The Cascales-Kadets-Rodríguez selection theorem

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I give an account of a theorem presented at the meeting on Integration, Vector Measures and Related Topics IV at La Manga, March 2 – March 5, 2011.

1 The selection theorem

1A Proposition Let U be a locally convex Hausdorff linear topological space and K a weakly compact subset of U. Write S for $\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, and suppose that $\langle A_\sigma\rangle_{\sigma\in S}$ is a family of non-empty subsets of K such that $A_\sigma\subseteq A_\tau$ whenever σ, $\tau\in S$ and σ extends τ. For each $\sigma\in S$, write C_σ for the convex hull of $A_{\sigma^\frown<1>}-A_{\sigma^\frown<0>}$. Then $0\in\overline{\bigcup_{\sigma\in S}C_\sigma}$.

proof (a) Consider first the case in which U is a Banach space. For each $\sigma\in S$, the norm-closed convex hull K_σ of A_σ is a weakly compact set (Fremlin 03, 461J), and $\overline{C}_\sigma=K_{\sigma^\frown<1>}-K_{\sigma^\frown<0>}$. So if there is any $\sigma\in S$ such that $K_{\sigma^\frown<1>}$ meets $K_{\sigma^\frown<0>}$, we can stop; henceforth, let us suppose that $K_{\sigma^\frown<1>}$ is disjoint from $K_{\sigma^\frown<0>}$ for every σ, so that $\langle K_\sigma\rangle_{\sigma\in\{0,1\}^n}$ is disjoint for every $n\in\mathbb{N}$.

For $\sigma\in S$, set $I_\sigma=\{y:\sigma\subseteq y\in\{0,1\}^{\mathbb{N}}$. Consider $R=\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\{0,1\}^n}K_\sigma\times I_\sigma$. Then $R\subseteq U\times\{0,1\}^{\mathbb{N}}$ is compact if U is given its weak topology and $\{0,1\}^{\mathbb{N}}$ is given its usual topology. For each $y\in\{0,1\}^{\mathbb{N}}$, $\langle K_{y\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty weakly compact subsets of U, so has non-empty intersection; thus $\pi_2[R]=\{0,1\}^{\mathbb{N}}$, where $\pi_2(u,y)=y$ for $(u,y)\in R$. Let ν be the usual measure on $\{0,1\}^{\mathbb{N}}$. By Fremlin 03, 418L, there is a Radon probability measure λ on R such that π_2 is inverse-measure-preserving for λ and ν. Now set $\pi_1(u,y)=u$ for $(u,y)\in\mathbb{R}$, and let μ be the image measure $\lambda\pi_1^{-1}$; then μ is a Radon probability measure for the weak topology of U (Fremlin 03, 418I). If $n\in\mathbb{N}$ and $\sigma\in\{0,1\}^{\mathbb{N}}$, then

$$\mu K_\sigma=\lambda(R\cap(K_\sigma\times\{0,1\}^{\mathbb{N}}))=\lambda(R\cap(U\times I_\sigma))$$

(because $K_\sigma\cap K_\tau=I_\sigma\cap I_\tau=\emptyset$ whenever $\tau\in\{0,1\}^n\setminus\{\sigma\}$)

$$=\nu I_\sigma=2^{-n}.$$

Since μ is a Radon probability measure for the weak topology on the complete metrizable locally convex space U, it is also Radon for the norm topology (Fremlin 03, 466A). Let L be a norm-compact subset of U such that $\mu L>0$. Take any $\epsilon>0$. Then L must be expressible as a finite union of compact sets of diameter less than ϵ, so we must have a compact set L_1 of diameter less than ϵ and measure greater than 0. Now, however, take $m\in\mathbb{N}$ such that $2^{-m}<\mu L_1$. Then there must be distinct σ, $\tau\in\{0,1\}^m$ such that L_1 meets both K_σ and K_τ; setting $\upsilon=\sigma\cap\tau$, L_1 meets both $K_{\upsilon^\frown<1>}$ and $K_{\upsilon^\frown<0>}$, so that $\overline{C}_\upsilon$ and C_υ meet $\{u:\|u\|<\epsilon\}$. As ϵ is arbitrary, $0\in\overline{\bigcup_{\sigma\in S}C_\sigma}$, as required.

(b) For the general case, let W be any neighbourhood of 0 in U. Then there is a continuous seminorm $\rho:U\to[0,\infty[$ such that $W\supseteq\{u:\rho(u)\le1\}$. Set $N_\rho=\{u:\rho(u)=0\}$, so that U/N_ρ is a normed space; write V for the completion of U/N_ρ and $T:U\to V$ for the canonical map, so that $\|Tu\|=\rho(u)$ for $u\in U$. Because T is weakly continuous, $T[K]$ is weakly compact in V. Applying (a) to $\langle T[A_\sigma]\rangle_{\sigma\in S}$, we see that there is a $\sigma\in S$ such that $T[C_\sigma]$ meets the unit ball of V, in which case C_σ meets W. As W is arbitrary, $\overline{\bigcup_{\sigma\in S}C_\sigma}$ contains 0.

