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Measure-centering ultrafilters

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Like pure mathematicians in general, measure theorists in the last hundred years have often used ultrafilters as a tool. I suppose that the first person to notice that ultrafilters have intrinsic properties expressible in terms of measure theory was Sierpiński (Sierpiński 45), who showed that if $\langle E_n\rangle_{n\in\mathbb{N}}$ is a stochastically independent sequence of measurable subsets of $[0,1]$, and $\mathcal{F}$ is a non-principal ultrafilter on $\mathbb{N}$, then $\lim_{n\to\mathcal{F}} E_n$ has inner measure 0 and outer measure 1. But if you are starting from an interest in ultrafilters rather than an interest in measure theory, your attention will be directed to ways in which measure theory can display differences between different classes of ultrafilter. In §538 of my book Fremlin 08, I looked at p-point filters, Ramsey ultrafilters, rapid filters, 'measure-converging' filters (an idea due to Matt Foreman), and filters with what I call the 'Fatou property'. Rather than try to cover such a range here, however, I will concentrate on a single class, the 'measure-centering' or 'property M' ultrafilters. The most interesting results are due to Michael Benedikt.

The plan of this note is to begin with statements of the principal definitions and results, with some discussion (§1). Proofs are given in §§3-6, after a preliminary section §2 examining 'dependently selective' filters. Finally I comment on some open questions in §7.

1 Definitions and results

1A I start by defining the class of ultrafilters I mean to study. Its nature will perhaps be clearer if I move to a slightly more general context than is strictly necessary for the main theorems to follow. If $\mathfrak{A}$ is a Boolean algebra, a functional $\nu:\mathfrak{A}\to[0,1]$ is **additive** if $\nu(a\cup b)=\nu a+\nu b$ whenever a, $b\in\mathfrak{A}$ and $a\cap b=0$. In this language, we can define measure-centering ultrafilters ('property M ultrafilters') as follows.

> **Definition** An ultrafilter $\mathcal{F}$ on a set I is **measure-centering** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu:\mathfrak{A}\to[0,1]$ is an additive functional such that $\nu(1_{\mathfrak{A}})=1$, and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i>0$, then there is a $J\in\mathcal{F}$ such that $\nu(\inf_{i\in K}a_i)>0$ for every finite $K\subseteq J$.

1B Principal ultrafilters are obviously measure-centering. There do not have to be any others (see Theorem 1Mj below). Subject to appropriate special axioms (in particular, the continuum hypothesis), we have a variety of types of measure-centering ultrafilter, which it is the purpose of this note to examine. To begin with, we have the following.

> **Theorem** (a)(see Henson & Wattenberg 81) A Ramsey ultrafilter is measure-centering.
>
> (b) If κ is an infinite cardinal and $\operatorname{cov}\mathcal{N}_\kappa=2^\kappa$, then there is a uniform measure-centering ultrafilter on κ.
>
> (c) If $\operatorname{cov}\mathcal{N}_{\rm Leb}=\mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ which contains no set of zero asymptotic density.

(For the proof, see 3E-3G.) As you see, we are going to need rather a lot of definitions. Most of them are to be found in Fremlin 08, but it will I expect help if I repeat some here. In particular:

> **Definitions** (α)(Comfort & Negrepontis 74) If I is an infinite set, an ultrafilter $\mathcal{F}$ on I is **Ramsey** (or 'selective') if it is uniform and for every $S\subseteq[I]^2$ there is a $J\in\mathcal{F}$ such that either $[J]^2\subseteq S$ or $[J]^2\cap S\neq\emptyset$.
>
> (β) If (X,Σ,μ) is a measure space, set $\mu^*A=\inf\{\mu E:E\in\Sigma,\,E\supseteq A\}$ for every $A\subseteq X$. The **null ideal** of μ is $\mathcal{N}=\{A:\mu^*A=0\}$. $\mathcal{N}_{\rm Leb}$ will be the null ideal of Lebesgue measure on $[0,1]$.
>
> (γ) For any set I, I will write ν_I for the usual probability measure on $\{0,1\}^I$, the completed product measure if each copy of $\{0,1\}$ is given the uniform probability in which each point has measure $\frac{1}{2}$; $\mathcal{N}_I$ will be its null ideal.

(δ) If X is a set and $\mathcal{I}$ is an ideal of subsets of X such that $X = \bigcup\mathcal{I}$, then its **covering number** $\operatorname{cov}\mathcal{I}$ will be the least cardinal of any set $\mathcal{A} \subseteq \mathcal{I}$ such that $X = \bigcup\mathcal{A}$.

(ϵ) If $A \subseteq \mathbb{N}$ then the **upper asymptotic density** of A is $d^*(A) = \limsup_{n\to\infty} \frac{1}{n}\#(A \cap n)$. A has **zero asymptotic density** if $d^*(A) = 0$.

Recall that the continuum hypothesis is sufficient to ensure that there are Ramsey ultrafilters on $\mathbb{N}$; in fact it is sufficient to suppose that $\operatorname{cov}\mathcal{M} = \mathfrak{c}$, where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$ (Fremlin 08, 538Fg). For an uncountable cardinal κ, there is a Ramsey ultrafilter on κ iff κ is two-valued-measurable, and in this case an ultrafilter on κ is Ramsey iff it is isomorphic to a normal ultrafilter (Comfort & Negrepontis 74, 9.6). There appears to be no bar to the number of such cardinals κ, but of course they must all be enormous. Note that if there is a Ramsey ultrafilter $\mathcal{F}$ on κ, then κ is regular and $\mathcal{F}$ is κ-complete.

To get a notion of the scope of (b) in this theorem, note that $\operatorname{cov}\mathcal{N}_\kappa \le \operatorname{cov}\mathcal{N}_{\text{Leb}} \le \mathfrak{c}$ for every infinite cardinal κ (Fremlin 08, 523F), with equalities if Martin's axiom is true (Fremlin 08, 524Na or Fremlin 84, 32C); moreover, Martin's axiom implies that $2^\kappa = \mathfrak{c}$ whenever $\omega \le \kappa < \mathfrak{c}$ (Fremlin 08, 517Rb or Fremlin 84, 21C). So we see that we can have many cardinals less than $\mathfrak{c}$ with uniform measure-centering ultrafilters.

Martin's axiom is sufficient to ensure that there are Ramsey ultrafilters on $\mathbb{N}$ (because it implies that $\operatorname{cov}\mathcal{M} = \mathfrak{c}$, or otherwise). But there is another important context in which (b) can be applied in the absence of any Ramsey ultrafilters at all. If we start with a model of ZFC and an uncountable regular cardinal λ such that $2^\kappa \le \lambda$ for every $\kappa < \lambda$, and add λ random reals, then in the resulting forcing language we shall have

$$\Vdash \operatorname{cov}\mathcal{N}_\kappa = 2^\kappa = \mathfrak{c} \text{ for every infinite } \kappa < \mathfrak{c},$$
$$\text{but there are no Ramsey ultrafilters on } \mathbb{N}$$

(Fremlin 08, 552B, 552G and 553H). The position is similar in any model in which $\mathfrak{c}$ is real-valued-measurable (Fremlin 93, 5E, 6B and 5G).

1C In 1A I gave a definition of 'measure-centering' ultrafilter in a context well removed from the ordinary concerns of elementary measure theory. The original conception derived, as you would expect, from ideas closer to home, looking at filters on $\mathbb{N}$ and Lebesgue measure. To relate Lebesgue measure to the next result, recall that the usual measure on $\{0,1\}^{\mathbb{N}}$ is isomorphic to Lebesgue measure on $[0,1]$ (Fremlin 01, 254K).

Proposition Let I be a set, and $\mathcal{F}$ an ultrafilter on I. Then the following are equiveridical, that is, if one is true so are the others:

(i) $\mathcal{F}$ is measure-centering;

(ii) whenever $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{B}_I$ such that $\inf_{i\in I} \bar\nu_I a_i > 0$, there is an $A \in \mathcal{F}$ such that $\{a_i : i \in A\}$ is centered in $\mathfrak{B}_I$;

(iii) whenever $\langle E_i\rangle_{i\in I}$ is a family of measurable subsets of $\{0,1\}^I$ such that $\inf_{i\in I} \nu_I E_i > 0$, there is an $A \in \mathcal{F}$ such that $\bigcap_{i\in A} E_i \ne \emptyset$;

(iv) whenever (X, Σ, μ) is a compact probability space and $\langle E_i\rangle_{i\in I}$ is a family in Σ, then $\mu^*(\lim_{i\to\mathcal{F}} E_i) \ge \lim_{i\to\mathcal{F}} \mu E_i$.

(For the proof, see 3C.) Of course this calls for some more definitions from modern abstract measure theory.

Definitions (α) If $\mathfrak{A}$ is a Boolean algebra, a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **centered** if $\inf_{i\in K} a_i \ne 0$ for every finite $K \subseteq I$.

(β) If (X, Σ, μ) is a measure space, and $\mathcal{N}$ the null ideal of μ, the **measure algebra** of μ is the quotient Boolean algebra $\mathfrak{A} = \Sigma/\Sigma \cap \mathcal{N}$ together with the functional $\bar\mu : \mathfrak{A} \to [0,\infty]$ defined by setting $\bar\mu E^\bullet = \mu E$ for every $E \in \Sigma$. For any cardinal κ, I will write $(\mathfrak{B}_I, \bar\nu_I)$ for the measure algebra of ν_I, the usual measure on $\{0,1\}^I$.

(γ) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap\mathcal{L}$ is non-empty whenever $\mathcal{L} \subseteq \mathcal{K}$ has the finite intersection property, that is, $\bigcap\mathcal{L}' \ne \emptyset$ for every finite $\mathcal{L}' \subseteq \mathcal{L}$. If (X, Σ, μ) is a measure space, μ is **inner regular** with respect to a family $\mathcal{K}$ of sets if whenever $E \in \Sigma$ and $0 \le \gamma < \mu E$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \geq \gamma$. A measure space (X, Σ, μ) is **compact** if μ is inner regular with respect to some compact class of sets.

(δ) If $\langle E_i \rangle_{i \in I}$ is a family of sets and $\mathcal{F}$ is an ultrafilter on I, I write $\lim_{i \to \mathcal{F}} E_i$ for

$$\bigcup_{A \in \mathcal{F}} \bigcap_{i \in A} E_i = \bigcap_{A \in \mathcal{F}} \bigcup_{i \in A} E_i = \{x : \{i : x \in E_i\} \in \mathcal{F}\},$$

the limit of $\langle E_i \rangle_{i \in I}$ along $\mathcal{F}$ in $\mathcal{P}X$ if X is any set including $\bigcup_{i \in I} E_i$ and $\mathcal{P}X \cong \{0,1\}^X$ is given its usual compact Hausdorff topology.

For basic results on compact measure spaces, see Fremlin 02, §342. I remark here that a family $\mathcal{K}$ of subsets of a set X is a compact class iff there is a compact (not necessarily Hausdorff) topology on X such that every member of $\mathcal{K}$ is closed (Fremlin 02, 342D); thus all Radon measures, and in particular Lebesgue measure, are compact measures in this sense.

1D It is natural to seek to explore the relationship of the class of measure-centering ultrafilters with the ordinary operations of the theory of ultrafilters. At an elementary level, we have the following.

Proposition (a) Let I and J be sets, $f : I \to J$ a function, and $\mathcal{F}$ a measure-centering ultrafilter on I. Then the image ultrafilter $f[[\mathcal{F}]]$ is a measure-centering ultrafilter on J.

(b)(Benedikt 98) If $\mathcal{F}$ is a non-principal ultrafilter, then $\mathcal{F} \ltimes \mathcal{F}$ is not measure-centering.

(Proof in 3H.) Some relevant definitions are as follows.

Definitions (α) If I and J are sets, $f : I \to J$ is a function and $\mathcal{F}$ is a filter on I, then the **image filter** $f[[\mathcal{F}]]$ is $\{B : B \subseteq J,\ f^{-1}[B] \in \mathcal{F}\}$, that is, the filter on J generated by $\{f[A] : A \in \mathcal{F}\}$.

(β) If $\mathcal{F}$, $\mathcal{G}$ are filters on sets I, J respectively, then I write $\mathcal{F} \ltimes \mathcal{G}$ for the filter

$$\{A : A \subseteq I \times J,\ \{i : i \in I,\ A[\{i\}] \in \mathcal{G}\} \in \mathcal{F}\};$$

here $A[\{i\}] = \{j : (i,j) \in A\}$.

1E Extension of measures The original impulse to study measure-centering ultrafilters arose because they give an interesting expression of an ultrapower construction which I will describe shortly. The first result is a theorem on extension of probability measures.

Theorem Let (X, Σ, μ) be a compact probability space, and $\mathcal{F}$ a measure-centering ultrafilter on a set I. Let $\mathcal{A}$ be the family of all sets of the form $\lim_{i \to \mathcal{F}} E_i$ where $\langle E_i \rangle_{i \in I}$ is a family in Σ. Then there is a unique complete probability measure λ on X such that λ is inner regular with respect to $\mathcal{A}$ and $\lambda(\lim_{i \to \mathcal{F}} E_i) = \lim_{i \to \mathcal{F}} \mu E_i$ for every family $\langle E_i \rangle_{i \in I}$ in Σ.

(Proof in 3I. I ought perhaps to note that a measure λ is **complete** if λA is defined whenever $\lambda^* A = 0$; thus Lebesgue measure is complete.) Note that in the context of this theorem, λ must extend μ, because we can apply the defining formula to constant families $\langle E_i \rangle_{i \in I}$.

1F Reduced products of probability algebras We now need an abstract construction from the theory of measure algebras.

(a) First, let me define measure algebras in the abstract, as opposed to those constructed from measure spaces as in Definition 1C above. A **measure algebra** is a pair $(\mathfrak{A}, \bar{\mu})$ where

$\mathfrak{A}$ is a Boolean algebra,

$\mathfrak{A}$ is **Dedekind σ-complete**, that is, every countable subset of $\mathfrak{A}$ has a least upper bound in $\mathfrak{A}$,

$\bar{\mu} : \mathfrak{A} \to [0, \infty]$ is **countably additive**, that is, $\bar{\mu}0 = 0$ and $\bar{\mu}(\sup_{n \in \mathbb{N}} a_n) = \sum_{n=0}^{\infty} \bar{\mu} a_n$ whenever $\langle a_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}$.

It is straightforward to check that the measure algebras of Definition 1Cb are measure algebras in this sense. Conversely, any measure algebra as defined here is isomorphic to the measure algebra of some measure space (Fremlin 02, 321J). A **probability algebra** is a measure algebra $(\mathfrak{A}, \bar{\mu})$ such that $\bar{\mu}1 = 1$, that is, $(\mathfrak{A}, \bar{\mu})$ is isomorphic to the measure algebra of a probability space.

(b) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $\mathcal{F}$ an ultrafilter on I. Write $\mathfrak{B}$ for the product Boolean algebra $\prod_{i\in I}\mathfrak{A}_i$, so that if $\boldsymbol{a}=\langle a_i\rangle_{i\in I}$ and $\boldsymbol{b}=\langle b_i\rangle_{i\in I}$ belong to $\mathfrak{B}$, then $\boldsymbol{a}*\boldsymbol{b}=\langle a_i*b_i\rangle_{i\in I}$ for all the Boolean operations $*=\triangle,\ \cap,\ \cup$ and $\setminus$. Define $\nu:\mathfrak{B}\to[0,1]$ by setting $\nu(\langle a_i\rangle_{i\in I})=\lim_{i\to\mathcal{F}}\bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I}\in\mathfrak{B}$. Then ν is additive. Set $\mathcal{I}=\{\boldsymbol{a}:\boldsymbol{a}\in\mathfrak{B},\ \nu\boldsymbol{a}=0\}$; then $\mathcal{I}\lhd\mathfrak{B}$. Let $\mathfrak{C}$ be the quotient Boolean algebra $\mathfrak{B}/\mathcal{I}$. Then we have a functional $\bar\nu:\mathfrak{C}\to[0,1]$ defined by saying that $\bar\nu(\boldsymbol{a}^\bullet)=\nu\boldsymbol{a}$ for every $\boldsymbol{a}\in\mathfrak{B}$; and it turns out that $(\mathfrak{C},\bar\nu)$ is a probability algebra. I will call it the **reduced product** $\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$. (See 4A below.)

1G Note that the construction in 1Fb does not depend on any property of the ultrafilter $\mathcal{F}$. When $\mathcal{F}$ is a measure-centering ultrafilter, however, we have the following result.

> **Theorem** Let (X,Σ,μ) be a compact probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let I be a set and $\mathcal{F}$ a measure-centering ultrafilter on I; write λ for the corresponding extension of μ as described in Theorem 1E, and $(\mathfrak{C},\bar\nu)$ for the reduced power $(\mathfrak{A},\bar\mu)^I|\mathcal{F}$ as described in 1Fb. Then we have a natural isomorphism between $(\mathfrak{C},\bar\nu)$ and the measure algebra $(\mathfrak{D},\bar\lambda)$ of λ defined by saying that $\langle E_i^\bullet\rangle_{i\in I}^\bullet\in\mathfrak{C}$ is matched with $(\lim_{i\to\mathcal{F}}E_i)^\bullet\in\mathfrak{D}$ for every family $\langle E_i\rangle_{i\in I}$ in Σ.

(Proof in 4C.)

1H Products of filters As will I hope become clear when we come to the proofs in §§3-4, all the results so far are more or less elementary, though some of them, naturally enough, demand graduate-level measure theory – in particular, Maharam's theorem and the lifting theorem – for their full strength. I want now to explain an astonishing theorem from Benedikt 98. This will depend on a construction of iterated products of filters, which may be of independent interest. I look at finite products of filters first.

> **Definition** For $n\in\mathbb{N}$ and filters $\mathcal{F}_0,\dots,\mathcal{F}_n$, define the product $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ inductively by saying that it is $\mathcal{F}_0$ when $n=0$ and $(\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_{n-1})\ltimes\mathcal{F}_n$ when $n\ge1$.

> **Proposition** If $0\le m<n$ and $\mathcal{F}_0,\dots,\mathcal{F}_n$ are filters on $I_0,\dots,I_n$ respectively, then the natural bijection between $((\ldots(I_0\times I_1)\times\ldots)\times I_m)\times((\ldots(I_{m+1}\times I_{m+2})\times\ldots)\times I_n)$ and $((\ldots(I_0\times I_1)\times\ldots)\times I_n)$ identifies $\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_n$ with $(\mathcal{F}_0\ltimes\ldots\ltimes\mathcal{F}_m)\ltimes(\mathcal{F}_{m+1}\ltimes\ldots\ltimes\mathcal{F}_n)$.

(The proof is a simple induction on n.)

1I Iterated products of filters The next bit works best for filters on $\mathbb{N}$ and countable iterations, but something can be done in a more general context.

(a) First, a scrap of notation. Let I be a set. If $m,\,n\in\mathbb{N}$, $\sigma\in I^m$ and $\tau\in I^n$, define the concatenation $\sigma^\frown\tau\in I^{m+n}$ by setting

$$\begin{aligned}(\sigma^\frown\tau)(k)&=\sigma(k)\text{ if }k<m,\\&=\tau(k-m)\text{ if }m\le k<m+n.\end{aligned}$$

For $i\in I$ write $<i>$ for the member of I^1 with value i.

(b) Now suppose that $\zeta>0$ is an ordinal, $\langle I_\xi\rangle_{1\le\xi\le\zeta}$ a family of sets, and $\mathcal{F}_\xi$ a filter on I_ξ for $1\le\xi\le\zeta$. Set $I=\bigcup_{1\le\xi\le\zeta}I_\xi$ and $S^*=\bigcup_{i\in\mathbb{N}}I^i$. Fix a function θ such that $\theta(\xi,i)<\xi$ for $1\le\xi\le\zeta$ and $i\in I_\xi$. For $\xi\le\zeta$, define $\mathcal{G}_\xi\subseteq\mathcal{P}S^*$ inductively, as follows. Start by taking $\mathcal{G}_0$ to be the principal filter generated by $\{\emptyset\}$. For $1\le\xi\le\zeta$, given that $\mathcal{G}_\eta$ has been defined for every $\eta<\xi$, set

$$\mathcal{G}_\xi=\{A:A\subseteq S^*,\ \{i:i\in I_\xi,\ \{\tau:<i>^\frown\tau\in A\}\in\mathcal{G}_{\theta(\xi,i)}\}\in\mathcal{F}_\xi\}.$$

It is elementary to check that every $\mathcal{G}_\xi$ is a filter. Moreover, if every $\mathcal{F}_\xi$ is an ultrafilter, so is every $\mathcal{G}_\xi$.

1J We are now ready for the statements of the main theorems.

> **Theorem** In the construction of 1Ib above, suppose that ζ is countable, $I_\xi=I$ whenever $1\le\xi\le\zeta$, $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ is a family of Ramsey ultrafilters on I, no two isomorphic, and $\{i:i\in I,\ \theta(\xi,i)\ge\eta\}\in\mathcal{F}_\xi$ whenever $\eta<\xi\le\zeta$. Then $\mathcal{G}_\zeta$ is measure-centering.

(Proof in 5B. You will lose very little by restricting yourself to the case in which every sequence $\langle\theta(\xi,i)\rangle_{i\in\mathbb{N}}$ is non-decreasing, and is constant with value η when $\xi=\eta+1$ is a successor ordinal.) The point of this theorem is that there will be for each $\xi\in[1,\zeta]$ a function $f:S^*\to I$ such that $f[[\mathcal{G}_\zeta]]=\mathcal{F}_\xi$; starting from the family $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ of Ramsey ultrafilters on I, we can find a single measure-centering ultrafilter on S^* from which they can all be derived. I ought to point out straight away that if ζ is infinite, then the Ramsey ultrafilter $\mathcal{F}_\omega$ contains all the sets $\{i:\theta(\omega,i)\ge n\}$, for $n<\omega$, but not their intersection, so is not ω_1-additive. In this case, of course, $I=I_\omega$ cannot be uncountable. Thus we have either a finite iteration in which $\mathcal{G}_\zeta$ is the extension to $\bigcup_{n\in\mathbb{N}}I^n$ of the filter $\mathcal{F}_m\ltimes\mathcal{F}_{m-1}\ltimes\ldots\ltimes\mathcal{F}_1$ on I^m, or a countably infinite iteration in which I can be identified with $\mathbb{N}$. The finite-iteration case is in fact the hard part of a more general result: the skew product of finitely many non-isomorphic Ramsey ultrafilters is always measure-centering (Proposition 5E).

1K The second theorem is a universal extension theorem for Ramsey ultrafilters on $\mathbb{N}$.

> **Theorem** Let (X,Σ,μ) be a compact probability space. Then there is a measure λ on X, extending μ, such that $\lambda(\lim_{i\to\mathcal{F}}E_i)$ is defined and equal to $\lim_{i\to\mathcal{F}}\mu E_i$ whenever $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$ and $\langle E_i\rangle_{i\in\mathbb{N}}$ is a sequence in Σ.

(Proof in 5J.) The idea here will be that for each countable family $\mathfrak{F}$ of Ramsey ultrafilters on $\mathbb{N}$, we can define a measure-centering ultrafilter $\mathcal{G}_{\mathfrak{F}}$ on S^* dominating every member of $\mathfrak{F}$, and that this can be done in such a way that the measures defined from the $\mathcal{G}_{\mathfrak{F}}$ by the process of Theorem 1E will have a common extension.

1L Perfect measure spaces Readers familiar with Benedikt 98 and Benedikt 99 may have noted that I speak of 'compact' measures where Benedikt deals with 'perfect' measures. The latter form a larger class, so it is not obvious that the results in this note really cover Benedikt's. The point is that a probability space (X,Σ,μ) is perfect iff $(X,\mathrm{T},\mu\restriction\mathrm{T})$ is compact for every countably generated σ-subalgebra T of Σ (Sazonov 66, or Fremlin 03, 451F). Using this, it is easy to check that we have a variant on condition (iv) of Proposition 1C for filters on $\mathbb{N}$:

> if $\mathcal{F}$ is an ultrafilter on $\mathbb{N}$, it is measure-centering iff whenever (X,Σ,μ) is a perfect probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ, then $\mu^*(\lim_{n\to\mathcal{F}}E_n)\ge\lim_{n\to\mathcal{F}}\mu E_n$.

Versions of Theorems 1E and 1G for perfect probability spaces and ultrafilters on $\mathbb{N}$ are now easy to deduce. With a little more trouble – it is probably easiest to check that the proof in 5J applies essentially unchanged – we can confirm that Theorem 1K is true for all perfect probability spaces (X,Σ,μ).

1M Of course there are many classes of ultrafilters, associated with those considered above, which have been studied over the years. Six of them are the following.

> **Definitions** Let $\mathcal{F}$ be an ultrafilter on a set I.
>
> (α)(Daguenet-Tessier 79) An ultrafilter $\mathcal{F}$ on a set I is **Hausdorff** (or has 'property C') if whenever J is a set and $f:I\to J$, $g:I\to J$ are functions such that $\{i:f(i)\ne g(i)\}\in\mathcal{F}$, then $f[[\mathcal{F}]]\ne g[[\mathcal{F}]]$.
>
> (β) $\mathcal{F}$ is **nowhere dense** if for every function $f:I\to\mathbb{R}$ the image filter $f[[\mathcal{F}]]$ contains a nowhere dense subset of $\mathbb{R}$.
>
> (γ)(Blass 74) $\mathcal{F}$ is **weakly Ramsey** if whenever S_0, S_1, S_2 are disjoint subsets of $[I]^2$ there is a $J\in\mathcal{F}$ such that $[J]^2$ is disjoint from at least one of S_0, S_1, S_2.
>
> (δ)(Baumgartner & Taylor 78) $\mathcal{F}$ is an **arrow ultrafilter** if whenever $S\subseteq[I]^2$ and $k\in\mathbb{N}$ then either there is a $K\in[I]^k$ such that $[K]^2\cap S=\emptyset$ or there is a $J\in\mathcal{F}$ such that $[J]^2\subseteq S$.
>
> (ϵ)(Benedikt 99) $\mathcal{F}$ is **measure-linking** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu:\mathfrak{A}\to[0,1]$ is an additive functional such that $\nu(1_{\mathfrak{A}})=1$, and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i>0$, then there is a $J\in\mathcal{F}$ such that $\nu(a_i\cap a_j)>0$ for all i, $j\in J$.
>
> (ζ)(Baumgartner 95) $\mathcal{F}$ is **closed Lebesgue null** if for every function $f:I\to[0,1]$ the image filter $f[[\mathcal{F}]]$ contains a closed Lebesgue negligible set.

An obvious strengthening of (ϵ) is

(η) $\mathcal{F}$ is **strongly measure-linking** if whenever $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0,1]$ is an additive functional such that $\nu(1_{\mathfrak{A}}) = 1$, and $\langle a_i \rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I} \nu a_i > 0$, then there is a $J \in \mathcal{F}$ such that $\inf_{i,j\in J} \nu(a_i \cap a_j) > 0$.

(I ought to remark that in all of Daguenet-Tessier 79, Blass 74, Baumgartner & Taylor 78, Baumgartner 95 and Benedikt 99 only non-principal ultrafilters on countable sets are considered; that what I call 'measure-linking' is what Benedikt 99 calls 'property M_2'; and that what I call 'closed Lebesgue null' is what Baumgartner 95 calls 'measure zero'.) Evidently Ramsey ultrafilters are weakly Ramsey, measure-centering ultrafilters are measure-linking, strongly measure-linking ultrafilters are measure-linking, and closed Lebesgue null filters are nowhere dense; it is also the case that p-point ultrafilters on $\mathbb{N}$ are closed Lebesgue null (Baumgartner 95). Subject to the continuum hypothesis, there are non-principal weakly Ramsey ultrafilters on $\mathbb{N}$ which are not Ramsey (Blass 74). The results I wish to present here are the following:

Theorem (a)(Benedikt 98) A measure-linking ultrafilter is Hausdorff.

(b)(Shelah 98) A measure-centering ultrafilter is nowhere dense.

(c)(see Baumgartner & Taylor 78, Corollary 2.5) A weakly Ramsey ultrafilter is an arrow ultrafilter.

(d)(i)(see Benedikt 99, p. 214, Proposition 3) An arrow ultrafilter is strongly measure-linking.

(ii) An arrow ultrafilter on $\mathbb{N}$ is nowhere dense.

(e) A strongly measure-linking ultrafilter on $\mathbb{N}$ contains a set of zero asymptotic density.

(f) A closed Lebesgue null ultrafilter on $\mathbb{N}$ contains a set of zero asymptotic density.

(g) If $\operatorname{cov}\mathcal{N}_{\text{Leb}} = \mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ which is neither strongly measure-linking nor closed Lebesgue null.

(h) If $\mathfrak{c} = \omega_1$, there is a strongly measure-linking ultrafilter on $\mathbb{N}$ which is not nowhere dense, so is neither measure-centering nor an arrow ultrafilter.

(i) If $\mathfrak{p} = \mathfrak{c}$, there is a Hausdorff p-point ultrafilter which is not measure-centering.

(j)(see Shelah 98) It is relatively consistent with ZFC to suppose that every measure-centering ultrafilter is a principal ultrafilter.

(Proof in 6A, 6C, 6E, 6H, 6L and 6O. Recall that $\mathfrak{p}$ is the least cardinal of any family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ such that $\bigcap \mathcal{A}_0$ is infinite for any finite $\mathcal{A}_0 \subseteq \mathcal{A}$, but there is no infinite $B \subseteq \mathbb{N}$ such that $B \setminus A$ is finite for every $A \in \mathcal{A}$.)

2 Dependently selective filters

A particularly important property of Ramsey ultrafilters is preserved under certain intersections of such ultrafilters, and it is in this form that it will be used in §5. I therefore isolate it in the next definition. The results which will be needed in the proofs of Theorems 1Ba, 1J and 1K are special cases of Propositions 2D and 2E, but I think it is worth while expressing the intermediate lemmas 2B and 2C in their full natural strength.

