Measurability in $C(2^\kappa)$ and Kunen cardinals

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I work through some results from Avilés Plebanek & Rodríguez p11.

1 Key lemma

1A Notation Generally I follow Fremlin 03 and Fremlin 08. In addition, I use the following.

(a) If $\langle A_i\rangle_{i\in I}$ is a family of sets and Σ_i is an algebra of subsets of A_i for each $i\in I$, then $\bigotimes_{i\in I}\Sigma_i$ will be the algebra of subsets of $\prod_{i\in I}A_i$ generated by sets of the form $\prod_{i\in I}E_i$ where $E_i\in\Sigma_i$ for every $i\in I$ and $\{i:E_i\ne A_i\}$ is finite. $\widehat{\bigotimes}_{i\in I}\Sigma_i$ will be the σ-algebra of subsets of $\prod_{i\in I}A_i$ generated by sets of the form $\prod_{i\in I}E_i$ where $E_i\in\Sigma_i$ for every $i\in I$ and $\{i:E_i\ne A_i\}$ is finite.
Similarly, if A, A' are sets, Σ is an algebra of subsets of A and Σ' is an algebra of subsets of A', $\Sigma\otimes\Sigma'$ will be the algebra of subsets of $A\times A'$ generated by $\{E\times E':E\in\Sigma,\,E'\in\Sigma'\}$, and $\Sigma\widehat{\otimes}\Sigma'$ the σ-algebra of subsets of $A\times A'$ generated by $\{E\times E':E\in\Sigma,\,E'\in\Sigma'\}$.

(b) If A is a set and $\mathcal{E}$ is a family of subsets of A, I will say that B, $B'\subseteq A$ are **$\mathcal{E}$-separated** if there are disjoint E, $E'\in\mathcal{E}$ such that $B\subseteq E$ and $B'\subseteq E'$.

(c) Suppose that A is a set and $\mathcal{E}$ is an algebra of subsets of A.

(i) Two subsets B, B' of A are $\mathcal{E}$-separated iff there is a member of $\mathcal{E}$ including one of B, B' and disjoint from the other.

(ii) If $\langle B_i\rangle_{i\in I}$ and $\langle B'_j\rangle_{j\in J}$ are finite families of subsets of A such that B_i, B'_j are $\mathcal{E}$-separated for all $i\in I$ and $j\in J$, then $\bigcup_{i\in I}B_i$ and $\bigcup_{j\in J}B'_j$ are $\mathcal{E}$-separated. (If $B_i\subseteq E_{ij}\subseteq A\setminus B'_j$ for all i and j, then $\bigcup_{i\in I}B_i\subseteq E\subseteq A\setminus\bigcup_{j\in J}B'_j$ where $E=\bigcup_{i\in I}\bigcap_{j\in J}E_{ij}$.)

(iii) If $\mathcal{E}$ is a σ-algebra of subsets of A and $\langle B_i\rangle_{i\in I}$ and $\langle B'_j\rangle_{j\in J}$ are countable families of subsets of A such that B_i, B'_j are $\mathcal{E}$-separated for all $i\in I$ and $j\in J$, then $\bigcup_{i\in I}B_i$ and $\bigcup_{j\in J}B'_j$ are $\mathcal{E}$-separated.

1B Kunen cardinals (a) A **Kunen cardinal** is a cardinal κ such that $\mathcal{P}\kappa\widehat{\otimes}\mathcal{P}\kappa=\mathcal{P}(\kappa\times\kappa)$.

(b) If κ is a Kunen cardinal then every cardinal less than or equal to κ is a Kunen cardinal. If $\langle\kappa_i\rangle_{i\in\mathbb{N}}$ is a sequence of Kunen cardinals, then $\sup_{i\in\mathbb{N}}\kappa_i$ is a Kunen cardinal. (Use 1A(b-iii).) Every cardinal less than or equal to $\mathfrak{p}$ is a Kunen cardinal (Kunen 68, or Fremlin 84, 21G); in particular, ω_1 is a Kunen cardinal. A Kunen cardinal is at most $\mathfrak{c}$. (Consider $\{(\xi,\xi):\xi<\kappa\}\in\mathcal{P}\kappa\widehat{\otimes}\mathcal{P}\kappa$.)

(c) Note that if $\langle\Gamma_i\rangle_{i\in I}$ is a finite family of sets such that $\#(\Gamma_i)$ is a Kunen cardinal for every $i\in I$, then every subset of $\Theta=\prod_{i\in I}\Gamma_i$ belongs to $\widehat{\bigotimes}_{i\in I}\mathcal{P}\Gamma_i$. (Induce on $\#(I)$.)

1C The context (I) Throughout this note, Z will be $\{0,1\}^{\mathbb{N}}$ and X will be $\{0,1\}^Z$. $\mathcal{J}$ will be a family of sets. For $m\ge1$, T_m will be the family of functions from $\{0,1\}^m$ to $\mathcal{J}$.

1D The context (II) In the present section, $m\ge1$ will be a fixed integer, τ a member of T_m, $\langle\Gamma_i\rangle_{i<m}$ a disjoint family of non-empty subsets of Z such that $\#(\Gamma_i)$ is a Kunen cardinal for every $i<m$, $\Theta=\prod_{i<m}\Gamma_i$, Ω a set, Σ a σ-algebra of subsets of Ω and $\langle W_\theta\rangle_{\theta\in\Theta}$ a family of subsets of Ω such that

(*) whenever $x\in X$ and J, $J'\in\mathcal{J}$ are disjoint, the sets

$$\bigcup\{W_\theta:\theta\in\Theta,\,\tau(x\theta)\subseteq J\},\quad\bigcup\{W_\theta:\theta\in\Theta,\,\tau(x\theta)\subseteq J'\}$$

are Σ-separated.

