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Give a penny, take a penny

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1 A game

1A Definition (Model 96, Abraham & Schipperus 07) For $k \in \mathbb{N}$, $l \leq m$ in $\mathbb{N}$ and a cardinal $\kappa \geq m$ consider the following game $\mathrm{MG}(\kappa, m, l, k)$:

start from $I_0 = m$;
given $I_n \in [\kappa]^m$, Remover plays $I'_n \in [I_n]^{m-l}$,
given $I'_n \in [\kappa]^{m-l}$, Adder plays $I_{n+1} \in [\kappa]^m$ such that $I_{n+1} \supseteq I'_n$,
Remover wins if $\#(\bigcup_{n\in\mathbb{N}} \bigcap_{j\geq n} I'_j) \leq k$.

1B Elementary remarks (a) On the ordinary rules for infinite games, Remover has a winning strategy whenever $l \geq 1$, since he can delete points in the order in which they are introduced.

(b) On the other hand, Remover can have a winning tactic only if $k \geq m - l$, since Adder can always replace the points which Remover has just deleted to make $I_n = m$, $I'_n = I'_0$ for every n. (Of course, if $k \geq m - l$, then Remover necessarily wins.)

(c) So we look at **time-dependent tactics** for Remover, that is, functions $f : \mathbb{N} \times [\kappa]^m \to [\kappa]^{m-l}$ such that $f(n, I) \subseteq I$ for every I and n, and a corresponding run of the game will have $I'_n = f(n, I_n)$ for every n.

2 Winning tactics

2A Proposition If $m \leq k + l$ then Remover has a winning time-dependent tactic in $\mathrm{MG}(\kappa, m, l, k)$.

proof Trivial.

2B Proposition (B.I.Model) If $\kappa \leq \mathfrak{c}$ and $m < (k+1)(l+1)$ then Remover has a winning time-dependent tactic in $\mathrm{MG}(\kappa, m, l, k)$.

proof Because $\kappa \leq \mathfrak{c}$ there is a family $\langle K_{ni}\rangle_{n\in\mathbb{N}, i\leq k}$ such that $K_{ni} \cap K_{nj} = \emptyset$ whenever $i, j \leq k$ are distinct and whenever $K \in [\kappa]^{k+1}$ there are infinitely many n such that K meets K_{ni} for every $i \leq k$. Now, given $J \in [\kappa]^m$ and $n \in \mathbb{N}$, there must be an $i \leq k$ such that $\#(J \cap K_{ni}) \leq l$; take $f(n, J) \subseteq J \setminus K_{ni}$ for such an i. **?** If $I_0, I'_0, \ldots$ is a run of the game, consistent with f, in which Remover loses, let $K \in [\kappa]^{k+1}$, $n_0 \in \mathbb{N}$ be such that $K \subseteq I'_n$ for every $n \geq n_0$. There is an $n \geq n_0$ such that $K \cap K_{ni}$ for every $i \leq k$; but there is an $i \leq k$ such that $I'_n \cap K_{ni} = \emptyset$. **X** So f is a winning time-dependent tactic.

2C Theorem (Abraham & Schipperus 07, 2.1) If k, l, $m \in \mathbb{N}$, $1 \leq l \leq m$, and $\kappa \leq \omega_k$, there is a winning time-dependent tactic for Remover in $\mathrm{MG}(\omega_k, m, l, k)$.

proof Fix $l \geq 1$.

(a) I show by induction on k, simultaneously for all $m \geq l$, that that there is a winning time-dependent tactic f_{mk} for Remover in $\mathrm{MG}(\omega_k, m, l, k)$ such that

(*) for every $K \in [\omega_k]^{k+1}$ there is a $q \in \mathbb{N}$ such that
whenever $I_0, I'_0, \ldots, I_q, I'_q$ is a finite sequence such that $I_n \in [\omega_k]^m$ and $I'_n = f_{mk}(n, I_n)$ for every $n \leq q$ and $I'_n \subseteq I_{n+1}$ for $n < q$
then $K \not\subseteq \bigcap_{n\leq q} I'_n$.

