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Distributive lattices and modular functions

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1 Two facts

1A Stone representation of distributive lattices: Theorem Let P be a distributive lattice and Z the set of surjective lattice homomorphisms from P to $\{0,1\}$. For $p \in P$ set $\pi(p) = \{z : z \in Z,\ z(p) = 1\}$. Then π is an injective lattice homomorphism, so P is isomorphic to the sublattice $\pi[P]$ of $\mathcal{P}Z$.

proof Just because Z consists of lattice homomorphisms, π is a lattice homomorphism. To see that it is injective, suppose that p_0, $q_0 \in P$ and $q_0 \not\leq p_0$. Consider the family $\mathcal{Q}$ of pairs (A, B) of subsets of P such that

$p_0 \in A$, $q_0 \in B$,

$p \vee p' \in A$ for all p, $p' \in A$, $q \wedge q' \in B$ for all q, $q' \in B$,

$q \not\leq p$ for all $p \in A$, $q \in B$.

Then $(\{p_0\}, \{q_0\}) \in \mathcal{Q}$. Ordering $\mathcal{Q}$ by saying that

$$(A, B) \leq (A', B') \text{ if } A \subseteq A',\ B \subseteq B',$$

any totally ordered subset of $\mathcal{Q}$ has an upper bound in $\mathcal{Q}$; so $\mathcal{Q}$ has a maximal element (A_0, B_0) say.

? If $A_0 \cup B_0 \neq P$, take $r \in P \setminus (A_0 \cup B_0)$. Set $A = A_0 \cup \{p \vee r : p \in A_0\}$, $B = B_0 \cup \{q \wedge r : q \in B_0\}$. Then $(A, B_0) \notin \mathcal{Q}$, so there must be $p \in A_0$, $q \in B_0$ such that $q \leq p \vee r$; also $(A_0, B) \notin \mathcal{Q}$, so there are $p' \in A_0$, $q' \in B_0$ such that $q' \wedge r \leq p'$. In this case,

$$q \wedge q' \leq (p \vee r) \wedge q' = (p \wedge q') \vee (r \wedge q')$$

(because P is distributive)

$$\leq p \vee p'.$$

But $q \wedge q' \in B_0$ and $p \vee p' \in A_0$, so this is impossible. **X**

Of course $A_0 \cap B_0 = \emptyset$, so we can define $z : P \to \{0,1\}$ by setting $z(p) = 0$ for $p \in A_0$ and $z(q) = 1$ for $q \in B_0$. If $p \in A_0$ and $p' \leq p$, then $p' \notin B_0$ so $p' \in A_0$; similarly, if $q \in B_0$ and $q \leq q'$, then $q' \in B_0$. Accordingly z is order-preserving. Because A_0 is closed under $\vee$, $z(p \vee q) = \max(z(p), z(q))$ for all p, $q \in P$; because B_0 is closed under $\wedge$, $z(p \wedge q) = \min(z(p), z(q))$ for all p, $q \in P$. Thus $z \in Z$, while $z(p_0) = 0 \neq 1 = z(q_0)$; and $\pi(p_0) \neq \pi(q_0)$.

Similarly, $\pi(p_0) \neq \pi(q_0)$ if $p_0 \not\leq q_0$, so π is injective.

1B Definition Let $(G, +)$ be a semigroup, $(H, +, \leq)$ a semigroup with a partial ordering $\leq$, and $f : G \to H$ a function.

(a) f is **superadditive** if $f(x + y) \geq f(x) + f(y)$ for all x, $y \in G$.

(b) f is **subadditive** if $f(x + y) \leq f(x) + f(y)$ for all x, $y \in G$.

1C Theorem (König 00) Let $(G, +)$ be a commutative semigroup with identity 0_G and U a Dedekind complete Riesz space. Let f, $g : G \to U$ be functions such that

$$f(0_G) = g(0_G) = 0, \quad f(x) \leq g(x) \text{ for every } x \in G,$$

$$f \text{ is superadditive}, \quad g \text{ is subadditive}.$$

Then there is an additive $h : G \to U$ such that $f \leq h \leq g$.

proof (a) Define nx, for $n \in \mathbb{N}$ and $x \in G$, by saying that $0x = 0_G$ and $(n+1)x = nx + x$ for every n; note that $n(x+y) = nx + ny$, $(m+n)x = mx + nx$ for $x, y \in G$ and $m, n \in \mathbb{N}$. Also $f_1(nx) \ge nf_1(x)$ and $g_1(nx) \le ng_1(x)$ whenever $n \in \mathbb{N}$, $x \in G$, $f_1 : G \to U$ is superadditive and zero at 0_G, and $g_1 : G \to U$ is subadditive and zero at 0_G.

(b) Give U^G the product partial order. If $Q \subseteq U^G$ is the set of superadditive functionals dominated by g, then any non-empty upwards-directed $R \subseteq Q$ has an upper bound in Q. **P** For each $x \in G$, $\{h(x) : h \in R\}$ is an upwards-directed set bounded above by $g(x)$, so has a supremum $h_0(x)$ say. If $x, y \in G$, then $A = \{h(x) : h \in R\}$, $B = \{h(y) : h \in R\}$ are upwards-directed and bounded above. Also, if $h', h'' \in R$, then there is an $h \in R$ such that $h' \le h$ and $h'' \le h$, and now

$$h'(x) + h''(y) \le h(x) + h(y) \le h(x+y) \le h_0(x+y).$$

So $h_0(x+y)$ is an upper bound for $A + B$ and

$$h_0(x+y) \ge \sup(A+B) = \sup A + \sup B = h_0(x) + h_0(y)$$

(Fremlin 02, 351Db). Accordingly h_0 belongs to Q and is an upper bound of R in Q. **Q**

By Zorn's Lemma, there is a maximal superadditive h such that $f \le h \le g$. The rest of the argument is devoted to showing that h is subadditive, therefore additive.

