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Invertive extensions

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1 Basics

1A Definition Let $(\mathfrak{A},\bar\mu)$ be a probability algebra and Φ a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. By an **invertive extension** of $(\mathfrak{A},\bar\mu,\Phi)$ I will mean a structure $(\mathfrak{C},\bar\lambda,\pi,\langle\tilde\phi\rangle_{\phi\in\Phi})$ such that

$(\mathfrak{C},\bar\lambda)$ is a probability algebra,

$\pi:\mathfrak{A}\to\mathfrak{C}$ is a measure-preserving Boolean homomorphism,

$\langle\tilde\phi\rangle_{\phi\in\Phi}$ is a family of measure-preserving Boolean automorphisms of $\mathfrak{C}$,

$(\phi\psi)^{\sim}=\tilde\phi\tilde\psi$ for all ϕ, $\psi\in\Phi$,

$\tilde\phi\pi=\pi\phi$ for every $\phi\in\Phi$;

that is, π embeds $\mathfrak{A}$ as a closed subalgebra of $\mathfrak{C}$ in such a way that the Boolean homomorphisms in Φ can be simultaneously and consistently extended to automorphisms of $\mathfrak{C}$.

1B The problem When can we expect such an extension to exist?

When $\Phi=\{\phi^n:n\in\mathbb{N}\}$ for some ϕ, we are looking for a 'natural extension' in the sense of Fremlin 02, 383Yc; when expressed in terms of inverse-measure-preserving functions on probability spaces, this is a classical construction (Petersen 83, 1.3G).

An obviously necessary condition is that Φ should be **right-cancellative** in the sense that if ϕ, ϕ', $\psi\in\Phi$ and $\phi\psi=\phi'\psi$ then $\phi=\phi'$. (Because Φ consists of injective functions, it is necessarily left-cancellative.)

Note that, writing ι for the identity automorphism of $\mathfrak{A}$, $(\mathfrak{A},\bar\mu,\Phi)$ has an invertive extension iff $(\mathfrak{A},\bar\mu,\Phi\cup\{\iota\})$ has an invertive extension. **P** Of course we need consider only the case in which $\iota\notin\Phi$. (i) If $(\mathfrak{A},\bar\mu,\Phi)$ has an invertive extension $(\mathfrak{C},\bar\lambda,\pi,\langle\tilde\phi\rangle_{\phi\in\Phi})$, let $\tilde\iota$ be the identity automorphism of $\mathfrak{C}$; then $(\mathfrak{C},\bar\lambda,\pi,\langle\tilde\phi\rangle_{\phi\in\Phi\cup\{\iota\}})$ is an invertive extension of $(\mathfrak{A},\bar\mu,\Phi\cup\{\iota\})$. (ii) The reverse implication is at least equally elementary. **Q**

When $(\mathfrak{C},\bar\lambda,\pi,\langle\tilde\phi\rangle_{\phi\in\Phi})$ is an invertive extension of $(\mathfrak{A},\bar\mu,\Phi)$, the function $\phi\mapsto\tilde\Phi$ is a semigroup homomorphism from Φ to the group $\mathrm{Aut}_{\bar\lambda}\,\mathfrak{C}$ of measure-preserving Boolean automorphisms of $\mathfrak{C}$. It follows that if Φ has an identity ι (which is necessarily the identity automorphism of $\mathfrak{A}$, since it is injective and idempotent), then $\tilde\iota$ is an idempotent in $\mathrm{Aut}_{\bar\lambda}(\mathfrak{C})$, and must be the identity automorphism of $\mathfrak{C}$. Similarly, if ϕ, ψ form an inverse pair in Φ (that is, $\phi\psi=\iota=\psi\phi$ is the identity in Φ), we must have $\tilde\phi\tilde\psi=\tilde\psi\tilde\phi=\tilde\iota$ and $\tilde\psi=\tilde\phi^{-1}$ in $\mathrm{Aut}_{\bar\lambda}(\mathfrak{C})$.

2 Amenable semigroups

2A Definition (see Fremlin 03, 449Ya) If S is a semigroup with identity e and X is a set, an **action** of S on X is a map $(s,x)\mapsto s\bullet x:S\times X\to X$ such that $s\bullet(t\bullet x)=(st)\bullet x$ and $e\bullet x=x$ for every s, $t\in S$ and $x\in X$. A topological semigroup S with identity is **amenable** if for every non-empty compact Hausdorff space X and every continuous action of S on X there is a Radon probability measure μ on X such that $\int f(s\bullet x)\mu(dx)=\int f(x)\mu(dx)$ for every $s\in S$ and $f\in C(X)$.

2B Proposition Abelian topological semigroups with identity are amenable.

proof (a) Let S be an abelian topological semigroup with identity e, and $\bullet$ a continuous action of S on a non-empty compact Hausdorff space X. Let $P_{\mathrm R}(X)$ be the compact Hausdorff space of Radon probability measure on X with the narrow topology (Fremlin 03, 437O). For $s\in S$ and $x\in X$, set $\hat s(x)=s\bullet x$, so that $\hat s:X\to X$ is continuous. For $\mu\in P_{\mathrm R}(X)$ and $s\in S$, the image measure $\mu\hat s^{-1}$ belongs to $P_{\mathrm R}(X)$ (Fremlin 03, 418I); call it $s\hat\bullet\mu$. If s, $t\in S$ then $\widehat{st}=\hat s\hat t$, so

$$s\hat{\bullet}(t\hat{\bullet}\mu) = s\hat{\bullet}(\mu\hat{t}^{-1}) = (\mu\hat{t}^{-1})\hat{s}^{-1} = \mu(\hat{s}\hat{t})^{-1}$$

(Fremlin 01, 234Ec[1])

$$= \mu(\widehat{st})^{-1} = (st)\hat{\bullet}\mu;$$

and of course $\hat{e}$ is the identity function on X, so $\hat{e}\hat{\bullet}\mu = \mu$. Thus $\hat{\bullet}$ is an action of S on $P_{\mathrm{R}}(X)$.