For $\Phi\subseteq\Theta$, write $\tilde W_\Phi$ for $\bigcup_{\phi\in\Phi}W_\Phi$.

1E Lemma Suppose that $\Phi=\prod_{i<m}\Delta_i$ and $\Psi=\prod_{i<m}\Delta'_i$ where, for each $i<m$, Δ_i and Δ'_i are subsets of Γ_i which are either identical or disjoint. If

($\dagger$) whenever $\phi\in\Phi$ and $\psi\in\Psi$ there is an $x\in X$ such that $\tau(x\phi)\cap\tau(x\psi)=\emptyset$,

then $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

proof If any Δ_i or Δ'_i is empty, the result is trivial; so suppose otherwise. Take $\phi_0\in\Phi$, $\psi_0\in\Psi$ such that $\phi_0(i)=\psi_0(i)$ whenever $i<m$ is such that $\Delta_i=\Delta'_i$. Let $x\in X$ be such that $\tau(x\phi_0)\cap\tau(x\psi_0)=\emptyset$. Let $y\in X$ be such that

$$\begin{aligned}y(t)&=x(\phi_0(i))\text{ whenever }i<m\text{ and }t\in\Delta_i,\\&=x(\psi_0(i))\text{ whenever }i<m\text{ and }t\in\Delta'_i.\end{aligned}$$

Then

$$y\phi=x\phi_0=y\phi_0\text{ for every }\phi\in\Phi,\quad y\psi=x\psi_0=y\psi_0\text{ for every }\psi\in\Psi.$$

Set $J=\tau(x\phi_0)$, $J'=\tau(x\psi_0)$. Then

$$\tilde W_\Phi\subseteq\bigcup_{\theta\in\Theta,\tau(y\theta)\subseteq J}W_\theta,$$

$$\bigcup_{\psi\in\Psi}W_\phi\subseteq\bigcup_{\theta\in\Theta,\tau(y\theta)\subseteq J'}W_\theta.$$

As $J\cap J'=\emptyset$, the hypothesis (*) of 1D tells us that these are Σ-separated, as required.

1F Notation (a) Let $\mathcal{R}$ be the family of sets $R\subseteq m$ such that

whenever σ_0, $\sigma_1\in\{0,1\}^m$ are such that $\tau(\sigma_0)\cap\tau(\sigma_1)=\emptyset$, there is an $i\in R$ such that $\sigma_0(i)\ne\sigma_1(i)$.

(b) For $R\in\mathcal{R}$ set $\pi_R(\phi)=\phi\restriction R$ for every $\phi\in\Theta$.

(c) If $R_0\in\mathcal{R}$ and $P_R\subseteq\prod_{i\in R}\Gamma_i$ for $R\in\mathcal{R}$, let $\mathfrak{Q}(\langle P_R\rangle_{R\in\mathcal{R}},R_0)$ be the family of those sets $Q\subseteq\prod_{i\in R_0}\Gamma_i$ such that

whenever Φ, $\Psi\subseteq\Theta$ are such that

$$\pi_R[\Phi]\subseteq P_R,\ \pi_R[\Psi]\cap P_R=\emptyset\text{ for every }R\in\mathcal{R}\setminus\{R_0\},$$

$$\pi_{R_0}[\Phi]\subseteq Q,\quad\pi_{R_0}[\Psi]\cap Q=\emptyset,$$

then $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

1G Lemma Suppose that Φ, $\Psi\subseteq\Theta$ are such that

($\dagger$) whenever $\phi\in\Phi$ and $\psi\in\Psi$ there is an $x\in X$ such that $\tau(x\phi)\cap\tau(x\psi)=\emptyset$.

Then $\pi_R[\Phi]$ and $\pi_R[\Psi]$ are disjoint for every $R\in\mathcal{R}$.

proof ? Otherwise, there are $\phi\in\Phi$ and $\psi\in\Psi$ such that $\phi\restriction R=\psi\restriction R$. On the other hand, there is supposed to be an $x\in X$ such that $\tau(x\phi)$ and $\tau(x\psi)$ are disjoint. So there ought to be an $i\in R$ such that $x\phi(i)\ne x\psi(i)$, by the definition of $\mathcal{R}$; and this is impossible. **X**

1H Lemma Suppose that $R_0\in\mathcal{R}$ and that $P_R\subseteq\prod_{i\in R}\Gamma_i$ for $R\in\mathcal{R}$. Then $\mathfrak{Q}(\langle P_R\rangle_{R\in\mathcal{R}},R_0)$ is closed under countable unions and intersections.

proof Let $\langle Q_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Q}=\mathfrak{Q}(\langle P_R\rangle_{R\in\mathcal{R}},R_0)$.