1B Definition Let U be a locally convex linear topological space, X a set and Σ a σ-algebra of subsets of X.

(a) A U-valued function ϕ defined on a subset of U is **scalarly measurable** if $f\phi$ is measurable (in the sense of Fremlin 00, 121C) for every $f\in U^*$.

(b) If $F \subseteq X \times U$ is a relation, I will call F **scalarly hull-measurable** if $x \mapsto \sup f[F[\{x\}]] : X \to [-\infty, \infty]$ is measurable for every $f \in U^*$, where here $\sup \emptyset$ is to be interpreted as $-\infty$.[1]

1C Lemma Let U be a locally convex Hausdorff linear topological space, X a set, Σ a σ-algebra of subsets of X and $F \subseteq X \times U$ a scalarly hull-measurable relation such that $F[\{x\}]$ is weakly compact for every $x \in X$. Suppose that $\psi : X \to U^*$ is a function of the form

$$\psi(x) = f_i \text{ for } x \in E_i$$

where $\langle E_i \rangle_{i \in I}$ is a countable partition of X into measurable sets and $f_i \in U^*$ for each $i \in I$. Set

$$G = \{(x, u) : (x, u) \in F,\ \psi(x)(u) = \sup \psi(x)[F[\{u\}]]\}.$$

Then G is scalarly hull-measurable.

proof (a) Suppose to begin with that $F[\{x\}]$ is non-empty for every $x \in X$ and that ψ is constant, with $\psi(x) = f$ for every $x \in X$. Take any $g \in U^*$. For $n \in \mathbb{N}$, set $h_n = f + 2^{-n} g$. Then

$$\sup g[G[\{x\}]] = \lim_{n\to\infty} 2^n (\sup h_n[F[\{x\}]] - \sup f[F[\{x\}]])$$

for every $x \in X$. **P** Set $\hat\alpha = \max f[F[\{x\}]]$, so that $G[\{x\}] = \{u : u \in F[\{x\}],\ f(u) = \hat\alpha\}$, and $\hat\beta = \max g[G[\{x\}]]$. For $u \in U$, set $Tu = (f(u), g(u))$, so that $T : U \to \mathbb{R}^2$ is a continuous linear operator. Set $K = T[F[\{x\}]]$, so that K is a compact subset of $\mathbb{R}^2$, $\hat\alpha = \max\{\alpha : (\alpha, \beta) \in K\}$ and $\hat\beta = \max\{\beta : (\hat\alpha, \beta) \in K\}$. Moreover, for each $n \in \mathbb{N}$,

$$\sup h_n[F[\{x\}]] = \max\{\alpha + 2^{-n}\beta : (\alpha, \beta) \in K\}.$$

For each $n \in \mathbb{N}$, let $(\alpha_n, \beta_n) \in K$ be such that $\alpha_n + 2^{-n}\beta_n = \max\{\alpha + 2^{-n}\beta : (\alpha, \beta) \in K\}$. Note that $(\hat\alpha, \hat\beta) \in K$, so that $\alpha_n + 2^{-n}\beta_n \ge \hat\alpha + 2^{-n}\hat\beta$ for each n, while also $\alpha_n \le \hat\alpha$ for every n. Let (α^*, β^*) be any cluster point of $\langle(\alpha_n, \beta_n)\rangle_{n\in\mathbb{N}}$. Then

$$\alpha^* \ge \liminf_{n\to\infty} \alpha_n = \liminf_{n\to\infty} \alpha_n + 2^{-n}\beta_n \ge \liminf_{n\to\infty} \hat\alpha + 2^{-n}\hat\beta = \hat\alpha \ge \alpha^*,$$

and $\alpha^* = \alpha$. Next, $2^{-n}\beta_n \ge 2^{-n}\hat\beta$ for every $n \in \mathbb{N}$, so $\beta^* \ge \hat\beta$; but $(\alpha^*, \beta^*) \in K$, so β^* must be equal to $\hat\beta$.

Thus $(\hat\alpha, \hat\beta)$ is the only cluster point of $\langle(\alpha_n, \beta_n)\rangle_{n\in\mathbb{N}}$; as K is compact, $\langle(\alpha_n, \beta_n)\rangle_{n\in\mathbb{N}} \to (\hat\alpha, \hat\beta)$. But this means that

$$\limsup_{n\to\infty} \beta_n + 2^n(\alpha_n - \hat\alpha) \le \limsup_{n\to\infty} \beta_n = \hat\beta.$$

On the other hand,

$$\alpha_n + 2^{-n}\beta_n \ge \hat\alpha + 2^{-n}\hat\beta, \quad \beta_n + 2^n(\alpha_n - \hat\alpha) \ge \hat\beta$$

for every n, so

$$2^n(\sup h_n[F[\{x\}]] - \sup f[F[\{x\}]]) = \beta_n + 2^n(\alpha_n - \hat\alpha) \to \hat\beta = \sup g[G[\{x\}]]$$

as $n \to \infty$. **Q**

Since $x \mapsto \sup f[F[\{x\}]]$ and $x \mapsto \sup h_n[F[\{x\}]]$ are measurable functions for every n, $x \mapsto \sup g[G[\{x\}]]$ is measurable. As g is arbitrary, G is scalarly hull-measurable.

