock integral.) And these are still multipliers for the vector-valued case, at least when $U = W$ is a complete locally convex space. (Of course I am taking it for granted that $V = \mathbb{R}$, νC is the length of C for every bounded interval $C \subseteq \mathbb{R}$, and that $\langle u|\alpha\rangle = \alpha u$ for $u \in U$ and $\alpha \in \mathbb{R}$.)

3E Returning to the questions raised in 3B, there are things which can be done.

The Henstock integral in $\mathbb{R}^2$ (a) Example (iii) of 1B/1D/1E has a two-dimensional form, as follows.

> **(iv)** Take $X = \mathbb{R}^2$, $\mathcal{C}$ the set of rectangles $C_1 \times C_2$ where C_1, $C_2 \subseteq \mathbb{R}$ are bounded intervals, $Q = \{(x, C) : x \in \mathbb{R}^2, C \in \mathcal{C}, x \in \overline{C}\}$, T_Q the corresponding set of tagged partitions, Δ the family of neighbourhood gauges on $\mathbb{R}^2$ (derived from families $\langle G_x\rangle_{x\in\mathbb{R}^2}$ of open sets in $\mathbb{R}^2$ such that $x \in G_x$ for every $x \in \mathbb{R}^2$), and $\mathfrak{R} = \{\mathcal{R}_C : C \in \mathcal{C}\}$ where $\mathcal{R}_C = \{\mathbb{R}^2 \setminus C' : C \subseteq C' \in \mathcal{C}\} \cup \{\emptyset\}$.

I should warn you that this is not quite standard. The big problem is that it's not rotation-invariant. To be much use in conventional physical applications, one wants structures which are both translation- and rotation-invariant, like Lebesgue measure. But it seems to be a useful idea as long as you are committed to a particular orientation of the coordinates.

[There is a most interesting construction, the 'Pfeffer integral', which *is* rotation-invariant, and gives a strong version of the divergence theorem in $\mathbb{R}^r$. But it's a limit of gauge integrals rather than a gauge integral itself, and it's a week's work to describe it. See Fremlin 03, §484.]

(b) We now have a result corresponding to Theorem 3Ac, but for a rather special kind of differentiation. I will say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is **cross-differentiable** at (a, b), with **cross-derivative** $\alpha = (D^\times f)(a, b)$, if for every $\epsilon > 0$ there is a neighbourhood G of (a, b) such that

$$|f(x, y) - f(x, b) - f(a, y) + f(a, b) - \alpha(x - a)(y - b)| \leq \epsilon|x - a||y - b|$$

whenever $(x, y) \in G$. If you have looked at the proof that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ when f has continuous second partial derivatives, you will find it easy to show that such a function is cross-differentiable, with $D^\times f = \frac{\partial^2 f}{\partial x \partial y}$. (The class of these functions is actually quite interesting – they correspond to the 'strongly derivable interval functions' of Laczkovich 82.)

(c) Theorem Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is cross-differentiable everywhere, and that

$$\gamma = \lim_{\substack{x\to\infty \\ y\to\infty}} f(x, y) - \lim_{\substack{x\to\infty \\ y\to-\infty}} f(x, y) - \lim_{\substack{x\to-\infty \\ y\to\infty}} f(x, y) + \lim_{\substack{x\to-\infty \\ y\to-\infty}} f(x, y)$$

is defined in $\mathbb{R}$. Then $⨎ D^\times f$ is defined and equal to γ.

(d) We have a similar result for functions $f : \mathbb{R}^2 \to W$ for any locally convex Hausdorff linear topological space W. And the same ideas work in $\mathbb{R}^r$ for any integer $r \geq 1$.

3F Fubini's theorem There is a version of Fubini's theorem for the Henstock integral.

Theorem (a) Suppose that $f:\mathbb{R}^2\to\mathbb{R}$ is zero outside some bounded set, and has a two-dimensional Henstock integral $\oint f(x,y)d(x,y)$. Let $g:\mathbb{R}\to\mathbb{R}$ be such that $g(x)=\oint f(x,y)dy$ whenever the one-dimensional Henstock integral $\oint f(x,y)dy$ is defined. Then the one-dimensional Henstock integral $\oint g(x)dx$ is defined and equal to $\oint f(x,y)d(x,y)$.

(b) Suppose that f, $g:\mathbb{R}\to\mathbb{R}$ have one-dimensional Henstock integrals. Then the two-dimensional Henstock integral $\oint f(x)g(y)d(x,y)$ is defined and equal to $\oint f(x)dx\cdot\oint g(y)dy$.

proof (a) See Fremlin 03, 482M.

