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Vector-valued Saks-Henstock indefinite integrals

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1 Basic definitions and results

1A The context (a) Let X be a set. A **tagged partition** on X will be a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$; in this case, $H_{\boldsymbol{t}}$ will be $\bigcup\{C : (x,C) \in \boldsymbol{t}\}$. A **gauge** on X is a subset δ of $X \times \mathcal{P}X$. For a gauge δ on X, a tagged partition $\boldsymbol{t}$ on X is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. If $\mathcal{R} \subseteq \mathcal{P}X$, and $\boldsymbol{t}$ is a tagged partition on X, $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus H_{\boldsymbol{t}} \in \mathcal{R}$. A **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x,C),\ (x',C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$.

(b)(i) A family Δ of gauges on a set X is **full** if whenever $\langle\delta_x\rangle_{x\in X}$ is a family in Δ, then

$$\{(x,A) : x \in X,\ (x,A) \in \delta_x\}$$

belongs to Δ. Δ is **countably full** if this is true whenever $\{\delta_x : x \in X\}$ is countable.

(ii) If X is a topological space, a **neighbourhood gauge** on X is a gauge of the form $\{(x,C) : x \in X,\ C \subseteq G_x\}$ where $\langle G_x\rangle_{x\in X}$ is a family of open subsets of X such that $x \in G_x$ for every $x \in X$.

For any topological space, the family of all neighbourhood gauges on X is full.

(c) A quadruple $(X, T, \Delta, \mathfrak{R})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set.

(ii) Δ is a non-empty downwards-directed family of gauges on X.

(iii)(α) $\mathfrak{R}$ is a non-empty downwards-directed collection of families of subsets of X, all containing $\emptyset$;

(β) for every $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint.

(iv) $\mathcal{C}$ is a family of subsets of X such that whenever C, $C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$.

(v) Whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is finite and $\mathcal{R} \in \mathfrak{R}$, there is a finite set $\mathcal{C}_1 \subseteq \mathcal{C}$, including $\mathcal{C}_0$, such that $X \setminus \bigcup\mathcal{C}_1 \in \mathcal{R}$.

(vi) $T \subseteq [X \times \mathcal{C}]^{<\omega}$ is a non-empty straightforward set of tagged partitions on X.

(vii) Whenever $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $H_{\boldsymbol{t}} \subseteq C$ and $C \setminus H_{\boldsymbol{t}} \in \mathcal{R}$.

(d) Given a set X, a non-empty set T of tagged-partitions on X, a non-empty family Δ of gauges on X, and a non-empty collection $\mathfrak{R}$ of families of subsets of X, consider sets of the form

$$T_{\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine and } \mathcal{R}\text{-filling}\}$$

for $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. If the collection of these sets has the finite intersection property, say that T is **compatible** with Δ and $\mathfrak{R}$, and write $\mathcal{F}(T, \Delta, \mathfrak{R})$ for the filter on T generated by the collection.

(e) For the basic theory of these structures, see Fremlin 03, §§481-482. In particular, we shall need the following facts. Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions.

(i) If $\mathcal{R} \in \mathfrak{R}$, there is a non-increasing sequence $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\le n} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint (Fremlin 03, 481He).

(ii) Let $\mathcal{E}_0$ be the subring of $\mathcal{P}X$ generated by $\mathcal{C}$. Then every member of $\mathcal{E}_0$ is expressible as a disjoint union of members of $\mathcal{C}$ (use (c-iv)).

(iii) Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. If $E \in \mathcal{E}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$, there is a δ-fine $\boldsymbol{t} \in T$ such that $H_{\boldsymbol{t}} \subseteq E$ and $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}$. **P** Let $\langle\mathcal{R}_i\rangle_{i\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i\le n} A_i \in \mathcal{R}$

whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i\rangle_{i\le n}$ is disjoint. (α) If $E = \emptyset$ we can take $\boldsymbol{t} = \emptyset$. ($\beta$) If $E \in \mathcal{E}_0 \setminus \{\emptyset\}$, express E as $\bigcup_{i\le n} C_i$ where $\langle C_i\rangle_{i\le n}$ is a disjoint family in $\mathcal{C}$. For each i, there is a δ-fine $\boldsymbol{t}_i \in T$ such that $H_{\boldsymbol{t}_i} \subseteq C_i$ and $C_i \setminus H_{\boldsymbol{t}_i} \in \mathcal{R}_{i+1}$, by (c-vii). Set $\boldsymbol{t} = \bigcup_{i\le n} \boldsymbol{t}_i$; then

$$E \setminus H_{\boldsymbol{t}} = \bigcup_{i\le n} C_i \setminus H_{\boldsymbol{t}_i} \in \mathcal{R}_1 \subseteq \mathcal{R}.$$

(γ) Otherwise, $X \setminus E \in \mathcal{E}_0$. By (c-v), there is an $F \in \mathcal{E}_0$, including $X \setminus E$, such that $X \setminus F \in \mathcal{R}_1$. By ($\alpha$)-($\beta$), there is a δ-fine $\boldsymbol{t} \in T$ such that $H_{\boldsymbol{t}} \subseteq E \cap F$ and $E \cap F \setminus H_{\boldsymbol{t}} \in \mathcal{R}_1$; in which case $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}$. **Q**

(iv) T is compatible with Δ and $\mathfrak{R}$. **P** Apply (iii) with $E = X$. **Q**

(f) Leading examples include the following.

(i) $X = [a, b] \subseteq \mathbb{R}$, $\mathcal{C}$ the family of subintervals of $[a, b]$ (open, closed, or half-open), T the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, Δ the set of gauges of the form $\{(x, C) : x \in X,\ C \subseteq X,\ \operatorname{diam} C \le \eta\}$ where $\eta > 0$, $\mathfrak{R} = \{\{\emptyset\}\}$. (This corresponds to the Riemann integral.)

(ii) $X = \mathbb{R}$, $\mathcal{C}$ the family of bounded subintervals of $\mathbb{R}$ (open, closed, or half-open), T the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$, Δ the set of neighbourhood gauges on X,

$$\mathfrak{R} = \{\{\mathbb{R} \setminus [a, b] : a \le a_0,\ b_0 \le b\} : a_0,\ b_0 \in \mathbb{R}\}.$$

(This corresponds to the Henstock integral.)

(iii) $X = \mathbb{R}$, $\mathcal{C}$ the family of bounded subintervals of $\mathbb{R}$ (open, closed, or half-open), T the straightforward set of tagged partitions generated by $\mathbb{R} \times \mathcal{C}$, Δ the set of neighbourhood gauges on X,

$$\mathfrak{R} = \{\{A : A \subseteq \mathbb{R},\ \mu^*(A \cap [a, b]) \le \eta\} : a \le b,\ \eta > 0\},$$

where μ is Lebesgue measure on $\mathbb{R}$. (This corresponds to McShane's description of the Lebesgue integral.)

1B Vector-valued gauge integrals Suppose that we are given a set X, a family Δ of gauges on X, a collection $\mathfrak{R}$ of families of subsets of X, a collection $\mathcal{C}$ of subsets of X, a family $T \subseteq X \times \mathcal{C}$ of tagged partitions on X which is compatible with Δ and $\mathfrak{R}$, Banach spaces U, V and W and a continuous bilinear operator $(u, v) \mapsto \langle u|v\rangle : U \times V \to W$. Let $f : X \to U$ and $\nu : \operatorname{dom}\nu \to V$ be functions, where $\mathcal{C} \subseteq \operatorname{dom}\nu$. For $\boldsymbol{t} \in T$, set

$$S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} \langle f(x)|\nu C\rangle \in W.$$

If $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined in W, call it $I_\nu(f)$, the **gauge integral** of f with respect to ν.

Evidently $\{(\nu, f) : I_\nu(f)$ is defined$\}$ is a linear subspace of $V^{\mathcal{C}} \times U^X$, and I_ν is a linear operator, just because every $S_{\boldsymbol{t}}$ is a linear operator on $V^{\mathcal{C}} \times U^X$.

In this context, if U or V is equal to $\mathbb{R}$, I will take it for granted that $\langle\,|\,\rangle$ is just scalar multiplication.

1C Lemma Suppose that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C} \subseteq \mathcal{P}X$, U, V and W are Banach spaces, and $\langle\,|\,\rangle : U \times V \to W$ is a continuous bilinear operator. Suppose that $f : X \to U$, $\nu : \mathcal{C} \to V$, $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\epsilon \ge 0$ are such that $\|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)\| \le \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $\mathcal{R}$-filling. Then

(a) $\|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)\| \le \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $H_{\boldsymbol{t}} = H_{\boldsymbol{t}'}$;

(b) whenever $\boldsymbol{t} \in T$ is δ-fine, $\delta' \in \Delta$ and $\mathcal{R}' \in \mathcal{R}$, there is a δ'-fine $\boldsymbol{s} \in T$ such that $H_{\boldsymbol{s}} \subseteq H_{\boldsymbol{t}}$, $H_{\boldsymbol{t}} \setminus H_{\boldsymbol{s}} \in \mathcal{R}'$ and $\|S_{\boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}}(f, \nu)\| \le \epsilon$.

proof (a) By Fremlin 03, 4A2Ab, there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \cap W_{\boldsymbol{t}} = \emptyset$ and $\boldsymbol{t} \cup \boldsymbol{s}$ is $\mathcal{R}$-filling. Now $H_{\boldsymbol{t}\cup\boldsymbol{s}} = H_{\boldsymbol{t}'\cup\boldsymbol{s}}$, so $\boldsymbol{t}' \cup \boldsymbol{s}$ also is $\mathcal{R}$-filling, and

$$\|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)\| = \|S_{\boldsymbol{t}\cup\boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}'\cup\boldsymbol{s}}(f, \nu)\| \le \epsilon.$$

(b) Replacing δ' by a lower bound of $\{\delta, \delta'\}$ in Δ and $\mathcal{R}'$ by a lower bound of $\{\mathcal{R}, \mathcal{R}'\}$ if necessary, we may suppose that $\delta' \subseteq \delta$ and $\mathcal{R}' \subseteq \mathcal{R}$. Enumerate $\boldsymbol{t}$ as $\langle(x_i, C_i)\rangle_{i<n}$. Let $\langle\mathcal{R}_k\rangle_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i\le k} A_i \in \mathcal{R}'$ whenever $\langle A_i\rangle_{i\le k}$ is disjoint and $A_i \in \mathcal{R}_i$ for every $i \le k$ (1A(e-i)). For each $i < n$, let $\boldsymbol{s}_i$ be a δ'-fine member of T such that $H_{\boldsymbol{s}_i} \subseteq C_i$ and $C_i \setminus H_{\boldsymbol{s}_i} \in \mathcal{R}_{i+1}$, and set $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$, so that

