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Maharam types of amoeba algebras

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1 Variable-measure amoeba algebras

1A Definitions (see Fremlin 08, §528) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will write $P^*(\mathfrak{A},\bar\mu)$ for the set $\{(a,\alpha):a\in\mathfrak{A},\ \bar\mu a<\alpha\le\bar\mu 1\}$, ordered by saying that $(a,\alpha)\le(b,\beta)$ if $a\subseteq b$ and $\beta\le\alpha$.

In this context, I will say that if $p=(a,\alpha)$ then $a=a_p$ and $\alpha=\alpha_p$. Note that p, $q\in P^*(\mathfrak{A},\bar\mu)$ are compatible upwards in $P^*(\mathfrak{A},\bar\mu)$ iff $(a_p\cup a_q,\min(\alpha_p,\alpha_q))\in P^*(\mathfrak{A},\bar\mu)$, that is, iff $\bar\mu(a_p\cup a_q)<\min(\alpha_p,\alpha_q)$.

$P^*(\mathfrak{A},\bar\mu)$ will always be given its up-topology (Fremlin 08, 514L); the **variable-measure amoeba algebra** $\mathrm{AM}^*(\mathfrak{A},\bar\mu)$ (Fremlin 08, 528Ab) is the regular open algebra $\mathrm{RO}^\uparrow(P^*(\mathfrak{A},\bar\mu))$. For $p\in P^*(\mathfrak{A},\bar\mu)$ I will write $V_p=\operatorname{int}\overline{[p,\infty[}\in\mathrm{RO}^\uparrow(P^*(\mathfrak{A},\bar\mu))$.

1B Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and P the partially ordered set $P^*(\mathfrak{A},\bar\mu)$. If $(a,\alpha)\in P$ and $C\subseteq\mathfrak{A}$ is a non-empty set with supremum a, then $V_{(a,\alpha)}=\inf_{c\in C}V_{(c,\alpha)}$ in $\mathrm{RO}^\uparrow(P)$.

proof Since $(c,\alpha)\le(a,\alpha)$, $V_{(a,\alpha)}\subseteq V_{(c,\alpha)}$ for every $c\in C$, and

$$V_{(a,\alpha)}\subseteq\inf\nolimits_{c\in C}V_{(c,\alpha)}=\bigcap\nolimits_{c\in C}V_{(c,\alpha)}$$

(Fremlin 08, 514M(d-ii)[1]). In the other direction, take any $p\in\bigcap_{c\in C}V_{(c,\alpha)}$. There is a $c_0\in C$; as p and (c_0,α) are compatible upwards, $\bar\mu a_p\le\bar\mu(a_p\cup c_0)<\min(\alpha_p,\alpha)$. Take β such that $\bar\mu a_p<\beta<\min(\alpha_p,\alpha)$ and consider (a_p,β). We know that $[(a_p,\beta),\infty[\,\subseteq\bigcap_{c\in C}V_{(c,\alpha)}$. Inducing on $\#(I)$, we see that for any finite $I\subseteq C$ there is a $q\ge(a_p,\beta)$ such that $c\subseteq a_q$ for every $c\in I$. But this means that $\bar\mu(a_p\cup\sup I)\le\beta$ for every finite $I\subseteq C$, so $\bar\mu(a_p\cup a)\le\beta$ and p is compatible upwards with (a,α), that is, $p\in\overline{[(a,\alpha),\infty[}$.

Thus the open set $\inf_{c\in C}V_{(c,\alpha)}$ is included in $\overline{[(a,\alpha),\infty[}$ and therefore in its interior $V_{(a,\alpha)}$, and $V_{(a,\alpha)}=\inf_{c\in C}V_{(c,\alpha)}$, as claimed.

1C Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $e\in\mathfrak{A}$; write $\mathfrak{A}_e$ for the principal ideal generated by e. Set

$$P=P^*(\mathfrak{A},\bar\mu),\quad P_e=\{p:p\in P,\ \alpha_p\le\bar\mu(a_p\cup e)\},$$

$$Q=P^*(\mathfrak{A}_e,\bar\mu\restriction\mathfrak{A}_e).$$

Then $P_e\in\mathrm{RO}^\uparrow(P)$. For $p\in P_e$, set

$$f(p)=(a_p\cap e,\alpha_p-\bar\mu(a_p\setminus e));$$

then f is an order-preserving function from P_e to Q. Now we have an order-continuous ring homomorphism $\pi:\mathrm{RO}^\uparrow(Q)\to\mathrm{RO}^\uparrow(P)$ defined by setting $\pi H=\operatorname{int}\overline{f^{-1}[H]}$ for every $H\in\mathrm{RO}^\uparrow(Q)$.

proof It is easy to check that P_e is up-open. To see that it is regular, take any $p\in P\setminus P_e$. Then $p'=(a_p\cup e,\alpha_p)$ belongs to P, and $[p',\infty[$ is disjoint from P_e, so $p\notin\operatorname{int}\overline{P_e}$. As p is arbitrary, $P_e=\operatorname{int}\overline{P_e}$ is a regular up-open set.

If $p\in P$, then

$$\bar\mu(a_p\cap e)=\bar\mu a_p-\bar\mu(a_p\setminus e)<\alpha_p-\bar\mu(a_p\setminus e)\le\bar\mu(a_p\cup e)-\bar\mu(a_p\setminus e)=\bar\mu e,$$

so $f(p)\in Q$; it is now easy to see that $f:P\to Q$ is order-preserving.

Suppose that $Q_0\subseteq Q$ is up-open and cofinal. Then $f^{-1}[Q_0]$ is cofinal with P_e. **P** Take any $p\in P_e$. Then there is a $q\in Q_0$ such that $q\ge f(p)$, that is, $a_q\supseteq a_p\cap e$ and $\alpha_q\le\alpha_p-\bar\mu(a_p\setminus e)$. In this case,

$$\bar\mu(a_p\cup a_q)=\bar\mu a_q+\bar\mu(a_p\setminus e)<\alpha_q+\bar\mu(a_p\setminus e)\le\alpha_p,$$

[1] Later editions only.

so $p' = (a_p \cup a_q, \alpha_q + \bar\mu(a_p \setminus e))$ belongs to P and $p' \ge p$. Because P_e is up-open, $p' \in P_e$. Now $f(p') = q$, so $p' \in f^{-1}[Q_0]$. As p is arbitrary, $f^{-1}[Q_0]$ is cofinal with P_e. **Q**

If $H \in \mathrm{RO}^\uparrow(Q)$ then $\mathrm{int}\,\overline{f^{-1}[H]}$ is the same whether we define closure and interior in P or in P_e, because P_e is a regular up-open set. By Fremlin 08, 514O, the given formula therefore defines an order-continuous Boolean homomorphism from $\mathrm{RO}^\uparrow(Q)$ to $\mathrm{RO}^\uparrow(P_e)$; since $\mathrm{RO}^\uparrow(P_e)$ is a principal ideal of $\mathrm{RO}^\uparrow(P)$, we have an order-continuous ring homomorphism from $\mathrm{RO}^\uparrow(Q)$ to $\mathrm{RO}^\uparrow(P)$.