(b) See Fremlin n11.

Remarks I have stated this result for real-valued f only; in the vector-valued case it will work well enough if $U=W$ is finite-dimensional, but not for infinite-dimensional spaces (see 3G). There is no difficulty in extending the idea of Example (iv) to give a Henstock integral on $\mathbb{R}^r$ for any integer $r\ge 1$. In Fubini's theorem, however, while (a) above extends to (real-valued) functions from $\mathbb{R}^{r+s}\cong\mathbb{R}^r\times\mathbb{R}^s$ for any r and s, I do not know whether (b) extends unless at least one of r and s is equal to 1. The following problem seems to be an obstacle if $\min(r,s)=2$:

> **Problem** Let $\mathcal{C}$ be the set of rectangles in $\mathbb{R}^2$, as in Example (iv), and $\mathcal{E}_0$ the set of finite unions of members of $\mathcal{C}$. Is there a constant $M\ge 0$ such that
>
> whenever $\phi:\mathcal{C}\to[0,1]$ is a function such that $\sum_{C\in\mathcal{C}_0}\phi C\le 1$ for every finite disjoint $\mathcal{C}_0\subseteq\mathcal{C}$, there is an additive function $\lambda:\mathcal{E}_0\to[0,M]$ such that $\phi C\le\lambda C$ for every $C\in\mathcal{C}$?

(For the one-dimensional version, the answer is 'yes', with $M=1$; but $M=1$ won't work in two dimensions. As of 4.7.11, the best example is due to Mircea Petrarche and shows that in two dimensions M must be at least $\frac{3}{2}$. See Fremlin n11.)

3G Example There is a Henstock integrable $f:\mathbb{R}^2\to\ell^2$ such that $\oint f(x,y)dy$ is undefined for every $y\in[0,1[$.

proof (a) Take orthonormal $e_{nj}\in\ell^2$ for $n\in\mathbb{N}$ and $j<4^n$. Set

$$\begin{aligned} f(x,y)&=2^ne_{nj}\text{ if }2^{-n-1}<x\le 3\cdot 2^{-n-2},\ 4^nj\le y<4^n(j+1),\\ &=-2^ne_{nj}\text{ if }3\cdot 2^{-n-2}<x\le 2^{-n},\ 4^nj\le y<4^n(j+1),\\ &=0\text{ for other }x,\ y\in\mathbb{R}.\end{aligned}$$

(b) If $y\in[0,1[$ then

$$\|\int_{2^{-n-1}}^{3\cdot 2^{-n-2}}f(x,y)dx\|=2^{-n-2}\cdot 2^n=\frac{1}{4}$$

for every n, so $\lim_{\delta\downarrow 0}\int_\delta^1 f(x,y)dx$ and $\oint_0^1 f(x,y)dx$ are undefined (see Fremlin 03, 483Bd).

(c) The bulk of the argument concerns the two-dimensional integral $\oint f(x,y)d(x,y)$. I aim to describe the associated Saks-Henstock indefinite integral. Taking $\mathcal{E}$ to be the algebra of subsets of $\mathbb{R}^2$ generated by the family $\mathcal{C}$ of products of bounded intervals, we can define F on $\mathcal{E}$ by setting

$$F(E)=\sum_{n=0}^\infty\sum_{j=0}^{4^n-1}\int_{E\cap(]2^{-n-1},2^{-n}]\times[4^{-n}j,4^{-n}(j+1)[)}fd\mu,$$

where the integrals here are the ordinary Lebesgue integral with respect to two-dimensional Lebesgue measure μ, because all but finitely many terms in this sum will be zero. The property (α) of 2D is immediate from the form of the definition of F, and (γ) there is equally elementary, since in fact $F(E)=0$ whenever E is disjoint from $[0,1]^2$.

As for (β), set

$$G=\{(x,y):f\text{ is constant on a neighbourhood of }(x,y)\},$$

$$E=\{(x,y):(x,y)\in\mathbb{R}^2\setminus G,\ f\text{ is bounded on a neighbourhood of }(x,y)\}.$$

Then $\mathbb{R}^2\setminus G$ is a closed set included in $(\mathbb{Q}\times\mathbb{R})\cup(\mathbb{R}\times\mathbb{Q})$, so is Lebesgue negligible, while $\mathbb{R}^2\setminus(G\cup E)$ is the line segment $\{0\}\times[0,1]$.