(b) For the general case, set $X_0 = \{x : F[\{x\}] \ne \emptyset\} = \{x : \sup \hat f_0[F[\{x\}]] \ne -\infty\}$, where $\hat f_0$ is the zero functional in U^*, and let $\langle E_i \rangle_{i\in I}$ be a countable partition of X into measurable sets such that ψ is constant on each E_i. Take any $g \in U^*$. Applying (a) to $X_0 \cap E_i$, with the subspace σ-algebra, and the relation $F \cap (E_i \times U)$ for each $i \in I$, we see that $x \mapsto \sup g[G[\{x\}]] : X_0 \cap E_i \to \mathbb{R}$ is measurable for each $i \in I$, so that $x \mapsto \sup g[G[\{x\}]] : X \to [-\infty, \infty[$ is measurable. As g is arbitrary, G is scalarly hull-measurable in this case also.

1D Theorem (Cascales Kadets & Rodríguez 10, Theorem 3.8) Let U be a metrizable locally convex linear topological space, (X, Σ, μ) a complete strictly localizable measure space and $F \subseteq X \times U$ a scalarly hull-measurable relation such that $F[\{x\}]$ is weakly compact for every $x \in X$. Then F has a scalarly measurable selector.

[1] Cascales Kadets & Rodríguez 10 would treat F as a function from X to $\mathcal{P}U$, and call it 'scalarly measurable'.

proof (a) To begin with (down to the end of (f) below) let us suppose that μ is totally finite and that $F[\{x\}]$ is non-empty for every $x \in X$. Let Φ be the set of scalarly hull-measurable sets $G \subseteq F$ such that all the vertical sections $G[\{x\}]$ are non-empty weakly compact sets, and Ψ the set of functions $\psi : X \to U^*$ such that ψ takes finitely many values and $\psi^{-1}[\{f\}] \in \Sigma$ for every $f \in U^*$. Note that if $G \in \Phi$ and $\psi \in \Psi$ then $x \mapsto \operatorname{diam}(\psi(x)[G[\{x\}]]) : X \to \mathbb{R}$ is measurable. **P** Let $\langle E_i \rangle_{i\in I}$ be a finite partition of X into measurable sets such that $\psi \restriction E_i$ is constant for each i; let $f_i \in U^*$ be such that $\psi(x) = f_i$ for $x \in E_i$. Then

$$x \mapsto \sup f_i[G[\{x\}]], \quad x \mapsto \sup(-f_i)[G[\{x\}]],$$

$$x \mapsto \operatorname{diam} f_i[G[\{x\}]] = \sup f_i[G[\{x\}] - \sup(-f_i)[G[\{x\}]$$

are measurable for each i, so $x \mapsto \operatorname{diam}(\psi(x)[G[\{x\}]])$ is measurable. **Q**

(b) For $G \in \Phi$ and $\psi \in \Psi$, set

$$h_{G\psi}(x) = \frac{\operatorname{diam}(\psi(x)[G[\{x\}]])}{1+\operatorname{diam}(\psi(x)[G[\{x\}]])}$$

for each $x \in X$, and $\delta(G,\psi) = \int h_{G\psi}$. Because U is metrizable, there is a $\langle W_n \rangle_{n\in\mathbb{N}}$ running over a base of neighbourhoods of 0 in U. For each $n \in \mathbb{N}$, set $\Psi_n = \{\psi : \psi \in \Psi,\ \psi[X] \subseteq W_n^\circ\}$. Note that if $G \in \Phi$ and ψ_0, $\psi_1 \in \Psi_n$, there is a $\psi \in \Psi_n$ such that $h_{G\psi} = h_{G\psi_0} \vee h_{G\psi_1}$. **P** Let $\langle E_i \rangle_{i\in I}$ be a finite partition of X into non-empty measurable sets such that both ψ_0 and ψ_1 are constant on E_i for every i; say that $\psi_0(x) = f_i$ and $\psi_1(x) = f_i'$ for each i. Set

$$E_i' = \{x : x \in E_i,\ \operatorname{diam} f_i[G[\{x\}]] \ge \operatorname{diam} f_i'[G[\{x\}]];$$

then E_i' is measurable for each i, so if we set

$$\begin{aligned}\psi(x) &= f_i \text{ if } i \in I \text{ and } x \in E_i',\\ &= f_i' \text{ if } i \in I \text{ and } x \in E_i \setminus E_i',\end{aligned}$$

we get $h_{G\psi}(x) = \max(h_{G\psi_0}(x), h_{G\psi_1}(x))$ for every x, while $\psi \in \Psi_n$. **Q**