$\boldsymbol{s} \in T$ is δ'-fine and $H_{\boldsymbol{s}} \subseteq H_{\boldsymbol{t}}$. By Fremlin 03, 482Aa, there is a δ-fine $\boldsymbol{u} \in T$ such that $H_{\boldsymbol{u}} \cap H_{\boldsymbol{t}} = \emptyset$ and $X \setminus (H_{\boldsymbol{t}} \cup H_{\boldsymbol{u}}) \in \mathcal{R}_0$. Set $\boldsymbol{t'} = \boldsymbol{t} \cup \boldsymbol{u}$, $\boldsymbol{s'} = \boldsymbol{s} \cup \boldsymbol{u}$; then $\boldsymbol{t'}$ and $\boldsymbol{s'}$ are δ-fine and $\mathcal{R}$-filling, because

$$X \setminus H_{\boldsymbol{s'}} = (X \setminus (H_{\boldsymbol{t}} \cup H_{\boldsymbol{u}})) \cup \bigcup_{i<n} (C_i \setminus H_{\boldsymbol{s}_i}) \in \mathcal{R}' \subseteq \mathcal{R},$$

by the choice of $\langle \mathcal{R}_k \rangle_{k \in \mathbb{N}}$. So

$$\|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{s}}(f,\nu)\| = \|S_{\boldsymbol{t'}}(f,\nu) - S_{\boldsymbol{s'}}(f,\nu)\| \le \epsilon,$$

as required. Also, of course,

$$H_{\boldsymbol{t}} \setminus H_{\boldsymbol{s}} = \bigcup_{i<n} C_i \setminus H_{\boldsymbol{s}_i} \in \mathcal{R}'.$$

1D Saks-Henstock Lemma Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U, V, W Banach spaces, $\langle\,|\,\rangle : U \times V \to W$ a continuous bilinear operator, and $f : X \to U$, $\nu : \mathcal{C} \to V$ functions such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined in W. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then there is a unique additive function $F : \mathcal{E} \to \mathbb{R}$ such that for every $\epsilon > 0$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

(α) $\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)\| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) $\|F(E)\| \le \epsilon$ whenever $E \in \mathcal{E} \cap \mathcal{R}$.

Moreover, $F(X) = I_\nu(f)$.

proof (a) For $E \in \mathcal{E}$, write T_E for the set of those $\boldsymbol{t} \in T$ such that, for every $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. For any $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and finite $\mathcal{D} \subseteq \mathcal{E}$ there is a δ-fine $\boldsymbol{t} \in \bigcap_{E \in \mathcal{D}} T_E$ such that $E \setminus H_{\boldsymbol{t}} \in \mathcal{R}$ for every $E \in \mathcal{D}$. **P** Let $\langle \mathcal{R}_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint then $\bigcup_{i \le n} A_i \in \mathcal{R}$. Let $\mathcal{E}_0$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{D}$, and enumerate the atoms of $\mathcal{E}_0$ as $\langle E_i \rangle_{i<n}$. By Fremlin 03, 482Aa, there is for each $i < n$ a δ-fine $\boldsymbol{s}_i \in T$ such that $H_{\boldsymbol{s}_i} \subseteq E_i$ and $E_i \setminus H_{\boldsymbol{s}_i} \in \mathcal{R}_i$. Set $\boldsymbol{t} = \bigcup_{i<n} \boldsymbol{s}_i$. If $E \in \mathcal{D}$ then $E = \bigcup_{i \in J} E_i$ for some $J \subseteq n$. For any $(x, C) \in \boldsymbol{t}$, there is some $i < n$ such that $C \subseteq E_i$, so that $C \subseteq E$ if $i \in J$, $C \cap E = \emptyset$ otherwise; thus $\boldsymbol{t} \in T_E$. Moreover, $E \setminus H_{\boldsymbol{t}} = \bigcup_{i \in J} (E_i \setminus H_{\boldsymbol{s}_i})$ belongs to $\mathcal{R}$. **Q**

(b) We therefore have a filter $\mathcal{F}^*$ on T generated by sets of the form

$$T_{E\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T_E \text{ is } \delta\text{-fine, } E \setminus H_{\boldsymbol{t}} \in \mathcal{R}\}$$

as δ runs over Δ, $\mathcal{R}$ runs over $\mathfrak{R}$ and E runs over $\mathcal{E}$. For $\boldsymbol{t} \in T$, $E \subseteq X$ set $\boldsymbol{t}_E = \{(x, C) : (x, C) \in \boldsymbol{t}, C \subseteq E\}$. Now $F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$ is defined for every $E \in \mathcal{E}$. **P** For any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that $\|I_\nu(f) - S_{\boldsymbol{t}}(f,\nu)\| \le \epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}'$. If $\boldsymbol{t}$, $\boldsymbol{t'}$ belong to $T_{E,\delta,\mathcal{R}'} = T_{X \setminus E,\delta,\mathcal{R}'}$, then set

$$\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t'}, C \subseteq E\} \cup \{(x, C) : (x, C) \in \boldsymbol{t}, C \cap E = \emptyset\}.$$

Then $\boldsymbol{s} \in T_E$ is δ-fine, and also $E \setminus H_{\boldsymbol{s}} = E \setminus H_{\boldsymbol{t'}}$, $(X \setminus E) \setminus H_{\boldsymbol{s}} = (X \setminus E) \setminus H_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$; so their union $X \setminus H_{\boldsymbol{s}}$ belongs to $\mathcal{R}$, and $\boldsymbol{s}$ is $\mathcal{R}$-filling. Accordingly

$$\begin{aligned}\|S_{\boldsymbol{t}_E}(f,\nu) - S_{\boldsymbol{t}'_E}(f,\nu)\| &= \|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{s}}(f,\nu)\| \\ &\le \|S_{\boldsymbol{t}}(f,\nu) - I_\nu(f)\| + \|I_\nu(f) - S_{\boldsymbol{s}}(f,\nu)\| \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary and W is complete, this is enough to show that $\lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$ is defined. **Q**

(c) If E, $E' \in \mathcal{E}$ are disjoint, then

$$S_{\boldsymbol{t}_{E \cup E'}}(f,\nu) = S_{\boldsymbol{t}_E}(f,\nu) + S_{\boldsymbol{t}_{E'}}(f,\nu)$$

for any $\boldsymbol{t} \in T_E \cap T_{E'}$; since both T_E and $T_{E'}$ belong to $\mathcal{F}^*$, $F(E \cup E') = F(E) + F(E')$. Thus F is additive.

(d) Now suppose that $\epsilon > 0$. Let $\delta \in \Delta$, $\mathcal{R}^* \in \mathfrak{R}$ be such that $\|I_\nu(f) - S_{\boldsymbol{t}}(f,\nu)\| \le \frac{1}{2}\epsilon$ for every δ-fine, $\mathcal{R}^*$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}^*$ for all disjoint A, $B \in \mathcal{R}$. If $\boldsymbol{t} \in T$ is δ-fine, then $\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)\| \le \epsilon$. **P** For any $\eta > 0$, there is a δ-fine $\boldsymbol{s} \in T$ such that

$\|I_\nu(f) - S_{\boldsymbol{s}}(f,\nu)\| \le \eta$,

for every $(x, C) \in \boldsymbol{s}$, either $C \subseteq H_{\boldsymbol{t}}$ or $C \cap H_{\boldsymbol{t}} = \emptyset$,

$(X \setminus H_{\boldsymbol{t}}) \setminus H_{\boldsymbol{s}} \in \mathcal{R}$, $H_{\boldsymbol{t}} \setminus H_{\boldsymbol{s}} \in \mathcal{R}$,

$\|F(H_{\boldsymbol{t}})-\sum_{(x,C)\in\boldsymbol{s},C\subseteq H_{\boldsymbol{t}}}\langle f(x)|\nu C\rangle\|\le\eta$

because the set of $\boldsymbol{s}$ with these properties belongs to $\mathcal{F}^*$. Now, setting $\boldsymbol{s}_1=\{(x,C):(x,C)\in\boldsymbol{s},\,C\subseteq H_{\boldsymbol{t}}\}$ and $\boldsymbol{t}'=\boldsymbol{t}\cup(\boldsymbol{s}\setminus\boldsymbol{s}_1)$, $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}^*$-filling, like $\boldsymbol{s}$, so

$$\begin{aligned}\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|&\le\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{s}_1}(f,\nu)\|+\|S_{\boldsymbol{s}_1}(f,\nu)-S_{\boldsymbol{t}}(f,\nu)\|\\&\le\eta+\|S_{\boldsymbol{s}}(f,\nu)-S_{\boldsymbol{t}'}(f,\nu)\|\\&\le\eta+\|S_{\boldsymbol{s}}(f,\nu)-I_\nu(f)\|+\|I_\nu(f)-S_{\boldsymbol{t}'}(f,\nu)\|\le\eta+\frac{1}{2}\epsilon.\end{aligned}$$

As η is arbitrary we have the result. **Q**

(ii) Now suppose that $E\in\mathcal{E}\cap\mathcal{R}$. Then $\|F(E)\|\le\epsilon$. **P** Let $\mathcal{R}'\in\mathfrak{R}$ be such that $A\cup B\in\mathcal{R}$ whenever A, $B\in\mathcal{R}'$ are disjoint. Let $\boldsymbol{t}$ be such that

$\boldsymbol{t}\in T_E$ is δ-fine,

$E\setminus H_{\boldsymbol{t}}$ and $(X\setminus E)\setminus H_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$,

$\|F(E)-S_{\boldsymbol{t}_E}(f,\nu)\|\le\frac{1}{2}\epsilon$;

once again, the set of candidates belongs to $\mathcal{F}^*$, so is not empty. Then $\boldsymbol{t}$ and $\boldsymbol{t}_{X\setminus E}$ are both $\mathcal{R}^*$-filling and δ-fine, so

$$\|F(E)\|\le\frac{1}{2}\epsilon+\|S_{\boldsymbol{t}_E}(f,\nu)\|=\frac{1}{2}\epsilon+\|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}_{X\setminus E}}(f,\nu)\|\le\epsilon.\ \mathbf{Q}$$

As ϵ is arbitrary, this shows that F has all the required properties.