(c) $h(nx) = nh(x)$ for every $n \in \mathbb{N}$, $x \in G$. **P** The case $n = 0$ is trivial (note that $f(0_G) \le h(0_G) \le g(0_G)$ so $h(0_G) = 0$). For $n \ge 1$, consider $f_1(x) = \frac{1}{n}h(nx)$ for $x \in G$. Then f_1 is superadditive because h is, and $h \le f_1$ because $h(nx) \ge nh(x)$ for every $x \in G$. Finally

$$nf_1(x) = h(nx) \le g(nx) \le ng(x)$$

for every x, so $f_1 \le g$. By the maximality of h, $f_1 = h$ and $h(nx) = nh(x)$ for every x. **Q**

(d) $h(x+z) \le h(x) + g(z)$ for all $x, z \in G$. **P** Set

$$f_1(x) = \sup\nolimits_{n\in\mathbb{N}} h(x+nz) - ng(z)$$

for $x \in G$. Then $f_1(x) \ge h(x)$ for every x; also

$$h(x+nz) - ng(z) \le g(x+nz) - ng(z) \le g(x) + g(nz) - ng(z) \le g(x)$$

for all n, so $f_1(x) \le g(x)$, for every x. If $x, y \in G$, then

$$f_1(x) + f_1(y) = \sup_{m,n\in\mathbb{N}} h(x+mz) + h(y+nz) - mg(z) - ng(z)$$

$$\le \sup_{m,n\in\mathbb{N}} h(x+mz+y+nz) - (m+n)g(z)$$

(because h is superadditive)

$$= \sup_{m,n\in\mathbb{N}} h(x+y+(m+n)z) - (m+n)g(z)$$

(because $(G,+)$ is commutative)

$$= f_1(x+y),$$

so f_1 is superadditive. By the maximality of h, $h = f_1$; in particular, $h(x) \ge h(x+z) - g(z)$, that is, $h(x+z) \le h(x) + g(z)$ for every x. **Q**

(e) $h(z+x) \le h(z) + h(x)$ for all $x, z \in G$. **P** This time, set

$$f_1(x) = \sup\nolimits_{n\in\mathbb{N}} h(nz+x) - nh(z)$$

for $x \in X$. Again, $f_1(x) \ge h(x)$, and

$$f_1(x) \le \sup_{n\in\mathbb{N}} h(nz) + g(x) - nh(z)$$

(by (d))

$$= g(x)$$

by (c). Also, as in (d),

$$\begin{aligned} f_1(x) + f_1(y) &= \sup_{m,n\in\mathbb{N}} h(mz + x) + h(nz + y) - mh(z) - nh(z) \\ &\le \sup_{m,n\in\mathbb{N}} h((m + n)z + x + y) - (m + n)h(z) = f_1(x + y), \end{aligned}$$

for all x, $y \in G$. So f_1 is superadditive; by the maximality of h, $f_1 = h$; in particular, $h(x) \ge h(z+x)-h(z)$. **Q**

(f) So h is subadditive and therefore additive, as required.

2 A note on additive functions

2A Lemma Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, $b \in \mathfrak{B}$ either 0 or an atom of $\mathfrak{B}$, I an ideal of $\mathfrak{A}$ and $\mathfrak{C}$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup I$.

(a) If $a \in \mathfrak{C}$, then either $b \cap a \in I$ or $b \setminus a \in I$.

(b) Now suppose that $(G, +)$ is an abelian group, $\nu : I \to G$ an additive function and $g \in G$. Define $\lambda : \mathfrak{C} \to G$ by setting

$$\begin{aligned} \lambda a &= \nu(b \cap a) \text{ if } b \cap a \in I, \\ &= g - \nu(b \setminus a) \text{ otherwise.} \end{aligned}$$

Then λ is an additive function.

proof (a)

$$\{a : a \in \mathfrak{A}, \text{ at least one of } b \cap a,\ b \setminus a \text{ belongs to } I\}$$

is a subalgebra of $\mathfrak{A}$ including $\mathfrak{B} \cup I$, so includes $\mathfrak{C}$.

(b) If $b \in I$ then $\lambda a = \nu(b \cap a)$ for every $a \in \mathfrak{C}$ and the result is immediate. Otherwise, if a, $a' \in \mathfrak{C}$ are disjoint, at most one of $b \setminus a$, $b \setminus a'$ can belong to I, so (by (a)) at least one of $b \cap a$, $b \cap a'$ belongs to I, and

$$\begin{aligned} \lambda a + \lambda a' &= \nu(b \cap a) + \nu(b \cap a') = \nu(b \cap (a \cup a')) = \lambda(a \cup a') \\ &\qquad \text{if } b \cap a \text{ and } b \cap a' \text{ belong to } I, \\ &= \nu(b \cap a) + g - \nu(b \setminus a') = g - \nu(b \setminus (a \cup a')) = \lambda(a \cup a') \\ &\qquad \text{if } b \cap a \in I \text{ and } b \cap a' \notin I, \\ &= \nu(b \cap a') + g - \nu(b \setminus a) = g - \nu(b \setminus (a \cup a')) = \lambda(a \cup a') \\ &\qquad \text{if } b \cap a' \in I \text{ and } b \cap a \notin I. \end{aligned}$$

2B Lemma Let $\mathfrak{A}$ be a Boolean algebra, I an ideal of $\mathfrak{A}$, $\mathfrak{B}$ a finite subalgebra of $\mathfrak{A}$, $\mathfrak{C}$ the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup I$, $(G, +)$ an abelian group and $\lambda : \mathfrak{C} \to G$ an additive function. Suppose that $d \in \mathfrak{A}$ and that $\mathfrak{B}'$, $\mathfrak{C}'$ are the subalgebras of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{d\}$, $\mathfrak{B}' \cup I$ respectively. Then there is an extension of λ to an additive function $\lambda' : \mathfrak{C}' \to G$.