2A Definition Let $\mathcal{F}$ be a filter on a set I. I will say that $\mathcal{F}$ is **dependently selective** if it has the following property:

whenever $\mathcal{S} \subseteq [I]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{i : K \cup \{i\} \in \mathcal{S}\} \in \mathcal{F}$ for every $K \in \mathcal{S}$, then there is a $J \in \mathcal{F}$ such that $[J]^{<\omega} \subseteq \mathcal{S}$.

In the present paper I will give only those results which are necessary for the applications in §5; for a fuller account of this class of filters, see my note Fremlin n09.

2B Lemma A uniform dependently selective ultrafilter is a Ramsey ultrafilter.

proof Let $\mathcal{F}$ be a uniform dependently selective ultrafilter on a set I. It will be enough to consider the case in which $I = \kappa$ is a cardinal. Take any $S \subseteq [\kappa]^2$. For $\xi < \kappa$ set $A_\xi = \{\eta : \{\xi,\eta\} \in S\}$. Let $\mathcal{S}$ be the family of finite subsets K of κ such that for all ξ, $\eta \in K$ such that $\xi < \eta$, $\{\xi,\eta\} \in S$ iff $A_\xi \in \mathcal{F}$. If $K \in \mathcal{S}$, then (because $\mathcal{F}$ is an ultrafilter) there is a $J \in \mathcal{F}$ such that, for every $\xi \in K$, J is either included in A_ξ or disjoint from A_ξ. Now $K\cup\{\eta\}\in\mathcal{S}$ whenever $\eta\in J$ and $\eta>\xi$ for every $\xi\in K$. So $\mathcal{S}$ satisfies the condition of 2A. Let $J\in\mathcal{F}$ be such that $[J]^{<\omega}\subseteq\mathcal{S}$. In this case, if ξ, $\eta\in J$ and $\xi<\eta$, $\{\xi,\eta\}\in S$ iff $A_\xi\in\mathcal{F}$. Now

$$J_1=\{\xi:\xi\in J,\,A_\xi\in\mathcal{F}\},\qquad J_0=\{\xi:\xi\in J,\,A_\xi\notin\mathcal{F}\}$$

have union J and one of them must belong to $\mathcal{F}$; while $[J_0]^2\cap S=\emptyset$ and $[J_1]^2\subseteq S$. As S is arbitrary, $\mathcal{F}$ is a Ramsey ultrafilter.

2C Lemma (a) Let I be an infinite set, $\mathcal{F}$ a Ramsey ultrafilter on I, and $\mathcal{A}\subseteq\mathcal{F}$ a set of size at most $\#(I)$. Then there is a $C\in\mathcal{F}$ such that $\#(C\setminus A)<\#(I)$ for every $A\in\mathcal{A}$.

(b) Let κ be an infinite cardinal, $\lambda\le\kappa$ another cardinal, and $\langle\mathcal{F}_\alpha\rangle_{\alpha<\lambda}$ a family of distinct Ramsey ultrafilters on κ. Then there is a disjoint family $\langle A_\alpha\rangle_{\alpha<\lambda}$ of subsets of κ such that $A_\alpha\in\mathcal{F}_\alpha$ for every $\alpha<\lambda$.

proof (a) Set $A^*=\kappa\cap\bigcap\mathcal{A}$. If $A^*\in\mathcal{F}$, we can set $C=A^*$ and stop. Otherwise, enumerate $\mathcal{A}$ as $\langle A_\alpha\rangle_{\alpha<\lambda}$, where $\lambda\le\kappa$. For $i\in I$, set $f(i)=\min\{\alpha:\alpha<\lambda,\,i\notin A_\alpha\setminus A^*\}$. Then there is a $C\in\mathcal{F}$ such that $f{\restriction}C$ is either constant or injective (Comfort & Negrepontis 74, 9.6). The former is impossible, because $\{i:f(i)=\alpha\}\subseteq A^*\cup(\kappa\setminus A_\alpha)$ never belongs to $\mathcal{F}$. So $f{\restriction}C$ is injective and $C\setminus A_\alpha\subseteq\{i:i\in C,\,f(i)\le\alpha\}$ has cardinal less than κ for every $\alpha<\lambda$.

(b) For $\alpha<\beta<\lambda$, take $A_{\alpha\beta}\in\mathcal{F}_\beta\setminus\mathcal{F}_\alpha$. For each $\alpha<\kappa$, there is a $B_\alpha\in\mathcal{F}_\alpha$ such that $\#(B_\alpha\cap A_{\alpha\beta})<\kappa$ for every $\beta>\alpha$ (apply (a) to $\{I\setminus A_{\alpha\beta}:\alpha<\beta<\lambda\}\subseteq\mathcal{F}_\alpha$)). Set

$$A_\beta=B_\beta\setminus\bigcup_{\alpha<\beta}B_\alpha$$

for $\beta<\lambda$. Of course $\langle A_\beta\rangle_{\beta<\lambda}$ is disjoint. On the other hand, for each $\beta<\lambda$, $A'_\beta=B_\beta\cap\bigcap_{\alpha<\beta}A_{\alpha\beta}$ belongs to $\mathcal{F}$ because $\mathcal{F}$ is κ-complete; and $A'_\beta\setminus A_\beta\subseteq\bigcup_{\alpha<\beta}A_{\alpha\beta}\cap B_\alpha$ has cardinal less than κ, so A_β also belongs to $\mathcal{F}$.

2D Proposition Let I be an infinite set, and $\mathfrak{F}$ a non-empty family of non-isomorphic Ramsey ultrafilters on I with $\#(\mathfrak{F})\le\#(I)$. Then $\mathcal{H}=\bigcap\mathfrak{F}$ is a dependently selective filter on I.

proof (a) It is enough to consider the case in which $I=\kappa$ is a cardinal. Let $\langle\mathcal{F}_\alpha\rangle_{\alpha<\lambda}$ be an enumeration of $\mathfrak{F}$.

(b) If $\langle A_\alpha\rangle_{\alpha<\lambda}$ is such that $A_\alpha\in\mathcal{F}_\alpha$ for $\alpha<\lambda$, then there is a family $\langle D_\alpha\rangle_{\alpha<\lambda}$ such that $D_\alpha\in\mathcal{F}_\alpha$ and $D_\alpha\subseteq A_\alpha$ for every $\alpha<\lambda$, and whenever $\xi<\eta<\kappa$, α, $\beta<\lambda$ are such that $\xi\in D_\alpha$ and $\eta\in D_\beta$, there is a $\zeta\in A_\beta$ such that $\xi\le\zeta<\eta$. **P** By 2Cb, we may suppose that $\langle A_\alpha\rangle_{\alpha<\lambda}$ is disjoint. For any $\zeta<\kappa$, $\{\alpha:\alpha<\lambda,\,A_\alpha\cap\zeta\ne\emptyset\}$ has cardinal less than κ; so there is a closed cofinal set $F\subseteq\kappa$, containing 0, such that $A_\alpha\cap\zeta'\setminus\zeta\ne\emptyset$ whenever $\zeta<\zeta'$ in F, $\alpha<\lambda$ and $A_\alpha\cap\zeta\ne\emptyset$. Set $f(\xi)=\max\{\zeta:\zeta\in F,\,\zeta\le\xi\}$ for $\xi<\kappa$. Then $\langle f[[\mathcal{F}_\alpha]]\rangle_{\alpha<\lambda}$ is a family of κ-complete uniform ultrafilters on F, so there must be a cofinal set $V\subseteq F$ not belonging to any of them. (We can easily build inductively a family $\langle V_\xi\rangle_{\xi<\kappa^+}$ of cofinal subsets of F such that $\#(V_\xi\cap V_\eta)<\kappa$ whenever $\xi<\eta<\kappa^+$, and now each $f[[\mathcal{F}_\alpha]]$ can contain V_ξ for at most one ξ, so there is a ξ left over for which we can set $V=V_\xi$.) Set $M=f^{-1}[V]$; then $A_\alpha\setminus M\in\mathcal{F}_\alpha$ for each α.

Define $g:\kappa\to\kappa$ by setting $g(\xi)=\min\{\zeta:\xi\le\zeta\in V\}$ for $\xi<\kappa$. For each $\alpha<\lambda$, there is a $B_\alpha\in\mathcal{F}_\alpha$ on which g is injective (Comfort & Negrepontis 74, 9.6), so that $g[[\mathcal{F}_\alpha]]$ is a Ramsey ultrafilter on κ isomorphic to $\mathcal{F}_\alpha$. Because the $\mathcal{F}_\alpha$ are non-isomorphic, all the $g[[\mathcal{F}_\alpha]]$ are different. By 2Cb again, there is a disjoint family $\langle G_\alpha\rangle_{\alpha<\lambda}$ of sets such that $G_\alpha\in g[[\mathcal{F}_\alpha]]$ for every α.

Set

$$C_\alpha=A_\alpha\cap B_\alpha\cap g^{-1}[G_\alpha]\setminus M,\qquad D_\alpha=C_\alpha\setminus\{\min C_\alpha\}\in\mathcal{F}_\alpha$$

for each $\alpha<\lambda$. Suppose that $\xi\in D_\alpha$, $\eta\in D_\beta$ and $\xi<\eta$. Then $g(\xi)<g(\eta)$. (If $\alpha=\beta$, this is because $g{\restriction}B_\alpha$ is injective; otherwise, it is because $G_\alpha\cap G_\beta$ is empty.) It follows that $g(\xi)<\eta$. Let η_0 be the least member of C_β. We have $\eta_0<\eta$. If $\xi\le\eta_0$, then η_0 is a member of $A_\beta\cap\eta\setminus\xi$. Otherwise, $A_\beta\cap g(\xi)\ne\emptyset$, so there is a $\zeta\in A_\beta\cap\gamma\setminus g(\xi)$, where γ is the next member of F above $g(\xi)$. Now $\gamma\setminus g(\xi)=f^{-1}[\{g(\xi)\}]\subseteq M$ is disjoint from D_β, so $\gamma\le\eta$ and $\zeta\in A_\beta\cap\eta\setminus\xi$.

Thus $\langle D_\alpha\rangle_{\alpha<\lambda}$ is a suitable family. **Q**

(c) Now suppose that $\mathcal{S}$ is a family of finite subsets of κ such that $\emptyset\in\mathcal{S}$ and $\{\xi:K\cup\{\xi\}\in\mathcal{S}\}\in\mathcal{H}$ for every $K\in\mathcal{S}$. Set

$$S = \{\{\xi, \eta\} : \xi < \eta < \kappa,\ K \cup \{\eta\} \in \mathcal{S} \text{ whenever } K \in \mathcal{S} \text{ and } K \subseteq \xi + 1\}.$$

For each $\alpha < \lambda$, there is an $A_\alpha \in \mathcal{F}_\alpha$ such that $[A_\alpha]^2$ is either included in or disjoint from S. But taking $\xi = \min A_\alpha$, we see that $\{\eta : \eta > \xi,\ K \cup \{\eta\} \in \mathcal{S}\}$ belongs to $\mathcal{H} \subseteq \mathcal{F}_\alpha$ for every $K \in \mathcal{S}$; because $\mathcal{F}_\alpha$ is κ-complete, there must be an $\eta \in A_\alpha$ such that $\eta > \xi$ and $K \cup \{\eta\} \in \mathcal{S}$ whenever $K \in \mathcal{S}$ and $K \subseteq \xi + 1$, in which case $\{\xi, \eta\} \in S$. So we must have $[A_\alpha]^2 \subseteq S$. Set $A'_\alpha = \{\xi : \xi \in A_\alpha,\ \{\xi\} \in \mathcal{S}\}$; then $A'_\alpha \in \mathcal{F}_\alpha$ because $\{\xi : \{\xi\} \in \mathcal{S}\} \in \mathcal{H} \subseteq \mathcal{F}_\alpha$.

By (b), we have a family $\langle D_\alpha \rangle_{\alpha<\lambda}$ of sets such that $D_\alpha \in \mathcal{F}_\alpha$ and $D_\alpha \subseteq A'_\alpha$ for every $\alpha < \lambda$, and whenever $\xi < \eta < \kappa$, α, $\beta < \lambda$ are such that $\xi \in D_\alpha$ and $\eta \in D_\beta$, there is a $\zeta \in A'_\beta$ such that $\xi \le \zeta < \eta$. Set $A = \bigcup_{\alpha<\lambda} D_\alpha \in \mathcal{H}$. Then $[A]^n \subseteq \mathcal{S}$ for every n. **P** Induce on n. The case $n = 0$ is trivial, and the case $n = 1$ has been dealt with when defining A'_α. For the inductive step to $n + 1 \ge 2$, suppose that $I \in [A]^{n+1}$. Let $\xi < \eta$ be the two greatest points of I; suppose that $\eta \in D_\beta$. Then there is a $\zeta \in A'_\beta$ such that $\xi \le \zeta < \eta$. In this case, $K = I \setminus \{\eta\}$ belongs to $[A]^n \subseteq \mathcal{S}$ and $K \subseteq \zeta + 1$. Also $\{\zeta, \eta\} \in [A_\beta]^2 \subseteq S$, so $I = K \cup \{\eta\} \in \mathcal{S}$. Thus the induction continues. **Q**

So $[A]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{F}$ is dependently selective.

Remark In particular, every Ramsey ultrafilter is dependently selective. Compare the 'weak T-ideals' of Grigorieff 71, and also §4 of Blass 88.

2E Proposition Let I be a set, and $\mathfrak{F}$ a non-empty countable family of non-isomorphic dependently selective ultrafilters on I. Then

(a) there is a disjoint family $\langle A_\mathcal{F} \rangle_{\mathcal{F}\in\mathfrak{F}}$ of sets such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$,

(b) $\mathcal{H} = \bigcap \mathfrak{F}$ is dependently selective.

proof (a) For each $\mathcal{F} \in \mathfrak{F}$, let $I_\mathcal{F} \in \mathcal{F}$ be a set of minimal size. Let K be the countable set $\{\#(I_\mathcal{F}) : \mathcal{F} \in \mathfrak{F}\}$; for $\kappa \in \mathrm{K}$, set $\mathfrak{F}_\kappa = \{\mathcal{F} : \mathcal{F} \in \mathfrak{F},\ \#(I_\mathcal{F}) = \kappa\}$ and $J_\kappa = \bigcup_{\mathcal{F}\in\mathfrak{F}_\kappa} I_\mathcal{F}$, so that $\#(J_\kappa) = \kappa$. (For if $\kappa = 1$, any member of $\mathfrak{F}_\kappa$ is a principal ultrafilter, and there can be at most one such.) Set $J'_\kappa = J_\kappa \setminus \bigcup_{\lambda\in\mathrm{K},\lambda<\kappa} J_\lambda$ for $\kappa \in \mathrm{K}$; then $\langle J'_\kappa \rangle_{\kappa\in\mathrm{K}}$ is disjoint and $J'_\kappa \in \mathcal{F}$ whenever $\kappa \in \mathrm{K}$ and $\mathcal{F} \in \mathfrak{F}_\kappa$.

For $\mathcal{F} \in \mathfrak{F}$, let $\mathcal{F}' = \mathcal{F} \cap \mathcal{P}J'_\kappa$ be the trace of $\mathcal{F}$ on J'_κ, where $\kappa \in \mathrm{K}$ is such that $\mathcal{F} \in \mathfrak{F}_\kappa$. It is easy to check that $\mathcal{F}'$ is dependently selective, so is either principal (if $\kappa = 1$) or a Ramsey ultrafilter (Lemma 2B). Moreover, $\mathcal{F}'$ and $\mathcal{G}'$ must be non-isomorphic whenever $\mathcal{F}$, $\mathcal{G}$ are distinct members of the same $\mathfrak{F}_\kappa$. So 2Cb tells us that we have for each $\kappa \in \mathrm{K}$ a disjoint family $\langle A_\mathcal{F} \rangle_{\mathcal{F}\in\mathfrak{F}_\kappa}$ of subsets of J'_κ such that $A_\mathcal{F} \in \mathcal{F}'$ for every $\mathcal{F} \in \mathfrak{F}_\kappa$, and 2D tells us that $\mathcal{H}_\kappa = \bigcap\{\mathcal{F}' : \mathcal{F} \in \mathfrak{F}_\kappa\}$ is dependently selective for every $\kappa \in \mathrm{K}$. Assembling the families $\langle A_\mathcal{F} \rangle_{\mathcal{F}\in\mathfrak{F}_\kappa}$, we have a disjoint family $\langle A_\mathcal{F} \rangle_{\mathcal{F}\in\mathfrak{F}}$ such that $A_\mathcal{F} \in \mathcal{F}$ for every $\mathcal{F} \in \mathfrak{F}$.

(b) Evidently

$$\mathcal{H} = \{A : A \subseteq I,\ A \cap J'_\kappa \in \mathcal{H}_\kappa \text{ for every } \kappa \in \mathrm{K}\}.$$

Now suppose that $\mathcal{S} \subseteq [I]^{<\omega}$ is such that $\emptyset \in \mathcal{S}$ and $\{i : K \cup \{i\} \in \mathcal{S}\} \in \mathcal{H}$ for every $K \in \mathcal{S}$. Choose $\langle B_\kappa \rangle_{\kappa\in\mathrm{K}}$ inductively, as follows. Given that $\kappa \in \mathrm{K}$, that $B_\lambda \in \mathcal{H}_\lambda$ has been defined for $\lambda \in \mathrm{K} \cap \kappa$ and that $[\bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda]^{<\omega} \subseteq \mathcal{S}$, note that $\#(\bigcup_{\lambda\in\mathrm{K}\cap\kappa} J'_\lambda) < \kappa$, because if $\kappa > \omega$ then κ is two-valued-measurable and certainly has uncountable cofinality. So $C_\kappa = \bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda$ and $[C_\kappa]^{<\omega}$ have cardinal less than κ.

Set

$$\mathcal{S}_\kappa = \{K : K \in [J'_\kappa]^{<\omega},\ K \cup L \in \mathcal{S} \text{ for every } L \in [C_\kappa]^{<\omega}\}.$$

Then $\emptyset \in \mathcal{S}_\kappa$, by the hypothesis on C_κ. If $K \in \mathcal{S}_\kappa$, then for each $L \in [C_\kappa]^{<\omega}$ the set $C_L = \{i : i \in J'_\kappa,\ K \cup L \cup \{i\} \in \mathcal{S}\}$ belongs to $\mathcal{H}_\kappa$; but $\mathcal{H}_\kappa$, being an intersection of κ-complete filters, is again κ-complete, so $C = \bigcap\{C_L : L \in [C_\kappa]^{<\omega}\} \in \mathcal{H}_\kappa$, and $K \cup \{i\} \in \mathcal{S}_\kappa$ for every $i \in C$. As $\mathcal{H}_\kappa$ is dependently selective, there is a $B_\kappa \in \mathcal{H}_\kappa$ such that $[B_\kappa]^{<\omega} \subseteq \mathcal{S}_\kappa$ and $[B_\kappa \cup C_\kappa]^{<\omega} \subseteq \mathcal{S}$.

The inductive hypothesis

$$[\textstyle\bigcup_{\lambda\in\mathrm{K}\cap\kappa} B_\lambda]^{<\omega} \subseteq \mathcal{S}$$

gives no difficulty when $\kappa \in \mathrm{K}$ is a limit in K, so the induction proceeds to the end. Setting $A = \bigcup_{\kappa\in\mathrm{K}} B_\kappa$, we have $A \in \mathcal{H}$ and $[A]^{<\omega} \subseteq \mathcal{S}$. As $\mathcal{S}$ is arbitrary, $\mathcal{H}$ is dependently selective.

3 Proofs of Theorems 1B-1E

The main work of this paper begins with proofs of the relatively elementary results up to Theorem 1E. These will not be done in exactly the order in which they were presented in §1; in the hope of minimising confusion, I will restate the results as I come to prove them. I begin with a review of basic fragments of measure theory which will be used later.

3A Measure spaces: definitions and facts

(a)(i) Let (X,Σ,μ) be a measure space. Its **completion** is the measure space $(X,\hat\Sigma,\hat\mu)$, where $\hat\Sigma = \{E\triangle F : E\in\Sigma,\ F$ belongs to the null ideal of $\mu\}$ and $\hat\mu$ is the unique monotonic extension of μ to $\hat\Sigma$ (Fremlin 01, 212C).

(ii) If a measure is inner regular with respect to a class $\mathcal{K}$ of sets, so is its completion (Fremlin 03, 412H).

(iii) If X is a set and μ_1, μ_2 are two complete probability measures on X such that μ_1 is inner regular with respect to $\{K : K\in\operatorname{dom}\mu_1\cap\operatorname{dom}\mu_2,\ \mu_1K=\mu_2K\}$, then μ_2 extends μ_1 (Fremlin 03, 412K).

(iv) If X is a set and μ_1, μ_2 are two complete probability measures on X both inner regular with respect to $\{K : K\in\operatorname{dom}\mu_1\cap\operatorname{dom}\mu_2,\ \mu_1K=\mu_2K\}$, they are equal (Fremlin 03, 412L).

(b) Let X be a set, and Λ a family of probability measures on X such that (α) for all λ_0, $\lambda_1\in\Lambda$ there is a $\lambda\in\Lambda$ which extends both λ_0 and λ_1 (β) for every countable $\Lambda_0\subseteq\Lambda$ there is a probability measure on X (not necessarily belonging to Λ) extending every measure in Λ_0. Then there is a probability measure on X extending every measure in Λ. (Fremlin 03, 457G.)

(c) A **Radon probability space** is a quadruple $(X,\mathfrak{T},\Sigma,\mu)$ where $(X,\mathfrak{T})$ is a Hausdorff topological space, μ is a complete probability measure on X with domain Σ, $\mathfrak{T}\subseteq\Sigma$ (so that μ measures every Borel subset of X), and μ is inner regular with respect to the family of compact subsets of X. The usual measure ν_I on $\{0,1\}^I$ is always a Radon probability measure (Fremlin 03, 416Ub).

(d) Suppose that Z is a zero-dimensional compact Hausdorff space and $\mathfrak{A}$ is the Boolean algebra of open-and-closed subsets of Z. If $\nu:\mathfrak{A}\to[0,1]$ is an additive functional such that $\nu Z=1$, there is a unique Radon probability measure on Z extending ν (Fremlin 03, 416Qa).

(e) Let I be an infinite set.

(i) $\#(\mathfrak{B}_I)$ is the cardinal power $\#(I)^\omega$ (Fremlin 08, 524Ma).

(ii) Let T_I be the domain of ν_I. If $F\in\mathrm{T}_I$ and $\nu_IF>0$, set $\Sigma=\mathrm{T}_I\cap\mathcal{P}F$ and $\mu E=\dfrac{1}{\nu_IF}\nu_IE$ for $E\in\Sigma$; then (F,Σ,μ) is isomorphic to $(\{0,1\}^I,\mathrm{T}_I,\nu_I)$ (Fremlin 02, 344L[1]). Writing $\mathcal{N}_I$ for the null ideal of ν_I, so that $\mathcal{N}_I\cap\mathcal{P}F$ is the null ideal of μ, we see that $\mathcal{N}_I$ and $\mathcal{N}_I\cap\mathcal{P}F$ are isomorphic, and have the same covering number.

(iii) Suppose that $\mathcal{E}\subseteq\mathrm{T}_I$ is a non-empty downwards-directed family of measurable sets such that $\#(\mathcal{E})<\operatorname{cov}\mathcal{N}_I$ and $\gamma=\inf_{E\in\mathcal{E}}\nu_IE>0$. Then $\bigcap\mathcal{E}\ne\emptyset$. **P** For $n\in\mathbb{N}$ we can find $F_n\in\mathcal{E}$ such that $\nu_IF_n\le\gamma+2^{-n}$; because $\mathcal{E}$ is downwards-directed, we can suppose that $\langle F_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so that $F=\bigcap_{n\in\mathbb{N}}F_n$ has measure $\lim_{n\to\infty}\nu_IF_n=\gamma$. If $E\in\mathcal{E}$, then $\nu_I(E\cap F_n)\ge\gamma$ for each n, so $\nu_I(E\cap F)=\gamma$ and $\nu_I(F\setminus E)=0$. As $\#(\mathcal{E})$ is less than $\operatorname{cov}\mathcal{N}_I$, which by (ii) is the covering number of the subspace measure on F, F cannot be covered by $\{F\setminus E:E\in\mathcal{E}\}$, and $\bigcap\mathcal{E}$ is non-empty. **Q**

(f) Let X be a set, Σ an algebra of subsets of X, and $\nu:\Sigma\to[0,1]$ an additive functional such that $\nu X=1$ and $\lim_{n\to\infty}\nu E_n=0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a complete probability measure on X which is inner regular with respect to the family Σ_δ of intersections of sequences in Σ (Fremlin 03, 413K).

3B Measure algebras: definitions and facts

(a) If $(\mathfrak{A},\bar\mu)$ is a probability algebra, then $\mathfrak{A}$ is ccc (Fremlin 02, 322G). So if $A\subseteq\mathfrak{A}$ is any set, there is a countable $B\subseteq A$ with the same upper bounds as A (Fremlin 02, 316E), and A has a least upper bound. (Thus $\mathfrak{A}$ is Dedekind complete.) $\bar\mu$ is order-continuous in the sense that

[1] Later editions only; obtainable through `http://www.essex.ac.uk/maths/staff/fremlin/mtcont.htm`. Note to reader: do you know of a more satisfactory reference for this result?

— if $A \subseteq \mathfrak{A}$ is non-empty and upwards-directed, with supremum c, then $\bar{\mu}c = \sup_{a \in A} \bar{\mu}a$,
— if $A \subseteq \mathfrak{A}$ is non-empty and downwards-directed, with infimum c, then $\bar{\mu}c = \inf_{a \in A} \bar{\mu}a$.
(Fremlin 02, 321C and 321F).

(b) If $\mathfrak{A}$ is a Boolean algebra and $A \subseteq \mathfrak{A}$, I say that A **τ-generates** $\mathfrak{A}$ if $\mathfrak{A}$ is the smallest order-closed subalgebra of itself including A. The **Maharam type** $\tau(\mathfrak{A})$ is the least cardinal of any set $A \subseteq \mathfrak{A}$ which τ-generates $\mathfrak{A}$. If $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, I is an infinite set and $\tau(\mathfrak{A}) \le \#(I)$, then $(\mathfrak{A}, \bar{\mu})$ can be embedded in $(\mathfrak{B}_I, \bar{\nu}_I)$ in the sense that there is an injective Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}_I$ such that $\bar{\nu}_I(\pi a) = \bar{\mu}a$ for every $a \in \mathfrak{A}$ (Fremlin 02, 332N).

(c) If (X, Σ, μ) is a measure space with measure algebra $(\mathfrak{A}, \bar{\mu})$, and $\mathcal{E} \subseteq \Sigma$ σ-generates Σ in the sense that Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, then $\{E^\bullet : E \in \mathcal{E}\}$ τ-generates $\mathfrak{A}$. (If $\mathfrak{A}'$ is an order-closed subalgebra of $\mathfrak{A}$ including $\{E^\bullet : E \in \mathcal{E}\}$, then $\{F : F^\bullet \in \mathfrak{A}'\}$ must be a σ-subalgebra of Σ including $\mathcal{E}$, so is the whole of Σ.)

(d) Let (X, Σ, μ) be a measure space with measure algebra $(\mathfrak{A}, \bar{\mu})$. A **lifting** for μ is a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$. Every complete probability measure has a lifting (Fremlin 02, 341K).

If (X, Σ, μ) is a probability space with measure algebra $(\mathfrak{A}, \bar{\mu})$, $\theta : \mathfrak{A} \to \Sigma$ is a lifting, and $A \subseteq \mathfrak{A}$ is a non-empty set with supremum c in $\mathfrak{A}$, then $\theta c \setminus \bigcup_{a \in A} \theta a$ is negligible. **P** There is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in A such that $c = \sup_{n \in \mathbb{N}} a_n$ ((a) above). Now

$$(\bigcup_{n \in \mathbb{N}} \theta(a_n))^\bullet = \sup_{n \in \mathbb{N}} \theta(a_n)^\bullet = \sup_{n \in \mathbb{N}} a_n = c = (\theta c)^\bullet,$$

so $\theta c \setminus \bigcup_{a \in A} \theta a \subseteq \theta c \setminus \bigcup_{n \in \mathbb{N}} \theta(a_n)$ is negligible. **Q**

(e) If I is an infinite set, I will say that the **standard generating family** in the probability algebra $\mathfrak{B}_I$ is the family $\langle e_i \rangle_{i \in I} = \langle E_i^\bullet \rangle_{i \in I}$ where $E_i = \{x : x \in \{0,1\}^I,\ x(i) = 1\}$ for each $i \in I$. Note that $\bar{\nu}_I e_i = \frac{1}{2}$ and $\bar{\nu}_I(e_i \setminus e_j) = \frac{1}{4}$ for all distinct i, $j \in I$.

3C We are ready to begin work on the proofs of results announced in §1.

Proof of Proposition 1C Let I be a set, and $\mathcal{F}$ an ultrafilter on I. Then the following are equiveridical:

(i) $\mathcal{F}$ is measure-centering;

(ii) whenever $\langle a_i \rangle_{i \in I}$ is a family in $\mathfrak{B}_I$ such that $\inf_{i \in I} \bar{\nu}_I a_i > 0$, there is an $A \in \mathcal{F}$ such that $\{a_i : i \in A\}$ is centered in $\mathfrak{B}_I$;

(iii) whenever $\langle E_i \rangle_{i \in I}$ is a family of measurable subsets of $\{0,1\}^I$ such that $\inf_{i \in I} \nu_I E_i > 0$, there is an $A \in \mathcal{F}$ such that $\bigcap_{i \in A} E_i \ne \emptyset$;

(iv) whenever (X, Σ, μ) is a compact probability space and $\langle E_i \rangle_{i \in I}$ is a family in Σ, then $\mu^*(\lim_{i \to \mathcal{F}} E_i) \ge \lim_{i \to \mathcal{F}} \mu E_i$.

proof The case in which $\mathcal{F}$ is a principal ultrafilter is trivial, so I shall assume henceforth that $\mathcal{F}$ is non-principal; in particular, that I is infinite.