1D Proposition Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra, and $E \subseteq \mathfrak{A}$ an upwards-directed family with supremum 1. Set $P = P^*(\mathfrak{A}, \bar\mu)$; for $e \in E$ set

$$P_e = \{p : p \in \mathbb{P},\ \alpha_p \le \bar\mu(a_p \cup e)\}, \quad Q_e = P^*(\mathfrak{A}_e, \bar\mu{\restriction}\mathfrak{A}_e);$$

for $e \in E$ and $p \in P_e$ set

$$f_e(p) = (a_p \cap e, \alpha_p - \bar\mu(a_p \setminus e));$$

and for $e \in E$ and $H \in \mathrm{RO}^\uparrow(Q_e)$ set $\pi_e H = \mathrm{int}\,\overline{f_e^{-1}[H]} \in \mathrm{RO}^\uparrow(P)$. Then $\bigcup_{e\in E} \pi_e[\mathrm{RO}^\uparrow(Q_e)]$ τ-generates $\mathrm{RO}^\uparrow(P)$ in the sense of Fremlin 02, 331E.

proof (a) As noted in Proposition 1C, P_e is a regular up open set and $f_e : P_e \to Q_e$ is order-preserving. The key to the proof is the following fact: if $p \in P$ and $\bar\mu a_p < \alpha < \alpha_p$, then there is an $e_0 \in E$ such that $p' = (a_p, \alpha)$ belongs to P_e for every $e \in E$ such that $e \supseteq e_0$, and

$$V_{p'} \subseteq \inf_{e\in E, e\supseteq e_0} \pi_e V_{f_e(p')} \subseteq V_p.$$

P(i) Of course $p' \ge p$. Because $\alpha_p \le \bar\mu 1 = \sup_{e\in E} \bar\mu e$, there is an $e_0 \in E$ such that $\alpha \le \bar\mu e_0$, and now $p \in P_e$ whenever $e \supseteq e_0$. Set $E_0 = \{e : e \in E,\ e \supseteq e_0\}$.

(ii) For any $e \in E_0$,

$$[p', \infty[\subseteq f_e^{-1}[\,[f_e(p'), \infty[\,] \subseteq f_e^{-1}[V_{f_e(p')}]$$

because f_e is order-preserving, so that

$$V_{p'} \subseteq \pi_e V_{f_e(p')}$$

for every $e \in E_0$; that is, $V_{p'} \subseteq \inf_{e\in E_0} \pi_e V_{f_e(p')}$.

(iii) On the other side, suppose that

$$q \in \inf_{e\in E_0} \pi_e V_{f_e(p')} = \bigcap_{e\in E_0} \pi_e V_{f_e(p')}$$

(Fremlin 08, 514M(d-ii) again). Then for any $q' \ge q$ and $e \in E_0$,

$$[q', \infty[\subseteq [q, \infty[\subseteq \overline{f_e^{-1}[V_{f_e(p')}]}.$$

So $[q', \infty[\cap f_e^{-1}[V_{f_e(p')}]$ is non-empty, and there is a $q'' \ge q'$ such that $f_e(q'') \in V_{f_e(p')} \subseteq \overline{[f_e(p'), \infty[}$. But this means that $f_e(q'')$ and $f_e(p')$, therefore also $f_e(q')$ and $f_e(p')$, are compatible upwards in Q_e, that is,

$$\bar\mu((a_{q'} \cap e) \cup (a_p \cap e)) < \min(\alpha_{q'} - \bar\mu(a_{q'} \setminus e), \alpha - \bar\mu(a_p \setminus e)).$$

Since E_0, like E, is upwards-directed and has supremum 1, we can take the limit as e increases and get

$$\bar\mu(a_{q'} \cup a_p) \le \min(\alpha_{q'}, \alpha);$$

and this is true whenever $q' \ge q$. In the first instance, this means that $\bar\mu(a_q \cup a_p) \le \beta$ whenever $\bar\mu a_q < \beta \le \alpha_q$; so in fact $\bar\mu(a_q \cup a_p) \le \bar\mu a_q$ and $a_q \supseteq a_p$. Next, $\bar\mu a_q \le \alpha$. But it follows that

$$\bar\mu(a_p \cup a_q) = \bar\mu a_q < \min(\alpha_p, \alpha_q)$$

and q and p are compatible upwards in P, that is, $q \in \overline{[p, \infty[}$.

As q is arbitrary, the open set $\inf_{e\in E} \pi_e V_{f_e(p')}$ is included in $\overline{[p, \infty[}$ and therefore in $\mathrm{int}\,\overline{[p, \infty[} = V_p$, as claimed. **Q**

(b) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^\uparrow(P)$ generated by $\bigcup_{e\in E} \pi_e[\mathrm{RO}^\uparrow(Q_e)]$. Then all the sets $\inf_{e\in E_0} \pi_e V_{f_e(a_p,\alpha)}$ examined in (a) belong to $\mathfrak{G}$. So (a) tells us that for every $p \in P$ there are a $p' \ge p$

and a $G \in \mathfrak{G}$ such that $V_{p'} \subseteq G \subseteq V_p$. As $\{[p,\infty[\, : p \in P\}$ is a base for the topology of P consisting of non-empty sets, $\{V_p : p \in P\}$ is a π-base of the Boolean algebra $\mathrm{RO}^{\uparrow}(P)$ consisting of non-zero elements, and $\mathfrak{G}$ includes a π-base of $\mathrm{RO}^{\uparrow}(P)$; being order-closed, it is the whole of $\mathrm{RO}^{\uparrow}(P)$, as required.

1E Lemma (compare Fremlin 08, 528B[2]) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Then $P = P^*(\mathfrak{A},\bar\mu)$ is separative upwards, so $[p,\infty[\, \in \mathrm{RO}^{\uparrow}(P)$ for every $p \in P$.

proof Let p, $q \in P$ be such that $p \not\leq q$. If $\bar\mu(a_p \cup a_q) \geq \min(\alpha_p,\alpha_q)$ then p and q are already incompatible upwards. So suppose that $\bar\mu(a_p \cup a_q) < \min(\alpha_p,\alpha_q)$. If $\alpha_p < \alpha_q$ there is a $c \supseteq a_q$ such that $\bar\mu c = \alpha_p$; now $q' = (c,\alpha_q) \geq q$ and p and q' are incompatible upwards. Otherwise, $\alpha_q \leq \alpha_p$ and $a_p \not\subseteq a_q$. As $\alpha_q \leq \bar\mu 1$, there is a d disjoint from $a_p \cup a_q$ such that $\bar\mu d = \alpha_q - \bar\mu(a_p \cup a_q)$; set $c = a_q \cup d$. Then

$$\bar\mu c < \bar\mu(a_p \cup c) = \alpha_q,$$

so, just as in (a), $(c,\alpha_q) \geq q$ and p, (c,α_q) are incompatible upwards.

