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Baker's product measures

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I discuss variations on the measures described in Baker 04. My notation generally follows Fremlin 03.

1 Infinite products

1A Definition Let I be a set and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$. For a family $\langle \alpha_i \rangle_{i \in I}$ in $[0, \infty]$ write $\prod_{i \in I}^{(\mathcal{F})} \alpha_i$ for $\lim_{J \to \mathcal{F}} \prod_{i \in J} \alpha_i$ if this is defined in $[0, \infty]$.

For definiteness, count $\prod_{i \in \emptyset} \alpha_i$ as 1.

When $\mathcal{F}$ is precisely the filter $\mathcal{F}([I]^{<\omega}\uparrow)$ generated by $\{\{J : i \in J \in [I]^{<\omega} : i \in I\}$, write $\prod_{i \in I} \alpha_i$ for $\prod_{i \in I}^{(\mathcal{F})} \alpha_i$.

1B Remarks Suppose that I and $\mathcal{F}$ are as in 1A.

(a) If $\langle \alpha_i \rangle_{i \in I}$ is a family in $[0, \infty]$ and $\alpha = \prod_{i \in I}^{(\mathcal{F})} \alpha_i$ is defined and finite, then

either $\alpha = 0$, in which case $\lim_{J \to \mathcal{F}} \prod_{i \in J \setminus K} \alpha_i = 0$ for every $K \subseteq I$ such that $\prod_{i \in K} \alpha_i$ is defined, finite and not 0,

or $\lim_{K \to \mathcal{F}} \lim_{J \to \mathcal{F}} \prod_{i \in J \setminus K} \alpha_i = 1$.

(b) If $0 \le \alpha_i \le \beta_i$ for every $i \in I$ and $\beta = \prod_{i \in I}^{(\mathcal{F})} \beta_i$ is defined and finite, then $\alpha = \prod_{i \in I}^{(\mathcal{F})} \alpha_i$ is defined; if $\beta = 0$, then $\alpha = 0$; if $\beta > 0$, then $\frac{\alpha}{\beta} = \prod_{i \in I} \frac{\alpha_i}{\beta_i}$. So if $\alpha > 0$ then $\{i : \alpha_i < \beta_i\}$ must be countable.

1C Lemma Let I be a set, $\langle I_k \rangle_{k \in K}$ a partition of I, and $\langle \alpha_i \rangle_{i \in I}$ a family in $]0, \infty[$ such that $\alpha = \prod_{i \in I} \alpha_i$ is defined and finite. Then $\prod_{k \in K} \prod_{i \in I_k} \alpha_i$ is defined and equal to α.

proof The point is that $\prod_{i \in I} \max(1, \alpha_i)$ must be finite, so it will be enough to deal separately with the cases (i) $\alpha_i \ge 1$ for every i (ii) $\alpha_i \le 1$ for every i.

2 The basic construction

2A Definitions Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of measure spaces, and $\mathcal{F}$ a filter on the set $[I]^{<\omega}$ of finite subsets of I containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$. Set $X = \prod_{i \in I} X_i$, and let $\mathcal{C} \subseteq \mathcal{P}X$ be the family of sets of the form $C = \prod_{i \in I} E_i$ where $E_i \in \Sigma_i$ for every $i \in I$ and $\tau C = \prod_{i \in I}^{(\mathcal{F})} \mu_i E_i$ is defined in $[0, \infty[$.

Note that τC is well-defined because $\prod_{i \in I} E_i$ can be equal to $\prod_{i \in I} F_i$ only if either some E_i is empty or if $E_i = F_i$ for every i. If there is any i such that $\mu_i E_i = \infty$ there must be a j such that $\mu_j E_j = 0$, and in this case $\tau C = 0$.

Let $\mathcal{D} \subseteq \mathcal{P}X$ be the family of sets of the form $D = \bigcup_{i \in I} \{x : x \in X,\, x(i) \in E_i\}$ where $\mu_i E_i = 0$ for every $i \in I$; set $\tau D = 0$ for every $D \in \mathcal{D}$. It is easy to check that in the exceptional case that there is a $D \in \mathcal{C} \cap \mathcal{D}$ then $\tau D = 0$ on either definition.

2B Lemma Suppose that I, $\mathcal{F}$, $\mathcal{C}$, $\mathcal{D}$ and τ are as in 2A. If $C \in \mathcal{C} \cup \mathcal{D}$, $\langle C_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{C} \cup \mathcal{D}$ and $C \subseteq \bigcup_{n \in \mathbb{N}} C_n$, then $\tau C \le \sum_{n=0}^{\infty} \tau C_n$.

proof (a) It is enough to consider the case $\tau C > 0$, so that $C \in \mathcal{C}$; express C as $\prod_{i \in I} E_i$. Set $L_0 = \{n : C_n \in \mathcal{D}\}$; for $n \in L_0$, let $\langle E_{ni} \rangle_{i \in I}$ be such that $C_n = \bigcup_{i \in I} \{x : x(i) \in E_{ni}\}$ and $\mu_i E_{ni} = 0$ for every i; for $n \in \mathbb{N} \setminus L_0$, let $\langle E_{ni} \rangle_{i \in I}$ be such that $C_n = \prod_{i \in I} E_{ni}$ and $E_{ni} \in \Sigma_i$ for every i. Set

$$L_1 = \{n : n \in \mathbb{N} \setminus L_0,\, \tau C_n > 0\},$$

$$L_2 = \{n : n \in \mathbb{N} \setminus (L_0 \cup L_1),\, \mu_i E_{ni} > 0 \text{ for every } i\},$$

$$E'_i = E_i \setminus \bigcup\{E_{ni} : n \in \mathbb{N},\, \mu_i E_{ni} = 0\} \text{ for each } i \in I, \quad C' = \prod_{i\in I} E'_i;$$

then $C' \cap C_n = \emptyset$ for every $n \in \mathbb{N} \setminus (L_1 \cup L_2)$, so $C' \subseteq \bigcup_{n\in L_1\cup L_2} C_n$, while $\tau C' = \tau C$.

