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Dependently selective filters

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I repeat and amplify the material of §2 of Fremlin p09.

1A Definitions Let $\mathcal{F}$ be a filter on a set X.

(a) I will say that $\mathcal{F}$ is **dependently selective** if it has the following property:

whenever $\mathcal{S} \subseteq [X]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{x : K \cup \{x\} \in \mathcal{S}\} \in \mathcal{F}$ for every $K \in \mathcal{S}$, then there is a $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}$.

(b) $\mathcal{F}$ is **uniform** if $\#(F) = \#(X)$ for every $F \in \mathcal{F}$.

(c) If $A \subseteq X$, set

$$\mathcal{F}\restriction A = \{F \cap A : F \in \mathcal{F}\} = \{B : B \subseteq A,\ B \cup (X \setminus A) \in \mathcal{F}\}.$$

Note that $\mathcal{F}\restriction A$ is $\mathcal{P}A$ if $X \setminus A \in \mathcal{F}$, and otherwise is a filter on A. If $A \in \mathcal{F}$ then $\mathcal{F}\restriction A = \mathcal{F} \cap \mathcal{P}A$.

1B Proposition Let X and Y be sets, $f : X \to Y$ a function, and $\mathcal{F}$ a dependently selective filter on X.
(a) The image filter $f[[\mathcal{F}]] = \{B : f^{-1}[B] \in \mathcal{F}\}$ is a dependently selective filter on Y.
(b) If $f[[\mathcal{F}]]$ is free, then there is an $F \in \mathcal{F}$ such that $f\restriction F$ is injective.
(c) If $\#(X) = \#(Y)$ and $f[[\mathcal{F}]]$ is free, then $\mathcal{F}$ and $f[[\mathcal{F}]]$ are isomorphic.

proof (a) Let $\mathcal{S} \subseteq [Y]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{y : K \cup \{y\} \in \mathcal{S}\} \in f[[\mathcal{F}]]$ for every $K \in \mathcal{S}$. Set $\mathcal{S}' = \{K : K \in [X]^{<\omega},\ f[K] \in \mathcal{S}\}$. Then $\emptyset \in \mathcal{S}'$. If $K \in \mathcal{S}'$, then $f[K] \in \mathcal{S}$, $\{y : f[K] \cup \{y\} \in \mathcal{S}\} \in f[[\mathcal{F}]]$ and

$$\{x : K \cup \{x\} \in \mathcal{S}'\} = \{x : x \in X,\ f[K] \cup \{f(x)\} \in \mathcal{S}\} = f^{-1}[\{y : f[K] \cup \{y\} \in \mathcal{S}\}]$$

belongs to $\mathcal{F}$. Because $\mathcal{F}$ is dependently selective, there is an $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}'$; now $f[F] \in f[[\mathcal{F}]]$ and $[f[F]\,]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitary, $f[[\mathcal{F}]]$ is dependently selective.

(b) If $f[[\mathcal{F}]]$ is free, consider

$$\mathcal{S} = \{K : K \in [X]^{<\omega},\ f\restriction K \text{ is injective}\}.$$

Of course $\emptyset \in \mathcal{S}$, and if $K \in \mathcal{S}$ then

$$\{x : K \cup \{x\} \in \mathcal{S}\} \supseteq f^{-1}[Y \setminus f[K]]$$

belongs to $\mathcal{F}$ because $Y \setminus f[K] \in f[[\mathcal{F}]]$. So there is an $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}$, that is, $f\restriction F$ is injective.

(c)(i) Set $\mathcal{G} = f[[\mathcal{F}]]$ and $B = f[F] \in \mathcal{G}$. Then there is a $C \subseteq B$ such that $C \in \mathcal{G}$ and $\#(B \setminus C) \geq \#(C)$. **P** As $\mathcal{G}$ is free, B is infinite. So we have a set Z and a function $g : Y \to Z$ such that $\#(B \cap g^{-1}[\{z\}]) = 2$ for every $z \in Z$. By (a), $\mathcal{G}$ is dependently selective; by (b), there is a $G \in \mathcal{G}$ such that $g\restriction G$ is injective. Now $C = B \cap G$ belongs to $\mathcal{G}$ and $g[B \setminus C] \supseteq g[C]$, so $\#(B \setminus C) \geq \#(C)$. **Q**

(ii) Now $F' = F \cap f^{-1}[C]$ belongs to $\mathcal{F}$, and $f[F'] = C$, $\#(X \setminus F') \geq \#(F')$, $\#(Y \setminus C) \geq \#(C)$. Consequently

$$\#(X \setminus F') = \#(X) = \#(Y) = \#(Y \setminus C),$$

and there is a bijection $h : X \to Y$ extending $f\restriction F'$; in which case $h[[\mathcal{F}]] = \mathcal{G}$ and $\mathcal{F} \cong \mathcal{G}$.

1C Proposition Let $\mathcal{F}$ be a filter on a set X, and A a subset of X such that $X \setminus A \notin \mathcal{F}$.
(a) If $\mathcal{F}$ is dependently selective, then $\mathcal{F}\restriction A$ is dependently selective.
(b) If $A \in \mathcal{F}$ and $\mathcal{F}\restriction A$ is dependently selective, then $\mathcal{F}$ is dependently selective.

proof (a) Let $\mathcal{S} \subseteq [A]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{x : K \cup \{x\} \in \mathcal{S}\} \in \mathcal{F}\lceil A$ for every $K \in \mathcal{S}$. Set

$$\mathcal{S}' = \{K : K \in [X]^{<\omega},\ K \cap A \in \mathcal{S}\}.$$

Then $\emptyset \in \mathcal{S}'$ and for $K \in \mathcal{S}'$

$$\{x : K \cup \{x\} \in \mathcal{S}'\} = (X \setminus A) \cup \{x : (K \cap A) \cup \{x\} \in \mathcal{S}\} \in \mathcal{F}.$$

So there is an $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}'$, and now $F \cap A \in \mathcal{F}\lceil A$ and $[F \cap A]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}\lceil A$ is dependently selective.