(Note that $I_0, I'_0, \ldots$ here need not be quite a finite partial run of $\mathrm{MG}(\omega_k, m, l, k)$ because there is no promise that $I_0 = m$.)

(b) To start the induction, with $k = 0$, all we need to do is take a function $f_{m0} : \mathbb{N} \times [\omega]^m \to \omega^{m-l}$ such that

$$f_{m0}(2^j(2i+1), I) \subseteq I \setminus \{j\}$$

for all i, $j \in \mathbb{N}$ and $I \in [\omega]^m$.

The rest of the proof will be devoted to the inductive step to $k > 1$, for a given $m \geq l$.

(c) Of course if $m = l$ then we have to take $f_{km}(I) = \emptyset$ for every $I \in [\omega_k]^m$, and this will do what we want. So we may suppose that $l < m$. In this case, if $\xi < \omega_k$ is an ordinal and $l \leq p < m$, then we can copy the function $f_{p,k-1}\restriction\mathbb{N} \times [\#(\xi)]^p$ into a function $g_{\xi p} : \mathbb{N} \times [\xi]^p \to [\xi]^{p-l}$ such that $g_{\xi p}(n, I) \subseteq I$ whenever $I \in [\xi]^p$ and $n \in \mathbb{N}$, and whenever $K \in [\xi]^k$ there is a $q \in \mathbb{N}$ such that $K \not\subseteq \bigcap_{n \leq q} I'_n$ whenever $I_0, I'_0, \dots, I_q, I'_q$ are such that $I_n \in [\xi]^p$ and $I'_n = g_{\xi p}(I_n)$ for $n \leq q$, while $I'_n \subseteq I_{n+1}$ for $n < q$.

(d) The tactic $f = f_{mk}$ will be based on 'cycles', 'phases' and 'steps', as follows. Cycle q, for $q \geq 1$, will consist of $m - l$ phases labelled with phase numbers $p = m-1, \dots, l$ in decreasing order. Phase p of cycle q will consist of q steps. So each cycle q will take $q(m-l)$ steps; accordingly cycle q will begin at time $\frac{1}{2}q(q-1)(m-l)$. Time n will therefore be step i of phase p of cycle q where $q \geq 1$, $m-1 \geq p \geq l$, $i < q$ and

$$n = \frac{1}{2}q(q-1)(m-l) + (m-1-p)q + i.$$

Next we need the notion of 'pivot'. If we are at step i of phase p of cycle q, with $n = \frac{1}{2}q(q-1)(m-l) + (m-1-p)q + i$, and $I \subseteq \omega_k$ has m members, the **pivot** of I will be that $\xi \in I$ such that $\#(I \cap \xi) = p$. In this case, Remover's move will be

$$f(n, I) = (I \setminus \xi) \cup g_{\xi p}(i, I \cap \xi) \in [I]^{m-l}.$$

(e) A run of $\mathrm{MG}(\omega_k, m, l, k)$ in which Remover follows this tactic will therefore see him using the tactics $g_{\xi p}$ in an irregular succession, because the pivots will change in response to the moves by Adder. So the key to the proof is the fact that these changes are limited in the presence of fixed points, as follows. Suppose that $\xi < \omega_k$ and that we have a finite sequence corresponding to a whole cycle, that is, $I_r, I'_r, I_{r+1}, I'_{r+1}, \dots, I_{r+(m-l)q}$, where $r = \frac{1}{2}q(q-1)$, such that $I_n \in [\omega_k]^m$ and $I'_n = f(n, I_n) \subseteq I_{n+1}$ for $r \leq n < r + (m-l)q$, such that $\xi \in \bigcap_{r \leq n < r+(m-l)q} I'_n$. Then there will be a phase of the cycle q during the whole of which ξ is the pivot. **P** For $m - 1 \geq p \geq m - l$ and $i < q$, set $n_{pi} = r + (m-1-p)q + i$, and let $\xi_{pi} \in I_{n_{pi}}$ be the corresponding pivot, so that $\#(I_{n_{pi}} \cap \xi_{pi}) = p$ and $I'_{n_{pi}} = (I_{n_{pi}} \setminus \xi_{pi}) \cup g_{\xi_{pi}p}(i, I_{n_{pi}} \cap \xi_{pi})$.