Let $\epsilon>0$. For each $n\in\mathbb{N}$, set

$$E_n=E\cap\operatorname{int}\{(x,y):\|f(x,y)\|\le 2^n\},$$

and let $\tilde G_n\supseteq E_n$ be an open set of measure at most $4^{-n}\epsilon$ such that $\|f(x,y)\|\le 2^n$ for every $(x,y)\in\tilde G_n$. Let $m\in\mathbb{N}$ be such that $2^{-m}\le\epsilon$. Let $\boldsymbol{G}=\langle G_{xy}\rangle_{(x,y)\in\mathbb{R}^2}$ be a family of open sets such that $(x,y)\in G_{xy}$ for every (x,y) and

— if $(x,y)\in G$, then f is constant on G_{xy},

— if $n\in\mathbb{N}$ and $(x,y)\in E_n$, then $G_{xy}\subseteq\tilde G_n$,

— if $y\in[0,1]$ then $\operatorname{diam}G_{0y}\le 2^{-m}$.

Then, writing μ for two-dimensional Lebesgue measure on $\mathbb{R}^2$, $\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\mu)\|\le 5\epsilon$ for every $\delta_{\boldsymbol{G}}$-fine $\boldsymbol{t}$.
P Set

$$\boldsymbol{s}=\{((x,y),C):((x,y),C)\in\boldsymbol{t},\ (x,y)\in G\},$$

$$\boldsymbol{s}_n=\{((x,y),C):((x,y),C)\in\boldsymbol{t},\ (x,y)\in E_n\}\text{ for }n\in\mathbb{N},$$

$$\boldsymbol{t}_n=\{((0,y),C):((0,y),C)\in\boldsymbol{t},\ y\in[0,1],\ 2^{-n-1}<\sup\pi_1[C]\le 2^{-n}\}\text{ for }n\ge m,$$

where $\pi_1(x,y)=x$ for $(x,y)\in\mathbb{R}^2$. Then, for any $n\ge m$ and $j<4^n$,

$$\sum_{((0,y),C)\in\boldsymbol{t}_n}\int_{C\cap(]2^{-n-1},2^{-n}]\times[4^{-n}j,4^{-n}(j+1)[)}fd\mu$$
$$=\sum_{\substack{((0,y),C)\in\boldsymbol{t}_n\\ \sup\pi_1[C]\le 3\cdot 2^{-n-2}}}2^n(\sup\pi_1[C]-2^{-n-1})\mu_1(\pi_2[C]\cap\big[4^{-n}j,4^{-n}(j+1)\big[)e_{nj}$$
$$+\sum_{\substack{((0,y),C)\in\boldsymbol{t}_n\\ 3\cdot 2^{-n-2}<\sup\pi_1[C]}}2^n(2^{-n}-\sup\pi_1[C])\mu_1(\pi_2[C]\cap\big[4^{-n}j,4^{-n}(j+1)\big[)e_{nj}$$

(writing $\pi_2(x,y)=y$ for $(x,y)\in\mathbb{R}^2$, and μ_1 for one-dimensional Lebesgue measure on $\mathbb{R}$)

$$=\beta_je_{nj}$$

where $|\beta_j|\le 2^{-n-2}\cdot 2^n\cdot 4^{-n}=4^{-n-1}$. Consequently

$$\|F(H_{\boldsymbol{t}_n})\|^2=\|\sum_{((0,y),C)\in\boldsymbol{t}_n}F(C)\|^2$$
$$=\|\sum_{j=0}^{4^n-1}\sum_{((0,y),C)\in\boldsymbol{t}_n}\int_{C\cap(]2^{-n-1},2^{-n}]\times[4^{-n}j,4^{-n}(j+1)[)}fd\mu\|^2$$
$$=\|\sum_{j=0}^{4^n-1}\beta_je_{nj}\|^2=\sum_{j=0}^{4^n-1}\beta_j^2\le 4^n\cdot 4^{-2n-2}=4^{-n-2}$$

and $\|F(H_{\boldsymbol{t}_n})\|\le 2^{-n-1}$ for every $n\ge m$. Now we can estimate