(c) For $G \in \Phi$ and $n \in \mathbb{N}$, set

$$\Delta_n(G) = \sup_{\psi\in\Psi_n} \delta(G,\psi).$$

From (b) we see, inducing on k, that if $\psi_0, \dots, \psi_k \in \Psi_n$, then there is a $\psi \in \Psi_n$ such that $h_{G\psi} = \sup_{i\le k} h_{G\psi_i}$ and $\int \sup_{i\le k} h_{G\psi_i} \le \Delta_n(G)$; so in fact if $\langle \psi_k \rangle_{k\in\mathbb{N}}$ is any sequence in Ψ_n and $h = \sup_{k\in\mathbb{N}} h_{G\psi_k}$, then $\int h \le \Delta_n(G)$.

(d) (The key.) If $G \in \Phi$ and $n \in \mathbb{N}$ are such that $\Delta_n(G) > 0$, there is a $G' \in \Phi$ such that $G' \subseteq G$ and $\Delta_n(G') < \Delta_n(G)$. **P?** Suppose, if possible, otherwise. Set $S = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ and let $\langle \epsilon_\sigma \rangle_{\sigma\in S}$ be a family of strictly positive real numbers such that $\sum_{\sigma\in S} \epsilon_\sigma < \Delta_n(G)$. Choose $\langle G_\sigma \rangle_{\sigma\in S}$ and $\langle \psi_\sigma \rangle_{\sigma\in S}$ inductively, as follows. $G_\emptyset = G$. Given that $\sigma \in S$, $G_\sigma \in \Phi$ and $G_\sigma \subseteq G$, take $\psi_\sigma \in \Psi_n$ such that

$$\delta(G_\sigma, \psi_\sigma) \ge \Delta_n(G_\sigma) - \epsilon_\sigma = \Delta_n(G) - \epsilon_\sigma.$$

Set

$$G_{\sigma^\frown<1>} = \{(x,u) : (x,u) \in G_\sigma,\ \psi_\sigma(x)(u) = \sup \psi_\sigma(x)[G_\sigma[\{x\}]]\},$$

$$G_{\sigma^\frown<0>} = \{(x,u) : (x,u) \in G_\sigma,\ \psi_\sigma(x)(u) = \inf \psi_\sigma(x)[G_\sigma[\{x\}]]\}.$$

By Lemma 3, $G_{\sigma^\frown<1>}$ and $G_{\sigma^\frown<0>}$ both belong to Φ. Continue.

At the end of the construction, let $\langle \psi_k' \rangle_{k\in\mathbb{N}}$ be a sequence in Ψ_n such that $\Delta_n(G) = \sup_{k\in\mathbb{N}} \delta(G, \psi_k')$, and set

$$h = \sup_{\sigma\in\Sigma} h_{G\psi_\sigma} \vee \sup_{k\in\mathbb{N}} h_{G\psi_k'},$$

so that $\int h \le \Delta_n(G)$, as noted in (c), while $h_{G_\sigma\psi_\sigma} \le h$ for every $\sigma \in S$. But this means that $\int h - h_{G_\sigma\psi_\sigma} \le \epsilon_\sigma$ for every $\sigma \in \Sigma$; also, of course, $\int h = \Delta_n(G) > \sum_{\sigma\in S} \epsilon_\sigma$.

There must therefore be an $x \in X$ such that $\gamma = \inf_{\sigma\in S} h_{G_\sigma\psi_\sigma}(x)$ is greater than 0. Set $K_\sigma = G_\sigma[\{x\}]$ and $f_\sigma = \psi_\sigma(x)$ for each σ, so that $K_\sigma \subseteq U$ is weakly compact, $f_\sigma \in W_n^\circ$, and

$$K_{\sigma^\frown<1>} = \{u : u \in K_\sigma,\ f_\sigma(u) = \sup f_\sigma[K_\sigma]\},$$

$$K_{\sigma^\frown<0>} = \{u : u \in K_\sigma,\ f_\sigma(u) = \inf f_\sigma[K_\sigma]\}.$$

Moreover, $\operatorname{diam}(f_\sigma[K_\sigma]) \ge \frac{\gamma}{1-\gamma}$ for every σ. But this means that if we take C_σ to be the convex hull of $K_{\sigma^\frown<1>} - K_{\sigma^\frown<0>}$, $f_\sigma(u) \ge \frac{\gamma}{1-\gamma}$ for every $u \in C_\sigma$, and C_σ does not meet $\frac{1-\gamma}{2\gamma}W$. Thus $0 \notin \overline{\bigcup_{\sigma\in S} C_\sigma}$; contradicting Proposition 1.