(a) Set $Q=\bigcup_{n\in\mathbb{N}}Q_n$, and take Φ, $\Psi\subseteq\Theta$ such that

$$\pi_R[\Phi]\subseteq P_R,\ \pi_R[\Psi]\cap P_R=\emptyset\text{ for every }R\in\mathcal{R}\setminus\{R_0\},$$

$$\pi_{R_0}[\Phi]\subseteq Q,\quad\pi_{R_0}[\Psi]\cap Q=\emptyset.$$

For $n\in\mathbb{N}$, set $\Phi_n=\Phi\cap\pi_{R_0}^{-1}[Q_n]$. Then $\Phi=\bigcup_{n\in\mathbb{N}}\Phi_n$. For each n we surely have

$$\pi_R[\Phi_n]\subseteq P_R,\ \pi_R[\Psi]\cap P_R=\emptyset\text{ for every }R\in\mathcal{R}\setminus\{R_0\},$$

$$\pi_{R_0}[\Phi_n]\subseteq Q_n,\quad \pi_{R_0}[\Psi]\cap Q_n=\emptyset,$$

so $\tilde W_{\Phi_n}$ and $\tilde W_\Psi$ are Σ-separated; by 1A(c-iii), $\tilde W_\Phi=\bigcup_{n\in\mathbb N}\tilde W_{\Phi_n}$ and $\tilde W_\Psi$ are Σ-separated. As Φ and Ψ are arbitrary, $Q\in\mathfrak Q$.

(b) Similarly, if $Q'=\bigcap_{n\in\mathbb N}Q_n$, Φ, $\Psi\subseteq\Theta$ and

$$\pi_R[\Phi]\subseteq P_R,\ \pi_R[\Psi]\cap P_R=\emptyset \text{ for every } R\in\mathcal R\setminus\{R_0\},$$

$$\pi_{R_0}[\Phi]\subseteq Q',\quad \pi_{R_0}[\Psi]\cap Q'=\emptyset,$$

set $\Psi_n=\Psi\setminus\pi_{R_0}^{-1}[Q_n]$ for each n. Then $\Psi=\bigcup_{n\in\mathbb N}\Psi_n$,

$$\pi_R[\Phi]\subseteq P_R,\ \pi_R[\Psi_n]\cap P_R=\emptyset \text{ for every } R\in\mathcal R\setminus\{R_0\},$$

$$\pi_{R_0}[\Phi]\subseteq Q_n,\quad \pi_{R_0}[\Psi_n]\cap Q_n=\emptyset,$$

so $\tilde W_\Phi$ and $\tilde W_{\Psi_n}$ are Σ-separated; consequently $\tilde W_\Phi$ and $\tilde W_\Psi=\bigcup_{n\in\mathbb N}\tilde W_{\Psi_n}$ are Σ-separated. As Φ and Ψ are arbitrary, $Q'\in\mathfrak Q$.

1I Lemma Let Φ, $\Psi\subseteq\Theta$ be such that for every $R\in\mathcal R$ the sets $\pi_R[\Phi]$, $\pi_R[\Psi]\subseteq\prod_{i\in R}\Gamma_i$ are $\bigotimes_{i\in R}\mathcal P\Gamma_i$-separated. Then $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

proof (a) For each $R\in\mathcal R$ let $Q_R\in\bigotimes_{i\in R}\mathcal P\Gamma_i$ be such that $\pi_R[\Phi]\subseteq Q_R$ and $\pi_R[\Psi]\cap Q_R$ is empty; let $\mathcal E_{Ri}$, for $i\in R$, be a finite subalgebra of $\mathcal P\Gamma_i$ such that $Q_R\in\bigotimes_{i\in R}\mathcal E_{Ri}$. For $i<m$ let $\mathcal E_i$ be the finite algebra of subsets of Γ_i generated by $\bigcup\{\mathcal E_{Ri}:i\in R\in\mathcal R\}$.

(b) If Δ_i, Δ_i' are atoms of $\mathcal E_i$ for each $i<m$, $Q=\prod_{i<m}\Delta_i$ and $Q'=\prod_{i<m}\Delta_i'$, then $\tilde W_{\Phi\cap Q}$ and $\tilde W_{\Psi\cap Q'}$ are Σ-separated. **P** If either $\Phi\cap Q$ or $\Psi\cap Q'$ is empty, this is trivial. Otherwise, set $R=\{i:\Delta_i=\Delta_i'\}$. **?** If $R\in\mathcal R$, then $\pi_R[\Phi]$ and $\pi_R[\Psi]$ both meet $\prod_{i\in R}\Delta_i$, which must be included in an atom of $\bigotimes_{i\in R}\mathcal E_{Ri}$, and is therefore either included in Q_R or disjoint from it; but Q_R separates $\pi_R[\Phi]$ from $\pi_R[\Psi]$. **X** There are therefore σ_0, $\sigma_1\in\{0,1\}^m$ such that $\tau(\sigma_0)\cap\tau(\sigma_1)=\emptyset$ and $\sigma_0(i)=\sigma_1(i)$ for every $i\in R$.

Suppose that $\phi\in Q$ and $\psi\in Q'$. Then $S=\{i:\phi(i)=\psi(i)\}$ is included in R. We can therefore find an $x\in X$ such that $x(\phi(i))=\sigma_0(i)$ and $x(\psi(i))=\sigma_1(i)$ for every $i<m$. So $\tau(x\phi)\cap\tau(x\psi)=\emptyset$. By Lemma 1E, $\tilde W_Q$ and $\tilde W_{Q'}$ are Σ-separated; *a fortiori*, $\tilde W_{\Phi\cap Q}$ and $\tilde W_{\Psi\cap Q'}$ are Σ-separated. **Q**

(c) By 1A(c-ii), $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated, as required.