By Fremlin 08, 514Me, it follows that $[p,\infty[$ is a regular up-open set for every $p \in P$.

1F Definition (Fremlin 08, 528S[3]) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. I will say that a **well-spread basis** for $\mathfrak{A}$ is a non-decreasing sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of subsets of $\mathfrak{A}$ such that

(i) setting $D = \bigcup_{n\in\mathbb{N}} D_n$, $\#(D) \leq \max(\omega, c(\mathfrak{A}), \tau(\mathfrak{A}))$;

(ii) if $a \in \mathfrak{A}$, $\gamma \in \mathbb{R}$ and $\bar\mu a < \gamma$, there is a set $D \subseteq \bigcup_{n\in\mathbb{N}} D_n$ such that $a \subseteq \sup D$ and $\bar\mu(\sup D) < \gamma$;

(iii) if $n \in \mathbb{N}$ and $\langle d_i\rangle_{i\in\mathbb{N}}$ is a sequence in D_n such that $\bar\mu(\sup_{n\in\mathbb{N}} d_n) < \infty$, there is an infinite set $J \subseteq \mathbb{N}$ such that $d = \sup_{i\in J} d_i$ belongs to D_n;

(iv) whenever $n \in \mathbb{N}$, $a \in \mathfrak{A}$ and $\bar\mu a \leq \gamma' < \gamma < \bar\mu 1$, there is a $b \in \mathfrak{A}$ such that $a \subseteq b$ and $\gamma' \leq \bar\mu b < \gamma$ and $\bar\mu(b \cup d) \geq \gamma$ whenever $d \in D_n$ and $d \not\subseteq a$.

1G Lemma (Fremlin 08, 528T[4]) (a) Let κ be an infinite cardinal, and $\langle e_\xi\rangle_{\xi<\kappa}$ the standard generating family in $\mathfrak{B}_\kappa$ (Fremlin 08, 525A). For $n \in \mathbb{N}$ let C_n be the set of elements of $\mathfrak{B}_\kappa$ expressible as $\inf_{\xi\in I} e_\xi \cap \inf_{\xi\in J}(1 \setminus e_\xi)$ where I, $J \subseteq \kappa$ are disjoint and $\#(I \cup J) \leq n$. Then $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$, with $C_0 = \{1\}$. Moreover,

(*) for each $n \geq 1$, there is a set $C'_n \subseteq C_n$, of cardinal κ, such that $\bar\mu c = 2^{-n}$ for every $c \in C'_n$, and whenever $a \in \mathfrak{B}_\kappa \setminus \{1\}$ and $I \subseteq C'_n$ is infinite, there is a $c \in I$ such that $c' \not\subseteq a \cup c$ whenever $c' \in C_n$ and $c \subset c'$.

(b) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra and $e \in \mathfrak{A}$. If $\langle C_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_e, \bar\mu\restriction\mathfrak{A}_e)$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{1\setminus e}, \bar\mu\restriction\mathfrak{A}_{1\setminus e})$, then $\langle C_n \cup D_n\rangle_{n\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A},\bar\mu)$.

1H Lemma (compare Fremlin 08, 528U[5]) Let $(\mathfrak{A},\bar\mu)$ be an atomless semi-finite measure algebra. Let E, ϵ, $\preccurlyeq$ and $\mathcal{F}$ be such that

E is a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \epsilon \leq \bar\mu e < \infty$ for every $e \in E$;

$\preccurlyeq$ is a well-ordering of E such that $\tau(\mathfrak{A}_e) \leq \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E;

$\mathcal{F}$ is a partition of E such that each member of $\mathcal{F}$ is either a singleton or a countable set with no $\preccurlyeq$-greatest member.

Set $P = P^*(\mathfrak{A},\bar\mu)$ and let P_0 be

$$\{p : p \in P,\ \alpha_p \leq \bar\mu(a_p \cup e) \text{ whenever } \{e\} \in \mathcal{F}\}.$$

Then $\mathrm{RO}^{\uparrow}(P_0)$ has countable Maharam type.

proof (a)(i) For every $e \in E$, $(\mathfrak{A}_e, \bar\mu\restriction\mathfrak{A}_e)$ is a non-zero atomless homogeneous totally finite measure algebra, so is isomorphic, up to a scalar multiple of the measure, to $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ for some infinite cardinal κ (Fremlin

[2] Later editions only.
[3] Later editions only.
[4] Later editions only.
[5] Later editions only.

02, 331L). So we can copy the well-spread basis for $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ described in 1Ga into a well-spread basis $\langle D_{en}\rangle_{n\in\mathbb{N}}$ for $(\mathfrak{A}_e,\bar\mu\restriction\mathfrak{A}_e)$ such that

$\#(\bigcup_{n\in\mathbb{N}}D_{en})=\tau(\mathfrak{A}_e)$,

$\bar\mu d\ge 2^{-n}\bar\mu e$ whenever $n\in\mathbb{N}$ and $d\in D_{en}$,

$D_{e0}=\{e\}$,

for each $n\ge 1$ there is a set $D'_{en}\subseteq D_{en}$, of cardinal $\tau(\mathfrak{A}_e)$, such that $\bar\mu d=2^{-n}\bar\mu e$ for every $d\in D'_{en}$, and whenever $a\in\mathfrak{A}_e\setminus\{e\}$ and $I\subseteq D'_{en}$ is infinite, there is a $d\in I$ such that $d'\not\subseteq a\cup d$ whenever $d'\in D_{en}$ and $d'\supset d$,

$(\bigcup_{n\in\mathbb{N}}D_{en})\setminus(\bigcup_{n\ge 1}D'_{en})$ has cardinal $\tau(\mathfrak{A}_e)$.

(The last item is not mentioned in 1G, but is clearly achievable by thinning the sets D'_{en} appropriately.) Note that $\langle D'_{en}\rangle_{n\ge 1}$ is a disjoint sequence of subsets of $\mathfrak{A}_e$ for each e, so $\langle D'_{en}\rangle_{e\in E,n\ge 1}$ is disjoint.

(ii) For $e\in F\in\mathcal{F}$, set

$$D_e=\bigcup_{n\in\mathbb{N}}D_{en}\setminus\bigcup_{n\ge 1}D'_{en},\quad D^*_e=\bigcup_{e'\in F,e'\preccurlyeq e}D_e.$$

Because F is countable and $\tau(\mathfrak{A}_{e'})\le\tau(\mathfrak{A}_e)$ whenever $e'\preccurlyeq e$, $\#(D^*_e)=\tau(\mathfrak{A}_e)=\#(D'_{en})$ for every $n\ge 1$. We therefore have a partition $\langle I_{ed}\rangle_{d\in D^*_e}$ of $\bigcup_{n\ge 1}D'_{en}$ into countably infinite sets such that $I_{ed}\cap D'_{en}$ is infinite whenever $d\in D^*_e$ and $n\ge 1$.