(i)⇒(ii) is trivial.

not-(iv)⇒not-(ii) Suppose there are a compact probability space (X, Σ, μ) and a family $\langle E_i \rangle_{i \in I}$ in Σ such that $\lim_{i \to \mathcal{F}} \mu E_i > \mu^*(\lim_{i \to \mathcal{F}} E_i)$. Let $F \in \Sigma$ be such that $\lim_{i \to \mathcal{F}} \mu E_i \subseteq F$ and $\mu F < \lim_{i \to \mathcal{F}} \mu E_i$; let $\gamma > 0$ be such that $\lim_{i \to \mathcal{F}} \mu E_i > \mu F + \gamma$, and set $C = \{i : \mu E_i > \gamma + \mu F\}$, so that $C \in \mathcal{F}$ and $\mu(E_i \setminus F) > \gamma$ for every $i \in C$.

Let $\mathcal{K}$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$. For $i \in C$, let $K_i \in \mathcal{K} \cap \Sigma$ be such that $K_i \subseteq E_i \setminus F$ and $\mu K_i \ge \gamma$. For $i \in I \setminus C$, set $K_i = X$. Observe that

$$\lim_{i \to \mathcal{F}} K_i \subseteq \lim_{i \to \mathcal{F}} (E_i \setminus F) = (\lim_{i \to \mathcal{F}} E_i) \setminus F = \emptyset.$$

Let T be the σ-subalgebra of Σ generated by $\{K_i : i \in I\}$, and $\nu = \mu \restriction \mathrm{T}$. Then the measure algebra $(\mathfrak{B}, \bar{\nu})$ of (X, T, ν) is a probability algebra of Maharam type at most $\#(I)$, by 3Bc. By 3Bb, there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{B}_I$. Set $a_i = \pi(K_i^\bullet)$ for $i \in C$; then $\bar{\nu}_I a_i = \mu K_i \ge \gamma$ for every $i \in I$.

If $A \in \mathcal{F}$, then $A \cap C \in \mathcal{F}$ so

$$\bigcap_{i\in A}K_i\subseteq\bigcap_{i\in A\cap C}E_i\setminus F\subseteq(\lim_{i\to\mathcal{F}}E_i)\setminus F=\emptyset.$$

As $\mathcal{K}$ is a compact class, there must be a finite set $J\subseteq A\cap C$ such that $\bigcap_{i\in J}K_i=\emptyset$. But this means that

$$\inf_{i\in J}a_i=\inf_{i\in J}\pi(K_i^\bullet)=\pi(\inf_{i\in J}K_i^\bullet)=\pi((\bigcap_{i\in J}K_i)^\bullet)$$
$$=\pi(\emptyset^\bullet)=\pi 0=0$$

in $\mathfrak{B}_I$. This shows that $\{a_i:i\in A\}$ is not centered. As A is arbitrary, $\langle a_i\rangle_{i\in I}$ witnesses that (ii) is false.

(iv)⇒(i) Suppose that (iv) is true. Take a Boolean algebra $\mathfrak{A}$, an additive functional $\nu:\mathfrak{A}\to[0,\infty[$ such that $\nu 1=1$, and a family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i>0$. Let Z be the Stone space of $\mathfrak{A}$, so that Z is a compact Hausdorff space and $\mathfrak{A}$ can be identified with the algebra of open-and-closed subsets of Z. Then there is an extension of ν to a Radon probability measure μ on Z (Fact 3Ad). Since μ is inner regular with respect to the compact class of compact subsets of Z, (Z,μ) is a compact probability space.

Let $\mathcal{G}$ be the family of μ-negligible open subsets of Z, and H its union. Then H is an open set, so μH is defined. If $K\subseteq H$ is compact, K is covered by finitely many members of $\mathcal{G}$, so $\mu K=0$; as μ is inner regular with respect to the compact sets, $\mu H=0$.

By (iv),

$$\mu^*(\lim_{i\to\mathcal{F}}(a_i\setminus H))\geq\lim_{i\to\mathcal{F}}\mu(a_i\setminus H)\geq\inf_{i\in I}\mu(a_i\setminus H)$$
$$=\inf_{i\in I}\mu a_i=\inf_{i\in I}\nu a_i>0.$$

So $\lim_{i\to\mathcal{F}}(a_i\setminus H)$ is non-empty and there is a $z\in Z\setminus H$ such that $A=\{i:z\in a_i\}\in\mathcal{F}$. If $J\subseteq A$ is finite and not empty, then $\bigcap_{i\in J}a_i$ is an open set containing z, so is not included in H and does not belong to $\mathcal{G}$, and

$$0<\mu(\bigcap_{i\in J}a_i)=\nu(\inf_{i\in J}a_i).$$

Thus $A\in\mathcal{F}$ has the property demanded in Definition 1A; as $\mathfrak{A}$, ν and $\langle a_i\rangle_{i\in I}$ are arbitrary, $\mathcal{F}$ is measure-centering.

(iv)⇒(iii) is elementary, once we know that every ν_I is a Radon measure (Fact 3Ac).

(iii)⇒(ii) Suppose that (iii) is true, and that $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{B}_I$ such that $\epsilon=\inf_{i\in I}\bar\nu_I a_i$ is greater than 0. Write T_I for the domain of ν_I, and let $\theta:\mathfrak{B}_I\to\mathrm{T}_I$ be a lifting (Fact 3Bd); set $E_i=\theta a_i$ for each $i\in I$. Then

$$\nu_I E_i=\bar\nu_I(E_i^\bullet)=\bar\nu_I a_i\geq\epsilon$$

for every $i\in I$. By (iii), there is an $A\in\mathcal{F}$ such that $\bigcap_{i\in A}E_i\neq\emptyset$. If $J\subseteq A$ is finite and not empty, then

$$\theta(\inf_{i\in J}a_i)=\bigcap_{i\in J}\theta a_i=\bigcap_{i\in J}E_i\neq\emptyset,$$

so $\inf_{i\in J}a_i\neq 0$; as J is arbitrary, $\{a_i:i\in A\}$ is centered in $\mathfrak{B}_I$; as $\langle a_i\rangle_{i\in I}$ is arbitrary, (ii) is true.

3D In Definition 1A, it is clear that we can expect to have more difficulty in finding a centering set in $\mathcal{F}$ if $\inf_{i\in I}\nu a_i$ is nearly 0, and it is natural to focus on that case as the essence of the definition. As it happens, however, it makes no difference.

Proposition Let $\mathcal{F}$ be an ultrafilter on a set I, and suppose that $\gamma<1$ is such that whenever $\mathfrak{A}$ is a Boolean algebra, $\nu:\mathfrak{A}\to[0,1]$ is an additive functional such that $\nu 1=1$, and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\nu a_i\geq\gamma$ for every i, then there is a $J\in\mathcal{F}$ such that $\nu(\inf_{i\in K}a_i)>0$ for every finite $K\subseteq J$. Then $\mathcal{F}$ is measure-centering.

proof Suppose that $\mathfrak{A}$ is a Boolean algebra, $\nu:\mathfrak{A}\to[0,1]$ is an additive functional such that $\nu 1=1$, and $\langle a_i\rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I}\nu a_i=\epsilon>0$. Let $m\in\mathbb{N}$ be such that $(1-\epsilon)^m\leq 1-\gamma$, and let $\mathfrak{C}=\bigotimes_m\mathfrak{A}$ be the free product of m copies of $\mathfrak{A}$, so that we have Boolean homomorphisms $\varepsilon_k:\mathfrak{A}\to\mathfrak{C}$, for $k<m$, such that $\inf_{k<m}\varepsilon_k b_k\ne 0$ whenever $b_k\in\mathfrak{A}\setminus\{0\}$ for every $k<m$ (Fremlin 02, §315). Let $\lambda:\mathfrak{C}\to[0,1]$ be the additive functional such that $\lambda(\inf_{k<m}\varepsilon_k b_k)=\prod_{k>m}\nu_k b_k$ whenever $\langle b_k\rangle_{k<m}\in\mathfrak{A}^m$ (Fremlin 02, 326Q). For $i\in I$, set $c_i=\sup_{k<m}\varepsilon_k a_i$; then

$$\lambda(1\setminus c_i)=(\nu(1\setminus a_i))^m\le(1-\epsilon)^m\le 1-\gamma$$

and $\lambda c_i\ge\gamma$ for every $i\in I$. By hypothesis, there is a $J\in\mathcal{F}$ such that $\lambda(\inf_{i\in K}c_i)>0$ for every finite $K\subseteq J$. Let $D\subseteq\mathfrak{C}$ be a maximal set such that $c_i\in D$ for every $i\in J$ and $\lambda(\inf D_0)>0$ for every finite $D_0\subseteq D$. Then for every $i\in J$ there is a $k_i<m$ such that $\varepsilon_{k_i}(a_i)\in D$. Because $\mathcal{F}$ is an ultrafilter, there is a $k<m$ such that $J'=\{i:i\in J,\ k_i=k\}$ belongs to $\mathcal{F}$; and now $\nu(\inf_{i\in K}a_i)=\lambda(\inf_{i\in K}\varepsilon_k a_i)>0$ for every finite $K\subseteq J'$. As $\mathfrak{A}$, ν and $\langle a_i\rangle_{i\in I}$ are arbitrary, $\mathcal{F}$ is measure-centering.

3E Proof of Theorem 1Ba A Ramsey ultrafilter is measure-centering.

proof Let $\mathcal{F}$ be a Ramsey ultrafilter on a set I, and $\langle a_i\rangle_{i\in I}$ a family in $\mathfrak{B}_I$ such that $\epsilon=\inf_{i\in I}\bar\nu_I a_i$ is greater than 0. For $C\subseteq I$ set $b_C=\sup_{i\in C}a_i$; then $\bar\nu_I b_C\ge\epsilon$ for every $C\in\mathcal{F}$. Set $b=\inf_{C\in\mathcal{F}}b_C$; because $\mathcal{F}$ is downwards-directed, so is $\{b_C:C\in\mathcal{F}\}$, and $\bar\nu_I b\ge\epsilon$ (Fact 3Ba). In particular, $b\ne 0$.

Let $\mathcal{S}$ be the set of those finite subsets K of I such that $b\cap\inf_{i\in K}a_i$ is non-zero, counting $\inf\emptyset$ as 1, so that $\emptyset\in\mathcal{S}$. If $K\in\mathcal{S}$, then $C=\{i:K\cup\{i\}\in\mathcal{S}\}$ belongs to $\mathcal{F}$. **P?** Otherwise, $I\setminus C\in\mathcal{F}$ and $b\subseteq b_{I\setminus C}$. Set

$$d=b\cap\inf_{k\in K}a_k\subseteq b_{I\setminus C}=\sup_{i\in I\setminus C}a_i;$$

as $d\ne 0$, there is a $j\in I\setminus C$ such that $d\cap a_j\ne 0$. But $d\cap a_j=b\cap\inf_{i\in K\cup\{j\}}a_i$, so $K\cup\{j\}\in\mathcal{S}$ and $j\in C$, which is absurd. **XQ**

So $\mathcal{S}$ satisfies the conditions of 2A. Since $\mathcal{F}$ is dependently selective (Proposition 2D), there is a $J\in\mathcal{F}$ such that $[J]^{<\omega}\subseteq\mathcal{S}$, that is, $\{a_i:i\in J\}$ is centered. As $\langle a_i\rangle_{i\in I}$ is arbitrary, $\mathcal{F}$ is measure-centering.

3F Proof of Theorem 1Bb If κ is an infinite cardinal and $\operatorname{cov}\mathcal{N}_\kappa=2^\kappa$, then there is a uniform measure-centering ultrafilter on κ.

proof (a) $\#(\mathfrak{B}_\kappa)=\kappa^\omega$ (Fact 3A(e-i)), so we can enumerate $\mathfrak{B}_\kappa^\kappa$ as $\langle\boldsymbol{a}_\zeta\rangle_{\zeta<2^\kappa}$. Let T_κ be the domain of ν_κ and $\theta:\mathfrak{B}_\kappa\to\mathrm{T}_\kappa$ a lifting. Set $\lambda=\operatorname{cf}\kappa$. If $\lambda=\kappa$, set $I_\alpha=\{\alpha\}$ for every $\alpha<\kappa$; otherwise, let $\langle\kappa_\alpha\rangle_{\alpha<\lambda}$ be a strictly increasing family of cardinals less than κ with supremum κ, and set $I_\alpha=\kappa_\alpha^+\setminus\bigcup_{\beta<\alpha}I_\beta$ for $\alpha<\lambda$, so that $\langle I_\alpha\rangle_{\alpha<\lambda}$ is a partition of κ and $\operatorname{otp}(I_\alpha)=\kappa_\alpha^+$ for every $\alpha<\lambda$.

(b) Construct families $\langle C_{\alpha\zeta}\rangle_{\alpha<\lambda,\zeta\le 2^\kappa}$, $\langle D_\zeta\rangle_{\zeta<2^\kappa}$, $\langle\mathcal{C}_{\alpha\zeta}\rangle_{\alpha<\lambda,\zeta\le 2^\kappa}$ and $\langle\mathcal{D}_\zeta\rangle_{\zeta<2^\kappa}$ inductively, as follows. Start by setting $\mathcal{C}_{\alpha 0}=\{I_\alpha\setminus\xi:\xi\in I_\alpha\}$, so that $\mathcal{C}_{\alpha 0}$ is a filter base of subsets of I_α of cardinal less than κ; let $\mathcal{D}_0$ be $\{\lambda\}$. Given that $\zeta<2^\kappa$, that $\mathcal{D}_\zeta$ is a filter base of subsets of λ of cardinal at most $\max(\omega,\#(\zeta))$, and $\mathcal{C}_{\alpha\zeta}$ is a filter base of subsets of I_α of cardinal at most $\max(\kappa,\#(\zeta))$ for each $\alpha<\lambda$, consider $\boldsymbol{a}_\zeta=\langle a_\xi\rangle_{\xi<\kappa}$ say. Set $\epsilon=\inf_{\xi<\kappa}\bar\nu_\kappa a_\xi$. If $\epsilon=0$, set $D_{\zeta+1}=\lambda$ and $C_{\alpha,\zeta+1}=I_\alpha$ for every $\alpha<\lambda$. Otherwise, set $b_C=\sup_{\xi\in C}a_\xi$ for $C\subseteq\kappa$, and $c_\alpha=\inf_{C\in\mathcal{C}_{\alpha\zeta}}b_C$ for $\alpha<\lambda$; as in 3E above, $\bar\nu_\kappa c_\alpha=\inf_{C\in\mathcal{C}_{\alpha\zeta}}\bar\nu_\kappa b_C\ge\epsilon$. Set $d_D=\sup_{\alpha\in D}c_\alpha$ for $D\in\mathcal{D}_\zeta$ and $e=\inf_{D\in\mathcal{D}_\zeta}d_D$; then the same arguments show that $\bar\nu_\kappa e\ge\epsilon$.

For each $C\subseteq\kappa$, $\theta(b_C)\setminus\bigcup_{\xi\in C}\theta(a_\xi)$ is negligible (Fact 3Bd). So $E_{\alpha C}=\theta(c_\alpha)\setminus\bigcup_{\xi\in C}\theta(a_\xi)$ is negligible whenever $\alpha<\lambda$ and $C\in\mathcal{C}_\alpha$. Similarly, $E_D=\theta(e)\setminus\bigcup_{\alpha\in D}\theta(c_\alpha)$ is negligible for every $D\in\mathcal{D}$. Now $\mathcal{D}_\zeta\cup\{(\alpha,C):\alpha<\lambda,\ C\in\mathcal{C}_{\alpha\zeta}\}$ has cardinal at most $\max(\omega,\#(\zeta),\#(\kappa))<2^\kappa$, so $\{E_D:D\in\mathcal{D}_\zeta\}\cup\{E_{\alpha C}:\alpha<\lambda,\ C\in\mathcal{C}_{\alpha\zeta}\}$ cannot cover the non-negligible measurable set $\theta(e)$ (Fact 3A(e-iii)), and there must be an $x_\zeta\in\theta(e)$ such that $x_\zeta\notin\bigcup_{D\in\mathcal{D}_\zeta}E_D\cup\bigcup_{\alpha<\lambda,C\in\mathcal{C}_\alpha}E_{\alpha C}$. Set $D_\zeta=\{\alpha:\alpha<\lambda,\ x_\zeta\in\theta(c_\alpha)\}$; then D_ζ meets every member of $\mathcal{D}_\zeta$. For $\alpha\in D_\zeta$, set $C_{\alpha\zeta}=\{\xi:\xi\in I_\alpha,\ x_\zeta\in\theta(a_\xi)\}$; for $\alpha\in\kappa\setminus D_\zeta$, set $C_{\alpha\zeta}=I_\alpha$; then $C_{\alpha\zeta}$ meets every member of $\mathcal{C}_{\alpha\zeta}$.

Now define $\mathcal{D}_{\zeta+1}$, $\mathcal{C}_{\alpha,\zeta+1}$ by setting

$$\mathcal{D}_{\zeta+1}=\mathcal{D}_\zeta\cup\{D\cap D_\zeta:D\in\mathcal{D}_\zeta\},$$

$$\mathcal{C}_{\alpha,\zeta+1}=\mathcal{C}_{\alpha\zeta}\cup\{C\cap C_{\alpha\zeta}:C\in\mathcal{C}_{\alpha\zeta}\}$$

for $\alpha<\lambda$. Then $\mathcal{D}_{\zeta+1}$ is a filter base of subsets of λ of cardinal at most $\max(\omega,\#(\zeta+1))$, and $\mathcal{C}_{\alpha,\zeta+1}$ is a filter base of subsets of I_α of cardinal at most $\max(\kappa,\#(\zeta+1))$ for every $\alpha<\lambda$.

For non-zero limit ordinals $\zeta \le 2^\kappa$, set $\mathcal{D}_\zeta = \bigcup_{\xi<\zeta} \mathcal{D}_\xi$ and $\mathcal{C}_{\alpha\zeta} = \bigcup_{\xi<\zeta} \mathcal{C}_{\alpha\xi}$; once again, $\mathcal{D}_\zeta$ will be a filter base of subsets of λ of cardinal at most $\max(\omega, \#(\zeta))$, and $\mathcal{C}_{\alpha\zeta}$ will a filter base of subsets of I_α of cardinal at most $\max(\kappa, \#(\zeta))$ for each $\alpha < \lambda$.

(c) At the end of the induction, let $\mathcal{F}$ be an ultrafilter on κ containing all sets of the form $\bigcup_{\alpha\in D} C_\alpha$ where $D \in \mathcal{D}_{2^\kappa}$ and $C_\alpha \in \mathcal{C}_{\alpha,2^\kappa}$ for every $\alpha \in D$. Then $\mathcal{F}$ is measure-centering. **P** Let $\boldsymbol{a} = \langle a_\xi\rangle_{\xi<\kappa} \in \mathfrak{B}_\kappa^\kappa$ be such that $\inf_{\xi<\kappa} \bar\nu_\kappa a_\xi > 0$. Then there is a $\zeta < 2^\kappa$ such that $\boldsymbol{a} = \boldsymbol{a}_\zeta$. In this case, x_ζ is defined and $J = \bigcup_{\alpha\in D_\zeta} C_{\alpha\zeta}$ belongs to $\mathcal{F}$. If $\alpha \in D_\zeta$ and $\xi \in C_{\alpha\zeta}$, then $x_\zeta \in \theta(a_\xi)$. But this means that if $K \subseteq J$ is finite and not empty,

$$x_\zeta \in \bigcap_{\xi\in K} \theta(a_\xi) = \theta(\inf_{\xi\in K} a_\xi),$$

and $\inf_{\xi\in K} a_\xi \ne 0$. Thus $\{a_\xi : \xi \in J\}$ is centered. As $\boldsymbol{a}$ is arbitrary, $\mathcal{F}$ is measure-centering, by Proposition 1C(ii). **Q**

(d) Of course I should note that $\mathcal{F}$ is uniform because if $D \in \mathcal{D}_{2^\kappa}$ and $C_\alpha \in \mathcal{C}_{\alpha,2^\kappa}$ for every $\alpha \in D$, then D meets every member of $\mathcal{D}_0$, so has cardinal λ. Set $B = \bigcup_{\alpha\in D} C_\alpha$. Since C_α meets every member of $\mathcal{C}_{\alpha 0}$, $\#(C_\alpha) = \#(I_\alpha)$ for every $\alpha \in D$, and $\#(B) = \#(\bigcup_{\alpha\in D} I_\alpha) = \kappa$.

3G Proof of Theorem 1Bc If $\operatorname{cov}\mathcal{N}_{\text{Leb}} = \mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ which contains no set of zero asymptotic density.

proof (a) I start with a general fact about upper asymptotic density $d^* : \mathcal{P}\mathbb{N} \to [0,1]$. Let (X, Σ, μ) be a probability space, $I \in \mathcal{P}\mathbb{N} \setminus \mathcal{Z}$, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ with $\inf_{n\in\mathbb{N}} \mu E_n = \gamma > 0$. For $x \in X$ set $J_x = \{n : n \in I,\ x \in E_n\}$. Then $\mu\{x : d^*(J_x) > 0\} \ge \gamma$.

P Set $\delta = d^*(I)$; then there is a disjoint sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}$, all of the form $\{i : k \le i < 2k\}$, such that $\#(I \cap K_n) \ge \frac{1}{3}\delta\#(K_n)$ for every n. If $n \in \mathbb{N}$ then

$$\gamma\#(I \cap K_n) \le \sum_{i\in I\cap K_n} \mu E_i = \int \sum_{i\in I\cap K_n} \chi E_i(x)\mu(dx) = \int \#(J_x \cap K_n)\mu(dx),$$

so if $\eta > 0$

$$\mu\{x : \#(J_x \cap K_n) \ge \eta\#(I \cap K_n)\} \ge \gamma - \eta.$$

Consequently

$$\begin{aligned}\mu\{x : d^*(J_x) \ge \tfrac{1}{6}\delta\eta\} \\ &\ge \mu\{x : \text{for infinitely many } n,\ \#(J_x \cap K_n) \ge \tfrac{1}{3}\delta\eta\#(K_n)\} \\ &\ge \mu\{x : \text{for infinitely many } n,\ \#(J_x \cap K_n) \ge \eta\#(I \cap K_n)\} \\ &\ge \gamma - \eta.\end{aligned}$$

As η is arbitrary, $\mu\{x : d^*(J_x) > 0\} \ge \gamma$. **Q**

(b) Now, given that $\operatorname{cov}\mathcal{N}_{\text{Leb}} = \mathfrak{c}$, enumerate the family of all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel subsets of $\{0,1\}^\omega$ such that $\inf_{n\in\mathbb{N}} \nu E_n > 0$ as $\langle\langle E_{\xi n}\rangle_{n\in\mathbb{N}}\rangle_{\xi<\mathfrak{c}}$. Build filter bases $\mathcal{E}_\xi \subseteq \mathcal{P}\mathbb{N}$, for $\xi \le \mathfrak{c}$, as follows. Start with $\mathcal{E}_0 = \{\mathbb{N}\}$. The inductive hypothesis will be that $\#(\mathcal{E}_\xi) \le \max(\omega, \#(\xi))$ and $\mathcal{E}_\xi \cap \mathcal{Z} = \emptyset$. For the inductive step to $\xi + 1$, set $\epsilon = \inf_{n\in\mathbb{N}} \nu E_{\xi n}$. For each $I \in \mathcal{E}_\xi$, set

$$F_I = \{x : d^*(J_x \cap I) > 0\},$$

where $J_x = \{n : x \in E_{\xi n}\}$. By (a), $\nu F_I \ge \epsilon$; because $\langle F_I\rangle_{I\in\mathcal{E}_\xi}$ is downwards-directed, and $\#(\mathcal{E}_\xi) < \operatorname{cov}\mathcal{N}_{\text{Leb}} = \operatorname{cov}\mathcal{N}_\omega$, there is a point $x \in \bigcap_{I\in\mathcal{E}_\xi} F_I$ (Fact 3A(e-iii)). Set $I_\xi = J_x$,

$$\mathcal{E}_{\xi+1} = \mathcal{E}_\xi \cup \{I \cap I_\xi : I \in \mathcal{E}_\xi\}.$$

Then $\mathcal{E}_{\xi+1}$ is a filter base, including $\mathcal{E}_\xi$, disjoint from $\mathcal{Z}$, of cardinal at most $\max(\omega, \#(\xi+1))$, and containing a set I_ξ such that $\bigcap_{n\in I_\xi} E_{\xi n}$ is non-empty. At non-zero limit ordinals $\xi \le \mathfrak{c}$, set $\mathcal{E}_\xi = \bigcup_{\eta<\xi} \mathcal{E}_\eta$.

Let $\mathcal{G}$ be the filter on $\mathbb{N}$ generated by $\mathcal{E}_\mathfrak{c}$. Then $\mathcal{G} \cap \mathcal{Z} = \emptyset$, so there is an ultrafilter $\mathcal{F}$ on $\mathbb{N}$, including $\mathcal{G}$, and still disjoint from $\mathcal{Z}$. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is any sequence of measurable subsets of $\{0,1\}^\mathbb{N}$ such that $\inf_{n\in\mathbb{N}} \nu F_n > 0$, there is a $\xi < \mathfrak{c}$ such that $E_{\xi n} \subseteq F_n$ for every n; now $I_\xi \in \mathcal{F}$ and $\bigcap_{n \in I_\xi} F_n \neq \emptyset$. By Proposition 1C(iii), $\mathcal{F}$ is measure-centering.

3H Proof of Proposition 1D (a) Let I and J be sets, $f : I \to J$ a function, and $\mathcal{F}$ a measure-centering ultrafilter on I. Then the image ultrafilter $f[[\mathcal{F}]]$ is a measure-centering ultrafilter on J.

(b) If $\mathcal{F}$ is a non-principal ultrafilter, then $\mathcal{F} \ltimes \mathcal{F}$ is not measure-centering.

proof (a) Let $\mathfrak{A}$ be a Boolean algebra, $\nu : \mathfrak{A} \to [0,1]$ an additive functional such that $\nu 1 = 1$, and $\langle a_j \rangle_{j \in J}$ a family in $\mathfrak{A}$ such that $\inf_{j \in J} \nu a_j > 0$. Set $b_i = a_{f(i)}$ for $i \in I$; then $\inf_{i \in I} \nu b_i > 0$, so there is an $A \in \mathcal{F}$ such that $\nu(\inf_{i \in K} b_i) > 0$ for every $K \in [A]^{<\omega}$. Now $f[A] \in f[[\mathcal{F}]]$, and if $L \subseteq f[A]$ is finite there is a finite $K \subseteq A$ such that $L = f[K]$, so that

$$\nu(\inf_{j \in L} a_j) = \nu(\inf_{i \in K} b_i) > 0.$$

This shows that $f[[\mathcal{F}]]$ has the property of Definition 1A and is measure-centering.

(b) Let I be a set and $\mathcal{F}$ a non-principal ultrafilter on I. Let $\langle e_i \rangle_{i \in I}$ be the standard generating family in $\mathfrak{B}_I$, and for i, $j \in I$ set

$$\begin{aligned} a_{ij} &= e_i \setminus e_j \text{ if } i \neq j, \\ &= 1 \text{ if } i = j. \end{aligned}$$

Then $\bar{\nu}_I a_{ij} \geq \frac{1}{4}$ for all i, $j \in I$. If $A \in \mathcal{F} \ltimes \mathcal{F}$, then $B = \{i : A[\{i\}] \in \mathcal{F}\}$ belongs to $\mathcal{F}$. Take any $i \in B$; then B must meet $A[\{i\}]$ in more than one point, because $\mathcal{F}$ is non-principal; take $j \in B \cap A[\{i\}] \setminus \{i\}$ and $k \in A[\{j\}] \setminus \{j\}$. The points (i,j) and (j,k) are distinct points of A and $a_{jk} \cap a_{ij} \subseteq e_j \setminus e_j = 0$. As A is arbitrary, $\mathcal{F} \ltimes \mathcal{F}$ cannot be measure-centering.

Remark I include (b) here to show that in Theorem 1J we really need to have non-isomorphic filters. In fact rather more can be said; the argument here already shows that $\mathcal{F} \ltimes \mathcal{F}$ is not measure-linking, and in fact it is not Hausdorff (Daguenet-Tessier 79).