1J Lemma Suppose that $\mathcal S\subseteq\mathcal R$ and that Φ, $\Psi\subseteq\Theta$ are such that
$\pi_R[\Phi]\cap\pi_R[\Psi]=\emptyset$ for every $R\in\mathcal S$,
$\pi_R[\phi]$ and $\pi_R[\Psi]$ are $\bigotimes_{i\in R}\mathcal P\Gamma_i$-separated for every $R\in\mathcal R\setminus\mathcal S$.
Then $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

proof Induce on $\#(\mathcal S)$. If $\mathcal S=\emptyset$, then this is just Lemma 1I. For the inductive step to $\#(\mathcal S)=k+1$, given Φ, $\Psi\subseteq\Theta$ such that
$\pi_R[\Phi]\cap\pi_R[\Psi]=\emptyset$ for every $R\in\mathcal S$,
$\pi_R[\Phi]$ and $\pi_R[\Psi]$ are $\bigotimes_{i\in R}\mathcal P\Gamma_i$-separated for every $R\in\mathcal R\setminus\mathcal S$,
then for each $R\in\mathcal R$ choose $P_R\subseteq\prod_{i\in R}\Gamma_i$ such that
$P_R=\pi_R[\Phi]$ if $R\in\mathcal S$,
$P_R\in\bigotimes_{i\in R}\mathcal P\Gamma_i$, $\pi_R[\Phi]\subseteq P_R$ and $\pi_R[\Psi]\cap P_R=\emptyset$ if $R\in\mathcal R\setminus\mathcal S$.
Fix $S\in\mathcal S$ and consider $\mathcal R'=\mathcal R\setminus\{S\}$, $\mathfrak Q=\mathfrak Q(\langle P_R\rangle_{R\in\mathcal R},S)$ as defined in 1Fc. If $\Delta_i\subseteq\Gamma_i$ for each $i\in S$, then $\prod_{i\in S}\Delta_i\in\mathfrak Q$. **P** If Φ', $\Psi'\subseteq\Theta$ are such that

$$\pi_R[\Phi']\subseteq P_R,\ \pi_R[\Psi']\cap P_R=\emptyset \text{ for every } R\in\mathcal R\setminus\{S\},$$

$$\pi_S[\Phi']\subseteq\textstyle\prod_{i\in S}\Delta_i,\quad \pi_S[\Psi']\cap\prod_{i\in S}\Delta_i=\emptyset,$$

then
$\pi_R[\Phi']\cap\pi_R[\Psi']=\emptyset$ for every $R\in\mathcal R'$,
$\pi_R[\Phi']$ and $\pi_R[\Psi']$ are $\bigotimes_{i\in R}\mathcal P\Gamma_i$-separated for every $R\in\mathcal R\setminus\mathcal R'$

(because $P_R \in \bigotimes_{i\in R} \mathcal{P}\Gamma_i$ separates $\pi_R[\Phi']$ from $\pi_R[\Psi']$ if $R \in \mathcal{R} \setminus \mathcal{S}$, while $\prod_{i\in S} \Delta_i$ separates $\pi_S[\Phi']$ from $\pi_S[\Psi']$). By the inductive hypothesis, $\bigcup_{\phi\in\Phi'} W_\phi$ and $\bigcup_{\psi\in\Psi'} W_\psi$ are Σ-separated. As Φ', Ψ' are arbitrary, $\prod_{i\in S} \Delta_i \in \mathfrak{Q}$. **Q**

Because $\mathfrak{Q}$ is closed under countable unions and intersections (Lemma 1H), and $\#(\Gamma_i)$ is a Kunen cardinal for every i, it follows that $\mathfrak{Q} = \mathcal{P}(\prod_{i\in S} \Gamma_i)$ (1Bc). In particular, $Q = \pi_S[\Phi]$ belongs to $\mathfrak{Q}$. Since

$$\pi_R[\Phi] \subseteq P_R,\ \pi_R[\Psi] \cap P_R = \emptyset \text{ for every } R \in \mathcal{R} \setminus \{S\},$$

$$\pi_S[\Phi] \subseteq Q, \quad \pi_S[\Psi] \cap Q = \emptyset,$$

$\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated. Thus the induction continues.

1K Lemma Suppose that Φ, $\Psi \subseteq \Theta$ are such that

($\dagger$) whenever $\phi \in \Phi$ and $\psi \in \Psi$, there is an $x \in X$ such that $\tau(x\phi) \cap \tau(x\psi) = \emptyset$.

Then $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

proof By Lemma 1G, $\pi_R[\Phi] \cap \pi_R[\Psi] = \emptyset$ for every $R \in \mathcal{R}$. So Lemma 1J, with $\mathcal{S} = \mathcal{R}$, tells us that $\tilde W_\Phi$ and $\tilde W_\Psi$ are Σ-separated.

2 Main theorem

2A The context (III) I continue from 1C.

(a) For this section, fix a family $\langle Z_{mi}\rangle_{i<m\in\mathbb{N}}$ of non-empty open-and-closed subsets of Z such that

$\langle Z_{mi}\rangle_{i<m}$ is a partition of Z for each $m \ge 1$,

whenever s, $t \in Z$ are distinct, the set $\{(m,i) : i < n \in \mathbb{N},\ Z_{mi}$ contains both s and $t\}$ is finite,

for every $m \ge 1$ and $i \le m$, there is a $j < m$ such that $Z_{m+1,i} \subseteq Z_{mj}$.

Γ will be a subset of Z, meeting Z_{mi} whenever $i < m$, such that $\#(\Gamma)$ is a Kunen cardinal. For $m \ge 1$ set $\Theta_m = \prod_{i<m} \Gamma \cap Z_{mi}$.