Let θ be a limit ordinal such that the set Ω of limit ordinals less than θ has cardinal $\#(\bigcup_{e\in E}D_e)$. (Of course we can take θ to be either an uncountable cardinal or the ordinal product $\omega\cdot\omega$ or 0.) Again because every member of $\mathcal{F}$ is countable, we have an enumeration $\langle d_\xi\rangle_{\xi<\theta}$ of $\bigcup_{e\in E,n\in\mathbb{N}}D_{en}$ such that whenever $\xi\in\Omega$ then there are $F\in\mathcal{F}$ and $e\in F$ such that

$$d_\xi\in D_e,\quad\{d_{\xi+i}:i\ge 1\}=\bigcup_{e'\in F,e'\succcurlyeq e}I_{e'd_\xi}.$$

This will mean that whenever $\xi\in\Omega$ and $F\in\mathcal{F}$, $e\in F$ are such that $d_\xi\in\mathfrak{A}_e$, then $\{i:d_{\xi+i}\in D'_{e'n}\}$ is infinite whenever $e'\in F$, $e\preccurlyeq e'$ and $n\in\mathbb{N}$.

(b)(i) $P_0\in\mathrm{RO}^\uparrow(P)$. **P** Evidently P_0 is up-open. If $p\in P\setminus P_0$, that is, there is some e such that $\{e\}\in\mathcal{F}$ and $\bar\mu(a_p\cup e)<\alpha_p$, set $q=(a_p\cup e,\alpha_p)$; then $p\le q\in P$, while

$$\bar\mu(a_{q'}\cup e)=\bar\mu a_{q'}<\alpha_{q'}\le\alpha_p$$

whenever $q'\in[q,\infty[$, so $[q,\infty[$ does not meet P_0. Accordingly $[p,\infty[\not\subseteq\overline{P}_0$ and $p\notin\operatorname{int}\overline{P}_0$. As p is arbitrary, $P_0=\operatorname{int}\overline{P}_0\in\mathrm{RO}^\uparrow(P)$. **Q**

It follows that $\mathrm{RO}^\uparrow(P_0)$ is the principal ideal of $\mathrm{RO}^\uparrow(P)$ generated by P_0 (Fremlin 02, 314R(b-ii)[6]). Moreover, for $p\in P_0$, $[p,\infty[$ is the same whether taken in P or P_0, and belongs to $\mathrm{RO}^\uparrow(P)$ by 1E above.

(ii) For $p\in P_0$ and $n\in\mathbb{N}$, set $A_n(p)=\{d:d\in\bigcup_{e\in E}D_{en},\ d\subseteq a_p\}$. Of course $A_n(p)\subseteq A_n(q)$ whenever $p\le q$. Also any sequence in $A_n(p)$ has a subsequence with an upper bound in $A_n(p)$. **P** Set $L=\{e:e\in E,\ \bar\mu(a_p\cap e)\ge 2^{-n}\epsilon\}$; then L is finite. If $e\in E\setminus L$ and $d\in D_{en}$, then $d\subseteq e$ and

$$\bar\mu d\ge 2^{-n}\bar\mu e\ge 2^{-n}\epsilon>\bar\mu(a_p\cap e)\ge\bar\mu(a_p\cap d),$$

so $d\not\subseteq a$. Thus $A_n(p)\subseteq\bigcup_{e\in L}D_{en}$. It follows that if $\langle c_i\rangle_{i\in\mathbb{N}}$ is any sequence in $A_n(p)$, there is an $e\in L$ such that $J=\{i:c_i\in D_{en}\}$ is infinite. Now there is an infinite $I\subseteq J$ such that $c=\sup_{i\in I}c_i$ belongs to D_{en}. In this case, $c\subseteq a$ so $c\in A_n(p)$ is an upper bound of $\{c_i:i\in I\}$. **Q**

It follows that $A_n(p)$ has only finitely many maximal elements, and any non-decreasing sequence in $A_n(p)$ has an upper bound in $A_n(p)$. Consequently, every member of $A_n(p)$ is included in a maximal element of $A_n(p)$. **P?** Otherwise, we should be able to find a strictly increasing family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $A_n(p)$; but now there must be a $\xi<\omega_1$ such that $\bar\mu c_\xi=\bar\mu c_{\xi+1}<\gamma$ and $c_\xi=c_{\xi+1}$. **XQ**

Set $E_n(p)=\{\xi:d_\xi$ is a maximal element of $A_n(p)\}$, so that $E_n(p)$ is a finite subset of θ.

(iii) For $n\in\mathbb{N}$ and $\gamma\in\mathbb{R}$, set

$$Q_{n\gamma}=\{q:q\in P_0,\ \alpha_q=\gamma,\ A_n(q)=A_n(q')\text{ whenever }q\le q'\in P_0\}.$$

Then whenever $p\in P_0$, $n\in\mathbb{N}$ and $\bar\mu a_p<\gamma<\alpha_p$ there is a $q\in Q_{n\gamma}$ such that $p\le q$ and $A_n(p)=A_n(q)$. **P** Let L be a finite subset of E including $\{e:\bar\mu(a_p\cap e)\ge 2^{-n-1}\epsilon\}$ and such that $\bar\mu(\sup L)>\gamma$. Then

[6]Later editions only.

$\langle\bigcup_{e\in L} D_{em}\rangle_{m\in\mathbb{N}}$ is a well-spread basis for $(\mathfrak{A}_{\sup L},\bar\mu\restriction\mathfrak{A}_{\sup L})$. (Induce on $\#(L)$, using 1Gb for the inductive step.) Since

$$\bar\mu(a_p\cap\sup L)<\gamma-\bar\mu(a_p\setminus\sup L)<\bar\mu(\sup L),$$

there is a a $b_0\in\mathfrak{A}_{\sup L}$, including $a_p\cap\sup L$, such that

$$\gamma-\bar\mu(a_p\setminus\sup L)-2^{-n-1}\epsilon\le\bar\mu b_0<\gamma-\bar\mu(a_p\setminus\sup L)\le\bar\mu(b_0\cup d)$$

whenever $d\in\bigcup_{e\in L}D_{en}$ and $d\not\subseteq a_p$. Then $\bar\mu(b_0\cup a_p)=\bar\mu b_0+\bar\mu(a\setminus\sup L)<\gamma$, so $q=(b_0\cup a_p,\gamma)$ belongs to P_0. If $q\le q'\in P_0$ and $d\in\bigcup_{e\in E}D_{en}\setminus A_n(p)$, then either $e\in L$ and

$$\bar\mu(a_{q'}\cup d)\ge\bar\mu(b_0\cup d)+\bar\mu(a_p\setminus\sup L)\ge\gamma>\bar\mu a_{q'},$$

or $e\notin L$,

$$\bar\mu(d\setminus a_p)\ge\bar\mu d-\bar\mu(a_p\cap e)\ge 2^{-n}\bar\mu e-2^{-n-1}\epsilon\ge 2^{-n-1}\epsilon$$

and

$$\bar\mu(a_{q'}\cup d)\ge\bar\mu b_0+\bar\mu(a_p\setminus\sup L)+2^{-n-1}\epsilon\ge\gamma>\bar\mu a_{q'};$$

in either case $d\not\subseteq a_{q'}$. Thus $A_n(q')=A_n(p)=A_n(q)$ whenever $q\le q'\in P_0$, and $q\in Q_{n\gamma}$. **Q**