3I Proof of Theorem 1E Let (X, Σ, μ) be a compact probability space, and $\mathcal{F}$ a measure-centering ultrafilter on a set I. Let $\mathcal{A}$ be the family of all sets of the form $\lim_{i \to \mathcal{F}} E_i$ where $\langle E_i \rangle_{i \in I}$ is a family in Σ. Then there is a unique complete probability measure λ on X such that λ is inner regular with respect to $\mathcal{A}$ and $\lambda(\lim_{i \to \mathcal{F}} E_i) = \lim_{i \to \mathcal{F}} \mu E_i$ for every family $\langle E_i \rangle_{i \in I}$ in Σ.

proof (a) The key fact is that if $\langle E_i \rangle_{i \in I}$ and $\langle F_i \rangle_{i \in I}$ are two families in Σ such that $\lim_{i \to \mathcal{F}} E_i = \lim_{i \to \mathcal{F}} F_i$, then $\lim_{i \to \mathcal{F}} \mu E_i = \lim_{i \to \mathcal{F}} \mu F_i$. **P** We have

$$\begin{aligned} |\lim_{i \to \mathcal{F}} \mu E_i - \lim_{i \to \mathcal{F}} \mu F_i| &= \lim_{i \to \mathcal{F}} |\mu E_i - \mu F_i| \leq \lim_{i \to \mathcal{F}} \mu(E_i \triangle F_i) \\ &\leq \mu^*(\lim_{i \to \mathcal{F}} (E_i \triangle F_i)) \end{aligned}$$

(by Proposition 1C(iv))

$$\leq \mu^*(\lim_{i \to \mathcal{F}} E_i \triangle \lim_{i \to \mathcal{F}} F_i) = \mu^* \emptyset = 0. \textbf{ Q}$$

(b) The formula $\phi(\lim_{i \to \mathcal{F}} E_i) = \lim_{i \to \mathcal{F}} \mu E_i$ therefore defines a functional $\phi : \mathcal{A} \to [0,1]$. If $*$ is any of the Boolean operations $\setminus$, $\cap$ and $\cup$, then

$$\lim_{i \to \mathcal{F}} E_i * \lim_{i \to \mathcal{F}} F_i = \lim_{i \to \mathcal{F}} (E_i * F_i)$$

for all families $\langle E_i \rangle_{i \in I}$ and $\langle F_i \rangle_{i \in I}$ in Σ, so $\mathcal{A}$ is an algebra of subsets of X. As noted in 1E, ϕ extends μ. If $\langle E_i \rangle_{i \in I}$ and $\langle F_i \rangle_{i \in I}$ are two families in Σ,

$$\begin{aligned}\phi(\lim_{i\to\mathcal{F}} E_i \cup \lim_{i\to\mathcal{F}} F_i) + \phi(\lim_{i\to\mathcal{F}} E_i \cap \lim_{i\to\mathcal{F}} F_i) \\ &= \phi(\lim_{i\to\mathcal{F}}(E_i \cup F_i)) + \phi(\lim_{i\to\mathcal{F}}(E_i \cap F_i)) \\ &= \lim_{i\to\mathcal{F}} \mu(E_i \cup F_i) + \lim_{i\to\mathcal{F}} \mu(E_i \cap F_i) \\ &= \lim_{i\to\mathcal{F}} \mu(E_i \cup F_i) + \mu(E_i \cap F_i) \\ &= \lim_{i\to\mathcal{F}} \mu E_i + \mu F_i = \phi(\lim_{i\to\mathcal{F}} E_i) + \phi(\lim_{i\to\mathcal{F}} F_i));\end{aligned}$$

so $\phi(A \cup B) + \phi(A \cap B) = \phi A + \phi B$ for all A, $B \in \mathcal{A}$. Because $\phi\emptyset = \mu\emptyset = 0$, this is enough to show that $\phi : \mathcal{A} \to [0, 1]$ is additive.

(c) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$ and $0 \le \gamma < \inf_{n\in\mathbb{N}} \phi A_n$, then there is an $A \in \mathcal{A}$ such that $A \subseteq \bigcap_{n\in\mathbb{N}} A_n$ and $\phi A \ge \gamma$. **P** For each n, express A_n as $\lim_{i\to\mathcal{F}} E_{ni}$, where $E_{ni} \in \Sigma$ for each $i \in I$; replacing E_{ni} by $\bigcap_{m\le n} E_{mi}$ if necessary, we may suppose that $E_{ni} \subseteq E_{mi}$ whenever $m \le n$ and $i \in I$. For each $i \in I$, define F_i by saying that

$$\begin{aligned}F_i &= X \text{ if } \mu E_{0i} < \gamma, \\ &= E_{ni} \text{ if } n \in \mathbb{N} \text{ and } \mu E_{ni} \ge \gamma > \mu E_{n+1,i}, \\ &= \bigcap_{n\in\mathbb{N}} E_{ni} \text{ if } \mu E_{ni} \ge \gamma \text{ for every } n \in \mathbb{N}.\end{aligned}$$

Then $\mu F_i \ge \gamma$ for every $i \in I$, and $F_i \subseteq E_{ni}$ whenever $n \in \mathbb{N}$, $i \in I$ and $\mu E_{ni} \ge \gamma$. For each $n \in \mathbb{N}$, therefore, $\{i : F_i \subseteq E_{ni}\}$ belongs to $\mathcal{F}$, and $A = \lim_{i\to\mathcal{F}} F_i \subseteq A_n$; while $\phi A \ge \gamma$. **Q**

(d) In particular, ϕ is countably additive in the sense that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{A}$ such that $\bigcap_{n\in\mathbb{N}} A_n$ is empty, then $\lim_{n\to\infty} \phi A_n = 0$. There is therefore an extension of ϕ to a complete probability measure λ on X which is inner regular with respect to $\mathcal{A}_\delta$, the family of subsets of X expressible as the intersection of a sequence in $\mathcal{A}$ (Fact 3Af). From (c) we see that if $B \in \mathcal{A}_\delta$ and $\gamma < \lambda B$, there must be an $A \in \mathcal{A}$ such that $A \subseteq B$ and $\phi A \ge \gamma$. So in fact λ is inner regular with respect to $\mathcal{A}$.

(e) Thus we have a suitable extension of μ. By 3A(a-iv), λ is uniquely defined.

Remark If you have seen a construction of Loeb measure (Loeb 75), you will recognise the method above; the special properties of measure-centering ultrafilters and compact measure spaces mean, in effect, that the original set X has full outer measure in the Loeb measure space.

4 Reduced products of probability algebras

The construction offered in 1F is straightforward enough, but requires some support, in particular in the assertion that the reduced product $(\mathfrak{C}, \bar\lambda)$ there is a probability algebra in the full sense of the phrase as used here. In this section I fill in the details, with a proof of Theorem 1G, and also of a further result, Proposition 4B, which will be needed for the proof of Theorem 1J in the next section.

4A I start with a slightly expanded version of 1Fb.

Proposition Let $\langle(\mathfrak{A}_i, \bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $\mathcal{F}$ an ultrafilter on I. Let $\mathfrak{B}$ be the product Boolean algebra $\prod_{i\in I} \mathfrak{A}_i$. Define $\nu : \mathfrak{B} \to [0, 1]$ by setting $\nu(\langle a_i\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \bar\mu a_i$ whenever $\langle a_i\rangle_{i\in I} \in \mathfrak{B}$, and set $\mathcal{I} = \{\boldsymbol{a} : \boldsymbol{a} \in \mathfrak{B}, \nu\boldsymbol{a} = 0\}$.

(a) ν is additive; $\mathcal{I}$ is an ideal of $\mathfrak{B}$; and if $\mathfrak{C}$ is the quotient Boolean algebra $\mathfrak{B}/\mathcal{I}$, we have an additive functional $\bar\nu : \mathfrak{C} \to [0, 1]$ defined by saying that $\bar\nu(\boldsymbol{a}^\bullet) = \nu\boldsymbol{a}$ for every $\boldsymbol{a} \in \mathfrak{B}$. If $\langle a_i\rangle_{i\in I}$, $\langle b_i\rangle_{i\in I} \in \mathfrak{B}$, then

— if $\{i : a_i \subseteq b_i\} \in \mathcal{F}$, then $\langle a_i\rangle^\bullet_{i\in I} \subseteq \langle b_i\rangle^\bullet_{i\in I}$;

— if $\{i : a_i = b_i\} \in \mathcal{F}$, then $\langle a_i\rangle^\bullet_{i\in I} = \langle b_i\rangle^\bullet_{i\in I}$.

(b) $(\mathfrak{C}, \bar\nu)$ is a probability algebra.

proof (a) is entirely elementary.

(b)(i) Perhaps I should begin by remarking that

$$\bar{\nu}(1_{\mathfrak{C}}) = \bar{\nu}(1_{\mathfrak{B}}^\bullet) = \nu(1_{\mathfrak{B}}) = \nu(\langle 1_{\mathfrak{A}_i}\rangle_{i\in I}) = \lim_{i\to\mathcal{F}} \mu_i(1_{\mathfrak{A}_i}) = 1.$$

Next, if $c \in \mathfrak{C}\setminus\{0\}$, then $c = \boldsymbol{b}^\bullet$ for some $\boldsymbol{b} \in \mathfrak{B}\setminus\mathcal{I}$, so $\bar{\nu}c = \nu\boldsymbol{b}$ is non-zero.

(ii) If $\langle c_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{C}$, there is a $c \in \mathfrak{C}$ such that $c_n \subseteq c$ for every $n \in \mathbb{N}$ and $\bar{\nu}c = \sup_{n\in\mathbb{N}} \bar{\nu}c_n$. **P** For each $n \in \mathbb{N}$, let $\langle a_{ni}\rangle_{i\in I} \in \mathfrak{B}$ be such that $c_n = \langle a_{ni}\rangle_{i\in I}^\bullet$. Set $a'_{ni} = \sup_{m\le n} a_{mi}$ for $n \in \mathbb{N}$ and $i \in I$; then $c_n = \langle a'_{ni}\rangle_{i\in I}^\bullet$ for each n, and $\langle a'_{ni}\rangle_{n\in\mathbb{N}}$ is non-decreasing for each $i \in I$. Let γ be $\sup_{n\in\mathbb{N}} \bar{\nu}c_n = \sup_{n\in\mathbb{N}} \lim_{i\to\mathcal{F}} \mu_i a'_{ni}$, and for $i \in I$ set

$$\begin{aligned} a_i &= a'_{ni} \text{ if } \mu_i a'_{ni} \le \gamma + 2^{-n} \text{ and } \mu_i a'_{n+1,i} > \gamma + 2^{-n-1}, \\ &= \sup_{n\in\mathbb{N}} a'_{ni} \text{ if } \mu_i a'_{ni} \le \gamma + 2^{-n} \text{ for every } n \in \mathbb{N}. \end{aligned}$$

Consider $c = \langle a_i\rangle_{i\in I}^\bullet$. For any $n \in \mathbb{N}$,

$$\begin{aligned} \{i : a'_{ni} \subseteq a_i,\ \mu_i a_i \le \gamma + 2^{-n}\} &\supseteq \{i : \mu_i a'_{ni} \le \gamma + 2^{-n}\} \\ &\supseteq \{i : \mu_i a'_{ni} \le \lim_{j\to\mathcal{F}} \mu_j a'_{nj} + 2^{-n}\} \in \mathcal{F}, \end{aligned}$$

so

$$c_n = \langle a'_{ni}\rangle_{i\in I}^\bullet \subseteq \langle a_i\rangle_{i\in I}^\bullet = c.$$

It follows at once that $\bar{\nu}c \ge \gamma$. At the same time, $\bar{\nu}c \le \gamma + 2^{-n}$ for every n, so $\bar{\nu}c \le \gamma$. **Q**

(iii) $\mathfrak{C}$ is Dedekind σ-complete. **P** If $C \subseteq \mathfrak{C}$ is a countable set, take a sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ running over $C \cup \{0_{\mathfrak{C}}\}$. Set $c'_n = \sup_{m\le n} c_m$ for each n. By (ii), there is a $c \in \mathfrak{C}$ such that $c'_n \subseteq c$ for every n and $\bar{\nu}c = \sup_{n\in\mathbb{N}} \bar{\nu}c'_n$. Because $c_n \subseteq c'_n \subseteq c$ for every n, c is an upper bound of C. If c' is any other upper bound of C, then $c' \supseteq c'_n$ for every $n \in \mathbb{N}$, so

$$\bar{\nu}(c\setminus c') \le \inf_{n\in\mathbb{N}} \bar{\nu}(c\setminus c'_n) \le \inf_{n\in\mathbb{N}} \bar{\nu}c - \bar{\nu}c'_n = 0,$$

$c\setminus c' = 0$ and $c \subseteq c'$. Thus c is the least upper bound of C; as C is arbitrary, $\mathfrak{C}$ is Dedekind σ-complete. **Q**

(iv) $\bar{\nu}$ is countably additive. **P** This time, let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathfrak{C}$. Again set $c'_n = \sup_{m\le n} c_m$ for each $n \in \mathbb{N}$. Re-running the argument of (iii), we see that if c is the least upper bound $\sup_{n\in\mathbb{N}} c'_n$ then

$$\bar{\nu}c = \sup_{n\in\mathbb{N}} \bar{\nu}c'_n = \sup_{n\in\mathbb{N}} \sum_{m=0}^n \bar{\nu}c_m = \sum_{n=0}^\infty \bar{\nu}c_n,$$

while c is also the least upper bound of $\{c_n : n \in \mathbb{N}\}$. **Q**

Putting these together, $(\mathfrak{C}, \bar{\nu})$ is a probability algebra.

4B Directed families Some further phenomena appear if $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ is appropriately linked, as follows.

Proposition Let $(I, \le)$ be a non-empty pre-ordered set (that is, $\le$ is a reflexive transitive relation on I), and $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ a family of probability algebras. Suppose that for $i \le j$ in I we are given a measure-preserving Boolean homomorphism $\pi_{ji} : \mathfrak{A}_i \to \mathfrak{A}_j$, and that $\pi_{ki} = \pi_{kj}\pi_{ji}$ whenever $i \le j \le k$ in I. Let $\mathcal{F}$ be an ultrafilter on I such that $\{j : i \le j\}$ belongs to $\mathcal{F}$ for every $i \in I$, and let $(\mathfrak{C}, \bar{\nu})$ be the reduced product $\prod_{i\in I}(\mathfrak{A}, \bar{\mu}_i)|\mathcal{F}$.

(a) For each $i \in I$ we have a measure-preserving Boolean homomorphism $\pi_i : \mathfrak{A}_i \to \mathfrak{C}$ defined by saying that $\pi_i a = \langle a_j\rangle_{j\in I}^\bullet$ whenever $\langle a_j\rangle_{j\in I} \in \prod_{j\in I} \mathfrak{A}_j$ is such that $a_j = \pi_{ji} a_i$ for every $j \ge i$.

(b) $\pi_i = \pi_j \pi_{ji}$ whenever $i \le j$ in I.

(c) $\langle a_i\rangle_{i\in I}^\bullet \subseteq \sup_{j\in A} \pi_j a_j$ whenever $\langle a_i\rangle_{i\in I} \in \prod_{i\in I} \mathfrak{A}_i$ and $A \in \mathcal{F}$.

proof (a) π_i is well-defined because $\{j : j \ge i\} \in \mathcal{F}$; now it is a measure-preserving Boolean homomorphism because every π_{ji} is.

(b) If $a_k=\pi_{ki}a$ for $k\ge i$, then $a_k=\pi_{kj}\pi_{ji}a$ for $k\ge j$, so $\pi_j\pi_{ji}a=\langle a_k\rangle^\bullet_{k\in I}=\pi_i a$.

(c) Set $c=\sup_{j\in A}\pi_j a_j$ in $\mathfrak{C}$. For every $\epsilon>0$, there is a finite $K\subseteq A$ such that $\bar\nu c\le\epsilon+\bar\nu(\sup_{j\in K}\pi_j a_j)$, because $\bar\nu$ is order-continuous (Fact 3Ba). The set $B=\{k:k\in I,\ j\le k\text{ for every }j\in K\}$ belongs to $\mathcal{F}$; fix $k\in B$, and set $b=\sup_{j\in K}\pi_{kj}a_j\in\mathfrak{A}_k$,

$$\begin{aligned}b_i&=\pi_{ik}b\text{ if }k\le i,\\&=0_{\mathfrak{A}_i}\text{ for other }i\in I.\end{aligned}$$

Then

$$\langle b_i\rangle^\bullet_{i\in I}=\pi_k b=\pi_k(\sup\nolimits_{j\in K}\pi_{kj}a_j)=\sup\nolimits_{j\in K}\pi_k\pi_{kj}a_j=\sup\nolimits_{j\in K}\pi_j a_j\subseteq c.$$

If $i\in A$ and $i\ge k$, then

$$\begin{aligned}\bar\mu_i(a_i\setminus b_i)&=\bar\nu(\pi_i a_i\setminus\pi_i b_i)=\bar\nu(\pi_i a_i\setminus\pi_i\pi_{ik}b)\\&=\bar\nu(\pi_i a_i\setminus\pi_k b)=\bar\nu(\pi_i a_i\setminus\sup_{j\in K}\pi_j a_j)\le\bar\nu(c\setminus\sup_{j\in K}\pi_j a_j)\le\epsilon\end{aligned}$$

by the choice of K. Because $\{i:i\in A,\ i\ge k\}\in\mathcal{F}$,

$$\begin{aligned}\bar\nu(\langle a_i\rangle^\bullet_{i\in I}\setminus c)&\le\bar\nu(\langle a_i\rangle^\bullet_{i\in I}\setminus\pi_k b)=\bar\nu(\langle a_i\setminus b_i\rangle^\bullet_{i\in I})\\&=\lim_{i\to\mathcal{F}}\bar\mu_i(a_i\setminus b_i)\le\sup_{i\in A,i\ge k}\bar\mu_i(a_i\setminus b_i)\le\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\bar\nu(\langle a_i\rangle^\bullet_{i\in I}\setminus c)=0$ and $\langle a_i\rangle^\bullet_{i\in I}\subseteq c$.

4C Proof of Theorem 1G Let (X,Σ,μ) be a compact probability space and $(\mathfrak{A},\bar\mu)$ its measure algebra. Let I be a set and $\mathcal{F}$ a measure-centering ultrafilter on I; write λ for the corresponding extension of μ as described in Theorem 1E, and $(\mathfrak{C},\bar\nu)$ for the reduced power $(\mathfrak{A},\bar\mu)^I|\mathcal{F}$. Then we have a natural isomorphism between $(\mathfrak{C},\bar\nu)$ and the measure algebra $(\mathfrak{D},\bar\lambda)$ of λ defined by saying that $\langle E_i^\bullet\rangle^\bullet_{i\in I}\in\mathfrak{C}$ is matched with $(\lim_{i\to\mathcal{F}}E_i)^\bullet\in\mathfrak{D}$ for every family $\langle E_i\rangle_{i\in I}$ in Σ.

proof If $\langle E_i\rangle_{i\in I}$ and $\langle F_i\rangle_{i\in I}$ are families in Σ, then $(\lim_{i\to\mathcal{F}}E_i)^\bullet=(\lim_{i\to\mathcal{F}}F_i)^\bullet$ in $\mathfrak{D}$ iff $\langle E_i^\bullet\rangle^\bullet_{i\in I}=\langle F_i^\bullet\rangle^\bullet_{i\in I}$ in $\mathfrak{C}$. **P**

$$\begin{aligned}(\lim_{i\to\mathcal{F}}E_i)^\bullet=(\lim_{i\to\mathcal{F}}F_i)^\bullet&\iff(\lim_{i\to\mathcal{F}}E_i)^\bullet\,\triangle\,(\lim_{i\to\mathcal{F}}F_i)^\bullet=0_{\mathfrak{D}}\\&\iff((\lim_{i\to\mathcal{F}}E_i)\,\triangle\,(\lim_{i\to\mathcal{F}}F_i))^\bullet=0_{\mathfrak{D}}\\&\iff(\lim_{i\to\mathcal{F}}E_i\,\triangle\,F_i)^\bullet=0_{\mathfrak{D}}\\&\iff\lambda(\lim_{i\to\mathcal{F}}E_i\,\triangle\,F_i)=0\\&\iff\lim_{i\to\mathcal{F}}\mu(E_i\,\triangle\,F_i)=0\\&\iff\lim_{i\to\mathcal{F}}\bar\mu(E_i\,\triangle\,F_i)^\bullet=0\\&\iff\lim_{i\to\mathcal{F}}\bar\mu(E_i^\bullet\,\triangle\,F_i^\bullet)=0\\&\iff\bar\nu(\langle E_i^\bullet\,\triangle\,F_i^\bullet\rangle^\bullet_{i\in I})=0\\&\iff\bar\nu(\langle E_i^\bullet\rangle^\bullet_{i\in I}\,\triangle\,\langle F_i^\bullet\rangle^\bullet_{i\in I})=0\\&\iff\langle E_i^\bullet\rangle^\bullet_{i\in I}\,\triangle\,\langle F_i^\bullet\rangle^\bullet_{i\in I}=0_{\mathfrak{C}}\\&\iff\langle E_i^\bullet\rangle^\bullet_{i\in I}=\langle F_i^\bullet\rangle^\bullet_{i\in I}.\ \mathbf{Q}\end{aligned}$$

We therefore have a function $\pi:\mathfrak{D}\to\mathfrak{C}$ defined by saying that $\pi(\langle E_i^\bullet\rangle_{i\in I}^\bullet)=(\lim_{i\to\mathcal{F}}E_i)^\bullet$ for every family $\langle E_i\rangle_{i\in I}$ in Σ. Following the formulae just above, we see that the same ideas tell us that π is a Boolean homomorphism and that $\bar\lambda\pi=\bar\nu$. Next, $\pi[\mathfrak{C}]=\{A^\bullet:A\in\mathcal{A}\}$, where $\mathcal{A}$ is the family of Theorem 1E. But this means that every $d\in\mathfrak{D}$ is the supremum of a non-decreasing sequence in $\pi[\mathfrak{C}]$ and is expressible as $\sup_{n\in\mathbb{N}}\pi c_n$ for some non-decreasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{C}$. In this case, if $c=\sup_{n\in\mathbb{N}}c_n$, we must have $d\subseteq\pi c$ and $\bar\lambda d=\bar\lambda(\pi c)$, so that $d=\pi c\in\pi[\mathfrak{C}]$. This shows that π is an isomorphism between $\mathfrak{C}$ and $\mathfrak{D}$, as required.

5 Products of filters

I come now to the proofs of Theorems 1J and 1K, based on the ideas of §§2-4. The first step is to find a way of simultaneously representing many reduced products inside a single probability algebra.

5A Lemma Let $\zeta\ge1$ be an ordinal, and suppose that for $1\le\xi\le\zeta$ we are given a set I_ξ, an ultrafilter $\mathcal{F}_\xi$ on I_ξ, and a function $i\mapsto\theta(\xi,i):I_\xi\to\xi$ such that $\{i:i\in I_\xi,\ \theta(\xi,i)\ge\eta\}\in\mathcal{F}_\xi$ for every $\eta<\xi$. Set $I=\bigcup_{1\le\xi\le\zeta}I_\xi$, and let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be the family of ultrafilters on $S^*=\bigcup_{i\in\mathbb{N}}I^i$ constructed from $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ and θ as in 1I. Let $(\mathfrak{A},\bar\mu)$ be a probability algebra. Then there are a probability algebra $(\mathfrak{C},\bar\nu)$, a family $\langle\mathfrak{C}_\xi\rangle_{\xi\le\zeta}$ of closed subalgebras of $\mathfrak{C}$, a measure-preserving Boolean homomorphism $\pi:\mathfrak{A}\to\mathfrak{C}$ such that $\pi[\mathfrak{A}]=\mathfrak{C}_0$, and a family $\langle\psi_\xi\rangle_{1\le\xi\le\zeta}$ such that, for $1\le\xi\le\zeta$, $\psi_\xi:\prod_{i\in I_\xi}\mathfrak{C}_{\theta(\xi,i)}\to\mathfrak{C}_\xi$ is a Boolean homomorphism and

$$\bar\nu\psi_\xi(\langle c_i\rangle_{i\in I_\xi})=\lim_{i\to\mathcal{F}_\xi}\bar\nu c_i,\quad \psi_\xi(\langle c_i\rangle_{i\in I_\xi})\subseteq\sup_{i\in A}c_i$$

whenever $\langle c_i\rangle_{i\in I_\xi}\in\prod_{i\in I_\xi}\mathfrak{C}_{\theta(\xi,i)}$ and $A\in\mathcal{F}_\xi$.

proof Define families $\langle(\mathfrak{A}_\xi,\bar\mu_\xi)\rangle_{\xi\le\zeta}$, $\langle\phi_{\xi\eta}\rangle_{\eta\le\xi\le\zeta}$ inductively, as follows. The inductive hypothesis will be that each $(\mathfrak{A}_\xi,\bar\mu_\xi)$ is a probability algebra and that each $\phi_{\xi\eta}$ is a measure-preserving Boolean homomorphism from $\mathfrak{A}_\eta$ to $\mathfrak{A}_\xi$ such that $\phi_{\xi\eta'}\phi_{\eta'\eta}=\phi_{\xi\eta}$ whenever $\eta\le\eta'\le\xi$.

Start with $(\mathfrak{A}_0,\bar\mu_0)=(\mathfrak{A},\bar\mu)$ and $\phi_{00}:\mathfrak{A}_0\to\mathfrak{A}_0$ the identity map. Given $\langle(\mathfrak{A}_\eta,\bar\mu_\eta)\rangle_{\eta<\xi}$ and $\langle\phi_{\eta'\eta}\rangle_{\eta\le\eta'<\xi}$, where $0<\xi\le\zeta$, let $(\mathfrak{A}_\xi,\bar\mu_\xi)$ be the reduced power $\prod_{i\in I_\xi}(\mathfrak{A}_{\theta(\xi,i)},\bar\mu_{\theta(\xi,i)})|\mathcal{F}_\xi$, as defined in 1Fb/4A. Let $\le_\xi$ be the pre-order on I_ξ defined by saying that $i\le_\xi j$ iff $\theta(\xi,i)\le\theta(\xi,j)$; then our hypothesis on θ ensures that $\{j:i\le_\xi j\}\in\mathcal{F}_\xi$ for every $i\in I_\xi$. For $i\le_\xi j\in I_\xi$, $\tilde\phi_{ji}=\phi_{\theta(\xi,j),\theta(\xi,i)}:\mathfrak{A}_{\theta(\xi,i)}\to\mathfrak{A}_{\theta(\xi,j)}$ is defined; and if $i\le_\xi j\le_\xi k$, then $\tilde\phi_{ki}=\tilde\phi_{kj}\tilde\phi_{ji}$. By 4B, we have measure-preserving Boolean homomorphisms $\tilde\phi_i:\mathfrak{A}_{\theta(\xi,i)}\to\mathfrak{A}_\xi$ such that $\tilde\phi_i=\tilde\phi_j\tilde\phi_{ji}$ for $i\le_\xi j$. If $i\le_\xi j$ and $\eta\le\theta(\xi,i)$, then

$$\tilde\phi_j\phi_{\theta(\xi,j),\eta}=\tilde\phi_j\phi_{\theta(\xi,j),\theta(\xi,i)}\phi_{\theta(\xi,i),\eta}=\tilde\phi_i\phi_{\theta(\xi,i),\eta},$$

so we can take this common value for $\phi_{\xi\eta}:\mathfrak{A}_\eta\to\mathfrak{A}_\xi$. If $\eta\le\eta'<\xi$, take $i\in I_\xi$ such that $\eta'\le\theta(\xi,i)$, and see that

$$\phi_{\xi\eta'}\phi_{\eta'\eta}=\tilde\phi_i\phi_{\theta(\xi,i),\eta'}\phi_{\eta'\eta}=\tilde\phi_i\phi_{\theta(\xi,i),\eta}=\phi_{\xi\eta},$$

so the induction proceeds.

At the end of the induction, set $\mathfrak{C}=\mathfrak{A}_\zeta$, $\bar\nu=\bar\mu_\zeta$, $\pi_\xi=\phi_{\zeta\xi}:\mathfrak{A}_\xi\to\mathfrak{C}$ and $\mathfrak{C}_\xi=\pi_\xi[\mathfrak{A}_\xi]\subseteq\mathfrak{C}$. If $\eta\le\xi\le\zeta$, then $\pi_\eta=\pi_\xi\phi_{\xi\eta}$, so $\mathfrak{C}_\eta\subseteq\mathfrak{C}_\xi$.

For each $\xi>0$, we have a canonical map $\langle a_i\rangle_{i\in I_\xi}\mapsto\langle a_i\rangle_{i\in I_\xi}^\bullet:\prod_{i\in I_\xi}\mathfrak{A}_{\theta(\xi,i)}\to\mathfrak{A}_\xi$. Since $\pi_\eta:\mathfrak{A}_\eta\to\mathfrak{C}_\eta$ is always a measure-preserving isomorphism, we have corresponding maps $\psi_\xi:\prod_{i\in I_\xi}\mathfrak{C}_{\theta(\xi,i)}\to\mathfrak{C}_\xi$. Reading off the basic facts from 4A-4B, we see that

$$\bar\nu\psi_\xi(\langle c_i\rangle_{i\in I_\xi})=\lim_{i\to\mathcal{F}_\xi}\bar\nu c_i \text{ whenever } \langle c_i\rangle_{i\in I_\xi}\in\prod_{i\in I_\xi}\mathfrak{C}_{\theta(\xi,i)},$$

$$\psi_\xi(\langle c_i\rangle_{i\in I_\xi})\subseteq\sup_{i\in A}c_i \text{ whenever } \langle c_i\rangle_{i\in I_\xi}\in\prod_{i\in\mathbb{N}}\mathfrak{C}_{\theta(\xi,i)} \text{ and } A\in\mathcal{F}_\xi$$

(we can take the suprema in $\mathfrak{C}$ because $\mathfrak{C}_\xi$, being a closed subalgebra, is regularly embedded in $\mathfrak{C}$, as noted in Fremlin 02, 314Ga).

5B Proof of Theorem 1J Let $\zeta\ge1$ be a countable ordinal, and I an infinite set. Suppose that for $1\le\xi\le\zeta$ we are given a Ramsey ultrafilter $\mathcal{F}_\xi$ on I, and a function $i\mapsto\theta(\xi,i):I\to\xi$ such that $\{i:\theta(\xi,i)\ge\eta\}$ belongs to $\mathcal{F}_\xi$ for every $\eta<\xi$. Let $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be the family of ultrafilters on $S^*=\bigcup_{n\in\mathbb{N}}I^n$ constructed from $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ and θ as in 1I. Suppose further that all the $\mathcal{F}_\xi$ are non-isomorphic. Then $\mathcal{G}_\zeta$ is measure-centering.

proof (a) It is enough to consider the case in which $I = \kappa$ is a cardinal. Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra, and $\langle a_\tau\rangle_{\tau\in S^*}$ a family in $\mathfrak{A}$ such that $\epsilon = \inf_{\tau\in S^*}\bar\mu a_\tau$ is greater than 0. By 5A, there are a probability algebra $(\mathfrak{C}, \bar\nu)$, a family $\langle\mathfrak{C}_\xi\rangle_{\xi\le\zeta}$ of closed subalgebras of $\mathfrak{C}$, a measure-preserving Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{C}$ such that $\pi[\mathfrak{A}] = \mathfrak{C}_0$, and a family $\langle\psi_\xi\rangle_{1\le\xi\le\zeta}$ such that, for $1 \le \xi \le \zeta$, $\psi_\xi : \prod_{\delta<\kappa}\mathfrak{C}_{\theta(\xi,\delta)} \to \mathfrak{C}_\xi$ is a Boolean homomorphism and

$$\bar\nu\psi_\xi(\langle c_\delta\rangle_{\delta<\kappa}) = \lim_{\delta\to\mathcal{F}_\xi}\bar\nu c_\delta, \quad \psi_\xi(\langle c_\delta\rangle_{\delta<\kappa}) \subseteq \sup_{\delta\in A} c_\delta$$

whenever $\langle c_\delta\rangle_{\delta<\kappa} \in \prod_{\delta<\kappa}\mathfrak{C}_{\theta(\xi,\delta)}$ and $A \in \mathcal{F}_\xi$.