proof (a) Let B be the set of atoms of $\mathfrak{B}$. For each $b \in B$, $b \cap d$ and $b \setminus d$ are either 0 or atoms of $\mathfrak{B}'$. Define λ'_b, $\lambda''_b : \mathfrak{C}' \to G$ by setting

$$\begin{aligned} \lambda'_b a &= \lambda(b \cap d \cap a) \text{ if } b \cap d \cap a \in I, \\ &= \lambda(b \cap d) - \lambda(b \cap d \setminus a) \text{ if } b \setminus d \in I \text{ and } b \cap d \cap a \notin I, \\ &= -\lambda(b \cap d \setminus a) \text{ otherwise }, \\ \lambda''_b a &= \lambda((b \setminus d) \cap a) \text{ if } (b \setminus d) \cap a \in I, \\ &= \lambda(b \setminus d) - \lambda((b \setminus d) \setminus a) \text{ if } b \cap d \in I \text{ and } (b \setminus d) \cap a \notin I, \\ &= \lambda b - \lambda((b \setminus d) \setminus a) \text{ otherwise }. \end{aligned}$$

By Lemma 2A, applied to $\lambda\restriction I$, λ'_b and λ''_b are additive. So we can define an additive $\lambda' : \mathfrak{C}' \to G$ by setting $\lambda' = \sum_{b\in B} \lambda'_b + \lambda''_b$.

(b) $\lambda'_b a + \lambda''_b a = \lambda(b \cap a)$ whenever $a \in \mathfrak{C}$ and $b \in B$. **P**

case 1 If $b \cap a \in I$, then

$$\lambda'_b a + \lambda''_b a = \lambda(b \cap d \cap a) + \lambda((b \setminus d) \cap a) = \lambda(b \cap a).$$

case 2 If $b \cap a \notin I$ and $b \cap d \in I$ then $(b \setminus d) \cap a \notin I$ and

$$\lambda'_b a + \lambda''_b a = \lambda(b \cap d \cap a) + \lambda(b \setminus d) - \lambda((b \setminus d) \setminus a) = \lambda(b \cap a).$$

case 3 If $b \cap a \notin I$ and $b \setminus d \in I$ then $b \setminus a \in I$ (by 2Aa) while $b \cap d \notin I$, so $b \cap d \cap a \notin I$ and

$$\lambda'_b a + \lambda''_b a = \lambda(b \cap d) - \lambda(b \cap d \setminus a) + \lambda((b \setminus d) \cap a) = \lambda(b \cap a).$$

case 4 If $b \cap a \notin I$ and neither $b \cap d$ nor $b \setminus d$ belongs to I, then again $b \setminus a \in I$, so neither $(b \cap d) \cap a$ nor $(b \setminus d) \cap a$ belongs to I, and

$$\lambda'_b a + \lambda''_b a = -\lambda(b \cap d \setminus a) + \lambda b - \lambda((b \setminus d) \setminus a) = \lambda(b \cap a).$$

Thus we have the required equality in all cases. **Q**

(c) Now, for any $a \in \mathfrak{C}$,

$$\lambda' a = \textstyle\sum_{b\in B} \lambda'_b a + \lambda''_b a = \sum_{b\in B} \lambda(b \cap a) = \lambda a,$$

so λ' extends λ, as required.

2C Corollary Let $\mathfrak{A}$ be a Boolean algebra, I an ideal of $\mathfrak{A}$, $(G, +)$ an abelian group and $\nu : I \to G$ an additive function. If $D \subseteq \mathfrak{A}$ is countable and $\mathfrak{C}$ is the subalgebra of $\mathfrak{A}$ generated by $D \cup I$, there is an additive function $\lambda : \mathfrak{C} \to G$ extending ν.

proof (a) If $A = \emptyset$, we can use Lemma 2A with $\mathfrak{B} = \{0, 1\}$ and $g = 0$.

(b) Otherwise, let $\langle d_n\rangle_{n\in\mathbb{N}}$ run over D, and for $n \in \mathbb{N}$ let $\mathfrak{B}_n$ be the subalgebra of $\mathfrak{A}$ generated by $\{d_i : i < n\}$, and $\mathfrak{C}_n$ the subalgebra generated by $\mathfrak{B}_n \cup I$. Using (a) to start, and Lemma 2B for the inductive step, we can construct inductively a sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ such that $\lambda_n : \mathfrak{C}_n \to G$ is additive for each n, λ_0 extends ν, and λ_{n+1} extends λ_n for each n; so that $\lambda = \bigcup_{n\in\mathbb{N}} \lambda_n$ is an additive function from $\mathfrak{C}$ to G extending ν.

2D Lemma Let $\mathfrak{A}$ be a countable Boolean algebra with a subalgebra $\mathfrak{B}$ and a $d \in \mathfrak{A}$ such that $\mathfrak{A}$ is the algebra generated by $\mathfrak{B} \cup \{d\}$. Suppose that $(G, +)$ is an abelian group and $\lambda : \mathfrak{B} \to G$ an additive function. Then there is an additive function $\lambda' : \mathfrak{A} \to G$ extending λ.

proof (a) Let I be the set

$$\{a : a \in \mathfrak{B},\ a \cap d \in \mathfrak{B}\}.$$

Then I is an ideal of $\mathfrak{A}$. **P** Of course I is closed under $\cup$ and contains 0. If $a \in I$ and $b \subseteq a$, then b is expressible as $(c \cap d) \cup (c' \setminus d)$ where c, $c' \in \mathfrak{B}$ (Fremlin 02, 312N). Now $a \setminus d \in \mathfrak{B}$, so

$$b \cap d = a \cap b \cap d = (a \cap d) \cap c \in \mathfrak{B},$$

$$b \setminus d = a \cap b \setminus d = (a \setminus d) \cap c' \in \mathfrak{B}.$$

So $b \in \mathfrak{B}$ and $b \in I$. **Q**

(b) By Lemma 2C, there is an additive function $\lambda_0 : \mathfrak{A} \to G$ such that $\lambda_0 a = \lambda(a \cap d)$ for every $a \in I$. Now observe that if b, $b' \in \mathfrak{B}$ and $b \cap d = b' \cap d$, both $b \setminus b'$ and $b' \setminus b$ are members of $\mathfrak{B}$ disjoint from d, so belong to I, and

$$\lambda_0 b - \lambda_0 b' = \lambda_0(b \setminus b') - \lambda_0(b' \setminus b) = 0.$$