In this case, for each p, we see that

$$\begin{aligned}\#(I_{n_{p,i+1}} \cap \xi_{pi}) &= \#(I_{n_{pi}+1} \cap \xi_{pi}) \leq l + \#(I'_{n_{pi}} \cap \xi_{pi}) = l + \#(f(n_{pi}, I_{n_{pi}}) \cap \xi_{pi}) \\ &= l + \#(g_{\xi_{pi}p}(i, I_{n_{pi}} \cap \xi_{pi})) = l + \#(I_{n_{pi}} \cap \xi_{pi}) - l = p\end{aligned}$$

whenever $i < q - 1$. But this will mean that $\xi_{p,i+1} \geq \xi_{pi}$, that is, the pivots are non-decreasing during the phase p.

In the final phase $p = l$, we have $g_{\xi_{li}l}(i, J) = \emptyset$ for every i and every $J \in [\xi_{li}]^l$, so that $\xi \geq \xi_{li}$ for every $i < q$. There is therefore a first (that is to say, greatest) p such that $\xi_{p,q-1} \leq \xi$. **?** If $\xi_{p0} < \xi$, we have $\xi \in I_{n_{p0}}$ and $\#(I_{n_{p0}} \cap \xi) > p$. Of course this means that $p < m - 1$ and $\xi' = \xi_{p+1,q-1}$ is defined and greater than ξ. But now

$$\begin{aligned}\#(I_{n_{p0}} \cap \xi) &\leq \#(I_{n_{p0}} \cap \xi') - 1 \\ &\leq l + \#(g_{\xi',p+1}(I_{n_{p+1,q-1}} \cap \xi')) - 1 = p\end{aligned}$$

and $\xi \leq \xi_{p0}$. **X** Thus $\xi \leq \xi_{p0}$ and $\xi_{pi} = \xi$ for every $i < q$, as required. **Q**

(f) Now suppose that $K \in [\omega_k]^{k+1}$. Set $\xi = \max K$. For each p such that $m - 1 \geq p \geq l$ there is a $q_p \in \mathbb{N}$ such that if $I_0, I'_0, \dots, I_{q_p}, I'_{q_p}$ are such that $I_i \in [\xi]^p$, $I'_i = g_{\xi p}(i, I_i)$ for $i \leq q_p$, $I_{i+1} \supseteq I'_i$ for $i < q_p$, then $K \cap \xi \not\subseteq \bigcap_{i < q_p} I'_i$. Take any $q \geq \max\{q_p : l \leq p < m - 1\}$. Then if $r = \frac{1}{2}q(q-1)$ and $I_r, I'_r, \dots, I_{r+(m-l)q}$ are such that $I_n \in [\omega_k]^m$ and $I'_n = f(n, I_n) \subseteq I_{n+1}$ for $r \leq n < r + (m-l)q$, then $K \not\subseteq \bigcap_{r \leq n < r+(m-l)q} I'_n$. **P?** Otherwise, there must be a phase p, with $l \leq p \leq m - 1$, such that ξ is the pivot throughout the phase, that is, setting $r' = r + (m-1-p)q$,

$$I'_n = f(n, I_n) = (I_n \setminus \xi) \cup g_{\xi p}(n - r', I_n \cap \xi)$$

for $r' \le n < r'+q$. But in this case $K\cap\xi \in [\xi]^k$ and $K\cap\xi \subseteq g_{\xi p}(i, I_{r'+i})$ for $i < q_p$, which is impossible. **XQ**

(g) In particular, taking $q = \max\{q_p : l \le p \le m-1\}$, we see that the hypothesis (*) is satisfied by f. But we also see that if $K \in [\omega_k]^{k+1}$ and $I_0, I'_0, \dots$ is a run of $\mathrm{MG}(\omega_k, m, l, k)$ consistent with f, then $K \not\subseteq \bigcap_{n\ge r} I'_n$ for any $r \in \mathbb{N}$, so f is a winning tactic for Remover, and the induction continues, with $f_{mk} = f$.