(b) By 2Ea, there is a disjoint family $\langle A_\xi\rangle_{1\le\xi\le\zeta}$ of subsets of I such that $A_\xi \in \mathcal{F}_\xi$ for every ξ. (This is where we need to know that ζ is countable.) Define $T \subseteq S^*$ and $\alpha : T \to [0, \zeta]$ as follows. Start by saying that $\emptyset \in T$ and that $\alpha(\emptyset) = \zeta$. Having determined $T \cap \kappa^n$ and $\alpha : T \cap \kappa^n \to [0, \zeta]$, where $n \in \mathbb{N}$, then for $\tau \in \kappa^{n+1}$ say that $\tau \in T$ iff τ is of the form $\sigma^\frown{<}\delta{>}$ where

$$\sigma \in T \cap \kappa^n, \quad \alpha(\sigma) > 0, \quad \delta \in A_{\alpha(\sigma)}, \quad \sigma(m) < \delta \text{ for every } m < n,$$

and in this case set $\alpha(\tau) = \theta(\alpha(\sigma), \delta)$. Continue. Observe that every member of T is a strictly increasing finite sequence in κ. For $D \subseteq \kappa$, set $T_D = T \cap \bigcup_{n\in\mathbb{N}} D^n$.

(c) Set $\mathcal{H} = \bigcap_{1\le\xi\le\zeta}\mathcal{F}_\xi$. Then $T^*_D = \{\tau : \tau \in T_D,\, \alpha(\tau) = 0\}$ belongs to $\mathcal{G}_\zeta$ for every $D \in \mathcal{H}$. **P** I aim to show by induction on ξ that if $\tau \in T_D$ and $\alpha(\tau) = \xi$ then $\{\sigma : \tau^\frown\sigma \in T^*_D\} \in \mathcal{G}_\xi$. The induction starts with $\alpha(\tau) = 0$ and $\{\sigma : \tau^\frown\sigma \in T^*_D\} = \{\emptyset\} \in \mathcal{G}_0$. For the inductive step to $\alpha(\tau) = \xi > 0$,

$$\begin{aligned}\{\delta : \{\sigma : \tau^\frown{<}\delta{>}^\frown\sigma \in T^*_D\} \in \mathcal{G}_{\theta(\xi,\delta)}\} \\ \supseteq \{\delta : \delta \in D,\, \tau^\frown{<}\delta{>} \in T,\, \alpha(\tau^\frown{<}\delta{>}) = \theta(\xi,\delta)\}\end{aligned}$$

(by the inductive hypothesis)

$$= \{\delta : \delta \in A_\xi \cap D,\, \tau(m) < \delta \text{ for every } m < \operatorname{dom}\tau\} \in \mathcal{F}_\xi,$$

so $\{\sigma : \tau^\frown\sigma \in T^*_D\} \in \mathcal{G}_\xi$. At the end of the induction, we can apply this to $\tau = \emptyset$ and $\xi = \zeta$. **Q**

(d) Set $c_\tau = \pi a_\tau$ for $\tau \in S^*\setminus T$. For $\tau \in T$ define $c_\tau \in \mathfrak{C}_{\alpha(\tau)}$ by induction on $\alpha(\tau)$, as follows. If $\alpha(\tau) = 0$, set $c_\tau = \pi a_\tau$. For the inductive step to $\alpha(\tau) = \xi > 0$, $\alpha(\tau^\frown{<}\delta{>}) = \theta(\alpha(\tau), \delta) < \alpha(\tau)$ whenever $\tau^\frown{<}\delta{>} \in T$. So $c_{\tau^\frown<\delta>}$ is defined and belongs to $\mathfrak{C}_{\theta(\xi,\delta)}$ for every $\delta < \kappa$. We can therefore set $c_\tau = \psi_\xi(\langle c_{\tau^\frown<\delta>}\rangle_{\delta<\kappa})$, and continue. Inducing on $\alpha(\tau)$ when $\tau \in T$, we see that $\bar\nu c_\tau \ge \epsilon$ for every $\tau \in S^*$.

(e) For $K \subseteq I$, set $e_K = \inf_{\tau\in T_K} c_\tau$; let $\mathcal{S}$ be the family of those finite sets $K \subseteq I$ such that $e_K \ne 0$. Interpreting $\inf\emptyset$ as $1_{\mathfrak{C}}$, $\emptyset \in \mathcal{S}$. Moreover, if $K \in \mathcal{S}$, then $\{\delta : K \cup \{\delta\} \in \mathcal{S}\}$ belongs to $\mathcal{H}$. **P** Set $\gamma = \sup K$. Take any ξ such that $1 \le \xi \le \zeta$. Set

$$d_\delta = \inf\{c_{\tau^\frown<\delta>} : \tau \in T_K,\, \alpha(\tau) = \xi\}$$

for $\delta < \kappa$,

$$B = \{\delta : \delta \in A_\xi,\, \delta > \gamma,\, d_\delta \cap e_K \ne 0\}.$$

If $\delta \in B$, then

$$T_{K\cup\{\delta\}} = T_K \cup \{\tau^\frown{<}\delta{>} : \tau \in T_K,\, \alpha(\tau) = \xi\},$$

because every member of T is strictly increasing and $\tau^\frown{<}\delta{>}$ can belong to T only when $\delta \in A_{\alpha(\tau)}$, that is, when $\alpha(\tau) = \xi$. So $e_{K\cup\{\delta\}} = d_\delta \cap e_K \ne 0$ and $K \cup \{\delta\} \in \mathcal{S}$.

? If $B \notin \mathcal{F}_\xi$, then $B' = \{\delta : \delta \in A_\xi,\, \delta > \gamma,\, d_\delta \cap e_K = 0\}$ belongs to $\mathcal{F}_\xi$. So

$$\begin{aligned}e_K &\subseteq \inf\{c_\tau : \tau \in T_K,\, \alpha(\tau) = \xi\} \\ &= \inf_{\substack{\tau\in T_K\\ \alpha(\tau)=\xi}}\psi_\xi(\langle c_{\tau^\frown<\delta>}\rangle_{\delta<\kappa}) = \psi_\xi(\langle \inf_{\substack{\tau\in T_K\\ \alpha(\tau)=\xi}} c_{\tau^\frown<\delta>}\rangle_{\delta<\kappa})\end{aligned}$$

(because ψ_ξ is a Boolean homomorphism and T_K is finite)

$$\subseteq \sup_{\delta\in B'} \inf_{\substack{\tau\in T_K\\ \alpha(\tau)=\xi}} c_{\tau^\frown<\delta>}$$

(as noted in (a) above)

$$= \sup_{\delta\in B'} d_\delta.$$

But $e_K \cap d_\delta = 0$ for every $\delta \in B'$ and $e_K \neq 0$. **X**

Thus $\{\delta : K \cup \{\delta\} \in \mathcal{S}\} \supseteq B$ belongs to $\mathcal{F}_\xi$. As ξ is arbitrary, $\{\delta : K \cup \{\delta\} \in \mathcal{S}\} \in \mathcal{H}$. **Q**

(f) At this point, recall that $\mathcal{H}$ is dependently selective, by 2D. So there is a $D \in \mathcal{H}$ such that $[D]^{<\omega} \subseteq \mathcal{S}$, that is, $e_K \neq 0$ for every $K \in [D]^{<\omega}$, that is, $\{c_\tau : \tau \in T_D\}$ is centered in $\mathfrak{C}$. It follows that $\{c_\tau : \tau \in T^*_D\}$ is centered; but for $\tau \in T^*_D$, $c_\tau = \pi a_\tau$, so $\{a_\tau : \tau \in T^*_D\}$ is centered, while $T^*_D \in \mathcal{G}_\zeta$, by (c). As $(\mathfrak{A}, \bar\mu)$ and $\langle a_\tau\rangle_{\tau\in S^*}$ are arbitrary, $\mathcal{G}_\zeta$ is measure-centering, by Proposition 1C(ii).

5C Lemma Suppose that I, J are sets and that $\mathcal{F}$, $\mathcal{G}$ are measure-centering ultrafilters on I, J respectively such that $\mathcal{F}$ is $\#(J)^+$-complete. If $(\mathfrak{A}, \bar\mu)$ is a probability algebra and $\langle a_{ij}\rangle_{i\in I,j\in J}$ is a family in $\mathfrak{A}$ such that $\inf_{i\in I,j\in J} \bar\mu a_{ij} > 0$, there are $A \in \mathcal{F}$, $B \in \mathcal{G}$ such that $\{a_{ij} : i \in A,\ j \in B\}$ is centered.

proof If J is finite then $\mathcal{G}$ is principal; taking $B = \{j\}$ to be the singleton belonging to $\mathcal{G}$, there is an $A \in \mathcal{F}$ such that $\{a_{ij} : i \in A\}$ is centered, and we're done. Otherwise, let κ be the greatest cardinal such that $\mathcal{F}$ is κ-complete. Then κ is two-valued-measurable and greater than $\#(J)$, so must be greater than $2^{\#(J)}$. For $K \in [J]^{<\omega}$ and $i \in J$ set $b_{iK} = \inf_{j\in K} a_{ij}$. Because $\mathcal{F}$ is κ-complete, there is a family $\langle\gamma_K\rangle_{K\in[J]^{<\omega}}$ in $[0, 1]$ such that $A^* = \{i : \bar\mu b_{iK} = \gamma_K$ for every finite $K \subseteq I\}$ belongs to $\mathcal{F}$. Take any $i_0 \in A^*$; then there must be a $B \in \mathcal{G}$ such that $\{a_{i_0j} : j \in B\}$ is centered, so that $\gamma_K > 0$ for every $K \in [B]^{<\omega}$. In this case, for each $K \in [B]^{<\omega}$, there is an $A_K \in \mathcal{F}$ such that $\{b_{iK} : i \in A_K\}$ is centered. Set $A = A^* \cap \bigcap_{K\subseteq B \text{ is finite}} A_K$; then $A \in \mathcal{F}$ and $\{a_{ij} : i \in A, j \in B\}$ is centered.

5D Lemma Suppose that $\mathcal{F}_0, \dots, \mathcal{F}_n$ are filters on sets $I_0, \dots, I_n$. Let (K, L) be a non-trivial partition of $\{0, \dots, n\}$, and $(i_0, \dots, i_k)$, $(j_0, \dots, j_l)$ the increasing enumerations of K, L respectively. For $A \subseteq I_{i_0} \times \dots \times I_{i_k}$ and $B \subseteq I_{j_0} \times \dots \times I_{j_l}$ set

$$A\#B = \{(x_0, \dots, x_n) : (x_{i_0}, \dots, x_{i_k}) \in A,\ (x_{j_0}, \dots, x_{j_l}) \in B\} \subseteq I_0 \times \dots \times I_n.$$

If $A \in \mathcal{F}_{i_0} \ltimes \dots \ltimes \mathcal{F}_{i_k}$ and $B \in \mathcal{F}_{j_0} \ltimes \dots \ltimes \mathcal{F}_{j_l}$ then $A\#B \in \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$.

proof Induce on n. The induction starts with $n = 1$ and $k = l = 0$ and either $A\#B = A \times B$ with $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_1$, or $A\#B = B \times A$ with $B \in \mathcal{F}_0$ and $A \in \mathcal{F}_1$; in either case the result is trivial.

For the inductive step to $n > 1$, suppose to begin with that $0 \in K$. If $K = \{0\}$, then we have $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_1 \ltimes \dots \ltimes \mathcal{F}_n$, so that $A\#B$ can be identified with $A \times B \in \mathcal{F}_0 \ltimes (\mathcal{F}_1 \ltimes \dots \ltimes \mathcal{F}_n)$, identified with $\mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$. Otherwise, set $K' = K \setminus \{0\}$, so that (K', L) is a non-trivial partition of $\{1, \dots, n\}$. For $x \in I_0$, it is easy to see that

$$(A\#B)[\{x\}] = \{(x_1, \dots, x_n) : (x, x_1, \dots, x_n) \in A\#B\}$$

and

$$A[\{x\}]\#B = \{(x_{i_1}, \dots, x_{i_k}) : (x, x_{i_1}, \dots, x_{i_k}) \in A\}\#B$$

are equal, where the interleaving $A[\{x\}]\#B$ is computed with regard to the partition (K', L) and the increasing enumerations $(i_1, \dots, i_k)$ and $(j_0, \dots, j_l)$. Now

$$\{x : (A\#B)[\{x\}] \in \mathcal{F}_1 \ltimes \dots \ltimes \mathcal{F}_n\} = \{x : A[\{x\}]\#B \in \mathcal{F}_1 \ltimes \dots \ltimes \mathcal{F}_n\}$$
$$\supseteq \{x : A[\{x\}] \in \mathcal{F}_{i_1} \ltimes \dots \ltimes \mathcal{F}_{i_k}\}$$

(by the inductive hypothesis, because $B \in \mathcal{F}_{j_0} \ltimes \dots \ltimes \mathcal{F}_{j_l}$)

$$\in \mathcal{F}_{i_0} = \mathcal{F}_0$$

because $A \in \mathcal{F}_{i_0} \ltimes \dots \ltimes \mathcal{F}_{i_k}$. So $A\#B \in \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$.

If $0 \in L$, the same argument applies, but looking at sections $B[\{x\}]$ rather than $A[\{x\}]$ and the partition $(K, L \setminus \{0\})$ of $\{1, \dots, n\}$. So the induction proceeds.

5E Proposition Suppose that $\mathcal{F}_0, \dots, \mathcal{F}_n$ are non-isomorphic Ramsey ultrafilters. Then $\mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$ is measure-centering.

proof It is enough to consider the case in which every $\mathcal{F}_i$ is based on a cardinal λ_i. Induce on n. If $n = 0$ we just use Theorem 1Ba. For the inductive step, set $\kappa = \max_{i \le n} \lambda_i$, $K = \{i : i \le n,\ \lambda_i = \kappa\}$, $L = \{i : i \le n,\ \lambda_i < \kappa\}$. If $K = \{0, \dots, n\}$ then we can use Theorem 1J with $I = \kappa$ and $\zeta = n + 1$. Otherwise, (K, L) is a proper partition of $\{0, \dots, n\}$. Let $(i_0, \dots, i_k)$ and $(j_0, \dots, j_l)$ be the increasing enumerations of K, L respectively. Then the inductive hypothesis tells us that $\mathcal{G} = \mathcal{F}_{i_0} \ltimes \dots \ltimes \mathcal{F}_{i_k}$ and $\mathcal{H} = \mathcal{F}_{j_0} \ltimes \dots \ltimes \mathcal{F}_{j_l}$ are measure-centering. Observe next that as every $\mathcal{F}_{i_m}$ is κ-complete, so is $\mathcal{G}$, while the base set of $\mathcal{H}$ is $\lambda_{j_0} \times \dots \times \lambda_{j_l}$, which has cardinal less than κ.

Let $(\mathfrak{A}, \bar{\mu})$ be a probability measure, and $\langle a_x \rangle_{x \in I}$ a family in $\mathfrak{A}$ such that $\inf_{x \in I} \bar{\mu} a_x > 0$, where $I = \lambda_0 \times \dots \times \lambda_n$. For $x \in I$, set $x' = (x_{i_0}, \dots, x_{i_k})$ and $x'' = \{x_{j_0}, \dots, x_{j_l}\}$, so that $x \mapsto (x', x'')$ is a bijection between I and $(I_{i_0} \times \dots \times I_{i_k}) \times (I_{j_0} \times \dots \times I_{j_l})$. By Lemma 5C, there are sets $A \in \mathcal{G}$, $B \in \mathcal{H}$ such that $\{a_x : x' \in A,\ x'' \in B\}$ is centered; that is, in the language of Lemma 5D, $\{a_x : x \in A \# B\}$ is centered. But 5D tells us that $A \# B \in \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$. As $\langle a_x \rangle_{x \in I}$ is arbitrary, $\mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$ is measure-centering, and the induction continues.

5F To prove Theorem 1K, we need to know a little more both about the extensions of measures described in 1E and about the iterated products of 1I. It will be convenient to have a name for a relation extending the Rudin-Keisler pre-ordering of the ultrafilters on a given set.

Definition If $\mathcal{F}$ and $\mathcal{G}$ are filters on sets I, J respectively, I will say that $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$ if there is a function $f : J \to I$ such that $\mathcal{F} = f[[\mathcal{G}]]$. Observe that $\le_{\mathrm{RK}}$ is a reflexive transitive relation on the class of all filters.

5G Proposition Let $\mathcal{F}$ and $\mathcal{G}$ be measure-centering ultrafilters such that $\mathcal{F} \le_{\mathrm{RK}} \mathcal{G}$. Let (X, Σ, μ) be a compact probability space, and $\lambda_{\mathcal{F}}$, $\lambda_{\mathcal{G}}$ the extensions of μ defined from $\mathcal{F}$ and $\mathcal{G}$ as in Theorem 1E. Then $\lambda_{\mathcal{G}}$ extends $\lambda_{\mathcal{F}}$.

proof Set $I = \bigcup \mathcal{F}$ and $J = \bigcup \mathcal{G}$, and let $f : J \to I$ be such that $\mathcal{F} = f[[\mathcal{G}]]$. Defining $\mathcal{A}_{\mathcal{F}}$ and $\mathcal{A}_{\mathcal{G}}$ from $\mathcal{F}$ and $\mathcal{G}$ as in 1E, $\mathcal{A}_{\mathcal{F}} \subseteq \mathcal{A}_{\mathcal{G}}$ and $\lambda_{\mathcal{G}} A = \lambda_{\mathcal{F}} A$ for every $A \in \mathcal{A}_{\mathcal{F}}$. **P** Express A as $\lim_{i \to \mathcal{F}} E_i$ where $\langle E_i \rangle_{i \in I}$ is a family in Σ. For $j \in J$, set $F_j = E_{f(j)}$; then $A = \lim_{j \to \mathcal{G}} F_j \in \mathcal{A}_{\mathcal{G}}$ and

$$\lambda_{\mathcal{F}} A = \lim_{i \to \mathcal{F}} \mu E_i = \lim_{j \to \mathcal{G}} \mu F_j = \lambda_{\mathcal{G}} A. \text{ } \mathbf{Q}$$

Since $\lambda_{\mathcal{F}}$ and $\lambda_{\mathcal{G}}$ are complete probability measures and $\lambda_{\mathcal{F}}$ is inner regular with respect to $\mathcal{A}_{\mathcal{F}}$, $\lambda_{\mathcal{G}}$ extends $\lambda_{\mathcal{F}}$, by 3A(a-iii).

5H Proposition Let $\mathcal{F}_0, \dots, \mathcal{F}_n$ be filters.

(a) $\mathcal{F}_0 \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \mathcal{F}_1$ and $\mathcal{F}_1 \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \mathcal{F}_1$.

(b) If $\mathcal{G}_0, \dots, \mathcal{G}_n$ are filters such that $\mathcal{G}_k \le_{\mathrm{RK}} \mathcal{F}_k$ for every $k \le n$, then $\mathcal{G}_0 \ltimes \dots \ltimes \mathcal{G}_n \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$.

(c) If $0 \le k_0 < k_1 < \dots < k_m \le n$, then $\mathcal{F}_{k_0} \ltimes \dots \ltimes \mathcal{F}_{k_m} \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$.

proof Let $I_k = \bigcup \mathcal{F}_k$ be the base set of $\mathcal{F}_k$ for each k.

(a) If $f_0 : I_0 \times I_1 \to I_0$ and $f_1 : I_0 \times I_1 \to I_1$ are the canonical projections, it is easy to see that $\mathcal{F}_0 = f_0[[\mathcal{F}_0 \ltimes \mathcal{F}_1]]$ and $\mathcal{F}_1 = f_1[[\mathcal{F}_0 \ltimes \mathcal{F}_1]]$.

(b)(i) If $n = 1$, set $J_k = \bigcup \mathcal{G}_k$ and let $f_k : I_k \to J_k$ be such that $\mathcal{G}_k = f_k[[\mathcal{F}_k]]$ for $k = 0$ and $k = 1$. Setting $h(i, j) = (f_0(i), f_1(j))$ for $i \in I_0$ and $j \in J_0$, it is easy to check that $\mathcal{G}_0 \ltimes \mathcal{G}_1 = h[[\mathcal{F}_0 \ltimes \mathcal{F}_1]] \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \mathcal{F}_1$.

(ii) For $n \ge 2$ the result now follows by induction.

(c)(i) Induce on n to see that $\mathcal{F}_k \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n$ whenever $0 \le k \le n$; for the case $n = k > 0$, apply (a) to the product $(\mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_{k-1}) \ltimes \mathcal{F}_k$; for the inductive step to $n + 1 > k$, observe that

$$\mathcal{F}_k \le_{\mathrm{RK}} \mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n \le_{\mathrm{RK}} (\mathcal{F}_0 \ltimes \dots \ltimes \mathcal{F}_n) \ltimes \mathcal{F}_{n+1}$$

by the other half of (a).

(ii) Now induce on m; the inductive step to $m+1$ is

$$\begin{aligned}(\mathcal{F}_{k_0} \ltimes \mathcal{F}_{k_1} \ltimes \ldots \ltimes \mathcal{F}_{k_m}) \ltimes \mathcal{F}_{k_{m+1}}&\\ &\le_{\mathrm{RK}} (\mathcal{F}_0 \ltimes \mathcal{F}_1 \ltimes \ldots \ltimes \mathcal{F}_{k_m}) \ltimes (\mathcal{F}_{k_m+1} \ltimes \ldots \ltimes \mathcal{F}_n)\\ &\cong \mathcal{F}_0 \ltimes \ldots \ltimes \mathcal{F}_n,\end{aligned}$$

using the inductive hypothesis, (i) here and (b) above for the first step.

5I These have been easy. For the next result we have to think a little harder, but the principles are the same.

Proposition Let ζ, $\langle I_\xi\rangle_{1\le\xi\le\zeta}$, $\langle \mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$, I, S^*, θ and $\langle\mathcal{G}_\xi\rangle_{\xi\le\zeta}$ be as in 1Ib, and suppose that $\{i : i \in I_\xi,\ \theta(\xi, i) \ge \eta\} \in \mathcal{F}_\xi$ whenever $\eta < \xi \le \zeta$. Then $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_\zeta$ whenever $1 \le \xi_0 < \xi_1 < \ldots < \xi_n \le \zeta$.

proof (a) The first step is to show that $\mathcal{F}_\xi \le_{\mathrm{RK}} \mathcal{G}_\xi$ whenever $1 \le \xi \le \zeta$. **P** Let $f : S^* \to I_\xi$ be such that $f(\tau) = \tau(0)$ whenever $\tau \ne \emptyset$ and $\tau(0) \in I_\xi$. For $A \subseteq I_\xi$,

$$\begin{aligned}f^{-1}[A] \in \mathcal{G}_\xi &\iff \{i : i \in I_\xi,\ \{\tau : <i>^\frown\tau \in f^{-1}[A]\} \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi\\ &\iff \{i : i \in I_\xi,\ \{\tau : f(<i>^\frown\tau) \in A\} \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi\\ &\iff \{i : i \in A,\ S^* \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi \iff A \in \mathcal{F}_\xi.\end{aligned}$$

So $\mathcal{F}_\xi = f[[\mathcal{G}_\xi]] \le_{\mathrm{RK}} \mathcal{G}_\xi$. **Q**

(b) Next, $\mathcal{G}_\eta \le_{\mathrm{RK}} \mathcal{G}_\xi$ whenever $\eta \le \xi \le \zeta$. **P** Induce on ξ. If $\xi = \eta$, the result is trivial. For the inductive step to $\xi > \eta$, set $J = \{i : i \in I_\xi,\ \theta(\xi, i) \ge \eta\}$; then $J \in \mathcal{F}_\xi$, by hypothesis. For $i \in J$, the inductive hypothesis tells us that $\mathcal{G}_\eta \le_{\mathrm{RK}} \mathcal{G}_{\theta(\xi,i)}$; let $g_i : S^* \to S^*$ be such that $g_i[[\mathcal{G}_{\theta(\xi,i)}]] = \mathcal{G}_\eta$. Now let $g : S^* \to S^*$ be such that $g(<i>^\frown\sigma) = g_i(\sigma)$ whenever $\sigma \in S^*$ and $i \in J$. For $A \subseteq S^*$,

$$\begin{aligned}g^{-1}[A] \in \mathcal{G}_\xi &\iff \{i : i \in I_\xi,\ \{\tau : g(<i>^\frown\tau) \in A\} \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi\\ &\iff \{i : i \in J,\ \{\tau : g_i(\tau) \in A\} \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi\\ &\iff \{i : i \in J,\ g_i^{-1}[A] \in \mathcal{G}_{\theta(\xi,i)}\} \in \mathcal{F}_\xi\\ &\iff \{i : i \in J,\ A \in \mathcal{G}_\eta\} \in \mathcal{F}_\xi \iff A \in \mathcal{G}_\eta,\end{aligned}$$

so $G_\eta = g[[\mathcal{G}_\xi]] \le_{\mathrm{RK}} g[[\mathcal{G}_\xi]]$. **Q**

(c) It follows that if $1 \le \xi_0 < \ldots < \xi_n \le \zeta$ then $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le \mathcal{G}_{\xi_n}$. **P** Induce on the pair (n, ξ_n). If $\xi_n = 1$ then $n = 0$ and we just have to know that $\mathcal{F}_1 \le_{\mathrm{RK}} \mathcal{G}_1$, as in (a). For the inductive step to $\xi_n > 1$, if $n = 0$ we again have only to know that $\mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_n}$, this time using (b). If $n > 0$, let J be $\{i : i \in I_{\xi_n},\ \theta(\xi_n, i) \ge \xi_{n-1}\} \in \mathcal{F}_{\xi_n}$. For $i \in J$,

$$\mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_{n-1}} \le_{\mathrm{RK}} \mathcal{G}_{\theta(\xi_n,i)}$$

by the inductive hypothesis; let $h_i : S^* \to I_{\xi_{n-1}} \times \ldots \times I_{\xi_0}$ be a function witnessing this. Now let $h : S^* \to I_{\xi_n} \times I_{\xi_{n-1}} \times \ldots \times I_{\xi_0}$ be such that $h(<i>^\frown\sigma) = (i, h_i(\sigma))$ whenever $i \in J$ and $\sigma \in S^*$. For $A \subseteq I_{\xi_n} \times \ldots \times I_{\xi_0}$,

$$\begin{aligned}h^{-1}[A] \in \mathcal{G}_{\xi_n} &\iff \{i : i \in I_{\xi_n},\ \{\tau : h(<i>^\frown\tau) \in A\} \in \mathcal{G}_{\theta(\xi_n,i)}\} \in \mathcal{F}_{\xi_n}\\ &\iff \{i : i \in J,\ \{\tau : h(<i>^\frown\tau) \in A\} \in \mathcal{G}_{\theta(\xi_n,i)}\} \in \mathcal{F}_{\xi_n}\\ &\iff \{i : i \in J,\ \{\tau : h_i(\tau) \in A[\{i\}]\} \in \mathcal{G}_{\theta(\xi_n,i)}\} \in \mathcal{F}_{\xi_n}\\ &\iff \{i : i \in J,\ A[\{i\}] \in h_i[[\mathcal{G}_{\theta(\xi_n,i)}\}]] \in \mathcal{F}_{\xi_n}\\ &\iff \{i : i \in J,\ A[\{i\}] \in \mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}\} \in \mathcal{F}_{\xi_n}\\ &\iff A \in \mathcal{F}_{\xi_n} \ltimes (\mathcal{F}_{\xi_{n-1}} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}) \cong \mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0}.\end{aligned}$$

So h witnesses that $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le_{\mathrm{RK}} \mathcal{G}_{\xi_n}$, and the induction proceeds. **Q**

(d) Finally, if $1 \le \xi_0 < \ldots < \xi_n \le \zeta$ then $\mathcal{F}_{\xi_n} \ltimes \ldots \ltimes \mathcal{F}_{\xi_0} \le \mathcal{G}_{\xi_n} \le_{\mathrm{RK}} \mathcal{G}_\zeta$.

5J Proof of Theorem 1K Let (X, Σ, μ) be a compact probability space. Then there is a measure λ on X, extending μ, such that $\lambda(\lim_{i\to\mathcal{F}} E_i)$ is defined and equal to $\lim_{i\to\mathcal{F}} \mu E_i$ whenever $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$ and $\langle E_i\rangle_{i\in\mathbb{N}}$ is a sequence in Σ.

proof (a) If there are no Ramsey ultrafilters on $\mathbb{N}$, we can set $\lambda = \mu$ and stop; so let us suppose that there is at least one Ramsey ultrafilter. Let $\mathfrak{F}$ be a family of Ramsey ultrafilters on $\mathbb{N}$ consisting of just one member of each isomorphism class. Fix a well-ordering $\preccurlyeq$ of $\mathfrak{F}$ with greatest member $\mathcal{F}^*$ and a ladder system $\langle\theta(\xi, i)\rangle_{1\le\xi<\omega_1, i\in\mathbb{N}}$ such that $\langle\theta(\xi, i)\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence in ξ, and $\{\theta(\xi, i) : i \in \mathbb{N}\}$ is cofinal with ξ, whenever $1 \le \xi < \omega_1$.

(b)(i) For any non-empty finite set $V \subseteq \mathfrak{F}$, list it in $\preccurlyeq$-increasing order as $\mathcal{F}_0 \prec \mathcal{F}_1 \prec \ldots \prec \mathcal{F}_n$, and set $\mathcal{H}_V = \mathcal{F}_n \ltimes \ldots \ltimes \mathcal{F}_0$.