(b) Let $\mathcal{S} \subseteq [X]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{x : K \cup \{x\} \in \mathcal{S}\} \in \mathcal{F}$ for every $K \in \mathcal{S}$. Set

$$\mathcal{S}' = \{K \cap A : K \in \mathcal{S}\}.$$

Then $\emptyset \in \mathcal{S}'$ and for $K \in \mathcal{S}'$

$$\{x : K \cup \{x\} \in \mathcal{S}'\} = A \cap \{x : K \cup \{x\} \in \mathcal{S}\} \in \mathcal{F}\lceil A.$$

So there is an $F \in \mathcal{F}\lceil A$ such that $[F]^{<\omega} \subseteq \mathcal{S}'$, and now $F \in \mathcal{F}$ and $[F]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}\lceil A$ is dependently selective.

1D Proposition Let X be a set and $\mathcal{F}$ a dependently selective filter on X.

(a) $\mathcal{F}$ is a rapid p-point filter in the sense that for every sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ there is a $F \in \mathcal{F}$ such that $\#(F \setminus F_n) \le n$ for every $n \in \mathbb{N}$.

(b) If $\mathcal{A} \subseteq \mathcal{F}$ there is an $F \in \mathcal{F}$ such that $\#(F \setminus A) < \#(\mathcal{A})$ for every $A \in \mathcal{A}$.

proof (a) Let $\mathcal{S}$ be

$$\{K : K \in [X]^{<\omega},\ \#(K \setminus F_n) \le n \text{ for every } n \in \mathbb{N}\}.$$

Of course $\emptyset \in \mathcal{S}$. If $K \in \mathcal{S}$, then

$$\{x : K \cup \{x\} \in \mathcal{S}\} \supseteq \textstyle\bigcap_{n\le\#(K)} F_n \in \mathcal{F},$$

so there is a $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}$ and $\#(F \setminus F_n) \le n$ for every n.

(b) If $\mathcal{A}$ is finite we can take $F = X \cap \bigcap \mathcal{A}$. Otherwise, enumerate $\mathcal{A}$ as $\langle F_\xi\rangle_{\xi<\kappa}$. Set $L = \bigcap \mathcal{A}$, and for $x \in X \setminus L$ set $f(x) = \min\{\xi : i \notin F_\xi\}$. Let $\mathcal{S}$ be the family of finite sets $K \subseteq X$ such that $f\restriction K \setminus L$ is injective. Of course $\emptyset \in \mathcal{S}$. If $K \in \mathcal{S}$ then

$$\{x : K \cup \{x\} \in \mathcal{S}\} \supseteq X \cap \textstyle\bigcap_{y\in K\setminus L} F_{f(y)}$$

belongs to $\mathcal{F}$. So there is a $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}$. Suppose that $\xi < \kappa$ and consider $C = F \setminus F_\xi$. If x, $y \in C$ then $f(x) \ne f(y)$, and both $f(x)$ and $f(y)$ are at most ξ; so $\#(C) \le \#(\xi + 1) < \kappa$, as required.

1E Proposition Let X be a set and $\mathcal{F}$ a dependently selective filter on X. Set $\kappa = \min\{\#(A) : A \subseteq X, X \setminus A \notin \mathcal{F}\}$.

(a) $\mathcal{F}$ is κ-complete.

(b) κ is either 1 or a regular infinite cardinal.

proof (a) If $\kappa \le \omega$ this is trivial. Otherwise, if $\mathcal{A} \in [\mathcal{F}]^{<\kappa}$, then by 1Db there is an $F \in \mathcal{F}$ such that $\#(F \setminus A) < \#(\mathcal{A})$ for every $A \in \mathcal{A}$. So $B = \bigcup_{F\in\mathcal{F}} F \setminus A$ has cardinal at most $\max(\omega, \#(\mathcal{A})) < \kappa$, and $X \setminus B$ and $F \setminus B$ belong to $\mathcal{F}$. But $F \setminus B \subseteq \bigcap \mathcal{A}$, so $X \cap \bigcap \mathcal{A} \in \mathcal{F}$.

(b) Of course $\kappa \ne 0$. If $\kappa > 1$, then $X \setminus \{x\} \in \mathcal{F}$ for every $x \in X$, so $\kappa \ge \omega$. If $\kappa = \omega$ it is certainly regular, so suppose that $\kappa > \omega$. Let $A \in [X]^\kappa$ be such that $X \setminus A \notin \mathcal{F}$, and enumerate A as $\langle x_\xi\rangle_{\xi<\kappa}$. If $C \subseteq \kappa$ is cofinal with κ, then for $\zeta \in C$ set $F_\zeta = X \setminus \{x_\xi : \xi \le \zeta\}$; then $F_\zeta \in \mathcal{F}$ for every $\zeta \in C$, while $\bigcap_{\zeta\in C} F_\zeta = X \setminus A$ does not belong to $\mathcal{F}$. By (a), $\#(C) \ge \kappa$. As C is arbitrary, κ is regular.

Remark If we take $\mathcal{I}$ to be the dual ideal $\{A : A \subseteq X,\ X \setminus A \in \mathcal{F}\}$, then κ is the uniformity $\operatorname{non}\mathcal{I}$ of $\mathcal{I}$ (Fremlin 08, 511F), and (b) can be restated as '$\operatorname{add}\mathcal{I} \ge \operatorname{non}\mathcal{I}$', where $\operatorname{add}\mathcal{I}$ is the additivity of $\mathcal{X}$ (Fremlin 08, 511B). If $\mathcal{F}$ is free (that is, contains all cofinite subsets of X) then we must have equality.

1F Proposition Let $\mathcal{F}$ and $\mathcal{G}$ be dependently selective filters on a set X such that $F \cap G$ is non-empty for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, and let $\mathcal{F} \vee \mathcal{G}$ be the filter on X generated by $\mathcal{F} \cup \mathcal{G}$. Then $\mathcal{F} \vee \mathcal{G}$ is dependently selective.

proof (a) To begin with, suppose that $\mathcal{F}$ and $\mathcal{G}$ are both uniform. The case of finite X is trivial, so we may suppose that $X = \kappa$ is an infinite cardinal; by 1E, κ is regular and both $\mathcal{F}$ and $\mathcal{G}$ are κ-additive.