Note that if $f \in C(X)$, $\mu \in P_{\mathrm{R}}(X)$ and $s \in S$, then

$$\int f d(s\hat{\bullet}\mu) = \int f d(\mu\hat{s}^{-1}) = \int f(\hat{s}(x))\mu(dx) = \int f(s\bullet x)\mu(dx).$$

So we are looking for a fixed point in $P_{\mathrm{R}}(X)$ under the action $\hat{\bullet}$.

(b) $\hat{\bullet} : S \times P_{\mathrm{R}}(X) \to P_{\mathrm{R}}(X)$ is continuous. **P** Suppose that $s \in S$, $\mu \in P_{\mathrm{R}}(X)$, $G \subseteq X$ is open and $(s\hat{\bullet}\mu)(G) > \alpha$ in $\mathbb{R}$. Let H be an open set such that $\mu H > \alpha$ and $\overline{H} \subseteq \hat{s}^{-1}[G]$, that is, $s\bullet x \in G$ for every $x \in \overline{H}$. Because $\overline{H}$ is compact, G is open and $\bullet$ is continuous, there is a neighbourhood U of s in S such that $t\bullet x \in G$ whenever $t \in U$ and $x \in \overline{H}$. Next, by the definition of the narrow topology, there is a neighbourhood V of μ in $P_{\mathrm{R}}(X)$ such that $\nu H > \alpha$ for every $\nu \in V$. If $t \in U$ and $\nu \in V$, then

$$(t\hat{\bullet}\nu)(G) = \nu\hat{t}^{-1}[G] \ge \nu H > \alpha.$$

As G and α are arbitrary, $\hat{\bullet}$ is continuous at (s, μ). **Q**

(c) Let $K \subseteq P_{\mathrm{R}}(X)$ be a minimal non-empty compact convex set such that $s\hat{\bullet}\mu \in K$ whenever $\mu \in K$ and $s \in S$. Then for any $s \in S$ there is a $\mu \in K$ such that $s\hat{\bullet}\mu = \mu$. **P** Start from any $\mu_0 \in K$ and take a cluster point of $\langle \frac{1}{n+1}\sum_{i=0}^{n} s^i\hat{\bullet}\mu_0\rangle_{n\in\mathbb{N}}$. **Q** At this point, recall that we are supposing that S is abelian. So if we set $K_s = \{\mu : \mu \in K,\ s\hat{\bullet}\mu = \mu\}$, we shall have $t\hat{\bullet}\mu \in K_s$ for every $t \in S$ and $\mu \in K_s$. Since also K_s is compact and convex, it must be equal to K. But this means that every member of K is a fixed point under $\hat{\bullet}$.

2C Proposition Let S be a semigroup with identity, endowed with its discrete topology. Then S is amenable iff there is an additive functional $\nu : \mathcal{P}S \to [0,1]$ such that $\nu S = 1$ and $\nu\{t : st \in A\} = \nu A$ whenever $A \subseteq S$ and $s \in S$.

proof (a) Suppose that S is amenable.

(i) Consider its Stone-Čech compactification βS. We can identify βS with the Stone space of $\mathcal{P}S$ (Fremlin 03, 4A2Ib); for $A \subseteq S$ let $\widehat{A}$ be the corresponding open-and-closed set in βS. For s, $t \in S$ set $\hat{s}(t) = st$; then the function $\hat{s} : S \to S$ gives us a Boolean homomorphism $A \mapsto \hat{s}^{-1}[A] : \mathcal{P}S \to \mathcal{P}S$ and a continuous function $\phi_s : \beta S \to \beta S$ such that $\phi_s^{-1}[\widehat{A}] = \widehat{\hat{s}^{-1}[A]}$ for every $A \subseteq S$ (Fremlin 02, 312Q). Now $\phi_{st} = \phi_s\phi_t$ for all s, $t \in S$. **P** If $A \subseteq S$, then

$$\phi_{st}^{-1}[\widehat{A}] = ((\widehat{st})^{-1}[A])\widehat{\ } = ((\hat{s}\hat{t})^{-1}[A])\widehat{\ }$$

(because $(\hat{s}\hat{t})(s') = \hat{s}(\hat{t}(s')) = \hat{s}(ts') = sts' = \widehat{st}(s')$ for every $s' \in S$)

$$= (\hat{t}^{-1}[\hat{s}^{-1}[A]])\widehat{\ } = \phi_t^{-1}[(\hat{s}^{-1}[A])\widehat{\ }] = \phi_t^{-1}[\phi_s^{-1}[\widehat{A}]] = (\phi_s\phi_t)^{-1}[\widehat{A}].$$ **Q**

It follows that if we set $s\bullet z = \phi_s(z)$ for $s \in S$ and $z \in \beta S$, $\bullet$ is an action of S on βS; as S is being given its discrete topology and every ϕ_s is continuous, $\bullet$ is a continuous action.

(ii) There is therefore a Radon probability measure μ on βS which is invariant in the sense that $\int f(s\bullet z)\mu(dz) = \int f\,d\mu$ for every $f \in C(\beta S)$ and $s \in S$. Set $\nu A = \mu(\widehat{A})$ for $A \subseteq S$. Then $\nu : \mathcal{P}S \to [0,1]$ is additive and $\nu S = 1$. Also, given $A \subseteq S$ and $s \in S$,

[1] Formerly Fremlin 00, 112Xd.

$$\nu\{t : st \in A\} = \nu(\hat{s}^{-1}[A]) = \mu(\widehat{\hat{s}^{-1}[A]}) = \mu(\phi_s^{-1}[\widehat{A}]) = \int \chi(\phi_s^{-1}[\widehat{A}])d\mu$$

$$= \int \chi(\widehat{A})(\phi_s(t))\mu(dt) = \int \chi(\widehat{A})(s\bullet t)\mu(dt) = \int \chi(\widehat{A})d\mu = \mu(\widehat{A}) = \nu A.$$

So ν is invariant in the sense required.