Similarly, if c, $c' \in \mathfrak{B}$ and $c \setminus d = c' \setminus d$, $c \triangle c'$ is included in d and belongs to I, and

$$\begin{aligned}\lambda c - \lambda_0 c &= \lambda(c \cap c') + \lambda(c \setminus c') - \lambda_0(c \cap c') - \lambda_0(c \setminus c') \\ &= \lambda(c \cap c') + \lambda(c \setminus c') - \lambda_0(c \cap c') - \lambda(c \setminus c') \\ &= \lambda(c \cap c') - \lambda_0(c \cap c') = \lambda c' - \lambda_0 c'.\end{aligned}$$

We can therefore define a function $\lambda' : \mathfrak{A} \to G$ by setting

$$\lambda'((b \cap d) \cup (c \setminus d)) = \lambda_0 b + \lambda c - \lambda_0 c$$

whenever b, $c \in \mathfrak{B}$. Just because $(b, c) \mapsto \lambda_0 b + \lambda c - \lambda_0 c : \mathfrak{B} \times \mathfrak{B} \to G$ is additive and $(b, c) \mapsto (b \cap d) \cup (c \cap d) : \mathfrak{B} \times \mathfrak{B} \to \mathfrak{A}$ is a Boolean homomorphism, λ' is additive. Now if $b \in \mathfrak{B}$,

$$\lambda' b = \lambda'((b \cap d) \cup (b \setminus d)) = \lambda_0 b + \lambda b - \lambda_0 b = \lambda b.$$

So λ' extends λ.

2E Theorem Let $\mathfrak{A}$ be a Boolean algebra of cardinal at most ω_1, $\mathfrak{B}$ a countable subalgebra of $\mathfrak{A}$, $(G, +)$ an abelian group and $\lambda : \mathfrak{B} \to G$ an additive function. Then there is an additive function $\lambda' : \mathfrak{A} \to G$ extending λ.

proof Let $\langle a_\xi \rangle_{\xi < \omega_1}$ run over $\mathfrak{A}$, and for $\xi \le \omega_1$ let $\mathfrak{B}_\xi$ be the subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B} \cup \{a_\eta : \eta < \xi\}$. Construct additive functions $\lambda_\xi : \mathfrak{B}_\xi \to G$ inductively, as follows. Start with $\mathfrak{B}_0 = \mathfrak{B}$ and $\lambda_0 = \lambda$. For the inductive step to $\xi + 1$, $\mathfrak{B}_{\xi+1}$ is countable and is generated by $\mathfrak{B}_\xi \cup \{a_\xi\}$, so Lemma 2D tells us that λ_ξ extends to an additive $\lambda_{\xi+1} : \mathfrak{B}_{\xi+1} \to G$. For the inductive step to a limit ordinal $\xi > 0$, set $\lambda_\xi = \bigcup_{\eta < \xi} \lambda_\eta$. Now $\lambda' = \lambda_{\omega_1}$ is an extension of λ to the whole of $\mathfrak{A}$.

3 Modular functions

3A Definition Let P be a lattice, $(G, +)$ a commutative semigroup and $\phi : P \to G$ a function. ϕ is **modular** if $\phi(p) + \phi(q) = \phi(p \vee q) + \phi(p \wedge q)$ for all p, $q \in P$.

3B Elementary facts (a) Let P be a lattice and $(G, +)$ a commutative semigroup.
(i) If ϕ, $\psi : P \to G$ are modular functions, then $\phi + \psi$ is a modular function.
(ii) Any constant function from P to G is modular.
(iii) If $(H, +)$ is another commutative semigroup and $T : G \to H$ is a homomorphism, then $T\phi$ is modular whenever $\phi : P \to G$ is modular.

(b) Let $\mathfrak{A}$ be a Boolean algebra and $(G, +)$ a commutative group. Then a function $\phi : \mathfrak{A} \to G$ is modular iff $a \mapsto \phi(a) - \phi(0)$ is additive. **P** Set $\nu a = \phi(a) - \phi(0)$. (α) If ϕ is modular and a, $b \in \mathfrak{A}$ are disjoint, then ν is modular (by (a)), so

$$\nu a + \nu b = \nu(a \cup b) + \nu(a \cap b) = \nu(a \cup b) + \nu 0 = \nu(a \cup b),$$

so ν is additive. (β) If ν is additive, and a, $b \in \mathfrak{A}$, then

$$\nu(a \cup b) + \nu(a \cap b) = \nu(a \setminus b) + \nu(a \cap b) + \nu(b \setminus a) + \nu(a \cap b) = \nu(a) + \nu(b),$$

so ν is modular; by (a) again, ϕ is modular. **Q**

3C Extension of modular functions: Proposition Let $\mathfrak{A}$ be a Boolean algebra, Q a sublattice of $\mathfrak{A}$, $(G, +)$ a commutative group and $\phi : Q \to G$ a modular function. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by Q. Then there is an extension of ϕ to a modular function from $\mathfrak{B}$ to G.

proof Let (Q^*, ϕ^*) be a maximal extension of (Q, ϕ) to a modular function from a sublattice of $\mathfrak{B}$ to G.

(a) $0 \in Q^*$. **P?** Otherwise, set $Q' = Q^* \cup \{0\}$ and extend ϕ^* to $\phi' : Q' \to V$ arbitrarily; then Q' is a sublattice of $\mathfrak{B}$, ϕ' is modular, and (Q', ϕ') is a proper extension of (Q^*, ϕ^*). **XQ** Similarly, $1 \in Q^*$.