Let $\mathcal{S} \subseteq [\kappa]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $\{\xi : K \cup \{\xi\} \in \mathcal{S}\} \in \mathcal{F} \vee \mathcal{G}$ for every $K \in \mathcal{S}$. For $K \in \mathcal{S}$ let $F_K \in \mathcal{F}$, $G_K \in \mathcal{G}$ be such that $K \cup \{x\} \in \mathcal{S}$ whenever $x \in F_K \cap G_K$. Set

$$\mathcal{S}' = \{K : K \in [\kappa]^{<\omega},\ \xi \in F_L \text{ whenever } \xi \in K \text{ and } L \subseteq K \cap \xi \text{ belongs to } \mathcal{S}\}.$$

Then $\emptyset \in \mathcal{S}'$. If $K \in \mathcal{S}'$ then

$$\{\xi : K \cup \{\xi\} \in \mathcal{S}'\} \supseteq \{\xi : K \subseteq \xi < \kappa,\ \xi \in F_L \text{ whenever } L \subseteq K \text{ belongs to } \mathcal{S}\}$$

belongs to $\mathcal{F}$. (This is where we need to know that $\mathcal{F}$ is uniform.) So there is an $F \in \mathcal{F}$ such that $[F]^{<\omega} \subseteq \mathcal{S}'$. Similarly, setting

$$\mathcal{S}'' = \{K : K \in [\kappa]^{<\omega},\ \xi \in G_L \text{ whenever } \xi \in K \text{ and } L \subseteq K \cap \xi \text{ belongs to } \mathcal{S}\},$$

there is a $G \in \mathcal{G}$ such that $[G]^{<\omega} \subseteq \mathcal{S}''$. Set $H = F \cap G \in \mathcal{F} \vee \mathcal{G}$; then $[H]^n \subseteq \mathcal{S}$ for every $n \in \mathbb{N}$. **P** Induce on n. The case $n = 0$ is trivial. For the inductive step to $n+1$, take $K \in [H]^{n+1}$ and set $\xi = \max K$, $L = K \setminus \{\xi\}$. By the inductive hypothesis, $L \in \mathcal{S}$. As $K \subseteq F$, $K \in \mathcal{S}'$ and $\xi \in F_L$; similarly, $\xi \in G_L$, so $K = L \cup \{\xi\} \in \mathcal{S}$ by the choice of F_L and G_L. Thus the induction proceeds. **Q**

So $[H]^{<\omega} \subseteq \mathcal{S}$; as $\mathcal{S}$ is arbitrary, $\mathcal{F} \vee \mathcal{G}$ is dependently selective.

(b) For the general case, let $A \in \mathcal{F} \vee \mathcal{G}$ be a set of minimal cardinality. Then $(\mathcal{F} \vee \mathcal{G})\lceil A = (\mathcal{F}\lceil A) \vee (\mathcal{G}\lceil A)$, so $\mathcal{F}\lceil A$ and $\mathcal{G}\lceil A$ are both uniform; by 1Ca, they are dependently selective. So (a) tells us that $(\mathcal{F} \vee \mathcal{G})\lceil A$ is dependently selective, and now 1Cb tells us that $\mathcal{F} \vee \mathcal{G}$ itself if dependently selective.

1G Proposition (a) Let κ be a regular uncountable cardinal and $\mathcal{F}$ a normal filter on κ (definition: Fremlin 03, 4A1Ic). Then $\mathcal{F}$ is dependently selective.

(b) If κ is any cardinal of uncountable cofinality, the filter generated by the closed cofinal subsets of κ is dependently selective.

proof (a) Let $\mathcal{S} \subseteq [\kappa]^{<\omega}$ be such that $\emptyset \in \mathcal{S}$ and $F_K = \{\xi : K \cup \{\xi\} \in \mathcal{S}\} \in \mathcal{F}$ for every $K \in \mathcal{S}$. For each $\xi < \kappa$, set $F'_\xi = \bigcap\{F_K : K \in \mathcal{S} \cap [\xi+1]^{<\omega}\}$; because $\mathcal{F}$ is κ-complete (Fremlin 03, 4A1J), $F'_\xi \in \mathcal{F}$. Let F be the diagonal intersection of $\langle F'_\xi \rangle_{\xi<\kappa}$. Because $\mathcal{F}$ is normal, F and $F \cap F_\emptyset$ belong to $\mathcal{F}$. Now $[F \cap F_\emptyset]^n \subseteq \mathcal{S}$ for every $n \in \mathbb{N}$. **P** Induce on n. The case $n = 0$ is trivial, and $\{\xi\} \in \mathcal{S}$ for every $\xi \in F_\emptyset$, which deals with the case $n = 1$. For the inductive step to $n+1 \ge 2$, take $K \in [F]^{n+2}$ and set $\eta = \max K$, $J = K \setminus \{\eta\}$ and $\xi = \max J$. By the inductive hypothesis, $J \in \mathcal{S}$, so $F'_\xi \subseteq F_J$; since $\eta \in F$ and $\eta > \xi$, $\eta \in F'_\xi$ and $K = J \cup \{\eta\}$ belongs to $\mathcal{S}$. Thus the induction proceeds. **Q** At the end of the induction, we have $[F \cap F_\emptyset]^{<\omega} \subseteq \mathcal{S}$; as $\mathcal{S}$ is arbitrary, $\mathcal{F}$ is dependently selective.

(b) Set $\lambda = \operatorname{cf}\kappa$. Then we have an order-continuous strictly increasing function $f : \lambda \to \kappa$ such that $f[\lambda]$ is cofinal with κ. The filter $\mathcal{F}$ on λ generated by the closed cofinal subsets of λ is normal (Fremlin 03, 4A1B(c-ii)), so is dependently selective, by (a); by 1Ba, $f[[\mathcal{F}]]$ is a dependently selective filter on κ; but $f[[\mathcal{F}]]$ is the filter generated by the closed cofinal subsets of κ.

1H Proposition Suppose that $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$. Let $\mathcal{A}$ be a family of fewer than $\mathfrak{c}$ infinite subsets of $\mathbb{N}$. Then there is a free dependently selective filter $\mathcal{F}$ on $\mathbb{N}$ such that $\mathbb{N} \setminus A \notin \mathcal{F}$ for every $A \in \mathcal{A}$.