(e) I have still to show that F is unique. Suppose that $F':\mathcal{E}\to\mathbb{R}$ is another function with the same properties, and take $E\in\mathcal{E}$ and $\epsilon>0$. Then there are δ, $\delta'\in\Delta$ and $\mathcal{R}$, $\mathcal{R}'\in\mathfrak{R}$ such that

$\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,

$\|F'(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le\epsilon$ for every δ'-fine $\boldsymbol{t}\in T$,

$\|F(R)\|\le\epsilon$ whenever $R\in\mathcal{E}\cap\mathcal{R}$,

$\|F'(R)\|\le\epsilon$ whenever $R\in\mathcal{E}\cap\mathcal{R}'$.

Now taking $\delta''\in\Delta$ such that $\delta''\subseteq\delta\cap\delta'$, and $\mathcal{R}''\in\mathfrak{R}$ such that $\mathcal{R}''\subseteq\mathcal{R}\cap\mathcal{R}'$, there is a δ''-fine $\boldsymbol{t}\in T$ such that $E'=H_{\boldsymbol{t}}$ is included in E and $E\setminus E'\in\mathcal{R}''$. In this case

$$\begin{aligned}\|F(E)-S_{\boldsymbol{t}}(f,\nu)\|&\le\|F(E)-F(E')\|+\|F(E')-S_{\boldsymbol{t}}(f,\nu)\|\\&=\|F(E\setminus E')\|+\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\end{aligned}$$

(because F is additive)

$$\le 2\epsilon$$

because $E\setminus E'\in\mathcal{R}''\subseteq\mathcal{R}$ and $\boldsymbol{t}$ is δ''-fine, therefore δ-fine. Similarly, $\|F'(E)-S_{\boldsymbol{t}}(f,\nu)\|\le 2\epsilon$ so $\|F'(E)-F(E)\|\le 4\epsilon$. As E and ϵ are arbitrary, $F=F'$.

(f) Finally, to calculate $F(X)$, take any $\epsilon>0$. Let $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ be such that $\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$ and $\|F(E)\|\le\epsilon$ whenever $E\in\mathcal{E}\cap\mathcal{R}$. Let $\boldsymbol{t}$ be any δ-fine $\mathcal{R}$-filling member of T such that $\|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)\|\le\epsilon$. Then, because F is additive,

$$\begin{aligned}\|F(X)-I_\nu(f)\|&\le\|F(X)-F(H_{\boldsymbol{t}})\|+\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|+\|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)\|\\&\le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, $F(X)=I_\nu(f)$.

1E Definition In the context of §1D, I will call the function F the **Saks-Henstock indefinite integral** of f with respect to ν; of course it depends on the whole structure $(X,T,\Delta,\mathfrak{R},\mathcal{C},U,V,W,\langle\,|\,\rangle,f,\nu)$, not just f and ν. You should *not* take it for granted that $F(E)=I_\nu(f\times\chi E)$, but see Proposition 2D below.

1F The Saks-Henstock lemma characterizes the gauge integral, as follows.

Theorem Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, U, V and W Banach spaces, $\langle\,|\,\rangle : U\times V\to W$ a continuous bilinear operator, and $\nu:\mathcal{C}\to V$, $f:X\to U$ two functions. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then the following are equiveridical:

(i) $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined in W;

(ii) there is an additive function $F:\mathcal{E}\to W$ such that

(α) for every $\epsilon>0$ there is a $\delta\in\Delta$ such that $\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le\epsilon$ for every δ-fine $\boldsymbol{t}\in T$,

(β) for every $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $\|F(E)\|\le\epsilon$ for every $E\in\mathcal{E}\cap\mathcal{R}$.

In this case, $F(X)=I_\nu(f)$.

proof (i)$\Rightarrow$(ii) is just the Saks-Henstock Lemma above; so let us assume (ii) and seek to prove (i). Given $\epsilon>0$, take $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that (α) and (β) are satisfied. Let $\boldsymbol{t}\in T$ be δ-fine and $\mathcal{R}$-filling. Then

$$\|F(X)-S_{\boldsymbol{t}}(f,\mu)\|\le\|F(X\setminus H_{\boldsymbol{t}})\|+\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le 2\epsilon.$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to $F(X)$.

2 Further properties

2A Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, U, V and W Banach spaces and $\langle\,|\,\rangle : U\times V\to W$ a continuous bilinear operator. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. For $\nu\in V^{\mathcal{C}}$ and $f\in U^X$, write $F_{f\nu}\in W^{\mathcal{E}}$ for the Saks-Henstock indefinite integral of f with respect to ν when this is defined. Then the operator $(\nu,f)\mapsto F_{f\nu}$ is bilinear.

proof Immediate from 1F.

2B Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. Suppose that U_0, V_0, W_0, U_1, V_1 and W_1 are Banach spaces, $\langle\,|\,\rangle_0 : U_0\times V_0\to W_0$, $\langle\,|\,\rangle_1 : U_1\times V_1\to W_1$ continuous bilinear operators, and $\pi:U_0\to U_1$, $\phi:V_0\to V_1$ and $\psi:W_0\to W_1$ continuous linear operators such that $\psi(\langle u|v\rangle_0)=\langle\pi(u)|\phi v\rangle_1$ for all $u\in U_0$ and $v\in V_0$. Let $f:X\to U_0$ and $\nu:\mathcal{C}\to V_0$ be such that $I_\nu(f)$ is defined and has Saks-Henstock indefinite integral F. Then $I_{\phi\nu}(\pi f)$ is defined and has Saks-Henstock indefinite integral ψF.

proof We just have to observe that $S_{\boldsymbol{t}}(\pi f,\phi\nu)=\psi(S_{\boldsymbol{t}}(f,\nu))$ for every $\boldsymbol{t}\in T$, and apply Theorem 1F.

2C Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, U, V and W Banach spaces and $\langle\,|\,\rangle : U\times V\to W$ a continuous bilinear operator. Suppose that Σ is a σ-algebra of subsets of X including $\mathcal{C}$, and $\nu:\Sigma\to V$ a vector measure; let $\mu:\Sigma\to[0,\infty]$ be the total variation of ν, and $f:X\to U$ a function which is Bochner integrable with respect to μ. Suppose further that

(i) X has a topology $\mathfrak{T}$ such that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets;

(ii) Δ contains every neighbourhood gauge on X;

(iii) whenever $E\in\Sigma$, $\mu E<\infty$ and $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $\mu^*(A\cap E)\le\epsilon$ for every $A\in\mathcal{R}$.

Then $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined.

proof Let $\gamma\ge 0$ be such that $\|\langle u|v\rangle\|\le\gamma\|u\|\|v\|$ for all $u\in U$ and $v\in V$.

(a) Consider first the case in which f is of the form $u\otimes\chi E$ where $E\in\Sigma$, $\mu E<\infty$ and $u\in U$, where $(u\otimes\chi E)(x)=\chi E(x)\cdot u$ for every $x\in X$. Then $I_\nu(f)=\langle u|\nu E\rangle$. **P** Let $\epsilon>0$. Let $G\supseteq E$ be an open set and $F\subseteq E$ a closed set such that $\mu(G\setminus F)\le\epsilon$, and $\mathcal{R}$ a member of $\mathfrak{R}$ such that $\mu^*(A\cap E)\le\epsilon$ for every $A\in\mathcal{R}$. Let $\delta\in\Delta$ be the neighbourhood gauge

$$\{(x,A):x\in E,\ A\subseteq G\}\cup\{(x,A):x\in X\setminus E,\ A\subseteq X\setminus F\}.$$

If $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}$-filling, then $S_{\boldsymbol{t}}(f)=\langle u|\nu H_{\boldsymbol{t}\restriction E}\rangle$, where $H_{\boldsymbol{t}\restriction E}=\{(x,C):(x,C)\in\boldsymbol{t},\ x\in E\}$. Now we know that $\mu(E\setminus H_{\boldsymbol{t}})\le\epsilon$, while $H_{\boldsymbol{t}\restriction E}\subseteq G$ and $H_{\boldsymbol{t}\restriction X\setminus E}$ does not meet F; so that $F\cap H_{\boldsymbol{t}}\subseteq H_{\boldsymbol{t}\restriction E}$, and

$$\mu(E\triangle H_{\boldsymbol{t}\restriction E})\le\mu(G\setminus F)+\mu(E\setminus H_{\boldsymbol{t}})\le 2\epsilon.$$

But this means that

$$\begin{aligned}\|S_{\boldsymbol{t}}(f) - \langle u|\nu E\rangle\| &= \|\langle u|\nu H_{\boldsymbol{t}\restriction E} - \nu E\rangle\| \le \gamma\|u\|\|\nu H_{\boldsymbol{t}\restriction E} - \nu E\| \\ &\le \gamma\|u\|(\|\nu(H_{\boldsymbol{t}\restriction E} \setminus E)\| + \|\nu(E \setminus H_{\boldsymbol{t}\restriction E})\|) \\ &\le \gamma\|u\|(\mu(H_{\boldsymbol{t}\restriction E} \setminus E) + \mu(E \setminus H_{\boldsymbol{t}\restriction E})) = \gamma\|u\|\mu(H_{\boldsymbol{t}\restriction E}\triangle E) \le 2\gamma\|u\|\epsilon.\end{aligned}$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to $\langle u|\nu E\rangle$. **Q**

(b) Consequently $I_\nu(f)$ is defined whenever $f : X \to U$ is a 'simple' function in the sense that it is expressible as $\sum_{i=0}^n u_i \otimes \chi E_i$ where each E_i has finite measure.

(c) Now suppose that $f : X \to U$ is any function. Then

$$\limsup_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \|S_{\boldsymbol{t}}(f)\| \le \gamma\overline{\int}\|f\|d\mu.$$

P If $\gamma = 0$, $S_{\boldsymbol{t}}(f,\nu) = 0$ for every $\boldsymbol{t}$ and we can stop. If $\gamma > 0$ and $\overline{\int}\|f\|d\mu = \infty$, the result is trivial. So suppose that $\gamma > 0$ and $\overline{\int}\|f\|d\mu$ is finite. Let $\hat\mu$ be the completion of μ and $\hat\Sigma$ its domain. Note that $\hat\mu$ is still inner regular with respect to the closed sets and outer regular with respect to the open sets. Let $g : X \to \mathbb{R}$ be a $\hat\Sigma$-measurable function such that $g(x) \ge \|f(x)\|$ for every x and $\int g\,d\mu = \overline{\int}\|f\|d\mu$.