(b) Now suppose that there is a functional ν as described, and that $\bullet$ is a continuous action of S on a non-empty compact Hausdorff space X. Take any $x_0 \in S$ and for $f \in C(X)$ set $\theta(f) = ⨍ f(s\bullet x_0)\nu(ds)$, defining $⨍ \ldots d\nu$ as in Fremlin 02, 363L. Then θ is a positive linear functional and $\theta(\chi X) = 1$, so there is a Radon probability measure μ on X such that $\theta(f) = \int f d\mu$ for every $f \in C(X)$ (Fremlin 03, 436J/436K). If $s \in S$ and $f \in C(X)^+$, set $g(x) = f(s\bullet x)$ for $x \in X$; then $g \in C(X)$ and

$$\int f(s\bullet x)\mu(dx) = \int g\,d\mu = \theta(g) = ⨍ g(t\bullet x_0)\nu(dt) = ⨍ f(s\bullet(t\bullet x_0))\nu(dt)$$

$$= ⨍ f(st\bullet x_0)\nu(dt) = \int_0^\infty \nu\{t : f(st\bullet x_0) \ge \alpha\}d\alpha$$

(Fremlin 02, 363Le)

$$= \int_0^\infty \nu\{t : f(t\bullet x_0) \ge \alpha\}d\alpha$$

(because if $A = \{t : f(t\bullet x_0) \ge \alpha\}$ then $\nu\{t : f(st\bullet x_0) \ge \alpha\} = \nu\{t : st \in A\} = \nu A$)

$$= ⨍ f(t\bullet x_0)\nu(dt) = \theta(f) = \int f d\mu.$$

Of course it follows at once that $\int f(s\bullet x)\mu(dx) = \int f d\mu$ for every $f \in C(X)$. As s is arbitrary, μ is invariant in the sense demanded by the definition in §2A. As X and $\bullet$ are arbitrary, S is amenable.

2D Definition I will say that a topological semigroup with identity S is **reverse-amenable** if $(S, \diamond)$ is amenable, where $s \diamond t = ts$ for s, $t \in S$. S is reverse-amenable in its discrete topology iff there is an additive functional $\nu : \mathcal{P}S \to [0, 1]$ such that $\nu S = 1$ and $\nu\{t : ts \in A\} = \nu A$ whenever $A \subseteq S$ and $s \in S$.

3 Sufficient conditions

3A Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and Φ a right-cancellative semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Suppose that $\Phi\phi_0 \cap \ldots \cap \Phi\phi_n$ is non-empty for all $\phi_0, \ldots, \phi_n \in \Phi$. Then $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension $(\mathfrak{C}^*, \bar{\lambda}^*, \pi^*, \langle\tilde{\phi}^*\rangle_{\phi\in\Phi})$ such that whenever $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\phi}\rangle_{\phi\in\Phi})$ is an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$ there is a unique measure-preserving Boolean homomorphism $\bar{\sigma} : \mathfrak{C}^* \to \mathfrak{C}$ such that $\bar{\sigma}\pi^* = \pi$ and $\bar{\sigma}\tilde{\phi}^* = \tilde{\phi}\bar{\sigma}$ for every $\phi \in \Phi$.

proof (a) It will be enough to deal with the case in which the identity automorphism belongs to Φ. Note that $\{\Phi\psi : \psi \in \Phi\}$ is a filter base, because if $\psi \in \Phi\psi_0 \cap \ldots \cap \Phi\psi_n$, then $\Phi\psi \subseteq \Phi\psi_0 \cap \ldots \cap \Phi\psi_n$; let $\mathcal{F}$ be the filter it generates.

(b) In the simple power Boolean algebra $\mathfrak{A}^\Phi$ let $\mathfrak{B}$ be the set of those families $\langle a_\phi\rangle_{\phi\in\Phi}$ with the property that there is a $\psi \in \Phi$ such that $a_{\phi\psi} = \phi a_\psi$ for every $\phi \in \Phi$. Then $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}^\Phi$. **P** Suppose that $\langle a_\phi\rangle_{\phi\in\Phi}$ and $\langle b_\phi\rangle_{\phi\in\Phi}$ belong to $\mathfrak{B}$, and that $*$ is either of the Boolean operations $\cap$, $\triangle$. There are ψ_0, $\psi_1 \in \Phi$ such that

$$a_{\phi\psi_0} = \phi a_{\psi_0}, \quad b_{\phi\psi_1} = \phi b_{\psi_1}$$

for every $\phi \in \Phi$. Now there is a $\psi \in \Phi\psi_0 \cap \Phi\psi_1$; suppose that $\psi = \psi_0'\psi_0 = \psi_1'\psi_1$ where ψ_0', ψ_1' belong to Φ. Then, for any $\phi \in \Phi$,

$$a_{\phi\psi} = a_{\phi\psi_0'\psi_0} = \phi\psi_0' a_{\psi_0} = \phi a_{\psi_0'\psi_0} = \phi a_\psi,$$

and similarly $b_{\phi\psi} = \phi b_\psi$. So

$$a_{\phi\psi} * b_{\phi\psi} = (\phi a_\psi) * (\phi b_\psi) = \phi(a_\psi * b_\psi).$$

As ϕ is arbitrary,

$$\langle a_\phi \rangle_{\phi\in\Phi} * \langle b_\phi \rangle_{\phi\in\Phi} = \langle a_\phi * b_\phi \rangle_{\phi\in\Phi}$$

belongs to $\mathfrak{B}$. Since $\langle 1 \rangle_{\phi\in\Phi} \in \mathfrak{B}$, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}^\Phi$. **Q**

(c) $\lambda(\langle a_\phi \rangle_{\phi\in\Phi}) = \lim_{\phi\to\mathcal{F}} \bar\mu a_\phi$ is defined in $[0,1]$ for every $\langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{B}$. **P** Let $\psi \in \Phi$ be such that $a_{\phi\psi} = \phi a_\psi$ for every $\phi \in \Phi$. Then $\bar\mu a_{\phi\psi} = \bar\mu a_\psi$ for every ϕ. As $\{\phi\psi : \phi \in \Phi\} \in \mathcal{F}$, this is enough to show $\lim_{\phi\to\mathcal{F}} \bar\mu a_\phi = \bar\mu a_\psi$. **Q**

It is easy to see that $\lambda : \mathfrak{B} \to [0,1]$ is additive, and that $\lambda 1_{\mathfrak{B}} = 1$.