Remark For the definition and basic properties of $\mathfrak{m}_{\text{countable}}$, see Fremlin 08, 517O-517Q and 522R.

proof Enumerate $\mathcal{P}([\mathbb{N}]^{<\omega})$ as $\langle \mathcal{S}_\xi \rangle_{\xi<\mathfrak{c}}$. For $\xi < \mathfrak{c}$ and $K \in [\mathbb{N}]^{<\omega}$ set $C_\xi(K) = \{n : K \cup \{n\} \in \mathcal{S}_\xi\}$. Choose a non-decreasing family $\langle \mathcal{E}_\xi \rangle_{\xi<\mathfrak{c}}$ of filter bases inductively, as follows. $\mathcal{E}_0 = \{\mathbb{N} \setminus n : n \in \mathbb{N}\}$. Given that $\#(\mathcal{E}_\xi) \le \max(\omega, \#(\xi))$ and that $E \cap A$ is non-empty for every $E \in \mathcal{E}_\xi$ and $A \in \mathcal{A}$, consider $\mathcal{S}_\xi$. If either $\emptyset \notin \mathcal{S}_\xi$ or there are $E \in \mathcal{E}_\xi$, $A \in \mathcal{A}$ and a finite family $\mathcal{K} \subseteq \mathcal{S}_\xi$ such that $\bigcap_{K \in \mathcal{K}} C_\xi(K) \cap E \cap A = \emptyset$, set $\mathcal{E}_{\xi+1} = \mathcal{E}_\xi$ and continue. Otherwise, set $\mathcal{S}'_\xi = \{K : \mathcal{P}K \subseteq \mathcal{S}_\xi\}$; then $\emptyset \in \mathcal{S}'_\xi$ and

$$\{K : K \in \mathcal{S}'_\xi,\ K \cap E \cap A \ne \emptyset\}$$

is cofinal with $\mathcal{S}'_\xi$ for every $E \in \mathcal{E}_\xi$ and $A \in \mathcal{A}$. Because $\#(\mathcal{E}_\xi \cup \mathcal{A}) < \mathfrak{m}_{\text{countable}}$, there is a J_ξ, meeting $E \cap A$ for every $E \in \mathcal{E}_\xi$ and $A \in \mathcal{A}$, such that $[J_\xi]^{<\omega} \subseteq \mathcal{S}'_\xi$. Set

$$\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{J_\xi \cap E : E \in \mathcal{E}_\xi\},$$

and continue.

At non-zero limit ordinals $\xi \le \mathfrak{c}$, set $\mathcal{E}_\xi = \bigcup_{\eta<\xi} \mathcal{E}_\eta$.

At the end of the induction, let $\mathcal{F}$ be the filter generated by $\mathcal{E}_\mathfrak{c}$. If $A \in \mathcal{A}$, then $F \cap A \neq \emptyset$ for every $F \in \mathcal{F}$, so $\mathbb{N} \setminus A \notin \mathcal{F}$. If $\mathcal{S} \subseteq [\mathbb{N}]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{n : K \cup \{n\} \in \mathcal{F}\}$ for every $K \in \mathcal{S}$, let $\xi < \mathfrak{c}$ be such that $\mathcal{S} = \mathcal{S}_\xi$. Then $\emptyset \in \mathcal{S}_\xi$ and if $\mathcal{K} \subseteq \mathcal{S}_\xi$ is finite, $\mathbb{N} \cap \bigcap_{K\in\mathcal{K}} C_\xi(K)$ belongs to $\mathcal{F}$ and must meet $E \cap A$ whenever $E \in \mathcal{E}_\xi$ and $A \in \mathcal{A}$. We therefore applied the second rule when determining $\mathcal{E}_{\xi+1}$, and $J_\xi \in \mathcal{F}$ is such that $[J_\xi]^{<\omega} \subseteq \mathcal{S}'_\xi \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}$ is dependently selective.

Remark In terms of the dual ideal $\mathcal{I}$ of $\mathcal{F}$, $\mathcal{A} \cap \mathcal{I} = \emptyset$. So if, for instance, $\mathcal{A}$ is almost disjoint, or we could otherwise arrange that $A \cap B \in \mathcal{E}_0$ for all distinct A, $B \in \mathcal{A}$, we get $\text{sat}(\mathcal{P}\mathbb{N}/\mathcal{I}) > \#(\mathcal{A})$, and in particular, $\mathcal{I}$ need not be ω_1-saturated, at least if $\mathfrak{m}_{\text{countable}} = \mathfrak{c}$.

Conceivably things are different in random real models. See Problem 3A.

2 Ramsey ultrafilters

2A Definition If X is an infinite set, a filter $\mathcal{F}$ on X is **Ramsey** if it is uniform and for every $S \subseteq [X]^2$ there is a $F \in \mathcal{F}$ such that either $[F]^2 \subseteq S$ or $[F]^2 \cap S \neq \emptyset$.

2B Theorem (see Comfort & Negrepontis 74, Theorem 9.6) Let $\mathcal{F}$ be a uniform Ramsey ultrafilter on an infinite cardinal κ.

(a) $\mathcal{F}$ is κ-complete.

(b) If κ is uncountable, then it is two-valued-measurable and there is a bijection $f : \kappa \to \kappa$ such that $f[[\mathcal{F}]]$ is a normal ultrafilter.

(c) If κ is uncountable, then for every $S \subseteq [\kappa]^{<\omega}$ there is an $X \in \mathcal{F}$ such that for each $n \in \mathbb{N}$ either $[X]^n \subseteq S$ or $[X]^n \cap S = \emptyset$.

proof (a) ? Otherwise, let $\lambda < \kappa$ be the least cardinal such that there is a non-empty family $\mathcal{E} \in [\mathcal{F}]^{\le\lambda}$ such that $\bigcap \mathcal{E} \notin \mathcal{F}$. Then there is a non-increasing family $\langle F_\alpha \rangle_{\alpha<\lambda}$ in $\mathcal{F}$ such that $L = \bigcap_{\alpha<\lambda} F_\alpha \notin \mathcal{F}$ and $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ if $\alpha < \lambda$ is a non-zero limit ordinal.