(ii) For any non-empty countable set $W \subseteq \mathfrak{F}$ containing $\mathcal{F}^*$, list it in $\preccurlyeq$-increasing order as $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$, where $\zeta \ge 1$ is a countable ordinal, and let $\mathcal{G}_W$ be the final ultrafilter on $S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ defined from $\langle\mathcal{F}_\xi\rangle_{1\le\xi\le\zeta}$ and $\langle\theta(\xi, i)\rangle_{1\le\xi\le\zeta, i\in\mathbb{N}}$ by the process of 1Ib. By Theorem 1J, $\mathcal{G}_W$ is measure-centering.

(iii) If $V \subseteq W \subseteq \mathfrak{F}$, W is countable and contains $\mathcal{F}^*$, and V is finite and not empty, then $\mathcal{H}_V \le_{\mathrm{RK}} \mathcal{G}_W$, by 5I; so $\mathcal{H}_V$ is measure-centering (Proposition 1Da). Let λ_V be the corresponding extension of μ as described in Theorem 1E.

(c) Consider the family $\Lambda = \{\lambda_V : V \in [\mathfrak{F}]^{<\omega} \setminus \{\emptyset\}\}$. This is a collection of probability measures on X. It is upwards-directed in the sense that if we have any two members of Λ they have a common extension belonging to Λ. **P** If V_0, V_1 are non-empty finite subsets of $\mathfrak{F}$ with union V, then $\mathcal{H}_{V_0} \le_{\mathrm{RK}} \mathcal{H}_V$, by 5Hc, so λ_V extends λ_{V_0}, by 5G; and similarly λ_V extends λ_{V_1}. **Q** Next, if $\Lambda_0 \subseteq \Lambda$ is countable, then there is a measure on X extending every member of Λ_0. **P** Let $W \subseteq \mathfrak{F}$ be a non-empty countable set, containing $\mathcal{F}^*$, such that $\Lambda_0 \subseteq \{\lambda_V : V \in [W]^{<\omega} \setminus \{\emptyset\}\}$. Let $\lambda^\#$ be the extension of μ corresponding to the measure-centering ultrafilter $\mathcal{G}_W$. If $V \subseteq W$ is finite and non-empty, then $\mathcal{H}_V \le_{\mathrm{RK}} \mathcal{G}_W$, so $\lambda^\#$ extends λ_V; thus $\lambda^\#$ extends every member of Λ_0. **Q**

By 3Ab, there is a measure λ on X extending every member of Λ.

(d) Suppose that $\langle E_i\rangle_{i\in\mathbb{N}}$ is a sequence in Σ and that $\mathcal{F}$ is a Ramsey ultrafilter on $\mathbb{N}$. Then there is a $\mathcal{F}' \in \mathfrak{F}$ such that $\mathcal{F}' = \mathcal{H}_{\{\mathcal{F}'\}}$ is isomorphic to $\mathcal{F}$. Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f[[\mathcal{F}']] = \mathcal{F}$, and set $F_j = E_{f(j)}$, as in the proof of 5G; then $\lim_{i\to\mathcal{F}} E_i = \lim_{j\to\mathcal{F}'} F_j$ and

$$\lambda(\lim_{i\to\mathcal{F}} E_i) = \lambda(\lim_{j\to\mathcal{F}'} F_j) = \lambda_{\{\mathcal{F}'\}}(\lim_{j\to\mathcal{F}'} F_j)$$

is defined and equal to $\lim_{j\to\mathcal{F}'} \mu F_j = \lim_{i\to\mathcal{F}} \mu E_i$. So λ has the required property.

6 Other kinds of ultrafilter

I return to the relatively concrete context of a proof of Theorem 1M, describing the relationships between various classes of ultrafilters.

6A Proof of Theorem 1M, parts (a), (c) and (j) As usual, I begin by repeating the statements of the results in question.

(a) A measure-linking ultrafilter is Hausdorff.

(b) A measure-centering ultrafilter is nowhere dense.

(j) It is relatively consistent with ZFC to suppose that every measure-centering ultrafilter is principal.

proof (a) Let $\mathcal{F}$ be a measure-linking ultrafilter on a set I, and $f : I \to J$, $g : I \to J$ two functions such that $A = \{i : f(i) \ne g(i)\} \in \mathcal{F}\}$. Let $\langle e_j\rangle_{j\in J}$ be the standard generating family in $\mathfrak{B}_J$ (Definition 3Be), and for $i \in I$ set

$$a_i = e_{f(i)} \setminus e_{g(i)} \text{ if } i \in A,$$
$$= 1 \text{ otherwise.}$$

Then $\inf_{i\in I}\bar\nu_J a_i\ge\frac14$, so there is a $B\in\mathcal{F}$ such that $a_i\cap a_j\ne0$ for all i, $j\in B$; we can suppose that $B\subseteq A$. Now $f[B]\cap g[B]=\emptyset$. **P?** Otherwise, there are i, $j\in B$ such that $f(i)=g(j)$. In this case,

$$0\ne a_i\cap a_j\subseteq e_{f(i)}\setminus e_{g(j)}=0.\ \textbf{XQ}$$

Since $f[B]\in f[[\mathcal{F}]]$ and $g[B]\in g[[\mathcal{F}]]$, $f[[\mathcal{F}]]\ne g[[\mathcal{F}]]$; as f and g are arbitrary, $\mathcal{F}$ is a Hausdorff ultrafilter.

(b) Let I be a set, $\mathcal{F}$ a measure-centering ultrafilter on I and $f:I\to\mathbb{R}$ a function. Let μ be Lebesgue measure on $\mathbb{R}$. Let $G\subseteq\mathbb{R}$ be an open set, including $\mathbb{Q}$, of measure at most $\frac12$; for $x\in\mathbb{R}$, write $<x>=x-\lfloor x\rfloor$ for the fractional part of x. For each $i\in I$, set

$$K_i=[0,1]\setminus(G+<f(i)>).$$

Then $\mu K_i\ge\frac12$. Applying the definition 1A with $\mathfrak{A}$ the algebra of measurable subsets of $[0,1]$, we see that there is an $A\in\mathcal{F}$ such that $\mu(\bigcap_{i\in L}K_i)>0$ for every non-empty finite $L\subseteq A$. Because all the sets K_i are closed subsets of the compact set $[0,1]$, there must be a point x of $\bigcap_{i\in A}K_i$. In this case, $x\notin G+<f(i)>$ for every $i\in A$, that is, $<f(i)>\notin x-G$ for every $i\in A$. Now $x-G$ is a dense open subset of $\mathbb{R}$, so $H=(x-G)\cap\,]0,1[$ is a dense open subset of $]0,1[$, and $H+\mathbb{Z}$ is a dense open subset of $\mathbb{R}$; while $f(i)\notin H+\mathbb{Z}$ for every $i\in A$. Thus $f[A]$ is nowhere dense in $\mathbb{R}$; as f is arbitrary, $\mathcal{F}$ is nowhere dense.

(j)($\boldsymbol\alpha$) The point is that if there is a non-principal measure-centering ultrafilter $\mathcal{F}$ on a set I, then either there is a non-principal nowhere dense ultrafilter on $\mathbb{N}$, or there is a two-valued-measurable cardinal. **P** If $\mathcal{F}$ is not closed under countable intersections, there is a partition of I into a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets not belonging to $\mathcal{F}$. Setting $f(i)=n$ for $i\in A_n$, we get a function $f:I\to\mathbb{N}$ such that $f[[\mathcal{F}]]$ is a non-principal ultrafilter on $\mathbb{N}$. By Proposition 1Da $f[[\mathcal{F}]]$ is measure-centering, and by (b) above it is nowhere dense. On the other hand, if $\mathcal{F}$ is closed under countable intersections, then there is a two-valued-measurable cardinal (Comfort & Negrepontis 74, 8.31; Jech 03, 10.2). **Q**

($\boldsymbol\beta$) In Shelah 98, Theorem 3.1, Shelah proved that if $\mathfrak{c}=\omega_1$ and $\Diamond_{\omega_2}(\{\gamma:\gamma<\omega_2,\ \operatorname{cf}\gamma=\omega_1\})$ is true, then there is a proper forcing notion $\mathbb{P}$ of cardinality ω_2 such that

$$\Vdash_{\mathbb{P}}\text{ there are no non-principal nowhere dense ultrafilters on }\mathbb{N}.$$

In particular, this is so if we start from a model of $V=L$ (Jech 03, 13.20 & Exercise 27.4). Now, if we begin with such a model, so that there are no two-valued-measurable cardinals (Jech 03, 17.1), then we shall certainly have

$$\Vdash_{\mathbb{P}}\text{ there are no two-valued-measurable cardinals}$$

(Jech 03, 21.2). So (α) tells us that

$$\Vdash_{\mathbb{P}}\text{ all measure-centering ultrafilters are principal.}$$

6B Lemma (a) (cf. Blass 74, Theorem 5) Let $\mathcal{F}$ be a uniform κ-complete weakly Ramsey ultrafilter on a regular infinite cardinal κ. If $\langle A_\xi\rangle_{\xi<\kappa}$ is any family in $\mathcal{F}$, there is an $A\in\mathcal{F}$ such that $\#(A\setminus A_\xi)<\kappa$ for every $\xi<\kappa$.

(b) Let $\mathcal{F}$ be a weakly Ramsey ultrafilter on a set I, and $\mathcal{D}$ a disjoint family of subsets of κ, none belonging to $\mathcal{F}$. Set $Q=\bigcup_{D\in\mathcal{D}}[D]^2$. Then for any $S\subseteq[\kappa]^2$ there is an $A\in\mathcal{F}$ such that $Q\cap[A]^2$ is either included in S or disjoint from S.

(c) Let $\mathcal{F}$ be a weakly Ramsey ultrafilter on a set I. If $A\in\mathcal{F}$, then $\mathcal{F}\restriction A$ is weakly Ramsey, where $\mathcal{F}\restriction A=\mathcal{F}\cap\mathcal{P}A$.

(d) Let $\mathcal{F}$ be an ultrafilter on a set I, and suppose that there is an $A\in\mathcal{F}$ such that $\mathcal{F}\restriction A$ is an arrow ultrafilter. Then $\mathcal{F}$ is an arrow ultrafilter.

proof (a) For $\alpha<\kappa$, set $h(\alpha)=\min\{\xi:\xi<\kappa,\ \alpha\notin A_\xi\setminus\xi\}$. Note that $h^{-1}[\xi]\notin\mathcal{F}$ for every $\xi<\kappa$, because $\mathcal{F}$ is uniform and κ-complete; so $\sup h[A]=\kappa$ for every $A\in\mathcal{F}$. Set

$$S_0=\{\{\alpha,\beta\}:\alpha<\beta<\kappa\text{ and }h(\alpha)>h(\beta)\},$$

$$S_1=\{\{\alpha,\beta\}:\alpha<\beta<\kappa\text{ and }h(\alpha)=h(\beta)\},$$

$$S_2=\{\{\alpha,\beta\}:\alpha<\beta<\kappa\text{ and }h(\alpha)<h(\beta)\},$$

If $A \in \mathcal{F}$ then $[A]^2 \cap S_2 \neq \emptyset$. **P?** Otherwise, $h(\alpha) \geq h(\beta)$ whenever α, $\beta \in A$ and $\alpha < \beta$. Set $\xi = \min h[A]$ and let $\alpha \in A$ be such that $h(\alpha) = \xi$; then $h(\beta) = \xi$ whenever $\beta \in A \setminus \alpha$. But this means that $A \setminus \alpha$ and $A_\xi \setminus \xi$ are disjoint sets both belonging to $\mathcal{F}$. **XQ**

Because $\mathcal{F}$ is weakly Ramsey, there is an $A \in \mathcal{F}$ such that $[A]^2$ is disjoint from at least one of S_0 and S_1.

case 0 If $[A]^2 \cap S_0 = \emptyset$, then $h(\alpha) \leq h(\beta)$ whenever α, $\beta \in A$ and $\alpha < \beta$. If $\xi < \kappa$, take $\alpha \in A$ such that $h(\alpha) > \xi$. If $\beta \in A \setminus \alpha$, then $h(\beta) \geq h(\alpha) > \xi$ so $\beta \in A_\xi \setminus \xi$; thus $A \setminus A_\xi \subseteq A \cap \alpha$ has cardinal less than κ.

case 1 If $[A]^2 \cap S_1 = \emptyset$, then $h{\restriction}A$ is injective. If $\xi < \kappa$, take α such that $h(\beta) > \xi$ for every $\beta \in A \setminus \alpha$; then again $A \setminus A_\xi \subseteq A \cap \alpha$ has cardinal less than κ, as required.

(b) There is an $A \in \mathcal{F}$ such that $[A]^2$ is disjoint from at least one of $Q \cap S$, $Q \setminus S$ and $[I]^2 \setminus Q$. Since $A \not\subseteq D$ for any $D \in \mathcal{D}$, $[A]^2 \setminus Q \neq \emptyset$. So $[A]^2 \cap Q$ is either disjoint from S or included in S.

(c) If S_0, S_1, S_2 are disjoint subsets of $[A]^2$, then there is a $B \in \mathcal{F}$ such that $[B]^2$ is disjoint from some S_j, and now $A \cap B \in \mathcal{F}{\restriction}A$ and $[A \cap B]^2 \cap S_j = \emptyset$.

(d) If $S \subseteq [I]^2$ and $k \in \mathbb{N}$, then either there is a $K \in [A]^k$ such that $[K]^2 \cap (S \cap [A]^2) = \emptyset$ (in which case $K \in [I]^k$ and $[K]^2 \cap S = \emptyset$) or there is a $B \in \mathcal{F}{\restriction}A$ such that $[B]^2 \subseteq S \cap [A]^2$ (in which case $B \in \mathcal{F}$ and $[B]^2 \subseteq S$).

6C Proof of Theorem 1Mc Every weakly Ramsey ultrafilter is an arrow ultrafilter.

proof (a)(i) To begin with, I will suppose that $\mathcal{F}$ is a uniform κ-complete weakly Ramsey ultrafilter on an infinite cardinal κ. Note that it follows at once that κ is regular, since if $A \in [\kappa]^{<\kappa}$ then $\bigcap_{\xi \in A} \kappa \setminus \xi$ belongs to $\mathcal{F}$ and cannot be empty. I aim to show, by induction on k, that if $S \subseteq [\kappa]^2$ is such that $[K]^2$ meets S for every $K \in [\kappa]^k$ then there is an $A \in \mathcal{F}$ such that $[A]^2 \subseteq S$.

(ii) The cases $k = 0$ and $k = 1$ are vacuous, since there is a $K \in [\kappa]^k$ and $[K]^2 = \emptyset$; and the case $k = 2$ is trivial, since if $[K]^2 \cap S \neq \emptyset$ for every $K \in [\kappa]^2$ then $S = [\kappa]^2$. For the inductive step to $k \geq 3$, take $S \subseteq [\kappa]^2$ such that $[K]^2$ meets S for every $K \in [\kappa]^k$. For each $\xi < \kappa$ set $A_\xi = \{\eta : \eta < \kappa, \{\xi, \eta\} \in S\} \cup \{\xi\}$. If there is any ξ such that $A_\xi \notin \mathcal{F}$, then there is an $A \in \mathcal{F}$ such that $[A]^2 \subseteq S$. **P** Set $B = \kappa \setminus A_\xi$, $S' = S \cup ([\kappa]^2 \setminus [B]^2)$; because $\mathcal{F}$ is an ultrafilter, $B \in \mathcal{F}$. If $K \in [\kappa]^{k-1}$ and $K \not\subseteq B$, then certainly $[K]^2$ meets S'. If $K \in [B]^{k-1}$, then $K' = K \cup \{\xi\}$ belongs to $[\kappa]^k$ and there is an $L \in S$ such that $L \subseteq K'$; now $\xi \notin L$ so $L \subseteq K$ and $[K]^2$ meets S'. By the inductive hypothesis, there is a $C \in \mathcal{F}$ such that $[C]^2 \subseteq S'$; now $A = B \cap C$ belongs to $\mathcal{F}$ and $[A]^2 \subseteq S$. **Q**

(iii) So we may suppose that $A_\xi \in \mathcal{F}$ for every $\xi < \kappa$. By 6Ba, there is an $A \in \mathcal{F}$ such that $\#(A \setminus A_\xi) < \kappa$ for every $\xi < \kappa$. Because κ is regular, we can define inductively a strictly increasing family $\langle \xi_\alpha \rangle_{\alpha < \kappa}$ in κ by saying that ξ_α is to be the least ordinal such that $\xi_\alpha > \xi_\beta$ for every $\beta < \alpha$ and $A \setminus A_\eta \subseteq \xi_\alpha$ whenever $\eta < \sup_{\beta < \alpha} \xi_\beta$. Note that $\xi_\alpha = \sup_{\beta < \alpha} \xi_\beta$ whenever $\alpha < \kappa$ is a limit ordinal. Set $D_\alpha = \xi_{\alpha+1} \setminus \xi_\alpha$ for each α; then $\langle D_\alpha \rangle_{\alpha \in \kappa}$ is a partition of κ. Write Q for $\bigcup_{\alpha < \kappa} [D_\alpha]^2$.

One of $\bigcup_{\alpha < \kappa \text{ is even}} D_\alpha$, $\bigcup_{\alpha < \kappa \text{ is odd}} D_\alpha$ belongs to $\mathcal{F}$. Call this B. Then $[A \cap B]^2 \subseteq S \cup Q$. **P** If ξ, $\eta \in A \cap B$ and $\xi < \eta$, let $\alpha \leq \beta$ be such that $\xi \in D_\alpha$ and $\eta \in D_\beta$. If $\alpha = \beta$ then $\{\xi, \eta\} \in [D_\alpha]^2 \subseteq Q$. Otherwise, $\xi < \alpha + 1 < \beta$ and $A \setminus A_\xi \subseteq \xi_{\alpha+2}$ does not meet D_β, so $\eta \in A_\xi$ and $\{\xi, \eta\} \in S$. **Q**

(iv) There is a $C \in \mathcal{F}$ such that $Q \cap [C]^2 \subseteq S$. **P** By 6Bb, there is a $C_0 \in \mathcal{F}$ such that $Q \cap [C_0]^2$ is either included in S or disjoint from S. In the former case, we can take $C = C_0$ and stop. In the latter case, $\#(C_0 \cap D_\alpha) < k$ for every $\alpha < \kappa$, so there is a $C \in \mathcal{F}$ such that $\#(C \cap D_\alpha) \leq 1$ for every $\alpha < \kappa$. But in this case $Q \cap [C]^2$ is empty so is included in S. **Q**

(v) Now $A \cap B \cap C \in \mathcal{F}$ and

$$[A \cap B \cap C]^2 = [A \cap B]^2 \cap [C]^2 \subseteq S \cup (Q \cap [C]^2) = S.$$

So we have a suitable member of $\mathcal{F}$. Thus the induction continues, and it is the case for every $k \in \mathbb{N}$ that if $S \subseteq [\kappa]^2$ is such that $S \cap [K]^2 \neq \emptyset$ for every $K \in [\kappa]^k$, then there is an $A \in \mathcal{F}$ such that $[A]^2 \subseteq S$; that is, $\mathcal{F}$ is an arrow ultrafilter.

(b) Accordingly the theorem is proved in the case of a uniform κ-complete ultrafilter on an infinite cardinal κ. Now suppose that $\mathcal{F}$ is a uniform weakly Ramsey ultrafilter on an infinite cardinal κ, and is not

κ-complete. Then there are a cardinal $\lambda < \kappa$ and a family $\langle A_\lambda \rangle_{\alpha<\lambda}$ in $\mathcal{F}$ such that $\bigcap_{\alpha<\lambda} A_\lambda$ is empty. For $\xi < \kappa$, set $h(\xi) = \min\{\alpha : \alpha < \lambda, \xi \notin A_\alpha\}$; for $\alpha < \lambda$, set $D_\alpha = h^{-1}[\{\alpha\}]$, so that $\langle D_\alpha \rangle_{\alpha<\lambda}$ is a partition of κ into sets not belonging to $\mathcal{F}$. Set $Q = \bigcup_{\alpha<\lambda} [D_\alpha]^2$.

Let $S \subseteq [\kappa]^2$ and $k \in \mathbb{N}$ be such that $[K]^2 \cap S \neq \emptyset$ whenever $K \in [\kappa]^k$. Set $S_0 = [\kappa]^2 \setminus (S \cup Q)$ and $S_1 = S \setminus Q$. Then there is a $B \in \mathcal{F}$ such that $[B]^2$ is disjoint from at least one of Q, S_0 and S_1. Since $\#(B) = \kappa > \lambda$, there is some $\alpha < \lambda$ such that $\#(B \cap D_\alpha) \geq 2$, and $[B]^2$ meets Q. Since B meets infinitely many D_α, there is a $K \in [B]^k$ such that $\#(K \cap D_\alpha) \leq 1$ for every α; now $[B]^2 \cap S_1 \supseteq [K]^2 \cap S$ is non-empty. So $[B]^2 \cap S_0 = \emptyset$, that is, $[B]^2 \subseteq Q \cup S$.

By 6Bb again, there is a $C \in \mathcal{F}$ such that $Q \cap [C]^2$ is either included in S or disjoint from S. Since $\#(C) = \kappa > \lambda$, there must be an $\alpha < \lambda$ such that $\#(C \cap D_\alpha) \geq k$, in which case $[C \cap D_\alpha]^2$ meets S; so we must have $Q \cap [C]^2 \subseteq S$. So if we set $A = B \cap C$, A belongs to $\mathcal{F}$ and $[A]^2 \subseteq S \cup (Q \cap [C]^2) = S$. As S and k are arbitrary, $\mathcal{F}$ is an arrow ultrafilter.

(c) Putting (a) and (b) together, we see that any uniform weakly Ramsey ultrafilter on an infinite set is an arrow ultrafilter. Of course any principal ultrafilter is an arrow ultrafilter, so all uniform weakly Ramsey ultrafilters are arrow ultrafilters. Finally, if $\mathcal{F}$ is any weakly Ramsey ultrafilter on any set I, let $A \in \mathcal{F}$ be a set of minimal size; then $\mathcal{F}\lceil A$ is uniform; by 6Bc, $\mathcal{F}\lceil A$ is weakly Ramsey, therefore an arrow ultrafilter; by 6Bd, $\mathcal{F}$ is an arrow ultrafilter. So the proof is complete.

6D The following lemma is very well known in essence, though it is usually expressed in less quantitative forms.

Lemma Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to [0,1]$ an additive functional such that $\nu 1 = 1$. Suppose that ϵ, $\delta \in [0,1]$ are such that $\delta < \epsilon^2$, and that $\langle a_i \rangle_{i\in I}$ is a family in $\mathfrak{A}$ such that $\nu a_i \geq \epsilon$ and $\nu(a_i \cap a_j) \leq \delta$ for all distinct i, $j \in I$. Then $\#(I) \leq \frac{1}{\epsilon^2-\delta}$.

proof It is enough to deal with the case in which I is finite and $\mathfrak{A}$ is generated by $\{a_i : i \in I\}$, so that $\mathfrak{A}$ is finite and can be identified with a power set $\mathcal{P}X$. Consider $u = \sum_{i\in I} \chi a_i \in \mathbb{R}^X$. By Cauchy's ineqality,

$$\begin{aligned}(\sum_{x\in X} u(x)\nu\{x\})^2 &\leq \sum_{x\in X} \nu\{x\} \cdot \sum_{x\in X} u(x)^2 \nu\{x\} \\ &= \nu X \cdot \sum_{x\in X} u(x)^2\nu\{x\} = \sum_{x\in X} u(x)^2 \nu\{x\}.\end{aligned}$$

Now, setting $m = \#(I)$,

$$\begin{aligned}\sum_{x\in X} u(x)\nu\{x\} &= \sum_{x\in X}\sum_{i\in I} \chi a_i(x)\nu\{x\} \\ &= \sum_{i\in I}\sum_{x\in X} \chi a_i(x)\nu\{x\} = \sum_{i\in I} \nu a_i \geq \epsilon m,\end{aligned}$$

$$\begin{aligned}\sum_{x\in X} u(x)^2\nu\{x\} &= \sum_{x\in X, i,j\in I} \chi a_i(x)\chi a_j(x)\nu\{x\} = \sum_{i,j\in I} \nu(a_i \cap a_j) \\ &= \sum_{i\in I} \nu a_i + \sum_{i\neq j} \nu(a_i \cap a_j) \leq m + \delta m(m-1).\end{aligned}$$

So we get

$$\epsilon^2 m^2 \leq m + \delta m^2, \quad m \leq \tfrac{1}{\epsilon^2-\delta},$$

as claimed.

6E Proof of Theorem 1Md (i) An arrow ultrafilter is strongly measure-linking.
(ii) An arrow ultrafilter on $\mathbb{N}$ is nowhere dense.

proof (i) Let $\mathcal{F}$ be an arrow ultrafilter on a set I, $\mathfrak{A}$ a Boolean algebra, $\nu : \mathfrak{A} \to [0,1]$ an additive functional such that $\nu(1_{\mathfrak{A}}) = 1$, and $\langle a_i \rangle_{i \in I}$ a family in $\mathfrak{A}$ such that $\inf_{i \in I} \nu a_i = \epsilon > 0$. Take any $\delta \in \left]0, \epsilon^2\right[$ and set $S = \{\{i,j\} : i,\, j \in I \text{ are distinct, } \nu(a_i \cap a_j) \geq \delta\}$. Let $k \in \mathbb{N}$ be such that $k > \frac{1}{\epsilon^2 - \delta}$. If $K \in [I]^k$ then by Lemma 6D there are distinct i, $j \in K$ such that $\nu(a_i \cap a_j) \geq \delta$, that is, $[K]^2$ meets S. Because $\mathcal{F}$ is an arrow ultrafilter, there is a $J \in \mathcal{F}$ such that $[J]^2 \subseteq S$, that is, $\nu(a_i \cap a_j) \geq \delta$ for all i, $j \in J$. As $\mathfrak{A}$, ν and $\langle a_i \rangle_{i \in I}$ are arbitrary, $\mathcal{F}$ is strongly measure-linking.

(ii)(*α*) The key to the proof is the following: if $D \subseteq]0,1]$ is a countable set, there is a set $S \subseteq [D]^2$ such that $[K]^2 \cap S \neq \emptyset$ for every $K \in [D]^3$ and A is nowhere dense whenever $A \subseteq D$, $[A]^2 \subseteq S$ and $0 \in \overline{A}$. **P** Let $\preccurlyeq$ be a well-ordering of D in order type at most ω. Let $\langle \epsilon_t \rangle_{t \in D}$ be a family of strictly positive real numbers with sum at most 1. For m, $k \in \mathbb{N}$ set $H_{mk} =]2^{-m}k, 2^{-m}(k+1)[$; for $t \in D$ let $m_t \in \mathbb{N}$ be such that $2^{-m_t - 1} < t \leq 2^{-m_t}$. Define G_t inductively, for $t \in D$, following the well-ordering $\preccurlyeq$, in such a way that, writing m for m_t,

$G_t \subseteq]2^{-m}, 1[$,
$s \notin G_t$ if $s \prec t$,
$G_t \cap G_s = \emptyset$ if $s \prec t$ and $m_s > m$,
for $1 \leq k < 2^m$, $G_t \cap H_{mk}$ is a non-empty open interval of length at most $2^{-m}\epsilon_t$.

To see that this is possible, note that when we come to choose G_t the forbidden points in $\bigcup_{1 \leq k < 2^m} H_{mk}$ consist of some of the the finitely many $s \prec t$, together with $\bigcup_{s \prec t, m_s > m} G_s$; and the latter meets each H_{mk} in a finite union of intervals of total length at most $\sum_{s \prec t} 2^{-m}\epsilon_s < 2^{-m}$, so there must be a gap remaining.

Note that $m_s < m_t$ whenever $s \in G_t$. On completing the inductive construction, set

$$S = \{\{s,t\} : s,\, t \in D \text{ are distinct, } s \notin G_t \text{ and } t \notin G_s\}.$$

If $K \in [D]^3$ then either there are distinct s, $t \in K$ such that $m_s = m_t$, in which case $\{s,t\} \in S \cap [K]^2$, or $K = \{s,t,u\}$ where $m_s < m_t < m_u$. If $\{t,u\} \notin S$, then $t \in G_u$ so $u \prec t$ and $G_t \cap G_u$ is empty. But this means that one of $\{s,t\}$, $\{s,u\}$ belongs to S. So in all cases we have $[K]^2 \cap S \neq \emptyset$.

Now suppose that $A \subseteq D$ is such that $0 \in \overline{A}$ and $[A]^2 \subseteq S$. For each $m \in \mathbb{N}$ and $k \geq 1$ there is a $t \in A$ such that $m_t > m$. In this case, A cannot meet G_t, so $\overline{A}$ cannot include H_{mk}. As m and k are arbitrary, A is nowhere dense. **Q**

(*β*) Let $\mathcal{F}$ be an arrow ultrafilter on $\mathbb{N}$, and $f : \mathbb{N} \to \mathbb{R}$ a function. Set $g(n) = \arctan f(n)$ for $n \in \mathbb{N}$; then g is bounded; set $z = \lim_{n \to \mathcal{F}} g(n)$. Set $h(n) = \frac{1}{\pi}|g(n) - z|$ for $n \in \mathbb{N}$; then h takes values in $[0,1]$ and $\lim_{n \to \mathcal{F}} h(n) = 0$.