$$\begin{aligned}\|F(H_{\boldsymbol t})-S_{\boldsymbol t}(f,\mu)\| &\le \|F(H_{\boldsymbol s})-S_{\boldsymbol s}(f,\mu)\|+\sum_{n=0}^{\infty}\|F(H_{\boldsymbol s_n})-S_{\boldsymbol s_n}(f,\mu)\|\\ &\qquad+\sum_{n=m}^{\infty}\|F(H_{\boldsymbol t_n})-S_{\boldsymbol t_n}(f,\mu)\|\\ &\le \sum_{((x,y),C)\in\boldsymbol s}\|F(C)-f(x,y)\mu C\|\\ &\qquad+\sum_{n=0}^{\infty}\sum_{((x,y),C)\in\boldsymbol s_n}(\|F(C)\|+\|f(x,y)\mu C\|)+\sum_{n=m}^{\infty}\|F(H_{\boldsymbol t_n})\|\\ &\le 0+\sum_{n=0}^{\infty}\sum_{((x,y),C)\in\boldsymbol s_n}2^{n+1}\mu C+\sum_{n=m}^{\infty}2^{-n-1}\\ &\le\sum_{n=0}^{\infty}2^{n+1}\mu\tilde G_n+2^{-m}\le\sum_{n=0}^{\infty}2^{n+1}\cdot 4^{-n}\epsilon+\epsilon=5\epsilon.\ \mathbf{Q}\end{aligned}$$

As ϵ is arbitrary, F satisfies the conditions of 2L, and f is Henstock integrable.

3H Line integrals We sometimes want to form a line integral, in two- or three-dimensional space, of a scalar- or vector-valued function. Ordinary approaches tend to insist on curves of finite length, that is, on rectifiable parametrizations of curves, and on continuous integrands. A version of the Henstock integral can do fractionally better than that, as follows.

Proposition Suppose that X is (the set of points of) a simple curve in a normed space V. Let $\mathcal{C}$ be the family of subintervals (open, closed or half-open) of X, Q the set of pairs $\{(v,C) : C\in\mathcal{C},\ v\in\overline{C}\}$, T_Q the corresponding set of tagged partitions, Δ the set of neighbourhood gauges on X, $\mathfrak{R}=\{\{\emptyset\}\}$. Set $\nu\emptyset=0$ and $\nu C=v^{\uparrow}(C)-v^{\downarrow}(C)$ if $C\in\mathcal{C}\setminus\{\emptyset\}$, writing $v^{\downarrow}(C)$ and $v^{\uparrow}(C)$ for the starting and finishing endpoints of C. Suppose that there is a countable closed set $D\subseteq X$ such that for every $v\in X\setminus D$ there is a neighbourhood G of v meeting X in a set of finite length[2]. Let $f : X\to W$ be a function with a derivative $(\nabla f)(v)\in\mathsf{B}(V;W)$ with respect to its domain at every point v of X.[3] Then $\int\nabla f\,d\nu$ is defined and equal to $f(v^{\uparrow}(X))-f(v^{\downarrow}(X))$.

Remark In the language of 1A, I am taking U to be the normed space $\mathsf{B}(V;W)$ of bounded linear operators from V to W, and $\langle T|v\rangle=Tv$ for $T\in\mathsf{B}(V;W)$, $v\in V$. I didn't work through exactly this structure X, $\mathcal{C}$, Q, Δ and $\mathfrak{R}$, but the arguments of §1 include all the ideas needed to show that it satisfies the requirements of 2A-2B. The point is that because X is a 'simple' curve there are no crossing points and X looks almost like the unit interval, except that it might start and finish in the same place.

proof (a)(i) Note that it will be enough to consider the case in which W is a Banach space, because if f is differentiable regarded as a function from V to W then it is still differentiable, with the same derivative, when regarded as a function from V to the completion $\hat W$; and if $\int\nabla f)\,d\nu$ is defined when we think of $\langle\,|\,\rangle$ as a function from $\mathsf{B}(V;W)\times V$ to $\hat W$, and belongs to W, then it is still defined, with the same value, if we think of $\langle\,|\,\rangle$ as a function from $\mathsf{B}(V;W)\times V$ to W, because both the target $f(v^{\uparrow}(X))-f(v^{\downarrow}(X))$ and the approximating sums $S_{\boldsymbol t}(f,\nu)$ belong to W.

(ii) The next thing to note is that there is a countable closed set $D'\subseteq X$, containing the endpoints $v^{\downarrow}(X)$ and $v^{\uparrow}(X)$, such that every component of $X\setminus D'$ has finite length. $\mathbf{P}$ For each component C of $X\setminus D$, take sequences $\langle a_{Cn}\rangle_{n\in\mathbb{N}}$, $\langle b_{Cn}\rangle_{n\in\mathbb{N}}$ in C converging to $v^{\downarrow}(C)$, $v^{\uparrow}(C)$ respectively, and set

$$\begin{aligned}D'=D\cup\{a_{Cn} : n\in\mathbb{N},\ C\text{ is a component of }X\setminus D\}\\ \cup\{b_{Cn} : n\in\mathbb{N},\ C\text{ is a component of }X\setminus D\}\cup\{v^{\downarrow}(X),v^{\uparrow}(X)\}.\end{aligned}$$

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[2]that is, there is an $M\ge 0$ such that $\sum_{i<n}\|v_{i+1}-v_i\|\le M$ whenever $v_0,\dots,v_n$ are taken in order along $X\cap G$.