Let $\epsilon > 0$. For $m \in \mathbb{Z}$, set $E_m = \{x : x \in X,\ (1+\epsilon)^m \le g(x) < (1+\epsilon)^{m+1}\}$. Then $E_m \in \hat\Sigma$ and $\hat\mu E_m < \infty$, so there is a measurable open set $G_m \supseteq E_m$ such that $(1+\epsilon)^{m+1}\mu(G_m \setminus E_m) \le 2^{-|m|}\epsilon$.

Define $\langle G'_x\rangle_{x\in X}$ by setting $G'_x = G_m$ if $m \in \mathbb{Z}$ and $x \in E_m$, $V_x = X$ if $g(x) = 0$. Let $\delta \in \Delta$ be the corresponding neighbourhood gauge $\{(x,C) : x \in X,\ C \subseteq G'_x\}$.

Suppose that $\boldsymbol{t}$ is any δ-fine member of T. For each $m \in \mathbb{Z}$, set $\boldsymbol{t}_m = \boldsymbol{t}\restriction E_m$. Then $H_{\boldsymbol{t}_m} \subseteq G_m$ for each m, so

$$\begin{aligned}S_{\boldsymbol{t}}(\|f\|,\mu) &= \sum_{m=-\infty}^{\infty} S_{\boldsymbol{t}_m}(\|f\|,\mu) \le \sum_{m=-\infty}^{\infty} (1+\epsilon)^{m+1}\mu H_{\boldsymbol{t}_m} \\ &\le \sum_{m=-\infty}^{\infty} (1+\epsilon)^{m+1}\mu G_m \le \sum_{m=-\infty}^{\infty} (1+\epsilon)^{m+2}\mu E_m + 2^{-|m|}\epsilon \\ &\le 3\epsilon + (1+\epsilon)^2 \sum_{m=-\infty}^{\infty} (1+\epsilon)^m \mu E_m \\ &\le 3\epsilon + (1+\epsilon)^2 \int g d\mu = 3\epsilon + (1+\epsilon)^2 \overline{\int} \|f\|d\mu\end{aligned}$$

and

$$\|S_{\boldsymbol{t}}(f,\nu)\| \le \gamma S_{\boldsymbol{t}}(\|f\|,\mu) \le 3\gamma\epsilon + (1+\epsilon)^2\gamma\overline{\int}\|f\|d\mu.$$

As ϵ is arbitrary, we have the result. **Q**

(d) Now suppose that $f : X \to U$ is Bochner integrable with respect to μ, and $\epsilon > 0$. Then there is a simple function $f_0 : X \to U$ such that $\int \|f - f_0\|d\mu \le \epsilon$. By (b) and (c), there are a $w \in W$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$\|S_{\boldsymbol{t}}(f_0,\nu) - w\| \le \epsilon, \quad \|S_{\boldsymbol{t}}(f - f_0,\nu)\| \le \epsilon + \gamma\epsilon$$

for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$. But this means that if $\boldsymbol{s}$, $\boldsymbol{t}$ are δ-fine and $\mathcal{R}$-filling members of T,

$$\begin{aligned}\|S_{\boldsymbol{s}}(f,\nu) - S_{\boldsymbol{t}}(f,\nu)\| &\le \|S_{\boldsymbol{s}}(f_0,\nu) - S_{\boldsymbol{t}}(f_0,\nu)\| + \|S_{\boldsymbol{s}}(f - f_0,\nu)\| + \|S_{\boldsymbol{t}}(f - f_0,\nu)\| \\ &\le 4\epsilon + 2\gamma\epsilon;\end{aligned}$$

as ϵ is arbitrary and W is complete, $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined.

2D Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U, V and W Banach spaces and $\langle\,|\,\rangle : U \times V \to W$ a continuous bilinear operator. Suppose that

(i) $\mathfrak{T}$ is a topology on X, and Δ is the set of neighbourhood gauges on X;

(ii) $\nu : \mathcal{C} \to V$ is a function which is additive in the sense that if $C_0, \dots, C_n \in \mathcal{C}$ are disjoint and have union $C \in \mathcal{C}$, then $\nu C = \sum_{i=0}^{n} \nu C_i$;

(iii) whenever $E \in \mathcal{C}$ and $\epsilon > 0$, there are closed sets $F \subseteq E$, $F' \subseteq X \setminus E$ such that $\sum_{(x,C)\in\boldsymbol{t}} \|\nu C\| \le \epsilon$ whenever $\boldsymbol{t} \in T$ and $H_{\boldsymbol{t}} \cap (F \cup F') = \emptyset$;

(iv) for every $E \in \mathcal{C}$ and $x \in X$ there is a neighbourhood G of x such that if $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a partition $\mathcal{D}$ of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$;

(v) for every $C \in \mathcal{C}$ and $\mathcal{R} \in \mathfrak{R}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap A \in \mathcal{R}$ whenever $A \in \mathcal{R}'$.

Let $f : X \to U$ be a function such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f. Then $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$ for every $E \in \mathcal{E}$.

proof (a) Because both F and I_ν are additive, and $F(X) = I_\nu(f)$, and either E or its complement is a finite disjoint union of members of $\mathcal{C}$ (see 1A(e-ii) above), it is enough to consider the case in which $E \in \mathcal{C}$. Let $\gamma \ge 0$ be such that $\|\langle u|v\rangle\| \le \gamma\|u\|\|v\|$ for all $u \in U$ and $v \in V$.

(b) Let $\epsilon > 0$. For each $x \in X$ let G_x be an open set containing x such that whenever $C \in \mathcal{C}$, $C \subseteq G$ and $\{(x, C)\} \in T$, there is a partition $\mathcal{D}$ of C into members of $\mathcal{C}$ such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}$ and every member of $\mathcal{D}$ is either included in E or disjoint from E. For each $n \in \mathbb{N}$, let $F_n \subseteq E$, $F'_n \subseteq X \setminus E$ be closed sets such that $\sum_{(x,C)\in\boldsymbol{t}} \|\nu C\| \le \frac{2^{-n}\epsilon}{n+1}$ whenever $\boldsymbol{t} \in T$ and $H_{\boldsymbol{t}} \cap (F_n \cup F'_n) = \emptyset$; now define G'_x, for $x \in X$, by saying that

$$\begin{aligned} G'_x &= G_x \setminus F'_n \text{ if } x \in E \text{ and } n \le \|f(x)\| < n+1, \\ &= G_x \setminus F_n \text{ if } x \in X \setminus E \text{ and } n \le \|f(x)\| < n+1. \end{aligned}$$

Let $\delta_0 \in \Delta$ be the neighbourhood gauge defined by the family $\langle G'_x\rangle_{x\in X}$. Let $\delta \in \Delta$ and $\mathcal{R}_1 \in \mathfrak{R}$ be such that $\delta \subseteq \delta_0$, $\|F(H_{\boldsymbol{t}}) - \sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C\| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$, and $|F(E)| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}_1$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $R \cap H \in \mathcal{R}_1$ whenever $R \in \mathcal{R}$.

(c) As in the proof of the Saks-Henstock Lemma, let T_E be the set of those $\boldsymbol{t} \in T$ such that, for each $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. The key to the proof is the following fact: if $\boldsymbol{t} \in T$ is δ-fine, then there is a δ-fine $\boldsymbol{s} \in T_E$ such that $W_{\boldsymbol{s}} = W_{\boldsymbol{t}}$ and $S_{\boldsymbol{s}}(g, \nu) = S_{\boldsymbol{t}}(g, \nu)$ for every $g : X \to U$. **P** For each $(x, C) \in \boldsymbol{t}$, we know that $C \subseteq G'_x \subseteq G_x$, because $\delta \subseteq \delta_0$. Let $\mathcal{D}_{(x,C)}$ be a finite partition of C into members of $\mathcal{C}$, each either included in E or disjoint from E, such that $\{(x, D)\} \in T$ for every $D \in \mathcal{D}_{(x,C)}$. Then $\boldsymbol{s} = \{(x, D) : (x, C) \in \boldsymbol{t}, D \in \mathcal{D}_{(x,C)}\}$ belongs to T_E. Because δ is a neighbourhood gauge, $(x, D) \in \delta$ whenever $(x, C) \in \boldsymbol{t}$ and $D \in \mathcal{D}_{(x,C)}$, so $\boldsymbol{s}$ is δ-fine.

If $g : X \to U$ is any function,

$$\begin{aligned} S_{\boldsymbol{s}}(g, \nu) &= \sum_{(x,C)\in\boldsymbol{t}} \sum_{D\in\mathcal{D}_{(x,C)}} \langle g(x)|\nu D\rangle \\ &= \sum_{(x,C)\in\boldsymbol{t}} \langle g(x)| \sum_{D\in\mathcal{D}_{(x,C)}} \nu D\rangle = \sum_{(x,C)\in\boldsymbol{t}} \langle g(x)|\nu C\rangle \end{aligned}$$

(because ν is additive)

$$= S_{\boldsymbol{t}}(g, \nu). \text{ \textbf{Q}}$$

(d) Now suppose that $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. Let $\boldsymbol{s} \in T_E$ be as in (c), and set

$$\boldsymbol{s}^* = \{(x, D) : (x, D) \in \boldsymbol{s}, x \in E, D \subseteq E\},$$

$$\boldsymbol{s}' = \{(x, D) : (x, D) \in \boldsymbol{s}, x \notin E, D \subseteq E\},$$

$$\boldsymbol{s}'' = \{(x, D) : (x, D) \in \boldsymbol{s}, x \in E, D \cap E = \emptyset\}.$$

Because $\boldsymbol{s} \in T_E$,

$$H_{\boldsymbol{s}^*\cup\boldsymbol{s}'} = E \cap H_{\boldsymbol{s}} = E \cap H_{\boldsymbol{t}}$$

and $E \setminus H_{\boldsymbol{s}^*\cup\boldsymbol{s}'} = E \setminus H_{\boldsymbol{t}}$ belongs to $\mathcal{R}_1$, by the choice of $\mathcal{R}$. Accordingly

$$\|F(E) - S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f,\nu)\| \le \|F(E) - F(H_{\boldsymbol{s}^*\cup\boldsymbol{s}'})\| + \|F(H_{\boldsymbol{s}^*\cup\boldsymbol{s}'}) - S_{\boldsymbol{s}'}(f,\nu)\| \le 2\epsilon$$

because $\boldsymbol{s}^* \cup \boldsymbol{s}' \subseteq \boldsymbol{s}$ is δ-fine.