(b) (Y, ρ) will be a separable metric space. $C(X;Y)$ will be the space of continuous functions from X to Y, and C_Γ will be

$$\begin{aligned}\{f : f \in C(X;Y),\ f \text{ is determined by coordinates in } \Gamma\}\\ = \{f : f \in C(X;Y),\ f(x) = f(x') \text{ whenever } x{\restriction}\Gamma = x'{\restriction}\Gamma\}.\end{aligned}$$

I will write $\mathfrak{T}_c$ for the topology on C_Γ induced by the compact-open topology of $C(X;Y)$ (Engelking 89, §3.4), that is, the topology generated by sets of the form

$$\{f : f \in C_\Gamma,\ f[K] \subseteq H\}$$

where $K \subseteq X$ is compact and $H \subseteq Y$ is open; because X is compact, this is actually the topology of uniform convergence defined by the metric $(f,g) \mapsto \sup_{x\in X} \rho(f(x), g(x))$ (Engelking 89, 4.2.17).

$\mathcal{B}$ will be the Borel σ-algebra of C_Γ under $\mathfrak{T}_c$, while Σ_p will be the σ-algebra of subsets of C_Γ generated by sets of the form

$$\{f : f \in C_\Gamma,\ f(x) \in H\}$$

where $x \in X$ and $H \subseteq Y$ is open. Of course $\Sigma_p \subseteq \mathcal{B}$. (The aim of this section is to show that they are equal.)

(c) Let D be a countable dense subset of Y and $\mathcal{J}$ the countable family $\{B(y,\delta) : y \in D,\ \delta \ge 0$ is rational$\}$, where $B(y,\delta) = \{y' : \rho(y', y) \le \delta\}$. As in 1C, $\mathrm{T}_m = \mathcal{J}^{\{0,1\}^m}$ for $m \ge 1$. For $m \ge 1$, $\tau \in \mathrm{T}_m$ and $\phi \in \Theta_m$ set

$$W_{\tau,\phi} = \{f : f \in C_\Gamma,\ f(x) \in \tau(x\phi) \text{ for every } x \in X\}.$$

2B Lemma If $\epsilon > 0$ and $f \in C_\Gamma$, there are $m \ge 1$, $\tau \in \mathrm{T}_m$ and $\phi \in \Theta_m$ such that $f \in W_{\tau,\phi}$ and $\operatorname{diam} \tau(\sigma) \le \epsilon$ for every $\sigma \in \{0,1\}^m$.

proof Take $\delta \in \mathbb{Q}\cap]0,\epsilon]$. For each $x \in X$, there is a finite set $I_x \subseteq \Gamma$ such that $\rho(f(x'), f(x)) \le \frac{1}{3}\delta$ whenever $x' \in U_x$, where $U_x = \{x' : x' \in X,\ x'\restriction I_x = x\restriction I_x\}$. **P?** Otherwise,

$$L_I = \{x' : x' \in X,\ x'\restriction I = x\restriction I,\ \rho(f(x'), f(x)) \ge \frac{1}{3}\delta\}$$

is non-empty for every finite $I \subseteq \Gamma$. Since every L_I is compact and $L_I \subseteq L_{I'}$ when $I \supseteq I'$, there is an $x' \in \bigcap\{L_I : I \in [\Gamma]^{<\omega}\}$; but now $x'\restriction\Gamma = x\restriction\Gamma$ and $f(x') \ne f(x)$. **XQ**

Let $L \subseteq X$ be a finite set such that $\bigcup_{x\in L} U_x = X$. Set $I = \bigcup_{x\in L} I_x$. Let $m \ge 1$ be such that no Z_{mi}, for $i < m$, contains more than one point of I; let $\phi \in \Theta_m$ be such that $\phi(i) \in I$ whenever $i < m$ and $I \cap Z_{mi}$ is not empty. Observe that $\rho(f(x), f(x') \le \frac{2}{3}\delta$ whenever $x,\ x' \in X$ and $x\phi = x'\phi$. **P** There is an $x_0 \in L$ such that $x \in U_{x_0}$. Now $I_{x_0} \subseteq I \subseteq \phi[m]$, so $x'\restriction I_{x_0} = x\restriction I_{x_0}$ and $x' \in U_{x_0}$. **Q** Next, there is a $\tau \in \mathrm{T}_m$ such that $\operatorname{diam}\tau(\sigma) \le \epsilon$ and $f(x) \in \tau(\sigma)$ whenever $\sigma \in \{0,1\}^m$, $x \in X$ and $x\phi = \sigma$. **P** Given $\sigma \in \{0,1\}^m$, $\{f(x) : x\phi = \sigma\}$ has diameter at most $\frac{2}{3}\delta$. Take any x such that $x\phi = \sigma$; then there is a $y \in D$ such that $\rho(f(x), y) \le \frac{1}{3}\delta$; set $\tau(\sigma) = B(y, \delta)$. **Q** And $f \in W_{\tau,\phi}$.

2C Proposition Take $m \ge 1$ and $\tau \in \mathrm{T}_m$. Then $E^* = \bigcup\{W_{\tau,\phi} : \phi \in \Theta_m\}$ belongs to Σ_p.

proof (a) For $n \ge 1$ let $X_n \subseteq X$ be

$$\{x : x \in X,\ x\restriction Z_{ni} \text{ is constant for each } i < n\}.$$

Then X_n is finite and each member of X_n is a continuous function from Z to $\{0,1\}$; also $\bigcup_{n\ge1} X_n$ is dense in X, and $X_n \subseteq X_{n+1}$ for each n. For $n \ge 1$ let Δ_n be a finite subset of Γ meeting $Z_{mi} \cap Z_{nj}$ whenever $i < m$ and $j < n$ and $Z_{mi} \cap Z_{nj} \ne \emptyset$, and set $\Psi_n = \prod_{i<m}(\Delta_n \cap Z_{mi}) \subseteq \Theta_m$. Set

$$A = \bigcap_{n\ge1} \bigcup_{\psi\in\Psi_n} \bigcap_{x\in X_n} \{f : f \in C_\Gamma,\ f(x) \in \tau(x\psi)\}.$$

Then $A \in \Sigma_p$, because $\tau(\sigma)$ is closed in Y for every $\sigma \in \{0,1\}^m$.