(b) If $d \in Q^*$, then $1 \setminus d \in Q^*$. **P** Set

$$Q' = \{(a \setminus d) \cup (b \cap d) : a,\, b \in Q^*\},$$

and define $\phi' : Q' \to V$ by setting

$$\phi'(c) = \phi^*(c \cup d) + \phi^*(c \cap d) - \phi^*(d)$$

for $c \in Q'$; this is well-defined because if a, $b \in Q^*$ and $c = (a \setminus d) \cup (b \cap d)$, then $(c \setminus d) \cup d = a \cup d$ and $c \cap d = b \cap d$ belong to Q^*. If $a \in Q^*$, then

$$\phi'(a) = \phi^*(a \cup d) + \phi^*(a \cap d) - \phi^*(d) = \phi^*(a)$$

because ϕ^* is modular. So ϕ' extends ϕ^*. Next, it is easy to check that Q' is a sublattice of $\mathfrak{B}$, and also that the functions

$$a \mapsto \phi^*(a \cup d), \quad a \mapsto \phi^*(a \cap d) : Q^* \to G$$

are modular. Consequently

$$c \mapsto \phi^*(c \cup d), \quad c \mapsto \phi^*(c \cap d) : Q' \to V$$

are modular, so their sum is modular, and the translated version ϕ' is still modular.

By the maximality of (Q^*, ϕ^*), $Q^* = Q'$ contains $1 \setminus d$. **Q**

(c) It follows that Q^* is a subalgebra of $\mathfrak{A}$, and must be $\mathfrak{B}$.

3D Proposition Let P be a distributive lattice, V a linear space over a field F, Q a sublattice of P, and $\phi : Q \to V$ a modular function. Then there is an extension of ϕ to a modular function from P to V.

proof (a) By Theorem 1A, P is isomorphic to a sublattice of $\mathcal{P}Z$ for some Z, so we can suppose that P is actually equal to $\mathcal{P}Z$. Let (Q^*, ϕ^*) be a maximal extension of (Q, ϕ) to a modular function from a sublattice of $\mathcal{P}Z$ to V. By Proposition 3C, Q^* must actually be the subalgebra of $\mathcal{P}Z$ it generates. Define $\nu : Q^* \to V$ by setting $\nu K = \phi^*(K) - \phi^*(\emptyset)$ for $K \in Q^*$; then ν is additive, by 3Bb.

(b)(i) Let $S \subseteq F^Z$ be the set of functions $u : Z \to F$ such that $u[Z]$ is finite and $\{z : u(z) = \alpha\} \in Q^*$ for every $\alpha \in F$. Then S is a linear subspace of the linear space F^Z. For $u \in S$, set

$$f(u) = \sum_{\alpha \in u[Z]} \alpha \nu(u^{-1}[\{\alpha\}]);$$

then $f : S \to V$ is linear. **P** (i) If u, $v \in S$, set $K_{\alpha\beta} = u^{-1}[\{\alpha\}] \cap v^{-1}[\{\beta\}]$ for $\alpha \in u[Z]$, $\beta \in v[Z]$. For each $\alpha \in u[Z]$, $\langle K_{\alpha\beta} \rangle_{\beta \in v[Z]}$ is a finite partition of $u^{-1}[\{\alpha\}]$, so

$$f(u) = \sum_{\alpha \in u[Z]} \alpha \nu(u^{-1}[\{\alpha\}]) = \sum_{\alpha \in u[Z]} \alpha \sum_{\beta \in v[Z]} \nu K_{\alpha\beta}.$$

Similarly,

$$f(v) = \sum_{\alpha \in u[Z], \beta \in v[Z]} \beta \nu K_{\alpha\beta}.$$

On the other hand, $(u + v)[Z] \subseteq u[Z] + v[Z]$, and for $\gamma \in (u + v)[Z]$,

$$\langle K_{\alpha\beta} \rangle_{\alpha \in u[Z], \beta \in v[Z], \alpha + \beta = \gamma}$$

is a finite partition of $(u + v)^{-1}[\{\gamma\}]$, so

$$f(u + v) = \sum_{\gamma \in (u+v)[Z]} \nu((u + v)^{-1}[\{\gamma\}]) = \sum_{\gamma \in (u+v)[Z]} \gamma \sum_{\substack{\alpha \in u[Z] \\ \beta \in v[Z] \\ \alpha + \beta = \gamma}} \nu K_{\alpha\beta}$$

$$= \sum_{\gamma \in (u+v)[Z]} \sum_{\substack{\alpha \in u[Z] \\ \beta \in v[Z] \\ \alpha + \beta = \gamma}} (\alpha + \beta) \nu K_{\alpha\beta} = \sum_{\substack{\alpha \in u[Z] \\ \beta \in v[Z]}} (\alpha + \beta) \nu K_{\alpha\beta}$$

(because $K_{\alpha\beta} = \emptyset$ if $\alpha + \beta \notin (u + v)[Z]$)

$$= f(u) + f(v).$$

(ii) If $u \in S$ and $\gamma \in F$ is non-zero, then $(\gamma u)[Z] = \{\gamma\alpha : \alpha \in u[Z]\}$, so

$$f(\gamma u)=\sum_{\beta\in(\gamma u)[Z]}\beta\nu((\gamma u)^{-1}[\{\beta\}]=\sum_{\alpha\in u[Z]}\gamma\alpha\nu((\gamma u)^{-1}[\{\gamma\alpha\}]$$
$$=\sum_{\alpha\in u[Z]}\gamma\alpha\nu(u^{-1}[\{\alpha\}])=\gamma f(u).$$

So f is linear. **Q**

(ii) There is therefore a linear function $g:F^Z\to V$ extending f. Set

$$\phi'(A)=\phi^*(\emptyset)+g(\chi A)$$

for $A\subseteq Z$, where $\chi A:Z\to F$ is defined by setting $\chi A(z)=1_F$ for $z\in A$, 0_F for $z\in Z\setminus F$. If $K\in Q^*$, $\chi K\in S$ and

$$\phi'(K)=g(\chi K)+\phi^*(\emptyset)=f(\chi K)+\phi^*(\emptyset)=\nu K+\phi^*(\emptyset)=\phi^*(K).$$

It is also easy to check that, for A, $B\subseteq Z$,

$$\chi A+\chi B=\chi(A\cap B)+\chi(A\cup B),$$

so ϕ' is modular. So in fact $\phi'=\phi^*$ and ϕ^* is an extension of ϕ to the whole of $\mathcal{P}Z$.