If $A_0 = \{i : h(i) = 0\}$ belongs to $\mathcal{F}$, then $f[A_0]$ is a singleton, and is certainly nowhere dense. Otherwise, set $D = h[\mathbb{N}] \setminus \{0\}$, and let $S \subseteq [D]^2$ be as in (α). Set

$$S_1 = \{\{i,j\} : i,\, j \in \mathbb{N}, \text{ either } i \in A_0 \text{ or } j \in A_0 \text{ or } h(i) = h(j) \text{ or } \{h(i), h(j)\} \in S\}.$$

If $K \subseteq \mathbb{N}$ and $[K]^2 \cap S_1 = \emptyset$, then $h{\restriction}K$ is injective, $h[K] \subseteq D$ and $[h[K]\,]^2 \cap S = \emptyset$, so $\#(K) \leq 2$. Because $\mathcal{F}$ is an arrow ultrafilter, there is an $A \in \mathcal{F}$ such that $[A]^2 \subseteq S_1$; we can suppose that $A \cap A_0$ is empty, so that $h[A] \subseteq D$ and $[h[A]\,]^2 \subseteq S$. But now recall that $\lim_{n \to \mathcal{F}} h(n) = 0$, so $0 \in \overline{h[A]}$ and $h[A]$ is nowhere dense, by the choice of S. In this case, $B = h[A] \cup (-h[A])$ is nowhere dense, $g[A] \subseteq z + \pi B$ is nowhere dense, and $f[A] = \tan[g[A]]$ is nowhere dense. As f is arbitrary, $\mathcal{F}$ is a nowhere dense filter.

6F The next lemma is a version of one which I learnt from Michel Talagrand when visiting him in 1987. He claims to have no recollection of it.

Lemma (M.Talagrand) (a) For finite I, $J \subseteq \mathbb{N}$, say that $I \preccurlyeq J$ if $\#(I) = \#(J)$ and $\#(I \setminus k) \leq \#(J \setminus k)$ for every $k \in \mathbb{N}$.

(i) $\preccurlyeq$ is a transitive relation.

(ii) If $I \preccurlyeq J$, there are $I_0, \ldots, I_r$ such that $I = I_0 \preccurlyeq I_1 \preccurlyeq \ldots \preccurlyeq I_r = J$ and $\#(I_k \triangle I_{k+1}) = 2$ for every $k < r$.

(iii) Suppose that $m \geq 1$. For $n \in \mathbb{N}$, set

$$\mathcal{I}_n = \{I : I \subseteq n,\ \#(I \cap J) \geq m \text{ whenever } J \subseteq n \text{ and } J \succcurlyeq I\}.$$

Then $\#(\mathcal{I}_n) \leq \frac{2^n\sqrt{n}}{m}$.

(b) If $1 \leq m \leq n \in \mathbb{N}$, $\mathcal{I} \subseteq \mathcal{P}n$ and $\#(I \cap J) \geq m$ for all I, $J \in \mathcal{I}$, then $\#(\mathcal{I}) \leq \frac{2^n\sqrt{n}}{m}$.

proof (a)(i) is trivial.

(ii) Induce on $r = \#(I \setminus J)$. If $r = 0$ then $I = J$ and we can stop. For the inductive step to $r + 1$, set $i_0 = \max(I \setminus J)$, $j_0 = \max(J \setminus I)$. Then $i_0 < j_0$. **P**

$$\#(I \setminus j_0) \leq \#(J \setminus j_0) = \#(I \cap J \setminus j_0)$$

so $I \setminus j_0 = I \cap J \setminus j_0$ does not contain i_0, and $i_0 \leq j_0$; but of course $i_0 \neq j_0$. **Q**

Set $J' = J \triangle \{i_0, j_0\}$. Then

$$\begin{aligned}\#(I \setminus k) &\leq \#(J \setminus k) = \#(J' \setminus k) \text{ if } k \leq i_0 \text{ or } j_0 < k,\\ &= \#(I \cap J \setminus k) \leq \#((J \setminus \{j_0\}) \setminus k) = \#(J' \setminus k) \text{ if } i_0 < k \leq j_0,\end{aligned}$$

so $I \preccurlyeq J' \preccurlyeq J$. Also $\#(I \setminus J') = r$ and $\#(J' \triangle J) = 2$. By the inductive hypothesis, there are $I_0, \ldots, I_r$ such that $I = I_0 \preccurlyeq I_1 \preccurlyeq \ldots \preccurlyeq I_r = J'$ and $\#(I_k \triangle I_{k+1}) = 2$ for every $k < r$; setting $I_{r+1} = J$, we have an appropriate chain to complete the inductive step.

(iii) For $I \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, set $h_n(I) = \min\{\#(I \cap J) : I \cap n \preccurlyeq J \subseteq n\}$. Then we find that

$$\begin{aligned}h_{n+1}(I) &= h_n(I) + 1 \text{ if } n \in I,\\ &= \max(0, h_n(I) - 1) \text{ otherwise.}\end{aligned}$$

P (α) If $n \in I$, then for $J \subseteq n+1$ we have $I \cap (n+1) \preccurlyeq J$ iff $n \in J$ and $I \cap n \preccurlyeq J \cap n$; so $h_{n+1}(I) = h_n(I)+1$. (β) If $n \notin I$, let J be such that $I \cap n \preccurlyeq J \subseteq n$ and $\#(I \cap J) = h_n(I)$. If $h_n(I) = 0$ then J witnesses that $h_{n+1}(I) = 0$. Otherwise, take any $i_0 \in I \cap J$, and consider $J' = J \triangle \{i_0, n\}$; we shall have

$$I \cap (n+1) = I \cap n \preccurlyeq J \preccurlyeq J' \subseteq n + 1,$$

so $h_{n+1}(I) \leq \#(I \cap J') = h_n(I) - 1$. Thus $h_{n+1}(I) \leq \max(0, h_n(I) - 1)$. (γ) Again supposing that $n \notin I$, take $J \subseteq n + 1$ such that $I \cap (n + 1) \preccurlyeq J$ and $\#(I \cap J) = h_{n+1}(I)$. If $n \notin J$, then

$$I \cap n = I \cap (n + 1) \preccurlyeq J \subseteq n$$

so $h_n(I) \leq \#(I \cap J) = h_{n+1}(I)$. If $n \subseteq J$, then $I \cap n \preccurlyeq J \cap n$ and again $h_n(I) \leq h_{n+1}(I)$. If $n \in J$ but $n \not\subseteq J$, set $j_0 = \max(n \setminus J)$ and $J' = J \triangle \{j_0, n\}$. For $k \in \mathbb{N}$,

$$\begin{aligned}\#(I \cap n \setminus k) &\leq \#(J \setminus k) = \#(J' \setminus k) \text{ if } k \leq j_0,\\ &\leq n - k = \#(J' \setminus k) \text{ if } j_0 \leq k \leq n,\\ &= 0 = \#(J' \setminus k) \text{ if } k \geq n.\end{aligned}$$

So $I \cap n \preccurlyeq J'$ and

$$h_n(I) \leq \#(I \cap J') \leq \#(I \cap J) + 1 = h_{n+1}(I) + 1.$$

Thus we must have $h_{n+1}(I) \geq h_n(I) - 1$, and of course $h_{n+1}(I) \geq 0$, so $h_{n+1}(I) \geq \max(0, h_n(I) - 1)$. **Q**

We have $h_{n+1} = h_n + g_n$ where

$$g_n(I) = 1 \text{ if } n \in I,$$
$$= -1 \text{ if } h_n(I) > 0 \text{ and } n \notin I,$$
$$= 0 \text{ if } h_n(I) = 0 \text{ and } n \notin I.$$

Giving $\mathcal{P}\mathbb{N}$ its usual probability measure, matching $\nu_{\mathbb{N}}$ on $\{0,1\}^{\mathbb{N}}$, and writing $\mathbb{E}$ for expectation with respect to this measure, we have

$$\mathbb{E}(h_{n+1}^2) = \mathbb{E}(h_n^2) + 2\mathbb{E}(h_n \times g_n) + \mathbb{E}(g_n^2).$$

Now $\mathbb{E}(h_n \times g_n) = 0$, because given that $h_n(I) \neq 0$ then $g_n(I)$ is equally likely to be ± 1; while $\mathbb{E}(g_n^2) \leq 1$. Since $h_0(I) = 0$ for every I, we see by induction that $\mathbb{E}(h_n^2) \leq n$ for every $n \in \mathbb{N}$. Consequently $\mathbb{E}(h_n) \leq \sqrt{n}$. Now $\mathcal{I}_n = \{I \cap n : h_n(I) \geq m\}$, so

$$\#(\mathcal{I}_n) = 2^n \Pr(h_n \geq m) \leq \frac{2^n}{m}\mathbb{E}(h_n) \leq \frac{2^n\sqrt{n}}{m}.$$

(b) Induce on $w(\mathcal{I}) = \sum_{I\in\mathcal{I}}\sum_{i\in I}(n-i)$. If $w(\mathcal{I}) = 0$ then $\mathcal{I}$ must be empty and the result is trivial. For the inductive step to $w(\mathcal{I}) = k+1$, then if $\mathcal{I} \subseteq \mathcal{I}_n$, as defined in (a-iii), we can stop. Otherwise, there must be an $I_0 \in \mathcal{I} \setminus \mathcal{I}_n$ and $J_0 \subseteq n$ such that $J_0 \not\succcurlyeq I_0$ and $\#(I_0 \cap J_0) < m$, so that $J_0 \notin \mathcal{I}$.

By (a-ii), $\mathcal{I}_0$ and $\mathcal{J}_0$ are linked by a $\preccurlyeq$-chain of sets each differing from the preceding one at just two points; so there must be $I_1 \in \mathcal{I}$, $J_1 \in \mathcal{P}n \setminus \mathcal{I}$ such that $I_1 \preccurlyeq J_1$ and $\#(I_1 \triangle J_1) = 2$. Let i_1 be the member of $I_1 \setminus J_1$ and j_1 the member of $J_1 \setminus I_1$; of course $i_1 < j_1$. Define $\phi : \mathcal{I} \to \mathcal{P}n$ by saying that

$$\phi(I) = I \triangle \{i_1, j_1\} \text{ if } i_1 \in I,\ j_1 \notin I \text{ and } I \triangle \{i_1, j_1\} \notin \mathcal{I},$$
$$= I \text{ otherwise.}$$

Then ϕ is injective. Set $\mathcal{J} = \phi[\mathcal{I}]$, so that $\#(\mathcal{I}) = \#(\mathcal{J})$, while $w(\mathcal{J}) < w(\mathcal{I})$. Now $\#(J \cap J') \geq m$ for all J, $J' \in \mathcal{J}$. **P** Set $I = \phi^{-1}(J)$, $I' = \phi^{-1}(J')$. Then

— if $I = J$ and $I' = J'$, surely $\#(J \cap J') = \#(I \cap I') \geq m$;

— if $I \neq J$ and $I' \neq J'$, then

$$\#(J \cap J') = \#((I \triangle \{i_1, j_1\}) \cap (I' \triangle \{i_1, j_1\}) = \#(I \cap I') \geq m$$

because $i_1 \in I \cap I'$ and $j_1 \notin I \cup I'$;

— if $I = J$ and $I' \neq J'$ and $j_1 \in I$, then $j_1 \notin I'$, $J' = I' \triangle \{i_1, j_1\}$ so

$$\#(J \cap J') = \#(I \cap J') \geq \#(I \cap I') \geq m;$$

— if $I = J$ and $I' \neq J'$ and $i_1 \notin I$, then $i_1 \in I'$, $J' = I' \triangle \{i_1, j_1\}$ so

$$\#(J \cap J') = \#(I \cap J') \geq \#(I \cap I') \geq m;$$

— if $I = J$ and $I' \neq J'$ and $i_1 \in I$ and $j_1 \notin I$, then $I'' = I \triangle \{i_1, j_1\} \in \mathcal{I}$ so

$$\#(J \cap J') = \#((I'' \triangle \{i_1, j_1\}) \cap (I' \triangle \{i_1, j_1\})) = \#(I'' \cap I') \geq m$$

because $i_1 \in I' \setminus I''$ and $j_1 \in I'' \setminus I'$.

Similarly, $\#(J \cap J') \geq m$ if $I' = J'$ and $I \neq J$. Thus in all cases we have $\#(J \cap J') \geq m$. **Q**

By the inductive hypothesis,

$$\frac{2^n\sqrt{n}}{m} \geq \#(\mathcal{J}) = \#(\mathcal{I})$$

and the induction continues.

6G Proposition (V.Bergelson-M.Talagrand) There are a probability algebra $(\mathfrak{A}, \bar{\mu})$ and a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\bar{\mu}a_n \geq \frac{1}{2}$ for every n and I has zero asymptotic density whenever $I \subseteq \mathbb{N}$ and $\inf_{m,n\in I} \bar{\mu}(a_m \cap a_n) > 0$.

proof For $r \in \mathbb{N}$, set $K_r = \{n : 2^r - 1 \leq n \leq 2^{r+1} - 2\}$ and

$$\mathcal{K}_r = \{I : I \subseteq r+1,\ \#(I) > \tfrac{r+1}{2}\} \cup \{I : 0 \in I \subseteq r+1,\ \#(I) = \tfrac{r+1}{2}\},$$

so that $\#(K_r) = \#(\mathcal{K}_r) = 2^r$. For each $r \in \mathbb{N}$ let $h_r : K_r \to \mathcal{K}_r$ be a bijection. Set $X = \prod_{r\in\mathbb{N}}(r+1)$, with the product μ of the uniform probabilities on the factors; let $(\mathfrak{A}, \bar\mu)$ be the measure algebra of μ. Define $A_n \subseteq X$, for $n \in \mathbb{N}$, by saying that $A_n = \{x : x \in X,\ x(r) \in h_r(n)\}$ for that r such that $n \in K_r$; as $\#(h_r(n)) \ge \frac{r+1}{2}$, $\mu A_n \ge \frac{1}{2}$. Set $a_n = A_n^\bullet \in \mathfrak{A}$, so that $\bar\mu a_n \ge \frac{1}{2}$ for every n.

Let $I \subseteq \mathbb{N}$ and $\epsilon > 0$ be such that $\bar\mu(a_m \cap a_n) \ge \epsilon$ for all m, $n \in I$. For $r \in \mathbb{N}$, set $\mathcal{J}_r = h_r[I \cap K_r]$. If J, $J' \in \mathcal{J}_r$, then $\#(J \cap J') \ge \epsilon(r+1)$. So, setting $m_r = \lceil \epsilon(r+1) \rceil$,

$$\#(I \cap K_r) = \#(\mathcal{J}_r) \le \frac{2^{r+1}\sqrt{r+1}}{m_r}$$

for each r, by 6Fb, and

$$2^{-r}\#(I \cap K_r) \le \frac{2\sqrt{r+1}}{\epsilon(r+1)} \to 0$$

as $r \to \infty$. So I has zero asymptotic density, as claimed.

6H Proof of Theorem 1M, parts (e)-(g)

(e) A strongly measure-linking ultrafilter on $\mathbb{N}$ contains a set of zero asymptotic density.

(f) A closed Lebesgue null ultrafilter on $\mathbb{N}$ contains a set of zero asymptotic density.

(g) If $\operatorname{cov}\mathcal{N}_{\text{Leb}} = \mathfrak{c}$, there is a measure-centering ultrafilter on $\mathbb{N}$ which is neither strongly measure-linking nor closed Lebesgue null.

proof For $I \subseteq \mathbb{N}$ write $d^*(I)$ for its upper asymptotic density; write $\mathcal{Z}$ for the asymptotic density ideal $\{I : d^*(I) = 0\}$.

(e) Let $(\mathfrak{A}, \bar\mu)$ and $\langle a_n \rangle_{n\in\mathbb{N}}$ be as in 6G. If $\mathcal{F}$ is a strongly measure-linking ultrafilter on $\mathbb{N}$, there is an $I \in \mathcal{F}$ such that $\inf_{m,n\in I} \bar\mu(a_m \cap a_n) > 0$, so $I \in \mathcal{Z}$.

(f) Let $\langle t_n \rangle_{n\in\mathbb{N}}$ be a sequence in $[0,1]$ which is equidistributed for Lebesgue measure, and $\mathcal{F}$ a closed Lebesgue null ultrafilter on $\mathbb{N}$. Then there is an $I \in \mathcal{F}$ such that $F = \overline{\{t_n : n \in I\}}$ is negligible. So $d^*(I) = 0$ (Fremlin 03, 491B).

(g) By Theorem 1Bc, we have a measure-centering ultrafilter on $\mathbb{N}$ which contains no set of zero asymptotic density, so can be neither strongly measure-linking nor closed Lebesgue null.

6I I turn now to the proofs of 1Mh-1Mi. While these are expected results, they seem to depend on some non-trivial combinatorial probability theory. For the rest of this section, I will write $\mathcal{N}\mathit{wd}$ for the ideal of nowhere dense subsets of $\mathbb{R}$ and $\mathcal{I}$ for the ideal $\{A : A \subseteq \mathbb{R},\ A \cap [0,\epsilon] \in \mathcal{N}\mathit{wd}$ for some $\epsilon > 0\}$.

Lemma Suppose that $D \subseteq \mathbb{R}$.

(a)(i) If $D \notin \mathcal{N}\mathit{wd}$ there is a $D' \subseteq D$, with no isolated points, such that $D' \notin \mathcal{N}\mathit{wd}$.

(ii) If $D \notin \mathcal{I}$ there is a $D' \subseteq D$, with no isolated points, such that $D' \notin \mathcal{I}$.

(b) Suppose that $S \subseteq [D]^2$ is such that $\{t : \{s,t\} \notin S\} \in \mathcal{N}\mathit{wd}$ for every $s \in D$.

(i) If $D \notin \mathcal{N}\mathit{wd}$ there is a $D' \subseteq D$ such that $[D']^2 \subseteq S$ and $D' \notin \mathcal{N}\mathit{wd}$.

(ii) If $D \notin \mathcal{I}$ there is a $D' \subseteq D$ such that $[D']^2 \subseteq S$ and $D' \notin \mathcal{I}$.

proof (a) Set $G = \operatorname{int}\overline{D}$, $D' = D \cap G$; then D' has no isolated points and $D \setminus D' \subseteq \overline{D} \setminus G$ is nowhere dense. So $D' \notin \mathcal{N}\mathit{wd}$ if $D \notin \mathcal{N}\mathit{wd}$ and $D' \notin \mathcal{I}$ if $D \notin \mathcal{I}$.

(b) By (a), we can suppose that D has no isolated points. Let $\langle V_n \rangle_{n\in\mathbb{N}}$ enumerate a base for the topology of $\mathbb{R}$ and choose $\langle s_n \rangle_{n\in\mathbb{N}}$ inductively so that

$$s_n \in D, \quad \{s_i, s_n\} \in S \text{ for } i < n, \quad \text{if } V_n \cap D \ne \emptyset \text{ then } s_n \in V_n$$

for each n. Set $D' = \{s_n : n \in \mathbb{N}\}$; then $[D']^2 \subseteq S$ and D' is dense in D. So $D' \notin \mathcal{N}\mathit{wd}$ if $D \notin \mathcal{N}\mathit{wd}$ and $D' \notin \mathcal{I}$ if $D \notin \mathcal{I}$.

6J Lemma Suppose that $D \subseteq \mathbb{R}$, and that $S \subseteq [D]^2$ and $k \in \mathbb{N}$ are such that $[K]^2 \cap S \ne \emptyset$ for every $K \in [D]^k$.

(a) If $D\notin\mathcal{N}\text{wd}$, then there is a $D'\subseteq D$ such that $D'\notin\mathcal{N}\text{wd}$ and $[D']^2\subseteq S$.
(b) If $D\notin\mathcal{I}$, there is a $D'\subseteq D$ such that $D'\notin\mathcal{I}$ and $\{t:t\in D',\,\{s,t\}\notin S\}\in\mathcal{I}$ for every $s\in D'$.

proof For $t\in D$ set $S'_t=\{s:s\in D\setminus\{t\},\,\{s,t\}\notin S\}$.

(a) Induce on k. If $k\le 1$ the result is vacuous and for $k=2$ it is trivial. For the inductive step to $k\ge 3$, if $S'_t\in\mathcal{N}\text{wd}$ for every $t\in D$, then 6I(b-i) tells us that there is a $D'\subseteq D$ such that $[D']^2\subseteq S$ and $D'\notin\mathcal{N}\text{wd}$. Otherwise, take t such that $S'_t\notin\mathcal{N}\text{wd}$. If $K\in[S'_t]^{k-1}$ then $K\cup\{t\}\in[D]^k$ so there is a $J\in[K\cup\{t\}]^2\cap S$; since $K\subseteq S'_t$, $t\notin J$ and $J\subseteq K$. By the inductive hypothesis, there is a $D'\subseteq S'_t$ such that $[D']^2\subseteq S$ and $D\notin\mathcal{N}\text{wd}$.

(b) Again induce on k; as before, the case $k\le 2$ is trivial. For the inductive step to $k\ge 3$, if there is a $t\in D$ such that S'_t does not belong to $\mathcal{I}$, then we can apply the inductive hypothesis to S'_t.

6K Lemma Suppose that $D\subseteq\mathbb{R}$, $(\mathfrak{A},\bar\mu)$ is a probability algebra and that $\langle a_t\rangle_{t\in D}$ is a family in $\mathfrak{A}$ such that $\epsilon=\inf_{t\in D}\bar\mu a_t$ is greater than 0.

(a) Take any $\delta<\epsilon^2$, and set $S=\{\{s,t\}:s,\,t\in D$ are distinct, $\bar\mu(a_s\cap a_t)\ge\delta\}$. If $k>\frac{1}{\epsilon^2-\delta}$, then $[K]^2\cap S\ne\emptyset$ for every $K\in[D]^k$.
(b) If $D\notin\mathcal{N}\text{wd}$ and δ, $\eta>0$ then there are an $e\in\mathfrak{A}$ and a $D'\subseteq D$ such that $D'\notin\mathcal{N}\text{wd}$, $\bar\mu(a_s\setminus e)\le\eta$ for every $s\in D'$, and whenever $d\subseteq e$, $\bar\mu d\ge\delta$ then $\{s:s\in D',\,\bar\mu(a_s\cap d)\le\eta\bar\mu d\}\in\mathcal{N}\text{wd}$.
(c) If $D\notin\mathcal{I}$ there are a $D'\subseteq D$ and an $\eta>0$ such that $D'\notin\mathcal{I}$ and $\{s:s\in D',\,s>t,\,\bar\mu(a_s\cap a_t)\le\eta\}\in\mathcal{N}\text{wd}$ for every $t\in D'$.
(d) If $D\notin\mathcal{I}$ there are a $D'\subseteq D$ and an $\eta>0$ such that $D'\notin\mathcal{I}$ and $\{t:t\in D',\,t<s,\,\bar\mu(a_s\cap a_t)\le\eta\}\in\mathcal{N}\text{wd}$ for every $s\in D'$.
(e) If $D\notin\mathcal{I}$ there are a $D'\subseteq D$ and an $\eta>0$ such that $D'\notin\mathcal{I}$ and $\bar\mu(a_s\cap a_t)\ge\eta$ for all s, $t\in D'$.

proof (a) This is immediate from Lemma 6D.

(b) Set

$$\gamma=\inf\{\bar\mu d:d\in\mathfrak{A},\,\{s:s\in D,\,\bar\mu(a_s\setminus d)\le\eta(1-\bar\mu d)\}\notin\mathcal{N}\text{wd}\}.$$

Let $e\in\mathfrak{A}$ be such that $D'=\{s:s\in D,\,\bar\mu(a_s\setminus e)\le\eta(1-\bar\mu e)\}$ is not in $\mathcal{N}\text{wd}$ and $\bar\mu e<\gamma+\delta$. If $d\subseteq e$ and $\bar\mu d\ge\delta$, set $C=\{s:s\in D',\,\bar\mu(a_s\cap d)\le\eta\bar\mu d\}$. For any $s\in C$,

$$\bar\mu(a_s\setminus(e\setminus d))=\bar\mu(a_s\setminus e)+\bar\mu(a_s\cap d)\le\eta(1-\bar\mu e)+\eta\bar\mu d=\eta(1-\bar\mu(e\setminus d)).$$

But also $\bar\mu(e\setminus d)<\gamma$; by the definition of γ, $C\in\mathcal{N}\text{wd}$, as required.

(c) The proof proceeds by induction on the least $n>\frac{1}{\epsilon}$. The case $n\le 2$ is trivial, because if $\epsilon>\frac12$ then we can take $D'=D$ and $\eta=\epsilon-\frac12$. For the inductive step to $n\ge 3$, take $\mathfrak{A}$, $\bar\mu$ and $\langle a_t\rangle_{t\in D}$ as described. Let $\delta>0$ be such that $\delta<\epsilon-\frac{1}{n}$; set $\eta_0=\delta^2$.

case 1 Suppose that whenever A, $B\subseteq D$, $A\notin\mathcal{N}\text{wd}$ and $B\notin\mathcal{I}$ there are $A'\subseteq A$, $B'\subseteq B$ such that $A'\notin\mathcal{N}\text{wd}$, $B'\notin\mathcal{I}$ and $\{s:s\in A',\,\bar\mu(a_s\cap a_t)\le\eta_0\}\in\mathcal{N}\text{wd}$ for every $t\in B'$. In this case, choose $\langle A_n\rangle_{n\in\mathbb{N}}$, $\langle B_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $B_0=D\cap[0,\infty[$. Given that $B_n\notin\mathcal{I}$, let $\alpha\in\,]0,2^{-n}]$ be such that $B_n\setminus[0,\alpha]\notin\mathcal{N}\text{wd}$. Then there are $A_n\subseteq B_n\setminus[0,\alpha]$ and $B_{n+1}\subseteq B_n\cap[0,\alpha]$ such that $A_n\notin\mathcal{N}\text{wd}$, $B_{n+1}\notin\mathcal{I}$ and $\{s:s\in A_n,\,\bar\mu(a_s\cap a_t)\le\eta_0\}\in\mathcal{N}\text{wd}$ for every $t\in B_n$. Continue.

At the end of the induction, choose for each $n\in\mathbb{N}$ an $A^*_n\subseteq A_n$ such that $A^*_n\notin\mathcal{N}\text{wd}$ and $\bar\mu(a_s\cap a_t)\ge\eta_0$ for every s, $t\in A^*_n$; this is possible by (a) above and Lemma 6Ja. Set $D'=\bigcup_{n\in\mathbb{N}}A^*_n$, $\eta=\eta_0$; this works.

case 2 Otherwise, take A, $B\subseteq D$ such that $A\notin\mathcal{N}\text{wd}$, $B\notin\mathcal{I}$ and whenever $A'\subseteq A$, $B'\subseteq B$, $A'\notin\mathcal{N}\text{wd}$ and $B'\notin\mathcal{I}$, then $\{s:s\in A',\,\bar\mu(a_s\cap a_t)\le\eta_0\}\notin\mathcal{N}\text{wd}$ for some $t\in B'$. By (b), there are $e\in\mathfrak{A}$, $A'\subseteq A$ such that $A'\notin\mathcal{N}\text{wd}$, $\bar\mu(a_s\setminus e)\le\delta$ for every $s\in A'$, and whenever $d\subseteq e$ and $\bar\mu d\ge\eta_0$ then $\{s:s\in A',\,\bar\mu(a_s\cap d)\le\delta\bar\mu d\}\in\mathcal{N}\text{wd}$.

Set $B'=\{t:t\in B,\,\bar\mu(a_t\cap e)\ge\delta\}$. **?** If $B'\notin\mathcal{I}$, then there is a $t\in B'$ such that $C=\{s:s\in A',\,\bar\mu(a_s\cap a_t)\le\eta_0\}\notin\mathcal{N}\text{wd}$. But $C\subseteq\{s:s\in A',\,\bar\mu(a_s\cap(a_t\cap e))\le\delta\bar\mu(a_t\cap e)\}$, which is nowhere dense, by the choice of A' and e. **X**

Accordingly $B' \in \mathcal{I}$; set $D_1 = B \setminus B' \notin \mathcal{I}$. Since there are $s \in A'$ and $t \in D_1$, we have

$$\epsilon - \delta \le \bar{\mu} a_s - \delta \le \bar{\mu} e = \bar{\mu}(e \cup a_t) + \bar{\mu}(e \cap a_t) - \bar{\mu} a_t \le 1 + \delta - \epsilon < 1,$$

and $1 \setminus e \ne 0$. Consider the principal ideal $\mathfrak{A}_{1 \setminus e}$ with the normalized measure $\bar{\nu}$ where $\bar{\nu} a = \frac{\bar{\mu} a}{1 - \bar{\mu} e}$ for $a \in \mathfrak{A}_{1 \setminus e}$; set $a'_t = a_t \setminus e$ for $t \in D_1$. Then

$$\bar{\nu} a'_t \ge \tfrac{\epsilon - \delta}{1 - \epsilon + \delta} > \tfrac{1}{n-1}$$

(by the choice of δ) for every $t \in D_1$. So the inductive hypothesis tells us that there are a $D' \subseteq D_1$ and an $\eta > 0$ such that $D' \notin \mathcal{I}$ and $\{s : s \in D',\ s > t,\ \bar{\mu}(a_s \cap a_t) \le \eta\} \in \mathcal{N}\mathcal{wd}$ for every $t \in D'$.

(d) Set $\delta = \frac{1}{3}\epsilon^2$, $\eta_0 = \frac{1}{2}\delta^2$. Putting (a) here and Lemma 6Jb together, there is a $D_1 \subseteq D$ such that $D_1 \notin \mathcal{I}$ and $\{t : t \in D_1,\ \bar{\mu}(a_s \cap a_t) \le \delta\} \in \mathcal{I}$ for every $s \in D_1$. Let $\langle \alpha_n \rangle_{n \in \mathbb{N}}$ be a strictly decreasing sequence in $]0, 1]$ such that $\alpha_{n+1} \le \frac{1}{2}\alpha_n$ and $A_n = D_1 \cap\,]\alpha_{n+1}, \alpha_n[\ \notin \mathcal{N}\mathcal{wd}$ for every $n \in \mathbb{N}$. By (b), we can find for each $n \in \mathbb{N}$ an $A'_n \subseteq A_n$ and an $e_n \in \mathfrak{A}$ such that $A'_n \notin \mathcal{N}\mathcal{wd}$, $\bar{\mu}(a_s \setminus e_n) \le \frac{1}{2}\delta$ for every $s \in A'_n$ and whenever $d \subseteq e_n$ and $\bar{\mu} d \ge \eta_0$ then $\{s : s \in A'_n,\ \bar{\mu}(a_s \cap d) \le \frac{1}{2}\delta\bar{\mu} d\} \in \mathcal{N}\mathcal{wd}$; moreover, by Lemma 6I(a-i), we can suppose that A'_n has no isolated points. Of course $\bar{\mu} e_n \ge \epsilon - \delta$ for every n; since $\delta < (\epsilon - \delta)^2$, there is an infinite $J \subseteq \mathbb{N}$ such that $\bar{\mu}(e_m \cap e_n) \ge \delta$ whenever m, $n \in J$ and $m < n$. (Apply Ramsey's theorem to $\{\{m, n\} : m < n,\ \bar{\mu}(e_m \cap e_n) < \delta\}$.)