(h) Finally, if $\kappa \le \omega_k$, then $f_{mk}\restriction \mathbb{N}\times[\kappa]^m$ is a winning time-dependent tactic for Remover in $\mathrm{MG}(\kappa, m, l, k)$.

3 When there is no winning tactic

3A Proposition (Model 96, Abraham & Schipperus 07, 3.1) If $\kappa \ge \omega_1$ and $l < m$ then Remover has no winning time-dependent tactic in $\mathrm{MG}(\kappa, m, l, 0)$.

proof Let $f : \mathbb{N}\times[\kappa]^m \to [\kappa]^{m-l}$ be a time-dependent tactic for Remover in $\mathrm{MG}(\kappa, m, l, 0)$. Let $D \subseteq \omega_1$ be the set of those $\delta < \omega_1$ such that whenever $n \in \mathbb{N}$, $J \in [\kappa]^{m-l}$ and $\max J = \delta$ there is an $I \in [\omega_1]^m$ such that $I \supseteq J$ and $\delta = \max f(n, I)$. Then D is infinite. **P?** Otherwise, for $\delta \in \omega_1\setminus D$, take $n_\delta \in \mathbb{N}$ and $J_\delta \in [\omega_1]^{m-l}$ such that $\delta = \max J_\delta$ and $\delta \ne \max f(n_\delta, I)$ whenever $J_\delta \subseteq I \in [\omega_1]^m$. By the Pressing-Down Lemma, there are $n^* \in \mathbb{N}$ and $J^* \in [\omega_1]^{m-l-1}$ such that $C = \{\delta : n^* = n_\delta,\ J_\delta = J^*\cup\{\delta\}\}$ is stationary. But now take any $I \in [\omega_1]^m$ such that $J^* \subseteq I \subseteq J^*\cup C$ and every member of $I\setminus J^*$ is greater than every member of J^*. Set $\delta = \max f(n^*, I)$. Then $\delta \in C$, $J_\delta = J^*\cup\{\delta\} \subseteq I$ and $\delta = \max f(n_\delta, I)$, which is supposed to be impossible. **XQ**

Now let $g : \mathbb{N}\times[\kappa]^{m-l} \to [\kappa]^m$ be a function such that

$g(n, J) \subseteq J \cup (D\setminus m)$,

if $\max J \in D$ then $\max f(n+1, g(n, J)) = \max J$.

Now consider a run $(I_0, I'_0, \dots)$ in $\mathrm{MG}(\kappa, m, l, 0)$ in which $I'_n = f(n, I_n)$ and, for every n,

$I'_n \subseteq I_n \subseteq I'_n \cup (D\setminus m)$,

if $\max I'_n \in D$ then $\max(f(n+1), I_n) = \max I'_n$;

such a run exists by the definition of D. In this case, either $I'_n = I'_0$ for every n, or there is an n such that $\max I'_j = \max I'_n$ for every $j \ge n$; in either case, Remover loses. As f is arbitrary, we have the result.

3B Theorem (see Abraham & Schipperus 07, §4) If $k, l \in \mathbb{N}$, $m = k+l+1$, and $\kappa > \beth_{k(k+1)/2}$, then there is no winning time-dependent tactic for Remover in $\mathrm{MG}(\kappa, m, l, k)$.

proof (a) Induce on k. The induction starts with $k = 0$, $\kappa > \omega$ and $m = l+1$, which is covered by Proposition 3A. For the inductive step to $k > 0$, given $l \in \mathbb{N}$ and $\kappa > \beth_{k(k+1)/2}$, let $f : \mathbb{N}\times[\kappa]^{k+l+1} \to [\kappa]^{k+1}$ be a time-dependent tactic for Remover in $\mathrm{MG}(\kappa, k+l+1, l, k)$. For $n \in \mathbb{N}$, set

$$S_n = \{J : J \in [\kappa]^{k+1},\ J \ne f(n, I) \text{ whenever } J \subseteq I \in [\kappa]^{k+l+1}\}.$$

Note that if $I \in [\kappa]^{k+l+1}$ then $f(n, I) \in [I]^{k+1}\setminus S_n$. By the Erdős-Rado theorem (Kanamori 03, 7.3) there is a $D \subseteq \kappa\setminus m$ such that $\#(D) > \beth_{(k(k+1)/2)-k}$ and either $[D]^{k+1}$ is included in some S_n or $[D]^{k+1}$ is disjoint from every S_n. The former is impossible, so $[D]^{k+1}\cap S_n = \emptyset$ for every n.