(b) Suppose that $f \in E^*_\tau$. Then there is a $\phi \in \Theta_m$ such that $f \in W_{\tau,\phi}$. If $n \ge 1$, there is a $\psi : m \to \Delta_n$ such that $\psi(i) \in Z_{nj} \cap Z_{mi}$ whenever $i < m$, $j < n$ and $\phi(i) \in Z_{nj}$; observe that $\psi \in \Psi_n$. If now $x \in X_n$, $x\psi = x\phi$ so $f(x) \in \tau(x\psi)$. Thus $f \in A$. As f is arbitrary, $E^*_\tau \subseteq A$.

(c) Now suppose that $f \in A$. For each $n \ge 1$ take $\psi_n \in \Psi_n$ such that $f(x) \in \tau(x\psi_n)$ for every $x \in X_n$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. Then $\phi(i) = \lim_{i\to\mathcal{F}} \psi_n(i)$ is defined for the topology of Z and belongs to Z_{mi} for each $i < m$. If $x \in X_l$, where $l \ge 1$, then for any $n \ge l$ we have $x\phi = \lim_{n\to\mathcal{F}} x\psi_n$ because x is continuous; accordingly $\{n : \tau(x\phi) = \tau(x\psi_n)\}$ belongs to $\mathcal{F}$ and $f(x) \in \tau(x\phi)$. Because f is continuous and $\bigcup_{l\ge1} X_l$ is dense in X, it follows that $f(x) \in \tau(x\phi)$ for every $x \in X$.

Let $\phi' \in \Theta_m$ be such that $\phi'(i) = \phi(i)$ whenever $i < m$ and $\phi(i) \in \Gamma$. If $x \in X$, there is an $x' \in X$ such that

$$\begin{aligned} x'(t) &= x(t) \text{ if } t \in \Gamma,\\ &= x(\phi'(i)) \text{ if } i < m,\ t = \phi(i) \notin \Gamma. \end{aligned}$$

Now, because f is determined by coordinates in Γ,

$$f(x) = f(x') \in \tau(x'\phi) = \tau(x\phi').$$

As x is arbitrary, $f \in W_{\tau,\phi'} \subseteq E^*_\tau$. As f is arbitrary, $A \subseteq E^*_\tau$.

(d) So $E^*_\tau = A \in \Sigma_p$.

2D Theorem $\Sigma_p = \mathcal{B}$.

proof Take any $G \in \mathfrak{T}_c$.

(a) Fix $m \ge 1$ and $\tau \in \mathrm{T}_m$ for the moment. I aim to apply the results of §1. Set $\Gamma_i = \Gamma \cap Z_{mi}$ for each $i < m$, so that Θ, as defined in 1D, is Θ_m of this section. Set $\Omega = C_\Gamma$ and $\Sigma = \Sigma_p$, and for $\theta \in \Theta = \Theta_m$ set $W_\theta = W_{\tau,\theta}$.

(i) The family $\langle W_\theta\rangle_{\theta\in\Theta_m}$ satisfies (*) of 1D. **P** If $x \in X$ and $J,\ J' \in \mathcal{J}$ are disjoint, then

$$\bigcup\{W_\theta : \theta \in \Theta,\, \tau(x\theta) \subseteq J\} \subseteq \{f : f \in C_\Gamma,\, f(x) \in J\},$$

$$\bigcup\{W_\theta : \theta \in \Theta,\, \tau(x\theta) \subseteq J'\} \subseteq \{f : f \in C_\Gamma,\, f(x) \in J'\};$$

since $\{f : f \in C_\Gamma,\, f(x) \in J\}$ and $\{f : f \in C_\Gamma,\, f(x) \in J'\}$ are disjoint members of $\Sigma = \Sigma_p$, $\bigcup\{W_\theta : \theta \in \Theta,$ $\tau(x\theta) \subseteq J\}$ and $\bigcup\{W_\theta : \theta \in \Theta,\, \tau(x\theta) \subseteq J'\}$ are Σ-separated. **Q**

(ii) Set

$$\Phi = \{\phi : \phi \in \Theta_m,\, W_\phi \not\subseteq G\}, \qquad \Psi = \{\psi : \psi \in \Theta_m,\, W_\psi \cap \tilde{W}_\Phi = \emptyset\},$$

writing $\tilde{W}_\Phi = \bigcup_{\phi\in\Phi} W_\phi$ as in 1D. Then Φ and Ψ satisfy (†) of Lemma 1K. **P?** If $\phi \in \Phi$, $\psi \in \Psi$ and $\tau(x\phi) \cap \tau(x\psi) \neq \emptyset$ for every $x \in X$, set $I = \phi[m] \cup \psi[m]$ and for each $z \in \{0,1\}^I$ choose $\alpha_z \in \tau(z\phi) \cap \tau(z\psi)$. Set $g(x) = \alpha_{x\restriction I}$ for $x \in X$; then $g \in C_\Gamma$, because $I \subseteq \Gamma$. Also $g \in W_\phi \cap W_\psi$. But this contradicts the definition of Ψ. **XQ**

(iii) By 1K, $\tilde{W}_\Phi$ and $\tilde{W}_\Psi$ are Σ-separated.

