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Nowhere dense filters

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1 Definition Let I be a set, $\mathcal{F}$ a filter on I and X a topological space. I will say that $\mathcal{F}$ is X**-nowhere dense** if the image filter $f[[\mathcal{F}]]$ contains a nowhere dense subset of X whenever $f : I \to X$ is a function.

Remark Thus the 'nowhere dense' filters of Shelah 98 are what I shall call '$\mathbb{R}$-nowhere dense'.

2 Lemma Let X, Y be topological spaces.

(a) If there is a surjective function $h : X \to Y$ such that $h[A]$ is nowhere dense in Y for every nowhere dense set $A \subseteq X$, then any X-nowhere dense filter is Y-nowhere dense.

(b) If there is a function $h : X \to Y$ such that $h^{-1}[B]$ is nowhere dense in X for every nowhere dense subset B of Y, then any Y-nowhere dense filter is X-nowhere dense.

proof (a) Let $\mathcal{F}$ be an X-nowhere dense filter on a set I, and $f : I \to Y$ a function. As h is surjective, there is a function $g : I \to X$ such that $f = hg$. Let $A \in g[[\mathcal{F}]]$ be a nowhere dense set; then $h[A] \in f[[\mathcal{F}]]$ is nowhere dense.

(b) Let $\mathcal{F}$ be a Y-nowhere dense filter on a set I, and $f : I \to X$ a function. Then there is a nowhere dense set $B \in (hf)[[\mathcal{F}]]$, and $h^{-1}[B] \in f[[\mathcal{F}]]$ is nowhere dense.

3 Proposition Let X be a compact Hausdorff space, and Z the Stone space of the regular open algebra $\mathfrak{G}$ of X. Then a filter is X-nowhere dense iff it is Z-nowhere dense.

proof We have a canonical map $h : Z \to X$ defined by saying that $h(z) = x$ iff $z \in \widehat{G}$ wheneve $x \in G \in \mathfrak{G}$. **P** If $z \in Z$, the set $\mathcal{G} = \{G : G \in \mathfrak{G},\ z \in \widehat{G}\}$ is downwards-directed, so there is an $x \in \bigcap\{\overline{G} : G \in \mathcal{G}\}$. If now $x \in H \in \mathfrak{G}$, $\widehat{H}$ meets $\widehat{G}$ for every $G \in \mathcal{G}$; as $\widehat{H}$ is closed, $z \in \widehat{H}$. If y is any point of X other than x, there are disjoint H_0, $H_1 \in \mathfrak{G}$ containing x, y respectively. Now $z \in \widehat{H_0}$ so $z \notin \widehat{H_1}$. Thus x is the only point such that $z \in \widehat{H}$ for every regular open set H containing x. **Q**

h is surjective. **P** If $x \in X$ then there is a $z \in \bigcap\{\widehat{G} : x \in G \in \mathfrak{G}\}$. **Q**

If $B \subseteq X$ is nowhere dense, $h^{-1}[B]$ is nowhere dense in Z. **P** If H is a non-empty open set in Z, then there is a non-empty $G_0 \in \mathfrak{G}$ such that $H \supseteq \widehat{G_0}$. Now there is a non-empty $G \in \mathfrak{G}$ such that $\overline{G} \subseteq G_0 \setminus B$. In this case, $\widehat{G}$ is a non-empty open subset of $H \setminus h^{-1}[B]$. **Q**

If $A \subseteq Z$ is nowhere dense, $h[A]$ is nowhere dense in X. **P** If G is a non-empty open set in X, there is a non-empty regular open set $G_0 \subseteq G$; now there is non-empty regular open set G_1 such that $\widehat{G_1} \subseteq \widehat{G_0} \setminus A$; in which case $G_1 \subseteq G \setminus h[A]$. **Q**

So Lemma 2 gives the result.

? Theorem Let $\mathcal{F}$ be an $\mathbb{R}$-nowhere dense filter.

(a) $\mathcal{F}$ is X-nowhere dense for every locally compact metrizable space X without isolated points.

(b) $\mathcal{F}$ is X-nowhere dense for every non-discrete locally compact Hausdorff topological group X.

proof (a) Let $\langle G_j\rangle_{j\in J}$ be a maximal disjoint family of non-empty relatively compact open sets in X. For $j \in J$ set $K_j = \overline{G}_j$. Set $Y = \bigcup_{j\in J} K_j$; then $\operatorname{int} Y \supseteq \bigcup_{j\in J} G_j$ is dense, so $X \setminus Y$ is nowhere dense. For each $j \in J$, K_j and $[0, 1]$ have isomorphic regular open algebras. So if Z is the Stone space of the regular open algebra of $[0, 1]$, we have for each $j \in J$ a function $g_j : X_j \to Z$ such that $g_j^{-1}[B]$ is nowhere dense in K_j for every nowhere dense $B \subseteq Z$. Also we have a function $h : Z \to [0, 1]$ such that $h^{-1}[A]$ is nowhere dense in Z for every nowhere dense $A \subseteq [0, 1]$. Let $g : X \to [0, 1]$ be a function such that for every $y \in Y$ there is a $j \in J$ such that $y \in K_j$ and $g(y) = hg_j(y)$. If $A \subseteq [0, 1]$ is nowhere dense, then $G_j \cap g^{-1}[A] \subseteq g_j^{-1}[h^{-1}[A]]$ is nowhere dense in K_j, therefore nowhere dense in G_j, for every $j \in J$; so $g^{-1}[A]$ is nowhere dense. As $\mathcal{F}$ is $[0, 1]$-nowhere dense, it is X-nowhere dense.

(b)(i) If X is metrizable this follows from (a).

(ii) If X is σ-compact, let W be a compact neighbourhood of the identity e, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a sequence of neighbourhoods of e such that for each $n \in \mathbb{N}$ there is a set $K \in [W]^n$ such that $xV_n \cap yV_n = \emptyset$ for all distinct x, $y \in K$. Then there is a compact normal subgroup Y of X such that $Y \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and X/Y is metrizable (Fremlin 03, 4A5S). Let $\pi : X \to X/Y$ be the canonical map; this is continuous and open, so X/Y is locally compact (Fremlin 03, 4A5J). $\pi[W]$ is an infinite compact neighbourhood of the identity in X/Y, so the topology of X/Y is not discrete, and X/Y has no isolated points; by (a), $\mathcal{F}$ is X/Y-nowhere dense. $\pi^{-1}[B]$ is nowhere dense in X for every nowhere dense $B \subseteq X/Y$ (Fremlin 08, 4A5K(b-v)), so $\mathcal{F}$ is X-nowhere dense, by 2b above.

(iii) In general, X has a σ-compact open subgroup Y. Of course the topology on Y is not discrete. By (ii), $\mathcal{F}$ is Y-nowhere dense. Let $M \subseteq X$ be such that $\#(M \cap xY) = 1$ for every $x \in X$, and define $h : X \to Y$ by setting $h(x) = z^{-1}x$ whenever $z \in M \cap xY$. Then $h^{-1}[B] = MB$ is nowhere dense in X for every nowhere dense $B \subseteq Y$, so $\mathcal{F}$ is X-nowhere dense, by 2b again.

References

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