Version of 26.11.08

Ergodic averages, following Austin

D.H.Fremlin

University of Essex, Colchester, England

I rewrite the main results of Austin p08a and Austin p08b, in which a version of the multiple recurrence theorem is proved by a new method based on ideas of T.Tao.

1 Useful facts

1A Lemma Let $(\mathfrak{A},\bar\mu)$ be a measure algebra. If $a\in\mathfrak{A}$ and $u\in L^\infty(\mathfrak{A})$ is such that $0\le u\le\chi 1$, there is an $\alpha\in\,]0,1[$ such that $\bar\mu(a\,\triangle\,[\![u>\alpha]\!])\le\int|\chi a-u|$.

proof Set $\gamma=\int|\chi a-u|$. If $\gamma=\infty$ we can stop. Otherwise, we may suppose that $(\mathfrak{A},\bar\mu)$ is the measure algebra of a measure space (X,Σ,μ). Express a as $E^\bullet$ and u as $f^\bullet$ where $E\in\Sigma$ and $f:X\to[0,1]$ is Σ-measurable. Then $\int|\chi E-f|d\mu=\gamma$ is finite, so $H=\{x:\chi E(x)\ne f(x)\}$ is expressible as a countable union of sets of finite measure. Set $\Omega'_f=\{(x,\alpha):x\in X,\,0\le\alpha<f(x)\}$ and $W=(E\times[0,1])\triangle\Omega'_f$. Then $W\subseteq H\times\mathbb{R}$ is measured by the product of the subspace measure μ_H on H and Lebesgue measure μ_L on $[0,1]$. Because μ_H is σ-finite, we have

$$\gamma=\int_H|\chi E(x)-f(x)|\mu(dx)=\int_H\mu_L W[\{x\}]\mu_H(dx)$$
$$=\int_0^1\mu_H W^{-1}[\{\alpha\}]\mu_L(d\alpha)=\int_0^1\mu_H(E\triangle\{x:f(x)>\alpha\})\mu_L(d\alpha),$$

and there must be an $\alpha\in\,]0,1[$ such that

$$\gamma\ge\mu_H(E\triangle\{x:f(x)>\alpha\})=\mu(E\triangle\{x:f(x)>\alpha\})=\bar\mu(a\,\triangle\,[\![u>\alpha]\!]).$$

1B Lemma Let G be a topological group, $(\mathfrak{A},\bar\mu)$ a measure algebra, and $\bullet$ a continuous action of G on $\mathfrak{A}$, where $\mathfrak{A}$ is given its measure-algebra topology (Fremlin 02, §323), such that $a\mapsto g\bullet a$ is a measure-preserving Boolean automorphism for every $g\in G$.

(a) We have an action of G on $L^0=L^0(\mathfrak{A})$ defined by saying that $[\![g\bullet u>\alpha]\!]=g\bullet[\![u>\alpha]\!]$ whenever $g\in G$, $u\in L^0$ and $\alpha\in\mathbb{R}$; for $g\in G$, $u\mapsto g\bullet u:L^0\to L^0$ is an f-algebra automorphism.

(b) For every $p\in[1,\infty]$, $L^p=L^p(\mathfrak{A},\bar\mu)$ and $\|\,\|_p$ are G-invariant. For $p\in[1,\infty[$, the action is continuous.

(c) Let B be the unit ball of $L^\infty=L^\infty(\mathfrak{A})$, with the topology $\mathfrak{T}_s(L^\infty,L^1)$ induced by the duality between L^∞ and $L^1=L^1(\mathfrak{A},\bar\mu)$. Then B is G-invariant and the action of G on B is continuous.

proof (a) For each $g\in G$, we have a measure-preserving automorphism π_g defined by saying that $\pi_g(a)=g\bullet a$ for $a\in\mathfrak{A}$, and a corresponding f-algebra isomorphism $R_g:L^0\to L^0$, where $L^0=L^0(\mathfrak{A})$, given by saying that $[\![R_g u>\alpha]\!]=\pi_g[\![u>\alpha]\!]$ for $u\in L^0$ and $\alpha\in\mathbb{R}$.

If g, $h\in G$, then

$$\pi_{gh}(a)=(gh)\bullet a=g\bullet(h\bullet a)=\pi_g(\pi_h(a))$$

for every $a\in\mathfrak{A}$, so $\pi_{gh}=\pi_g\pi_h$, $R_{gh}=R_gR_h$ (Fremlin 02, 364Re) and $g\bullet(h\bullet u)=(gh)\bullet u$ for every $u\in L^0(\mathfrak{A})$. So we have an action of G on $L^0(\mathfrak{A})$.

(b) Every R_g acts on every L^p as a Banach lattice automorphism (Fremlin 02, 364R, 365O and 366H). If $p<\infty$, this action is continuous for the norm topology on L^p. **P** Suppose that $g_0\in G$, $v_0\in L^p$ and $\epsilon>0$. Then we can find a $v_1\in L^p$ such that $\|v_1-v_0\|_p\le\epsilon$ and v_1 is expressible as $\sum_{i=0}^n\alpha_i\chi a_i$ where $\bar\mu a_i<\infty$ for every $i\le n$.

Let $\eta>0$ be such that $(2\eta)^{1/p}\sum_{i=0}^n|\alpha_i|\le\epsilon$. Because the action of G on $\mathfrak{A}$ is continuous, there is a neighbourhood V of g_0 such that $\bar\mu(g\bullet a_i\cap g_0\bullet a_i)\ge\bar\mu(g_0\bullet a_i)-\eta$ whenever $i\le n$ and $g\in V$. Since π_g is measure-preserving for every g, we see that $\bar\mu(g\bullet a_i\,\triangle\,g_0\bullet a_i)\le 2\eta$ whenever $g\in V$ and $i\le n$, so that $\|g\bullet v_1-g_0\bullet v_1\|_p\le\epsilon$ whenever $g\in V$. Now if $g\in V$ and $v\in L^1$ is such that $\|v-v_0\|_p\le\epsilon$, we shall have

$$\|g\bullet v - g_0\bullet v_0\|_p \le \|g\bullet v - g\bullet v_1\|_p + \|g\bullet v_1 - g_0\bullet v_1\|_p + \|g_0\bullet v_1 - g_0\bullet v_0\|_p$$
$$\le \|v - v_1\|_p + \epsilon + \|v_1 - v_0\|_p \le 4\epsilon.$$

As g_0, v_0 and ϵ are arbitrary, the action is continuous. **Q**

(c) $R_g\restriction L^\infty$ is a norm-preserving automorphism of L^∞, so we have an action of G on B. Now suppose that $u_0 \in B$, $g_0 \in G$, $v \in L^1$ and $\epsilon > 0$. Then there is a neighbourhood V of g_0 such that $\|g^{-1}\bullet v - g_0^{-1}\bullet v\|_1 \le \epsilon$ whenever $g \in V$. Suppose that $u \in B$ is such that $|\int u \times (g_0^{-1}\bullet v) - \int u_0 \times (g_0^{-1}\bullet v)| \le \epsilon$. Then, for any $g \in V$,

$$|\int (g\bullet u - g_0\bullet u_0) \times v| = |\int (g\bullet u) \times v - \int (g_0\bullet u_0) \times v|$$
$$= |\int g^{-1}\bullet((g\bullet u) \times v) - \int g_0^{-1}\bullet((g_0\bullet u_0) \times v)|$$
$$= |\int u \times (g^{-1}\bullet v) - \int u_0 \times (g_0^{-1}\bullet v)|$$

(because R_g, R_{g_0} are multiplicative)

$$\le |\int u \times (g^{-1}\bullet v) - \int u \times (g_0^{-1}\bullet v)|$$
$$+ |\int u \times (g_0^{-1}\bullet v) - \int u_0 \times (g_0^{-1}\bullet v)|$$
$$\le \|g^{-1}\bullet v - g_0^{-1}\bullet v\|_1 + \epsilon \le 2\epsilon.$$

As u_0, g_0, v and ϵ are arbitrary, the action of G on B is continuous.

1C Remark In this context, the following remark will be useful. Suppose that G is a topological group, $(\mathfrak{A}, \bar\mu)$ a probability algebra, and $\bullet$ an action of G on $\mathfrak{A}$ such that $a \mapsto g\bullet a$ is a measure-preserving Boolean automorphism for every $g \in G$. If $D \subseteq \mathfrak{A}$ is such that the subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ generated by D is dense for the measure-algebra topology of $\mathfrak{A}$, and $g \mapsto g\bullet d : G \to \mathfrak{A}$ is continuous for every $d \in D$, then $\bullet$ is continuous. **P** (i) $\{d : d \in \mathfrak{A}, g \mapsto g\bullet d \text{ is continuous}\}$ is a subalgebra of $\mathfrak{A}$ because the Boolean operations are uniformly continuous (Fremlin 02, 323B). So it includes $\mathfrak{D}$. (ii) Suppose that $g_0 \in G$, $a_0 \in \mathfrak{A}$ and $\epsilon > 0$. Let $d \in \mathfrak{D}$ be such that $\bar\mu(d \triangle a) \le \epsilon$, and $H \subseteq G$ a neighbourhood of g_0 such that $\bar\mu(g\bullet d \triangle g_0\bullet d) \le \epsilon$ for every $g \in H$. Then if $g \in H$ and $\bar\mu(a \triangle a_0) \le \epsilon$,

$$\bar\mu(g\bullet a \triangle g_0\bullet a_0) \le \bar\mu(g\bullet a \triangle g\bullet d) + \bar\mu(g\bullet d \triangle g_0\bullet d) + \bar\mu(g_0\bullet d \triangle g_0\bullet a_0)$$
$$\le \bar\mu(a \triangle d) + \epsilon + \bar\mu(d \triangle a_0) \le 4\epsilon.$$

As g_0, a_0 and ϵ are arbitrary, $\bullet$ is continuous. **Q**

1D Proposition Let U and V be Hausdorff locally convex linear topological spaces, $A \subseteq U$ a convex set and $\phi : A \to V$ a continuous function such that $\phi[A]$ is bounded and $\phi(\alpha x + (1-\alpha)y) = \alpha\phi(x) + (1-\alpha)\phi(y)$ for all x, $y \in A$ and $\alpha \in [0,1]$. Let μ be a topological probability measure on A with a barycenter x^* in A. Then $\phi(x^*)$ is the barycenter of the image measure $\mu\phi^{-1}$ on V.

proof (a) Suppose that $\langle E_i\rangle_{i\in I}$ is a finite partition of A into non-empty convex sets measured by μ, and set $\alpha_i = \mu E_i$ for each $i \in I$. Set $C = \{\sum_{i\in I} \alpha_i x_i : x_i \in E_i \text{ for every } i \in I\}$. Then $x^* \in \overline{C}$. **P** Because each E_i is convex, so is C. If $g \in U^*$, then

$$g(x^*) = \int_A g(x)\mu(dx) = \sum_{i\in I} \int_{E_i} g(x)\mu(dx)$$
$$\le \sum_{i\in I} \alpha_i \sup_{x\in E_i} g(x) = \sup\{\sum_{i\in I} \alpha_i g(x_i) : x_i \in E_i \text{ for every } i \in I\}$$
$$= \sup\{g(\sum_{i\in I} \alpha_i x_i) : x_i \in E_i \text{ for every } i \in I\} = \sup_{z\in C} g(z).$$

By the Hahn-Banach theorem, $x^* \in \overline{C}$. **Q**

(b) Now suppose that $h \in V^*$ and $\epsilon > 0$. Then $h[\phi[A]]$ is bounded; take $\alpha \in \mathbb{R}$ and $n \ge 1$ such that $h[\phi[A]] \subseteq [\alpha, \alpha + n\epsilon[$. For $i < n$ set $F_i = \{y : y \in V,\ \alpha + i\epsilon \le h(y) < \alpha + (i+1)\epsilon\}$ and $E_i = \phi^{-1}[F_i]$; set $I = \{i : i < n,\ E_i \ne \emptyset\}$. Then $\langle E_i \rangle_{i \in I}$ is a partition of A into relatively Borel sets. As in (a), set $\alpha_i = \mu E_i$ for $i \in I$ and $C = \{\sum_{i\in I} \alpha_i x_i : x_i \in E_i$ for every $i \in I\}$. Then $C \subseteq A$ and $x^* \in \overline{C}$; there must therefore be a $z \in C$ such that $|h(\phi(z)) - h(\phi(x^*))| \le \epsilon$. Express z as $\sum_{i \in I} \alpha_i x_i$ where $x_i \in E_i$ for each $i \in I$. Then

$$
\begin{aligned}
|h(\phi(x^*)) - \int h\, d(\mu\phi^{-1})| &\le \epsilon + |h(\phi(z)) - \sum_{i\in I} \int_{F_i} h\, d(\mu\phi^{-1})| \\
&= \epsilon + |h(\sum_{i\in I} \alpha_i \phi(x_i)) - \sum_{i\in I} \int_{F_i} h\, d(\mu\phi^{-1})| \\
&\le \epsilon + \sum_{i\in I} |\alpha_i h(\phi(x_i)) - \int_{F_i} h\, d(\mu\phi^{-1})| \\
&\le \epsilon + \sum_{i\in I} \alpha_i \sup_{y\in F_i} |h(\phi(x_i)) - h(y)|
\end{aligned}
$$

(because $\mu\phi^{-1}[F_i] = \alpha_i$ for each i)

$$\le \epsilon + \sum_{i\in I} \alpha_i \epsilon$$

(by the choice of the F_i)

$$= 2\epsilon.$$

As h and ϵ are arbitrary, $\phi(x^*)$ is the barycenter of $\mu\phi^{-1}$.

1E Lemma Let U be a uniformly convex Banach space, $A \subseteq U$ a non-empty bounded set, and $C \subseteq U$ a non-empty closed convex set. Set

$$\delta_0 = \inf\{\delta : \text{there is some } w \in C \text{ such that } A \subseteq B(w, \delta)\}.$$

Then there is a unique $w^* \in C$ such that $A \subseteq B(w^*, \delta_0)$.

proof (a) For $\delta \ge \delta_0$, set

$$C_\delta = C \cap \bigcap\nolimits_{u\in A} B(u, \delta)\},$$

so that C_δ is closed, and is non-empty if $\delta > \delta_0$. Now $\lim_{\delta\downarrow\delta_0} \operatorname{diam} C_\delta = 0$. **P** Of course $\operatorname{diam} C_\delta \le 2\delta$, so if $\delta_0 = 0$ the result is trivial. Otherwise, let $\epsilon > 0$. Then there is an $\eta > 0$ such that $\|\frac{1}{2}(v_0 + v_1)\| < \frac{1-\eta}{1+\eta}$ whenever $\|v_0\|, \|v_1\| \le 1$ and $\|v_0 - v_1\| \ge \epsilon\delta_0$. **?** Suppose that $\delta \le (1+\eta)\delta_0$ and $\operatorname{diam} C_\delta > \epsilon$. Let $w_0, w_1 \in C_\delta$ be such that $\|w_0 - w_1\| \ge \epsilon$. Then $\frac{1}{2}(w_0 + w_1) \in C$, so there is a $u \in A$ such that $\|u - \frac{1}{2}(w_0 + w_1)\| \ge (1-\eta)\delta_0$, while $\|u - w_0\| \le (1+\eta)\delta_0$ and $\|u - w_1\| \le (1+\eta)\delta_0$; setting $v_j = \frac{1}{(1+\eta)\delta_0}(u - w_j)$ for $j = 0$ and $j = 1$, we see that this contradicts the choice of η. **X**

So $\operatorname{diam} C_\delta \le \epsilon$ whenever $\delta \le (1+\eta)\delta_0$; as ϵ is arbitrary, we have the result. **Q**

(b) $\{C_\delta : \delta > \delta_0\}$ generates a Cauchy filter, which has a limit $w^* \in \bigcap_{\delta>\delta_0} C_\delta$. Now $w^* \in C_{\delta_0}$; since C_{δ_0} has zero diameter, w^* is its only member, that is, is the unique element of U such that $A \subseteq B(w^*, \delta_0)$.

1F Proposition (T.Austin, e-mail of 8.10.08) Let G be a group, $(\mathfrak{A}, \bar\mu)$ a probability algebra, and $\bullet$ an action of G on $\mathfrak{A}$ such that $a \mapsto g\bullet a$ is a measure-preserving Boolean automorphism for every $g \in G$. Let $\mathfrak{C}$ be the fixed-point algebra $\{c : c \in \mathfrak{A},\ g\bullet c = c$ for every $g \in G\}$. Then for every $a \in \mathfrak{A}$, there is a $c \in \mathfrak{C}$ such that $\bar\mu(a \triangle c) \le \sup_{g\in G} \bar\mu(a \triangle g\bullet a)$.

proof (a) Set $\gamma = \sup_{g\in G} \bar\mu(a \triangle g\bullet a)$. As in Lemma 1B, we have an action of G on $L^0(\mathfrak{A})$ defined by saying that $[\![g\bullet u > \alpha]\!] = g\bullet[\![u > \alpha]\!]$ whenever $g \in G$, $u \in B$ and $\alpha \in \mathbb{R}$. Set

$$A = \{\chi(g\bullet a) : g \in G\} = \{g\bullet\chi a : g \in G\}, \quad C = \{u : 0 \le u \le \chi 1 \text{ in } L^0\}.$$

If $p \in [1,\infty[$, $L^p = L^p(\mathfrak{A},\bar\mu)$ is invariant under this action, and $u \mapsto g\bullet u : L^p \to L^p$ is a Banach lattice automorphism for every $g \in G$. If $p \in]1,\infty[$, L^p is uniformly convex (Fremlin 01, 244P[1], or Clarkson 36), so there is a unique $w_p \in C$ such that

$$\sup_{u\in A}\|u - w_p\|_p = \inf_{w\in C}\sup_{u\in A}\|u - w\|_p \le \sup_{u\in A}\|u - \chi a\|_p = \gamma^{1/p}$$

(1E). Because A and C and $\|\ \|_p$, are G-invariant, so is w_p, and $w_p \in L^0(\mathfrak{C})$.