(b) ? Suppose, if possible, that $\tau C > \sum_{n=0}^{\infty} \tau C_n$. Then there is a $\gamma < 1$ such that $\sum_{n=0}^{\infty} \tau C_n < \gamma\tau C'$. Set $\beta = \frac{1}{2}\min(1, \tau C')$. Let $\langle\gamma_m\rangle_{m\in\mathbb{N}}$ be a strictly increasing sequence, with limit γ, such that $\sum_{n=0}^{\infty} \tau C_n \le \gamma_0 \tau C'$. By 1Ba, there is a non-decreasing sequence $\langle J_m\rangle_{m\in\mathbb{N}}$ in $[I]^{<\omega}$, starting from $J_0 = \emptyset$, such that

$$\lim_{J\to\mathcal{F}} \prod_{i\in J\setminus J_m} \mu_i E'_i > \beta \text{ for every } m \in \mathbb{N},$$

$$\prod_{i\in J_{m+1}\setminus J_m} \mu_i E'_i \ge \beta \text{ for every } m \in \mathbb{N},$$

$$\lim_{J\to\mathcal{F}} \prod_{i\in J\setminus J_m} \mu_i E_{ni} \ge 1 - 2^{-m} \text{ whenever } m \ge 1,\, n \in L_1 \text{ and } n \le m,$$

$$\lim_{J\to\mathcal{F}} \prod_{i\in J\setminus J_m} \mu_i E'_i \le 1 + 2^{-m} \text{ whenever } m \ge 1,$$

$$\prod_{i\in J_{m+1}\setminus J_m} \mu_i E_{mi} \le \beta\Bigl(1 - \frac{\gamma_m}{\gamma_{m+1}}\Bigr) \text{ whenever } m \in L_2.$$

For $m \in \mathbb{N}$ and $z \in \prod_{i\in J_m} X_i$ set

$$\alpha_m = \lim_{J\to\mathcal{F}} \prod_{i\in J\setminus J_m} \mu_i E'_i,$$

$$\alpha_{mn} = \lim_{J\to\mathcal{F}} \prod_{i\in J\setminus J_m} \mu_i E_{ni} \text{ for } n \in L_1 \cup L_2,$$

$$K_m(z) = \{n : n \in L_1 \cup L_2,\, z(i) \in E_{ni} \text{ for every } i \in J_m\},$$

$$g_m(z) = \sum_{n\in K_m(z)} \alpha_{mn}.$$

We are supposing that $g_m(\emptyset) \le \gamma_0\alpha_0$. Now, for each $m \in \mathbb{N}$, write λ_m for the c.l.d. product measure on $Z_m = \prod_{i\in J_{m+1}\setminus J_m} X_i$, and set

$$\begin{aligned} H_m &= \prod_{i\in J_{m+1}\setminus J_m} E_{mi} \text{ if } m \in L_2, \\ &= \emptyset \text{ otherwise,} \end{aligned}$$

$$F_m = \prod_{i\in J_{m+1}\setminus J_m} E'_i, \quad F'_m = F_m \setminus H_m, \quad F_{mn} = \prod_{i\in J_{m+1}\setminus J_m} E_{ni}$$

for $n \in \mathbb{N}$. Then we shall always have

$$\lambda_m H_m \le \beta\Bigl(1 - \frac{\gamma_m}{\gamma_{m+1}}\Bigr) \le \Bigl(1 - \frac{\gamma_m}{\gamma_{m+1}}\Bigr)\lambda_m F_m,$$

$$\gamma_m \lambda_m F_m \le \gamma_{m+1}(\lambda_m F_m - \lambda_m H_m) \le \gamma_{m+1}\lambda_m F'_m.$$

In this case, for $m \in \mathbb{N}$ and $z \in \prod_{i\in J_m} X_i$,

$$\begin{aligned} g_m(z) &= \sum_{n\in K_m(z)} \alpha_{mn} = \sum_{n\in K_m(z)} \lambda_0 F_{mn}\alpha_{m+1,n} \\ &= \int_{Z_m} \sum_{n\in K_m(z)} \alpha_{m+1,n}\chi F_{mn}(w)\lambda_m(dw) \\ &= \int_{Z_m} \sum_{n\in K_{m+1}(z^\frown w)} \alpha_{m+1,n}\lambda_m(dw) = \int_{Z_m} g_{m+1}(z^\frown w)\lambda_m(dw). \end{aligned}$$

So if $z \in \prod_{i\in J_m} X_i$ is such that

$$g_m(z) \le \gamma_m\alpha_m = \gamma_m\lambda_m F_m \cdot \alpha_{m+1} \le \gamma_{m+1}\lambda_m F'_m \cdot \alpha_{m+1},$$

there must be a $w \in F'_m$ such that

$$g_{m+1}(z^\frown w) \le \gamma_{m+1}\alpha_{m+1}.$$

We can therefore choose $\langle z_m\rangle_{m\in\mathbb{N}}$, $\langle w_m\rangle_{m\in\mathbb{N}}$ inductively such that

$$z_m\in\prod_{i\in J_m}E'_i,\quad g_m(z_m)\le\gamma_m\alpha_m,\quad z_{m+1}=z_m{}^\frown w_m,\quad w_m\notin H_m$$

for each m, starting with $z_0=\emptyset$.