(e) Note next that if $\langle G_m\rangle_{m\in\mathbb{N}}$ is any non-increasing sequence in Φ, $G = \bigcap_{m\in\mathbb{N}} G_m$ belongs to Φ. **P** For any $x \in X$, $\langle G_m[\{x\}]\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence of non-empty weakly compact sets, so $G[\{x\}] = \bigcap_{m\in\mathbb{N}} G_m[\{x\}]$ is a non-empty weakly compact set. Moreover, if $f \in U^*$, $\sup f[G[\{x\}]] = \inf_{m\in\mathbb{N}} \sup f[G_m[\{x\}]]$, so $x \mapsto \sup f[G[\{x\}]]$ is measurable; as f is arbitrary, G is scalarly hull-measurable and belongs to Φ. **Q**

Of course $\Delta_n(G') \le \Delta_n(G)$ whenever $n \in \mathbb{N}$, G', $G \in \Phi$ and $G' \subseteq G$. There is therefore a $G \in \Phi$ such that $\Delta_n(G) = 0$ for every $n \in \mathbb{N}$. **P** Let $\langle(n_k, q_k)\rangle_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{N}\times\mathbb{Q}$. Choose $\langle G_k\rangle_{k\in\mathbb{N}}$ inductively so that

$G_0 = F$,

if there is a $G \subseteq G_k$ such that $G \in \Phi$ and $\Delta_{n_k}(G) \le q_k$, then $G_{k+1} \subseteq G_k$, $G_{k+1} \in \Phi$ and $\Delta_{n_k}(G_{k+1}) \le q_k$,

otherwise, $G_{k+1} = G_k$.

At the end of the induction, set $G = \bigcap_{k\in\mathbb{N}} G_k$. Then $G \in \Phi$. **?** If $n \in \mathbb{N}$ is such that $\Delta_n(G) > 0$, (d) tells us that there is a $G' \subseteq G$ such that $G' \in \Phi$ and $\Delta_n(G') < \Delta_n(G)$. Let $k \in \mathbb{N}$ be such that $n_k = n$ and $\Delta_n(G') \le q_k < \Delta_n(G)$. Then $G' \subseteq G_k$ and $\Delta_{n_k}(G') \le q_k$, so

$$q_k < \Delta_n(G) = \Delta_{n_k}(G) \le \Delta_{n_k}(G_{k+1}) \le q_k$$

(by the choice of G_{k+1}), which is absurd. **X** Thus $\Delta_n(G) = 0$ for every n, as required. **Q**

(f) Let $\phi : X \to U$ be any selector for G. Then of course ϕ is a selector for F. Also ϕ is scalarly measurable. **P** Take any $f \in U^*$. Then there is an $n \in \mathbb{N}$ such that $f \in W_n^\circ$, so that the constant function ψ on X with value f belongs to Ψ_n. Accordingly $\delta(G, \psi) = 0$ and $h_{G\psi} = 0$ a.e. and $\operatorname{diam}(f[G[\{x\}]]) = 0$ for almost every x. But this means that $f(\phi(x)) = \sup f[G[\{x\}]]$ for almost every x; as $G \in \Phi$ and μ is complete, $f\phi$ is measurable; as f is arbitrary, ϕ is scalarly measurable.

(g) This deals with the case in which μ is totally finite and all the vertical sections of F are non-empty. For the general case, let $\langle X_i\rangle_{i\in I}$ be a decomposition of X (Fremlin 01, 211E), and set $Y = \{x : F[\{x\}] \ne \emptyset\}$. As in part (b) of the proof of Lemma 1C, Y is measurable. Applying (a)-(f) to $Y\cap X_i$ and $F_i = F\cap((Y\cap X_i)\times U$, we have for each i a scalarly measurable selector ϕ_i for F_i; now $\phi = \bigcup_{i\in I} \phi_i$ is a scalarly measurable selector for F.

2 Set selectors

2A The problem Theorem 1D offers a process for choosing a selector ϕ for a relation $F \subseteq X \times U$ which depends on examining the whole relation F. By contrast, such selection theorems as the von Neumann-Jankow theorem (Fremlin 03, 423M-423O) and the Kuratowski-Ryll-Nardzewski theorem (Kuratowski & Ryll-Nardzewski 65) can be reduced to procedures for choosing $\phi(x) \in F[\{x\}]$ directly from the set $F[\{x\}]$ alone, without considering other sections. V.Kadets therefore asked the following:

> Let U be a metrizable locally convex linear topological space, and $\mathcal{K}$ the family of non-empty weakly compact subsets of U. Is there a function $\theta : \mathcal{K} \to U$ such that $\theta(K) \in K$ for every $K \in \mathcal{K}$ and whenever (X, Σ, μ) is a complete strictly localizable space and $F \subseteq X \times U$ is a scalarly hull-measurable function such that $F[\{x\}] \in \mathcal{K}$ for every $x \in X$, then $x \mapsto \theta(F[\{x\}])$ is scalarly measurable?