(b) Recall now that there is a $w^* \in L^1_{\mathfrak{C}} = L^1(\mathfrak{C},\bar\mu)$ such that $\|\chi a - w^*\|_1 = \inf\{\|\chi a - w\| : w \in L^1_{\mathfrak{C}}$ (use Bukhvalov's theorem, Fremlin 02, 367V/367Xx, or Komlós' theorem, Fremlin 01, 276H). Replacing w^* by $\mathrm{med}(0,w^*,\chi 1)$ if necessary, we may suppose that $w^* \in C$. In this case,

$$\|\chi a - w^*\|_1 \le \|\chi a - w_p\|_1 \le \|\chi a - w_p\|_p \le \gamma^{1/p}$$

for every $p > 1$, and $\|\chi a - w^*\|_1 \le \gamma$. By Lemma 1A, there is an $\alpha \in]0,1[$ such that $\bar\mu(a \triangle [\![v > \alpha]\!]) \le \gamma$. Set $c = [\![v > \alpha]\!]$; then $c \in \mathfrak{C}$ and $\bar\mu(a \triangle c) \le \gamma$, so we have the result.

1G Proposition (Austin p08a, 2.1) Let $(T,\le)$ be an upwards-directed partially ordered set, $\langle(\mathfrak{A}_t,\bar\mu_t)\rangle_{t\in T}$ a family of probability algebras and G a group; suppose that $\phi_{ji} : \mathfrak{A}_t \to \mathfrak{A}_j$ and $\bullet^{(t)} : G \times \mathfrak{A}_t \to \mathfrak{A}_t$ are such that

(i) ϕ_{st} is a measure-preserving Boolean homomorphism whenever $s \le t$ in T,
(ii) $\phi_{su} = \phi_{tu}\phi_{st}$ whenever $i \le j \le k$ in T,
(iii) $\bullet^{(t)}$ is an action of G on $\mathfrak{A}_t$ for each $t \in T$,
(iv) $g\bullet^{(t)}(\phi_{st}a) = \phi_{st}(g\bullet^{(s)}a)$ whenever $s \le t$ in T, $a \in \mathfrak{A}_t$ and $g \in G$,
(v) $a \mapsto g\bullet^{(t)}a : \mathfrak{A}_t \to \mathfrak{A}_t$ is a measure-preserving Boolean automorphism for each $t \in T$.

(a) Writing $(\mathfrak{A},\bar\mu,\langle\phi_t\rangle_{i\in I})$ for the inductive limit of $(\langle(\mathfrak{A}_t,\bar\mu_t)\rangle_{i\in I},\langle\phi_{st}\rangle_{s\le t})$ as in Fremlin 02, 328G[2], we have a unique action $\bullet$ of G on $\mathfrak{A}$ such that

$a \mapsto g\bullet a : \mathfrak{A} \to \mathfrak{A}$ is a measure-preserving Boolean automorphism for every $g \in G$,
$g\bullet(\phi_t a) = \phi_t(g\bullet^{(t)}a)$ whenever $t \in T$, $a \in \mathfrak{A}_t$ and $g \in G$.

(b) For each $t \in T$, let $\mathfrak{C}_t = \{c : c \in \mathfrak{A}_t,\ g\bullet^{(t)}c = c$ for every $g \in G\}$ be the fixed-point subalgebra of the action $\bullet^{(t)}$. Then the fixed-point subalgebra $\mathfrak{C}$ of the action $\bullet$ is the closure of $\bigcup_{t\in T}\phi_t[\mathfrak{C}_t]$.

(c) If G is a topological group and $\bullet^{(t)}$ is continuous for every $t \in T$, then $\bullet$ is continuous.

proof (a) For $g \in G$ and $t \in T$, set $\psi_{gt}(a) = \phi_t(g\bullet^{(t)}a)$ for every $a \in \mathfrak{A}_t$. Then $\psi_{gt} = \psi_{gt}\phi_{st}$ whenever $s \le t$, so by the defining property of probability algebra inductive limit, there is a unique measure-preserving Boolean homomorphism $\psi_g : \mathfrak{A} \to \mathfrak{A}$ such that $\psi_g\phi_t = \psi_{gt}$ for every t. It is now elementary to verify that $(g,a) \to \phi_g(a)$ is an action of G on $\mathfrak{A}$, as required.

(b) If $i \in I$ and $a \in \phi_t[\mathfrak{C}_t]$, set $c = \phi_t^{-1}a$; then

$$g\bullet a = \phi_t(g\bullet^{(t)}c) = a$$

so $a \in \mathfrak{C}$. Now suppose that $c \in \mathfrak{C}$ and $\epsilon > 0$. Then there are a $t \in T$ and an $a \in \mathfrak{A}_t$ such that $\bar\mu(c \triangle \phi_t a) \le \epsilon$. If $g \in G$, then

$$\begin{aligned}\bar\mu_t(a \triangle g\bullet^{(t)}a) &= \bar\mu\phi_t(a \triangle g\bullet^{(t)}a) = \bar\mu(\phi_t a \triangle g\bullet\phi_t a)\\ &\le \bar\mu(\phi_t a \triangle c) + \bar\mu(g\bullet c \triangle g\bullet\phi_t a) = \bar\mu(\phi_t a \triangle c) + \bar\mu(c \triangle \phi_t a) \le 2\epsilon.\end{aligned}$$

By Lemma 1F, there is a $b \in \mathfrak{C}_t$ such that $\bar\mu_t(a \triangle b) \le 2\epsilon$, so that $\phi_t b \in \phi_t[\mathfrak{C}_t]$ and $\bar\mu(c \triangle \phi_t b) \le 2\epsilon$. As c and ϵ are arbitrary, $\mathfrak{C} = \overline{\bigcup_{t\in T}\phi_t[\mathfrak{C}_t]}$.

(c) Because $\{0,1\} \cup \bigcup_{t\in T}\phi_t[\mathfrak{A}_t]$ is dense in $\mathfrak{A}$ (Fremlin 02, 328G), it will be enough to show that $g \mapsto g\bullet(\phi_t a) : G \to \mathfrak{A}$ is continuous whenever $t \in T$ and $a \in \mathfrak{A}_t$ (1C). But this is just the function $g \mapsto \phi_t(g\bullet^{(t)}a)$, which is continuous because $\bullet^{(t)}$ and ϕ_t are continuous.

[1]Later editions only; see `http://www.essex.ac.uk/maths/staff/fremlin/mtcont.htm`.
[2]Later editions only.

1H Well-distributed limits (Fremlin n08) Let G be an amenable discrete group (Fremlin 03, §449) and U a Banach space.

(a) The **left Følner filter** of G is the filter $\mathcal{F}\text{ø}$ on $[G]^{<\omega}\setminus\{\emptyset\}$ generated by sets of the form

$$\{K : K\subseteq G \text{ is finite and not empty and } \#(K\triangle hK)\le\epsilon\#(K)\}$$

where $h\in G$ and $\epsilon>0$. If U is a Banach space and $f:G\to U$ is a bounded function, I write

$$\mathrm{WDL}_{g\to G}\, f(g)=\lim_{L\to\mathcal{F}\text{ø}}\frac{1}{\#(L)}\sum_{g\in L}f(g)$$

if the limit exists in U for the norm of U. Of course $\mathrm{WDL}_{g\to G}\, f(g)$, if defined, must belong to the closed convex hull of the image $f[G]$, and we have

$$\mathrm{WDL}_{g\to G}(f_1+f_2)(g)=\mathrm{WDL}_{g\to G}\, f_1(g)+\mathrm{WDL}_{g\to G}\, f_2(g),$$

$$\mathrm{WDL}_{g\to G}(Tf)(g)=T(\mathrm{WDL}_{g\to G}\, f(g))$$

whenever the right-hand sides are defined and $T:U\to V$ is a bounded linear operator to another Banach space. Also

$$\|\mathrm{WDL}_{g\to G}\, f(g)\|\le\mathrm{WDL}_{g\to G}\,\|f(g)\|$$

whenever both sides are defined.

(b) If $f:G\to\mathbb{R}$ is any function I will write

$$\overline{\mathrm{WDL}}_{g\to G}f(g)=\limsup_{L\to\mathcal{F}\text{ø}}\frac{1}{\#(L)}\sum_{g\in L}f(g).$$

Observe that of U is a Banach space and $f:G\to U$ is a bounded function such that $\overline{\mathrm{WDL}}_{g\to G}\|f(g)\|=0$ then $\mathrm{WDL}_{g\to G}\, f(g)=0$.

(c) For a bounded function $f:G\to\mathbb{R}$,

$$\overline{\mathrm{WDL}}(f)=\sup\{\int f\,d\mu:\mu \text{ is a translation-invariant finitely additive functional from } \mathcal{P}G \text{ to } [0,1],\text{ and } \mu G=1\}.$$

(Here the 'integral' $\int f\,d\mu$ must be interpreted as in Fremlin 02, 363L.) **P** For $f\in\mathbb{R}^G$ and $g\in G$, define $g\bullet_l f\in\mathbb{R}^G$ by setting $(g\bullet_l f)(h)=f(g^{-1}h)$ for every $h\in G$. Writing P for the set of positive linear functionals $p:\ell^\infty(G)\to\mathbb{R}$ such that $p(\chi G)=1$ and $p(g\bullet_l f)=p(f)$ whenever $f\in\ell^\infty(G)$ and $g\in G$,

$$\overline{\mathrm{WDL}}(f)=\sup_{p\in P}p(f)$$

for every $f\in\ell^\infty(G)$ (Fremlin n08, 6Ia). On the other hand, it is easy to check that we have a one-to-one correspondence between positive linear functionals on $\ell^\infty(X)$ and the set of finitely additive measures $\mu:\mathcal{P}G\to[0,\infty[$, given by setting

$$\mu A=p(\chi A)\text{ for } A\subseteq X,\quad p(f)=\int f d\mu \text{ for } f\in\ell^\infty(G)$$

(see the discussion in Fremlin 02, 363L); and $p\in P$ iff μ is translation-invariant and $\mu G=1$. So we get

$$\overline{\mathrm{WDL}}(f)=\sup_{p\in P}p(f)$$
$$=\sup\{\int f\,d\mu:\mu \text{ is a translation-invariant finitely additive functional from } \mathcal{P}G \text{ to } [0,1],\text{ and } \mu G=1\}.$$

(d) If G is infinite, and $f:G\to U$ is a bounded function such that $\#(\{g:f(g)\ne0\})<\#(G)$, then $\mathrm{WDL}_{g\to G}\, f(g)=0$. **P** Setting $A=\#(\{g:f(g)\ne0\})$, we can choose inductively a sequence $\langle g_n\rangle_{n\in\mathbb{N}}$ in G such that $g_nA\cap\bigcup_{i<n}g_iA=\emptyset$ for every n. (When we come to choose g_n, only $\bigcup_{i<n}g_iAA^{-1}$ is forbidden, and this has cardinal less than $\#(G)$.) By (c), $\overline{\mathrm{WDL}}_{g\to G}\chi A(g)=0$, so $\mathrm{WDL}_{g\to G}\,\|f(g)\|=0$ and $\mathrm{WDL}_{g\to G}\, f(g)=0$. **Q**

1I Theorem Let G be an abelian group, and $\bullet$ an action of G on a Banach space U such that $u\mapsto g\bullet u$ is a linear operator of norm at most 1 for every $g\in G$. If $u\in U$ is such that $\{g\bullet u:g\in G\}$ is relatively weakly compact, then $w=\mathrm{WDL}_{g\to G}\,g\bullet u$ is defined in U and $g\bullet w=w$ for every $g\in G$.

proof Fremlin 08, 6M. (To match between the definition of WDL in 1H with that in Fremlin 08, apply Fremlin 08, 6Ic to the discrete topology on G.)

1J Notation (a) We shall have a very large number of conditional expectation operators in the work to follow. It will be convenient to reserve a letter for these. If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$,[3] I will write $Q_{\mathfrak{B}}$ for the associated conditional expectation operator from $L^1(\mathfrak{A},\bar\mu)$ to $L^1(\mathfrak{B},\bar\mu\restriction\mathfrak{B})\subseteq L^1(\mathfrak{A},\bar\mu)$ (Fremlin 02, 365R).

(b) It will also be convenient to have some notation for lattices of closed subalgebras. If $(\mathfrak{A},\bar\mu)$ is a probability algebra and $\langle\mathfrak{B}_t\rangle_{t\in T}$ is a family of closed subalgebras of $\mathfrak{A}$, then I will write $\bigvee_{t\in T}\mathfrak{B}_t$ for the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{t\in T}\mathfrak{B}_t$. Similarly, if $\mathfrak{B}$ and $\mathfrak{C}$ are two closed subalgebras of $\mathfrak{A}$, $\mathfrak{B}\vee\mathfrak{C}$ will be the smallest closed subalgebra including both $\mathfrak{B}$ and $\mathfrak{C}$.

2 Measure-automorphism action systems

2A Definitions (a) An **action system** is a triple $(X,G,\langle\bullet_i\rangle_{i\in I})$ where X is a set, G is a group and $\bullet_i$ is an action of G on X for each $i\in I$.

(b) An action system $(X,G,\langle\bullet_i\rangle_{i\in I})$ is **commuting** if G is abelian and $g\bullet_i(h\bullet_j x)=h\bullet_j(g\bullet_i x)$ whenever g, $h\in G$, i, $j\in I$ and $x\in X$.

(c) A **measure-automorphism action system** is a quadruple $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ such that

$(\mathfrak{A},\bar\mu)$ is a probability algebra,
$(\mathfrak{A},G,\langle\bullet_i\rangle_{i\in I})$ is an action system,
$a\mapsto g\bullet_i a$ is a measure-preserving Boolean automorphism for every $i\in I$ and $g\in G$.

2B Construction Let $(\mathfrak{A},G,\langle\bullet_i\rangle_{i\in I})$ be an action system. Suppose that $\mathfrak{A}$ is a Boolean algebra and that $\mu:\mathfrak{A}\to[0,1]$ an additive functional; suppose that

$a\mapsto g\bullet_i a$ is a Boolean automorphism whenever $g\in G$ and $i\in I$,
$\mu 1=1$,
$\mu(g\bullet_i a)=\mu a$ whenever $a\in\mathfrak{A}$, $g\in G$ and $i\in I$.

Set $\mathcal{I}=\{a:a\in\mathfrak{A},\ \mu a=0\}$; then $\mathcal{I}\lhd\mathfrak{A}$. Let $\mathfrak{C}_0$ be the quotient $\mathfrak{A}/\mathcal{I}$. Then we can define $\bullet'_i:G\times\mathfrak{C}_0\to\mathfrak{C}_0$, for $i\in I$, by saying that $g\bullet'_i a^\bullet=(g\bullet_i a)^\bullet$ whenever $a\in\mathfrak{A}$, $g\in G$ and $i\in I$. Each $\bullet'_i$ is an action of G on $\mathfrak{C}_0$.

There is a strictly positive additive functional $\bar\nu_0:\mathfrak{C}_0\to[0,1]$ defined by saying that $\bar\nu_0 a^\bullet=\mu a$ for every $a\in\mathfrak{A}$. Let $\mathfrak{C}$ be the completion of $\mathfrak{C}_0$ under the metric $(c,c')\mapsto\bar\nu_0(c\mathbin{\triangle}c')$, and $\bar\nu$ the continuous extension of $\bar\nu_0$ to $\mathfrak{C}$; then $(\mathfrak{C},\bar\nu)$ is a probability algebra. Each $\bullet'_i$ has a unique extension to a function $\tilde\bullet_i:G\times\mathfrak{C}\to\mathfrak{C}$ such that $c\mapsto g\tilde\bullet_i c$ is a measure-preserving Boolean automorphism for every $g\in G$.

$(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I})$ is a measure-preserving action system. Setting $\phi a={}^\bullet$ for $a\in\mathfrak{A}$, $\phi:\mathfrak{A}\to\mathfrak{C}$ is a Boolean homomorphism and

$$g\tilde\bullet_i\phi(a)=g\bullet'_i\phi(a)=\phi(g\bullet_i a)$$

whenever $a\in\mathfrak{A}$, $i\in I$ and $g\in G$.