At the end of this construction, take $x\in\prod_{i\in I}E'_i$ such that $x\restriction J_m=z_m$ for every m. Then there is an $n\in L_1\cup L_2$ such that $x\in\prod_{i\in I}E_{ni}$. Note that n cannot belong to L_2, because $w_n=x\restriction J_{n+1}\setminus J_n$ never belongs to H_n; so $n\in L_1$. We therefore have

$$\lim_{m\to\infty}\alpha_m\le 1,$$

$$\lim_{m\to\infty}\alpha_{mn}\ge 1,$$

$$\alpha_{mn}\le\gamma\alpha_m$$

for every m, which is absurd. **X**

2C Definition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J:i\in J\in[I]^{<\omega}\}$ for every $i\in I$. Defining X, $\mathcal{C}$, $\mathcal{D}$ and τ as in 2A, define $\theta:\mathcal{P}X\to[0,\infty]$ by setting

$$\theta A=\inf\{\sum_{n=0}^{\infty}\tau C_n:C_n\in\mathcal{C}\cup\mathcal{D}\text{ for every }n\in\mathbb{N},\ A\subseteq\bigcup_{n\in\mathbb{N}}C_n\},$$

counting $\inf\emptyset$ as 0. Then θ is an outer measure. Let λ_0 be the measure defined from θ by Carathéodory's method, and λ the c.l.d. version of λ; I will call λ the **Baker $\mathcal{F}$-product** of $\langle\mu_i\rangle_{i\in I}$, or just the **Baker product** if $\mathcal{F}$ is the filter generated by $\{\{J:i\in J\in[I]^{<\omega}:i\in I\}$.

2D Theorem Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J:i\in J\in[I]^{<\omega}\}$ for every $i\in I$, and λ the Baker $\mathcal{F}$-product of $\langle\mu_i\rangle_{i\in I}$ on $X=\prod_{i\in I}X_i$; write Λ for the domain of λ.

(a) $\widehat{\bigotimes}_{i\in I}\Sigma_i\subseteq\Lambda$.

(b) If $A_i\subseteq X_i$ is μ_i-conegligible for every i, then $\prod_{i\in I}A_i$ is λ-conegligible.

(c) If $\langle E_i\rangle_{i\in I}\in\prod_{i\in I}\Sigma_i$, then $\prod_{i\in I}E_i\in\Lambda$; if $\alpha=\prod^{(\mathcal{F})}_{i\in I}\mu_iE_i$ is defined and finite, then $\lambda(\prod_{i\in I}E_i)=\alpha$.

(d) Let $\mathcal{C}$ be the family of subsets of X expressible as $\prod_{i\in I}E_i$ where $E_i\in\Sigma_i$ for every $i\in I$ and $\prod^{(\mathcal{F})}_{i\in I}\mu_iE_i$ is defined and finite. Let $\mathcal{K}$ be the set of countable intersections of finite unions of members of $\mathcal{C}$. Then λ is inner regular with respect to $\mathcal{K}$.

proof Let $\mathcal{D}$, τ, θ and λ_0 be as in 1C-1D. By 2B, $\theta C=\tau C$ for every $C\in\mathcal{C}\cup\mathcal{D}$. Let Λ_0 be the domain of λ_0.

(a) If $j\in I$ and $E\in\Sigma_j$, λ_0 measures $W=\{x:x\in X,\ x(j)\in E\}$. **P** If $C\in\mathcal{C}$, express C as $\prod_{i\in I}E_i$; set

$$\begin{aligned}E'_i&=E_j\cap E\text{ if }i=j,\\&=E_i\text{ for other }i\in I,\\E''_i&=E_j\setminus E\text{ if }i=j,\\&=E_i\text{ for other }i\in I.\end{aligned}$$

Then $C\cap W=\prod_{i\in I}E'_i$ and $C\setminus W=\prod_{i\in I}E''_i$, so $\tau C=\tau(C\cap W)+\tau(C\setminus W)$. It follows that $\theta A=\theta(A\cap W)+\theta(A\setminus W)$ for every $A\subseteq X$, so that λ_0 measures W. **Q**

Consequently $\widehat{\bigotimes}_{i\in I}\Sigma_i\subseteq\Lambda_0\subseteq\Lambda$.

(b) For each $i\in I$ let $E_i\subseteq A_i$ be a measurable conegligible set; then $D=X\setminus\prod_{i\in I}E_i$ belongs to $\mathcal{D}$, so $\lambda D=\lambda_0D=\theta D=\tau D=0$ and $\prod_{i\in I}A_i\supseteq X\setminus D$ is λ_0-conegligible and λ-conegligible.

(c) Let $\mathcal{C}^*$ be the set of all products $\prod_{i\in I}E_i$ such that $E_i\in\Sigma_i$ for every $i\in I$. If $C\in\mathcal{C}$ and $C^*\in\mathcal{C}^*$ then there is a $W\in\Lambda_0$ such that $C\cap C^*=C\cap W$. **P** If $I=\emptyset$ this is trivial; suppose otherwise. Express C^* as $\prod_{i\in I}E_i$ and C as $\prod_{i\in I}F_i$, so that $C\cap C^*=\prod_{i\in I}(E_i\cap F_i)$; set $\alpha_i=\mu_i(E_i\cap F_i)$, $\beta_i=\mu_iF_i$ for each i. If $\tau(C\cap C^*)=0$ then $C\cap C^*$ itself belongs to Λ_0. Otherwise, as remarked in 1Bb, $K=\{i:\alpha_i\ne\beta_i\}$ is countable. Let $\langle i_n\rangle_{n\in\mathbb{N}}$ run over a subset K' of I including K. For $n\in\mathbb{N}$ set

$$W_n=\{x:x\in X,\ x(i_m)\in E_{i_m}\text{ for }m<n,\ x(i_n)\notin E_{i_n}\};$$

set

$$D = \bigcup_{i\in I\setminus K'}\{x : x(i) \in F_i \setminus E_i\}.$$

Then

$$C \setminus C^* = \bigcup_{n\in\mathbb{N}}(C \cap W_n) \cup (C \cap (D \setminus C^*)) = C \cap W'$$

where $W' = (D \setminus C^*) \cup \bigcup_{n\in\mathbb{N}} W_n$ belongs to Λ_0. Now $C \cap C^* = C \cap W$ where $W = X \setminus W'$ belongs to Λ_0. **Q**

It follows that $\mathcal{C}^* \subseteq \Lambda_0$ and $\lambda_0 C = \tau C$ for every $C \in \mathcal{C}$; consequently $\lambda C = \tau C$ for every $C \in \mathcal{C}$.