(b) Re-writing this result in the language of the present section, we see that if $m \geq 1$, $\tau \in \mathrm{T}_m$ and we set

$$\Phi_\tau = \{\phi : \phi \in \Theta_m,\, W_{\phi,\tau} \not\subseteq G\},$$

$$\Psi_\tau = \{\psi : \psi \in \Theta_m,\, W_{\psi,\tau} \cap W_{\phi,\tau} = \emptyset \text{ for every } \phi \in \Phi_\tau\},$$

then $\bigcup_{\phi\in\Phi_\tau} W_{\phi,\tau}$ and $\bigcup_{\psi\in\Psi_\tau} W_{\psi,\tau}$ are Σ_p-separated. Let $E_\tau \in \Sigma_p$ be such that

$$\bigcup_{\psi\in\Psi_\tau} W_{\psi,\tau} \subseteq E_\tau \subseteq C_\Gamma \setminus \bigcup_{\phi\in\Phi_\tau} W_{\phi,\tau}.$$

Because $E^*_\tau = \bigcup_{\phi\in\Theta_m} W_{\tau,\phi}$ belongs to Σ_p (2C), we can take it that $E_\tau \subseteq E^*_\tau$.

(c) Set $E = \bigcup_{m\geq 1,\tau\in\mathrm{T}_m} E_\tau$; because $\mathcal{J}$ is countable, every T_m is countable and $E \in \Sigma_p$.

(i) $G \subseteq E$. **P** Let $f \in G$. Then there is an $\epsilon > 0$ such that $g \in G$ whenever $g \in C_\Gamma$ and $\rho(g(x), f(x)) \leq 2\epsilon$ for every $x \in X$. By Lemma 2B, there are $m \geq 1$, $\tau \in \mathrm{T}_m$ and $\psi \in \Theta_m$ such that $f \in W_{\tau,\psi}$ and $\operatorname{diam}\tau(\sigma) \leq \epsilon$ for every $\sigma \in \{0,1\}^m$. **?** If $\psi \notin \Psi_\tau$, let $\phi \in \Phi_\tau$ be such that there is a $g \in W_{\tau,\phi} \cap W_{\tau,\psi}$. There is also an $h \in W_{\tau,\phi} \setminus G$. But for any $x \in X$,

$$f(x) \in \tau(x\psi),\ g(x) \in \tau(x\psi) \text{ so } \rho(f(x), g(x)) \leq \epsilon,$$

$$g(x) \in \tau(x\phi),\ h(x) \in \tau(x\phi) \text{ so } \rho(g(x), h(x)) \leq \epsilon,$$

and $\rho(h(x), f(x)) \leq 2\epsilon$; as x is arbitrary, $h \in G$, which is absurd. **X**

Thus $\psi \in \Psi_\tau$ and

$$f \in W_{\tau,\psi} \subseteq E_\tau \subseteq E.$$

As f is arbitrary, $G \subseteq E$. **Q**

(ii) $E \subseteq G$. **P?** Otherwise, there are $m \geq 1$, $\tau \in \mathrm{T}_m$ and $f \in E_\tau \setminus G$. As $E_\tau \subseteq Y_\tau$, there is a $\phi \in \Theta_m$ such that $f \in W_{\tau,\phi}$. But in this case $W_{\tau,\phi} \not\subseteq G$, $\phi \in \Phi_\tau$ and $f \notin E_\tau$. **XQ**

(c) Thus $G = E$ belongs to Σ_p. As G is arbitrary, $\mathcal{B} \subseteq \Sigma_p$ and $\mathcal{B} = \Sigma_p$.

3 Tidying up

3A Theorem Let K be a compact dyadic space such that its weight $w(K)$ is a Kunen cardinal, and Y a separable metrizable space. On the space $C(K;Y)$ of continuous functions from K to Y, let Σ_p be the σ-algebra generated by sets of the form $\{f : f \in C(K;Y),\, f(v) \in H\}$ where $v \in K$ and $H \subseteq Y$ is open. Then Σ_p is the Borel σ-algebra of the compact-open topology $\mathfrak{T}_c$ on $C(K;Y)$.

proof (a) If K is finite, $C(K;Y) = Y^K$, the compact-open topology is just the product topology, and the result is elementary, because Y is second-countable. So suppose from now on that K is infinite.

(b) If K is infinite, it is a continuous image of $\{0,1\}^{w(K)}$ (Engelking 89, 3.12.12 or Fremlin 03, 4A2Dd). As $w(K)$ is an infinite Kunen cardinal, it is at most $\mathfrak{c}$ (1Bb), and there is a dense set $\Gamma \subseteq Z$ of cardinal $w(K)$. $\{0,1\}^\Gamma$ is homeomorphic to $\{0,1\}^{w(K)}$, so there is a continuous surjection $q : \{0,1\}^\Gamma \to K$.

For $x \in X$, set $\pi(x) = x{\restriction}\Gamma$; then $\pi : X \to \{0,1\}^\Gamma$ and $q\pi : X \to K$ are continuous surjections, so we have an injection $\tilde{\pi} : C(K;Y) \to C(X;Y)$ defined by setting $\tilde{\pi}(f) = fq\pi$ for $f \in C(K;Y)$.