For $n \in \mathbb{N}$ set

$$\boldsymbol{s}'_n = \{(x,D) : (x,D) \in \boldsymbol{s}',\, n \le \|f(x)\| < n+1\},$$

$$\boldsymbol{s}''_n = \{(x,D) : (x,D) \in \boldsymbol{s}'',\, n \le \|f(x)\| < n+1\}.$$

Then $H_{\boldsymbol{s}'_n} \subseteq E \setminus F_n$. **P** If $(x,D) \in \boldsymbol{s}'_n$, there is a $C \in \mathcal{C}$ such that $D \subseteq E \cap C$ and $(x,C) \in \boldsymbol{t}$, while $x \notin E$, so that $C \subseteq G'_x$ and $C \cap F_n = \emptyset$. **Q** Similarly, $H_{\boldsymbol{s}''_n} \subseteq (X \setminus E) \setminus F'_n$. Thus $H_{\boldsymbol{s}'_n\cup\boldsymbol{s}''_n}$ is disjoint from $F_n \cup F'_n$ and

$$\begin{aligned}\|S_{\boldsymbol{s}'_n}(f,\nu) - S_{\boldsymbol{s}''_n}(f,\nu)\| &= \|\sum_{(x,D)\in\boldsymbol{s}'_n} \langle f(x_i)|\nu D\rangle - \sum_{(x,D)\in\boldsymbol{s}''_n} \langle f(x_i)|\nu D\rangle\| \\ &\le \sum_{(x,D)\in\boldsymbol{s}'_n\cup\boldsymbol{s}''_n} \gamma\|f(x_i)\|\|\nu D\| \\ &\le \gamma(n+1) \sum_{(x,D)\in\boldsymbol{s}'_n\cup\boldsymbol{s}''_n} \|\nu D\| \le 2^{-n}\gamma\epsilon\end{aligned}$$

by the choice of F_n and F'_n.

Consequently,

$$\|F(E) - S_{\boldsymbol{t}}(f \times \chi E, \nu)\| = \|F(E) - S_{\boldsymbol{s}}(f \times \chi E, \nu)\| = \|F(E) - S_{\boldsymbol{s}^*\cup\boldsymbol{s}''}(f,\nu)\|$$

(because $\boldsymbol{s}^* \cup \boldsymbol{s}'' = \{(x,D) : (x,D) \in \boldsymbol{s},\, x \in E\}$)

$$\le \|F(E) - S_{\boldsymbol{s}^*\cup\boldsymbol{s}'}(f,\nu)\| + \|S_{\boldsymbol{s}'}(f,\nu) - S_{\boldsymbol{s}''}(f,\nu)\|$$

(because $\boldsymbol{s}^*$, $\boldsymbol{s}'$ and $\boldsymbol{s}''$ are disjoint subsets of $\boldsymbol{s}$)

$$\le 2\epsilon + \|\sum_{n=0}^{\infty} S_{\boldsymbol{s}'_n}(f,\nu) - \sum_{n=0}^{\infty} S_{\boldsymbol{s}''_n}(f,\nu)\|$$

(the infinite sums are well-defined because $\boldsymbol{s}$ is finite, so that all but finitely many terms are zero)

$$\le 2\epsilon + \sum_{n=0}^{\infty} \|S_{\boldsymbol{s}'_n}(f,\nu) - S_{\boldsymbol{s}''_n}(f,\nu)\|$$

$$\le 2\epsilon + \sum_{n=0}^{\infty} 2^{-n}\gamma\epsilon = 2(1+\gamma)\epsilon.$$

As ϵ is arbitrary, $I_\nu(f \times \chi E)$ is defined and equal to $F(E)$, as required.

2E Proposition Suppose that X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ, $\mathfrak{R}$, U, V, W, $\langle\,|\,\rangle$ and ν satisfy the conditions of 2D, and that $f : X \to U$, $\langle G_n\rangle_{n\in\mathbb{N}}$, G and w are such that

(vi) $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of X with union G,

(vii) $I_\nu(f \times \chi G_n)$ is defined for every $n \in \mathbb{N}$,

(viii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} I_\nu(f \times \chi H_{\boldsymbol{t}\restriction G})$ is defined and equal to w,

where $\boldsymbol{t}\restriction G = \{(x,C) : (x,C) \in \boldsymbol{t},\, x \in G\}$ for $\boldsymbol{t} \in T$. Then $I_\nu(f \times \chi G)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let F_n be the Saks-Henstock indefinite integral of $f \times \chi G_n$. Let $\delta_n \in \Delta$ be such that

$$\|F_n(H_{\boldsymbol{s}}) - S_{\boldsymbol{s}}(f \times \chi G_n, \nu)\| \le 2^{-n}\epsilon$$

whenever $\boldsymbol{s} \in T$ is δ_n-fine. Set

$$\tilde{\delta} = \{(x, A) : x \in X \setminus G,\ A \subseteq X\} \cup \bigcup_{n\in\mathbb{N}} \{(x, A) : x \in G_n \setminus \bigcup_{i<n} G_i,\ A \subseteq G_n,\ (x, A) \in \delta_n\},$$

so that $\tilde{\delta} \in \Delta$. Note that if $x \in G$ and $C \in \mathcal{C}$ and $(x, C) \in \tilde{\delta}$, then there is some $n \in \mathbb{N}$ such that $x \in G_n$ and $C \subseteq G_n$, so that

$$I_\nu(f \times \chi C) = I_\nu((f \times \chi G_n) \times \chi C) = F_n(C)$$

is defined, by 2D; this means that $I_\nu(f \times \chi H_{\boldsymbol{t}\restriction G})$ will be defined for every $\tilde{\delta}$-fine $\boldsymbol{t} \in T$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $\|w - I_\nu(f \times \chi H_{\boldsymbol{t}\restriction G})\| \le \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling.

Let $\boldsymbol{t} \in T$ be $(\delta \cap \tilde{\delta})$-fine and $\mathcal{R}$-filling. For $n \in \mathbb{N}$, set $\boldsymbol{t}_n = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in G_n \setminus \bigcup_{i<n} G_i\}$. Then $\boldsymbol{t}\restriction G = \bigcup_{n\in\mathbb{N}} \boldsymbol{t}_n$, and $\boldsymbol{t}_n$ is δ_n-fine and $H_{\boldsymbol{t}_n} \subseteq G_n$ for every n. So

$$\|w - S_{\boldsymbol{t}}(f \times \chi G, \nu)\| = \|w - \sum_{n=0}^{\infty} S_{\boldsymbol{t}_n}(f \times \chi G_n, \nu)\|$$

$$\le \|w - I_\nu(f \times \chi H_{\boldsymbol{t}\restriction G})\| + \sum_{n=0}^{\infty} \|I_\nu(f \times \chi H_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi G_n, \nu)\|$$

$$\le \epsilon + \sum_{n=0}^{\infty} \|I_\nu(f \times \chi G_n \times \chi H_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi G_n, \nu)\|$$

$$= \epsilon + \sum_{n=0}^{\infty} \|F_n(H_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f \times \chi G_n, \nu)\|$$

(2D)

$$\le \epsilon + \sum_{n=0}^{\infty} 2^{-n}\epsilon$$

(because every $\boldsymbol{t}_n$ is δ_n-fine)

$$= 3\epsilon.$$

As ϵ is arbitrary, $w = I_\nu(f \times \chi G)$, as claimed.

2F Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U and V Banach spaces, $\langle\,|\,\rangle : U \times V \to \mathbb{R}$ a continuous bilinear functional, and $\nu : \mathcal{C} \to V$ a function. Suppose that $\langle f_i \rangle_{i\in I}$ is a family of functions from X to U such that

(i) $w_i = I_\nu(f_i, \nu)$ is defined for every $i \in I$,

(ii) $\inf_{\delta\in\Delta,\mathcal{R}\in\mathfrak{R}} \sum_{i\in I} \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine and } \mathcal{R}\text{-filling}} \|S_{\boldsymbol{t}}(f_i, \nu)\|$ is finite,

(iii) $f(x) = \sum_{i\in I} f_i(x)$ is defined in U for every $x \in X$.

Then $I_\nu(f, \nu)$ and $\sum_{i\in I} w_i$ are defined in W and equal.

proof (a) Let $\delta_0 \in \Delta$, $\mathcal{R}_0 \in \mathfrak{R}$ be such that

$$M = \sum_{i\in I} \sup\{\|S_{\boldsymbol{t}}(f_i, \nu)\| : \boldsymbol{t} \in T \text{ is } \delta_0\text{-fine and } \mathcal{R}_0\text{-filling}\}$$

is finite. Then $\sum_{i\in I} \|w_i\| \le M$. **P** If $J \subseteq I$ is finite and $\epsilon > 0$, there is a δ_0-fine $\mathcal{R}_0$-filling $\boldsymbol{t} \in T$ such that $\sum_{i\in J} \|w_i - S_{\boldsymbol{t}}(f_i, \nu)\| \le \epsilon$, so that $\sum_{i\in J} \|w_i\| \le M + \epsilon$. **Q**

So $w = \sum_{i\in I} w_i$ is defined.

(b) Now take any $\epsilon > 0$. Let $J \subseteq I$ be a finite set such that

$$\sum_{i\in I\setminus J} \sup\{\|S_{\boldsymbol{t}}(f_i, \nu)\| : \boldsymbol{t} \in T \text{ is } \delta_0\text{-fine and } \mathcal{R}_0\text{-filling}\} \le \epsilon;$$

then the argument of (a) tells us that $\sum_{i\in I\setminus J} \|w_i\| \le \epsilon$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $\delta \subseteq \delta_0$, $\mathcal{R} \subseteq \mathcal{R}_0$ and $\sum_{i\in J} \|w_i - S_{\boldsymbol{t}}(f_i, \nu)\| \le \epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$. In this case, for any such $\boldsymbol{t}$,

$$S_{\boldsymbol{t}}(f,\nu)=\sum_{(x,C)\in\boldsymbol{t}}\langle f(x)|\nu C\rangle=\sum_{(x,C)\in\boldsymbol{t}}\langle\sum_{i\in I}f_i(x)|\nu C\rangle$$
$$=\sum_{(x,C)\in\boldsymbol{t}}\sum_{i\in I}\langle f_i(x)|\nu C\rangle=\sum_{i\in I}S_{\boldsymbol{t}}(f_i,\nu),$$

so

$$\|w-S_{\boldsymbol{t}}(f,\nu)\|\le\sum_{i\in J}\|w_i-S_{\boldsymbol{t}}(f_i,\nu)\|+\sum_{i\in I\setminus J}\|S_{\boldsymbol{t}}(f_i,\nu)\|+\sum_{i\in I\setminus J}\|w_i\|\le 3\epsilon.$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to w.

