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Linked sets in probability algebras

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1 Definition For a cardinal κ, write $(\mathfrak{B}_\kappa,\bar\nu_\kappa)$ for the measure algebra of the usual measure on $\{0,1\}^\kappa$. For a cardinal κ and real $\alpha\in\,]0,1]$, write $\mathrm{lk}(\kappa,\alpha)$ for the least cardinal of any family of linked subsets of $\mathfrak{B}_\kappa$ covering $C_{\kappa\alpha}=\{a:a\in\mathfrak{B}_\kappa,\,\bar\nu_\kappa a\ge\alpha\}$. (Recall that a set $A\subseteq\mathfrak{B}_\kappa$ is linked if $a\cap b\ne 0$ for all a, $b\in A$; see Fremlin 08, 511Dd.)

2 Lemma Let $r\ge 1$ be an integer, and $S_r\subseteq\mathbb{R}^{r+1}$ the unit sphere in $(r+1)$-dimensional Euclidean space. If $\langle F_i\rangle_{i\le r}$ is a cover of S_r by closed sets, then there are a $u\in S_r$ and an $i\le r$ such that u and $-u$ both belong to S_r.

proof ? Otherwise, there is a $\delta>0$ such that $\|u+v\|\ge\delta$ whenever $i\le r$ and u, $v\in F_i$. For each $i\le r$, set $f_i(v)=\max(0,1-\frac{2}{\delta}\rho(v,F_i))$ for every $v\in S_r$, setting $\rho(v,F_i)=\inf_{u\in F_i}\|u-v\|$ (or ∞ if $F_i=\emptyset$). Then $f(v)=\sum_{i=0}^r f_i(v)\ge 1$ for every $v\in S_r$. Set $g(v)=\langle\frac{f_i(v)}{f(v)}\rangle_{i\le r}$ for $v\in S_r$; then g is a continuous function from S_r to $\{x:x\in\mathbb{R}^{r+1},\,\sum_{i=0}^r x(i)=1\}\cong\mathbb{R}^r$. By the Borsuk-Ulam theorem (Hatcher 02, 2B.7), there is a $w\in S_r$ such that $g(w)=g(-w)$. In this case, there is an $i\le r$ such that $f_i(w)>0$, and we must also have $f_i(-w)>0$. So there are u, $v\in F_i$ such that $\|u-w\|\le\frac{1}{2}\delta$ and also $\|v+w\|\le\frac{1}{2}\delta$; but it follows that $\|u+v\|\le\delta$, contrary to the definition of δ. **X**

3 Theorem (a) If $\alpha\in\left]\frac{1}{2},1\right]$ then $\mathrm{lk}(\kappa,\alpha)=1$ for every κ.
(b) $\mathrm{lk}(\kappa,\frac{1}{2})=2$ for every $\kappa\ge 1$.
(c) If $\kappa\le\lambda$ are cardinals and $\alpha\le\beta$, then $\mathrm{lk}(\kappa,\beta)\le\mathrm{lk}(\lambda,\alpha)$.
(d) If $\omega\le\kappa\le\mathfrak{c}$ and $0<\alpha<\frac{1}{2}$, then $\mathrm{lk}(\kappa,\alpha)=\omega$.

proof (a) In this case, $C_{\kappa\alpha}$ itself is linked.

(b) Since $C_{\kappa,1/2}$ is not linked, $\mathrm{lk}(\kappa,\frac{1}{2})\ge 2$. In the other direction, let A_0, A_1 be a partition of $C_{\kappa\alpha}$ such that $1\setminus a\in A_1$ for every $a\in A_0$; then A_0 and A_1 are both linked, so $\mathrm{lk}(\kappa,\frac{1}{2})\le 2$.

(c) There is a measure-preserving Boolean homomorphism $\pi:\mathfrak{B}_\kappa\to\mathfrak{B}_\lambda$, and $\pi[C_{\kappa\beta}]\subseteq C_{\lambda\alpha}$. So if $\mathcal{A}$ is a family of linked subsets of $\mathfrak{B}_\lambda$, covering $C_{\lambda\alpha}$, of cardinal $\mathrm{lk}(\lambda,\beta)$, $\{\pi^{-1}[A]:A\in\mathfrak{A}\}$ is a family of linked subsets of $\mathfrak{B}_\lambda$, covering $C_{\kappa\beta}$, of cardinal at most $\mathrm{lk}(\lambda,\beta)$.

(d) Since $\mathfrak{B}_\kappa$ is σ-linked (Fremlin 08, 524L), $\mathrm{lk}(\kappa,\alpha)\le\omega$. In the other direction, $\mathrm{lk}(\kappa,\alpha)>r+1$ for every integer $r\ge 1$. **P?** If $\mathrm{lk}(\kappa,\alpha)\le r+1$, let $\langle A_i\rangle_{i\le r}$ be a family of linked subsets of $\mathfrak{B}_\kappa$ covering $C_{\omega\alpha}$. As in §2, let $S_r\subseteq\mathbb{R}^{r+1}$ be the unit sphere. Let μ_{Hr} be Hausdorff r-dimensional measure on S_r; set $\mu=\frac{1}{\mu_{Hr}S_r}\mu_{Hr}$. Because μ is an atomless Radon probability measure on a compact metrizable space, μ and ν_ω are isomorphic (Fremlin 02, 344K), and the measure algebra $(\mathfrak{A},\bar\mu)$ of μ is isomorphic to $(\mathfrak{B}_\omega,\bar\nu_\omega)$; there is therefore a measure-preserving homomorphism $\pi:\mathfrak{A}\to\mathfrak{B}_\kappa$. Let $\delta>0$ be such that $\mu\{x:x\in S_r,\,x(0)\ge\delta\}=\alpha$, and for $u\in S_r$ set

$$D_u=\{x:x\in S_r,\,x\centerdot u\ge\delta\},\quad d_u=\pi D_u^\bullet\in C_{\omega\alpha}.$$