(c)(i) For m, n, $i\in\mathbb{N}$, $\gamma\in\mathbb{Q}$ and $\xi\in\Omega$, set

$$Q_{nmi\gamma\xi}=\{q:q\in Q_{n\gamma},\ \xi+i\in E_n(q),\ \#(E_n(q)\cap\xi)=m\},$$

$$G_{nmi\gamma\xi}=\sup\{[b,\infty[\,:b\in Q_{nmi\gamma\xi}\}\in\mathrm{RO}^{\uparrow}(P_0).$$

(ii) For any m, n, $i\in\mathbb{N}$ and $\gamma\in\mathbb{Q}$, $\langle G_{nmi\gamma\xi}\rangle_{\xi\in\Omega}$ is disjoint. **P** Suppose that $\xi<\eta$ in Ω. If $p\in Q_{nmi\gamma\xi}$ and $q\in Q_{nmi\gamma\eta}$, we see that $\xi+i<\eta$, $\xi+i\in E_n(p)$ and

$$\#(E_n(q)\cap\eta)=m=\#(E_n(p)\cap\xi)<\#(E_n(p)\cap\eta).$$

So $E_n(p)\ne E_n(q)$ and $A_n(p)\ne A_n(q)$. But both p and q are supposed to belong to $Q_{n\gamma}$, so $[p,\infty[$ must be disjoint from $[q,\infty[$. As q is arbitrary, $[p,\infty[\,\cap G_{nmi\gamma\eta}=\emptyset$; as p is arbitrary, $G_{nmi\gamma\xi}\cap G_{nmi\gamma\eta}=\emptyset$. **Q**

(iii) For any $\xi\in\Omega$ and $p\in P_0$, there are m, n, $i\in\mathbb{N}$, $\gamma\in\mathbb{Q}$ and $q\in Q_{nmi\gamma\xi}$ such that $p\le q$. **P** Let $e\in E$ be such that $d_\xi\subseteq e$; let F be the member of $\mathcal{F}$ containing e. If $F=\{e\}$, then $\bar\mu(a_p\cup e)\ge\alpha_p>\bar\mu a_p$; set $e_0=e$, so that $e_0\in F$, $e_0\succcurlyeq e$ and $a_p\cap e_0\ne e_0$. Otherwise, there are infinitely many members of F greater than e for the ordering $\preccurlyeq$, because F has no greatest member, so $\bar\mu(\sup_{e'\in F,e'\succcurlyeq e}e')=\infty$, and there must be an $e_0\in F$ such that $e_0\succcurlyeq e$ and $a_p\cap e_0\ne e_0$.

Take $\gamma\in\mathbb{Q}\cap]\bar\mu a_p,\alpha_p]$. Let $n\in\mathbb{N}$ be such that $2^{-n}\bar\mu e_0<\min(\gamma-\bar\mu a,\bar\mu(e_0\setminus a))$. Then $\{d_{\xi+i}:i\ge 1\}$ meets D'_{e_0n} in an infinite set. So there is an $i\in\mathbb{N}$ such that $d_{\xi+i}\in D'_{e_0n}$, $\bar\mu d_{\xi+i}=2^{-n}\bar\mu e_0$, and $d\not\subseteq(a_p\cap e_0)\cup d_{\xi+i}$ whenever $d\in D_{e_0n}$ and $d\supset d_{\xi+i}$. Set $p'=(a_p\cup d_{\xi+i},\gamma)$; then $p\le p'\in P_0$ and $d_{\xi+i}$ is a maximal member of $A_n(p')$. Let $q\in Q_{n\gamma}$ be such that $p'\le q$ and $A_n(q)=A_n(p')$. Then $\xi+i\in E_n(q)$. Set $m=\#(E_n(q)\cap\xi)$. Then $q\in Q_{nmi\gamma\xi}$ and $p\le q$. **Q**

Accordingly $q\in[p,\infty[\cap G_{nmi\gamma\xi}$. As p is arbitrary, $\bigcup_{m,n,i\in\mathbb{N},\gamma\in\mathbb{Q}}G_{nmi\gamma\xi}$ is dense in P_0 and $\sup_{m,n,i\in\mathbb{N},\gamma\in\mathbb{Q}}G_{nmi\gamma\xi}=P_0$ in $\mathrm{RO}^{\uparrow}(P_0)$.

(d)(i) Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{RO}^{\uparrow}(P_0)$ generated by $\{G_{nmi\gamma\xi}:m,\ n,\ i\in\mathbb{N},\ \gamma\in\mathbb{Q},\ \xi\in\Omega\}$. By (c-ii) and (c-iii), the conditions of Fremlin 08, 514F are satisfied, and $\mathfrak{G}$ has countable Maharam type.

(ii) If $p\in P_0$ and $a_p\in\bigcup_{e\in E,n\in\mathbb{N}}D_{en}$, then $[p,\infty[\,\in\mathfrak{G}$. **P** Set

$$H=\sup\{G_{nmi\gamma\xi}:m,\ n,\ i\in\mathbb{N},\ \gamma\in\mathbb{Q},\ \xi\in\Omega\text{ and }G_{nmi\gamma\xi}\subseteq[p,\infty[\}\in\mathrm{RO}^{\uparrow}(P_0).$$

Then $H\in\mathfrak{G}$ and $H\subseteq[p,\infty[$. Suppose that $p'\in P_0$ and $p'\ge p$. Let $n\in\mathbb{N}$ be such that $a_p\in\bigcup_{e\in E}D_{en}$. Take $\gamma\in\mathbb{Q}\cap]\bar\mu a_{p'},\alpha_{p'}]$ and set $p''=(a_{p'},\gamma)$. Then $p\le p'\le p''$ and there is a $q\in Q_{n\gamma}$ such that $p''\le q$. In this case, $a_p\in A_n(q)$ so there is a maximal $d\in A_n(q)$ including a_p; let $\xi\in\Omega$, $i\in\mathbb{N}$ be such that $d=d_{\xi+i}$, and set $m=\#(E_n(q)\cap\xi)$. Then $q\in Q_{nmi\gamma\xi}$. On the other hand, for any $q'\in Q_{nmi\gamma\xi}$, $a_p\subseteq d_{\xi+i}\subseteq a_{q'}$, while $\alpha_{q'}=\gamma\le\alpha_{p'}\le\alpha_p$, so $[q',\infty[\,\subseteq[p,\infty[$; as q' is arbitrary, $G_{nmi\gamma\xi}\subseteq[p,\infty[$ and $G_{nmi\gamma\xi}\subseteq H$. Accordingly