Set

$$S = \bigcup_{\alpha<\lambda} \{\{\xi, \eta\} : \xi \in \kappa \setminus F_\alpha,\ \eta \in F_\alpha,\ \xi < \eta\} \subseteq [\kappa]^2.$$

Then there is an $F \in \mathcal{F}$ such that either $[F]^2 \subseteq S$ or $[F]^2 \cap S = \emptyset$.

In fact $[F]^2 \subseteq S$. **P** Take any $\xi \in F \setminus L$. Then there is an $\alpha < \lambda$ such that $\xi \notin F_\alpha$. Now $F \cap F_\alpha$ belongs to $\mathcal{F}$, so has cardinal κ, and there must be an $\eta \in F \cap F_\alpha$ such that $\xi < \eta$; in which case $\{\xi, \eta\} \in [F]^2 \cap S$. Thus $[F]^2 \cap S \neq \emptyset$ and $[F]^2 \subseteq S$. **Q**

Since $\#(F \setminus L) = \kappa > \lambda$, there must be a $\beta < \kappa$ such that $F \cap F_\beta \setminus F_{\beta+1}$ has more than one member. Suppose that ξ, $\eta \in F \cap F_\beta \setminus F_{\beta+1}$ and $\xi < \eta$. Then there is an $\alpha < \lambda$ such that $\xi \notin F_\alpha$ (so $\alpha > \beta$) and $\eta \in F_\alpha$ (so $\alpha \le \beta$); which is absurd. **X**

(b) Part (a) tells us immediately that κ is regular. By Fremlin 08, 541F, there are a set $Y \subseteq \kappa$ and a function $g : Y \to \kappa$ such that $\{B : B \subseteq \kappa,\ g^{-1}[B] \notin \mathcal{F}\}$ is a normal principal ideal of $\mathcal{P}\kappa$. Of course it follows that $Y \in \mathcal{F}$ and that $\mathcal{G} = \{B : B \subseteq \kappa,\ g^{-1}[B] \in \mathcal{F}\}$ is a normal ultrafilter on κ. Extending g to the whole of κ by setting $g(\xi) = 0$ for $\xi \in \kappa \setminus Y$, we have $g : \kappa \to \kappa$ such that $\mathcal{G} = g[[\mathcal{F}]]$.

Consider the set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa,\ g(\xi) = g(\eta)\}.$$

If $F \in \mathcal{F}$ then $g[F] \in \mathcal{G}$ has cardinal κ, so $[F]^2 \not\subseteq S$; it follows that there is an $F \in \mathcal{F}$ such that $[F]^2 \cap S = \emptyset$, that is, $g{\restriction}F$ is injective. Next, there is certainly a partition of F into two sets of cardinal κ, just one of which belongs to κ; so we can suppose that both $\kappa \setminus F$ and $\kappa \setminus g[F]$ have cardinal κ. In this case, there is an extension of $g{\restriction}F$ to a bijection $f : \kappa \to \kappa$, and $f[[\mathcal{F}]] = \mathcal{G}$ is a normal ultrafilter on κ.

(c) This is true for normal ultrafilters by Rowbottom's theorem (Fremlin 03, 4A1L); by (b), it is true for Ramsey ultrafilters.

2C Proposition If X is an infinite set, an ultrafilter on X is Ramsey iff it is uniform and dependently selective.

proof It is enough to consider the case in which $X = \kappa$ is a cardinal.

(a)(i) If $\mathcal{F}$ is a Ramsey ultrafilter on κ, then it is uniform (by definition) and κ-complete, by 2Ba. It follows that if $\langle F_\xi \rangle_{\xi<\kappa}$ is any family in $\mathcal{F}$, there is an $F \in \mathcal{F}$ such that $F \setminus F_\xi \subseteq \xi + 1$ for every $\xi \in F$. **P** Set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa, \, \eta \in F_\xi\}.$$

If $F \in \mathcal{F}$ and $\xi \in F$, then $F \cap F_\xi \setminus (\xi + 1)$ belongs to $\mathcal{F}$, so there is an $\eta \in F \cap F_\xi$ such that $\eta > \xi$ and $\{\xi, \eta\} \in S$. Thus $[F]^2 \cap S \neq \emptyset$ for every $F \in \mathcal{F}$; because $\mathcal{F}$ is a Ramsey ultrafilter, there is $F \in \mathcal{F}$ such that $[F]^2 \subseteq S$. Now $F \setminus F_\xi \subseteq \xi + 1$ for every $\xi \in F$. **Q**

(ii) Now suppose that $\mathcal{S} \subseteq [\kappa]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{\xi : K \cup \{\xi\} \in \mathcal{S}\} \in \mathcal{F}$ for every $K \in \mathcal{S}$. For $\xi < \kappa$ set

$$F_\xi = \{\eta : K \cup \{\eta\} \in \mathcal{S} \text{ whenever } K \in [\xi + 1]^{<\omega} \text{ and } K \in \mathcal{S}\}.$$

Then F_ξ is the intersection of fewer than κ members of $\mathcal{F}$ and belongs to $\mathcal{F}$. By (i), there is a $F \in \mathcal{F}$ such that $F \setminus F_\xi \subseteq \xi + 1$ for every $\xi \in F$; and we can suppose that $F \subseteq F_0$. Now $K \in \mathcal{S}$ whenever $n \in \mathbb{N}$ and $K \in [F]^n$. **P** Induce on n. If $n = 0$ we just have to recall that $\emptyset \in \mathcal{S}$. If $n = 1$, then $K = \{\eta\}$ for some $\eta \in F_0$, so $\{\eta\} \in \mathcal{S}$. For the inductive step to $n \geq 2$, set $\eta = \max K$, $K' = K \setminus \{\eta\}$ and $\xi = \max K'$. Because $\xi, \eta \in F$ and $\xi < \eta$, $\eta \in F_\xi$; $K' \subseteq \xi + 1$ and $K' \in \mathcal{S}$, by the inductive hypothesis; so $K = K' \cup \{\eta\} \in \mathcal{S}$ and the induction proceeds. **Q**

So $[F]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}$ is dependently selective.