[3]that is, for every $v\in X$ and $\epsilon>0$ there is an $\eta>0$ such that $\|f(v')-f(v)-((\nabla f)(v))(v'-v)\|\le\epsilon\|v'-v\|$ whenever $v'\in X$ and $\|v'-v\|\le\eta$.

Now if I is a component of $X\setminus D'$, then its closure $\overline{I}=I\cup\{v^\downarrow(I),v^\uparrow(I)\}$ is included in $X\setminus D$, so every point of $\overline{I}$ has a neighbourhood meeting X in a set of finite length; because $\overline{I}$ is compact, it can be covered by finitely many such neighbourhoods, and is itself of finite length. **Q**

(b) Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then we have an additive function $F:\mathcal{E}\to W$ defined by setting

$$F(\bigcup_{i\le n}C_i)=\sum_{i=0}^n f(v^\uparrow(C_i))-f(v^\downarrow(C_i))$$

whenever $C_0,\dots,C_n\in\mathcal{C}$ are disjoint and non-empty. (There is a little bit of geometry here; we have to know that every member of $\mathcal{E}$ is a finite disjoint union of members of $\mathcal{C}$, that the intersection of two members of $\mathcal{C}$ belongs to $\mathcal{C}$, and that when we have two or more disjoint non-empty members $C_0,\dots,C_n$ of $\mathcal{C}$ with union in $\mathcal{C}$, then two of them must be abutting in such a way that $v^\uparrow(C_i)=v^\downarrow(C_j)$, $C_i\cup C_j\in\mathcal{C}$, $v^\downarrow(C_i\cup C_j)=v^\downarrow(C_i)$ and $v^\uparrow(C_i\cup C_j)=v^\uparrow(C_j)$.)

(c) If I is a component of $X\setminus D'$ and $\epsilon>0$, there is a $\delta\in\Delta$ such that

$$\sum_{(v,C)\in\boldsymbol{t}}\|f(v^\uparrow(C))-f(v^\downarrow(C))-(\nabla f(v)(v^\uparrow(C)-v^\downarrow(C))\|\le\epsilon$$

whenever $\boldsymbol{t}\in T$ is δ-fine and $H_{\boldsymbol{t}}\subseteq I$. **P** Set $M=\text{lh}(I)$; by the choice of D', M is finite. For each $v\in X$, let $G_v\subseteq V$ be an open set containing v such that

$$\|f(v')-f(v)-\nabla f(v)(v'-v)\|\le\frac{\epsilon}{M}\|v'-v\|$$

for every $v'\in X\cap\overline{G}_v$. Then

$$\delta=\{(v,A):v\in X,\ A\subseteq X\cap G_v\}$$

belongs to Δ. Now suppose that $\boldsymbol{t}\in T$ is δ-fine and $\overline{H}_{\boldsymbol{t}}\subseteq I$. Then

$$\begin{aligned}&\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(\nabla f,\nu)\|\\ &\quad=\|\sum_{(v,C)\in\boldsymbol{t}}f(v^\uparrow(C))-f(v^\downarrow(C))-\sum_{(v,C)\in\boldsymbol{t}}\nabla f(v)(\nu C)\|\\ &\quad\le\sum_{(v,C)\in\boldsymbol{t}}\|f(v^\uparrow(C))-f(v^\downarrow(C))-\nabla f(v)(v^\uparrow(C)-v^\downarrow(C))\|\\ &\quad\le\sum_{(v,C)\in\boldsymbol{t}}\|f(v^\uparrow(C))-f(v)-\nabla f(v)(v^\uparrow(C)-v)\|\\ &\qquad\quad+\sum_{(v,C)\in\boldsymbol{t}}\|f(v)-f(v^\downarrow(C))-\nabla f(v)(v-v^\downarrow(C))\|\\ &\quad\le\sum_{(x,C)\in\boldsymbol{t}}\frac{\epsilon}{M}\|v^\uparrow(C)-v\|+\sum_{(x,C)\in\boldsymbol{t}}\frac{\epsilon}{M}\|v-v^\downarrow(C)\|\end{aligned}$$