(d)(i) Note first that $\mathcal{K}$ is closed under finite unions and countable intersections.

(ii) If C, $C' \in \mathcal{C}$ and $\epsilon > 0$, then there is a $K \in \mathcal{K}$ such that $K \subseteq C \setminus C'$ and $\lambda K \geq \lambda(C \setminus C') - \epsilon$. **P** Express C, C' as $\prod_{i\in I} E_i$ and $\prod_{i\in I} F_i$ respectively. Set $\beta_i = \mu_i E_i$, $\alpha_i = \mu_i(E_i \cap F_i)$ for each i, $\beta = \prod^{(\mathcal{F})}_{i\in I} \beta_i = \lambda C$. If $\beta = 0$ we can stop. Otherwise, $\lambda(C \setminus C') = \beta(1 - \prod_{i\in I} \frac{\alpha_i}{\beta_i})$ and there is a finite $L \subseteq I$ such that $\beta(1 - \prod_{j\in L} \frac{\alpha_j}{\beta_j}) \geq \beta - \epsilon$. For $j \in L$ set

$$C_j = \{x : x \in C,\, x(j) \notin F_j\};$$

then

$$\lambda(C \setminus \bigcup_{j\in L} C_j) = \beta\prod_{j\in L} \frac{\alpha_j}{\beta_j}$$

so $\lambda(\bigcup_{j\in L} C_j) \geq \beta - \epsilon$. Set $K = \bigcup_{j\in L} C_j$. **Q**

Note that if $C \in \mathcal{C}$ and $D \in \mathcal{D}$ then there is a $C_0 \in \mathcal{C}$ such that $C_0 \subseteq C \setminus D$ and $\lambda C_0 = \lambda C$.

(iii) Suppose that $W \in \Lambda$ and $\lambda W > 0$. Then there is a $K \in \mathcal{K}$ such that $K \subseteq W$ and $\lambda K > 0$. **P** There is a $W_1 \in \Lambda_0$ such that $W_1 \subseteq W$ and $0 < \lambda_0 W_1 < \infty$. In this case, θW_1 is finite, so there is a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{C} \cup \mathcal{D}$ such that $W_1 \subseteq \bigcup_{n\in\mathbb{N}} C_n$. Now there is an $n \in \mathbb{N}$ such that $W_2 = \lambda_0(W_1 \cap C_n) > 0$. Of course C_n must belong to $\mathcal{C}$. Let $\langle C'_m\rangle_{m\in\mathbb{N}}$ be a sequence in $\mathcal{C} \cup \mathcal{D}$ such that $C_n \setminus W_2 \subseteq \bigcup_{m\in\mathbb{N}} C'_m$ and $\sum_{m=0}^{\infty} \lambda_0 C'_n < \lambda_0 C_n$; set $W_3 = \bigcap_{m\in\mathbb{N}} C_m \setminus C'_m$, so that $\lambda W_3 > 0$.

For each $m \in \mathbb{N}$ we have a $K_m \in \mathcal{K}$ such that $K_m \subseteq C_n \setminus C'_m$ and $\lambda K_m \geq \lambda(C_n \setminus C'_m) - 2^{-m-2}\lambda W_3$. Now $K = \bigcap_{m\in\mathbb{N}} K_m \subseteq W_3 \subseteq W$, $K \in \mathcal{K}$ and $\lambda K > 0$. **Q**

(iv) As $\mathcal{K}$ is closed under finite unions, this is enough to show that λ is inner regular with respect to $\mathcal{K}$.

2E Two special cases: Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$, $\mathcal{F}$, X and λ be as in 2D.

(a) If I is finite, then λ is the c.l.d. product measure on X.

(b) If I is countable and $\beta = \prod^{(\mathcal{F})}_{i\in I} \mu_i X_i$ is defined, finite and not zero, set $\mu'_i = \frac{1}{\mu_i X_i}\mu_i$ for each i, and let λ' be the product probability measure on X; then $\lambda = \beta\lambda'$.

proof In both parts, because I is countable, we see that the outer measure θ of 2C can be defined from $\mathcal{C}$ alone, since every member of $\mathcal{D}$ is included in the union of countably many sets $C \in \mathcal{C}$ with $\tau C = 0$.

(a) If I is finite, then, looking at the construction in Fremlin 01, §251, we see that λ_0, as defined in 2C, is just the primitive product measure, so its c.l.d. version λ is the c.l.d. product measure.

(b) A direct calculation, using 1B, shows that $\beta\prod_{i\in I} \mu'_i E_i = \theta C$ whenever $E_i \in \Sigma_i$ for every i and $C = \prod_{i\in I} E_i$. So if we write θ' for the outer measure described in Fremlin 01, 251A-251B, we shall have $\theta = \beta\theta'$ and $\lambda_0 = \beta\lambda'$. Since λ_0 is totally finite, we now have $\lambda = \beta\lambda'$.

2F Subspaces: Proposition Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$. For each $i \in I$ take a $Y_i \in \Sigma_i$ and write T_i, ν_i for the subspace σ-algebra and measure on Y_i. Let λ be the Baker $\mathcal{F}$-product measure on $X = \prod_{i\in I} X_i$. Then the Baker $\mathcal{F}$-product measure of $\langle\nu_i\rangle_{i\in I}$ is the subspace measure λ_Y induced on $Y = \prod_{\in I} Y_i$ by λ.

proof Defining $\mathcal{C}$, τ, θ and λ_0 as in 2C, and $\mathcal{C}'$, τ', θ' and λ'_0 by the same process applied to $\langle(Y_i, \mathrm{T}_i, \nu_i)\rangle_{i\in I}$, we see that

$\mathcal{C}' = \{C \cap Y : C \in \mathcal{C}\}$,
$\tau' C' = \min\{\tau C : C \in \mathcal{C},\, C' = C \cap Y\}$ for $C' \in \mathcal{C}'$,
$\theta' = \theta \restriction \mathcal{P}Y$,
$Y \in \operatorname{dom}\lambda_0$ (see the proof of 2Dc).