3E Proposition Let P be a distributive lattice of cardinal at most ω_1, $(G,+)$ an abelian group, Q a countable sublattice of P, and $\phi:Q\to G$ a modular function. Then there is an extension of ϕ to a modular function from P to G.

proof By Theorem 1A again, P is isomorphic to a sublattice of a Boolean algebra; the subalgebra it generates is again of cardinal at most ω_1, so it is enough to consider the case in which P is itself a sublattice of a Boolean algebra $\mathfrak{A}$ of cardinal at most ω_1.

Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by Q. Then Proposition 3C tells us that there is an extension of ϕ to a modular function $\phi_1:\mathfrak{B}\to G$. Set $\lambda b=\phi_1 b-\phi_1 0$ for $b\in\mathfrak{B}$; then λ is additive. By Theorem 2E, λ has an extension to an additive function $\lambda':\mathfrak{A}\to G$. Setting $\psi a=\lambda' a+\phi_1 0$ for $a\in\mathfrak{A}$, $\psi\restriction P:P\to G$ is a modular function extending ϕ.

3F Problem Set $\phi(K)=\#(K)$ for $K\in[\omega_1]^{<\omega}$. Is there an extension of ϕ to a finitely additive functional from $\mathcal{P}\omega_1$ to $\mathbb{Z}$?

4 Submodular and supermodular functions

4A Definitions (a) Let P be a lattice, $(G,+)$ a commutative semigroup with a partial ordering $\le$ and $\phi:P\to G$ a function.

(i) ϕ is **supermodular** if $\phi(p)+\phi(q)\le\phi(p\vee q)+\phi(p\wedge q)$ for all p, $q\in P$.

(ii) ϕ is **submodular** if $\phi(p)+\phi(q)\ge\phi(p\vee q)+\phi(p\wedge q)$ for all p, $q\in P$.

(b) A **partially ordered semigroup** is a semigroup $(G,+)$ such that $x+y\le x'+y'$ whenever x, x', y, $y'\in G$, $x\le x'$ and $y\le y'$.

4B Lemma Let $\mathfrak{A}$ be a Boolean ring and $K\subseteq\mathfrak{A}$ a sublattice containing 0. Set

$$W=\{\textstyle\sum_{i=0}^n\chi a_i:a_i\in K\text{ for every }i\le n\}\subseteq S(\mathfrak{A}).$$

(a) If $u\in W$ then $[\![u>t]\!]=[\![u>\lfloor t\rfloor]\!]\in K$ for every $t\ge 0$.
(b) If $u\in W$ then u is expressible as $\sum_{i=0}^n\chi a_i$ where $a_i\in K$ for each i, and moreover $a_0\supseteq a_1\supseteq\ldots\supseteq a_n$.

proof (For basic facts about the Riesz space $S(\mathfrak{A})$, see Fremlin 02, §361.)

(a) If $u=\sum_{i=0}^n\chi a_i$ where $a_i\in K$ for each i, then

$$[\![u>t]\!]=\sup\{\inf_{i\in J}a_i:J\subseteq\{0,\ldots,n\},\ \#(J)>t\}$$
$$=\sup\{\inf_{i\in J}a_i:J\subseteq\{0,\ldots,n\},\ \#(J)>\lfloor t\rfloor\}\in K.$$

(b) If $u=0$, this is trivial. Otherwise, express u as $\sum_{i=0}^{n}\alpha_i\chi b_i$ where $\alpha_i>0$ and $b_i\in\mathfrak{A}$ for each i and $b_0\supset\ldots\supset b_n\neq 0$ (Fremlin 02, 361Ec). Then

$$b_i=[\![u>\sum_{j=0}^{i-1}\alpha_j]\!]\in K$$

for each i, while $[\![u>\sum_{j=0}^{n}\alpha_j]\!]=0$. We know also that the function $t\mapsto[\![u>t]\!]$ is constant on each interval $[k,k+1[$. So all the sums $\sum_{j=0}^{i}\alpha_j$ must be integers, and every α_j is an integer. Accordingly we can get the desired form by taking $a_k=b_i$ if $\sum_{j=0}^{i-1}\alpha_j\le k<\sum_{j=0}^{n}\alpha_j$.

4C Proposition (König 97) Let $\mathfrak{A}$ be a Boolean ring, K a sublattice of $\mathfrak{A}$ containing 0, $(G,+,\le)$ a partially ordered commutative semigroup with identity 0_G, and $\phi:K\to U$ a functional such that $\phi 0=0_G$. Let $W\subseteq S(\mathfrak{A})$ be the sub-semigroup generated by $\{\chi a: a\in K\}$, as in 4B, and define $⨍ d\phi:W\to G$ by setting

$$⨍ u\,d\phi=\sum_{n=0}^{\infty}\phi[\![u>n]\!]$$

for $u\in W$.

(a) If $u=\sum_{i=0}^{n}\chi a_i$ where $a_i\in K$ for every i and $a_0\supseteq a_1\supseteq\ldots\supseteq a_n$, then $⨍ u\,d\phi=\sum_{i=0}^{n}\alpha_i\phi a_i$.

(b) $⨍ nu\,d\phi=n⨍ u\,d\phi$ for every $u\in W$ and $n\in\mathbb{N}$.

(c) If $\phi(a)\le\phi(b)$ whenever $a\subseteq b$, then $⨍ u\,d\phi\le⨍ v\,d\phi$ whenever $u\le v$.

(d) If ϕ is supermodular, then $⨍ u+v\,d\phi\ge⨍ u\,d\phi+⨍ v\,d\phi$ for all u, $v\in G$.