2G The scalar-valued case: Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U and V Banach spaces, $\langle\,|\,\rangle:U\times V\to\mathbb{R}$ a continuous bilinear functional, $f:X\to U$, $\nu:\mathcal{C}\to V$ functions such that $I_\nu(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$ is defined in $\mathbb{R}$, $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$ and $F:\mathcal{E}\to\mathbb{R}$ the Saks-Henstock indefinite integral of f with respect to ν. Then for every $\epsilon>0$ there is a $\delta\in\Delta$ such that

$$\sum_{(x,C)\in\boldsymbol{t}}|F(C)-\langle f(x)|\nu C\rangle|\le\epsilon$$

for every δ-fine $\boldsymbol{t}\in T$,

proof (See Fremlin 03, 482B.) Let $\delta\in\Delta$ be such that

$$|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)|\le\frac{\epsilon}{2}$$

for every δ-fine $\boldsymbol{t}\in T$. For any such $\boldsymbol{t}$, any subset $\boldsymbol{s}$ of $\boldsymbol{t}$ is also a δ-fine member of T, so

$$|\sum_{(x,C)\in\boldsymbol{s}}F(C)-\langle f(x)|\nu C\rangle|=|F(H_{\boldsymbol{s}})-S_{\boldsymbol{s}}(f,\nu)|\le\frac{\epsilon}{2}.$$

Applying this to $\boldsymbol{s}=\{(x,C):(x,C)\in\boldsymbol{t},\ F(C)>\langle f(x)|\nu C\rangle\}$ and $\boldsymbol{s}'=\{(x,C):(x,C)\in\boldsymbol{t},\ F(C)<\langle f(x)|\nu C\rangle\}$, we get

$$\sum_{(x,C)\in\boldsymbol{t}}|F(C)-\langle f(x)|\nu C\rangle|$$
$$=\sum_{(x,C)\in\boldsymbol{s}}(F(C)-\langle f(x)|\nu C\rangle)-\sum_{(x,C)\in\boldsymbol{s}'}(F(C)-\langle f(x)|\nu C\rangle)\le\epsilon,$$

as required.

2H In the case of real-valued set functions ν, many problems can be reduced to the case in which ν is additive, as in the following.

Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, U a Banach space, and $\nu:\mathcal{C}\to\mathbb{R}$ a function; let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Suppose that $I_\nu(\chi X)$ is defined, and that $F_1:\mathcal{E}\to\mathbb{R}$ is the Saks-Henstock indefinite integral of χX with respect to ν. Then for a bounded function $f:X\to U$, $I_\nu(f)=I_{F_1}(f)$ if either is defined, and in this case f has the same Saks-Henstock indefinite integral with respect to either ν or F_1.

proof (a) Suppose that f has Saks-Henstock indefinite integral F with respect to ν. Given $\epsilon>0$, there is a $\delta\in\Delta$ such that

$$\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|\le\epsilon,\quad\sum_{(x,C)\in\boldsymbol{t}}|F_1(C)-\nu C|\le\epsilon$$

for every δ-fine $\boldsymbol{t}\in T$ (2G). Now, given such a $\boldsymbol{t}$,

$$\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,F_1)\|\le\|F(H_{\boldsymbol{t}})-S_{\boldsymbol{t}}(f,\nu)\|+\|S_{\boldsymbol{t}}(f,\nu)-S_{\boldsymbol{t}}(f,F_1)\|$$
$$\le\epsilon+\sum_{(x,C)\in\boldsymbol{t}}\|\nu C\cdot f(x)-F_1(C)f(x)\|$$
$$\le\epsilon+\gamma\|f\|_\infty\sum_{(x,C)\in\boldsymbol{t}}|\nu C-F_1(C)|\le(1+\gamma\|f\|_\infty)\epsilon.$$

Also, of course, there is an $\mathcal{R} \in \mathfrak{R}$ such that $\|F(E)\| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$. So F is the Saks-Henstock indefinite integral of f with respect to F_1.

(b) Conversely, suppose that f has Saks-Henstock indefinite integral F with respect to F_1. Given $\epsilon > 0$, there is a $\delta \in \Delta$ such that

$$\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f, F_1)\| \le \epsilon, \quad \sum_{(x,C)\in\boldsymbol{t}} |F_1(C) - \nu C| \le \epsilon$$

for every δ-fine $\boldsymbol{t} \in T$ (2G). This time, for such a $\boldsymbol{t}$,

$$\begin{aligned}\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)\| &\le \|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,F_1)\| + \|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{t}}(f,F_1)\| \\ &\le \epsilon + \sum_{(x,C)\in\boldsymbol{t}} \|\nu C \cdot f(x) - F_1(C) f(x)\| \\ &\le \epsilon + \gamma\|f\|_\infty \sum_{(x,C)\in\boldsymbol{t}} |\nu C - F_1(C)| \le (1 + \gamma\|f\|_\infty)\epsilon.\end{aligned}$$

As before, there is an $\mathcal{R} \in \mathfrak{R}$ such that $\|F(E)\| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$. So F is the Saks-Henstock indefinite integral of f with respect to ν.

2I Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U a Banach space, $f : X \to U$, $\nu : \mathcal{C} \to \mathbb{R}$ functions such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\nu)$ is defined in U, $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$ and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f with respect to ν. Suppose further that

(α) Δ is countably full,

(β) $I_\nu(\chi X) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} \sum_{(x,C)\in\boldsymbol{t}} \nu C$ is defined in $\mathbb{R}$ and the Saks-Henstock indefinite integral of χX with respect to ν is F_0.

Then $I_{F_0}(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, F_0)$ is defined and equal to $I_\nu(f)$, and F is the Saks-Henstock indefinite integral of f with respect to F_0.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$ there is a $\delta_n \in \Delta$ such that

$$\sum_{(x,C)\in\boldsymbol{t}} |F_0(C) - \nu C| \le \tfrac{2^{-n-1}\epsilon}{n+1}$$

for every δ_n-fine $\boldsymbol{t} \in T$ (2G). Because Δ is countably full, there is a $\delta' \in \Delta$ such that $(x, C) \in \delta_n$ whenever $(x, C) \in \delta$ and $n \le \|f(x)\| < n + 1$; now there is a $\delta \in \Delta$, included in δ', such that $\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)\| \le \epsilon$ for every δ-fine $\boldsymbol{t} \in T$. In this case, for such $\boldsymbol{t}$,

$$\begin{aligned}\|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,F_0)\| &\le \|F(H_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f,\nu)\| + \|S_{\boldsymbol{t}}(f,\nu) - S_{\boldsymbol{t}}(f,F_0)\| \\ &\le \epsilon + \sum_{(x,C)\in\boldsymbol{t}} \|\nu C \cdot f(x) - F_0(C) f(x)\| \\ &= \epsilon + \sum_{(x,C)\in\boldsymbol{t}} |\nu C - F_0(C)|\|f(x)\| \\ &\le \epsilon + \sum_{n=0}^{\infty} \frac{2^{-n-1}\epsilon}{n+1} \cdot (n+1) = 2\epsilon.\end{aligned}$$

At the same time, there is certainly an $\mathcal{R} \in \mathfrak{R}$ such that $\|F(E)\| \le \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$. By 1F, $I_{F_0}(f)$ is defined; by 1D, F is the Saks-Henstock indefinite integral of f with respect to F_0.

2J Dominated convergence: Proposition Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, U, V and W Banach spaces, $\langle\,|\,\rangle : U \times V \to W$ a continuous bilinear operator, and $\nu : \mathcal{C} \to V$ a function. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Suppose that

(i) Δ is countably full,

(ii) whenever $\langle h_n \rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence of functions from X to V^* such that $I_\nu(h_n)$ is defined for every n and $\lim_{n\to\infty} h_n(x) = 0$ for every x, then the Saks-Henstock indefinite integrals of the h_n converge uniformly to 0,

(iii) there is an $M \ge 0$ such that $\sum_{(x,C)\in\boldsymbol{t}} \|\nu C\| \le M$ for every $\boldsymbol{t} \in T$.

Then whenever U is a Banach space and $\langle f_n\rangle_{n\in\mathbb{N}}$ a uniformly bounded sequence of functions from X to U such that $I_\nu(f_n)$ is defined for every n and $f(x) = \lim_{n\to\infty} f_n(x)$ is convergent for every $x \in [0,1]$, $I_\nu(f)$ is defined, and the Saks-Henstock indefinite integrals of the f_n converge uniformly to the Saks-Henstock indefinite integral of f.

Remark When speaking of $I_\nu(h_n)$ in the hypothesis (ii), I mean to use the natural bilinear operator $(w,v) \mapsto w(v) : V^* \times V \to \mathbb{R}$, so that $I_\nu(h_n)$ is a real number and the Saks-Henstock indefinite integral of h_n is real-valued; while for $I_\nu(f_n)$ and $I_\nu(f)$ in the conclusion of the proposition, I mean to use the bilinear operator $\langle\,|\,\rangle$ of the first sentence.

proof (a) For each $n \in \mathbb{N}$ let F_n be the Saks-Henstock indefinite integral of f_n. Then $\langle F_n\rangle_{n\in\mathbb{N}}$ is uniformly convergent to $F : \mathcal{E} \to U$ say. **P?** Otherwise, there is an $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there are k_n, $l_n \ge n$ and $E_n \in \mathcal{E}$ such that $\|F_{k(n)}(E_n) - F_{l(n)}(E_n)\| \ge \epsilon$. Note that $F_{k(n)} - F_{l(n)}$ is the Saks-Henstock indefinite integral of $f_{k(n)} - f_{l(n)}$, by 2A. For each n, let $\psi_n \in W^*$ be such that $\|\psi_n\| \le 1$ and $\psi_n(F_{k(n)}(E_n) - F_{l(n)}(E_n)) \ge \epsilon$; define $\pi_n : U \to V^*$ by setting $\pi_n(u)(v) = \psi_n(\langle u|v\rangle)$ for $u \in U$ and $v \in V$, and $h_n : X \to V^*$ by setting $h_n(x) = \pi_n(f_{k(n)}(x) - f_{l(n)}(x))$ for $x \in X$. Then $\langle h_n(x)|v\rangle = \psi_n(\langle f_{k(n)}(x) - f_{l(n)}(x)|v\rangle)$ for every $x \in X$ and $v \in V$, so 2B tells us that h_n has Saks-Henstock indefinite integral $E \mapsto \psi_n(F_{k(n)}(E) - F_{l(n)}(E))$. Also $\langle h_n\rangle_{n\in\mathbb{N}}$ is uniformly bounded and converges pointwise to the zero function. So $\lim_{n\to\infty} \psi_n(F_{k(n)}(E_n) - F_{l(n)}(E_n)) = 0$, by hypothesis (ii). **XQ**