It follows that λ_0' is the subspace measure on Y induced by λ_0 (Fremlin 01, 214H(b-ii)), and it is now easy to check that λ_Y is the c.l.d. version of λ_0', so is the Baker $\mathcal{F}$-product of $\langle\nu_i\rangle_{i\in I}$.

3 Associative law

3B Theorem (cf. Baker 04, Theorem II) Let I be a set, $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ a family of measure spaces, and (X, Λ, λ) the Baker product of $\langle\mu_i\rangle_{i\in I}$. Let (I_0, I_1) be a partition of I; for each $k \le 1$, let λ_{0k} be the Baker product measure on $Y_k = \prod_{i\in I_k} X_i$. Let ν be the c.l.d. product measure of λ_0 and λ_1 on $Z = Y_0 \times Y_1$. Let $\phi : X \to Z$ be the natural bijection. Then ϕ is an isomorphism between λ and ν.

proof (a) Let $\mathcal{C}$ be the family of subsets of X expressible in the form $\prod_{i\in I} E_i$ where $E_i \in \Sigma_i$ for every $i \in I$ and $\prod_{i\in I} \mu_i E_i$ is finite, and $\mathcal{K}$ the family of countable intersections of finite unions of members of $\mathcal{C}$, as in 2Dd, so that λ is inner regular with respect to $\mathcal{K}$ Now $\nu\phi[K] = \lambda K$ for every $K \in \mathcal{K}$. **P** If $K = \emptyset$ this is trivial. If $K \in \mathcal{C} \setminus \{\emptyset\}$, express it as $\prod_{i\in I} E_i$ where $\prod_{i\in I} \mu_i E_i$ is finite; then

$$\lambda K = \lambda_0(\textstyle\prod_{i\in I_0} E_i) \cdot \lambda_1(\prod_{i\in I_1} E_i) = \nu\phi[K].$$

As $\mathcal{C}$ is closed under finite intersections, $\nu\phi[K] = \lambda K$ for every $K \in \mathcal{K}$ (see Fremlin 01, 136Xc[1]). **Q**

(b) Now, for $k = 0$ and $k = 1$, let $\mathcal{C}_k$ be the family of subsets of Z_k expressible in the form $\prod_{i\in I} E_i$ where $E_i \in \Sigma_i$ for every $i \in I_k$ and $\prod_{i\in I_k} \mu_i E_i$ is finite, and $\mathcal{K}_k$ the family of countable intersections of finite unions of members of $\mathcal{C}_k$. Then λ_{0k} is inner regular with respect to $\mathcal{K}_k$. Writing $\mathcal{L} = \{\phi[K] : K \in \mathcal{K}\}$, we see that $\mathcal{L}$ is closed under finite unions and countable intersections and contains $C_0 \times C_1$ whenever $C_k \in \mathcal{C}_k$ for both k. It therefore contains $K_0 \times K_1$ whenever $K_k \in \mathcal{K}_k$ for both k, and ν must be inner regular with respect to $\mathcal{L}$, by Fremlin 03, 412R. By Fremlin 03, 412L, ϕ must be a measure space isomorphism.

4 Topological Baker products

4A Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Radon measure spaces, and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$. Let $\mathcal{C}_k$ be the family of sets $C \subseteq X$ expressible in the form $C = \prod_{i\in I} E_i$ where $E_i \subseteq X_i$ is compact for every i and $\tau C = \prod_{i\in I}^{(\mathcal{F})} \mu_i E_i$ is defined and finite. Then there is a unique complete locally determined topological measure $\tilde\lambda$ on $X = \prod_{i\in I} X_i$, inner regular with respect to the compact sets, such that

$\tilde\lambda C = \tau C$ for every $C \in \mathcal{C}_k$;
$\tilde\lambda A = 0$ whenever $A \subseteq X$ is such that $\tilde\lambda(A \cap C) = 0$ for every $C \in \mathcal{C}_k$.

proof (a) Define $\mathcal{C} \subseteq \mathcal{P}X$ and $\tau : \mathcal{C} \to [0, \infty[$ as in 2A. Let $\mathcal{C}'_{\mathrm{kss}} \subseteq \mathcal{C}$ be the family of sets expressible in the form $C = \prod_{i\in I} \mu_i E_i$ where $E_i \in \Sigma_i$ for every i, E_i is compact and self-supporting (Fremlin 03, 411Na) for all but countably many i, and $\tau C = \prod_{i\in I}^{(\mathcal{F})} E_i$ is defined and finite. Set

$$\theta_k A = \inf\{\textstyle\sum_{n=0}^{\infty} \tau C_n : C_n \in \mathcal{C}'_{\mathrm{kss}} \text{ for every } n \in \mathbb{N},\ C \subseteq \bigcup_{n\in\mathbb{N}} C_n\}$$

for $A \subseteq X$ (cf. 2C); as before, interpret $\inf\emptyset$ as ∞, so that θ_k is an outer measure. By 2B, $\theta_k C = \tau C$ for every $C \in \mathcal{C}'_{\mathrm{kss}}$. Let λ_{0k} be the measure on X defined from θ_k by Carathéodory's method, and λ_k its c.l.d. version; write Λ_{0k} and Λ_k for their respective domains.