If $(\mathfrak{A},G,\langle\bullet_i\rangle_{i\in I})$ is commuting, so is $(\mathfrak{C},G,\langle\tilde\bullet_i\rangle_{i\in I})$.

proof The verifications are all elementary. We have to confirm, for instance, that if a, $b\in\mathfrak{A}$ and $a^\bullet=b^\bullet$ in $\mathfrak{C}_0$, then $(g\bullet_i a)^\bullet=(g\bullet_i b)^\bullet$ whenever $g\in G$ and $i\in I$. But for this all we need to know is that

$$\mu((g\bullet_i a)\mathbin{\triangle}(g\bullet_i b))=\mu(g\bullet_i(a\mathbin{\triangle}b))=\mu(a\mathbin{\triangle}b)=0.$$

Because $a\mapsto g\bullet_i a:\mathfrak{A}\to\mathfrak{A}$ is always a Boolean automorphism, so is $c\mapsto g\bullet'_i c:\mathfrak{C}_0\to\mathfrak{C}_0$. We see at the same time that

[3] As noted in Fremlin 02, 323H, a subalgebra of $\mathfrak{A}$ is order-closed iff it is topologically closed; so we can use the word 'closed' without qualification in this context.

$$\bar\nu_0(g\bullet'_i a^\bullet) = \bar\nu_0(g\bullet_i a)^\bullet = \mu(g\bullet_i a) = \mu a = \bar\nu_0 a^\bullet$$

whenever $a \in \mathfrak{A}$, $g \in G$ and $i \in I$. So all the maps $c \mapsto g\bullet'_i c$ are isometries on $\mathfrak{C}_0$, and extend uniquely to isometries on the completion $\mathfrak{C}$, which are again Boolean automorphisms. (See Fremlin 02, 392H[4] for the construction of $(\mathfrak{C}, \bar\nu)$ from $(\mathfrak{C}_0, \bar\nu_0)$.) Now the confirmation that all the $\bullet'_i$ and $\tilde\bullet_i$ are actions is just a matter of writing out the relevant formulae with their interpretations, and the same is true of the confirmation that if the original system $(\mathfrak{A}, G, \langle\bullet_i\rangle_{i\in I})$ is commuting, so are $(\mathfrak{C}_0, G, \langle\bullet'_i\rangle_{i\in I})$ and $(\mathfrak{C}, G, \langle\tilde\bullet_i\rangle_{i\in I})$.

2C Definition Let $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ be a measure-preserving action system. A **factor** of the system is a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ which is G-invariant in the sense that $g\bullet_i b \in \mathfrak{B}$ whenever $b \in \mathfrak{B}$, $g \in G$ and $i \in I$.

2D Lemma Let $\mathbb{A} = (\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ be a commuting measure-preserving action system.

(a) If $\mathfrak{B}$ is a factor of $\mathbb{A}$, then $(\mathfrak{B}, \bar\mu\restriction\mathfrak{B}, G, \langle\bullet_i\restriction G\times\mathfrak{B}\rangle_{i\in I})$ is a commuting measure-preserving action system.

(b) If $\langle\mathfrak{B}_t\rangle_{t\in T}$ is a non-empty family of factors of $\mathbb{A}$, then $\bigvee_{t\in T}\mathfrak{B}_t$ and $\bigcap_{t\in T}\mathfrak{B}_t$ are factors of $\mathbb{A}$.

(c) If $J \subseteq I$, then $\mathfrak{B}_J = \{a : a \in \mathfrak{A},\ g\bullet_i b = g\bullet_j b$ for all $g \in G$ and $i, j \in J\}$ is a factor of $\mathbb{A}$.

(d) Let $\mathfrak{B}$ be a factor of $\mathbb{A}$. Then

$$g\bullet_i(Q_{\mathfrak{B}}u) = Q_{\mathfrak{B}}(g\bullet_i u)$$

for all $g \in G$, $i \in I$ and $u \in L^1(\mathfrak{A}, \bar\mu)$.

(e) Suppose that $J \subseteq I$ and that $\mathfrak{B}$ is any factor of $\mathbb{A}$. Then $Q_{\mathfrak{B}}Q_{\mathfrak{B}_J} = Q_{\mathfrak{B}\cap\mathfrak{B}_J}$.

proof (a)-(b) Elementary.

(c) Elementary, recalling that $\mathbb{A}$ is supposed to be commuting.

(d) Because $Q_{\mathfrak{B}}u \in L^0(\mathfrak{B})$, $g\bullet_i(Q_{\mathfrak{B}}u) \in L^0(\mathfrak{B})$. **P** For any $\alpha \in \mathbb{R}$,

$$[\![g\bullet_i(Q_{\mathfrak{B}}u) > \alpha]\!] = g\bullet_i[\![Q_{\mathfrak{B}}u > \alpha]\!] \in \mathfrak{B}$$

because $[\![Q_{\mathfrak{B}}u > \alpha]\!] \in \mathfrak{B}$. **Q** Also, for any $b \in \mathfrak{B}$,

$$\int_b g\bullet_i(Q_{\mathfrak{B}}u)\,d\bar\mu = \int_{g^{-1}\bullet_i b} Q_{\mathfrak{B}}u\,d\bar\mu$$
$$= \int_{g^{-1}\bullet_i b} u\,d\bar\mu = \int_b g\bullet_i u\,d\bar\mu;$$

as b is arbitrary, $g\bullet_i(Q_{\mathfrak{B}}u) = Q_{\mathfrak{B}}(g\bullet_i u)$.

(e) If $u \in L^1(\mathfrak{A}, \bar\mu)$, then $Q_{\mathfrak{B}}Q_{\mathfrak{B}_J}u \in L^0(\mathfrak{B}_J)$. **P** Set $v = Q_{\mathfrak{B}_J}u$. For any $g \in G$, $\alpha \in \mathbb{R}$ and $i, j \in J$,

$$[\![g\bullet_i v > \alpha]\!] = g\bullet_i[\![v > \alpha]\!] = g\bullet_j[\![v > \alpha]\!] = [\![g\bullet_j v > \alpha]\!];$$

so $g\bullet_i v = g\bullet_j v$. It follows that, for any $\alpha \in \mathbb{R}$, $g \in G$ and $i, j \in J$,

$$g\bullet_i[\![Q_{\mathfrak{B}}v > \alpha]\!] = [\![g\bullet_i(Q_{\mathfrak{B}}v) > \alpha]\!] = [\![Q_{\mathfrak{B}}(g\bullet_i v) > \alpha]\!]$$

(by (d))

$$= [\![Q_{\mathfrak{B}}(g\bullet_j v) > \alpha]\!] = g\bullet_j[\![Q_{\mathfrak{B}}v > \alpha]\!],$$

so that $[\![Q_{\mathfrak{B}}v > \alpha]\!] \in \mathfrak{B}_J$; as α is arbitrary, $Q_{\mathfrak{B}}v \in L^0(\mathfrak{B}_J)$. **Q**

So in fact $Q_{\mathfrak{B}}Q_{\mathfrak{B}_J}u \in L^0(\mathfrak{B}\cap\mathfrak{B}_J)$. Now if $b \in \mathfrak{B}\cap\mathfrak{B}_J$,

$$\int_b Q_{\mathfrak{B}}Q_{\mathfrak{B}_J}u\,d\bar\mu = \int_b Q_{\mathfrak{B}_J}u\,d\bar\mu = \int_b u\,d\bar\mu,$$

so $Q_{\mathfrak{B}}Q_{\mathfrak{B}_J}u = Q_{\mathfrak{B}\cap\mathfrak{B}_J}u$.

[4] Formerly 393B.

2E Definition Let $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I})$ be a measure-automorphism action system. An **extension** of the system $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I})$ will be a quintuple $(\mathfrak{A}', \bar{\mu}', G, \langle\bullet'_i\rangle_{i\in I}, \phi)$ such that $(\mathfrak{A}', \bar{\mu}', G, \langle\bullet'_i\rangle_{i\in I})$ is a measure-automorphism action system, $\phi : \mathfrak{A} \to \mathfrak{A}'$ is a measure-preserving homomorphism and $g\bullet'_i(\phi a) = \phi(g\bullet_i a)$ whenever $a \in \mathfrak{A}$, $g \in G$ and $i \in I$.

In this case, $\phi[\mathfrak{A}]$ is a factor of $(\mathfrak{A}', \bar{\mu}', G, \langle\bullet'_i\rangle_{i\in I})$.

2F Inductive limits Elaborating on 1G, we have the following. Let us say that an **inductive system** of measure-automorphism action systems is an object of the form $(\langle(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I}\rangle_{t\in T}, \langle\phi_{st}\rangle_{s\le t\in T})$ where

T is an upwards-directed set,

I is a set, G is a group,

$(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I})$ is a measure-automorphism action system for each $t \in T$,

$(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I}, \phi_{st})$ is an extension of $(\mathfrak{A}_s, \bar{\mu}_s, G, \langle\bullet^{(s)}_i\rangle_{i\in I})$ whenever $s \le t$ in T,

$\phi_{tu}\phi_{st} = \phi_{su}$ whenever $s \le t \le u$ in T.

In this case, if $(\mathfrak{A}, \bar{\mu}, \langle\phi_t\rangle_{t\in T})$ is the inductive limit of $(\langle(\mathfrak{A}_t, \bar{\mu}_t)\rangle_{t\in T}, \langle\phi_{st}\rangle_{s\le t})$, we have a unique family $\langle\bullet_i\rangle_{i\in I}$ of actions of G on $\mathfrak{A}$ such that $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I}, \phi_t)$ is an extension of $(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I})$ for every $t \in T$ (1Ga).

In this case I will call $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I}, \langle\phi_t\rangle_{t\in T})$ the **inductive limit** of $(\langle(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I})\rangle_{t\in T}, \langle\phi_{st}\rangle_{s\le t\in T})$.

2G Proposition Let $(\langle(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I})\rangle_{t\in T}, \langle\phi_{st}\rangle_{s\le t\in T})$ be an inductive system of measure-automorphism action systems, with inductive limit $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I}, \langle\phi_t\rangle_{t\in T})$.

(a) Suppose that $J \subseteq I$. Set

$$\mathfrak{B}^{(t)}_J = \{a : a \in \mathfrak{A}_t,\ g\bullet^{(t)}_i a = g\bullet^{(t)}_j a \text{ whenever } g \in G \text{ and } i, j \in J\} \text{ for } t \in T,$$

$$\mathfrak{B}_J = \{a : a \in \mathfrak{A},\ g\bullet_i a = g\bullet_j a \text{ whenever } g \in G \text{ and } i, j \in J\}.$$

Then $\mathfrak{B}_J = \overline{\bigcup_{t\in T} \phi_t[\mathfrak{B}^{(t)}_J]}$.

(b) Suppose that $\mathcal{J} \subseteq \mathcal{P}I$. Then

$$\bigvee_{J\in\mathcal{J}} \mathfrak{B}_J = \overline{\bigcup_{t\in T} \phi_t[\bigvee_{J\in\mathcal{J}} \mathfrak{B}^{(t)}_J]}.$$

(c) If $(\mathfrak{A}_t, \bar{\mu}_t, G, \langle\bullet^{(t)}_i\rangle_{i\in I})$ is commuting for every $t \in T$, then $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I})$ is commuting.

proof (a) It is easy to check that $\phi_t[\mathfrak{B}^{(t)}_J] = \mathfrak{B}_J \cap \phi_t[\mathfrak{A}_t]$ for every $t \in T$. If $a \in \mathfrak{B}_J$ and $\epsilon > 0$, there are a $t \in T$ and a $b \in \phi_t[\mathfrak{A}_t]$ such that $\bar{\mu}(a \triangle b) \le \epsilon$. Let $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{A}, \bar{\mu})$ be the conditional expectation defined by the factor $\phi_t[\mathfrak{A}_t]$. Then

$$\|P(\chi a) - \chi b\|_1 = \|P(\chi a - \chi b)\|_1 \le \epsilon.$$

By Lemma 1A, there is an $\alpha \in \,]0,1[\,$ such that $\bar{\mu}(a \triangle b') \le \epsilon$, where $b' = [\![P\chi a > \alpha]\!]$. Now recall from Lemma 2De that $P(\chi a) \in L^0(\mathfrak{B}_J)$, so that b' belongs to $\mathfrak{B}_J$ and therefore to $\phi_t[\mathfrak{B}^{(t)}_J]$. As ϵ is arbitrary, $a \in \overline{\bigcup_{t\in T} \phi_t[\mathfrak{B}^{(t)}_J]}$; as a is arbitrary, $\mathfrak{B}_J = \overline{\bigcup_{t\in T} \phi_t[\mathfrak{B}^{(t)}_J]}$.

(b) Of course $\phi_s[\bigvee_{J\in\mathcal{J}} \mathfrak{B}^{(s)}_J] \subseteq \phi_t[\bigvee_{J\in\mathcal{J}} \mathfrak{B}^{(t)}_J]$ whenever $s \le t$ in T, so $\mathfrak{D} = \bigcup_{t\in T} \phi_t[\bigvee_{J\in\mathcal{J}} \mathfrak{B}^{(t)}_J]$ is a subalgebra of $\mathfrak{A}$ and $\overline{\mathfrak{D}}$ is a closed subalgebra included in $\bigvee_{J\in\mathcal{J}} \mathfrak{B}_J$. By (a), it includes $\mathfrak{B}_J$ for each $J \in \mathcal{J}$, so we have equality.

(c) If $g, h \in G$ and $i, j \in I$, then $\{a : g\bullet_i(h\bullet_j a) = h\bullet_j(g\bullet_i a)\}$ is a closed subalgebra of $\mathfrak{A}$ including

$$\bigcup_{t\in T} \phi_t[\{a : a \in \mathfrak{A}_t,\ g\bullet^{(t)}_i(h\bullet^{(t)}_j a) = h\bullet^{(t)}_j(g\bullet^{(t)}_i a)\} = \bigcup_{t\in T} \mathfrak{A}_t$$

so is the whole of $\mathfrak{A}$.

2H Definitions (a) A measure-automorphism action system $(\mathfrak{A}, \bar{\mu}, G, \langle\bullet_i\rangle_{i\in I})$ is **measure-averaging** if

G is an abelian group,

I is finite,

$\mathrm{WDL}_{g\to G}(\prod_{i\in I} g\bullet_i u_i)$ is defined, for the norm $\|\ \|_1$, for every family $\langle u_i\rangle_{i\in I}$ in $L^\infty(\mathfrak{A})$.

(If $I=\emptyset$, so that we need to interpret an empty product $\prod_{i\in I} g^{-1}\bullet_i u_i$, I will take it to be the multiplicative identity $\chi 1$ of $L^0(\mathfrak{A})$.)

(b) A measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is **weakly measure-averaging** if

G is an abelian group,

I is finite,

$\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I} g\bullet_i a_i)$ is defined in $\mathbb{R}$ for every family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$.

2I Remark A measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is measure-averaging whenever G is an abelian group and $\#(I)=1$, by Theorem 1I, since $\|\ \|_\infty$-bounded sets are relatively weakly compact in $L^1(\mathfrak{A},\bar\mu)$.

2J Definition (Austin p08a, 4.1-4.2) Let $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ be a commuting measure-automorphism action system, with I finite, and $j\in I$. I will say that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is ***j*-pleasant** if, taking $\mathfrak{B}$ to be the closed subalgebra of $\mathfrak{A}$ generated by

$$\{a: g\bullet_j a=a \text{ for every } g\in G\}\cup\bigcup_{i\in I}\{a: g\bullet_i a=g\bullet_j a \text{ for every } g\in G\},$$

then

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j(u_j-Q_{\mathfrak{B}}u_j)\times\textstyle\prod_{i\in I\setminus\{j\}} g\bullet_i u_i\big)=0$$

in $L^1(\mathfrak{A},\bar\mu)$ whenever $\langle u_i\rangle_{i\in I}$ is a family in $L^\infty(\mathfrak{A})$.

2K Lemma In the context of Definition 2J,

$$\|\frac{1}{\#(L)}\textstyle\sum_{g\in L}\prod_{i\in I} g\bullet_i u_i\|_1\le\|u_j\|_1\cdot\prod_{i\in I\setminus\{j\}}\|u_i\|_\infty$$

for every non-empty finite set $L\subseteq G$.

proof For each $g\in L$,

$$\|\textstyle\prod_{i\in I} g\bullet_i u_i\|_1\le\|g\bullet_j u_j\|_1\cdot\prod_{i\in I\setminus\{j\}}\|g\bullet_i u_i\|_\infty\le\|u_j\|_1\cdot\prod_{i\in I\setminus\{j\}}\|u_i\|_\infty.$$

2L Lemma (Austin p08a, 4.5) Suppose that I is a finite set, $j\in I$, and that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is a j-pleasant system such that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I\setminus\{j\}})$ is measure-averaging. Then $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is measure-averaging.

proof Take $\mathfrak{B}$ as in 2J. Take $u_i\in L^\infty(\mathfrak{A})$ for $i\in I$. Set $v=Q_{\mathfrak{B}}u_j$. Set

$$\mathfrak{B}_j=\{a: g\bullet_j a=a \text{ for every } g\in G\},\quad \mathfrak{B}_i=\{a: g\bullet_i a=g\bullet_j a \text{ for every } g\in G\}$$

for $i\in I\setminus\{j\}$, so that every $\mathfrak{B}_i$ is a closed subalgebra of $\mathfrak{A}$ and $\mathfrak{B}=\bigvee_{i\in I}\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$. Taking $\mathfrak{D}$ to be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I}\mathfrak{B}_i$, $\mathfrak{B}$ is the closure of $\mathfrak{D}$ for the measure-algebra topology. Let $E\subseteq\mathfrak{A}$ be the family of elements expressible as $\inf_{i\in I} b_i$ where $b_i\in\mathfrak{B}_i$ for every $i\in I$. Then every element of $\mathfrak{D}$ is expressible as the supremum of a finite disjoint subset of E. Let $\epsilon>0$. Then we have disjoint $e_0,\dots,e_m\in E$ and $\alpha_0,\dots,\alpha_m\in\mathbb{R}$ such that $\|v-w\|_1\le\epsilon$, where $w=\sum_{k=0}^m\alpha_k\chi e_k$.