(b) Let $\gamma \ge 0$ be such that $\|\langle u|v\rangle\| \le \gamma\|u\|\|v\|$ for all $u \in U$ and $v \in V$. Let $\epsilon > 0$. Then there is a neighbourhood gauge δ such that $\|S_{\boldsymbol{t}}(f,\nu) - F(H_{\boldsymbol{t}})\| \le (4+\gamma M)\epsilon$ for every δ-fine $\boldsymbol{t}$. **P** Let $\langle r_n\rangle_{n\in\mathbb{N}}$ be strictly increasing and such that $\|F_{r_n}(E) - F(E)\| \le 2^{-n}\epsilon$ for every $n \in \mathbb{N}$ and $E \in \mathcal{E}$. For each $n \in \mathbb{N}$, let δ_n be a gauge such that $\|S_{\boldsymbol{t}}(f_{r_n},\nu) - F_{r_n}(H_{\boldsymbol{t}})\| \le 2^{-n}\epsilon$ for every δ_n-fine $\boldsymbol{t}$. Let δ be the gauge

$$\bigcup_{n\in\mathbb{N}}\{(x,C) : \|f_{r_n}(x) - f(x)\| \le \epsilon,\ (x,C) \in \delta_n\}.$$

If $\boldsymbol{t}$ is δ-fine, express it as a disjoint union $\bigcup_{n\le m} \boldsymbol{t}_n$ where $(x,C) \in \delta_n$ and $\|f_{r_n}(x) - f(x)\| \le \epsilon$ for $(x,C) \in \boldsymbol{t}_n$. Then each $\boldsymbol{t}_n$ is δ_n-fine, so

$$\begin{aligned}
\|S_{\boldsymbol{t}}(f,\nu) - F(H_{\boldsymbol{t}})\| &= \|\sum_{n=0}^m S_{\boldsymbol{t}_n}(f,\nu) - \sum_{n=0}^m F(H_{\boldsymbol{t}_n})\| \\
&\le \sum_{n=0}^m \|S_{\boldsymbol{t}_n}(f,\nu) - F(H_{\boldsymbol{t}_n})\| \\
&\le \sum_{n=0}^m \|S_{\boldsymbol{t}_n}(f,\nu) - S_{\boldsymbol{t}_n}(f_{r_n},\nu)\| + \sum_{n=0}^m \|S_{\boldsymbol{t}_n}(f_{r_n},\nu) - F_{r_n}(H_{\boldsymbol{t}_n})\| \\
&\qquad + \sum_{n=0}^m \|F_{r_n}(H_{\boldsymbol{t}_n}) - F(H_{\boldsymbol{t}_n})\| \\
&\le \sum_{n=0}^m \sum_{(x,C)\in\boldsymbol{t}_n} \|\langle f(x) - f_{r(n)}(x)|\nu C\rangle\| + \sum_{n=0}^m 2^{-n}\epsilon + \sum_{n=0}^m 2^{-n}\epsilon \\
&\le \sum_{n=0}^m \sum_{(x,C)\in\boldsymbol{t}_n} \gamma\epsilon\|\nu C\| + 4\epsilon \\
&= \sum_{(x,C)\in\boldsymbol{t}} \gamma\epsilon\|\nu C\| + 4\epsilon \le (4+\gamma M)\epsilon. \ \mathbf{Q}
\end{aligned}$$

(c) By (a),

$$\inf_{\mathcal{R}\in\mathfrak{R}} \sup_{E\in\mathcal{E}\cap\mathcal{R}} \|F(E)\| = \lim_{n\to\infty} \inf_{\mathcal{R}\in\mathfrak{R}} \sup_{E\in\mathcal{E}\cap\mathcal{R}} \|F_n(E)\| = 0.$$

By 1F, f is $(X,T,\Delta,\mathfrak{R},\nu)$-integrable and its Saks-Henstock indefinite integral is F.

Remark To have (i) and (ii) true but (iii) false, or anyway so false that the argument of (b) won't work, something a little odd has to be happening. I do not have an example in which (i) and (iii) are true but (ii) is false.

2K Proposition Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$, $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$, and $\nu:\mathcal{E}\to[0,1]$ an additive functional such that $\nu X=1$. Set $\mathcal{N}=\{E:E\in\mathcal{E},\,\nu E=0\}$, $\mathfrak{A}_0=\mathcal{E}/\mathcal{N}$ and $\bar\nu_0E^\bullet=\nu E$ for $E\in\mathcal{E}$; let $(\mathfrak{A},\bar\nu)$ be the probability algebra metric completion of $(\mathfrak{A}_0,\bar\nu_0)$ (Fremlin 02, 392H[1]). Let $\mathcal{F}^*$ be the filter on T described in part (b) of the proof of 1D. For $A\subseteq X$, set

$$\nu^*A=\limsup_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}}(\chi A,\nu),$$

and let Q_A be the set of those $a\in\mathfrak{A}$ such that

$$\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a)=0,$$

where $\boldsymbol{t}\restriction A=\{(x,C):(x,C)\in\boldsymbol{t},\,x\in A\}$. Then Q_A has a least member a_A, and $\bar\nu a_A=\nu^*A$.

proof For a finite set $\mathcal{E}_0\subseteq\mathcal{E}$, say that $\boldsymbol{t}\in T$ is $\mathcal{E}_0$**-respecting** if whenever $E\in\mathcal{E}_0$ and $(x,C)\in\boldsymbol{t}$ then either $C\subseteq E$ or $C\cap E=\emptyset$.

(a) If a, $b\in Q_A$, then

$$\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus(a\cap b))\le\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a)+\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus b)$$

for every $\boldsymbol{t}\in T$, so

$$\begin{aligned}\limsup_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus(a\cap b))&\le\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a)+\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus b)\\&=0.\end{aligned}$$

Thus Q_A is downwards-directed. Setting $a_A=\inf Q_A$, we have

$$\limsup_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a_A)\le\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a)+\bar\nu(a\setminus a_A)=\bar\nu(a\setminus a_A)$$

for every $a\in Q_A$, while $\inf_{a\in Q_A}\bar\nu(a\setminus a_A)=0$ (Fremlin 02, 321F), so $\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a_A)=0$ and $a_A\in Q_A$ is the least member of Q_A.

(b) We have

$$\begin{aligned}\nu^*A&=\limsup_{\boldsymbol{t}\to\mathcal{F}^*}S_{\boldsymbol{t}}(\chi A,\nu)=\limsup_{\boldsymbol{t}\to\mathcal{F}^*}\nu H_{\boldsymbol{t}\restriction A}\\&=\limsup_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu H^\bullet_{\boldsymbol{t}\restriction A}\le\lim_{\boldsymbol{t}\to\mathcal{F}^*}\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus a_A)+\bar\nu a_A\ =\bar\nu a_A.\end{aligned}$$

(c) In the other direction, choose $\langle\mathcal{E}_n\rangle_{n\in\mathbb{N}}$, $\langle\delta_n\rangle_{n\in\mathbb{N}}$, $\langle\mathcal{R}_n\rangle_{n\in\mathbb{N}}$ and $\langle\boldsymbol{t}_n\rangle_{n\in\mathbb{N}}$ inductively in such a way that, for each n,

$\mathcal{E}_n\in[\mathcal{E}]^{<\omega}$, $\delta_n\in\Delta$, $\mathcal{R}_n\in\mathfrak{R}$, $\boldsymbol{t}_n\in T$,

$\bar\nu(H_{\boldsymbol{t}^\bullet\restriction A}\setminus a_A)\le2^{-n}$, $\nu H_{\boldsymbol{t}\restriction A}\le\nu^*A+2^{-n}$ whenever $\boldsymbol{t}\in T$ is δ_n-fine, $\mathcal{R}_n$-filling and $\mathcal{E}_n$-respecting,

$\boldsymbol{t}_n$ is δ_n-fine, $\mathcal{R}_n$-filling and $\mathcal{E}_n$-respecting, and $S_{\boldsymbol{t}_n}(\chi A,\nu)\ge\nu^*A-2^{-n}$,

$\delta_{n+1}\subseteq\delta_n$, $\mathcal{R}_{n+1}\subseteq\mathcal{R}_n$ and $\mathcal{E}_n\cup\{C:(x,C)\in\boldsymbol{t}_n\}\subseteq\mathcal{E}_{n+1}$.

If $\boldsymbol{t}\in T$ is δ_n-fine and $\mathcal{E}_{n+1}$-respecting, then $\nu(H_{\boldsymbol{t}\restriction A}\setminus H_{\boldsymbol{t}_n\restriction A})\le2^{-n+1}$. **P** Set

$$\boldsymbol{s}=(\boldsymbol{t}_n\restriction A)\cup\{(x,C):(x,C)\in\boldsymbol{t}\restriction A,\,C\cap H_{\boldsymbol{t}_n\restriction A}=\emptyset\};$$

then $\boldsymbol{s}\in T$ is δ_n-fine and $\mathcal{E}_n$-respecting, so extends to a δ_n-fine, $\mathcal{E}_n$-respecting and $\mathcal{R}_n$-filling $\boldsymbol{s}'\in T$ (see the proof of 1D). Now, because $\boldsymbol{t}$ is $\mathcal{E}_{n+1}$-respecting and $C\in\mathcal{E}_{n+1}$ whenever $(x,C)\in\boldsymbol{t}_n$,

$$\begin{aligned}\nu(H_{\boldsymbol{t}_n\restriction A}\cup H_{\boldsymbol{t}\restriction A})&=\nu H_{\boldsymbol{s}\restriction A}\le\nu H_{\boldsymbol{s}'\restriction A}=S_{\boldsymbol{s}'}(\chi A,\nu)\\&\le\nu^*A+2^{-n}\le\nu H_{\boldsymbol{t}_n\restriction A}+2^{-n+1},\end{aligned}$$

[1] Formerly 393B.

so $\nu(H_{\boldsymbol{t}\restriction A}\setminus H_{\boldsymbol{t}_n\restriction A})\le 2^{-n+1}$. **Q**