(b) Let us say that a **sensible partial run** of $\mathrm{MG}(\kappa, k+l+1, l, k)$ is a finite sequence $(I_0, I'_0, \dots, I_n, I'_n)$ such that $I_0 = m$, $I'_j = f(j, I_j)$ for every $j \le n$, and $I_{j+1} \subseteq I'_j \cup D$ for every $j < n$. Among the sensible partial runs, choose one, $(\hat I_0, \dots, \hat I'_{\hat n})$ say, for which $K = \hat I'_{\hat n}\setminus m$ has as many elements as possible. Note that as $\hat I'_{\hat n} \subseteq \hat I_{\hat n} \subseteq m\cup D$, $K \subseteq D$. Set $L = m\cap \hat I'_{\hat n}$. The argument now divides.

(c) Suppose that $L = \emptyset$. In this case, $K = \hat I'_{\hat n}$ belongs to $[D]^{k+1}$. Choose $(I_0, I'_0, \dots)$ such that

if $j \le \hat n$ then $I_j = \hat I_j$ and $I'_j = \hat I'_j$,

if $j > \hat n$ then $I_j \in [\kappa]^{k+l+1}$ is such that $K \subseteq I_j$ and $f(j, I_j) = K$, while $I'_j = K$.

Then $(I_0, I'_0, \dots)$ is a run of $\mathrm{MG}(\kappa, k+l+1, l, k)$, consistent with f, such that $\#(\bigcap_{j\ge\hat n} I'_j) = K$ has $k+1$ members, so Remover loses.

(d) Now suppose that $L \ne \emptyset$.

(i) Set $k' = \#(K)-1 < k$, so that $\#(L) = k-k'$. Take $M \in [D]^{k'+l+1}$ including K. Let $\mathrm{MG}_M(D, k'+l+1, l, k')$ be the variant of $\mathrm{MG}(\#(D), k'+l+1, l, k')$ in which

the players start from $J_0 = M$;
given $J_n \in [D]^{k'+l+1}$, Remover plays $J'_n \in [J_n]^{k'+1}$,
given $J'_n \in [D]^{k'+1}$, Adder plays $J_{n+1} \in [D]^{k'+l+1}$ such that $J_{n+1} \supseteq J'_n$,
Remover wins if $\#(\bigcup_{n\in\mathbb{N}} \bigcap_{j\ge n} J'_j) \le k'$.

Since $\#(D) > \beth_{(k(k+1)/2)-k} = \beth_{(k-1)k/2} \ge \beth_{k'(k'+1)/2}$, the inductive hypothesis tells us that Remover has no winning time-dependent tactic in either $\mathrm{MG}(\#(D), k'+l+1, l, k')$ or in the isomorphic game $\mathrm{MG}_M(D, k'+l+1, l, k')$.

Let $g : \mathbb{N} \times [D]^{k'+l+1} \to [D]^{k'+1}$ be such that

$$g(i, J) \in [J]^{k'+1}, \quad g(i, J) \subseteq f(\hat{n}+i+1, J \cup L) \setminus L$$

for every $i \in \mathbb{N}$ and $J \in [D]^{k'+l+1}$. Then g is a time-dependent tactic for Remover in $\mathrm{MG}_M(D, k'+l+1, l, k')$. It cannot be a winning tactic, so there is a run $(J_0, J'_0, \dots)$ of $\mathrm{MG}_M(D, k'+l+1, l, k')$, consistent with g, and an $n \in \mathbb{N}$ such that $\bigcap_{j\ge n} J'_j$ has more than k' members.