(because $v^\uparrow(C)$, $v^\downarrow(C)\in\overline{G}_v$ whenever $(v,C)\in\delta$)

$$\le\frac{\epsilon}{M}\,\text{lh}(I)$$

(because the intervals $[v^\downarrow(C),v]$, $[v,v^\uparrow(C)]$, as (v,C) runs over $\boldsymbol{t}$, are non-overlapping subintervals of $\overline{C}_0$)

$$=\epsilon.$$ **Q**

(d) If $\epsilon>0$ there is a $\delta\in\Delta$ such that $\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(\nabla f,\nu)\|\le\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $H_{\boldsymbol{t}}=X$. **P** Let $\mathcal{I}$ be the family of components of $X\setminus D'$. Because $\mathcal{I}$ and D' are both countable, we have families $\langle\epsilon_C\rangle_{C\in\mathcal{I}}$ and $\langle\epsilon_v\rangle_{v\in D'}$ of strictly positive real numbers such that $\sum_{C\in\mathcal{I}}\epsilon_C+\sum_{v\in D'}\epsilon_v\le\epsilon$. For each $C\in\mathcal{I}$, (c) tells us that there is a $\delta_C\in\Delta$ such that

$$\sum_{(v,C)\in\boldsymbol{t}}\|f(v^\uparrow(C))-f(v^\downarrow(C))-\nabla f(v)(v^\uparrow(C)-v^\downarrow(C))\|\le\epsilon_C$$

whenever $\boldsymbol{t} \in T_Q$ is δ_C-fine and $H_{\boldsymbol{t}} \subseteq C$. For each $v \in D'$ let G_v be an open neighbourhood of v in V such that $\|\nabla f(v)\|\|v' - v''\| + \|f(v'') - f(v')\| \le \epsilon_v$ whenever v', $v'' \in X \cap \overline{G}_v$. Now let δ be a neighbourhood gauge included in

$$\bigcup_{C\in\mathcal{I}}\{(v,A) : v \in C,\ (v,A) \in \delta_C,\ A \subseteq C\} \cup \{(v,A) : v \in D',\ A \subseteq X \cap G_v\}.$$

Suppose that $\boldsymbol{t}$ is δ-fine and $H_{\boldsymbol{t}} = X$. Then

$$\begin{aligned}
&\sum_{(v,C)\in\boldsymbol{t}} \|F(C) - \langle\nabla f(v)|\nu C\rangle\| \\
&\quad= \sum_{(v,C)\in\boldsymbol{t}} \|f(v^\uparrow(C)) - f(v^\downarrow(C)) - \nabla f(v)(v^\uparrow(C) - v^\downarrow(C))\| \\
&\quad= \sum_{I\in\mathcal{I}} \sum_{\substack{(v,C)\in\boldsymbol{t}\\ C\subseteq I}} \|f(v^\uparrow(C)) - f(v^\downarrow(C)) - \nabla f(v)(v^\uparrow(C) - v^\downarrow(C))\| \\
&\qquad\qquad + \sum_{\substack{(v,C)\in\boldsymbol{t}\\ v\in D'}} \|f(v^\uparrow(C)) - f(v^\downarrow(C)) - \nabla f(v)(v^\uparrow(C) - v^\downarrow(C))\| \\
&\quad\le \sum_{I\in\mathcal{I}} \epsilon_I + \sum_{v\in D'} \epsilon_v \le \epsilon. \ \mathbf{Q}
\end{aligned}$$

(e) By 2D, $\int \nabla f\, d\nu$ is defined, and F is the Saks-Henstock integral of ∇f with respect to ν, so $\int \nabla f\, d\nu = f(v^\uparrow(X)) - f(v^\downarrow(X))$.

3I Corollary In Proposition 3H, suppose that $V = W = \mathbb{C}$, and that f is an analytic function with domain including X. Then $\int f\, d\nu$ is defined. Let F be the Saks-Henstock indefinite integral of f with respect to ν. If g is an analytic function such that $g'(z) = f(z)$ for every $z \in X \cap \operatorname{dom} g$ and $C \in \mathcal{C}$ is a closed interval included in $\operatorname{dom} g$, then $F(C) = g(v^\uparrow(C) - g(v^\downarrow(C))$.

proof We need to know enough about analytic functions (e.g., that an analytic function has Taylor series valid near each point of its domain) to know that we shall be able to cover X by the domains of primitives g of f. So we can apply the result to finitely many primitives g, each relevant to some subinterval of X.