$q \in H \cap [p', \infty[$. As p' is arbitrary, H is dense in $[p, \infty[$ and must be the whole of $[p, \infty[$; thus we have $[p, \infty[= H \in \mathfrak{G}$. **Q**

(iii) If $p \in P_0$ there is a $q \in P_0$ such that $p \le q$ and $[q, \infty[\in \mathfrak{G}$. **P** Take $\gamma \in \mathbb{Q} \cap]\bar\mu a_p, \alpha_p[$. Let E_0 be a countable subset of E such that $a_p \subseteq \sup E_0$ and $\bar\mu(\sup E_0) > \gamma$. Set $L = \{e : e \in E_0,\ a_p \supseteq e\}$. Then $E_0 \setminus L$ is non-empty, and

$$\sum_{e \in E_0 \setminus L} \bar\mu(a_p \cap e) = \bar\mu a_p - \bar\mu(\sup L) < \gamma - \bar\mu(\sup L).$$

We therefore have a family $\langle \gamma_e \rangle_{e \in E_0 \setminus L}$ such that $\bar\mu(a \cap e) < \gamma_e \le \bar\mu e$ for every $e \in E_0 \setminus L$ and $\sum_{e \in E_0 \setminus L} \gamma_e < \gamma - \bar\mu(\sup L)$. For each $e \in E_0$ there is a $B_e \subseteq \bigcup_{n \in \mathbb{N}} D_{en}$ such that $a \cap e \subseteq \sup B_e$ and $\bar\mu \sup B_e \le \gamma_e$, by 1F(ii). Set

$$B = L \cup \bigcup_{e \in E_0 \setminus L} B_e \subseteq \bigcup_{e \in E, n \in \mathbb{N}} D_{en}$$

and $b = \sup B$. Then $a_p \subseteq b$ and

$$\bar\mu b = \bar\mu(\sup L) + \sum_{e \in E_0 \setminus L} \bar\mu(\sup B_e) \le \bar\mu(\sup L) + \sum_{e \in E_0 \setminus L} \gamma_e < \gamma,$$

so $q = (b, \gamma) \in P_0$. On the other hand, Lemma 1B tells us that

$$[q, \infty[= \inf_{d \in B} [(d, \gamma), \infty[\in \mathfrak{G}$$

as required. **Q**

(iv) As p is arbitrary, $\mathfrak{G}$ includes a π-base for the Boolean algebra $\mathrm{RO}^\uparrow(P_0)$ and must be the whole of $\mathrm{RO}^\uparrow(P_0)$. Accordingly

$$\tau(\mathrm{RO}^\uparrow(P_0)) = \tau(\mathfrak{G}) \le \omega.$$

This completes the proof.

1I Theorem (compare Fremlin 08, 528V[7]) Let $(\mathfrak{A}, \bar\mu)$ be a semi-finite measure algebra with at most $\mathfrak{c}$ atoms. Then $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ has countable Maharam type.

proof Throughout the proof, P will stand for $P^*(\mathfrak{A}, \bar\mu)$.

(a) Suppose that there are a partition E of unity in $\mathfrak{A}$ and an $\epsilon > 0$ such that $\mathfrak{A}_e$ is homogeneous and $\epsilon \le \bar\mu e < \infty$ for every $e \in E$.

(i) Set $E_a = \{e : e \in E,\ e \text{ is an atom}\}$; then $\#(E_a) \le \mathfrak{c}$. Set $E_c = E \setminus E_a$. Let $\preccurlyeq$ be a well-ordering of E_c such that $\tau(\mathfrak{A}_e) \le \tau(\mathfrak{A}_{e'})$ whenever $e \preccurlyeq e'$ in E_c. Let $\mathcal{F}_0$ be a maximal disjoint family of subsets of E_c of order type ω. Then $M_0 = E_c \setminus \bigcup \mathcal{F}_0$ must be finite; set $\mathcal{F} = \mathcal{F}_0 \cup \{\{e\} : e \in M_0\}$.

(ii) Set $M = M_0 \cup E_a$. For $L \in [M]^{<\omega}$, set

$$P_L = \{p : p \in P,\ a_p \supseteq \sup L,\ \bar\mu(a_p \cup e) \ge \alpha_p \text{ for } e \in M \setminus L\}.$$

Then $\langle P_L \rangle_{L \subseteq M \text{ is finite}}$ is a disjoint family of open subsets of P. Also $\bigcup_{L \in [M]^{<\omega}} P_L$ is dense in P. **P** If $p \in P$, there is a maximal finite $L \subseteq M$ such that $\bar\mu(a_p \cup \sup L) < \alpha_p$, because $\bar\mu e \ge \epsilon$ for every $e \in E$. Set $q = (a_p \cup \sup L, \alpha_p)$; then $p \le q \in P_L$. **Q** So $\mathrm{RO}^\uparrow(P)$ is isomorphic to the simple product $\prod_{L \in [M]^{<\omega}} \mathrm{RO}^\uparrow(P_L)$ (Fremlin 02, 315S[8]).

(iii) If $L \in [M]^{<\omega}$, then $\mathrm{RO}^\uparrow(P_L)$ has countable Maharam type. **P** If $P_L = \emptyset$ this is trivial. Otherwise there is a $p \in P_L$ and $\bar\mu(\sup L) \le \bar\mu a_p < \alpha_p \le \bar\mu 1$. Consider $\mathfrak{A}' = \mathfrak{A}_{1 \setminus \sup(L \cup E_a)}$, $E' = E \setminus (L \cup E_a)$, $\mathcal{F}' = \mathcal{F} \setminus \{\{e\} : e \in L \cap M_0\}$ and $\preccurlyeq' = \preccurlyeq \cap (E' \times E')$. Then $(\mathfrak{A}', \bar\mu \restriction \mathfrak{A}')$, E', ϵ, $\preccurlyeq'$ and $\mathcal{F}'$ satisfy the conditions of Lemma 1H. Setting

$$Q_0 = \{q : q \in P^*(\mathfrak{A}', \bar\mu \restriction \mathfrak{A}'),\ \alpha_q \le \bar\mu(a_q \cup e) \text{ for every } e \in M_0 \setminus L\},$$

$\mathrm{RO}^\uparrow(Q_0)$ has countable Maharam type, by 1H. But the map $q \mapsto (a_q \cup \sup L, \alpha_q + \bar\mu(\sup L))$ is an order-isomorphism between Q_0 and P_L, so $\mathrm{RO}^\uparrow(P_L)$ has countable Maharam type. **Q**

[7] Later editions only.
[8] Later editions only.