For each $k\le m$, express e_k as $\inf_{i\in I} b_{ki}$ where $b_{ki}\in\mathfrak{B}_i$ for each i. Then

$$\begin{aligned}g\bullet_j\chi e_k\times\prod_{i\in I\setminus\{j\}} g\bullet_i u_i&=g\bullet_j(\prod_{i\in I}\chi b_{ki})\times\prod_{i\in I\setminus\{j\}} g\bullet_i u_i=\prod_{i\in I} g\bullet_k\chi b_{ki}\times\prod_{i\in I\setminus\{j\}} g\bullet_i u_i\\&=\chi b_{kj}\times\prod_{i\in I\setminus\{j\}} g\bullet_i\chi b_{ki}\times\prod_{i\in I\setminus\{j\}} g\bullet_i u_i\\&=\chi b_{kj}\times\prod_{i\in I\setminus\{j\}} g\bullet_i(\chi b_{ki}\times u_i)\end{aligned}$$

for each g, so

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j\chi e_k\times\prod_{i\in I\setminus\{j\}}g\bullet_i u_i\big)=\chi b_{kj}\times\mathrm{WDL}_{g\to G}\big(\prod_{i\in I\setminus\{j\}}g\bullet_i(\chi b_{ki}\times u_i)\big)$$

is defined for $\|\,\|_1$ because $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I\setminus\{j\}})$ is measure-averaging. Consequently

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j w\times\prod_{i\in I\setminus\{j\}}g\bullet_i u_i\big)$$

is defined. As ϵ is arbitrary,

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j v\times\prod_{i\in I\setminus\{j\}}g\bullet_i u_i\big)$$

is defined (use 2K). Because $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is a j-pleasant system,

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j(v-u_j)\times\prod_{i\in I\setminus\{j\}}g\bullet_i u_i\big)=0$$

for $\|\,\|_1$. So

$$\mathrm{WDL}_{g\to G}\big(g\bullet_j u_j\times\prod_{i\in I\setminus\{j\}}g\bullet_i u_i\big)$$

is defined. As $\langle u_i\rangle_{i\in I}$ is arbitrary, $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is measure-averaging.

2M Lemma (Austin p08a, §3) Let I be a finite set, j an element of I, and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system such that $(\mathfrak{A},\bar\mu,G,\langle\bullet'_i\rangle_{i\in I\setminus\{j\}})$ is measure-averaging whenever $\langle\bullet'_i\rangle_{i\in I\setminus\{j\}}$ is such that $(\mathfrak{A},\bar\mu,G,\langle\bullet'_i\rangle_{i\in I\setminus\{j\}})$ is a commuting measure-automorphism action system. Then $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is weakly measure-averaging.

proof Let $\langle a_i\rangle_{i\in I}$ be a family in $\mathfrak{A}$. For $i\in I\setminus\{j\}$, define $\bullet'_i:G\times\mathfrak{A}\to\mathfrak{A}$ by setting $g\bullet'_i a=g^{-1}\bullet_j(g\bullet_i a)$ for $g\in G$ and $a\in\mathfrak{A}$. If g, $h\in G$ and $a\in\mathfrak{A}$, then

$$\begin{aligned}(gh)\bullet'_i a&=(gh)^{-1}\bullet_j((gh)\bullet_i a)=h^{-1}\bullet_j g^{-1}\bullet_j g\bullet_i h\bullet_i a\\&=g^{-1}\bullet_j g\bullet_i h^{-1}\bullet_j h\bullet_i a=g\bullet'_i h\bullet'_i a,\end{aligned}$$

so $\bullet'_i$ is an action. Similarly direct calculation shows that $(\mathfrak{A},\bar\mu,G,\langle\bullet'_i\rangle_{i\in I\setminus\{j\}})$ is a commuting measure-automorphism action system. Accordingly

$$\mathrm{WDL}_{g\to G}\prod_{i\in I\setminus\{j\}}g\bullet'_i\chi a_i$$

is defined in $L^1(\mathfrak{A},\bar\mu)$, and

$$\begin{aligned}\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_i a_i)&=\mathrm{WDL}_{g\to G}\int\prod_{i\in I}g\bullet_i\chi a_i\,d\bar\mu\\&=\mathrm{WDL}_{g\to G}\int g^{-1}\bullet_j(\prod_{i\in I}g\bullet_i\chi a_i)d\bar\mu\\&=\mathrm{WDL}_{g\to G}\int\prod_{i\in I}g^{-1}\bullet_j(g\bullet_i\chi a_i)d\bar\mu\\&=\mathrm{WDL}_{g\to G}\int_{a_j}\prod_{i\in I\setminus\{j\}}g\bullet'_i\chi a_i\,d\bar\mu\\&=\int_{a_j}\mathrm{WDL}_{g\to G}\big(\prod_{i\in I\setminus\{j\}}g\bullet'_i\chi a_i\big)\,d\bar\mu\end{aligned}$$

is defined in $\mathbb{R}$.

3 Furstenberg self-joinings

3A Construction (Austin p08a, §3) Let G be an abelian group and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system. Suppose that $J\subseteq I$ is a non-empty finite set such that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in J})$ is weakly measure-averaging.

(a) Let $(\mathfrak{B},\langle\varepsilon_j\rangle_{i\in J})$ be the free power $\bigotimes_J\mathfrak{A}$ of J copies of $\mathfrak{A}$ (Fremlin 03, §315). Then we have an additive functional $\nu:\mathfrak{B}\to[0,1]$ defined by saying that

$$\nu(\inf_{j\in J}\varepsilon_j a_j)=\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J}g\bullet_j a_j)$$

whenever $\langle a_j\rangle_{j\in J}$ is a family in $\mathfrak{A}$, writing $\mathcal{F}\text{ø}$ for the Følner filter of G. Note that $\nu\varepsilon_j(a)=\bar\mu a$ for every $a\in\mathfrak{A}$ and $j\in J$.

(b) Let $\mathfrak{C}_0$ be the quotient Boolean algebra $\mathfrak{B}/\{b:\nu b=0\}$, $\bar\nu_0$ the strictly positive finitely additive functional on $\mathfrak{C}_0$ defined by saying that $\bar\nu_0 b^\bullet=\nu b$ for every $b\in\mathfrak{B}$, and $\mathfrak{C}$ the metric completion of $\mathfrak{C}_0$ under the associated metric; let $\bar\nu$ be the continuous extension of $\bar\nu_0$ to $\mathfrak{C}$, so that $(\mathfrak{C},\bar\nu)$ is a probability algebra. For each $j\in J$, we have a measure-preserving Boolean homomorphism $\pi_j:\mathfrak{A}\to\mathfrak{C}$ defined by saying that $\pi_j a=(\varepsilon_j a)^\bullet$ for $a\in\mathfrak{A}$.

(c)(i) $\bar\nu(\inf_{i\in J}a_i)=\text{WDL}_{g\to G}\,\bar\mu(\inf_{i\in J}g\bullet_i a_i)$ for any family $\langle a_j\rangle_{i\in J}$ in $\mathfrak{A}$.

(ii) For $j\in J$ let $R_j:L^0(\mathfrak{A})\to L^0(\mathfrak{C})$ be the multiplicative Riesz homomorphism corresponding to the Boolean homomorphism $\pi_j:\mathfrak{A}\to\mathfrak{C}$. Then for any family $\langle u_j\rangle_{j\in J}$ in $L^\infty(\mathfrak{A})$,

$$\int\prod_{j\in J}R_j u_j\,d\bar\nu=\text{WDL}_{g\to G}\int\prod_{j\in J}g\bullet_j u_j\,d\bar\mu.$$

(d) We have a commuting measure-automorphism action system $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}})$ defined by saying that

$$g\tilde\bullet_i(\pi_j a)=\pi_j(g\tilde\bullet_i a),$$

$$g\tilde\bullet_\infty(\pi_j a)=\pi_j(g\bullet_j a)$$

whenever $i\in I$, $j\in J$ and $a\in\mathfrak{A}$.[5] The corresponding actions on $L^0(\mathfrak{C})$ are defined by the formulae

$$g\tilde\bullet_i(R_k u)=R_k(g\bullet_i u),$$

$$g\tilde\bullet_\infty(R_k u)=R_k(g\bullet_k u)$$

for $i\in I$, $k\in J$ and $u\in L^0(\mathfrak{A})$.

(e) Now fix on a member j of J, and for $i\in I$ set

$$\begin{aligned}\hat\bullet_i&=\tilde\bullet_\infty\text{ if }i=j,\\&=\tilde\bullet_i\text{ otherwise.}\end{aligned}$$

Then $(\mathfrak{C},\bar\nu,G,\langle\hat\bullet_i\rangle_{i\in I},\pi_j)$ is an extension of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$.

proof (a) We know that the limit

$$\text{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J}g\bullet_j a_j)$$

is always defined, so we have a well-defined functional on the set $\mathfrak{A}^J$. Since this is clearly additive in each variable separately, it uniquely defines an additive functional on $\mathfrak{B}$ (Fremlin 02, 326Q).

Taking $a_j=a$, $a_i=1$ for $i\in J\setminus\{j\}$ in the formula, we get the correct value for $\nu\varepsilon_j(a)$.

(b) Elementary, in view of the results in Fremlin 02.

(c)(i) This is just the definition of ν translated into terms of $\bar\nu$.

(ii) Both sides of the equation correspond to $\|\,\|_\infty$-continuous multilinear functionals on $L^\infty(\mathfrak{A})^J$, which agree on families of the form $\langle u_i\rangle_{i\in J}=\langle\chi a_i\rangle_{i\in J}$.

(d)(i) The defining universal mapping property of $\bigotimes_J\mathfrak{A}$ tells us that we have functions $\bullet_i^*$, $\bullet_\infty^*$ from $G\times\mathfrak{B}$ to $\mathfrak{B}$ defined by saying that

$$g\bullet_i^*(\varepsilon_j a)=\varepsilon_j(g\bullet_i a),$$

$$g\bullet_\infty^*(\varepsilon_j a)=\varepsilon_j(g\bullet_j a)$$

for $a\mathfrak{A}$, $g\in G$, $i\in I$ and $j\in J$, and that all the maps $b\mapsto g\bullet_i^* b$ (for $i\in I\cup\{\infty\}$) are Boolean homomorphisms. Direct calculation shows that $\bullet_i^*$ is an action of G on $\mathfrak{B}$ for every $i\in I\cup\{\infty\}$.

(ii) ν is G-invariant for all these actions. **P** If $i\in I$, $h\in G$ and $a_j\in\mathfrak{A}$ for $j\in J$,

[5] Here, and later, I use the symbol ∞ unscrupulously to denote an object not belonging to any relevant set previously mentioned.

$$\begin{aligned}\nu(h\bullet_i^*(\inf_{j\in J}\varepsilon_j a_j)) &= \nu(\inf_{j\in J}\varepsilon_j(h\bullet_i a_j))\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J} g\bullet_j h\bullet_i a_j)\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J} h\bullet_i g\bullet_j a_j)\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(h\bullet_i(\inf_{j\in J} g\bullet_j a_j))\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J} g\bullet_j a_j) = \nu(\inf_{j\in J}\varepsilon_j a_j),\end{aligned}$$

$$\begin{aligned}\nu(h\bullet_\infty^*(\inf_{j\in J}\varepsilon_j a_j)) &= \nu(\inf_{j\in J}\varepsilon_j(h\bullet_j a_j))\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J} g\bullet_j h\bullet_j a_j)\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J}(gh)\bullet_j a_j)\\ &= \lim_{L\to\mathcal{F}\text{ø}}\frac{1}{\#(Lh)}\sum_{g\in Lh}\bar\mu(\inf_{j\in J} g\bullet_j a_j) = \nu(\inf_{j\in J}\varepsilon_j a_j)\end{aligned}$$

because $\mathcal{F}\text{ø}$ is invariant under translation. Since an additive functional on $\mathfrak{B}$ is determined by its values on the basic elements $\inf_{j\in J}\varepsilon_j a_j$, $\nu(h\bullet_i^* b) = \nu b$ for every $b\in\mathfrak{B}$, $h\in G$ and $i\in I\cup\{\infty\}$. **Q**

(iii) Of course

$$\begin{aligned}g\bullet_i^*(h\bullet_k^*(\inf_{j\in J}\varepsilon_j a_j)) &= g\bullet_i^*(\inf_{j\in J}\varepsilon_j(h\bullet_k a_j)) = \inf_{j\in J}\varepsilon_j(g\bullet_i h\bullet_k a_j))\\ &= \inf_{j\in J}\varepsilon_j(h\bullet_k g\bullet_i a_j)) = h\bullet_k^*(g\bullet_i^*(\inf_{j\in J}\varepsilon_j a_j)),\end{aligned}$$

$$\begin{aligned}g\bullet_i^*(h\bullet_\infty^*(\inf_{j\in J}\varepsilon_j a_j)) &= g\bullet_i^*(\inf_{j\in J}\varepsilon_j(h\bullet_j a_j)) = \inf_{j\in J}\varepsilon_j(g\bullet_i h\bullet_j a_j))\\ &= \inf_{j\in J}\varepsilon_j(h\bullet_j g\bullet_i a_j)) = h\bullet_\infty^*(g\bullet_i^*(\inf_{j\in J}\varepsilon_j a_j))\end{aligned}$$

whenever g, $h\in G$, i, $k\in I$ and $\langle a_j\rangle_{j\in J}\in\mathfrak{A}^J$. So the $\bullet_i^*$, for $i\in I\cup\{\infty\}$, are commuting actions.

(iv) Applying the method of 2B to the system $(\mathfrak{B},\nu,G,\langle\bullet_i^*\rangle_{i\in I\cup\{\infty\}})$, we see that the declared formulae define actions $\tilde\bullet_i$ of G on $\mathfrak{C}$ such that $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}})$ is a commuting measure-automorphism action system.

(v) The other formulae are now elementary.

(e) All we have to check is that, for $g\in G$ and $a\in\mathfrak{A}$,

$$\begin{aligned}g\hat\bullet_i(\pi_j a) &= g\tilde\bullet_\infty(\pi_j a) = \pi_j(g\bullet_j a) \text{ if } i=j,\\ &= g\tilde\bullet_i(\pi_j a) = \pi_j(g\bullet_i a) \text{ otherwise.}\end{aligned}$$

3B Definition In the context of 3A, I will call $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_j\rangle_{j\in J})$ the **Furstenberg self-joining** of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ over J; in addition, I will call $(\mathfrak{C},\bar\nu,G,\langle\hat\bullet_i\rangle_{i\in I},\pi_j)$ the (J,j)**-Furstenberg extension** of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$. (See Austin p08a for some of the history of this construction.)

3C Proposition Let $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ be a commuting measure-automorphism action system with an extension $(\mathfrak{A}',\bar\mu',G,\langle\bullet_i'\rangle_{i\in I},\phi)$, and $J\subseteq I$ a non-empty finite set. Suppose that both $(\mathfrak{A},\bar\mu,G,\langle\bullet_j\rangle_{j\in J})$ and $(\mathfrak{A}',\bar\mu',G,\langle\bullet_j'\rangle_{j\in J})$ are weakly measure-averaging, with Furstenberg self-joinings $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_j\rangle_{j\in J})$ and $(\mathfrak{C}',\bar\nu',G,\langle\tilde\bullet_i'\rangle_{i\in I\cup\{\infty\}},\langle\pi_j'\rangle_{j\in J})$ respectively. Then there is a unique measure-preserving Boolean homomorphism $\psi:\mathfrak{C}\to\mathfrak{C}'$ such that $\psi\pi_j=\pi_j'\phi$ for every $i\in I$, and $(\mathfrak{C}',\bar\nu',G,\langle\tilde\bullet_i'\rangle_{i\in I\cup\{\infty\}},\psi)$ is an extension of $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}})$.

proof (a) Taking $\mathfrak{B}=\bigotimes_J\mathfrak{A}$ and $\mathfrak{B}'=\bigotimes_J\mathfrak{A}'$, we have a Boolean homomorphism $\theta:\mathfrak{B}\to\mathfrak{B}'$ defined by saying that $\theta\varepsilon_j=\varepsilon_j'\phi$ for every $j\in J$. Now, writing $\mathcal{F}\text{ø}$ for the Følner filter of G,

$$\nu'\theta(\inf_{j\in J}\varepsilon_j a_j) = \nu'(\inf_{j\in J}\varepsilon'_j\phi a_j) = \text{WDL}_{g\to G}\,\bar\mu'(\inf_{j\in J} g\bullet'_j\phi a_j)$$
$$= \text{WDL}_{g\to G}\,\bar\mu'(\inf_{j\in J}\phi(g\bullet_j a_j)) = \text{WDL}_{g\to G}\,\bar\mu'\phi(\inf_{j\in J} g\bullet_j a_j)$$
$$= \text{WDL}_{g\to G}\,\bar\mu(\inf_{j\in J} g\bullet_j a_j) = \nu(\inf_{j\in J}\varepsilon_j a_j)$$

whenever $a_j \in \mathfrak{A}$ for $j \in J$. So $\nu'\theta b = \nu b$ for every $b \in \mathfrak{B}$. It follows that θ induces a Boolean homomorphism $\psi_0 : \mathfrak{C}_0 \to \mathfrak{C}'_0$ such that $\psi_0(b^\bullet) = (\theta b)^\bullet$ for every $b \in \mathfrak{B}$, taking $\mathfrak{C}_0$, $\mathfrak{C}'_0$ to be the quotient algebras as in 3Ab; and $\bar\nu'_0\psi_0 c = \bar\nu_0 c$ for every $c \in \mathfrak{C}_0$. Accordingly ψ_0 extends to a measure-preserving Boolean homomorphism $\psi : \mathfrak{C} \to \mathfrak{C}'$. Tracing the definitions, we have

$$\psi\pi_j a = \psi_0\pi_j a = \psi_0(\varepsilon_j a)^\bullet = (\varepsilon'_j\phi a)^\bullet = \pi'_j\phi a$$

for every $a \in \mathfrak{A}$ and $j \in J$, and clearly this defines ψ. Similarly, examining the actions of G on these structures,

$$g\tilde\bullet'_i(\psi\pi_j a) = g\tilde\bullet'_i(\pi'_j\phi a) = \pi'_j(g\bullet'_i(\phi a))$$
$$= \pi'_j\phi(g\bullet_i a) = \psi\pi_j(g\bullet_i a) = \psi(g\tilde\bullet_i(\pi_j a)),$$
$$g\tilde\bullet'_\infty(\psi\pi_j a) = g\tilde\bullet'_\infty(\pi'_j\phi a) = \pi'_j(g\bullet'_j(\phi a))$$
$$= \pi'_j\phi(g\bullet_j a) = \psi\pi_j(g\bullet_j a) = \psi(g\tilde\bullet_\infty(\pi_j a))$$

whenever $a \in \mathfrak{A}$, $g \in G$, $i \in I$ and $j \in J$; consequently

$$g\tilde\bullet'_i(\psi c) = \psi(g\tilde\bullet_i c)$$

whenever $c \in \mathfrak{C}$, $g \in G$ and $i \in I\cup\{\infty\}$. So $(\mathfrak{C}', \bar\nu', G, \langle\tilde\bullet'_i\rangle_{i\in I\cup\{\infty\}}, \psi)$ is an extension of $(\mathfrak{C}, \bar\nu, G, \langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}})$.