(e) If ϕ is submodular, then $⨍ u+v\,d\phi\le⨍ u\,d\phi+⨍ v\,d\phi$ for all u, $v\in G$.

proof (a) I ought to explain what I mean by $\int_0^\infty\phi[\![u>t]\!]dt$. We can take this together with the calculation in hand. As in the proof of 4Bb, $[\![u>t]\!]=a_i$ for $\sum_{j=0}^{i-1}\alpha_j\le t<\sum_{j=0}^{i}\alpha_j$, while $[\![u>t]\!]=0$ for $t\ge\sum_{j=0}^{n}\alpha_i$. Thus we are integrating a U-valued function which is constant on each of finitely many half-open intervals and zero above a certain level; of course the integral is the sum of the (scalar) products of the lengths of the intervals with the values on those intervals. The reason for expressing it in this form is to make (c) transparent and (d)-(e) natural. There is a check to be performed. If, as is natural, we define the integral in terms of intervals on which our function takes different values, then the expression $u=\sum_{i=0}^{n}\alpha_i\chi a_i$ may subdivide some of these intervals if the a_i take repeated values; we have to observe that we can collapse these together without changing the sum $\sum_{i=0}^{n}\alpha_i\phi(a_i)$.

(b) Immediate from (a) and 4Bb.

(c) $[\![u>t]\!]\subseteq[\![v>t]\!]$ for every t, so $\phi[\![u>n]\!]\le\phi[\![v>n]\!]$ for every n.

(d)(i) If $u=\chi a+v$ where $a\in K$, $v\in G$ and $[\![v>0]\!]\subseteq a$, then

$$⨍ u+v\,d\phi=\phi a+⨍ v\,d\phi=⨍ u\,d\phi+⨍ v\,d\phi$$

by (a).

(ii) For the moment, suppose that $\mathfrak{A}$ is finite. If u, $v\in W$, then $⨍ u+v\,d\phi\ge⨍ u\,d\phi+⨍ v\,d\phi$. **P** Induce on $m=\int u\,d\bar\mu+\int v\,d\bar\mu$ where $\bar\mu$ is the measure on $\mathfrak{A}$ such that $\bar\mu a=1$ for every atom a. (See Fremlin 02, §365 for the elementary theory of such integrals.) We start with the trivial case $m=0$ in which $u=v=0$. For the inductive step to $m>0$, set $a=[\![u>0]\!]$ and $b=[\![v>0]\!]$. Then $u'=u-\chi a$ and $v'=v-\chi b$ belong to W (use 4Bb once more). Now

$$⨍ u+v\,d\phi=⨍ \chi a+\chi b+u'+v'd\phi=⨍ \chi(a\cup b)+\chi(a\cap b)+u'+v'd\phi$$
$$=\phi(a\cup b)+⨍ \chi(a\cap b)+u'+v'd\phi$$

(by (i))

$$\ge\phi(a\cup b)+⨍ \chi(a\cap b)d\phi+⨍ u'+v'd\phi$$

(by the inductive hypothesis, because $\int\chi(a\cap b)d\bar\mu+\int u'+v'd\bar\mu=m-\bar\mu(a\cup b)<m$)

$$\geq \phi(a \cup b) + \phi(a \cap b) + \int u' d\phi + \int v' d\phi$$

(by the inductive hypothesis again)

$$\geq \phi a + \phi b + \int u' d\phi + \int v' d\phi$$

(because ϕ is supermodular)

$$= \int u\, d\phi + \int v\, d\phi$$

by (i) again, and the induction proceeds. **Q**

(iii) For the general case, given u, $v \in W$, express them as $\sum_{i=0}^{m} \chi a_i$ and $\sum_{j=0}^{m} \chi b_j$ where a_i, $b_j \in K$ for all i and j; then there is a finite subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ containing all the a_i and b_j. Applying (ii) to $\phi{\restriction}K\cap\mathfrak{B}$, we get $\int u + v\, d\phi \geq \int u\, d\phi + \int v\, d\phi$, as required.

(e) We can copy the proof of (d), with each $\geq$ replaced by $\leq$.

4D Theorem (König 00) Let P be a distributive lattice, U a Dedekind complete Riesz space, $\phi : P \to U$ a supermodular function and $\psi : P \to U$ a submodular function. Suppose that $\phi \leq \psi$.

(a) There is a modular function $\theta : P \to U$ such that $\phi \leq \theta \leq \psi$.

(b) If either ϕ or ψ is non-decreasing, we can take θ to be non-decreasing.

proof (a)(i) Suppose to begin with that P is a sublattice of a Boolean algebra $\mathfrak{A}$ containing 0, and that $\phi(0) = \psi(0) = 0$. Let $W \subseteq S(\mathfrak{A})^+$ be the semigroup generated by $\{\chi a : a \in P\}$ as in 4B, and consider the functionals $\int d\phi$, $\int d\psi : W \to U$ as described in 4C. By 4C(d-e) these are respectively superadditive and subadditive, and

$$\int u\, d\phi = \sum_{n=0}^{\infty} \phi[\![u > n]\!] \leq \sum_{n=0}^{\infty} \psi[\![u > n]\!] = \int u\, d\psi$$

for every $u \in W$. By 1C, there is an additive $h : W \to U$ such that $\int u\, d\phi \leq h(u) \leq \int u\, d\psi$ for every $u \in W$; setting $\theta a = h(\chi a)$ for $a \in P$, θ is modular (see 3Bb) and

$$\phi a = \int \chi a\, d\phi \leq \theta a \leq \int \chi a\, d\psi = \psi a$$

for every $a \in P$.

(ii) Now suppose that P is any distributive lattice with least element 0, and that $\phi(0) = \psi(0) = 0$. Taking Z to be the set of surjective lattive homomorphisms from P to $\{0, 1\}$, $p \mapsto \pi(p) = \{z : z(p) = 1\}$ is an injective lattice homomorphism from P to $\mathcal{P}(Z)$, matching 0 with $\emptyset$. So we can apply (a) to the sublattice $\pi[P]$ of the Boolean algebra $\mathcal{P}Z$ and the functionals $\phi\pi^{-1}$, $\psi\pi^{-1} : \pi[P] \to U$ to find a modular function $\theta_0 : \pi[P] \to U$ bracketed between $\phi\pi^{-1}$ and $\psi\pi^{-1}$, and now $\theta\pi : P \to U$ will be a modular function between ϕ and ψ.