(iv) As $\#([M]^{<\omega}) \le \mathfrak{c}$, $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma) = \mathrm{RO}^{\uparrow}(P)$ is isomorphic to the product of at most $\mathfrak{c}$ Boolean algebras with countable Maharam type, and has countable Maharam type (Fremlin 08, 514Ef).

(b) Now suppose that $(\mathfrak{A}, \bar\mu)$ is localizable.

(i) In this case, let E be a partition of unity in $\mathfrak{A}$ such that $\mathfrak{A}_e$ is homogeneous and $0 < \bar\mu e < \infty$ for every $e \in E$. For each $k \in \mathbb{N}$, set

$$E_k = \{e : e \in E,\, \bar\mu e \ge 2^{-k}\}, \quad e_k^* = \sup E_k.$$

By (a), $\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu \restriction \mathfrak{A}_{e_k^*})$ has countable Maharam type for every k.

(ii) Now Proposition 1D tells us that we have a sequence $\langle \pi_k\rangle_{k\in\mathbb{N}}$ such that π_k is an order-continuous ring homomorphism from $\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})$ into $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ for each k, and $\bigcup_{k\in\mathbb{N}} \pi_k[\mathrm{AM}(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})]$ τ-generates $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$. So $\mathrm{AM}(\mathfrak{A}, \bar\mu, \gamma)$ has countable Maharam type. **P** For each k, we have a countable τ-generating set $D_k \subseteq \mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})$. Let $\mathfrak{G}$ be the order-closed subalgebra of $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ generated by $D = \bigcup_{k\in\mathbb{N}} \pi_k[D_k \cup \{1_k\}]$, where 1_k here is the greatest element of $\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})$. For each $k \in \mathbb{N}$, $\pi_k^{-1}[\mathfrak{G}]$ is an order-closed subalgebra of $\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})$ including D_k, so is the whole of $\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})$, that is, $\pi_k[\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})] \subseteq \mathfrak{G}$. Since $\bigcup_{k\in\mathbb{N}} \pi_k[\mathrm{AM}^*(\mathfrak{A}_{e_k^*}, \bar\mu\restriction\mathfrak{A}_{e_k^*})]$ τ-generates $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$, $\mathfrak{G} = \mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ and $\tau(\mathrm{AM}^*(\mathfrak{A}, \bar\mu)) \le \#(D) \le \omega$. **Q**

(c) Thus we have the result when $(\mathfrak{A}, \bar\mu)$ is localizable. For the general case of atomless semi-finite $(\mathfrak{A}, \bar\mu)$, let $(\widehat{\mathfrak{A}}, \tilde\mu)$ be the localization of $(\mathfrak{A}, \bar\mu)$ (Fremlin 02, 322Q[9]). Since the embedding $\mathfrak{A} \subsetneq \widehat{\mathfrak{A}}$ identifies $\mathfrak{A}^f$ with $\widehat{\mathfrak{A}}^f$ (Fremlin 02, 322P[10]), $P^*(\widehat{\mathfrak{A}}, \tilde\mu)$ can be identified with P, and the regular open algebras $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ and $\mathrm{AM}^*(\widehat{\mathfrak{A}}, \tilde\mu)$ are isomorphic. Again because $\mathfrak{A}^f$ and $\widehat{\mathfrak{A}}^f$ are isomorphic, $\widehat{\mathfrak{A}}$ has at most $\mathfrak{c}$ atoms. By (b), the common Maharam type of $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ and $\mathrm{AM}^*(\widehat{\mathfrak{A}}, \tilde\mu)$ is countable.

1J Example Let X be a set, and μ counting measure on X. Then $\mathrm{AM}^*(\mathcal{P}X, \mu)$ is purely atomic, with $\#([X]^{<\omega})$ atoms. **P** (i) For $I \in [X]^{<\omega}$, set $p_I = (I, \frac{1}{2} + \mu I)$. Then $\langle p_I\rangle_{I\in[X]^{<\omega}}$ is an up-antichain in $P = P^*(\mathcal{P}X, \mu)$. (ii) If $p \in P$, then $I = a_p$ is a finite subset of X, and $\mu(a_p \cup a_{p_I}) = \mu I < \min(\alpha_p, \alpha_{p_I})$; thus p and p_I are compatible upwards; as p is arbitrary, $\langle p_I\rangle_{I\in[X]^{<\omega}}$ is a maximal up-antichain. (iii) If $I \in [X]^{<\omega}$ and p, $q \in [p_I, \infty[$, then $a_p = a_q = I$ so p and q are compatible upwards; thus V_{p_I} is an atom in $\mathrm{RO}^{\uparrow}(P)$. So $\mathrm{RO}^{\uparrow}(P) = \mathrm{AM}^*(\mathcal{P}X, \mu)$ is purely atomic, and we have a listing of its atoms. **Q**

Accordingly $\tau(\mathrm{AM}^*(\mathcal{P}X, \mu)) = \#(X)$ if X is finite and otherwise is $\min\{\lambda : \#(X) \le 2^\lambda\}$ (Fremlin 08, 514Xr[11]). In particular, it will be uncountable if $\#(X) > \mathfrak{c}$.

1K Theorem Let $(\mathfrak{A}, \bar\mu)$ be a purely atomic semi-finite measure algebra, and E the set of its atoms. Suppose that

for every $\epsilon > 0$ there is a $\delta > 0$ such that $\#(\{e : e \in E,\, \delta \le \bar\mu e < \epsilon\}) = \#(E)$.

Then $\mathrm{AM}^*(\mathfrak{A}, \bar\mu)$ has countable Maharam type.

proof (a) Set $\kappa = \#(E)$; let $\langle \epsilon_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence with limit 0 such that $E^{(n)} = \{e : e \in E, \epsilon_{n+1} \le \bar\mu e < \epsilon_n\}$ has cardinal κ for every $n \in \mathbb{N}$. Then we have a partition $\langle E_\xi\rangle_{\xi<\kappa}$ of E into countable sets such that $E_\xi \cap E^{(n)}$ is infinite for every $\xi < \kappa$ and $n \in \mathbb{N}$. Enumerate each E_ξ as $\langle e_{\xi i}\rangle_{e\in\mathbb{N}}$. Let $\preccurlyeq$ be the well-ordering of E corresponding to the lexicographic ordering of $\mathbb{N} \times \kappa$; then $E_\xi \cap E^{(n)}$ is cofinal for every ξ and n.