Remark I went to the trouble, in 3H-3I, to leave the curve X unparametrized. Of course it would be unusual to have a simple curve which didn't come with a reasonably natural parametrization. But I like the idea of a definition which doesn't involve choosing a parametrization and then showing that it didn't matter which parametrization you chose.

Of course this means that I have sacrificed curves which repeat parts of themselves, forwards or backwards, which are often useful in the theory of complex variables. To allow this we should need to change the notion of 'tagged partition', which would force us to re-negotiate the Saks-Henstock indefinite integral.

3J Classical integrals of vector-valued functions Let (X, Σ, μ) be a σ-finite measure space and $U = W$ a Hausdorff linear topological space, $V = \mathbb{R}$, $\langle w|\alpha\rangle = \alpha w$ for $\alpha \in \mathbb{R}$ and $w \in W$. Let $\mathcal{C}$ be the family of measurable sets of finite measure, $Q = X \times \mathcal{C}$, T_Q the corresponding set of tagged partitions, $\delta_{\mathcal{E}} = \bigcup_{E\in\mathcal{E}} E \times \mathcal{P}E$, $\mathcal{R}_{F\epsilon} = \{E : E \in \Sigma,\ \mu(E \cap F) \le \epsilon\}$ as in Example (ii).

(a) When W is a separable Banach space, we get an extension of the Bochner integral. (It is possible to have a function such that $\int f\, d\mu$ is defined but $\int \|f\| d\mu = \infty$.)

(b) When the topology on W is a weak topology, we get an extension of the Pettis integral. (In this case, W will not normally be complete, so we cannot be sure of having a Saks-Henstock indefinite integral as defined in 2D. However, if $f : X \to W$ is such that there *is* a function $F : \Sigma \to W$ with the properties (α)-(γ) of 2D, then f will be integrable.)

3K Definition A set Δ of gauges on a set X is **countably full** if whenever $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ and $x \mapsto n(x)$ is a function from X to $\mathbb{N}$, then there is a $\delta \in \Delta$ such that $(x, A) \in \delta_{n(x)}$ whenever $(x, A) \in \delta$.

Observe that for any topological space X the set of neighbourhood gauges on X is countably full. However, the family of gauges described in 1D for the classical integral is not a countably full family.

3L Definition If X is a set, $\mathcal{C}$ is a family of subsets of X, V is a normed space and $\delta \subseteq X \times \mathcal{P}X$ is a gauge, a function $\nu : \mathcal{C} \to V$ is **δ-moderate** if there is a strictly positive $h : X \to]0, \infty[$ such that $\sum_{(x,C)\in \boldsymbol{t}} h(x)\|\nu C\| \le 1$ whenever $\boldsymbol{t} \in T_{X\times\mathcal{C}}$ is δ-fine.

3M A convergence theorem Suppose that $X, \mathcal{C}, Q, \Delta$ and $\mathfrak{R}$ are as in 2A-2B, and that Δ is countably full in the sense of 3K. Let U, V and W be Banach spaces, and $\langle\,|\,\rangle : U \times V \to W$ a continuous bilinear operator; let $\nu : \mathcal{C} \to V$ be a function which is $\tilde{\delta}$-moderate in the sense of 3L for some $\tilde{\delta} \in \Delta$. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of functions from X to U such that $\int f_n d\nu$ is defined for every n, and F_n the Saks-Henstock indefinite integral of f_n for each n. Suppose that

— $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in U for every $n \in \mathbb{N}$;

— $F(E) = \lim_{n\to\infty} F_n(E)$ is defined in W for every E in the algebra $\mathcal{E}$ of subsets of X generated by $\mathcal{C}$;

— $\langle F_n\rangle_{n\in\mathbb{N}}$ is uniformly convergent to F.

Then $\int f d\nu$ is defined, and the Saks-Henstock indefinite integral of f with respect to ν is F; in particular, $\int f d\nu = \lim_{n\to\infty} \int f_n d\nu$.

proof (a) Let $\gamma \ge 0$, $h : X \to]0, \infty[$ and $\tilde{\delta} \in \Delta$ be such that

$$\|\langle u|v\rangle\| \le \gamma\|u\|\|v\| \text{ for every } u \in U,\ v \in V,$$

$$\textstyle\sum_{(x,C)\in\boldsymbol{t}} h(x)\|\nu C\| \le 1 \text{ for every } \tilde{\delta}\text{-fine } \boldsymbol{t} \in T_Q.$$

Note that because every F_n is additive, $F : \mathcal{E} \to W$ is certainly additive.