For $n\in\mathbb{N}$, set $b_n=\sup_{m\ge n}H^\bullet_{\boldsymbol{t}_m\restriction A}$. Then, for any $m\ge n$,

$$\bar\nu(H^\bullet_{\boldsymbol{t}\restriction A}\setminus b_n)\le\nu(H_{\boldsymbol{t}\restriction A}\setminus H_{\boldsymbol{t}_m\restriction A})\le 2^{-m+1}$$

whenever $\boldsymbol{t}\in T$ is δ_m-fine, $\mathcal{R}_m$-filling and $\mathcal{E}_{m+1}$-respecting, so $b_n\in Q_A$ and $b_n\supseteq a_A$. Thus

$$\begin{aligned}\bar\nu a_A\le\bar\nu b_n&\le\bar\nu H^\bullet_{\boldsymbol{t}_n\restriction A}+\sum_{m=n}^\infty\bar\nu(H^\bullet_{\boldsymbol{t}_{m+1}\restriction A}\setminus H^\bullet_{\boldsymbol{t}_m\restriction A})\\&=\nu H_{\boldsymbol{t}_n\restriction A}+\sum_{m=n}^\infty\nu(H_{\boldsymbol{t}_{m+1}\restriction A}\setminus H_{\boldsymbol{t}_m\restriction A})\\&\le\nu^*A+2^{-n}+\sum_{m=n}^\infty 2^{-m+1}=\nu^*A+5\cdot 2^{-n}.\end{aligned}$$

As n is arbitrary, $\bar\nu a_A\le\nu^*A$ and we have equality.

3 The problem Characterise the functions which can arise as Saks-Henstock indefinite integrals.

(Compare the ACG$_*$ functions for the ordinary Henstock integral, see Fremlin 03, §483 or Gordon 94.)

3A Example Let $(X,T,\Delta,\mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, W a Banach space, $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$, and $F:\mathcal{E}\to W$ an additive functional such that

for every $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $\|F(E)\|\le\epsilon$ for every $E\in\mathcal{R}\cap\mathcal{E}$.

Then there are Banach spaces U and V, a continuouous bilinear operator $\langle\,|\,\rangle:U\times V\to W$, and functions $f:X\to U$, $\nu:\mathcal{C}\to V$ such that $I_\nu(f)$ is defined and F is the Saks-Henstock indefinite integral of f with respect to ν. **P** Set $U=\mathbb{R}$, $V=W$, $\langle\alpha|w\rangle=\alpha w$ for $\alpha\in\mathbb{R}$ and $w\in W$, $f(x)=1$ for every $x\in X$, $\nu C=F(C)$ for every $C\in\mathcal{C}$. Then $S_{\boldsymbol{t}}(f,\nu)=F(H_{\boldsymbol{t}})$ for every $\boldsymbol{t}\in T$, so $I_\nu(f)=F(X)$ and F is the Saks-Henstock indefinite integral of f with respect to ν. **Q**

Remark Thus any non-trivial answer to the problem of this section (e.g., giving conditions for a Saks-Henstock indefinite integral to be countably additive) will demand hypotheses on the other elements U, V, $\langle\,|\,\rangle$, ν and f of the structure.

3B Example Let X be a set, $\mathcal{E}$ an algebra of subsets of X, W a Banach space and $F:\mathcal{E}\to W$ an additive function. Set $T=\{(x,C):x\in C\in\mathcal{E}\}$, $\Delta=\{X\times\mathcal{P}X\}$, $\mathfrak{R}=\{\{\emptyset\}\}$; then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{E}$, so we can apply the construction of 3A.

3C Example Let $([0,1],T,\mathcal{C},\mathfrak{R})$ be the Henstock tagged-partition structure allowing subdivisions, as in 1A(f-ii), and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Define $\nu:\mathcal{C}\to\mathbb{R}$ by saying that

$$\begin{aligned}\nu C&=1 \text{ if }]\gamma,1[\ \subseteq C \text{ for some } \gamma<1,\\&=0 \text{ otherwise.}\end{aligned}$$

If $f:[0,1]\to\mathbb{R}$ is any function, $I_\nu(f)=f(1)$ is defined for every $f:[0,1]\to\mathbb{R}$, and the Saks-Henstock indefinite integral F of f is defined by

$$\begin{aligned}F(E)&=f(1) \text{ if }]\gamma,1[\ \subseteq E \text{ for some } \gamma<1,\\&=0 \text{ otherwise.}\end{aligned}$$

On the other hand,

$$\begin{aligned}I_\nu(f\times\chi E)&=f(1) \text{ if } 1\in E,\\&=0 \text{ otherwise.}\end{aligned}$$

3D Example Let $X=\{x_0,x_1,x_2\}$ be a set with three members, $\mathcal{C}=\{X\}\cup\{\{x\}:x\in X\}$, $Q=\{(x,\{x\}):x\in X\}\cup\{(x_1,X)\}$, T the straightforward set of tagged partitions generated by Q, $\Delta=\{X\times\mathcal{P}X\}$, $\mathfrak{R}=\{\{\emptyset\}\}$. Then $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions witnessed by $\mathcal{C}$; the $\{\emptyset\}$-filling members of T are $\boldsymbol{t}_0=\{(x,\{x\}):x\in X\}$ and $\boldsymbol{t}_1=\{(x_1,X)\}$. Set $\nu C=\#(C)$ for $C\in\mathcal{C}$, $f(x_i)=i-1$ for $i\le 2$; then $S_{\boldsymbol{t}_0}(f,\nu)=S_{\boldsymbol{t}_1}(f,\nu)=0$ so $I_\nu(f)=0$. But $S_{\boldsymbol{t}_0}(|f|,\nu)=2$ and $S_{\boldsymbol{t}_1}(|f|,\nu)=0$ so $I_\nu(|f|)$ is undefined.

3E The Pfeffer integral In Fremlin 03, §484, I describe a special integral on Euclidean space which is the basis of a very general divergence theorem. Here I briefly recapitulate the definition to show that the same idea can be used to give a class of vector-valued integrals. Let $r\ge 1$ be an integer. For a Lebesgue measurable set $E\subseteq\mathbb{R}^r$ write $\operatorname{per}E$ for its perimeter, and let $\mathcal{C}$ be the algebra of subsets of $\mathbb{R}^r$ with locally finite perimeters (Fremlin 03, 474D). For $\alpha>0$ set

$$\mathcal{C}_\alpha=\{C:C\in\mathcal{C}\text{ is bounded, }\mu C\ge\alpha(\operatorname{diam}C)^r\},\ \alpha\operatorname{per}C\le(\operatorname{diam}C)^{r-1},$$

where μ is Lebesgue measure on $\mathbb{R}^r$, and

$$Q_\alpha=\{(x,C):C\in\mathcal{C}_\alpha,\ x\in\operatorname{cl}^*C\},$$

where cl^*C is the essential closure of C (Fremlin 03, 475B); let T_α be the straightforward set of tagged partitions generated by Q_α. Let $\mathcal{I}$ be the σ-ideal of subsets of $\mathbb{R}^r$ generated by the sets of finite $(r-1)$-dimensional Hausdorff measure, and set

$$\Delta=\{\delta\setminus(D\times\mathcal{P}\mathbb{R}^r):\delta\text{ is a neighbourhood gauge on }\mathbb{R}^r,\ D\in\mathcal{I}\}.$$

Then Δ is a countably full family of gauges on $\mathbb{R}^r$. Let $\mathrm{H}\subseteq\mathbb{R}^{\mathbb{N}}$ be the family of strictly positive sequences. For $\eta\in\mathrm{H}$, write $\mathcal{M}_\eta$ for the set of disjoint sequences $\langle E_i\rangle_{i\in\mathbb{N}}$ of Lebesgue measurable subsets of $\mathbb{R}^r$ such that $\mu E_i\le\eta(i)$ and $\operatorname{per}E_i\le 1$ for every $i\in\mathbb{N}$, and E_i is empty for all but finitely many i. For $\eta\in\mathrm{H}$ and $C\in\mathcal{C}$ set

$$\mathcal{R}_\eta=\{\textstyle\bigcup_{i\in\mathbb{N}}E_i:\langle E_i\rangle_{i\in\mathbb{N}}\in\mathcal{M}_\eta\}\subseteq\mathcal{C},\qquad \mathcal{R}_\eta^{(C)}=\{R:R\subseteq\mathbb{R}^r,\ R\cap C\in\mathcal{R}_\eta\};$$

set

$$\mathfrak{R}=\{R_\eta^{(C)}:C\in\mathcal{C}\text{ is bounded, }\eta\in\mathrm{H}\}.$$

Then there is an $\alpha^*>0$ such that $(\mathbb{R}^r,T_\alpha,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, whenever $0<\alpha\le\alpha^*$ (Fremlin 03, 484F).

Suppose now that we are given Banach spaces U, V and W, a continuous bilinear operator $\langle\,|\,\rangle:U\times V\to W$, a function $f:\mathbb{R}^r\to U$, a $\beta>0$ and a function $\nu:\mathcal{C}_\beta\to V$. For $0<\alpha\le\min(\alpha^*,\beta)$, set

$$I_\nu^{(\alpha)}(f)=\lim_{\boldsymbol{t}\to\mathcal{F}(T_\alpha,\Delta,\mathfrak{R})}S_{\boldsymbol{t}}(f,\nu)$$

if this is defined. It is easy to show that if $I_\nu^{(\alpha)}(f)$ is defined, and $F_\alpha:\mathcal{C}\to W$ is the corresponding Saks-Henstock indefinite integral, then for any $\alpha'\in[\alpha,\min(\alpha^*,\beta)]$ we also have the integral $I_\nu^{(\alpha')}(f)$, and the indefinite integrals $F_{\alpha'}$ and F_α coincide (Fremlin 03, 484H). We can therefore define a 'Pfeffer integral' by saying that

$$\not\!\!\int f\,d\nu=\lim_{\alpha\downarrow 0}I_\nu^{(\alpha)}(f)$$

whenever f and ν are such that the limit is defined, that is, there is a $\beta\in\,]0,\alpha^*]$ such that $\operatorname{dom}\nu\supseteq\mathcal{C}_\beta$ and $I_\nu^{(\alpha)}(f)$ is defined for every $\alpha\in\,]0,\beta]$; the common value of F_α for $\alpha\in\,]0,\beta]$ can now be called the Saks-Henstock indefinite integral of f with respect to ν.

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