(b) If $j \in I$ and $E \in \Sigma_j$, λ_{0k} measures $W = \{x : x \in X,\ x(j) \in E\}$. **P** Argue as in 2Da. If $C \in \mathcal{C}'_{\mathrm{kss}}$, express it as $\prod_{i\in I} E_i$; set

$$\begin{aligned} E_i' &= E_j \cap E \text{ if } i = j,\\ &= E_i \text{ for other } i \in I,\\ E_i'' &= E_j \setminus E \text{ if } i = j,\\ &= E_i \text{ for other } i \in I. \end{aligned}$$

[1]Later editions only.

Then $C \cap W = \prod_{i\in I} E'_i$ and $C \setminus W = \prod_{i\in I} E''_i$ both belong to $\mathcal{C}'_{\mathrm{kss}}$, and $\tau C = \tau(C \cap W) + \tau(C \setminus W)$. It follows that $\theta_k A = \theta_k(A \cap W) + \theta_k(A \setminus W)$ for every $A \subseteq X$, so that $W \in \Lambda_{0k}$. **Q**

Consequently $\widehat{\bigotimes}_{i\in I}\Sigma_i \subseteq \Lambda_{0k}$.

(c) Let $\mathcal{C}'_c$ be the family of sets expressible in the form $\prod_{i\in I} E_i$ where $E_i \in \Sigma_i$ for every $i \in I$ and E_i is closed for all but countably many i.

(i) If $C^* \in \mathcal{C}'_c$ and $C \in \mathcal{C}'_{\mathrm{kss}}$, then $\theta_k(C\cap C^*)+\theta_k(C\setminus C^*) \le \tau C$. **P** If $\tau C = 0$, this is trivial. Otherwise, express C^* as $\prod_{i\in I} E_i$ and C as $\prod_{i\in I} F_i$ where E_i is closed and F_i is compact and self-supporting for all but countably many i. Set

$$L = \{i : E_i \text{ is not closed}\} \cup \{i : \mu_i(E_i \cap F_i) < \mu_i F_i\} \\ \cup \{i : F_i \text{ is not a compact self-supporting set}\}.$$

(α) If L is uncountable, then there must be a $\delta > 0$ such that $\{i : \mu_i(E_i\cap F_i) \le (1-\delta)\mu_i F_i\}$ is infinite, and a countable $L' \subseteq I$ such that $\prod_{i\in L'} \frac{\mu_i(E_i\cap F_i)}{\mu_i F_i} = 0$. Setting $F'_i = E_i\cap F_i$ for $i \in L'$, F_i for $i \in I\setminus L'$, we see that $C' = \prod_{i\in I} F'_i$ belongs to $\mathcal{C}'_{\mathrm{kss}}$ and $\tau C' = 0$. As $C' \supseteq C\cap C^*$, $\theta_k(C\cap C^*) = 0$ and $\theta_k(C\cap C^*)+\theta_k(C\setminus C^*) \le \tau C$.
(β) If L is countable, set $W = \{x : x \in X,\ x(i) \in E_i \text{ for every } i \in L\}$; then $W \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ and $C\cap C^* = C\cap W$, so

$$\theta_k(C \cap C^*) + \theta_k(C \setminus C^*) = \theta_k(C \cap W) + \theta_k(C \setminus W) = \theta_k C = \tau C. \textbf{ Q}$$

(ii) It follows that $\mathcal{C}'_c \subseteq \Lambda_{0k}$; in particular, $\mathcal{C}'_{\mathrm{kss}} \subseteq \Lambda_{0k}$, so $\lambda_{0k}C = \theta_k C = \tau C$ for every $C \in \mathcal{C}'_{\mathrm{kss}}$.

(iii) In fact $\lambda_{0k}(C \cap C^*) = \tau(C \cap C^*)$ whenever $C \in \mathcal{C}'_{\mathrm{kss}}$ and $C^* \in \mathcal{C}'_c$. **P** Express C, C^* as $\prod_{i\in I} F_i$, $\prod_{i\in I} E_i$ respectively, as in (i) above. If $\tau C = 0$ then of course $\tau(C \cap C^*) = \lambda_{0k}(C \cap C^*) = 0$. Otherwise, again set

$$L = \{i : E_i \text{ is not closed}\} \cup \{i : \mu_i(E_i \cap F_i) < \mu_i F_i\} \\ \cup \{i : F_i \text{ is not a compact self-supporting set}\}.$$

As in (i), if L is uncountable then $\tau(C \cap C^*) = \lambda_{0k}(C \cap C^*) = 0$. But if L is countable, then $C \cap C^* \in \mathcal{C}'_{\mathrm{kss}}$, so surely $\tau(C \cap C^*) = \lambda_{0k}(C \cap C^*)$. **Q**

(iv) We find also that $\lambda_k C = \tau C$ for every $C \in \mathcal{C}_k$. **P** We know from (ii) that C belongs to Λ_{0k} so is measured by λ_k. Express C as $\prod_{i\in I} E_i$ where every E_i is compact. For each i, let $\hat{E}_i \subseteq E_i$ be a self-supporting compact set with the same measure as E_i; then $\hat{C} = \prod_{i\in I} \hat{E}_i$ belongs to $\mathcal{C}'_{\mathrm{kss}}$ and is measured by λ_{0k}. We also have

$$\lambda_{0k}(C' \cap \hat{C}) = \tau(C' \cap \hat{C}) = \tau(C' \cap C) = \lambda_{0k}(C' \cap C), \quad \lambda_{0k}(C' \cap C \setminus \hat{C}) = 0$$

for every $C \in \mathcal{C}'_{\mathrm{kss}}$, so $\lambda_k(C \setminus \hat{C}) = 0$ and

$$\lambda_k C = \lambda_k \hat{C} = \tau \hat{C} = \tau C. \textbf{ Q}$$

(d) Let $\mathcal{C}_{\mathrm{kss}} \subseteq \mathcal{C}'_{\mathrm{kss}}$ be the family of subsets of X of the form $\prod_{i\in I} E_i$ where E_i is a compact self-supporting subset of X_i for every i and $\prod^{(\mathcal{F})}_{i\in I} \mu_i E_i$ is defined and finite. Let $\mathcal{K}_k$ be the family of countable intersections of finite unions of members of $\mathcal{C}_{\mathrm{kss}}$. Then $\mathcal{K}_k$ is closed under finite unions and countable intersections, and every member of $\mathcal{K}_k$ is compact.