(d)(i) For $\theta \in \Phi$, define $\hat\theta : \mathfrak{A}^\Phi \to \mathfrak{A}^\Phi$ by setting $\hat\theta(\langle a_\phi \rangle_{\phi\in\Phi}) = \langle a_{\phi\theta} \rangle_{\phi\in\Phi}$ whenever $\langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{A}^\Phi$. It is easy to see that $\hat\theta$ is a Boolean homomorphism.

(ii) $\hat\theta\hat\psi = \widehat{\theta\psi}$ for all θ, $\psi \in \Phi$. **P** If $\langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{A}^\Phi$, then

$$\hat\theta\hat\psi(\langle a_\phi \rangle_{\phi\in\Phi}) = \hat\theta(\langle a_{\phi\psi} \rangle_{\phi\in\Phi}) = \langle a_{\phi\theta\psi} \rangle_{\phi\in\Phi} = \widehat{\theta\psi}(\langle a_\phi \rangle_{\phi\in\Phi}). \quad \mathbf{Q}$$

(iii) $\hat\theta[\mathfrak{B}] \subseteq \mathfrak{B}$ for every $\theta \in \Phi$. **P** If $\langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{B}$, let $\psi \in \Phi$ be such that $a_{\phi\psi} = \phi a_\psi$ for every $\phi \in \Phi$. Then $\Phi\psi \cap \Phi\theta$ is not empty; suppose that ψ_0, $\psi_1 \in \Phi$ are such that $\psi_0\psi = \psi_1\theta$. In this case, setting $b_\phi = a_{\phi\theta}$ for $\phi \in \Phi$,

$$\begin{aligned} b_{\phi\psi_1} &= a_{\phi\psi_1\theta} = a_{\phi\psi_0\psi} = \phi\psi_0 a_\psi \\ &= \phi a_{\psi_0\psi} = \phi a_{\psi_1\theta} = \phi b_{\psi_1}. \end{aligned}$$

As ϕ is arbitrary, $\langle b_\phi \rangle_{\phi\in\Phi} = \hat\theta(\langle a_\phi \rangle_{\phi\in\Phi})$ belongs to $\mathfrak{B}$. **Q**

(iv) If $\theta \in \Phi$, then $\lambda\hat\theta = \lambda$. **P** Take $\boldsymbol{a} = \langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{B}$; set $\alpha = \lambda\boldsymbol{a}$ and $\langle b_\phi \rangle_{\phi\in\Phi} = \hat\theta\boldsymbol{a}$. There is a $\psi \in \Phi$ such that $\bar\mu a_{\phi\psi} = \alpha$ for every $\phi \in \Phi$ (see (c) above). Once again, take ψ_0, $\psi_1 \in \Phi$ such that $\psi_0\psi = \psi_1\theta$. Then

$$\bar\mu b_{\phi\psi_1} = \bar\mu a_{\phi\psi_1\theta} = \bar\mu a_{\phi\psi_0\psi} = \alpha$$

for every $\phi \in \Phi$. Thus $\lambda(\langle b_\phi \rangle_{\phi\in\Phi}) = \alpha$, that is, $\lambda\hat\theta\boldsymbol{a} = \lambda\boldsymbol{a}$. **Q**

(v) Define $\hat\pi : \mathfrak{A} \to \mathfrak{A}^\Phi$ by setting $\hat\pi a = \langle \phi a \rangle_{\phi\in\Phi}$ for every $a \in \mathfrak{A}$. Then $\hat\pi$ is a Boolean homomorphism. If $a \in \mathfrak{A}$, then $\hat\pi a \in \mathfrak{B}$, $\lambda\hat\pi a = \bar\mu a$ and

$$\hat\theta\hat\pi a = \hat\theta(\langle \phi a \rangle_{\phi\in\Phi}) = \langle \phi\theta a \rangle_{\phi\in\Phi} = \hat\pi\theta a$$

for every $\theta \in \Phi$. So $\hat\theta\hat\pi = \hat\pi\theta$ for every $\theta \in \Phi$.

(e)(i) Taking $\mathcal{I}$ to be the ideal $\{\boldsymbol{b} : \boldsymbol{b} \in \mathfrak{B},\ \lambda\boldsymbol{b} = 0\}$, we have a quotient algebra $\mathfrak{C}_0^* = \mathfrak{B}/\mathcal{I}$ with a strictly positive additive functional $\bar\lambda_0^*$ defined by setting $\bar\lambda_0^*\boldsymbol{b}^\bullet = \lambda\boldsymbol{b}$ for every $\boldsymbol{b} \in \mathfrak{B}$. Because $\lambda\hat\theta = \lambda$ for $\theta \in \Phi$, we have Boolean homomorphisms $\theta^* : \mathfrak{C}_0^* \to \mathfrak{C}_0^*$ defined by setting $\theta^*\boldsymbol{b}^\bullet = (\hat\theta\boldsymbol{b})^\bullet$ for every $\boldsymbol{b} \in \mathfrak{B}$, and $\bar\lambda_0^*\theta^* = \bar\lambda_0^*$ for every θ. If θ, $\psi \in \Phi$, then $(\theta\psi)^* = \theta^*\psi^*$ because $\widehat{\theta\psi} = \hat\theta\hat\psi$. Setting $\pi^* a = (\hat\pi a)^\bullet$ for $a \in \mathfrak{A}$, $\pi^* : \mathfrak{A} \to \mathfrak{C}_0^*$ is a Boolean homomorphism, $\bar\lambda_0^*\pi^* = \bar\mu$ and $\theta^*\pi^* = \pi^*\theta$ for every $\theta \in \Phi$.