(ii) Consider the sequence $(I_0, I'_0, \dots)$ where
if $j \le \hat{n}$ then $I_j = \hat{I}_j$ and $I'_j = \hat{I}'_j$,
if $j > \hat{n}$ then $I_j = J_{j-\hat{n}-1} \cup L$ and $I'_j = J'_{j-\hat{n}-1} \cup L$.

Since

$$I_{\hat{n}+1} = J_0 \cup L = M \cup L \supseteq K \cup L = I'_{\hat{n}},$$

this is a run of $\mathrm{MG}(\kappa, k+l+1, l, k)$, and since $\bigcap_{j>n+\hat{n}} I'_j = L \cup \bigcap_{j\ge n} J'_j$ has more than k members, Remover loses the run.

(iii) However, the run is consistent with the time-dependent tactic f. **P?** Otherwise, there is a first $j \in \mathbb{N}$ such that $I'_j \ne f(j, I_j)$; in this case, since these sets both have $k+1$ members, $I'_j \not\subseteq f(j, I_j)$. We know that $\hat{I}'_i = f(i, \hat{I}_i)$ for every i, so $j > \hat{n}$ and

$$\begin{aligned} I'_j &= L \cup J'_{j-\hat{n}-1} = L \cup g(j-\hat{n}-1, J_{j-\hat{n}-1}) \\ &\subseteq L \cup (f(j, J_{j-\hat{n}-1} \cup L) \setminus L) \subseteq L \cup f(j, J_{j-\hat{n}-1} \cup L) = L \cup f(j, I_j). \end{aligned}$$

So $f(j, I_j)$ does not include L. On the other hand,

$$f(j, I_j) \subseteq I_j \subseteq D \cup L, \quad \#(D \cap f(j, I_j)) > \#(f(j, I_j)) - \#(L) = k'+1.$$

Now $(I_0, I'_0, \dots, I_{j-1}, I'_{j-1}, I_j, f(j, I_j))$ is a sensible partial run of $\mathrm{MG}(\kappa, k+l+1, l, k)$ in which the final term meets D in more than $k'+1 = \#(K)$ elements, contradicting the choice of $(\hat{I}_0, \dots, \hat{I}'_{\hat{n}})$. **XQ**

(e) Thus in both cases we have a run of $\mathrm{MG}(\kappa, k+l+1, l, k)$, consistent with f, in which Remover loses, and f is not a winning time-dependent tactic for Remover. As f is arbitrary, the induction proceeds.

3C Proposition If $m > k+l$ and $\kappa > \beth_m$ then Remover has no winning time-dependent tactic in $\mathrm{MG}(\kappa, m, l, k)$.

proof (a) Fix a time-dependent tactic $f : \mathbb{N} \times [\kappa]^m \to [\kappa]^{m-l}$ tactic for Remover in $\mathrm{MG}(\kappa, m, l, k)$. We need a re-coding of f, as follows. Let F be the set of functions from $\mathbb{N} \times [2m]^m$ to $[2m]^{m-l}$. For $I \subseteq \kappa$, let $\phi_I : \mathrm{otp}(I \cup m) \to I \cup m$ be the order-isomorphism between the ordinal $\mathrm{otp}(I \cup m)$ and the well-ordered set $I \cup m$. Define $g : [\kappa \setminus m]^m \to F$ by saying that

$$g(I)(n, J) = \phi_I^{-1}[f(n, \phi_I[J])]$$

whenever $I \in [\kappa \setminus m]^m$, $n \in \mathbb{N}$ and $J \in [2m]^m$. Now $\#(F) = \mathfrak{c} = \beth_1$, so the Erdős-Rado theorem tells us that if $\kappa > \beth_m$ there must be an uncountable set $L^* \subseteq \kappa \setminus m$ such that g is constant on $[L^*]^m$. Let $L \subseteq L^*$ be a set of size m such that between any two members of L, and also below the first member of L and above the last member of L, there are at least l members of L^*.