(b) For $p \in P = P^*(\mathfrak{A}, \bar\mu)$ and $n \in \mathbb{N}$, set $A_n(p) = \{e : e \in E,\, e \subseteq a_p,\, \bar\mu e \ge \epsilon_n\}$; then $A_n(p)$ is finite. For $n \in \mathbb{N}$, $J \in [\mathbb{N}]^{<\omega}$, $\gamma \in \mathbb{Q}$ and $\xi < \kappa$ set

$$Q_n = \{q : q \in P,\, A_n(q') = A_n(q) \ne \emptyset \text{ whenever } q' \ge q\},$$

[9] Formerly 322P.

[10] Formerly 322O.

[11] Later editions only.

$$Q_{nJ\gamma\xi} = \{q : q \in Q_n, \text{ the } \preccurlyeq\text{-largest member of } A_n(q) \text{ belongs to } E_\xi,$$
$$\alpha_q = \gamma \text{ and } \{i : e_{\xi i} \in A_n(q)\} = J\},$$

$$G_{nJ\gamma\xi} = \sup\{V_q : q \in Q_{nJ\gamma\xi}\} \in \mathrm{RO}^\uparrow(P).$$

(c)(i) If $n \in \mathbb{N}$, $J \in [\mathbb{N}]^{<\omega}$, $\gamma \in \mathbb{Q}$, $\xi < \eta < \kappa$, $q \in Q_{nJ\gamma\xi}$ and $q' \in Q_{nJ\gamma\eta}$ then $A_n(q) \ne A_n(q')$, while both q and q' belong to Q_n. So $[q, \infty[$ cannot meet $[q', \infty[$ and $V_q \cap V_{q'} = \emptyset$. As q and q' are arbitrary, $G_{nJ\gamma\xi} \cap G_{nJ\gamma\eta} = \emptyset$.

(ii) If $p \in P$, $\alpha_p = \gamma \in \mathbb{Q}$, $\xi < \kappa$ and $m \in \mathbb{N}$, there are $n \ge m$, $J \in [\mathbb{N}]^\omega$ and $q \in Q_{nJ\gamma\xi}$ such that $p \le q$. **P** Take $n > m$ such that $\epsilon_{n-1} + \bar\mu a_p \le \gamma$. Let $i \in \mathbb{N}$ be such that $\epsilon_n \le \bar\mu e_{\xi i} < \epsilon_{n-1}$ and $e \preccurlyeq e_{\xi i}$ for every $e \in A_n(p)$. Then $\bar\mu(a_p \cup e_{\xi i}) < \bar\mu a_p + \epsilon_{n-1} \le \gamma$. Let $L \subseteq E^{(n)}$ be a maximal set such that $\bar\mu(a_p \cup e_{\xi i} \cup \sup L) < \gamma$ (such exists because $\bar\mu e \ge \epsilon_{n+1}$ for every $e \in E^{(n)}$); set $q = (a_p \cup e_{\xi i} \cup \sup L, \gamma) \in P$. Of course $p \le q$. Because $E^{(n)}$ is infinite and $\bar\mu e \le \epsilon_n$ for every $e \in E^{(n)}$, $\bar\mu a_q \ge \gamma - \epsilon_n$. It follows that if $q' \ge q$, $A_n(q') = A_n(q) = A_n(p) \cup \{e_{\xi i}\}$. Set $J = \{j : e_{\xi j} \in A_n(q)\}$. By the choice of i, $e_{\xi i}$ is the $\preccurlyeq$-greatest member of $A_n(q)$, and $q \in Q_{nJ\gamma\xi}$, as required. **Q**

It follows that $\bigcup_{n\in\mathbb{N}, J\in[\mathbb{N}]^{<\omega}, \gamma\in\mathbb{Q}} G_{nJ\gamma\xi}$ is dense in P for every $\xi < \kappa$. **P** Given $p \in P$, take $\gamma \in \mathbb{Q}$ such that $\bar\mu a_p < \gamma \le \alpha_p$; then we have $n \in \mathbb{N}$, $J \in [N]^{<\omega}$ and $q \in Q_{nJ\gamma\xi}$ such that $(a_p, \gamma) \le q$, in which case $[p, \infty[\cap G_{nJ\gamma\xi} \ne \emptyset$. **Q**

(iii) Suppose that $p \in P$ and $a_p \in E$. Set

$$H = \sup\{G_{nJ\gamma\xi} : n \in \mathbb{N},\ J \in [\mathbb{N}]^{<\omega},\ \gamma \in \mathbb{Q},\ \xi < \kappa,\ G_{nJ\gamma\xi} \subseteq V_p\} \in \mathrm{RO}^\uparrow(P).$$

Then $H = V_p$. **P** Of course $H \subseteq V_p$. In the other direction, take any $p' \ge p$ in P. Let $\gamma \in \mathbb{Q}$ be such that $\bar\mu a_{p'} < \gamma \le \alpha_{p'}$. Let $m \in \mathbb{N}$ be such that $\bar\mu a_p \ge \epsilon_m$; let $\xi < \kappa$, $i \in \mathbb{N}$ be such that $a_p = e_{\xi i}$. By (ii), there are $n \ge m$, $J \in [\mathbb{N}]^{<\omega}$ and $q \in Q_{nJ\gamma\xi}$ such that $p' \le q$. Now $e_{\xi i} \in A_m(p) \subseteq A_m(q) \subseteq A_n(q)$, so $i \in J$ and $e_{\xi i} \in A_n(q')$ whenever $q' \in Q_{nJ\gamma\xi}$; thus $p \le (e_{\xi i}, \gamma) \le q'$ for every $q' \in Q_{nJ\gamma\xi}$, $V_{q'} \subseteq V_p$ for every $q' \in Q_{nJ\gamma\xi}$, $G_{nJ\gamma\xi} \subseteq V_p$ and $G_{nJ\gamma\xi} \subseteq H$. Accordingly $q \in [p', \infty[\cap H$; as p' is arbitrary, H is dense in V_p and $H = V_p$. **Q**

(d) By (c-i) and (c-ii) and Fremlin 08, 514F, as in (d-i) of the proof of 1H, the order-closed subalgebra $\mathfrak{G}$ of $\mathrm{RO}^\uparrow(P)$ generated by $\{G_{nJ\gamma\xi} : n \in \mathbb{N},\ J \in [\mathbb{N}]^{<\omega},\ \gamma \in \mathbb{Q},\ \xi < \kappa\}$ has countable Maharam type; by (c-iii), $\mathfrak{G}$ contains V_p whenever $p \in P$ and $a_p \in E$; by Lemma 1B, $\mathfrak{G}$ contains V_p for every $p \in P$; as in (d-iv) of the proof of 1H, $\mathfrak{G} = \mathrm{RO}^\uparrow(P)$ and $\mathrm{RO}^\uparrow(P) = \mathrm{AM}(\mathfrak{A}, \bar\mu)$ has countable Maharam type.

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