3D Lemma (Bergelson McCutcheon & Zhang 97, 4.2) Let G be an abelian group, $\mathcal{F}\text{ø}$ its Følner filter, U an inner product space and $g \mapsto u_g : G \to U$ a bounded function such that

$$\inf_{\emptyset\ne M\in[G]^{<\omega}}\frac{1}{\#(M)^2}\overline{\text{WDL}}_{g\to G}\textstyle\sum_{h,h'\in M}(u_{hg}|u_{h'g}) \le 0.$$

Then $\text{WDL}_{g\to G}\,u_g = 0$.

proof Set $\gamma = \sup_{g\in G}\|u_g\|$. Let $\epsilon > 0$. Let $M \in [G]^{<\omega}\setminus\{\emptyset\}$ be such that

$$\frac{1}{\#(M)^2}\overline{\text{WDL}}_{g\to G}\textstyle\sum_{h,h'\in M}(u_{hg}|u_{h'g}) \le \epsilon.$$

For non-empyt finite sets $L \subseteq G$ set

$$v_L = \textstyle\sum_{g\in L}\frac{1}{\#(M)}\sum_{h\in M}\frac{1}{\#(L)}u_{hg}.$$

Then

$$\limsup_{L\to\mathcal{F}\text{ø}}\|v_L - \frac{1}{\#(L)}\sum_{g\in L}u_g\| \le \frac{1}{\#(M)}\sum_{h\in M}\limsup_{L\to\mathcal{F}\text{ø}}\|\frac{1}{\#(L)}(\sum_{g\in L}u_g - \sum_{g\in L}u_{hg})\|$$
$$\le \sup_{h\in M}\limsup_{L\to\mathcal{F}\text{ø}}\frac{1}{\#(L)}\|\sum_{g\in L}u_g - ⨏_{hL}u_g\|$$
$$\le \sup_{h\in M}\limsup_{L\to\mathcal{F}\text{ø}}\frac{1}{\#(L)}\gamma\#(L\triangle hL) = 0.$$

On the other hand, for every non-empty finite $L \subseteq G$,

$$\|v_L\| \le \frac{1}{\#(L)}\sum_{g\in L}\|\frac{1}{\#(M)}\sum_{h\in M}u_{hg}\|$$
$$\le \frac{1}{\#(L)}\cdot\sqrt{\#(L)}\cdot\sqrt{\sum_{g\in L}\|\frac{1}{\#(M)}\sum_{h\in M}u_{hg}\|^2}$$

(by the Cauchy-Schwartz inequality), so

$$\|v_L\|^2 \le \frac{1}{\#(L)}\sum_{g\in L}\|\frac{1}{\#(M)}\sum_{h\in M}u_{hg}\|^2$$
$$= \frac{1}{\#(L)}\sum_{g\in L}\frac{1}{\#(M)^2}\sum_{h,h'\in M}(u_{hg}|u_{h'g})$$

and

$$\limsup_{L\to\mathcal{F}\emptyset}\|v_L\|^2 \le \frac{1}{\#(M)^2}\limsup_{L\to\mathcal{F}\emptyset}\frac{1}{\#(L)}\sum_{g\in L}\sum_{h,h'\in M}(u_{hg}|u_{h'g})$$
$$= \frac{1}{\#(M)^2}\overline{\mathrm{WDL}}_{g\to G}\sum_{h,h'\in M}(u_{hg}|u_{h'g}) \le \epsilon.$$

Putting these together,

$$\limsup_{L\to\mathcal{F}\emptyset}\|\frac{1}{\#(L)}\sum_{g\in L}u_g\| \le \sqrt{\epsilon};$$

as ϵ is arbitrary, the limit is zero, and

$$\mathrm{WDL}_{g\to G}\,u_g = \lim_{L\to\mathcal{F}\emptyset}\frac{1}{\#(L)}\sum_{g\in L}u_g = 0.$$

3E Lemma (Austin p08a, 4.7) Let $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ be a commuting measure-automorphism action system, $J\subseteq I$ a finite non-empty set such that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in J})$ is weakly measure-averaging, and $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_j\rangle_{j\in J})$ the Furstenberg self-joining of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ over J. Let $\mathfrak{D}$ be the fixed-point algebra $\{c: c\in\mathfrak{C}, g\tilde\bullet_\infty c = c$ for every $g\in G\}$. For $j\in J$ let $R_j: L^0(\mathfrak{A})\to L^0(\mathfrak{C})$ be the multiplicative Riesz homomorphism corresponding to $\pi_j:\mathfrak{A}\to\mathfrak{C}$.

If $\langle u_j\rangle_{j\in J}$ is a family in $L^\infty(\mathfrak{A})$ such that $Q_{\mathfrak{D}}(\prod_{j\in J}R_ju_j)=0$, then

$$\mathrm{WDL}_{g\to G}\prod_{j\in J}g\bullet_j u_j = 0$$

in $L^1(\mathfrak{A},\bar\mu)$.

proof Set $w=\prod_{j\in J}R_ju_j$; for $h\in G$, set $w_h = h\tilde\bullet_\infty w$. Set $\gamma=\prod_{j\in J}\|u_j\|_\infty$; note that $\|w_h\|_\infty\le\gamma$ for every h.

(a) Note first that $Q_{\mathfrak{D}}w_h=0$ for every $h\in G$. **P** If $d\in\mathfrak{D}$,

$$\int_d h\tilde\bullet_\infty w\,d\bar\nu = \int_{h^{-1}\tilde\bullet_\infty d} w\,d\bar\nu = 0$$

because $h^{-1}\tilde\bullet_\infty d = d\in\mathfrak{D}$. As d is arbitrary, $Q_{\mathfrak{D}}w_h=0$. **Q**

(b) For any h, $h'\in G$,

$$\int w_h\times w_{h'}\,d\bar\nu = \int\prod_{j\in J}R_j(h\bullet_j u_j)\times\prod_{j\in J}R_j(h'\bullet_j u_j)\,d\bar\nu$$
$$= \int\prod_{j\in J}R_j(h\bullet u_j\times h'\bullet_j u_j)\,d\bar\nu$$
$$= \mathrm{WDL}_{g\to G}\int\prod_{j\in J}g\bullet_j(h\bullet u_j\times h'\bullet_j u_j)\,d\bar\mu$$

by 3Ac. Now

$$w^* = \mathrm{WDL}_{h\to G}\,w_h$$

is defined for $\|\,\|_2$ and belongs to $L^\infty(\mathfrak{D})$ (1I).

(c) For $g\in G$ set $v_g=\prod_{j\in J}g\bullet_j u_j$, We find that

$$\inf_{\emptyset\ne M\in[G]^{<\omega}}\frac{1}{\#(M)^2}\overline{\mathrm{WDL}}_{g\to\infty}\sum_{h,h'\in M}\int v_{hg}\times v_{h'g}\,d\bar\mu\le 0.$$

P Let $\epsilon>0$. Then there is an non-empty finite $M\subseteq G$ such that $\|w^*-\frac{1}{\#(M)}\sum_{h\in M}w_h\|_2\le\epsilon$. Now

$$\frac{1}{\#(M)^2}\overline{\mathrm{WDL}}_{g\to G}\sum_{h,h'\in M}\int v_{hg}\times v_{h'g}\,d\bar\mu$$

$$=\frac{1}{\#(M)^2}\overline{\mathrm{WDL}}_{g\to G}\sum_{h,h'\in M}\int\prod_{j\in J}(hg)\bullet_j u_j\times\prod_{j\in J}(h'g)\bullet_j u_j)\,d\bar\mu$$

$$=\frac{1}{\#(M)^2}\overline{\mathrm{WDL}}_{g\to G}\sum_{h,h'\in M}\int g\bullet_j\prod_{j\in J}(h\bullet_j u_j\times h'\bullet_j u_j)\,d\bar\mu$$

(because the system is commuting)

$$=\frac{1}{\#(M)^2}\sum_{h,h'\in M}\int w_h\times w_{h'}\,d\bar\nu$$

(by (b) above)

$$=\frac{1}{\#(M)}\sum_{h\in M}\int w_h\times(\frac{1}{\#(M)}\sum_{h\in M}w_{h'})\,d\bar\nu$$

$$\le\frac{1}{\#(M)}\sum_{h\in M}\int w_h\times w^*d\bar\nu+\frac{1}{\#(M)}\sum_{h\in M}\|w_h\|_2\|w^*-\frac{1}{\#(M)}\sum_{h\in M}w_{h'}\|_2$$

$$\le\frac{1}{\#(M)}\sum_{h\in M}\gamma\epsilon$$

(because $w^*\in L^\infty(\mathfrak{D})$ and $Q_{\mathfrak{D}}w_h=0$, so $\int w_h\times w^*d\bar\nu=0$ for every h)

$$\le\gamma\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(d) By 3D, $\lim_{L\to\mathcal{F}_\emptyset}\|\frac{1}{\mu L}\sum_{g\in L}v_g\|_2=0$. But $\{v_g:g\in G\}$ is $\|\,\|_\infty$-bounded, so $\{\frac{1}{\mu L}\sum_{g\in L}v_g:L\in[G]^{<\omega}\setminus\{0\}\}$ also is, and $\lim_{L\to\mathcal{F}_\emptyset}\|\frac{1}{\mu L}\sum_{g\in L}v_g\|_1=0$, as required.

3F Lemma (Austin p08a, 4.6) Let G be an abelian group, and suppose that I is a non-empty finite set such that every commuting measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is weakly measure-averaging. If $j\in I$, every commuting measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ has a j-pleasant extension.

proof (a) Set $\mathfrak{A}_0=\mathfrak{A}$, $\bar\mu_0=\bar\mu$ and $\bullet_{0i}=\bullet_i$ for $i\in I$. Given that $(\mathfrak{A}_m,\bar\mu_m,G,\langle\bullet_i^{(m)}\rangle_{i\in I})$ is a commuting measure-automorphism action system, then our hypothesis tells us that it is weakly measure-averaging; let $(\mathfrak{C}_m,\bar\nu_m,G,\langle\tilde\bullet_i^{(m)}\rangle_{i\in I\cup\{\infty\}},\langle\pi_i^{(m)}\rangle_{i\in I})$ be its Furstenberg self-joining over I, and $(\mathfrak{A}_{m+1},\bar\mu_{m+1},G,\langle\bullet_i^{(m+1)}\rangle_{i\in I},\phi_{m,m+1})$ the (I,j)-Furstenberg extension of $(\mathfrak{A}_m,\bar\mu_m,G,\langle\bullet_i^{(m)}\rangle_{i\in I})$. Continue.

For $l\le m$, define $\phi_{lm}:\mathfrak{A}_l\to\mathfrak{A}_m$ by taking ϕ_{ll} to be the identity on $\mathfrak{A}_l$ and $\phi_{l,m+1}=\phi_{m,m+1}\phi_{lm}$. Let $(\mathfrak{A}',\bar\mu',\langle\phi_m\rangle_{m\in\mathbb{N}})$ be the inductive limit of $\langle(\mathfrak{A}_m,\bar\mu_m)\rangle_{m\in\mathbb{N}},\langle\phi_{lm}\rangle_{l\le m})$. For each $i\in I$ we have an action $\bullet'_i$ of G on $\mathfrak{A}'$ defined by saying that $g\bullet'_i(\phi_m a)=\phi_m(g\bullet_i^{(m)}a)$ for $g\in G$, $m\in\mathbb{N}$ and $a\in\mathfrak{A}_m$ (1Ga); now $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I})$ is a measure-automorphism action system. Because all the systems $(\mathfrak{A}_m,G,\langle\bullet_i^{(m)}\rangle_{i\in I})$ and $(\mathfrak{C}_m,G,\langle\tilde\bullet_i^{(m)}\rangle_{i\in I\cup\{\infty\}})$ are commuting, so is $(\mathfrak{A}',G,\langle\bullet'_i\rangle_{i\in I})$. Of course $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I},\phi_0)$ is an extension of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$.

(b) Once again, the hypothesis of this lemma ensure that $(\mathfrak{A}',G,\langle\bullet'_i\rangle_{i\in I})$ is weakly measure-averaging and has a Furstenberg self-joining $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_i\rangle_{i\in I})$ over I. Now we can identify $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}})$

with the inductive limit of $(\langle(\mathfrak{C}_m,\bar\nu_m,G,\langle\tilde{\bullet}_i^{(m)}\rangle_{i\in I\cup\{\infty\}})\rangle_{m\in\mathbb{N}},\langle\phi_{l+1,m+1}\rangle_{l\le m})$. **P** By Proposition 3C, we have measure-preserving Boolean homomorphisms $\psi_{lm}:\mathfrak{C}_l\to\mathfrak{C}_m$ and $\psi_m:\mathfrak{C}_m\to\mathfrak{C}$, for $l\le m$, such that

$$\psi_{lm}\pi_i^{(l)}=\pi_i^{(m)}\phi_{lm},\quad \psi_m\pi_i^{(m)}=\pi_i\phi_m$$

for $l\le m$ and $i\in I$; and these homomorphisms are consistent with the actions, that is,

$$g\tilde{\bullet}_i^{(m)}(\psi_{lm}d)=\psi_{lm}(g\tilde{\bullet}_i^{(l)}d)$$

whenever $l\le m$, $i\in I\cup\{\infty\}$, $g\in G$ and $d\in\mathfrak{C}_l$. We need to check that $\bigcup_{m\in\mathbb{N}}\psi_m[\mathfrak{C}_m]$ is metrically dense in $\mathfrak{C}$, but this is easy; the closure of $\bigcup_{m\in\mathbb{N}}\psi_m[\mathfrak{C}_m]$ must include

$$\bigcup_{m\in\mathbb{N},i\in I}\psi_m[\pi_i^{(m)}[\mathfrak{A}]]=\bigcup_{i\in I}\pi_i[\bigcup_{m\in\mathbb{N}}\phi_m[\mathfrak{A}]]$$

and therefore includes $\bigcup_{i\in I}\pi_i[\mathfrak{A}']$ and the subalgebra it generates, which is dense in $\mathfrak{C}$ (see the construction in 3Ab). **Q**

(c) The formulae of the rest of this proof will be easier to read if I give names to the multiplicative Riesz homomorphisms corresponding to the measure-preserving Boolean homomorphisms here:

$$S_{lm}:L^0(\mathfrak{A}_l)\to L^0(\mathfrak{A}_m)\text{ from }\phi_{lm}:\mathfrak{A}_l\to\mathfrak{A}_m,$$

$$S_m:L^0(\mathfrak{A}_m)\to L^0(\mathfrak{A}')\text{ from }\phi_m:\mathfrak{A}_m\to\mathfrak{A}',$$

$$R_i^{(m)}:L^0(\mathfrak{A}_m)\to L^0(\mathfrak{C}_m)=L^0(\mathfrak{A}_{m+1})\text{ from }\pi_i^{(m)}:\mathfrak{A}_m\to\mathfrak{C}_m,$$

$$R_i:L^0(\mathfrak{A}')\to L^0(\mathfrak{C})\text{ from }\pi_i:\mathfrak{A}'\to\mathfrak{C},$$

$$T_{lm}:L^0(\mathfrak{C}_l)\to L^0(\mathfrak{C}_m)\text{ from }\psi_{lm}:\mathfrak{C}_l\to\mathfrak{C}_m,$$

$$T_m:L^0(\mathfrak{C}_m)\to L^0(\mathfrak{C})\text{ from }\psi_m:\mathfrak{C}_m\to\mathfrak{C}$$

for $l\le m$ and $i\in I$. The identities above become

$S_{m,m+1}=R_n^{(m)}$ because $\phi_{m,m+1}=\pi_n^{(m)}$,
$S_l=S_mS_{lm}$ because $\phi_l=\phi_m\phi_{lm}$,
$T_l=T_mT_{lm}$ because $\psi_l=\psi_m\psi_{lm}$,
$T_mR_i^{(m)}=R_iS_m$ because $\psi_m\pi_i^{(m)}=\pi_i\phi_m$.