If C, $C' \in \mathcal{C}'_{\mathrm{kss}}$ and $\epsilon > 0$, there is a $K \subseteq C \setminus C'$ such that $K \in \mathcal{K}_k$ and $\lambda_{0k}K \ge \lambda_{0k}(C \setminus C') - 3\epsilon$. **P** Express C, C' as $\prod_{i\in I} E_i$, $\prod_{i\in I} E'_i$ respectively; set $\beta_i = \mu_i E_i$ for each i, $\beta = \tau C$. If $\beta = 0$ we can take $K = \emptyset$ and stop. Otherwise, since $L_0 = \{i : E_i \text{ is not a compact self-supporting set}\}$ is countable, we can find $\langle\beta'_i\rangle_{i\in L_0}$ such that $0 < \beta'_i < \beta_i$ for $i \in L_0$ and $\beta \cdot \prod_{i\in L_0} \frac{\beta'_i}{\beta_i} \ge \beta - \epsilon$. For $i \in L_0$ choose a compact self-supporting $F_i \subseteq E_i$ such that $\mu_i F_i \ge \beta'_i$; for $i \in I \setminus L_0$ set $F_i = E_i$, and set $\hat{C} = \prod_{i\in I} F_i \in \mathcal{C}_{\mathrm{kss}}$. For every $i \in I$ set $\gamma_i = \mu_i F_i$, and $\gamma = \tau\hat{C} = \prod^{(\mathcal{F})}_{i\in I} \gamma_i$. Then $\lambda_{0k}\hat{C} = \gamma \ge \beta - \epsilon$ and $\lambda_{0k}(C \setminus \hat{C}) \le \epsilon$.

Next, for $i \in I$, set $\alpha_i = \mu_i(F_i \cap E'_i)$. By (c-iii) above,

$$\lambda_{0k}(\hat{C}\cap C') = \prod^{(\mathcal{F})}_{i\in I}\alpha_i = \gamma\cdot\prod_{i\in I}\frac{\alpha_i}{\gamma_i},$$

and

$$\gamma(1-\prod_{i\in I}\frac{\alpha_i}{\gamma_i}) = \lambda_{0k}(\hat{C}\setminus C') \ge \lambda_{0k}(C\setminus C')-\epsilon.$$

There is now a finite $L\subseteq I$ such that $\gamma(1-\prod_{j\in L}\frac{\alpha_j}{\gamma_j}) \ge \lambda_{0k}(C\setminus C')-2\epsilon$. Let $\langle\alpha'_j\rangle_{j\in L}$ be such that $\gamma(1-\prod_{j\in L}\frac{\alpha'_j}{\gamma_j}) \ge \lambda_{0k}(C\setminus C')-3\epsilon$ and $\alpha'_j>\alpha_j$ for each $j\in L$. For $j\in L$ take a compact self-supporting set $H_j\subseteq F_j\setminus E'_j$ such that $\mu_j H_j\ge\gamma_j-\alpha'_j$, and set

$$C_j = \{x : x\in\hat{C},\, x(j)\in H_j\};$$

then $C_j\in\mathcal{C}_{\mathrm{kss}}$ for each j, $\hat{C}\setminus\bigcup_{j\in L}C_j\in\mathcal{C}'_{\mathrm{kss}}$, and

$$\lambda_{0k}(\hat{C}\setminus\bigcup_{j\in L}C_j) = \tau(\{x : x\in\hat{C},\, x(j)\notin H_j \text{ for every } j\in L\})$$

$$= \gamma\cdot\prod_{j\in L}\frac{\mu_j(F_j\setminus H_j)}{\mu_j F_j} \le \gamma\cdot\prod_{j\in L}\frac{\alpha'_j}{\gamma_j}$$

so $K=\bigcup_{j\in L}C_j$ belongs to $\mathcal{K}_k$ and

$$\lambda_{0k}K = \lambda_{0k}(\bigcup_{j\in L}C_j) \ge \gamma(1-\prod_{j\in L}\frac{\alpha'_j}{\gamma_j}) \ge \lambda_{0k}(C\setminus C')-3\epsilon$$

as required. **Q**

Now we can use the argument in (d-iii) and (d-iv) of the proof of 2D to see that λ_k is inner regular with respect to $\mathcal{K}_k$.

(e) Let $\mathcal{K}^f$ be the family of those compact sets $K\subseteq X$ included in some member of $\mathcal{K}_k$. By Fremlin 03, 413O, there is a complete locally determined measure $\tilde\lambda$ on X, extending λ_k, and inner regular with respect to $\mathcal{K}^f$. By Fremlin 03, 412Ja, $\tilde\lambda$ measures every closed set and is a topological measure; of course it is inner regular with respect to the compact sets. By (c-iv), $\tilde\lambda C=\tau C$ for every $C\in\mathcal{C}_k$.

If $A\subseteq X$ is such that $\tilde\lambda(A\cap C)=0$ for every $C\in\mathcal{C}_k$, then of course $\tilde\lambda(A\cap C)=0$ for every $C\in\mathcal{C}_{\mathrm{kss}}$, and $\tilde\lambda(A\cap K)=0$ for every $K\in\mathcal{K}^f$; it follows that A is $\tilde\lambda$-negligible (Fremlin 03, 412Jb).