(ii) (The key.) For every $\theta \in \Phi$, $\theta^* : \mathfrak{C}_0^* \to \mathfrak{C}_0^*$ is surjective. **P** If $c \in \mathfrak{C}_0^*$, let $\boldsymbol{a} = \langle a_\phi \rangle_{\phi\in\Phi} \in \mathfrak{B}$ be such that $c = \boldsymbol{a}^\bullet$. Define $\boldsymbol{b} = \langle b_\phi \rangle_{\phi\in\Phi}$ by saying that

$$\begin{aligned} b_\phi &= a_\psi \text{ if } \phi = \psi\theta, \\ &= 0 \text{ if } \phi \in \Phi \setminus \Phi\psi; \end{aligned}$$

this definition is acceptable because Φ is right-cancellative, so that $\psi \mapsto \psi\theta$ is injective. In this case, $b_{\phi\theta} = a_\phi$ for every ϕ, so $\hat\theta\boldsymbol{b} = \boldsymbol{a}$. We know that there is a $\psi \in \Phi$ such that $a_{\phi\psi} = \phi a_\psi$ for every $\phi \in \Phi$; in this case,

$$b_{\phi\psi\theta} = a_{\phi\psi} = \phi a_\psi = \phi b_{\psi\theta}$$

for every ϕ. Thus $\boldsymbol{b} \in \mathfrak{B}$. Now

$$\theta^*\boldsymbol{b}^\bullet = (\hat{\theta}\boldsymbol{b})^\bullet = \boldsymbol{a}^\bullet = c.$$

As c is arbitrary, θ^* is surjective. **Q**

(f)(i) Taking $\mathfrak{C}^*$ to be the metric completion of $\mathfrak{C}_0^*$, with the corresponding continuous extension $\bar{\lambda}^*$ of λ^*, $(\mathfrak{C}^*, \bar{\lambda}^*)$ is a probability algebra (Fremlin 02, 392H[2]). The Boolean homomorphisms $\theta^* : \mathfrak{C}_0^* \to \mathfrak{C}_0^*$, for $\theta \in \Phi$, extend continuously to measure-preserving Boolean homomorphisms $\tilde{\theta}^* : \mathfrak{C}^* \to \mathfrak{C}^*$. Of course $\pi^* : \mathfrak{A} \to \mathfrak{C}_0^*$ can now be regarded as a measure-preserving Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{C}^*$. Because $\theta^*\psi^* = (\theta\psi)^*$ and $\theta^*\pi^* = \pi^*\theta$, $\tilde{\theta}^*\tilde{\psi}^* = (\widetilde{\theta\psi})^*$ and $\tilde{\theta}^*\pi = \pi^*\theta$ for all θ, $\psi \in \Phi$. If $\theta \in \Phi$, $\tilde{\theta}^*[\mathfrak{C}^*]$ is a closed subalgebra of $\mathfrak{C}^*$ (Fremlin 02, 324Kb) including the topologically dense subalgebra $\mathfrak{C}_0^*$, so it is the whole of $\mathfrak{C}^*$; thus $\tilde{\theta}^*$ is surjective, therefore a measure-preserving automorphism.

So $(\mathfrak{C}^*, \bar{\lambda}^*, \pi^*, \langle\tilde{\phi}^*\rangle_{\phi\in\Phi})$ is an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$.

(ii) $\mathfrak{C}_0^* = \bigcup_{\theta\in\Phi}(\tilde{\theta}^*)^{-1}[\pi^*[\mathfrak{A}]]$. **P** If $c \in \mathfrak{C}_0^*$, it is of the form $\boldsymbol{a}^\bullet$ where $\boldsymbol{a} = \langle a_\phi\rangle_{\phi\in\Phi} \in \mathfrak{B}$. Let $\theta \in \Phi$ be such that $a_{\phi\theta} = \phi a$ for every $\phi \in \Phi$, where $a = a_\theta$. In this case,

$$\begin{aligned}\tilde{\theta}^* c\theta^*\boldsymbol{a}^\bullet &= \theta^*\boldsymbol{a}^\bullet = (\hat{\theta}\boldsymbol{a})^\bullet \\ &= \langle a_{\phi\theta}\rangle^\bullet_{\phi\in\Phi} = \langle\phi a\rangle^\bullet_{\phi\in\Phi} = (\hat{\pi}a)^\bullet = \pi^* a,\end{aligned}$$

so

$$c = (\tilde{\theta}^*)^{-1}(\pi^* a) \in (\tilde{\theta}^*)^{-1}[\pi^*[\mathfrak{A}]]. \textbf{ Q}$$

(g)(i) Now let $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\phi}\rangle_{\phi\in\Phi})$ be another invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$. In this case, for any $\boldsymbol{a} = \langle a_\phi\rangle_{\phi\in\Phi} \in \mathfrak{B}$, $\sigma\boldsymbol{a} = \lim_{\phi\to\mathcal{F}} \tilde{\phi}^{-1}\pi a_\phi$ is defined in $\mathfrak{C}$, and in fact there is a $\psi \in \Phi$ such that $\sigma\boldsymbol{a} = \tilde{\phi}^{-1}\pi a_\phi$ for every $\phi \in \Phi\psi$. **P** There is a $\psi \in \Phi$ such that $a_{\phi\psi} = \phi a_\psi$ for every $\phi \in \Phi$. In this case

$$\begin{aligned}(\widetilde{\phi\psi})^{-1}\pi a_{\phi\psi} &= \tilde{\psi}^{-1}\tilde{\phi}^{-1}\pi\phi a_\psi \\ &= \tilde{\psi}^{-1}\tilde{\phi}^{-1}\tilde{\phi}\pi a_\psi = \tilde{\psi}^{-1}\pi a_\psi\end{aligned}$$

for every $\phi \in \Phi$, and $\sigma\boldsymbol{a} = \tilde{\psi}^{-1}\pi a_\psi$. **Q**