2B Proposition Suppose that U is a locally convex Hausdorff linear topological space and that there is a family $\langle f_\xi\rangle_{\xi<\omega_1}$ in U^* such that $U^* = \bigcup_{\xi<\omega_1} \overline{\{f_\eta : \eta < \xi\}}$, the closures here being taken for the weak* topology on U^*. Let $\mathcal{K}$ be the family of non-empty weakly compact subsets of U. Then there is a function

$\theta : \mathcal{K} \to U$ such that $\theta(K) \in K$ for every $K \in \mathcal{K}$ and whenever X is a set, Σ is a σ-algebra of subsets of X and $F \subseteq X \times U$ is a scalarly hull-measurable function such that $F[\{x\}] \in \mathcal{K}$ for every $x \in X$, then $x \mapsto \theta(F[\{x\}])$ is scalarly measurable.

proof (See Cascales Kadets & Rodríguez 09, Theorem 5.4.) For $K \in \mathcal{K}$, define a family $\langle \theta_\xi(K)\rangle_{\xi\le\omega_1}$ in $\mathcal{K}$ inductively by saying that $\theta_0(K) = K$ and

$$\theta_{\xi+1}(K) = \{u : u \in \theta_\xi(K),\ f_\xi(u) = \sup f_\xi[\theta_\xi(K)]\}$$

for each $\xi < \omega_1$,

$$\theta_\xi(K) = \bigcap_{\eta<\xi} \theta_\eta(K)$$

for non-zero limit ordinals $\xi \le \omega_1$. If $u, v \in \theta_{\omega_1}(K)$, then $\{f : f \in U^*,\ f(u) = f(v)\}$ is a weak*-closed subset of U^* containing every f_ξ, so is the whole of U^*, and $u = v$; thus $\theta_{\omega_1}(K)$ is a singleton. We can therefore define $\theta : \mathcal{K} \to U$ by taking $\theta(K)$ to be the member of $\theta_{\omega_1}(K)$ for each $K \in \mathcal{K}$, and we shall have $\theta(K) \in K$ for every K.

Now suppose that X is a set, Σ is a σ-algebra of subsets of X and $F \subseteq X\times U$ is a scalarly hull-measurable function such that $F[\{x\}] \in \mathcal{K}$ for every $x \in X$. For each $\xi < \omega_1$ set

$$F_\xi = \{(x, u) : x \in X,\ u \in \theta_\xi(F[\{x\}])\}.$$

Then F_ξ is scalarly hull-measurable for every $\xi < \omega_1$. **P** Induce on ξ. The induction starts with $F_0 = F$. For the inductive step to $\xi + 1$, use Lemma 1C. For the inductive step to a non-zero countable limit ordinal ξ, observe that for any $f \in U^*$ and $x \in X$, $\langle F_\eta[\{x\}]\rangle_{\eta<\xi}$ is a non-increasing family of non-empty weakly compact sets with intersection $F_\xi[\{x\}]$. So

$$\sup f[F_\xi[\{x\}]] = \inf_{\eta<\xi} \sup f[F_\eta[\{x\}]];$$

as $x \mapsto \sup f[F_\eta[\{x\}]]$ is measurable for every $\eta < \xi$, so is $x \mapsto \sup f[F_\xi[\{x\}]]$. **Q**

Set $\phi(x) = \theta(F[\{x\}])$ for each x. Then ϕ is scalarly measurable. **P** If $f \in U^*$, let $\xi < \omega_1$ be such that $f \in \overline{\{f_\eta : \eta < \xi\}}$. For any $x \in X$ and $u, v \in F_\xi[\{x\}] = \theta_\xi(F[\{x\}])$, $f_\eta(u) = f_\eta(v)$ for every $\eta < \xi$, so $f(u) = f(v)$. But

$$\phi(x) = \theta(F[\{x\}]) \in \theta_{\omega_1}[F\{x\}] \subseteq \theta_\xi[F\{x\}]$$

so $f(\phi(x)) = \sup f[F_\xi[\{x\}]]$. And $x \mapsto \sup f[F_\xi[\{x\}]]$ is measurable; as f is arbitrary, ϕ is scalaraly measurable. **Q**

2C Lemma Let κ be an infinite cardinal, and $\theta : [\kappa^+]^2 \to \kappa^+$ a function such that $\theta(I) \in I$ for every $I \in [\kappa^+]^2$. Then there is a $\xi < \kappa^+$ such that $\{\eta : \theta(\{\xi, \eta\}) = \xi\}$ and $\{\eta : \theta(\{\xi, \eta\}) = \eta\}$ both have cardinal at least κ.

proof ? Otherwise, then for each $\xi < \kappa^+$ there is an α_ξ such that $\xi < \alpha_\xi < \kappa^+$ and

either $\theta(\{\xi, \eta\}) = \xi$ for every $\eta \ge \alpha_\xi$

or $\theta(\{\xi, \eta\}) = \eta$ for every $\eta \ge \alpha_\xi$.