(b) The aim of the argument is to devise a tactic for player Adder in $\mathrm{MG}(\kappa, m, l, k)$ which will defeat the time-dependent tactic f. What we find is that for any $J \in [L \cup m]^{m-l}$ and $n \in \mathbb{N}$ there is a $\overline{I} \in [L^* \cup m]^m$ such that $\overline{I} \supseteq J$, $f(n+1, \overline{I}) \subseteq L \cup (J \cap m)$ and if $f(n+1, \overline{I}) \cap m = J \cap m$ then $f(n+1, \overline{I}) = J$. **P** Set

$r = m - \#(J \cap m)$, and let $N \in [2m]^{m-l}$ be the constant value of $g(I)(n+1, (J \cap m) \cup ((m+r) \setminus m))$ for $I \in [L^*]^m$. Let $\mathcal{I}$ be the set of those $I \in [L^* \cup m]^m$ such that $I \supseteq J$ and $I \cap m = J \cap m$. For each $I \in \mathcal{I}$, there is an $I' \in [L^*]^m$ such that $I \setminus m$ is an initial segment of I', and now

$$\phi_I^{-1}[f(n+1, I)] = \phi_{I'}^{-1}[f(n+1, I)]$$

(because $I \cup m$ is an initial segment of $I' \cup m$)

$$= g(I')(n+1, \phi_{I'}^{-1}[I]) = g(I')(n+1, (J \cap m) \cup ((m+r) \setminus m)) = N,$$

and $f(n+1, I) = \phi_I[N]$. We have $\phi_I(i) = i$ for $i < m$, so

$$N \cap m = f(n+1, I) \cap m \subseteq I \cap m \subseteq J$$

for every $I \in \mathcal{I}$.

If $N \cap m$ is a proper subset of $J \cap m$, then we can take $\overline{I}$ to be any member of $\mathcal{I}$ included in $L \cup m$. Otherwise, $N \setminus m$ and $J \setminus m$ both have $r - l$ members, so there is an order-preserving bijection $\psi : N \setminus m \to J \setminus m$. Since $J \setminus m \subseteq L$, and there are many points of $L^* \setminus L$, we can extend ψ to an order-preserving bijection between subsets M_0 of $2m \setminus m$ and M_1 of L^*, both of cardinal r. Setting $\overline{I} = M_1 \cup (J \cap m)$, we have $\overline{I} \supseteq J$, $\overline{I} \cap m = J \cap m$ and $f(n+1, \overline{I}) = \phi_{\overline{I}}[N] = J$, as required. **Q**

(c) Now consider the run $I_0, I'_0, \dots$ of the game $\mathrm{MG}(\kappa, m, l, k)$ in which $I'_n = f(n, I_n)$ and $I_{n+1} = \overline{I'_n}$, as defined in (b), for every n. Then

$$I'_{n+1} \cap m = f(n+1, \overline{I'_n}) \cap m \subseteq I'_n \cap m$$

for every n, and if $I'_{n+1} \cap m = I'_n \cap m$ then $I'_{n+1} = I'_n$. But this means that $\langle I'_n \rangle_{n \in \mathbb{N}}$ is eventually constant and $\bigcup_{n \in \mathbb{N}} \bigcap_{j \ge n} I'_j$ has $m - l > k$ members, so Remover loses and f is not a winning time-dependent tactic. As f is arbitrary, we have the result.

4 Problems

4A Does Remover have a winning time-dependent tactic in $\mathrm{MG}(\omega_2, 4, 1, 1)$?

By 2B, Remover has winning time-dependent tactics in $\mathrm{MG}(\omega_2, 4, 2, 1)$ and $\mathrm{MG}(\omega_2, 4, 1, 2)$; by 2C, he has a winning time-dependent tactic in $\mathrm{MG}(\omega_1, 4, 1, 1)$. Remover does not have a winning time-dependent tactic in $\mathrm{MG}(\omega_2, 4, 1, 0)$, by 3A, or in $\mathrm{MG}(\beth_5, 4, 1, 1)$, by 3C; if $\mathfrak{c} = \omega_1$, then he does not have a winning time-dependent tactic in $\mathrm{MG}(\omega_2, 3, 1, 1)$, by 3B.

4B Generally, there is a large gap between the positive results of §2, dealing with games $\mathrm{MG}(\aleph_\bullet, m, l, k)$, and the negative results of §3, dealing with games $\mathrm{MG}(\beth_\bullet^+, m, l, k)$.

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