Evidently $\sigma : \mathfrak{B} \to \mathfrak{C}$ is a Boolean homomorphism. If $a \in \mathfrak{A}$, then

$$\sigma\hat{\pi}a = \lim_{\phi\to\mathcal{F}} \tilde{\phi}^{-1}\pi\phi a = \lim_{\phi\to\mathcal{F}} \tilde{\phi}^{-1}\tilde{\phi}\pi a = \pi a.$$

If $\boldsymbol{a} = \langle a_\phi\rangle_{\phi\in\Phi} \in \mathfrak{B}$, then

$$\begin{aligned}\bar{\lambda}\sigma\boldsymbol{a} &= \bar{\lambda}(\lim_{\phi\to\mathcal{F}} \tilde{\phi}^{-1}\pi a_\phi) = \lim_{\phi\to\mathcal{F}} \bar{\lambda}(\tilde{\phi}^{-1}\pi a_\phi) \\ &= \lim_{\phi\to\mathcal{F}} \bar{\lambda}(\pi a_\phi) = \lim_{\phi\to\mathcal{F}} \bar{\mu}a_\phi = \lambda\boldsymbol{a}\end{aligned}$$

(all the limits here being of eventually-constant functions, so we do not even need to appeal to continuity). So $\bar{\lambda}\sigma = \lambda$.

(ii) If $\theta \in \Phi$, then $\tilde{\theta}\sigma = \sigma\hat{\theta}$. **P** Take $\boldsymbol{a} = \langle a_\phi\rangle_{\phi\in\Phi} \in \mathfrak{B}$. Then

$$\sigma\hat{\theta}\boldsymbol{a} = \sigma(\langle a_{\phi\theta}\rangle_{\phi\in\Phi}) = \lim_{\phi\to\mathcal{F}} \tilde{\phi}^{-1}\pi a_{\phi\theta},$$

so

$$\tilde{\theta}^{-1}\sigma\hat{\theta}\boldsymbol{a} = \lim_{\phi\to\mathcal{F}} \tilde{\theta}^{-1}\tilde{\phi}^{-1}\pi a_{\phi\theta} = \lim_{\phi\to\mathcal{F}} (\widetilde{\phi\theta})^{-1}\pi a_{\phi\theta}.$$

By (g-i) and (a) we can find a $\psi \in \Phi$ such that

$$\sigma\boldsymbol{a} = \tilde{\phi}^{-1}\pi a_\phi, \quad \tilde{\theta}^{-1}\sigma\hat{\theta}\boldsymbol{a} = (\widetilde{\phi\theta})^{-1}\pi a_{\phi\theta}$$

whenever $\phi \in \Phi\psi$. At this point observe that $\Phi\psi$ and $\Phi\psi\theta$ meet, so there are ψ_0, $\psi_1 \in \Phi$ such that $\psi_1\psi\theta = \psi_0\psi$. In this case, taking $\phi = \psi_1\psi$, both ϕ and $\phi\theta$ belong to $\Phi\psi$, and

[2]Formerly 393B.

$$\sigma\boldsymbol{a} = (\widetilde{\phi\theta})^{-1}\pi a_{\phi\theta}, \quad \tilde{\theta}^{-1}\sigma\hat{\theta}\boldsymbol{a} = (\widetilde{\phi\theta})^{-1}\pi a_{\phi\theta},$$

so $\sigma\boldsymbol{a} = \tilde{\theta}^{-1}\sigma\hat{\theta}\boldsymbol{a}$. As $\boldsymbol{a}$ is arbitrary, $\tilde{\theta}\sigma = \sigma\hat{\theta}$. **Q**

(iii) Because $\bar{\lambda}\sigma = \lambda$, σ induces a Boolean homomorphism $\bar{\sigma}_0 : \mathfrak{C}_0^* \to \mathfrak{C}$ defined by saying that $\bar{\sigma}_0\boldsymbol{a}^\bullet = \sigma\boldsymbol{a}$ whenever $\boldsymbol{a} \in \mathfrak{B}$; now

$$\bar{\sigma}_0\pi^* a = \sigma\hat{\pi}a = \pi a$$

for every $a \in \mathfrak{A}$. Next,

$$\bar{\lambda}\bar{\sigma}_0\boldsymbol{a}^\bullet = \bar{\lambda}\sigma\boldsymbol{a} = \lambda\boldsymbol{a} = \bar{\lambda}_0^*\boldsymbol{a}^\bullet,$$

$$\tilde{\theta}\bar{\sigma}_0\boldsymbol{a}^\bullet = \tilde{\theta}\sigma\boldsymbol{a} = \sigma\hat{\theta}\boldsymbol{a} = \bar{\sigma}_0(\hat{\theta}\boldsymbol{a})^\bullet = \bar{\sigma}_0\theta^*\boldsymbol{a}^\bullet$$

whenever $\boldsymbol{a} \in \mathfrak{B}$ and $\theta \in \Phi$, so

$$\bar{\lambda}\bar{\sigma}_0 = \bar{\lambda}_0^*, \quad \tilde{\theta}\bar{\sigma}_0 = \bar{\sigma}_0\theta^*$$

for every $\theta \in \Phi$.

(iv) Because $\bar{\lambda}\bar{\sigma}_0 = \bar{\lambda}_0^*$, and $\mathfrak{C}_0^*$ is a topologically dense subalgebra of $\mathfrak{C}^*$, $\bar{\sigma}_0$ has a unique extension to a measure-preserving Boolean homomorphism $\bar{\sigma} : \mathfrak{C}^* \to \mathfrak{C}$; and we shall have

$$\bar{\sigma}\pi^* a = \pi a, \quad \bar{\lambda}\bar{\sigma} = \bar{\lambda}^*, \quad \tilde{\theta}\bar{\sigma} = \bar{\sigma}_0\tilde{\theta}^*$$

for every $\theta \in \Phi$, by continuity.