Set $A = \{\xi : \theta(\{\xi, \eta\}) = \xi$ for every $\eta \ge \alpha_\xi\}$. Let $C \subseteq \kappa^+$ be a cofinal set such that $\alpha_\xi \le \eta$ whenever ξ, $\eta \in C$ and $\xi < \eta$ (Fremlin 03, 4A1Bd). Then at least one of $C \cap A$, $C \setminus A$ has cardinal κ^+.

Suppose that $\#(C \cap A) = \kappa^+$. Then there is a $\xi \in C \cap A$ such that $\#(C \cap A \cap \xi) = \kappa$, in which case $C \cap A \setminus (\xi + 1)$ has cardinal κ^+. If $\eta \in C \cap A \cap \xi$, then $\alpha_\eta \le \xi$ so $\theta(\{\xi, \eta\}) = \eta$; if $\eta \in C \cap A \setminus (\xi + 1)$ then $\alpha_\xi \le \eta$ so $\theta(\{\xi, \eta\}) = \xi$. But this means that $\{\eta : \theta(\{\xi, \eta\}) = \xi\}$ and $\{\eta : \theta(\{\xi, \eta\}) = \eta\}$ both have cardinal at least κ, and we were supposing that this was impossible.

Similarly, if $\#(C \setminus A) = \kappa^+$, there is a $\xi \in C \setminus A$ such that $\#(C \cap \xi \setminus A) = \kappa$, in which case $C \setminus (A \cup (\xi + 1))$ has cardinal κ^+. If $\eta \in C \cap \xi \setminus A$, then $\alpha_\eta \le \xi$ so $\theta(\{\xi, \eta\}) = \xi$; if $\eta \in C \setminus (A \cup (\xi + 1))$ then $\alpha_\xi \le \eta$ so $\theta(\{\xi, \eta\}) = \eta$. So our counter-hypothesis is contradicted in this case also. **X**

2D Example (a) Let U be the Hilbert space $\ell^2(\omega_2)$. Let $\mathcal{K}$ be the family of non-empty weakly compact subsets of U and $\theta : \mathcal{K} \to U$ any function such that $\theta(K) \in K$ for every $K \in \mathcal{K}$. Let $\langle e_\xi\rangle_{\xi<\omega_2}$ be the standard orthonormal basis of U, and define $\theta_0 : [\omega_2]^2 \to \omega_2$ by saying that $\theta_0(\{\xi, \eta\}) = \xi$ whenever $\xi \ne \eta$ and $\theta(\{e_\xi, e_\eta\}) = e_\xi$. By Lemma 2C, there is a $\xi < \omega_2$ such that $\{\eta : \theta_0(\{\xi, \eta\}) = \xi\}$ and $\{\eta : \theta_0(\{\xi, \eta\}) = \eta\}$ are both uncountable.

Let μ be the countable-cocountable measure on ω_2 (Fremlin 01, 211R), and Σ its domain, so that (ω_2, Σ, μ) is a complete probability space. Set

$$F = \{(\eta, e_\xi) : \eta < \omega_2\} \cup \{(\eta, e_\eta) : \eta < \omega_2\} \subseteq \omega_2 \times U.$$

Then $F[\{\eta\}] = \{e_\xi, e_\eta\} \in \mathcal{K}$ for every $\eta < \omega_2$. Now F is scalarly hull-measurable. **P** If $f \in U^*$, there is a $v \in U$ such that $f(u) = (u|v)$ for every $u \in U$. Then $f[F[\{\eta\}]] = \{v(\eta), v(\xi)\}$ for every η. Let J be the countable set $\{\eta : v(\eta) \neq 0\}$. For $\eta \in \omega_2 \setminus J$, $\sup f[F[\{\eta\}]] = \max(0, v(\xi))$; so $\eta \mapsto \sup f[F[\{\eta\}]]$ is constant on the coneglible set $\omega_2 \setminus J$, and is measurable. **Q**

On the other hand, if we set $\phi(\eta) = \theta(F[\{\eta\}])$ for $\eta < \omega_2$, and $g(u) = u(\xi)$ for $u \in U$, then

$$\{\eta : g(\phi(\eta)) = 0\} = \{\eta : \eta \neq \xi,\, \phi(\eta) = e_\eta\} = \{\eta : \eta \neq \xi,\, \theta_0(\{\eta, \xi\} = \eta\},$$

$$\{\eta : g(\phi(\eta)) = 1\} = \{\eta : \phi(\eta) = e_\xi\} = \{\xi\} \cup \{\eta : \eta \neq \xi,\, \theta_0(\{\eta, \xi\}) = \xi\}$$

are both uncountable, so $g\phi$ is not measurable and ϕ is not scalarly measurable.

This shows that there is no selector θ for $\mathcal{K}$ which will generate scalarly measurable selectors of the type found in Theorem 1D and Proposition 2B.