In addition, we shall have

$$\int_d v\,d\bar\nu_m=\int_{\psi_md}T_mv\,d\bar\mu'\text{ whenever }d\in\mathfrak{C}_m\text{ and }u\in L^1(\mathfrak{C}_m,\bar\nu_m),$$

$$\int_a u\,d\bar\mu_m=\int_{\phi_md}S_mu\,d\bar\mu'\text{ whenever }a\in\mathfrak{A}_m\text{ and }u\in L^1(\mathfrak{A}_m,\bar\mu_m).$$

(d) For each $m\in\mathbb{N}$, let $\mathfrak{B}_m$ be the closed subalgebra of $\mathfrak{A}_m$ generated by

$$\{a:a\in\mathfrak{A}_m,\ g\bullet_j^{(m)}a=a\text{ for every }g\in G\}$$
$$\cup\bigcup_{i\in I\setminus\{j\}}\{a:a\in\mathfrak{A}_m,\ g\bullet_i^{(m)}a=g\bullet_j^{(m)}a\text{ for every }g\in G\},$$

and $P_m=Q_{\mathfrak{B}_m}$. Similarly, let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}'$ generated by

$$\{a:a\in\mathfrak{A}',\ g\bullet'_ja=a\text{ for every }g\in G\}$$
$$\cup\bigcup_{i\in I\setminus\{j\}}\{a:a\in\mathfrak{A}',\ g\bullet'_ia=g\bullet'_ja\text{ for every }g\in G\},$$

and $P=Q_{\mathfrak{B}}$. If $l\in\mathbb{N}$ and $u\in L^1(\mathfrak{A}_l,\bar\mu_l)$, then $PS_lu=\lim_{m\to\infty}S_mP_mS_{lm}u$ for $\|\,\|_1$. **P** By 1Fb, $\{a:a\in\mathfrak{A}',\ g\bullet'_ja=a$ for every $g\in G\}$ is the metric closure of

$$\bigcup_{m\in\mathbb{N}}\{\phi_ma:a\in\mathfrak{A}_m,\ g\bullet_j^{(m)}a=a\text{ for every }g\in G\};$$

applying the same result to the actions $(g,a)\mapsto g^{-1}\bullet_j^{(m)}(g\bullet_i^{(m)}a)$, we see that $\{a : a\in\mathfrak{A}',\ g\bullet'_i a = g\bullet'_j a$ for every $g\in G\}$ is the metric closure of

$$\bigcup_{m\in\mathbb{N}}\{\phi_m a : a\in\mathfrak{A}_m,\ g\bullet_i^{(m)}a = g\bullet_j^{(m)}a \text{ for every } g\in G\}.$$

So $\mathfrak{B}$ is the closure of $\bigcup_{m\in\mathbb{N}}\phi_m[\mathfrak{B}_m]$. Of course $\phi_{m,m+1}[\mathfrak{B}_m]\subseteq\mathfrak{B}_{m+1}$ for every m, so $\langle\phi_m[\mathfrak{B}_m]\rangle_{m\in\mathbb{N}}$ is non-decreasing. For each $m\ge l$, $S_mP_mS_{lm}u$ is the conditional expectation of S_lu on $\phi_m[\mathfrak{B}_m]$; the result follows at once, by the martingale convergence theorem (Fremlin 02, 367Qb). **Q**

(e) Suppose that $m\in\mathbb{N}$, that $u_i\in L^\infty(\mathfrak{A}_m)$ for $i\in I$ and that $d\in\mathfrak{C}_m=\mathfrak{A}_{m+1}$ is such that $g\tilde{\bullet}_\infty^{(m)}d=d$ for every $g\in G$. Then

$$\begin{aligned}\int_{\psi_m d}\prod_{i\in I}R_iS_mu_i\,d\bar\nu &= \int_{\psi_m d}\prod_{i\in I}T_mR_i^{(m)}u_i\,d\bar\nu = \int_{\psi_m d}T_m(\prod_{i\in I}R_i^{(m)}u_i)\,d\bar\nu\\ &= \int_d\prod_{i\in I}R_i^{(m)}u_i\,d\bar\nu_m = \int_d\prod_{i\in I}R_i^{(m)}u_i\,d\bar\mu_{m+1}\\ &= \int_d S_{m,m+1}u_j\times\prod_{i\in I\setminus\{j\}}R_i^{(m)}u_i\,d\bar\mu_{m+1}.\end{aligned}$$

Now $\tilde{\bullet}_\infty^{(m)}=\bullet_j^{(m+1)}$, so $d\in\mathfrak{B}_{m+1}$. While if $i\in I\setminus\{j\}$, then

$$g\bullet_i^{(m+1)}(\pi_i^{(m)}a)=g\tilde{\bullet}_i^{(m)}(\pi_i^{(m)}a)=g\tilde{\bullet}_\infty^{(m)}(\pi_i^{(m)}a)=g\bullet_j^{(m+1)}(\pi_i^{(m)}a)$$

for every $a\in\mathfrak{A}_m$ and $g\in G$, so that $\pi_i^{(m)}[\mathfrak{A}_m]\subseteq\mathfrak{B}_{m+1}$ and $P_{m+1}R_i^{(m)}u=R_i^{(m)}u$ for every $u\in L^1(\mathfrak{A}_m,\bar\mu_m)$. Accordingly

$$\begin{aligned}\int_{\psi_m d}\prod_{i\in I}R_iS_mu_i\,d\bar\nu &= \int_d S_{m,m+1}u_j\times\prod_{i\in I\setminus\{j\}}R_i^{(m)}u_i\,d\bar\mu_{m+1}\\ &= \int_d P_{m+1}(S_{m,m+1}u_j\times\prod_{i\in I\setminus\{j\}}R_i^{(m)}u_i)\,d\bar\mu_{m+1}\\ &= \int_d P_{m+1}S_{m,m+1}u_j\times\prod_{i\in I\setminus\{j\}}R_i^{(m)}u_i\,d\bar\mu_{m+1}\\ &= \int_{\phi_{m+1}d}S_{m+1}P_{m+1}S_{m,m+1}u_j\times\prod_{i\in I\setminus\{j\}}S_{m+1}R_i^{(m)}u_i\,d\bar\mu'.\end{aligned}$$

(f) Re-casting the formulae in (e) we get the following. Suppose that $l\in\mathbb{N}$, that $u_i\in L^\infty(\mathfrak{A}_l)$ for $i\in I$ and that $d\in\mathfrak{C}_l=\mathfrak{A}_{l+1}$ is such that $g\bullet_j^{(l+1)}d=g\tilde{\bullet}_\infty^{(l)}d=d$ for every $g\in G$. Then

$$\begin{aligned}\int_{\psi_l d}\prod_{i\in I}R_iS_lu_i\,d\bar\nu &= \lim_{m\to\infty}\int_{\psi_m\psi_{lm}d}\prod_{i\in I}R_iS_mS_{lm}u_i\,d\bar\nu\\ &= \lim_{m\to\infty}\int_{\phi_{m+1}\psi_{lm}d}S_{m+1}P_{m+1}S_{l,m+1}u_j\times\prod_{i\in I\setminus\{j\}}S_{m+1}R_i^{(m)}S_{lm}u_i\,d\bar\mu'\end{aligned}$$

(by (e), because $g\tilde{\bullet}_\infty^{(m)}(\psi_{lm}d)=\psi_{lm}(g\tilde{\bullet}_\infty^{(l)}d)=\psi_{lm}d$ for every $g\in G$)

$$= \lim_{m\to\infty}\int_{\phi_{m+1}\psi_{lm}d}S_mP_mS_{lm}u_j\times\prod_{i\in I\setminus\{j\}}S_{m+1}R_i^{(m)}S_{lm}u_i\,d\bar\mu'$$

(because $\lim_{m\to\infty}S_mP_mS_{lm}u_j=PS_lu_j=\lim_{m\to\infty}S_{m+1}P_{m+1}S_{l,m+1}u_j$ for the norm $\|\ \|_1$, by (d))

$$= \lim_{m\to\infty}\int_{\phi_{m+1}\psi_{lm}d}S_{m+1}P_{m+1}S_{m,m+1}P_mS_{lm}u_j\times\prod_{i\in I\setminus\{j\}}S_{m+1}R_i^{(m)}S_{lm}u_i\,d\bar\mu'$$

(because $\phi_{m,m+1}[\mathfrak{B}_m]\subseteq\mathfrak{B}_{m+1}$, so $P_{m+1}S_{m,m+1}P_m=S_{m,m+1}P_m$)

$$= \lim_{m\to\infty} \int_{\psi_l d} R_j S_m P_m S_{lm} u_j \times \prod_{i\in I\setminus\{j\}} R_i S_l u_i \, d\bar\nu$$

(by (e) again, applied to $\psi_{lm} d$, $P_m S_{lm} u_j$ and $\langle S_{lm} u_i\rangle_{i\in I\setminus\{j\}}$)

$$= \int_{\psi_l d} R_j P S_l u_j \times \prod_{i\in I\setminus\{j\}} R_i S_l u_i \, d\bar\nu.$$

(g) It follows that if $v_i \in L^\infty(\mathfrak{A}')$ for $i \in I$ and $c \in \mathfrak{C}$ is such that $g\tilde{\bullet}_\infty c = c$ for every $g \in G$, then

$$\int_c \prod_{i\in I} R_i v_i \, d\bar\nu = \int_c R_j P v_j \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu.$$

P Set $\gamma = \max_{i\in I} \|v_i\|_\infty$. Let $\epsilon > 0$. Then c belongs to the metric closure of $\{\psi_m d : m \in \mathbb{N},\ d \in \mathfrak{C}_m,\ g\tilde{\bullet}^{(m)}_\infty d = d$ for every $g \in G\}$ (1Gb). Also every v_i belongs to the $\| \|_1$-closure of $\{S_m u : m \in \mathbb{N},\ u \in L^\infty(\mathfrak{A}_m),\ \|u\|_\infty \le \gamma\}$. So there are an $l \in \mathbb{N}$, a $d \in \mathfrak{C}_l$ and $u_i \in L^\infty(\mathfrak{A}_l)$, for $i \in I$, such that

$$g\tilde{\bullet}^{(l)}_\infty d = d \text{ for every } g \in G, \quad \bar\nu(c \,\triangle\, \psi_l d) \le \epsilon,$$

$$\|u_i\|_\infty \le \gamma, \quad \|v_i - S_l u_i\|_1 \le \epsilon \text{ for every } i \in I.$$

It follows that

$$\|\prod_{i\le j} R_i v_i - \prod_{i\le j} R_i S_l u_i\|_1 \le (j+1)\epsilon\gamma^j$$

for every $j \le n$ (induce on j, recalling that S_l and every R_i are both $\| \|_1$-non-expanding and $\| \|_\infty$-non-expanding), so that

$$\|\prod_{i\in I} R_i v_i - \prod_{i\in I} R_i S_l u_i\|_1 \le (n+1)\epsilon\gamma^n.$$

Consequently

$$|\int_c \prod_{i\in I} R_i v_i \, d\bar\nu - \int_{\psi_l d} \prod_{i\in I} R_i S_l u_i \, d\bar\nu| \le (n+1)\epsilon\gamma^n + \epsilon.$$

Similarly,

$$|\int_c R_j v_j \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu - \int_{\psi_l d} R_j S_l u_j \times \prod_{i\in I\setminus\{j\}} R_i S_l u_i \, d\bar\nu| \le (1 + (n+1)\gamma^n)\epsilon.$$

Putting these together with (f), we get

$$|\int_c R_j P v_j \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu - \int_c R_j v_j \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu| \le 2\epsilon(1 + (n+1)\gamma^n).$$

As ϵ is arbitrary,

$$\int_c \prod_{i\in I} R_i v_i \, d\bar\nu = \int_c R_j P v_j \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu. \quad \mathbf{Q}$$

(h) We are nearly home. Take any $v_i \in L^\infty(\mathfrak{A}')$ for $i \in I$. Let $\mathfrak{D}$ be the fixed-point algebra $\{c : c \in \mathfrak{C},\ g\tilde{\bullet}_\infty c = c$ for every $g \in G\}$. We know that

$$\int_c R_j(v_j - P v_j) \times \prod_{i\in I\setminus\{j\}} R_i v_i \, d\bar\nu = 0$$

for every $c \in \mathfrak{D}$, that is, that

$$Q_{\mathfrak{D}}(R_j(v_j - P v_j) \times \prod_{i\in I\setminus\{j\}} R_i v_i) = 0.$$

By Lemma 3E,

$$\mathrm{WDL}_{g\to G}\big(g\bullet'_j(v_j - P v_j) \times \prod_{i\in I\setminus\{j\}} g\bullet'_i v_i\big) = 0$$

But this means that $(\mathfrak{A}', \bar\mu', G, \langle\bullet'_i\rangle_{i\in I})$ is a j-pleasant system. And we have known since (a) above that it is an extension of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$.

3G Theorem (Austin p08a, 1.1) Let G be an abelian group, I a non-empty finite set and $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system. Then $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ is measure-averaging.

proof We may suppose that $I = n+1$ for some $n \in \mathbb{N}$. Induce on n. If $n = 0$ the result is a special case of Proposition 1I. For the inductive step to $n \ge 1$, the inductive hypothesis tells us that the conditions of Lemma

2M are satisfied, so $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\le n})$ is weakly measure-averaging whenever it is a commuting measure-automorphism action system. Consequently, if $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\le n})$ is a commuting measure-automorphism action system, it has an n-pleasant extension $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\le n})$, by 3F. Take $\phi:\mathfrak{A}\to\mathfrak{A}'$ witnessing the extension, and $S:L^0(\mathfrak{A})\to L^0(\mathfrak{A}')$ the associated multiplicative Riesz homomorphism. By the inductive hypothesis and Lemma 2L, $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\le n})$ is measure-averaging. Let $\mathcal{F}\text{ø}$ be the Følner filter of G. If $u_0,\dots,u_n$ belong to $L^\infty(\mathfrak{A})$,

$$\mathrm{WDL}_{g\to G}\prod_{i\le n}g\bullet'_iSu_i$$

is defined in $L^1(\mathfrak{A}',\bar\mu')$, so

$$\begin{aligned}\|\frac{1}{\mu L}\sum_{g\in L}\prod_{i\le n}g\bullet_iu_i\,d\bar\mu-\frac{1}{\mu M}\not\!\!\int_M\prod_{i\le n}g\bullet_iu_i\,d\bar\mu\|_1&\\
=\|S(\frac{1}{\mu L}\sum_{g\in L}\prod_{i\le n}g\bullet_iu_i\,d\bar\mu-\frac{1}{\mu M}\not\!\!\int_M\prod_{i\le n}g\bullet_iu_i)\,d\bar\mu\|_1&\\
=\|\frac{1}{\mu L}\sum_{g\in L}\prod_{i\le n}S(g\bullet_iu_i)\,d\bar\mu-\frac{1}{\mu M}\not\!\!\int_M\prod_{i\le n}S(g\bullet_iu_i)\,d\bar\mu\|_1&\\
=\|\frac{1}{\mu L}\sum_{g\in L}\prod_{i\le n}g\bullet'_iSu_i\,d\bar\mu-\frac{1}{\mu M}\not\!\!\int_M\prod_{i\le n}g\bullet'_iSu_i\,d\bar\mu\|_1\to0&\end{aligned}$$

as L, $M\to\mathcal{F}\text{ø}$, and

$$\mathrm{WDL}_{g\to G}\prod_{i\le n}g\bullet_iu_i$$

is defined in $L^1(\mathfrak{A},\bar\mu)$. As $u_0,\dots,u_n$ are arbitrary, $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\le n})$ is measure-averaging, and the induction proceeds.

3H Corollary Let G be an abelian group and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system. Then $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ has a Furstenberg self-joining over J for any finite set $J\subseteq I$.

4 Agreeable and isotropized extensions

4A Definition (Austin p08b, 4.1)

(a) Let I be a set, J a finite subset of I and j a member of J and G an abelian group. A commuting measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is (J,j)**-agreeable** if, writing $\mathfrak{B}$ for the closed subalgebra of $\mathfrak{A}$ generated by

$$\bigcup_{i\in J\setminus\{j\}}\{a:a\in\mathfrak{A},\ g\bullet_ia=g\bullet_ja\text{ for every }g\in G\},$$

we have

$$\mathrm{WDL}_{g\to G}\int g\bullet_j(u_j-Q_{\mathfrak{B}}u_j)\times\prod_{i\in J\setminus\{j\}}g\bullet_iu_id\bar\mu=0$$

whenever $\langle u_i\rangle_{i\in J}$ is a family in $L^\infty(\mathfrak{A})$.

(Compare, but do not confuse, with 2J.)

(b) A commuting measure-automorphism action system $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is **fully agreeable** if it is (J,j)-agreeable whenever $j\in J\in[I]^{<\omega}$.