If $m \in J$ and $J' \subseteq J$ is infinite, there are an infinite $J'' \subseteq J'$ and a set $A''_m \subseteq A'_m$ such that A''_m is dense in A'_m and $\bar{\mu}(a_s \cap e_n) \ge \eta_0$ whenever $s \in A''_m$ and $n \in J''$. **P** Let $\langle V_n \rangle_{n \in \mathbb{N}}$ enumerate a base for the topology of $\mathbb{R}$. Choose $\langle s_n \rangle_{n \in \mathbb{N}}$, $\langle j_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. j_0 is to be any member of J' greater than m. Given that $j_i > m$ for $i \le n$,

$$\{s : s \in A'_m,\ \bar{\mu}(a_s \cap e_{j_i}) \le \eta_0\} \subseteq \{s : s \in A'_m,\ \bar{\mu}(a_s \cap e_{j_i}) \le \frac{1}{2}\delta\bar{\mu}(e_m \cap e_{j_i})\}$$
$$\in \mathcal{N}\mathcal{wd}$$

for every $i \le n$, so there is an $s_n \in A_m$ such that $\bar{\mu}(a_{s_n} \cap e_{j_i}) \ge \eta_0$ for every $i \le n$, and if $A_m \cap V_n$ is not empty then we can take $s_n \in V_n$. Given s_i for $i < n$, where $n \ge 1$, then there is an $r \in \mathbb{N}$ such that $\{t : t \in D_1 \cap [0, \alpha_r],\ \bar{\mu}(a_{s_i} \cap a_t) \le \delta\}$ is nowhere dense for each $i < n$; so if we take $j_n \in J'$ such that $j_n > \max(j_{n-1}, r)$, there is a $t \in A'_{j_n}$ such that $\bar{\mu}(a_{s_i} \cap a_t) \ge \delta$ for every $i < n$, in which case $\bar{\mu}(a_{s_i} \cap e_{j_n}) \ge \delta - \frac{1}{2}\delta \ge \eta_0$ for every $i < n$. Continue; at the end, set $A''_m = \{s_i : i \in \mathbb{N}\}$ and $J'' = \{j_i : i \in \mathbb{N}\}$. **Q**

We can therefore find an infinite $I \subseteq J$ and a family $\langle A''_m \rangle_{m \in I}$ of sets such that A''_m is dense in A'_m and $\bar{\mu}(a_s \cap e_n) \ge \eta_0$ whenever $m < n$ in I and $s \in A''_m$. But this will mean that $\{t : t \in A''_n,\ \bar{\mu}(a_s \cap a_t) \le \frac{1}{2}\delta\eta_0\} \in \mathcal{N}\mathcal{wd}$ whenever $m < n$ in I and $s \in A''_m$.

Finally, by Lemma 6Ja, there is for each $m \in I$ an $A^*_m \subseteq A''_m$ such that $A^*_m \notin \mathcal{N}\mathcal{wd}$ and $\{t : t \in A^*_m,\ \bar{\mu}(a_s \cap a_t) \le \delta\} \in \mathcal{N}\mathcal{wd}$ for every $s \in A^*_m$. So setting $D' = \bigcup_{m \in I} A^*_m$ we shall have a suitable set, with $\eta = \frac{1}{2}\delta\eta_0$.

(e) Putting (c) and (d) together, we see that there are an $\eta > 0$ and a $D_1 \subseteq D$ such that $D_1 \notin \mathcal{I}$ and $\{t : t \in D_1,\ \bar{\mu}(a_s \cap a_t) \le \eta\} \in \mathcal{N}\mathcal{wd}$ for every $s \in D_1$. By Lemma 6I(b-ii) there is a $D' \subseteq D_1$ such that $D' \notin \mathcal{I}$ and $\bar{\mu}(a_s \cap a_t) \ge \eta$ for all s, $t \in D'$.

6L Proof of Theorem 1Mh If $\mathfrak{c} = \omega_1$, there is a strongly measure-linking ultrafilter on $\mathbb{N}$ which is not nowhere dense, therefore not measure-centering nor an arrow ultrafilter.

proof (a) Set $D = \mathbb{Q} \cap [0, 1]$. Let $\langle \langle a_{\xi t} \rangle_{t \in D} \rangle_{\xi < \omega_1}$ run over all families $\langle a_t \rangle_{t \in D}$ in $\mathfrak{B}_\omega$ such that $\inf_{t \in D} \bar{\nu}_\omega a_t > 0$. Choose a family $\langle D_\xi \rangle_{\xi < \omega_1}$ of subsets of D inductively, as follows. The inductive hypothesis will be that $\bigcap_{\eta \in J} D_\eta \notin \mathcal{I}$ for every finite $J \subseteq \xi$. Start with $D_0 = D$. Given $\langle D_\eta \rangle_{\eta < \xi}$, where $1 \le \xi < \omega_1$, let $\langle \theta(\xi, n) \rangle_{n \in \mathbb{N}}$ run over ξ, and take a strictly decreasing sequence $\langle \alpha_n \rangle_{n \in \mathbb{N}}$, with infimum 0, such that $C_n = \,]\alpha_{n+1}, \alpha_n[\, \cap \bigcap_{i \le n} D_{\theta(\xi, i)} \notin \mathcal{N}\mathcal{wd}$ for every $n \in \mathbb{N}$. Then $C = \bigcup_{n \in \mathbb{N}} C_n \notin \mathcal{I}$. By Lemma 6Ke, there is a $D_\xi \subseteq C$ such that $D_\xi \notin \mathcal{I}$ and $\inf_{s, t \in D_\xi} \bar{\nu}_\omega(a_{\xi s} \cap a_{\xi t}) > 0$. Since $D_\xi \cap [0, \alpha_n[\ \subseteq \bigcap_{i \le n} D_{\theta(\xi, i)}$ for every n, $D_\xi \cap \bigcap_{\eta \in J} D_\eta \notin \mathcal{I}$ for every finite $J \subseteq \xi$, and $\bigcap_{\eta \in J} D_\eta \notin \mathcal{I}$ for every finite $J \subseteq \xi + 1$. Continue.

(b) At the end of the induction, let $\mathcal{F}$ be an ultrafilter on D containing every D_ξ and no nowhere dense set; then $\mathcal{F}$ is not nowhere dense. But $\mathcal{F}$ is strongly measure-linking. **P** If $\mathfrak{A}$ is a Boolean algebra, $\nu : \mathfrak{A} \to [0,1]$ an additive functional such that $\nu 1 = 1$, and $\langle a_t \rangle_{t\in D}$ is a family in $\mathfrak{A}$ such that $\inf_{t\in D} \nu a_t > 0$, then (as in (iv)⇒(i) of the proof in 3C) there is a Radon probability measure μ on the Stone space Z of $\mathfrak{A}$, identified with the algebra of open-and-closed subsets of Z, which extends ν. Taking T to be the σ-algebra of subsets of Z generated by $\{a_t : t \in D\}$, $(\mathfrak{B}, \bar\mu)$ to be the measure algebra of $(Z, \mathrm{T}, \mu\restriction\mathrm{T})$ and $a_t^\bullet \in \mathfrak{B}$ the equivalence class of $a_t \in \mathrm{T}$ for each $t \in D$, we see that $\inf_{t\in D} \bar\mu a_t^\bullet > 0$. As in part (ii)⇒(iv) of the proof in 3C, there is a measure-preserving Boolean homomorphism $\pi : \mathfrak{B} \to \mathfrak{B}_\omega$, and now there must be a $\xi < \omega_1$ such that $\pi a_t^\bullet = a_{\xi t}$ for every $t \in D$. In this case, $D_\xi \in \mathcal{F}$ and

$$\inf_{s,t\in D_\xi} \nu(a_s \cap a_t) = \inf_{s,t\in D_\xi} \bar\nu_\omega(a_{\xi s} \cap a_{\xi t}) > 0.$$

As $\mathfrak{A}$, ν and $\langle a_t \rangle_{t\in D}$ are arbitrary, $\mathcal{F}$ is strongly measure-linking. **Q**

(c) Thus we have a strongly measure-linking ultrafilter on the countably infinite set D which is not nowhere dense. Of course it follows at once that there is such an ultrafilter on $\mathbb{N}$. By parts (b) and (d-ii) of Theorem 1M it cannot be either measure-centering or an arrow ultrafilter.

6M To convert the last result into a proof that there can be a Hausdorff p-point ultrafilter which is not measure-centering, I use the language of 'game strategies'. Let $G^{\mathrm{H|m\text{-}c}}$ be the game for two players, Empty and Non-empty, in which

Empty chooses $m \ge 1$,
Non-empty chooses $k \in \mathbb{N}$,
Empty chooses $n \ge 1$, a set B with mn members, and a set $L_0 \subseteq [B]^n$,
given $i < k$ and L_i, Non-empty chooses f_i, $g_i : L_i \to \mathbb{N}$ with $f_i(a) \ne g_i(a)$ for every $a \in L_i$,
given $i < k$ and L_i, f_i and g_i, Empty chooses $L_{i+1} \subseteq L_i$ such that $f_i[L_{i+1}] \cap g_i[L_{i+1}] = \emptyset$.

A run of the game ends when Empty has chosen L_k; Empty wins if $\bigcap L_k = \emptyset$; otherwise Non-empty wins.

Note that the game is determined, that is, one of the players has a winning strategy. (Since the game always terminates after finitely many moves, it is an 'open' game in the usual terminology of infinite games.)

6N Lemma Empty has a winning strategy in the game $G^{\mathrm{H|m\text{-}c}}$.

proof (a) To begin with, suppose that $\mathfrak{c} = \omega_1$.

(i) By Theorem 1Mh, there is a strongly measure-linking ultrafilter $\mathcal{F}$ on $\mathbb{N}$ which is not measure-centering. By Theorem 1Ma, $\mathcal{F}$ is a Hausdorff ultrafilter. Let $\langle a_j \rangle_{j\in\mathbb{N}}$ be a sequence in $\mathfrak{B}_\omega$ such that $\epsilon = \inf_{j\in\mathbb{N}} \bar\nu_\omega a_i$ is greater than 0, but there is no $A \in \mathcal{F}$ such that $\{a_j : j \in A\}$ is centered

(ii) Let $\mathcal{L}_0$ be the family of finite subsets L of $\mathbb{N}$ such that $\inf_{j\in L} a_j = 0$ in $\mathfrak{B}_\omega$, and for $i \in \mathbb{N}$ set

$\mathcal{L}_{i+1} = \{L : L \in \mathcal{L}_i$ and for every pair f, g of nowhere equal functions defined on L there is an $L' \in \mathcal{L}_i$ such that $L' \subseteq L$ and $f[L'] \cap g[L'] = \emptyset\}$.

By the choice of $\langle a_j \rangle_{j\in\mathbb{N}}$, every member of $\mathcal{F}$ has a finite subset belonging to $\mathcal{L}_0$. In fact, if $A \in \mathcal{F}$ and $i \in \mathbb{N}$, A has a finite subset belonging to $\mathcal{L}_i$. **P** Induce on i. For the inductive step to $i+1$, **?** suppose, if possible, that $A \in \mathcal{F}$ has no finite subset belonging to $\mathcal{L}_{i+1}$. For each $r \in \mathbb{N}$, $A \cap r \notin \mathcal{L}_{i+1}$, so there are nowhere equal functions f_r, g_r defined on $A \cap r$ such that if $L \subseteq A_r$ is such that $f_r[L] \cap g_r[L]$ is empty, then $L \notin \mathcal{L}_i$. Adjusting f_r, g_r if necessary, we can suppose that $f_r(j) \le 2j$ and $g_r(j) \le 2j+1$ for every $j \in A \cap r$. In this case, there will be functions $f : A \to \mathbb{N}$, $g : A \to \mathbb{N}$ such that for every finite $L \subseteq A$ there is an $r \in \mathbb{N}$ such that $L \subseteq r$, $f\restriction L = f_r\restriction L$ and $g\restriction L = g_r\restriction L$; of course f and g are nowhere equal. Because $\mathcal{F}$ is a Hausdorff ultrafilter, there is an $A' \subseteq A$ such that $A' \in \mathcal{F}$ and $f[A'] \cap g[A'] = \emptyset$. Now A' has a finite subset $L \in \mathcal{L}_i$, by the inductive hypothesis. Take r such that $f_r\restriction L = f\restriction L$ and $g_r\restriction L = g\restriction L$; since $f_r[L] \cap g_r[L] = \emptyset$, $L \notin \mathcal{L}_i$, which is absurd. **XQ**

(iii) I am now in a position to describe a winning strategy for Empty. His first move should be $m \ge 1$ such that $\frac{1}{m} < \epsilon$. Suppose that Non-empty responds with $k \in \mathbb{N}$. By (ii), $\mathcal{L}_k$ is not empty; take $L_0 \in \mathcal{L}_k$. Let $\mathfrak{B}$ be the finite subalgebra of $\mathfrak{B}_\omega$ generated by $\{a_j : j \in L_0\}$; let r be the number of atoms of $\mathfrak{B}$. Let $n \ge r$ be such that $\dfrac{r}{mn} \le \epsilon - \dfrac{1}{m}$ and every atom of $\mathfrak{B}$ has measure at least $\dfrac{1}{mn}$, and let C be a partition of unity in $\mathfrak{B}_\omega$ such that every member of C has measure $\frac{1}{mn}$ and for every atom b of $\mathfrak{B}$ the number of members of C included in b is the maximum possible value $\lfloor mn\bar{\mu}(b)\rfloor$. For $j \in L_0$, set $K_j = \{c : c \in C, c \subseteq a_j\}$; then $\#(K_j) \geq n$ (because the number of members of C not included in any atom of $\mathfrak{B}$ is at most $r \leq mn\epsilon - n$, so $\#(K_j) \geq mn\bar{\mu}(a_j) - (mn\epsilon - n) \geq n$). Take $K'_j \subseteq K_j$ to be a set of size n for each $j \in L_0$; because $n \geq r$, we can suppose that whenever b is an atom of $\mathfrak{B}$ included in a_j, there is a $c \in K'_j$ included in b. Consequently $j \mapsto K'_j : L_0 \to [C]^n$ is injective. Finally, Empty plays (n, C, L'_0) for his second move, where $L'_0 = \{K'_j : j \in L_0\}$.

For subsequent moves, given that $i < k$, Empty has played $L'_i \subseteq L'_0$ and Non-empty has played nowhere-equal functions f'_i, g'_i, the rule for Empty is as follows. The inductive hypothesis will be that $L_i = \{j : K'_j \in L'_i\}$ belongs to $\mathcal{L}_{k-i}$. Define f_i, g_i on L_i by saying that $f_i(j) = f'_i(K'_j)$, $g_i(j) = g'_i(K'_j)$ for $j \in L_i$; because $L_i \in \mathcal{L}_{k-i}$, there is an $L_{i+1} \in \mathcal{L}_{k-i-1}$ such that $L_{i+1} \subseteq L_i$ and $f_i[L_{i+1}] \cap g_i[L_{i+1}]$ is empty. Now Empty plays $L'_{i+1} = \{K'_j : j \in L_{i+1}\}$, and the run continues.

At the end of the run, we get $L_k \in \mathcal{L}_0$. But this means that $\inf_{j\in L_k} a_j = 0$, so $\bigcap_{j\in L_k} K'_j$ must be empty, and Empty has won the run. Thus we have a winning strategy for Empty.

(b) This proves the result on the assumption that $\mathfrak{c} = \omega_1$. But now look at the logical nature of the statement 'Empty has a winning strategy in $G^{\text{H|m-c}}$'. It makes no difference if Empty is required to choose a member of $\mathbb{N}$ for the set B in his second move, following which all Non-empty's moves will have to belong to the countable set $\mathbb{N}^{[[\mathbb{N}]^{<\omega}]^{<\omega}}$, all Empty's moves will be in the countable set $[[\mathbb{N}]^{<\omega}]^{<\omega}$, and the deciding move L_k wins iff it too belongs to a specific countable set (the family of finite subsets of $[\mathbb{N}]^{<\omega}$ with empty intersection). We therefore have in fact a Borel code for the set of winning strategies for Empty. By Shoenfield's theorem (Jech 78, Theorem 98, or Jech 03, 25.20), the assertion that it is non-empty is absolute for inner models of ZFC. Consequently it is absolute for forcing, that is, if $\mathbb{P}$ is any forcing notion, then

$$\text{Empty has a winning strategy in } G^{\text{H|m-c}} \quad \text{iff } \Vdash_{\mathbb{P}} \text{Empty has a winning strategy in } G^{\text{H|m-c}}.$$

Now take any forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \mathfrak{c} = \omega_1;$$

for instance, take $\mathbb{P}$ to be the partially ordered set of functions from countable ordinals to $\mathbb{R}$, active upwards. Then we shall have

$$\Vdash_{\mathbb{P}} \mathfrak{c} = \omega_1, \text{ so Empty has a winning strategy in } G^{\text{H|m-c}},$$

and it follows that Empty has a winning strategy in $G^{\text{H|m-c}}$ in the ordinary universe.

Remark No doubt there is a more illuminating proof of this lemma which does not employ considerations of absoluteness.

6O Proof of Theorem 1Mi If $\mathfrak{p} = \mathfrak{c}$, there is a Hausdorff p-point ultrafilter on $\mathbb{N}$ which is not measure-centering.

proof (a) By Lemma 6N, Empty has a winning strategy in $G^{\text{H|m-c}}$; let $m \geq 1$ be such that Empty has a winning strategy with first move m. For each $n \in \mathbb{N}$, fix a set B_n of size mn. For $n \in \mathbb{N}$, set $I_n = [B_n]^n$; set $I = \bigcup_{n\in\mathbb{N}} I_n$,

$$\mathcal{L}_0 = \bigcup_{n\geq 1}\{L : L \subseteq I_n, \bigcap L = \emptyset\},$$

and for $i \in \mathbb{N}$ set

$$\mathcal{L}_{i+1} = \{L : L \in \mathcal{L}_i \text{ and whenever } f, g \text{ are nowhere equal functions defined on } L, \text{ there is an } L' \in \mathcal{L}_i \text{ such that } L' \subseteq L \text{ and } f[L'] \cap g[L'] = \emptyset\}.$$

An easy induction shows that if i, $n \in \mathbb{N}$, $L \subseteq L' \subseteq I_n$ and $L \in \mathcal{L}_i$, then $L' \in \mathcal{L}_i$.

For $L \in \mathcal{L}_0$, say that the **depth** $\text{depth}(L)$ of L is the greatest k such that $L \in \mathcal{L}_k$; observe that $\text{depth}(L) \leq \#(L)$ for every L. (The point is that every member of $\mathcal{L}_0$ has at least two members, so that for every $L \in \mathcal{L}_0$ there are nowhere equal functions f and g defined on L such that if $L' \subseteq L$ and $f[L']$ is disjoint from $g[L']$, L' is a proper subset of L.) Say that $A \subseteq I$ is **deep** if $\sup_{n\ge 1, A\cap I_n\in\mathcal{L}_0} \operatorname{depth}(A\cap I_n)$ is infinite. Note that if $A \subseteq B \subseteq I$, then $\operatorname{depth}(B\cap I_n) \ge \operatorname{depth}(A\cap I_n)$ whenever the latter is defined, so B will be deep if A is.

Now let σ be a winning strategy for Empty in $G^{\text{H|m-c}}$ with first move m; we can suppose that σ is such that whenever Empty plays $(n, B, L_0) = \sigma(k)$ for his second move, he actually chooses $B = B_n$.

(b) Suppose that $(m, k, (n, B_n, L_0), (f_0, g_0), L_1, (f_1, g_1), \dots , L_i)$, where $i \le k$, is any partial run of the game in which Empty follows the strategy σ. Then $L_i \in \mathcal{L}_{k-i}$. **P** Induce downwards on i, starting with $i = k$. At the end of the run, Empty must win, so we certainly have L_k a subset of $L_0 \subseteq I_n$ with empty intersection, in which case $L_k \in \mathcal{L}_0$. For the inductive step down to $i < k$, given that Empty has just played L_i, let f, g be any nowhere-equal functions defined on L_i. Then Empty will reply with $L_{i+1} = \sigma(k, (f_0, g_0), \dots , (f, g)) \subseteq L_i$ such that $f[L_{i+1}] \cap g[L_{i+1}] = \emptyset$; by the inductive hypothesis, $L_{i+1} \in \mathcal{L}_{k-i-1}$; as f and g are arbitrary, $L_i \in \mathcal{L}_{k-i}$, as required. **Q**

It follows that I is deep. **P** For every $k \in \mathbb{N}$ there is a partial run, following Empty's strategy, of the form $(m, k, (n, B_n, L_0))$, where $(n, B_n, L_0) = \sigma(k)$, so that $L_0 \in \mathcal{L}_k$ and the depth of I_n is at least k. **Q**

(c) If $A \subseteq I$ is deep and $A' \subseteq A$, then at least one of A', $A \setminus A'$ is deep. **P** Define f, $g : I \to \{0, 1\}$ by setting

$$\begin{aligned} f(a) &= 1 \text{ if } a \in A', \\ &= 0 \text{ otherwise}, \\ g(a) &= 1 - f(a) \text{ for every } a \in I. \end{aligned}$$

For every k, there is an $n_k \in \mathbb{N}$ such that $A\cap I_{n_k} \in \mathcal{L}_{k+1}$. Since f and g are nowhere equal on $A\cap I_{n_k}$, there is an $L_k \subseteq A \cap I_{n_k}$ such that $L_k \in \mathcal{L}_k$ and $f[L_k] \cap g[L_k] = \emptyset$, that is, L_k is either included in A' or disjoint from A'. If $L_k \subseteq A'$ for infinitely many k, then A' is deep; otherwise, $L_k \subseteq A \setminus A'$ for infinitely many k and $A \setminus A'$ is deep. **Q**

(d) If $A \subseteq I$ is deep and f, g are nowhere equal functions defined on I, then there is a deep $A' \subseteq A$ such that $f[A']$ and $g[A']$ are disjoint. **P** (i) If there is some j such that $A \cap f^{-1}[\{j\}]$ is deep, we can take this for A'; similarly, if there is some j such that $A \cap g^{-1}[\{j\}]$ is deep, this will serve for A'. By (c), we can therefore restrict our attention to the case in which $A \setminus (f^{-1}[M] \cup g^{-1}[M])$ is deep for every finite set M. (ii) Choose $\langle n_k\rangle_{k\in\mathbb{N}}$, $\langle M_k\rangle_{k\in\mathbb{N}}$ inductively in such a way that, for each $k \in \mathbb{N}$,

$M_k = \bigcup_{i<k} f[I_{n_i}] \cup g[I_{n_i}]$,

$n_k > n_i$ whenever $i < k$,

$\operatorname{depth}(A \cap I_{n_k} \setminus (f^{-1}[M_k] \cup g^{-1}[M_k])) \ge k + 1$.

Then we have for each k an $L_k \subseteq A \cap I_{n_k} \setminus (f^{-1}[M_k] \cup g^{-1}[M_k])$ such that $\operatorname{depth}(L_k) \ge k$ and $f[L_k]$ is disjoint from $g[L_k]$. Setting $A' = \bigcup_{k\in\mathbb{N}} L_k$, we see that $A' \subseteq A$ is deep and $f[A'] \cap g[A'] = \emptyset$. **Q**

(e) If $\xi < \mathfrak{p}$ and $\langle A_\eta\rangle_{\eta<\xi}$ is a family of deep subsets of I such that $A_\eta \setminus A_\zeta$ is finite whenever $\zeta \le \eta < \xi$, then there is a deep set $A \subseteq I$ such that $A \setminus A_\eta$ is finite for every $\eta < \xi$. **P** Let P be the set of pairs (J, D) where $J \subseteq I$, $D \subseteq \xi$ are finite, ordered by saying that $(J, D) \le (J', D')$ if $J \subseteq J'$, $D \subseteq D'$ and $J' \setminus J \subseteq A_\eta$ for every $\eta \in D$. Then P is a partially ordered set, σ-centered upwards. For every $k \in \mathbb{N}$, $Q_k = \{(J, D) : (J, D) \in P$, there is an $n \in \mathbb{N}$ such that $\operatorname{depth}(J \cap I_n) \ge k\}$ is cofinal with P, because if $D \subseteq \xi$ is finite then $I \cap \bigcap_{\eta\in D} A_\eta$ is deep. For every $\eta < \xi$, $Q'_\eta = \{(J, D) : (J, D) \in P, \eta \in D\}$ is cofinal with P. By Bell's theorem (Fremlin 84, 14C) there is an upwards-directed $R \subseteq P$ meeting every Q_k and every Q'_η; set $A = \bigcup\{J : (J, D) \in R\}$. **Q**

(f) Let $\langle(f_\xi, g_\xi)\rangle_{\xi<\mathfrak{c}}$ enumerate the set of pairs (f, g) of nowhere equal functions from I to $\mathbb{N}$. Because $\mathfrak{p} = \mathfrak{c}$, we can use (d) and (e) to find a family $\langle A_\xi\rangle_{\xi<\mathfrak{c}}$ of deep subsets of I such that

$A_\xi \setminus A_\eta$ is finite whenever $\eta \le \xi < \mathfrak{c}$,

$f_\xi[A_{\xi+1}]$ is disjoint from $g_\xi[A_{\xi+1}]$ for every $\xi < \mathfrak{c}$.

Let $\mathcal{F}$ be the filter on I generated by $\{A_\xi \setminus J : \xi < \mathfrak{c}, J \in [I]^{<\omega}\}$. If f, g are any two nowhere-equal functions on I, then (because $f[I] \cup g[I]$ must be countable) there is a $\xi < \mathfrak{c}$ such that for all a, $b \in I$, $f(a) = g(b)$ iff

$f_\xi(a)=g_\xi(b)$; so $f[A_{\xi+1}]\cap g[A_{\xi+1}]=\emptyset$. It follows that $\mathcal{F}$ is a Hausdorff ultrafilter. (It is an ultrafilter by the argument in (c) above.) By construction, $\mathcal{F}$ is a p-point ultrafilter.

(g) $\mathcal{F}$ is not measure-centering. **P** Let $(\mathfrak{A},\bar\mu)$ be any atomless probability algebra. For each $n\in\mathbb{N}$ let $\pi_n:\mathcal{P}B_n\to\mathfrak{A}$ be a Boolean homomorphism which is measure-preserving for the uniform probability measure on $\mathcal{P}B_n$. For $n\in\mathbb{N}$ and $a\in I_n$ set $d_a=\pi_n a$; then $\bar\mu d_a=\dfrac{\#(a)}{\#(B_n)}=\dfrac{1}{m}$. If $A\in\mathcal{F}$, there is a $\xi<\mathfrak{c}$ such that $A_\xi\setminus A$ is finite, so A is deep; in particular, there is some $n\in\mathbb{N}$ such that $A\cap I_n\in\mathcal{L}_0$, that is, $A\cap I_n$ has empty intersection. But now $\{d_a:a\in A\}\supseteq\pi_n[A\cap I_n]$ is not centered. So $\langle d_a\rangle_{a\in I}$ witnesses that $\mathcal{F}$ is not measure-centering. **Q**

Transferring $\mathcal{F}$ from the countably infinite set I to $\mathbb{N}$, we have the required example.

7 Problems

7A If Martin's axiom is true, the constructions used in the proof of Theorem 1B, given in 3F-3G, and in the proof of existence of Ramsey ultrafilters in Fremlin 08, provide three essentially different classes of free measure-centering ultrafilters on $\mathbb{N}$, and further measure-centering ultrafilters on κ for $\omega<\kappa<\mathfrak{c}$; moreover, since there are $2^{\mathfrak{c}}$ Ramsey ultrafilters (Fremlin 84, 26Ed), we get further measure-centering ultrafilters on $\mathbb{N}$ from Theorem 1J. I know of no construction for a uniform measure-centering ultrafilter on $\mathfrak{c}$. Indeed, Martin's axiom ensures that there is no such ultrafilter, for the following reason. A cardinal κ is a **measure-precaliber of probability algebras** if whenever $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\bar\mu a_\xi>0$, there is a set $J\subseteq\kappa$, of cardinal κ, such that $\{a_\xi:\xi\in J\}$ is centered. If $\mathcal{F}$ is a uniform measure-centering ultrafilter on κ, there will always be such a set belonging to $\mathcal{F}$, so κ will certainly be a measure-precaliber of probability algebras. Now Martin's axiom (or much less) ensures that $\mathfrak{c}$ is not a measure-precaliber of probability algebras (see Fremlin 08, 525D and 525O); so it is certainly consistent to suppose that there is no uniform measure-centering ultrafilter on $\mathfrak{c}$. On the other hand, it is also consistent to suppose that $\mathfrak{c}$ is a measure-precaliber of probability algebras (this happens, for instance, if $\mathfrak{c}=\omega_2$ and there is a subset of $\mathbb{R}$ of cardinal ω_1 which is not Lebesgue negligible; see Fremlin 08, 525L). So my question is: is it consistent with ZFC to suppose that there is a uniform measure-centering ultrafilter on $\mathfrak{c}$? Could it even be consistent to suppose that for every infinite cardinal κ there is a uniform measure-centering ultrafilter on κ?

7B I have been unable to answer the following question from Andreas Blass: is a weakly Ramsey ultrafilter on $\mathbb{N}$ necessarily measure-centering?

7C Under what circumstances is the product $\mathcal{F}\ltimes\mathcal{G}$ of two measure-centering ultrafilters measure-centering? For instance, when the product is Hausdorff, will it always be measure-centering?

7D Is every Hausdorff ultrafilter measure-linking?

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