(b) Let U be the Hilbert space $\ell^2(\mathfrak{c}^+)$. Let $\mathcal{K}$ be the family of non-empty weakly compact subsets of U and $\theta : \mathcal{K} \to U$ any function such that $\theta(K) \in K$ for every $K \in \mathcal{K}$. Let $\langle e_\xi \rangle_{\xi < \mathfrak{c}^+}$ be the standard orthonormal basis of U, and define $\theta_0 : [\mathfrak{c}^+]^2 \to \mathfrak{c}^+$ by saying that $\theta_0(\{\xi, \eta\}) = \xi$ whenever $\xi \neq \eta$ and $\theta(\{e_\xi, e_\eta\}) = e_\xi$. By Lemma 2C, there is a $\xi < \mathfrak{c}^+$ such that $D_1 = \{\eta : \theta_0(\{\xi, \eta\}) = \xi\}$ and $D_0 = \{\eta : \theta_0(\{\xi, \eta\}) = \eta\}$ both have cardinal at least $\mathfrak{c}$.

Let μ be Lebesgue measure on $[0,1]$ and Σ its domain, so that $([0,1], \Sigma, \mu)$ is a complete probability space. Let $A \subseteq [0,1]$ be a non-measurable set (Fremlin 00, 134B) and $q : [0,1] \to \mathfrak{c}^+$ an injective function such that $q[A] \subseteq D_1$ and $q[\,[0,1] \setminus A] \subseteq D_0$. Set

$$F = \{(x, e_\xi) : x \in [0,1]\} \cup \{(x, e_{q(x)}) : x \in [0,1]\} \subseteq [0,1] \times U.$$

Then $F[\{x\}] = \{e_\xi, e_{q(x)}\} \in \mathcal{K}$ for every $x \in [0,1]$. Now F is scalarly hull-measurable. **P** If $f \in U^*$, there is a $v \in U$ such that $f(u) = (u|v)$ for every $u \in U$. In this case $f[F[\{x\}]] = \{v(q(x)), v(\xi)\}$ for every x. Let J be the countable set $\{\eta : v(\eta) \neq 0\}$. For $x \in [0,1] \setminus q^{-1}[J]$, $\sup f[F[\{x\}]] = \max(0, v(\xi))$; so $x \mapsto \sup f[F[\{x\}]]$ is constant on the co-countable set $[0,1] \setminus q^{-1}[J]$, and is measurable. **Q**

On the other hand, if we set $\phi(x) = \theta(F[\{x\}])$ for $x \in [0,1]$, and $g(u) = u(\xi)$ for $u \in U$, then

$$\{x : g(\phi(x)) = 1\} = \{x : \phi(x) = e_\xi\} = q^{-1}[\{\xi\} \cup D_1] = A \cup q^{-1}[\{\xi\}]$$

is non-measurable, so $g\phi$ is not measurable and ϕ is not scalarly measurable.

This shows that there is no selector θ for $\mathcal{K}$ which will generate scalarly measurable selectors of the type found in Theorem 1D and Proposition 2B.

2E Supplementary problems The example in 2D leaves the following questions open.

(a) Let U be a metrizable locally convex linear topological space, and $\mathcal{K}_c$ the family of non-empty convex weakly compact subsets of U. Must there be a function $\theta : \mathcal{K}_c \to U$ such that $\theta(K) \in K$ for every $K \in \mathcal{K}$ and whenever (X, Σ, μ) is a complete strictly localizable space and $F \subseteq X \times U$ is a scalarly hull-measurable function such that $F[\{x\}] \in \mathcal{K}_c$ for every $x \in X$, then $x \mapsto \theta(F[\{x\}])$ is scalarly measurable?

(b) Let U be a metrizable locally convex linear topological space of density ω_1, and $\mathcal{K}$ the family of all non-empty weakly compact subsets of U. Must there be a function $\theta : \mathcal{K} \to U$ such that $\theta(K) \in K$ for every $K \in \mathcal{K}$ and whenever (X, Σ, μ) is a complete strictly localizable space and $F \subseteq X \times U$ is a scalarly hull-measurable function such that $F[\{x\}] \in \mathcal{K}_c$ for every $x \in X$, then $x \mapsto \theta(F[\{x\}])$ is scalarly measurable?

Remark In both problems, it would be interesting if one could find a positive answer for Banach spaces U; and in (a), for Hilbert spaces U. Note that in (b) we already have an answer for separable U and for reflexive Banach spaces of density ω_1, since they satisfy the conditions of Proposition 2B. Of course only probability spaces (X, Σ, μ) need to be considered in either case. One would anticipate however that a positive answer to either question would be associated with a generalization to the case of a set X with a σ-algebra Σ not involving any measure, as in Proposition 2B.

Acknowledgements Hospitality of the conference 'Integration, Vector Measures and Related Topics IV', Murcia, 2-5 March 2011; correspondence with V.Kadets and J.Rodríguez.

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