Because π is determined by coordinates in Γ, so is $fq\pi$ for every $f : K \to Y$; accordingly $\tilde{\pi}$ can be thought of as a map from $C(K;Y)$ to C_Γ as defined in 2A. Let $\mathfrak{T}'_c$ be the topology on C_Γ induced by the compact-open topology of $C(X;Y)$.

Let ρ be a metric on Y defining its topology. Then $\mathfrak{T}_c$ and $\mathfrak{T}'_c$ are defined by the metrics $\tilde{\rho}$, $\tilde{\rho}'$ where

$$\tilde{\rho}(f_0, f_1) = \sup_{v \in K} \rho(f_0(v), f_1(v)) \text{ for } f_0,\ f_1 \in C(K;Y),$$

$$\tilde{\rho}'(g_0, g_1) = \sup_{x \in X} \rho(g_0(x), g_1(x)) \text{ for } g_0,\ g_1 \in C_\Gamma.$$

Because $q\pi : X \to K$ is a surjection, $\tilde{\pi}$ is an isometry for $\tilde{\rho}$ and $\tilde{\rho}'$.

Suppose now that $G \subseteq C(K;Y)$ is $\mathfrak{T}_c$-open. Then $\tilde{\pi}[G]$ is relatively open in the isometric copy $\tilde{\pi}[C(K;Y)] \subseteq C_\Gamma$, and there is a $\mathfrak{T}'_c$-open $G' \subseteq C_\Gamma$ such that $\tilde{\pi}[G] = G' \cap \tilde{\pi}[C(K;Y)]$, that is, $G = \tilde{\pi}^{-1}[G']$. By Theorem 2D, G' belongs to the σ-algebra of subsets of C_Γ generated by sets of the form $V_{xH} = \{g : g \in C_\Gamma,\ g(x) \in H\}$ where $x \in X$ and $H \subseteq Y$ is open. Consequently $G = \tilde{\pi}^{-1}[G']$ belongs to the σ-algebra of subsets of $C(K;Y)$ generated by $\{\tilde{\pi}^{-1}[V_{xH}] : x \in X,\ H \subseteq Y \text{ is open}\}$. But $\tilde{\pi}^{-1}[V_{xH}] = \{f : f(q\pi(x)) \in H\}$ belongs to Σ_p whenever $x \in X$ and $H \subseteq Y$ is open, so $G \in \Sigma_p$. As G is arbitrary, $\mathfrak{T}_c$ and $\mathcal{B}$ are included in Σ_p.

On the other hand, of course, $\Sigma_p \subseteq \mathcal{B}$, just as in 2Ab. So $\Sigma_p = \mathcal{B}$, as claimed.

3B Proposition Let κ be a cardinal, $C(\{0,1\}^\kappa)$ the Banach space of continuous real-valued functions on $\{0,1\}^\kappa$, $\mathcal{B}$ the Borel σ-algebra of $C(\{0,1\}^\kappa)$ for the norm topology, and Σ_p the σ-algebra generated by sets of the form $V_{xH} = \{f : f \in C(\{0,1\}^\kappa),\ f(x) \in H\}$ where $x \in \{0,1\}^\kappa$ and $H \subseteq \mathbb{R}$ is open. Then κ is a Kunen cardinal iff $\Sigma_p = \mathcal{B}$.

proof (a) If κ is a Kunen cardinal then $\Sigma_p = \mathcal{B}$ by Theorem 3A.

(b) If κ is not a Kunen cardinal, let Γ_0, Γ_1 be disjoint subsets of κ of cardinal κ. Let $A \subseteq C(\{0,1\}^\kappa)$ be the set of continuous $\{0,1\}$-valued functions; then A is closed and discrete for the norm topology, so every subset of A belongs to $\mathcal{B}$. For $\xi \in \Gamma_0$, $\eta \in \Gamma_1$ set

$$f_{\xi\eta}(x) = x(\xi)x(\eta)$$

for $x \in \{0,1\}^\kappa$. Then $(\xi, \eta) \mapsto f_{\xi\eta} : \Gamma_0 \times \Gamma_1 \to C(\{0,1\}^\kappa)$ is injective. So any subset of $\Gamma_0 \times \Gamma_1$ is of the form $\{(\xi, \eta) : f_{\xi\eta} \in E\}$ for some $E \in \mathcal{B}$.

On the other hand, if $x \in \{0,1\}^\kappa$ and $H \subseteq \mathbb{R}$, then $\{(\xi, \eta) : f_{\xi\eta}(x) \in H\}$ is either $\{(\xi, \eta) : x(\xi) = 1$ and $x(\eta) = 1\}$ or $\{(\xi, \eta) : x(\xi) = 0$ or $x(\eta) = 0\}$ or $\Gamma_0 \times \Gamma_1$ or $\emptyset$; in any case, it belongs to $\mathcal{P}\Gamma_0 \otimes \mathcal{P}\Gamma_1$. So if $E \in \Sigma_p$, $\{(\xi, \eta) : f_{\xi\eta} \in E\} \in \mathcal{P}\Gamma_0 \widehat{\otimes} \mathcal{P}\Gamma_1$.

Since we are supposing that $\mathcal{P}(\kappa \times \kappa) \neq \mathcal{P}\kappa \widehat{\otimes} \mathcal{P}\kappa$, and therefore $\mathcal{P}(\Gamma_0 \times \Gamma_1) \neq \mathcal{P}\Gamma_0 \widehat{\otimes} \mathcal{P}\Gamma_1$, Σ_p cannot be the whole of $\mathcal{B}$.

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