4B Lemma (Austin p08b, §4) Let G be an abelian group, κ an ordinal and $(\langle(\mathfrak{A}_\xi,\bar\mu_\xi,G,\langle\bullet_i^{(\xi)}\rangle_{i\in I})\rangle_{\xi<\kappa}$, $\langle\phi_{\eta\xi}\rangle_{\eta\le\xi<\kappa})$ an inductive system of commuting measure-automorphism action systems with inductive limit $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I},\langle\phi_\xi\rangle_{\xi<\kappa})$. Suppose that $J\in[I]^{<\omega}$, $j\in J$ and a cofinal set $M\subseteq\kappa$ are such that, for $\xi\in M$, $(\mathfrak{A}_{\xi+1},\bar\mu_{\xi+1},G,\langle\bullet_i^{(\xi+1)}\rangle_{i\in I},\phi_{\xi,\xi+1})$ is the (J,j)-Furstenberg extension of $(\mathfrak{A}_\xi,\bar\mu_\xi,G,\langle\bullet_i^{(\xi)}\rangle_{i\in I})$. Then $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is (J,j)-agreeable.

proof (a) For $\xi<\kappa$ let $\mathfrak{B}_\xi$ be the closed subalgebra of $\mathfrak{A}_\xi$ generated by

$$\bigcup_{i\in J\setminus\{j\}}\{a:a\in\mathfrak{A}_\xi,\ g\bullet_i^{(\xi)}a=g\bullet_j^{(\xi)}a\text{ for every }g\in G\},$$

and let $\mathfrak{B}$ be the closed subalgebra of $\mathfrak{A}$ generated by

$$\bigcup_{i\in J\setminus\{j\}}\{a : a\in\mathfrak{A},\ g\bullet_i a = g\bullet_j a \text{ for every } g\in G\};$$

set $P_\xi = Q_{\mathfrak{B}_\xi}$ and $P = Q_{\mathfrak{B}}$. Then $\phi_{\eta\xi}[\mathfrak{B}_\eta]\subseteq\mathfrak{B}_\xi$ whenever $\eta\le\xi$, and $\mathfrak{B}$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{\xi<\kappa}\phi_\xi[\mathfrak{B}_\xi]$ (2Gb), so $PS_\eta u = \lim_{\xi\to\kappa} S_\xi P_\xi S_{\eta\xi}u$ for $\|\,\|_1$ whenever $\eta<\kappa$ and $u\in L^1(\mathfrak{A}_\eta,\bar\mu_\eta)$, writing $S_{\eta\xi} : L^0(\mathfrak{A}_\eta)\to L^0(\mathfrak{A}_\xi)$ and $S_\xi : L^0(\mathfrak{A}_\xi)\to L^0(\mathfrak{A})$ for the multiplicative Riesz homomorphisms corresponding to $\phi_{\eta\xi}:\mathfrak{A}_\eta\to\mathfrak{A}_\xi$ and $\phi_\xi:\mathfrak{A}_\xi\to\mathfrak{A}$, as in the proof of 3H.

(b) Suppose that $v_i\in L^\infty(\mathfrak{A})$ for each $i\in J$; set $\gamma=\max_{i\in J}\|v_i\|_\infty$. Let $\epsilon>0$.

(i) There are a $\xi\in M$ and $u_i\in L^\infty(\mathfrak{A}_\xi)$, for $i\in J$, such that

$$\|u_i\|_\infty\le\gamma,\ \|v_i - S_\xi u_i\|_1\le\epsilon \text{ for every } i\in J,$$

$$\|PS_\xi u_j - S_\xi P_\xi u_j\|_1\le\epsilon,\quad \|PS_\xi u_j - S_{\xi+1}P_{\xi+1}S_{\xi,\xi+1}u_j\|_1\le\epsilon.$$

P First, there are an $\eta<\kappa$ and $u_i'\in L^\infty(\mathfrak{A}_\eta)$, for $i\in J$, such that $\|v_i - S_\eta u_i\|_1\le\epsilon$ for every i; replacing u_i' by $\text{med}(-\gamma\chi1, u_i', \gamma\chi1)$ if necessary, we can arrange that $\|u_i'\|_\infty\le\gamma$ for every i. Next, by the martingale convergence theorem, there is a $\zeta<\kappa$ such that $\eta\le\zeta$ and $\|S_\xi P_\xi S_{\eta\xi}u_i' - PS_\eta u_i'\|_1\le\epsilon$ whenever $i\in J$ and $\zeta\le\xi<\kappa$. Since M is cofinal with κ, there is a $\xi\in M$ such that $\xi\ge\zeta$; set $u_i = S_{\eta\xi}u_i'$ for each i. **Q**

(ii) It follows that

$$\begin{aligned}\Big|\int\prod_{i\in J} g\bullet_i v_i\,d\bar\mu - \int\prod_{i\in J} g\bullet_i^{(\xi)} u_i\,d\bar\mu_\xi\Big| &= \Big|\int\prod_{i\in J} g\bullet_i v_i - \prod_{i\in J} g\bullet_i S_\xi u_i\,d\bar\mu\Big|\\ &\le \gamma^{\#(J)-1}\sum_{i\in J}\|g\bullet_i v_i - g\bullet_i S_\xi u_i\|_1\\ &\le\gamma^{\#(J)-1}\#(J)\epsilon\end{aligned}$$

for every $g\in G$, so that

$$\Big|\text{WDL}_{g\to G}\int\prod_{i\in J} g\bullet_i v_i\,d\bar\mu - \text{WDL}_{g\to G}\int\prod_{i\in J} g\bullet_i^{(\xi)}u_i\,d\bar\mu_\xi\Big|\le\gamma^{\#(J)-1}\#(J)\epsilon.$$

(iii) Writing $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in J\cup\{\infty\}},\langle\pi_i\rangle_{i\in J})$ for the Furstenberg self-joining of $(\mathfrak{A}_\xi,\bar\mu_\xi,G,\langle\bullet_i^{(\xi)}\rangle_{i\in I})$, and $R_i : L^0(\mathfrak{A}_\xi)\to L^0(\mathfrak{C})$ for the Riesz homomorphism corresponding to $\pi_i:\mathfrak{A}_\xi\to\mathfrak{C}$,

$$\begin{aligned}\text{WDL}_{g\to G}\int\prod_{i\in J} g\bullet_i^{(\xi)}u_i\,d\bar\mu_\xi &= \int\prod_{i\in J}R_iu_i\,d\bar\nu = \int\prod_{i\in J}R_iu_i\,d\bar\mu_{\xi+1}\\ &=\int P_{\xi+1}(R_ju_j\times\prod_{i\in I\setminus\{j\}}R_iu_i)\,d\bar\mu_{\xi+1}\\ &=\int P_{\xi+1}R_ju_j\times\prod_{i\in I\setminus\{j\}}R_iu_i\,d\bar\mu_{\xi+1}\end{aligned}$$

(because $g\bullet_i^{(\xi+1)}R_iu_i = g\tilde\bullet_iR_iu_i = R_i(g\bullet_i^{(\xi)}u_i) = g\tilde\bullet_\infty(R_iu_i) = g\bullet_j^{(\xi+1)}R_iu_i$ for every $g\in G$, $i\in J\setminus\{j\}$, so $P_{\xi+1}R_iu_i = R_iu_i$ for every $i\in J\setminus\{j\}$)

$$=\int P_{\xi+1}S_{\xi,\xi+1}u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\mu_{\xi+1}.$$

(iv)

$$\|P_{\xi+1}S_{\xi,\xi+1}u_j - S_{\xi,\xi+1}P_\xi u_j\|_1 = \|S_{\xi+1}P_{\xi+1}S_{\xi,\xi+1}u_j - S_\xi P_\xi u_j\|_1\le 2\epsilon,$$

so

$$\Bigl|\int P_{\xi+1}S_{\xi,\xi+1}u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\mu_{\xi+1}$$
$$-\int S_{\xi,\xi+1}P_\xi u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\mu_{\xi+1}\Bigr|\le 2\epsilon\gamma^{\#(J)-1}.$$

(v)

$$\begin{aligned}\int S_{\xi,\xi+1}P_\xi u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\mu_{\xi+1}&=\int R_jP_\xi u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\nu\\&=\mathrm{WDL}_{g\to G}\int g\bullet_j^{(\xi)}P_\xi u_j\times\prod_{i\in J\setminus\{j\}}g\bullet_i^{(\xi)}u_i\,d\bar\mu_\xi\\&=\mathrm{WDL}_{g\to G}\int g\bullet_jS_\xi P_\xi u_j\times\prod_{i\in J\setminus\{j\}}g\bullet_iS_iu_i\,d\bar\mu.\end{aligned}$$

(vi) Since

$$\begin{aligned}\|g\bullet_j(Pv_j)-g\bullet_jS_\xi P_\xi u_j\|_1&=\|Pv_j-S_\xi P_\xi u_j\|_1\\&\le\|Pv_j-PS_\xi u_j\|_1+\|PS_\xi u_j-S_\xi P_\xi u_j\|_1\\&\le\|v_j-S_\xi u_j\|_1+\epsilon\le 2\epsilon,\\\|g\bullet_iv_i-g\bullet_iS_\xi u_i\|_1&=\|v_i-S_\xi u_i\|_1\le\epsilon\end{aligned}$$

for every $g\in G$ and $i\in I\setminus\{j\}$,

$$\Bigl|\mathrm{WDL}_{g\to G}\int g\bullet_jS_\xi P_\xi u_j\times\prod_{i\in J\setminus\{j\}}g\bullet_iS_iu_i\,d\bar\mu$$
$$-\mathrm{WDL}_{g\to G}\int g\bullet_jPv_j\times\prod_{i\in J\setminus\{j\}}g\bullet_iv_i\,d\bar\mu\Bigr|\le\gamma^{\#(J)-1}(\#(J)+1)\epsilon.$$

(vii) Assembling (ii)-(vi), we get

$$\Bigl|\mathrm{WDL}_{g\to G}\int\prod_{i\in J}g\bullet_iv_i\,d\bar\mu-\mathrm{WDL}_{g\to G}\int g\bullet_jPu_j\times\prod_{i\in J\setminus\{j\}}g\bullet_iv_i\,d\bar\mu\Bigr|$$
$$\le\gamma^{\#(J)-1}(\epsilon\#(J)+2\epsilon+(\#(J)+1)\epsilon).$$

As ϵ is arbitrary,

$$\mathrm{WDL}_{g\to G}\int\prod_{i\in J}g\bullet_iv_i\,d\bar\mu=\mathrm{WDL}_{g\to G}\int g\bullet_jPu_j\times\prod_{i\in J\setminus\{j\}}g\bullet_iv_i\,d\bar\mu.$$

As $\langle v_i\rangle_{i\in J}$ is arbitrary, $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is (J,j)-agreeable.

4B Definition (Austin p08b, 5.1) Let G be a group, I a set, and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a measure-automorphism action system. If $J\subseteq I$, write

$$\mathfrak{B}_J=\{a:a\in\mathfrak{A},\ g\bullet_ia=g\bullet_ja\text{ for all }i,\ j\in J\text{ and }g\in G\}.$$

If $j\in J\subseteq I$, we say that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is **(J,j)-isotropized** if

$$\mathfrak{B}_J\cap\bigvee_{i\in I\setminus J}\mathfrak{B}_{\{i,j\}}=\bigvee_{i\in I\setminus J}\mathfrak{B}_{J\cup\{i\}}.$$

$(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is **fully isotropized** if it is (J,j)-isotropized whenever $j\in J\subseteq I$.

4D Construction (a) Let $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ be a measure-automorphism action system, and $j\in J\subseteq I$. Set $\mathfrak{B}_J=\{a:a\in\mathfrak{A},\ g\bullet_ia=g\bullet_ja$ whenever $i,\ j\in J$ and $g\in G\}$. The **(J,j)-isotropizing extension** of

$(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I},\psi_0)$, constructed as follows. $(\mathfrak{A}',\bar\mu',\psi_0,\psi_1)$ is the relative free product of $(\mathfrak{A},\bar\mu)$ with itself over $\mathfrak{B}_J$ (Fremlin 03, 458N[6]). For $i\in I$ and $g\in G$, we can define $g\bullet'_i b$, for $b\in\mathfrak{A}'$, by setting

$$g\bullet'_i(\psi_0 a)=\psi_0(g\bullet_i a),$$
$$g\bullet'_i(\psi_1 a)=\psi_1(g\bullet_i a) \text{ if } i\in I\setminus J,$$
$$=\psi_1(g\bullet_j a) \text{ if } i\in J$$

whenever $g\in G$ and $a\in\mathfrak{A}$, and requiring that $b\mapsto g\bullet'_i b:\mathfrak{A}'\to\mathfrak{A}'$ is a measure-preserving Boolean homomorphism for every $g\in G$. **P** The point is that if $i\in J$ and $a\in\mathfrak{B}_J$ then $g\bullet_j a=g\bullet_i a\in\mathfrak{B}_J$, so $\psi_0(g\bullet_i a)=\psi_1(g\bullet_j a)$. We can therefore apply the defining universal mapping theorem for the relative free product (Fremlin 03, 458O[7]) to see that there is indeed a (unique) measure-preserving Boolean homomorphism from $\mathfrak{A}'$ to itself satisfying the given formulae. **Q**

It is now elementary to check that every $\bullet'_i$ is an action of G on $\mathfrak{A}'$, so that $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I})$ is a measure-automorphism action system. And the formula for $g\bullet'_i(\psi_0 a)$ is just what we need to ensure that $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I},\psi_0)$ is an extension of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$.

(b) If $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is a commuting system, then a similar calculation shows that $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I})$ is also commuting.

4E Lemma Let $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ be a measure-automorphism action system, and $j\in J\subseteq I$. Let $(\mathfrak{A}',\bar\mu',G,\langle\bullet'_i\rangle_{i\in I},\psi_0)$ be the (J,j)-isotropizing extension of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$. For $K\subseteq I$, set

$$\mathfrak{B}_K=\{a:a\in\mathfrak{A},\ g\bullet_i a=g\bullet_k a \text{ for all } i,\ k\in K \text{ and } g\in G\},$$

$$\mathfrak{B}'_K=\{a:a\in\mathfrak{A}',\ g\bullet'_i a=g\bullet'_k a \text{ for all } i,\ k\in K \text{ and } g\in G\};$$

set

$$\mathfrak{D}=\mathfrak{B}_J\cap\bigvee\nolimits_{i\in I\setminus J}\mathfrak{B}_{\{i,j\}}\subseteq\mathfrak{A},\quad \mathfrak{C}=\bigvee\nolimits_{i\in I\setminus J}\mathfrak{B}'_{J\cup\{i\}}\subseteq\mathfrak{A}'.$$

Then $\psi_0[\mathfrak{D}]\subseteq\mathfrak{C}$.

proof Take $d\in\mathfrak{D}$ and $\epsilon>0$. Then there are $n\in\mathbb{N}$, a finite set $K\subseteq I\setminus J$ and a family $\langle c_{rk}\rangle_{r\le n,k\in K}$ such that $c_{rk}\in\mathfrak{B}_{\{k,j\}}$ for $r\le n$ and $k\in K$ and $\bar\mu(d\triangle d')\le\epsilon$, where $d'=\sup_{r\le n}\inf_{k\in K}c_{rk}$. Now if $r\le n$, $k\in K$, $i\in J$ and $g\in G$,

$$g\bullet'_k(\psi_1 c_{rk})=\psi_1(g\bullet_k c_{rk})=\psi_1(g\bullet_j c_{rk})=g\bullet'_i(\psi_1 c_{rk}),$$

so $\psi_1 c_{rk}\in\mathfrak{B}'_{J\cup\{k\}}\subseteq\mathfrak{C}$; accordingly $\psi_1 d'\in\mathfrak{C}$. Also

$$\bar\mu'(\psi_0 d\triangle\psi_1 d')=\bar\mu'(\psi_1 d\triangle\psi_1 d')$$

(because $d\in\mathfrak{B}_J$)

$$=\bar\mu(d\triangle d')\le\epsilon.$$

As ϵ is arbitrary and $\mathfrak{C}$ is closed, $\psi_0 d\in\mathfrak{C}$; as d is arbitrary, we have the result.