(b) If $\mathcal{F}$ is a uniform dependently selective ultrafilter on κ, take any $S \subseteq [\kappa]^2$. For $\xi < \kappa$ set $A_\xi = \{\eta : \{\xi, \eta\} \in S\}$. Let $\mathcal{S}$ be the family of finite subsets K of κ such that for all $\xi, \eta \in K$ such that $\xi < \eta$, $\{\xi, \eta\} \in S$ iff $A_\xi \in \mathcal{F}$. If $K \in \mathcal{S}$, then (because $\mathcal{F}$ is an ultrafilter) there is a $F \in \mathcal{F}$ such that, for every $\xi \in K$, F is either included in A_ξ or disjoint from A_ξ. Now $K \cup \{\eta\} \in \mathcal{S}$ whenever $\eta \in F$ and $\eta > \xi$ for every $\xi \in K$. So $\mathcal{S}$ satisfies the condition of 1A. Let $F \in \mathcal{F}$ be such that $[F]^{<\omega} \subseteq \mathcal{S}$. In this case, if $\xi, \eta \in F$ and $\xi < \eta$, $\{\xi, \eta\} \in S$ iff $A_\xi \in \mathcal{F}$. Now

$$F_1 = \{\xi : \xi \in F, \, A_\xi \in \mathcal{F}\}, \quad F_0 = \{\xi : \xi \in F, \, A_\xi \notin \mathcal{F}\}$$

have union F and one of them must belong to $\mathcal{F}$; while $[F_0]^2 \cap S = \emptyset$ and $[F_1]^2 \subseteq S$. As S is arbitrary, $\mathcal{F}$ is a Ramsey ultrafilter.

2D Lemma (a) Let X be an infinite set, $\mathcal{F}$ a Ramsey ultrafilter on X, and $\mathcal{A} \subseteq \mathcal{F}$ a set of size at most $\#(X)$. Then there is a $C \in \mathcal{F}$ such that $\#(C \setminus A) < \#(X)$ for every $A \in \mathcal{A}$.

(b) Let κ be an infinite cardinal, $\lambda \leq \kappa$ another cardinal, and $\langle \mathcal{F}_\alpha \rangle_{\alpha<\lambda}$ a family of distinct Ramsey ultrafilters on κ. Then there is a disjoint family $\langle A_\alpha \rangle_{\alpha<\lambda}$ of subsets of κ such that $A_\alpha \in \mathcal{F}_\alpha$ for every $\alpha < \lambda$.

proof (a) Set $A^* = \kappa \cap \bigcap \mathcal{A}$. If $A^* \in \mathcal{F}$, we can set $C = A^*$ and stop. Otherwise, enumerate $\mathcal{A}$ as $\langle A_\alpha \rangle_{\alpha<\lambda}$. For $i \in X$, set $f(i) = \min\{\alpha : \alpha < \lambda, \, i \notin A_\alpha \setminus A^*\}$. Then there is a $C \in \mathcal{F}$ such that $f \restriction C$ is either constant or injective (Comfort & Negrepontis 74, 9.6). The former is impossible, because $\{i : f(i) = \alpha\}$ never belongs to $\mathcal{F}$. So $f \restriction C$ is injective and $C \setminus A_\alpha = \{i : i \in C, \, f(i) \leq \alpha\}$ has cardinal less than κ for every $\alpha < \lambda$.

(b) For $\alpha < \beta < \lambda$, take $A_{\alpha\beta} \in \mathcal{F}_\beta \setminus \mathcal{F}_\alpha$. For each $\alpha < \kappa$, there is a $B_\alpha \in \mathcal{F}_\alpha$ such that $\#(B_\alpha \cap A_{\alpha\beta}) < \kappa$ for every $\beta > \alpha$ (apply (a) to $\{X \setminus A_{\alpha\beta} : \alpha < \beta < \lambda\} \subseteq \mathcal{F}_\alpha$)). Set

$$A_\beta = B_\beta \setminus \bigcup_{\alpha<\beta} B_\alpha$$

for $\beta < \lambda$. Of course $\langle A_\beta \rangle_{\beta<\lambda}$ is disjoint. On the other hand, for each $\beta < \lambda$, $A'_\beta = B_\beta \cap \bigcap_{\alpha<\beta} A_{\alpha\beta}$ belongs to $\mathcal{F}$ because $\mathcal{F}$ is κ-complete; and $A_\beta \setminus A'_\beta \subseteq \bigcup_{\alpha<\beta} A_{\alpha\beta} \cap B_\alpha$ has cardinal less than κ, so A_β also belongs to $\mathcal{F}$.

2E Proposition Let X be an infinite set, and $\mathfrak{F}$ a non-empty family of non-isomorphic Ramsey ultrafilters on X with $\#(\mathfrak{F}) \le \#(X)$. Then $\mathcal{H} = \bigcap \mathfrak{F}$ is a dependently selective filter on X.

proof (a) It is enough to consider the case in which $X = \kappa$ is a cardinal. Let $\langle \mathcal{F}_\alpha \rangle_{\alpha<\lambda}$ be an enumeration of $\mathfrak{F}$.