(v) Thus $\bar{\sigma}$ has the required properties. To see that it is uniquely defined, recall from (f-ii) that $\bigcup_{\theta\in\Phi}(\tilde{\theta}^*)^{-1}\pi^*[\mathfrak{A}] = \mathfrak{C}_0^*$. Now if $\bar{\sigma}' : \mathfrak{C}^* \to \mathfrak{C}$ is another Boolean homomorphism such that $\bar{\sigma}'\pi^* = \pi$ and $\bar{\sigma}'\tilde{\theta}^* = \tilde{\theta}\bar{\sigma}'$ for every $\theta \in \Phi$, we see that

$$\begin{aligned}\bar{\sigma}'(\theta^*)^{-1}\pi^* &= (\tilde{\theta})^{-1}\tilde{\theta}\bar{\sigma}'(\theta^*)^{-1}\pi^* = (\tilde{\theta})^{-1}\bar{\sigma}'\theta^*(\theta^*)^{-1}\pi^* \\ &= (\tilde{\theta})^{-1}\bar{\sigma}'\pi^* = (\tilde{\theta})^{-1}\pi = \bar{\sigma}(\theta^*)^{-1}\pi^*\end{aligned}$$

for every $\theta \in \Phi$. But this means that $\bar{\sigma}'$ and $\bar{\sigma}$ agree on $\mathfrak{C}_0^*$ and are equal.

This completes the proof.

3B Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and Φ a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself, with identity, which is right-cancellative and reverse-amenable in its discrete topology. Then $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension.

proof Let $\nu : \mathcal{P}\Phi \to [0,1]$ be an additive functional such that $\nu\Phi = 1$ and $\nu\{\phi : \phi\psi \in A\} = \nu A$ whenever $A \subseteq \Phi$ and $\psi \in \Phi$ (2D). Then, in particular, $\nu\Phi = \nu(\Phi\psi)$ for every $\psi \in \Phi$, so $\nu(\Phi\psi_0 \cap \ldots \cap \Phi\psi_n) = 1$ for all $\psi_0, \ldots, \psi_n \in \Phi$, and Φ satisfies the conditions of Theorem 3A.

3C Corollary Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, and Φ a commutative semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Then $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension.

proof Φ is right-cancellative (because it is left-cancellative), and $\Phi\psi_0 \cap \ldots \cap \Phi\psi_n$ contains the product $\psi_0 \ldots \psi_n$ whenever $\psi_0, \ldots, \psi_n \in \Phi$, so again the conditions of Theorem 2E are satisfied.

3D Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, and Φ a semigroup of measure-preserving Boolean automorphisms of $\mathfrak{A}$. Suppose that $(\mathfrak{A}, \bar{\mu}, \Psi)$ has an invertive extension for every finitely generated sub-semigroup Ψ of Φ. Then $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension.

proof For $I \in [\Phi]^{<\omega}$ let Ψ_I be the sub-semigroup of Φ generated by I and $(\mathfrak{C}_I, \bar{\lambda}_I, \pi_I, \langle\tilde{\phi}_I\rangle_{\phi\in\Psi_I})$ an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Psi_I)$. For $\phi \in \Phi \setminus \Psi_I$, take $\tilde{\phi}_I$ to be the identity automorphism of $\mathfrak{C}_I$. Let $\mathcal{F}$ be an ultrafilter on $[\Phi]^\omega$ containing $\{J : I \subseteq J \in [\Phi]^{<\omega}\}$ for every $I \in [\Phi]^{<\omega}$. Let $(\mathfrak{C}, \bar{\lambda})$ be the probability algebra reduced product $\prod_{I\in[\Phi]^{<\omega}}(\mathfrak{C}_I, \bar{\lambda}_I)|\mathcal{F}$ (Fremlin 02, 328C[3]).

[3]Later editions only.

For $\phi \in \Phi$ we have a function $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ defined by saying that

$$\tilde{\phi}(\langle c_I \rangle^\bullet_{I \in [\Phi]^{<\omega}}) = \langle \tilde{\phi}_I c_I \rangle^\bullet_{I \in [\Phi]^{<\omega}}$$

whenever $c_I \in \mathfrak{C}_I$ for every finite $I \subseteq \Phi$. Because every $\tilde{\phi}_I$ is a measure-preserving Boolean homomorphism, so is $\tilde{\phi}$. If ϕ, $\psi \in \Phi$, then $\{I : \tilde{\phi}_I \tilde{\psi}_I = (\widetilde{\phi\psi})_I\}$ belongs to $\mathcal{F}$, so $\tilde{\phi}_I \tilde{\psi}_I = (\phi\psi)^{\sim}$. If $\phi \in \Phi$, then $\{I : \tilde{\phi}_I$ is surjective$\}$ belongs to $\mathcal{F}$, so $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ is surjective, therefore is a measure-preserving automorphism. For $a \in \mathfrak{A}$, set $\pi a = \langle \pi_I a \rangle^\bullet_{I \in [\Phi]^{<\omega}}$; then $\pi : \mathfrak{A} \to \mathfrak{C}$ is a measure-preserving Boolean homomorphism. If $\phi \in \Phi$, then $\{I : \pi_I \phi = \phi_I \pi\}$ belongs to $\mathcal{F}$, so $\pi\tilde{\phi} = \tilde{\phi}\pi$.

Thus $(\mathfrak{C}, \bar{\lambda}, \pi, \langle \tilde{\phi} \rangle_{\phi \in \Phi})$ is an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$.