(b) Let $\epsilon > 0$. Let $\langle n_k\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $\|F_{n_k}(E) - F(E)\| \le 2^{-k}\epsilon$ for every $k \in \mathbb{N}$ and $E \in \mathcal{E}$. For each $k \in \mathbb{N}$, let $\delta_k \in \Delta$ be such that $\|S_{\boldsymbol{t}}(f_{n_k}, \nu) - F_{n_k}(H_{\boldsymbol{t}})\| \le 2^{-k}\epsilon$ for every δ_k-fine $\boldsymbol{t} \in T_Q$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $\|F_{n_0}(E)\| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$.

(i) For $x \in X$, let $k(x) \in \mathbb{N}$ be such that $\gamma\|f_{n_{k(x)}}(x) - f(x)\| \le \epsilon h(x)$. Let $\delta \in \Delta$ be such that $(x, A) \in \delta_{k(x)} \cap \tilde{\delta}$ whenever $(x, A) \in \delta$. Then $\|S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}})\| \le 6\epsilon$ for every δ-fine $\boldsymbol{t}$. **P** For each $l \in \mathbb{N}$, set $\boldsymbol{t}_l = \{(x, C) : (x, C) \in \boldsymbol{t},\ k(x) = l\}$; note that $\boldsymbol{t}_l$ belongs to T_Q and is δ_l-fine. Then

$$\|S_{\boldsymbol{t}}(f, \nu) - F(H_{\boldsymbol{t}})\| = \|\sum_{l=0}^{\infty} S_{\boldsymbol{t}_l}(f, \nu) - F(H_{\boldsymbol{t_l}})\|$$

(of course all but finitely many terms in the sum are zero)

$$\le \sum_{l=0}^{\infty} \|S_{\boldsymbol{t}_l}(f, \nu) - F(H_{\boldsymbol{t_l}})\|$$

$$\le \sum_{l=0}^{\infty} \|S_{\boldsymbol{t}_l}(f, \nu) - S_{\boldsymbol{t}_l}(f_{n_l}, \nu)\| + \|S_{\boldsymbol{t}_l}(f_{n_l}, \nu) - F_{n_l}(H_{\boldsymbol{t}_l}\|$$

$$+ \|F_{n_l}(H_{\boldsymbol{t}_l} - F(H_{\boldsymbol{t_l}})\|$$

$$\le \sum_{l=0}^{\infty} \sum_{(x,C)\in\boldsymbol{t}_l} \|\langle f(x)|\nu C\rangle - \langle f_{n_l}(x)|\nu C\rangle\| + 2^{-l}\epsilon + 2^{-l}\epsilon$$

$$\le 4\epsilon + \sum_{l=0}^{\infty} \sum_{(x,C)\in\boldsymbol{t}_l} \gamma\|f(x) - f_{n_l}(x)\|\|\nu C\|$$

$$\le 4\epsilon + \sum_{l=0}^{\infty} 2^{-l}\epsilon \sum_{(x,C)\in\boldsymbol{t}_l} h(x)\|\nu C\| \le 4\epsilon + \sum_{l=0}^{\infty} 2^{-l}\epsilon = 6\epsilon. \textbf{ Q}$$

(ii) If $E \in \mathcal{E} \cap \mathcal{R}$ then

$$\|F(E)\| \leq \|F(E) - F_{n_0}(E)\| + \|F_{n_0}(E)\| \leq 2\epsilon.$$

(c) Thus F satisfies (α)-(γ) of Theorem 2D, and f is integrable with respect to ν, with Saks-Henstock indefinite integral F.

3N Example Suppose that X, $\mathcal{C}$, Q, T_Q, Δ and $\mathfrak{R}$ are as in 2A-2B, and that Δ is countably full. Let $\nu : \mathcal{C} \to [0, \infty[$ be a non-negative functional which is δ-moderate for some $\delta \in \Delta$, and $\langle f_n \rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of functions such that $\int f_n d\nu$ is defined for every $n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \int f_n d\nu = \alpha < \infty$, and $f(x) = \sup_{n\in\mathbb{N}} f_n(x)$ is finite for every $x \in X$. Then $\int f d\nu$ is defined and equal to α.

$\bigotimes$**Remark** Note that this does not have the full strength of B.Levi's theorem in the classical theory. Asking for ν to be 'moderate' in the sense of 3M corresponds to having a σ-finite measure space. But more importantly, we have to assume that $\langle f_n \rangle_{n\in\mathbb{N}}$ is convergent at every point, which is not required in the standard theorem.

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