(b) If $\langle A_\alpha \rangle_{\alpha<\lambda}$ is such that $A_\alpha \in \mathcal{F}_\alpha$ for $\alpha < \lambda$, then there is a family $\langle D_\alpha \rangle_{\alpha<\lambda}$ such that $D_\alpha \in \mathcal{F}_\alpha$ and $D_\alpha \subseteq A_\alpha$ for every $\alpha < \lambda$, and whenever $\xi < \eta < \kappa$, α, $\beta < \lambda$ are such that $\xi \in D_\alpha$ and $\eta \in D_\beta$, there is a $\zeta \in A_\beta$ such that $\xi \le \zeta < \eta$. **P** By 2Db, we may suppose that $\langle A_\alpha \rangle_{\alpha<\lambda}$ is disjoint. For any $\zeta < \kappa$, $\{\alpha : \alpha < \lambda,\ A_\alpha \cap \zeta \ne \emptyset\}$ has cardinal less than κ; so there is a closed cofinal set $F \subseteq \kappa$, containing 0, such that $A_\alpha \cap \zeta' \setminus \zeta \ne \emptyset$ whenever $\zeta < \zeta'$ in F, $\alpha < \lambda$ and $A_\alpha \cap \zeta \ne \emptyset$. Set $f(\xi) = \max\{\zeta : \zeta \in F,\ \zeta \le \xi\}$ for $\xi < \kappa$. Then $\langle f[[\mathcal{F}_\alpha]] \rangle_{\alpha<\lambda}$ is a family of κ-complete uniform ultrafilters on F, so there must be a cofinal set $V \subseteq F$ not belonging to any of them. (We can easily build inductively a family $\langle V_\xi \rangle_{\xi<\kappa^+}$ of cofinal subsets of F such that $\#(V_\xi \cap V_\eta) < \kappa$ whenever $\xi < \eta < \kappa^+$, and now each $f[[\mathcal{F}_\alpha]]$ can contain V_ξ for at most one ξ, so there is a ξ left over for which we can set $V = V_\xi$.) Set $M = f^{-1}[V]$; then $A_\alpha \setminus M \in \mathcal{F}_\alpha$ for each α.

Define $g : \kappa \to \kappa$ by setting $g(\xi) = \min\{\zeta : \xi \le \zeta \in V\}$ for $\xi < \kappa$. By 1Bc, or otherwise, $g[[\mathcal{F}_\alpha]]$ is isomorphic to $\mathcal{F}_\alpha$, and is surely a Ramsey ultrafilter. Because the $\mathcal{F}_\alpha$ are non-isomorphic, all the $g[[\mathcal{F}_\alpha]]$ are different. By 2Db again, there is a disjoint family $\langle G_\alpha \rangle_{\alpha<\lambda}$ of sets such that $G_\alpha \in g[[\mathcal{F}_\alpha]]$ for every α.

Set

$$C_\alpha = A_\alpha \cap B_\alpha \cap g^{-1}[G_\alpha] \setminus M, \quad D_\alpha = C_\alpha \setminus \{\min C_\alpha\} \in \mathcal{F}_\alpha$$

for each $\alpha < \lambda$. Suppose that $\xi \in D_\alpha$, $\eta \in D_\beta$ and $\xi < \eta$. Then $g(\xi) < g(\eta)$. **P** If $\alpha = \beta$, this is because $g \restriction B_\alpha$ is injective; otherwise, it is because $G_\alpha \cap G_\beta$ is empty. **Q** Let η_0 be the least member of C_β. We have $\eta_0 < \eta$. If $\xi \le \eta_0$, then η_0 is a member of $A_\beta \cap \eta \setminus \xi$. Otherwise, $A_\beta \cap g(\xi) \ne \emptyset$, so there is a $\zeta \in A_\beta \cap \gamma \setminus g(\xi)$, where γ is the next member of F above $g(\xi)$. Now $\gamma \setminus g(\xi) = f^{-1}[\{g(\xi)\}] \subseteq M$ is disjoint from D_β, so $\gamma \le \eta$ and $\zeta \in A_\beta \cap \eta \setminus \xi$.

Thus $\langle D_\alpha \rangle_{\alpha<\lambda}$ is a suitable family. **Q**

(c) Now suppose that $\mathcal{S}$ is a family of finite subsets of κ such that $\emptyset \in \mathcal{S}$ and $\{\xi : K \cup \{\xi\} \in \mathcal{S}\} \in \mathcal{H}$ for every $K \in \mathcal{S}$. For each $\alpha < \lambda$, set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa,\ K \cup \{\eta\} \in \mathcal{S} \text{ whenever } K \in \mathcal{S} \text{ and } K \subseteq \xi + 1\}.$$

Then there is an $A_\alpha \in \mathcal{F}_\alpha$ such that $[A_\alpha]^2$ is either included in or disjoint from S_α. But taking $\xi = \min A_\alpha$, we see that $\{\eta : \eta > \xi,\ K \cup \{\eta\} \in \mathcal{S}\}$ belongs to $\mathcal{H} \subseteq \mathcal{F}_\alpha$ for every $K \in \mathcal{S}$; because $\mathcal{F}_\alpha$ is κ-complete, there must be an $\eta \in A_\alpha$ such that $\eta > \xi$ and $K \cup \{\eta\} \in \mathcal{S}$ whenever $K \in \mathcal{S}$ and $K \subseteq \xi + 1$, in which case $\{\xi, \eta\} \in S$. So we must have $[A_\alpha]^2 \subseteq S$. Set $A'_\alpha = \{\xi : \xi \in A_\alpha,\ \{\xi\} \in \mathcal{S}\}$; then $A'_\alpha \in \mathcal{F}_\alpha$ because $\{\xi : \{\xi\} \in \mathcal{S}\} \in \mathcal{H} \subseteq \mathcal{F}_\alpha$.

By (b), we have a family $\langle D_\alpha \rangle_{\alpha<\lambda}$ of sets such that $D_\alpha \in \mathcal{F}_\alpha$ and $D_\alpha \subseteq A'_\alpha$ for every $\alpha < \lambda$, and whenever $\xi < \eta < \kappa$, α, $\beta < \lambda$ are such that $\xi \in D_\alpha$ and $\eta \in D_\beta$, there is a $\zeta \in A'_\beta$ such that $\xi \le \zeta < \eta$. Set $A = \bigcup_{\alpha<\lambda} D_\alpha \in \mathcal{H}$. Then $[A]^n \subseteq \mathcal{S}$ for every n. **P** Induce on n. The case $n = 0$ is trivial, and the case $n = 1$ has been dealt with when defining A'_α. For the inductive step to $n+1 \ge 2$, suppose that $X \in [A]^{n+1}$. Let $\xi < \eta$ be the two greatest points of X; suppose that $\eta \in D_\beta$. Then there is a $\zeta \in A'_\beta$ such that $\xi \le \zeta < \eta$. In this case, $K = X \setminus \{\eta\}$ belongs to $[A]^n \subseteq \mathcal{S}$ and $K \subseteq \zeta + 1$. Also $\{\zeta, \eta\} \in [A_\beta]^2 \subseteq S$, so $X = K \cup \{\eta\} \in \mathcal{S}$. Thus the induction continues. **Q**

So $[A]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}$ is dependently selective.