3E Lemma Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and Φ a semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Let $\kappa \geq \max(\omega, \tau(\mathfrak{A}), \#(\Phi))$ be a cardinal, and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ the measure algebra of the usual measure on $\{0,1\}^\kappa$. Suppose that $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension. Then it has an invertive extension $(\mathfrak{C}, \bar{\lambda}, \pi, \langle \tilde{\phi} \rangle_{\phi \in \Phi})$ where $(\mathfrak{C}, \bar{\lambda}, \pi, \varepsilon)$ is the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ (Fremlin 02, 325K).

proof Let $(\mathfrak{C}^*, \bar{\lambda}^*, \pi^*, \langle \tilde{\phi}^* \rangle_{\phi \in \Phi})$ be an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$. Let Ψ^* be the subgroup of the automorphism group $\operatorname{Aut} \mathfrak{C}^*$ generated by $\{\tilde{\phi}^* : \phi \in \Phi\}$, and $\mathfrak{D}$ the closed subalgebra of $\mathfrak{C}^*$ generated by $\bigcup_{\theta \in \Psi^*} \theta[\pi^*[\mathfrak{A}]]$. Then $\tilde{\phi}^* \restriction \mathfrak{D}$ is a measure-preserving Boolean automorphism of $\mathfrak{D}$ for every $\phi \in \Phi$. Let $(\mathfrak{C}, \bar{\lambda}, \varepsilon_0, \varepsilon_1)$ be the probability algebra free product of $(\mathfrak{D}, \bar{\lambda}^* \restriction \mathfrak{D})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$. For $\phi \in \Phi$, let $\tilde{\phi} : \mathfrak{C} \to \mathfrak{C}$ be the measure-preserving Boolean automorphism such that

$$\tilde{\phi}\varepsilon_0 = \varepsilon_0 \tilde{\phi}^* \restriction \mathfrak{D}, \quad \tilde{\phi}\varepsilon_1 = \varepsilon_1.$$

(See the defining universal mapping theorem 325J in Fremlin 02.) If ϕ, $\psi \in \Phi$ then $(\widetilde{\phi\psi})^* = \tilde{\phi}^* \tilde{\psi}^*$, so $(\widetilde{\phi\psi})^* \restriction \mathfrak{D} = (\tilde{\phi}^* \restriction \mathfrak{D})(\tilde{\psi}^* \restriction \mathfrak{D})$ and $(\phi\psi)^{\sim} = \tilde{\phi}\tilde{\psi}$.

Set $\pi = \varepsilon_0 \pi^* : \mathfrak{A} \to \mathfrak{C}$. Then

$$\pi\phi = \varepsilon_0 \pi^* \phi = \varepsilon_0 \tilde{\phi}^* \pi^* = \tilde{\phi} \varepsilon_0 \pi^* = \tilde{\phi}\pi$$

for every $\phi \in \Phi$. So $(\mathfrak{C}, \bar{\lambda}, \pi, \langle \tilde{\phi} \rangle_{\phi \in \Phi})$ is an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$.

Now we come to the point. Consider $\mathfrak{A}' = \pi[\mathfrak{A}]$. This is a closed subalgebra of $\mathfrak{C}$ included in $\mathfrak{D}' = \varepsilon_0[\mathfrak{D}^*]$. If $c \in \mathfrak{C} \setminus \{0\}$, the relative Maharam type $\tau_{\mathfrak{D}'_c}(\mathfrak{C}_c)$ of the principal ideal $\mathfrak{C}_c$ of $\mathfrak{C}$ over $\mathfrak{D}'_c = \{c \cap d : d \in \mathfrak{D}'\}$ is κ (Fremlin 02, 333E). Setting $\mathfrak{A}'_c = \{c \cap a : a \in \mathfrak{A}'\}$,

$$\kappa = \tau_{\mathfrak{D}'_c}(\mathfrak{C}_c) \leq \tau_{\mathfrak{A}'_c}(\mathfrak{C}_c) \leq \tau(\mathfrak{C}_c)$$

(because $\mathfrak{A}'_c \subseteq \mathfrak{D}'_c \subseteq \mathfrak{C}_c$, see Fremlin 02, 333Be)

$$\leq \tau(\mathfrak{C})$$

(Fremlin 02, 332Tb)

$$\leq \max(\omega, \tau(\mathfrak{D}^*), \tau(\mathfrak{B}_\kappa))$$

(Fremlin 02, 334B)

$$= \kappa.$$

Thus $\tau_{\mathfrak{A}'_c}(\mathfrak{C}_c) = \kappa$ for every non-zero $c \in \mathfrak{C}$. But this means that we have a measure algebra isomorphism between $\mathfrak{C}$ and the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$ which makes $\pi : \mathfrak{A} \to \mathfrak{C}$ correspond to the canonical embedding of $\mathfrak{A}$ in the free product (Fremlin 02, 333F(ii)); which is what we needed to know.

3F Theorem Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and Φ a free semigroup of measure-preserving Boolean homomorphisms from $\mathfrak{A}$ to itself. Then $(\mathfrak{A}, \bar{\mu}, \Phi)$ has an invertive extension.

proof Set $\kappa = \max(\omega, \tau(\mathfrak{A}))$, and let $(\mathfrak{C}, \bar{\lambda}, \pi, \varepsilon)$ be the probability algebra free product of $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}_\kappa, \bar{\nu}_\kappa)$. Let $\Theta \subseteq \Phi$ be a set such that Φ is the free semigroup generated by Θ. For $\theta \in \Theta$, let Φ_θ be the semigroup $\{\theta^n : n \geq 1\}$. By 3C, $(\mathfrak{A}, \bar{\mu}, \Phi_\theta)$ has an invertive extension; by 3E, we can base this extension on

$(\mathfrak{C}, \bar{\lambda}, \pi)$, so that there is a measure-preserving automorphism $\tilde{\theta} : \mathfrak{C} \to \mathfrak{C}$ such that $\tilde{\theta}\pi = \pi\theta$. Writing $\mathrm{Aut}_{\bar{\lambda}}\,\mathfrak{C}$ for the group of measure-preserving automorphisms of $\mathfrak{C}$, $\theta \mapsto \tilde{\theta} : \Theta \to \mathrm{Aut}_{\bar{\lambda}}\,\mathfrak{C}$ must extend to a semigroup homomorphism $\phi \mapsto \tilde{\phi} : \Phi \to \mathrm{Aut}_{\bar{\lambda}}\,\mathfrak{C}$. The set $\{\phi : \phi \in \Phi, \ \tilde{\phi}\pi = \pi\phi\}$ is a sub-semigroup of Φ including Θ, so is the whole of Φ; thus $(\mathfrak{C}, \bar{\lambda}, \pi, \langle\tilde{\phi}\rangle_{\phi\in\Phi})$ is an invertive extension of $(\mathfrak{A}, \bar{\mu}, \Phi)$.

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