(f) Thus $\tilde\lambda$ satisfies the conditions of the theorem. Now suppose that ν is another measure on X with these properties. Let $\mathcal{E}$ be the ring of sets generated by $\mathcal{C}_k$. Because $\mathcal{C}_k$ is closed under finite intersections and ν and $\tilde\lambda$ agree on $\mathcal{C}_k$, they agree on $\mathcal{E}$. It follows that they agree on any compact set K included in a member of $\mathcal{E}$. **P** Every member of $\mathcal{E}$ is included in a finite union of members of $\mathcal{C}_k$, which will always be compact; so there is a compact $V_0\in\mathcal{E}$ including K. Let $\mathcal{V}$ be the family of compact members of $\mathcal{E}$ including K and included in V_0. Then $\mathcal{V}$ is downwards-directed and $K\subseteq\bigcap\mathcal{V}$. If $x\in X\setminus K$, there is an open set U, of the form $\{y : y(i)\in U_i \text{ for } i\in J\}$ where $J\subseteq X$ is finite and $U_i\subseteq X_i$ is open for every $i\in J$, such that $x\in U\subseteq X\setminus K$; in which case $V_0\setminus U$ is a compact set, belonging to $\mathcal{E}$, including K and not containing x. Thus $K=\bigcap\mathcal{V}$. Because both ν and $\tilde\lambda$ are topological measures inner regular with respect to the compact sets and finite on V_0,

$$\nu K = \inf_{V\in\mathcal{V}}\nu V = \inf_{V\in\mathcal{V}}\tilde\lambda V = \tilde\lambda K.\ \mathbf{Q}$$

On the other hand, the final clause in the specifications for $\tilde\lambda$ implies that both $\tilde\lambda$ and ν are inner regular with respect to the family of compact sets included in members of $\mathcal{E}$. Since they are both complete locally determined topological measures, they are identical (Fremlin 03, 412L).

4B There are significant simplifications for countable products, as follows.

Proposition Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of measure spaces, and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J : i\in J\in[I]^{<\omega}\}$ for every $i\in I$. Let λ be the Baker $\mathcal{F}$-product measure on $X=\prod_{i\in I}X_i$. Suppose that each X_i is endowed with a topology, and that X has the product topology.

(i) If every μ_i is inner regular with respect to the closed sets, so is λ.

(ii) If every μ_i is inner regular with respect to the zero sets, so is λ.
(iii) If every μ_i is inner regular with respect to the closed compact sets, so is λ.

proof (a) If for each $i \in I$ we are given a family $\mathcal{K}_i \subseteq \mathcal{P}X_i$ such that μ_i is inner regular with respect to $\mathcal{K}_i$, and if $\mathcal{M} \subseteq \mathcal{P}X$ is closed under finite unions and countable intersections and contains $\prod_{i\in I} K_i$ whenever $K_i \in \mathcal{K}_i$ for every i, then λ is inner regular with respect to $\mathcal{M}$. **P** As in 2D, let $\mathcal{C}$ be the family of sets expressible as $\prod_{i\in I} E_i$ where $E_i \in \Sigma_i$ for every i and $\prod_{i\in I}^{(\mathcal{F})} \mu_i E_i$ is defined and finite, and $\mathcal{K}$ the family of sets expressible as countable intersections of finite unions of members of $\mathcal{C}$. Because I is countable, we find that whenever $C \in \mathcal{C}$ and $\epsilon > 0$ there is an $M \in \mathcal{M}$ such that $M \subseteq C$ and $\lambda M \geq \lambda C - \epsilon$. It follows that whenever $K \in \mathcal{K}$ and $\epsilon > 0$ there is an $M \in \mathcal{M}$ such that $M \subseteq K$ and $\lambda M \geq \lambda K - \epsilon$. As λ is inner regular with respect to $\mathcal{K}$ (2Dd), it is inner regular with respect to $\mathcal{M}$. **Q**

(b) Now all we have to do is apply (a) with suitable families $\mathcal{K}_i$, as in the proof of 412T in FREMLIN 03.

4C Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of Radon measure spaces, and $\mathcal{F}$ a filter on $[I]^{<\omega}$ containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$. Let λ be the Baker $\mathcal{F}$-product measure on $X = \prod_{i\in I} X_i$, and $\tilde{\lambda}$ be the product topological measure defined in Theorem 4A. Then $\tilde{\lambda}$ extends λ.

proof In the language of the proof of 4A, $\mathcal{C}'_{\text{kss}} = \mathcal{C}$. So θ_k, λ_{0k} and λ_k, as defined there, coincide with θ, λ_0 and λ as defined in 2C. But we saw in part (e) of the proof of 4A that $\tilde{\lambda}$ extends λ_k.

4D Let I be any set and suppose that (X_i, Σ_i, μ_i) is $\mathbb{R}$ with Lebesgue measure for every $i \in I$. Let $\mathcal{F}$ be a filter on $[I]^{<\omega}$ containing $\{J : i \in J \in [I]^{<\omega}\}$ for every $i \in I$, and λ the Baker $\mathcal{F}$-product measure on $\mathbb{R}^I$. Then λ is translation-invariant. **P** In the construction of 2A, $\mathcal{C}$, $\mathcal{D}$ and τ are all translation-invariant. **Q** Similarly, $\lambda(-W) = \lambda W$ whenever either is defined. By 2Dd, $\{x : x \geq y\}$ and $\{x : x \leq y\}$ are measured by λ, for every $y \in \mathbb{R}^I$.

The subspace measure on $\ell^\infty(I)$ induced by λ is a translation-invariant measure in which a ball $B(x, \alpha)$ has measure 0 if $\alpha < \frac{1}{2}$, 1 if $\alpha = \frac{1}{2}$ and ∞ if $\alpha > \frac{1}{2}$.

The magnitude of λ is $\mathfrak{c}$ (because there are just $\mathfrak{c}$ compact subsets of X) and its additivity is the additivity of Lebesgue measure (use 2F and 2Eb). So if these are equal, λ is strictly localizable (FREMLIN 08, 521K[2]).

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[2]Later editions only.