2F Proposition Let X be a set, and $\mathfrak{F}$ a non-empty countable family of non-isomorphic dependently selective ultrafilters on X. Then

(a) there is a disjoint family $\langle A_\mathcal{F} \rangle_{\mathcal{F}\in\mathfrak{F}}$ of sets such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$;

(b) $\mathcal{H} = \bigcap \mathfrak{F}$ is dependently selective.

proof (a) For each $\mathcal{F} \in \mathfrak{F}$, let $X_\mathcal{F} \in \mathcal{F}$ be a set of minimal size. Let K be the countable set $\{\#(X_\mathcal{F}) : \mathcal{F} \in \mathfrak{F}\}$; for $\kappa \in \mathrm{K}$, set $\mathfrak{F}_\kappa = \{\mathcal{F} : \mathcal{F} \in \mathfrak{F},\ \#(X_\mathcal{F}) = \kappa\}$ and $F_\kappa = \bigcup_{\mathcal{F}\in\mathfrak{F}_\kappa} X_\mathcal{F}$, so that $\#(F_\kappa) \le \kappa$. (For if $\kappa = 1$, any member of $\mathfrak{F}_\kappa$ is a principal ultrafilter, and there can be at most one such.) Set $F'_\kappa = F_\kappa \setminus \bigcup_{\lambda\in\mathrm{K},\lambda<\kappa} F_\lambda$ for $\kappa \in \mathrm{K}$; then $\langle F'_\kappa \rangle_{\kappa\in\mathrm{K}}$ is disjoint and $F'_\kappa \in \mathcal{F}$ whenever $\kappa \in \mathrm{K}$ and $\mathcal{F} \in \mathfrak{F}_\kappa$.

For $\mathcal{F} \in \mathfrak{F}$, let $\mathcal{F}' = \mathcal{F} \cap \mathcal{P}F'_\kappa$ be the trace of $\mathcal{F}$ on F'_κ, where $\kappa \in \mathrm{K}$ is such that $\mathcal{F} \in \mathfrak{F}_\kappa$. Then $\mathcal{F}'$ is either a principal ultrafilter or a Ramsey ultrafilter. Moreover, $\mathcal{F}'$ and $\mathcal{G}'$ must be non-isomorphic whenever $\mathcal{F}$, $\mathcal{G}$ are distinct members of the same $\mathfrak{F}_\kappa$. So 2Db tells us that we have for each $\kappa \in \mathrm{K}$ a disjoint family $\langle A_\mathcal{F}\rangle_{\mathcal{F}\in\mathfrak{F}_\kappa}$ of subsets of F'_κ such that $A_\mathcal{F} \in \mathcal{F}'$ for every $\mathcal{F} \in \mathfrak{F}_\kappa$, and 2E tells us that $\mathcal{H}_\kappa = \bigcap\{\mathcal{F}' : \mathcal{F} \in \mathfrak{F}_\kappa\}$ is dependently selective for every $\kappa \in \mathrm{K}$. Assembling the families $\langle A_\mathcal{F}\rangle_{\mathcal{F}\in\mathfrak{F}_\kappa}$, we have a disjoint family $\langle A_\mathcal{F}\rangle_{\mathcal{F}\in\mathfrak{F}}$ such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(b) Evidently

$$\mathcal{H} = \{A : A \subseteq X,\ A \cap F'_\kappa \in \mathcal{H}_\kappa \text{ for every } \kappa \in \mathrm{K}\}.$$

Now suppose that $\mathcal{S} \subseteq [X]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{i : K \cup \{i\} \in \mathcal{S}\} \in \mathcal{H}$ for every $K \in \mathcal{S}$. Choose $\langle B_\kappa\rangle_{\kappa\in\mathrm{K}}$ inductively, as follows. Given that $\kappa \in \mathrm{K}$, that $B_\lambda \in \mathcal{H}_\lambda$ has been defined for $\lambda \in \mathrm{K} \cap \kappa$ and that $[\bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda]^{<\omega} \subseteq \mathcal{S}$, note that $\#(\bigcup_{\lambda\in\mathrm{K}\cap\kappa} F'_\lambda) < \kappa$, because if $\kappa > \omega$ then κ is two-valued-measurable and certainly has uncountable cofinality. So $C_\kappa = \bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda$ and $[C_\kappa]^{<\omega}$ have cardinal less than κ.

Set

$$\mathcal{S}_\kappa = \{K : K \in [F'_\kappa]^{<\omega},\ K \cup L \in \mathcal{S} \text{ for every } L \in [C_\kappa]^{<\omega}\}.$$

Then $\emptyset \in \mathcal{S}_\kappa$, by the hypothesis on C_κ. If $K \in \mathcal{S}_\kappa$, then for each $L \in [C_\kappa]^{<\omega}$ the set $C_L = \{i : i \in F'_\kappa,\ K \cup L \cup \{i\} \in \mathcal{S}\}$ belongs to $\mathcal{H}_\kappa$; but $\mathcal{H}_\kappa$, being an intersection of κ-complete filters, is again κ-complete, so $C = \bigcap\{C_L : L \in [C_\kappa]^{<\omega}\} \in \mathcal{H}_\kappa$, and $K \cup \{i\} \in \mathcal{S}_\kappa$ for every $i \in C$. As $\mathcal{H}_\kappa$ is dependently selective, there is an $B_\kappa \in \mathcal{H}_\kappa$ such that $[B_\kappa]^{<\omega} \subseteq \mathcal{S}_\kappa$ and $[B_\kappa \cup C_\kappa]^{<\omega} \subseteq \mathcal{S}$.

The inductive hypothesis

$$[\textstyle\bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda]^{<\omega} \subseteq \mathcal{S}$$

gives no difficulty when $\kappa \in \mathrm{K}$ is a limit in K, so the induction proceeds to the end. Setting $A = \bigcup_{\kappa\in\mathrm{K}} B_\kappa$, we have $A \in \mathcal{H}$ and $[A]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{H}$ is dependently selective.

3 Problems

3A Is it relatively consistent with ZFC to suppose that there are no free dependently selective filters on $\mathbb{N}$?

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