(iii) For the general case, we can adjoin a least element to P by taking any $0^* \notin P$ and setting $P^* = P \cup \{0^*\}$, $0^* < p$ for every $p \in P$; P^* is again a distributive lattice, and if we extend ϕ and ψ by setting $\phi(0^*) = \psi(0^*) = 0$, these are still respectively supermodular and submodular. So we can interpolate them with a modular function on P^* whose restriction to P is a modular function on P, as required.

(b)(i) Suppose that ψ is non-decreasing. Then there is a maximal modular function θ such that $\phi \leq \theta \leq \psi$. (As in part (b) of the proof of 1C, if R is a non-empty upwards-directed family of modular functions dominated by ψ, the supremum of R in U^P is a modular function.) Now set

$$\theta^+(p) = \sup\{\theta q : q \leq p\}$$

for $p \in P$. Because ψ is non-decreasing, $\theta^+ \leq \psi$, and of course $\theta \leq \theta^+$. If p, $q \in P$, then

$$\theta^+(p) + \theta^+(q) = \sup_{p' \leq p} \theta(p') + \sup_{q' \leq q} \theta(q') = \sup_{p' \leq p, q' \leq q} \theta(p') + \theta(q')$$

(Fremlin 02, 351Db again)

$$= \sup_{p' \le p, q' \le q} \theta(p' \vee q') + \theta(p' \wedge q') \le \theta^+(p \vee q) + \theta^+(p \wedge q),$$

because if $p' \le p$ and $q' \le q$ then $p' \vee q' \le p \vee q$ and $p' \wedge q' \le p \wedge q$, so $\theta(p' \vee q') \le \theta^+(p \vee q)$ and $\theta(p' \wedge q') \le \theta^+(p \wedge q)$. As p, q and ϵ are arbitrary, θ^+ is supermodular. But now (a) tells us that there is a modular $\theta_1 : P \to \mathbb{R}$ such that $\theta^+ \le \theta_1 \le \psi$, in which case $\theta \le \theta_1 \le \psi$ and $\theta_1 = \theta$. Accordingly $\theta^+ = \theta$; but this means that θ is non-decreasing.

(ii) If ϕ is non-decreasing, argue similarly but with a minimal modular function θ dominating ϕ, and $\theta_-(p) = \inf\{\theta(q) : p \le q\}$. Or apply (i) to the reversed lattice $(P, \ge)$ and the functions $-\psi$, $-\phi$.

4E Corollary (König 00) Let P be a distributive lattice, $\phi : P \to \,]-\infty, \infty]$ a supermodular function and $\psi : P \to \,]-\infty, \infty]$ a non-decreasing submodular function. If $\phi \le \psi$, there is a non-decreasing modular function $\theta : P \to \,]-\infty, \infty]$ such that $\phi \le \theta \le \psi$.

proof Set $Q = \{p : p \in P, \psi(p) < \infty\}$. Then Q is a sublattice of P, and $p \in Q$ whenever $p \in P$ and $p \le q \in Q$. Applying 4D to $\phi \restriction Q$ and $\psi \restriction Q$, we get a non-decreasing modular $\theta_0 : Q \to \mathbb{R}$ between ϕ and ψ. Extending θ_0 to $\theta : P \to \,]-\infty, \infty]$ by setting $\theta(p) = \infty$ for $p \in P \setminus Q$, we find that we have a suitable interpolation between ϕ and ψ.

4F Corollary Let P be a distributive lattice, Q a sublattice of P, and $\theta_0 : Q \to [0, \infty]$ a non-decreasing modular function. Then there is a non-decreasing modular function $\theta : P \to [0, \infty]$ extending θ_0.

proof (a) For $p \in P$ set $\phi(p) = \sup\{\theta_0(q) : q \in Q,\ q \le p\}$, $\psi(p) = \inf\{\theta_0(q) : p \le q \in Q\}$, counting $\inf \emptyset$ as ∞ and $\sup \emptyset$ as 0. Then ϕ and ψ are both non-decreasing functions from P to $[0, \infty]$. Because θ_0 is non-decreasing, $\phi(q) = \theta_0(q) = \psi(q)$ for every $q \in Q$ and $\phi(p) \le \psi(p)$ for every $p \in P$.

(b)(i) ϕ is supermodular. **P?** Otherwise, there are p, $p' \in P$ such that $\phi(p \vee p') + \phi(p \wedge p') < \phi(p) + \phi(p')$. In this case, $\phi(p)$ and $\phi(p')$ must both be non-zero, so there are q, $q' \in Q$ such that $q \le p$, $q' \le p'$ and

$$\theta(q) + \theta(q') > \phi(p \vee p') + \phi(p \wedge p') \ge \theta(q \vee q') + \theta(q \wedge q'). \quad \mathbf{XQ}$$

(ii) ψ is submodular. **P?** Otherwise, there are p, $p' \in P$ such that $\psi(p \vee p') + \psi(p \wedge p') > \phi(p) + \phi(p')$. In this case, $\phi(p)$ and $\phi(p')$ must both be finite, so there are q, $q' \in Q$ such that $q \ge p$, $q' \ge p'$ and

$$\theta(q) + \theta(q') < \phi(p \vee p') + \phi(p \wedge p') \le \theta(q \vee q') + \theta(q \wedge q'). \quad \mathbf{XQ}$$

(c) By 4E, there is a non-decreasing modular function $\theta : P \to \,]-\infty, \infty]$ such that $\phi \le \theta \le \psi$; as ϕ is non-negative, so is θ; as ϕ and ψ both extend θ_0, so does θ.

Acknowledgments Correspondence with H.König and J.Bagaria.

References

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.
Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.
König H. [97] *Measure and Integration.* Springer, 1997.
König H. [00] 'Upper envelopes of inner premeasures', Ann. Inst. Fourier 50 (2000) 401-422 (`http://www.numdam.org/i`