For each $i<r$, set $H_i=\{u:u\in S_r,\,c_u\in A_i\}$. Then $S_r=\bigcup_{i<r}H_i$, so there are $w\in S_r$ and $i<r$ such that w and $-w$ both belong to $\overline{H}_i$, by Lemma 2. Let u, $v\in H_i$ be such that $\|u-w\|<\delta$ and $\|v+w\|<\delta$. Then $D_u\cap D_v\ne\emptyset$; take $x\in D_u\cap D_v$; we have

$$x\centerdot w>x\centerdot u-\delta\ge 0,\quad x\centerdot(-w)>x\centerdot v-\delta\ge 0,$$

which is impossible. **X** Thus $\mathrm{lk}(\omega,\alpha)>r+1$. **Q**

Since r is arbitrary, $\mathrm{lk}(\kappa,\alpha)\ge\omega$, as required.

4 Problem What about $\kappa > \mathfrak{c}$, $\alpha < \frac{1}{2}$?

5 Lemma Let κ be an infinite cardinal and $k \ge 1$ an integer; set $n = 4k$. For $I \in [\kappa]^n$, enumerate I in increasing order as $\langle \xi_i \rangle_{i<n}$ and set $I_e = \{\xi_{2i} : i < 2k\}$, $I_o = \{\xi_{2i+1} : i < 2k\}$. Set

$$E_I = \{x : x \in \{0,1\}^\kappa, \#(I_o \cap x^{-1}[\{1\}]) > \#(I_e \cap x^{-1}[\{1\}])\}.$$

Then

(a) $\nu_\kappa E_I \ge \dfrac{1}{2} - 2^{-k}$ for every $I \in [\kappa]^n$,

(b) if $J \in [\kappa]^{n+1}$ then $E_{J\setminus\{\max J\}}$ and $E_{J\setminus\{\min J\}}$ are disjoint.

proof (a) Let $\langle X_i \rangle_{i<n}$ be independent random variables, each taking the values 0, 1 with probability $\frac{1}{2}$; set $X_e = \sum_{i=0}^{2k-1} X_{2i}$, $X_o = \sum_{i=0}^{2k-1} X_{2i+1}$. Then

$$\nu_\kappa E_I = \Pr(X_e - X_o > 0) = \frac{1}{2}(1 - \Pr(X_o = X_e)) \ge \frac{1}{2}(1 - \max_{r \le 2k} \Pr(X_o = r))$$

(because X_o, X_e are independent, so $\Pr(X_o = X_e) = \sum_{r=0}^{2k} \Pr(X_e = r)\Pr(X_o = r)$)

$$= \frac{1}{2}\big(1 - \tfrac{(k!)^2}{(2k)!}\big) \ge \frac{1}{2} - 2^{-k}.$$

(b) Setting $I = J \setminus \{\max J\}$ and $I' = J \setminus \{\min J\}$, we see that

$$I'_e = I_o, \quad I'_o = (I_e \setminus \{\min J\}) \cup \{\max J\}.$$

So if $x \in E_I$,

$$\#(I'_o \cap x^{-1}[\{1\}]) - \#(I'_e \cap x^{-1}[\{1\}]) \le 1 + \#(I_e \cap x^{-1}[\{1\}]) - \#(I_o \cap x^{-1}[\{1\}]) < 1$$

and $x \notin E'_I$.

6 An arrow relation If κ, λ are cardinals and n is an integer, consider the statement

$P(\kappa, n, \lambda)$: whenever $f : [\kappa]^n \to \lambda$ is a function, there is a $J \in [\kappa]^{n+1}$ such that $f(J \setminus \{\max J\}) = f(J \setminus \{\min J\})$.

7 Proposition Suppose that $k \ge 1$ is an integer and that κ, λ are infinite cardinals such that $P(\kappa, 4k, \lambda)$ is true. Then $\mathrm{lk}(\kappa, \frac{1}{2} - 2^{-k}) > \lambda$.

proof ? Otherwise, let $\langle A_\eta \rangle_{\eta<\lambda}$ be a family of linked sets in $\mathfrak{B}_\kappa$ covering $C_{\kappa\alpha}$, where $\alpha = \frac{1}{2} - 2^{-k}$. For $I \in [\kappa]^n$, define E_I as in Lemma 5; then $E_I^\bullet \in C_{\kappa\alpha}$, by 5a. Set $f(I) = \min\{\eta : E_I^\bullet \in A_\eta\}$ for $I \in [\kappa]^n$. Let $J \in [\kappa]^{n+1}$ be such that $f(J \setminus \{\max J\}) = f(J \setminus \{\min J\})$; then $E_{J\setminus\{\max J\}} \cap E_{J\setminus\{\min J\}} \ne \emptyset$; but this contradicts 5b. **X**

8 Problem Is there any $n \in \mathbb{N}$ such that $P(\mathfrak{c}^+, n, \omega)$ is true? By the Erdős-Rado theorem (Just & Weese 97, 15.13), $P(\mathfrak{c}^+, 2, \omega)$ is false.

References

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.

Fremlin D.H. [08] *Measure Theory, Vol. 5: Set-theoretic Measure Theory.* Torres Fremlin, 2008 (`http://www.lulu.com`)

Hatcher, A. [02] *Algebraic Topology.* Cambridge U.P., 2002; http://www.math.cornell.edu/~hatcher/AT/ATpage.html.

Just W. & Weese M. [97] *Discovering Modern Set Theory II.* Amer. Math. Soc., 1997 (Graduate Studies in Mathematics 18).