4F Lemma (Austin p08b, §5) Let G be an abelian group, κ an ordinal of uncountable cofinality, and $(\langle(\mathfrak{A}_\xi,\bar\mu_\xi,G,\langle\bullet^{(\xi)}_i\rangle_{i\in I})\rangle_{\xi<\kappa},\langle\phi_{\eta\xi}\rangle_{\eta\le\xi<\kappa})$ an inductive system of commuting measure-automorphism action systems with inductive limit $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I},\langle\phi_\xi\rangle_{\xi<\kappa})$. Suppose that $J\subseteq I$, $j\in J$ and a cofinal set $M\subseteq\kappa$ are such that, for $\xi\in M$, $(\mathfrak{A}_{\xi+1},\bar\mu_{\xi+1},G,\langle\bullet^{(\xi+1)}_i\rangle_{i\in I},\phi_{\xi,\xi+1})$ is the (J,j)-isotropizing extension of $(\mathfrak{A}_\xi,\bar\mu_\xi,G,\langle\bullet^{(\xi)}_i\rangle_{i\in I})$. Then $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is (J,j)-isotropized.

proof (a) For $K\subseteq I$ and $\xi<\kappa$ set

[6] Formerly 458J.
[7] Formerly 458K.

$$\mathfrak{C}_K^{(\xi)} = \{a : a \in \mathfrak{A}_\xi,\ g\bullet_i^{(\xi)} a = g\bullet_k^{(\xi)} a \text{ for all } i,\ k \in K \text{ and } g \in G\},$$

$$\mathfrak{C}_K = \{a : a \in \mathfrak{A},\ g\bullet_i a = g\bullet_k a \text{ for all } i,\ k \in K \text{ and } g \in G\};$$

set

$$\mathfrak{D}_\xi = \mathfrak{C}_J^{(\xi)} \cap \bigvee_{i\in I\setminus J} \mathfrak{C}_{\{i,j\}}^{(\xi)}, \quad \mathfrak{E}_\xi = \bigvee_{i\in I\setminus J} \mathfrak{C}_{J\cup\{i\}}^{(\xi)} \subseteq \mathfrak{A}_\xi$$

for $\xi < \kappa$ and

$$\mathfrak{D} = \mathfrak{C}_J \cap \bigvee_{i\in I\setminus J} \mathfrak{C}_{\{i,j\}}, \quad \mathfrak{E} = \bigvee_{i\in I\setminus J} \mathfrak{C}_{J\cup\{i\}} \subseteq \mathfrak{A}.$$

Because $\operatorname{cf}\kappa > \omega$, $\mathfrak{A} = \bigcup_{\xi<\kappa} \phi_\xi[\mathfrak{A}_\xi]$; consequently $\mathfrak{C}_K = \bigcup_{\xi<\kappa} \phi_\xi[\mathfrak{C}_K^{(\xi)}]$ for every $K \subseteq I$, and $\mathfrak{D} = \bigcup_{\xi<\kappa} \phi_\xi[\mathfrak{D}_\xi]$.

(b) Take any $d \in \mathfrak{D}$. Then there is a $\xi \in M$ such that $d \in \phi_\xi[\mathfrak{D}_\xi]$; set $d' = \phi_\xi^{-1}(d)$. By 4E, $\phi_{\xi,\xi+1} d' \in \mathfrak{E}_{\xi+1}$. But this means that

$$d = \phi_{\xi+1}\phi_{\xi,\xi+1} d' \in \phi_{\xi+1}[\mathfrak{E}_{\xi+1}] \subseteq \mathfrak{E}.$$

As d is arbitrary, $\mathfrak{D} \subseteq \mathfrak{E}$. It is elementary to check from their definitions that $\mathfrak{D}$ includes $\mathfrak{E}$, so they are equal, that is, $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ is (J, j)-isotropized.

4G Proposition Let G be an abelian group and $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system. Then it has an extension which is commuting, fully isotropized and fully agreeable.

proof Set $\kappa = \max(\omega_1, 2^{\#(I)})$. Then we can build inductively an inductive system $(\langle(\mathfrak{A}_\xi, \bar\mu_\xi, G, \langle\bullet_i^{(\xi)}\rangle)_{i\in I}\rangle_{\xi<\kappa}, \langle\phi_{\eta\xi}\rangle_{\eta\le\xi<\kappa})$ of commuting measure-automorphism action systems such that $(\mathfrak{A}_0, \bar\mu_0, G, \langle\bullet_i^{(0)}\rangle_{i\in I}) = (\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ and

$$\{\xi : (\mathfrak{A}_{\xi+1}, \bar\mu_{\xi+1}, G, \langle\bullet_i^{(\xi+1)}\rangle_{i\in I}, \phi_{\xi,\xi+1}) \text{ is the } (J,j)\text{-Furstenberg extension of } (\mathfrak{A}_\xi, \bar\mu_\xi, G, \langle\bullet_i^{(\xi)}\rangle_{i\in I})\}$$

is cofinal with κ whenever $j \in J \in [I]^{<\omega}$, and

$$\{\xi : (\mathfrak{A}_{\xi+1}, \bar\mu_{\xi+1}, G, \langle\bullet_i^{(\xi+1)}\rangle_{i\in I}, \phi_{\xi,\xi+1}) \text{ is the } (J,j)\text{-isotropizing extension of } (\mathfrak{A}_\xi, \bar\mu_\xi, G, \langle\bullet_i^{(\xi)}\rangle_{i\in I})\}$$

is cofinal with κ whenever $j \in J \subseteq I$. Now if $(\mathfrak{A}', \bar\mu', G, \langle\bullet_i'\rangle_{i\in I}, \langle\phi_\xi\rangle_{\xi<\kappa})$ is the inductive limit of this system, Lemmas 4B and 4F tell us that $(\mathfrak{A}', \bar\mu', G, \langle\bullet_i'\rangle_{i\in I})$ is fully agreeable and fully isotropized, and of course $(\mathfrak{A}', \bar\mu', G, \langle\bullet_i'\rangle_{i\in I}, \phi_0)$ is an extension of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$.

5 More about Furstenberg self-joinings

5A Alternative description of agreeable systems Let G be an abelian group, $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ a commuting measure-preserving action system, J a finite subset of I, and j a member of J. Let $(\mathfrak{C}, \bar\nu, G, \langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}}, \langle\pi_i\rangle_{i\in J})$ be the Furstenberg self-joining of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ over J. Set

$$\mathfrak{B} = \bigvee_{i\in J\setminus\{j\}} \{a : a \in \mathfrak{A},\ g\bullet_i a = g\bullet_j a \text{ for every } g \in G\} \subseteq \mathfrak{A}.$$

Then $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ is (J, j)-agreeable iff $\pi_j[\mathfrak{A}]$ and $\bigvee_{i\in J\setminus\{j\}} \pi_i[\mathfrak{A}]$ are relatively independent over $\pi_j[\mathfrak{B}]$.

proof For $j \in J$, let $R_j : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ be the Riesz homomorphism defined from $\pi_j : \mathfrak{A} \to \mathfrak{C}$. Set $\mathfrak{D} = \bigvee_{i\in I\setminus\{j\}} \pi_i[\mathfrak{A}] \subseteq \mathfrak{C}$. We have

$$(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I}) \text{ is } (J,j)\text{-agreeable}$$
$$\iff \mathrm{WDL}_{g\to G} \int g\bullet_j(Q_{\mathfrak{B}} u_j) \times \prod_{i\in J\setminus\{j\}} g\bullet_i u_i \, d\bar\mu = \mathrm{WDL}_{g\to G} \int \prod_{i\in J} g\bullet_i u_i \, d\bar\mu$$
$$\text{whenever } \langle u_i\rangle_{i\in J} \in L^\infty(\mathfrak{A})^J$$

(4Aa)

$$\iff \int R_jQ_{\mathfrak{B}}u_j\times\prod_{i\in J\setminus\{j\}}R_iu_i\,d\bar\nu=\int\prod_{i\in J}R_iu_i\,d\bar\nu$$

whenever $\langle u_i\rangle_{i\in J}\in L^\infty(\mathfrak{A})^J$

(3Ac)

$$\iff \int R_jQ_{\mathfrak{B}}\chi a_j\times\prod_{i\in J\setminus\{j\}}R_i\chi a_i\,d\bar\nu=\int\prod_{i\in J}R_i\chi a_i d\bar\nu$$

whenever $\langle a_i\rangle_{i\in J}\in\mathfrak{A}^J$

$$\iff \int_d R_jQ_{\mathfrak{B}}\chi a_j\,d\bar\nu=\bar\nu(d\cap\pi_ja_j)$$

whenever $\langle a_i\rangle_{i\in J}\in\mathfrak{A}^J$ and $d=\inf_{i\in J\setminus\{j\}}\pi_ia_i$

$$\iff \int_d R_jQ_{\mathfrak{B}}\chi a_j\,d\bar\nu=\bar\nu(d\cap\pi_ja_j)\text{ whenever }a_j\in\mathfrak{A}\text{ and }d\in\mathfrak{D}$$

(because $\{\inf_{i\in J\setminus\{j\}}\pi_ia_i:a_i\in\mathfrak{A}$ for every $i\in J\setminus\{j\}\}$ is closed under finite infima and generates $\mathfrak{D}$)

$$\iff \int_d Q_{\pi_j[\mathfrak{B}]}R_j\chi a_j\,d\bar\nu=\bar\nu(d\cap\pi_ja_j)\text{ whenever }a_j\in\mathfrak{A}\text{ and }d\in\mathfrak{D}$$

(because $R_jQ_{\mathfrak{B}}=Q_{\pi_j[\mathfrak{B}]}R_j$ (Fremlin 02, 365Xq[8]))

$$\iff \int(Q_{\pi_j[\mathfrak{B}]}\chi c)\times\chi d\,d\bar\nu=\int\chi c\times\chi d\,d\bar\nu\text{ whenever }c\in\pi_j[\mathfrak{A}]\text{ and }d\in\mathfrak{D}$$

$$\iff \int(Q_{\pi_j[\mathfrak{B}]}\chi c)\times(Q_{\pi_j[\mathfrak{B}]}\chi d)\,d\bar\nu=\int\chi c\times\chi d\,d\bar\nu$$

whenever $c\in\pi_j[\mathfrak{A}]$ and $d\in\mathfrak{D}$

$\iff$ $\pi_j[\mathfrak{A}]$ and $\mathfrak{D}$ are relatively independent over $\pi_j[\mathfrak{B}]$.

5B Lemma (Austin p08b, 3.2) Let G be an abelian group, $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system and J a finite subset of I. Let $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_i\rangle_{i\in J})$ be the Furstenberg self-joining of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ over J.

(a) If j, $k\in J$ and $a\in\mathfrak{A}$ is such that $g\bullet_ja=g\bullet_ka$ for every $g\in G$, then $\pi_ja=\pi_ka$.

(b) If $K\subseteq J$ and $\mathfrak{B}_K=\{a:a\in\mathfrak{A},\ g\bullet_ja=g\bullet_ka$ for all $g\in G$ and $j,\ k\in K\}$, then $\pi_j[\mathfrak{B}_K]=\pi_k[\mathfrak{B}_K]$ for all j, $k\in K$.

proof (a) If $j=k$ this is trivial. Otherwise, by 3A(c-i),

$$\bar\nu(\pi_ja\cap\pi_ka)=\mathrm{WDL}_{g\to G}\,\bar\mu(g\bullet_ja\cap g\bullet_ka)=\mathrm{WDL}_{g\to G}\,\bar\mu(g\bullet_ja)=\bar\nu\pi_ja$$

and $\pi_ja\subseteq\pi_ka$; similarly, $\pi_ka\subseteq\pi_ja$ and the two are equal.

(b) follows at once.

5C Definition (Austin p08b, 3.3) In the context of part (b) of 5B, I will call the common value $\pi_j[\mathfrak{B}_K]$ the **divaricate copy** of $\mathfrak{B}_K$ in $\mathfrak{C}$. For definiteness, if K is empty, I will say that the divaricate copy of $\mathfrak{B}_\emptyset=\mathfrak{A}$ is $\mathfrak{C}$.

5D Lemma (Austin p08b, 6.1) Let G be an abelian group, I a finite set and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system which is fully isotropized and fully agreeable. Let $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_i\rangle_{i\in I})$ be the Furstenberg self-joining of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ over I. For $J\subseteq I$ set

$$\mathfrak{B}_J=\{a:a\in\mathfrak{C},\ g\bullet_ia=g\bullet_ja\text{ for all }i,\ j\in J\text{ and }g\in G\},$$

and let $\mathfrak{B}^*_J\subseteq\mathfrak{C}$ be the divaricate copy of $\mathfrak{B}$ (5C). Let $\mathcal{J}\subset\mathcal{P}I$ be such that $K\in\mathcal{J}$ whenever $J\in\mathcal{J}$ and $J\subseteq K\subseteq I$, and L a maximal element of $\mathcal{P}I\setminus\mathcal{J}$. Set

[8] Later editions only.

$$\mathfrak{D} = \bigvee_{J\in\mathcal{J}} \mathfrak{B}^*_J,$$

$$\mathfrak{E} = \bigvee_{L\subseteq J\in\mathcal{J}} \mathfrak{B}^*_J.$$

Then $\mathfrak{D}$ and $\mathfrak{B}^*_L$ are relatively independent over $\mathfrak{E}$.

proof (a) If L is empty, then $\mathfrak{D} = \mathfrak{E}$ and $\mathfrak{B}^*_L = \mathfrak{C}$, so the result is trivial. If $\mathcal{J} = \emptyset$ then $\mathfrak{D} = \mathfrak{E} = \{0, 1\}$ and again the result is trivial. Otherwise, fix $j \in L$. Set $\mathfrak{B} = \bigvee_{i\in I\setminus L} \mathfrak{B}_{\{i,j\}}$ and $\mathfrak{B}' = \bigvee_{i\in I\setminus L} \mathfrak{B}_{L\cup\{i\}}$. Because $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ is (L, j)-isotropized,

$$\mathfrak{B}_L \cap \mathfrak{B} = \mathfrak{B}',$$

so $Q_{\mathfrak{B}'} = Q_{\mathfrak{B}}Q_{\mathfrak{B}_L}$ (2De).

Because $L \notin \mathcal{J}$, $\bigvee_{L\subseteq J\in\mathcal{J}} \mathfrak{B}_J \subseteq \mathfrak{B}'$; on the other hand, by the maximality of L, $\mathfrak{B}' \subseteq \bigvee_{L\subseteq J\in\mathcal{J}} \mathfrak{B}_J$. Now

$$\mathfrak{E} = \bigvee_{L\subseteq J\in\mathcal{J}} \mathfrak{B}^*_J = \bigvee_{L\subseteq J\in\mathcal{J}} \pi_j[\mathfrak{B}_J] = \pi_j[\bigvee_{L\subseteq J\in\mathcal{J}} \mathfrak{B}_J] = \pi_j[\mathfrak{B}'].$$

Set $I' = (I\setminus L)\cup\{j\}$, and let $(\mathfrak{C}', \bar\nu', G, \langle\tilde\bullet'_i\rangle_{i\in I'\cup\{\infty\}}, \langle\pi'_i\rangle_{i\in I'})$ be the Furstenberg self-joining of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ over I'.

Let $J_0, \dots, J_n$ enumerate the minimal elements of $\mathcal{J}$. Since $L \notin \mathcal{J}$, we can find $i_m \in J_m \setminus L$ for each $m \le n$. If $J \in \mathcal{J}$, there is an $m \le n$ such that $J \supseteq J_m$ and $\mathfrak{B}_J \subseteq \mathfrak{B}_{J_m}$. So $\mathfrak{D} = \bigvee_{m\le n} \mathfrak{B}^*_{J_m}$. Suppose that $a_m \in \mathfrak{B}_{J_m}$ for $m \le n$, and that $b \in \mathfrak{B}_L$. Then

$$\bar\nu(\pi_j b \cap \inf_{m\le n} \pi_{i_m} a_m) = \mathrm{WDL}_{g\to G}\, \bar\mu(g\bullet_j b \cap \inf_{m\le n} g\bullet_{i_m} a_m)$$

(note that it makes no difference if the i_m are not all distinct)

$$= \bar\nu'(\pi'_j b \cap \inf_{m\le n} \pi'_{i_m} a_m)$$

$$= \int R'_j Q_{\mathfrak{B}}\chi b \times \chi(\inf_{m\le n} \pi'_{i_m} a_m) d\bar\nu'$$

(because $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ is (I', j)-agreeable)

$$= \int R_j Q_{\mathfrak{B}}\chi b \times \chi(\inf_{m\le n} \pi_{i_m} a_m) d\bar\nu$$

$$= \int R_j Q_{\mathfrak{B}} Q_{\mathfrak{B}_L}\chi b \times \chi(\inf_{m\le n} \pi_{i_m} a_m) d\bar\nu$$

$$= \int R_j Q_{\mathfrak{B}'}\chi b \times \chi(\inf_{m\le n} \pi_{i_m} a_m) d\bar\nu$$

$$= \int Q_{\mathfrak{E}} R_j\chi b \times \chi(\inf_{m\le n} \pi_{i_m} a_m) d\bar\nu$$

(because $\mathfrak{E} = \pi_j[\mathfrak{B}']$)

$$= \int Q_{\mathfrak{E}}\chi(\pi_j b) \times \chi(\inf_{m\le n} \pi_{i_m} a_m) d\bar\nu.$$

Because $\pi_j[\mathfrak{B}_L] = \mathfrak{B}^*_L$ and $\pi_{i_m}[\mathfrak{B}_{J_m}] = \mathfrak{B}^*_{J_m}$ for each m, we have

$$\bar\nu(c \cap \inf_{m\le n} c_m) = \int Q_{\mathfrak{E}}(\chi c) \times \chi(\inf_{m\le n} c_m) d\bar\nu$$

whenever $c \in \mathfrak{B}^*_L$ and $c_m \in \mathfrak{B}^*_{J_m}$ for each m. Because $\mathfrak{D} = \bigvee_{m\le n} \mathfrak{B}^*_{J_m}$,

$$\bar\nu(c \cap d) = \int Q_{\mathfrak{E}}(\chi c) \times \chi d\, d\bar\nu = \int Q_{\mathfrak{E}}(\chi c) \times Q_{\mathfrak{E}}(\chi d)\, d\bar\nu$$

whenever $c \in \mathfrak{B}^*_L$ and $d \in \mathfrak{D}$. But this is just what is required to ensure that $\mathfrak{B}^*_L$ and $\mathfrak{D}$ are relatively independent over $\mathfrak{E}$ (Fremlin 03, 458Lc[9]).

5E Lemma (Austin p08b, 6.2) Let G be an abelian group, I a finite set and $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system which is fully isotropized and fully agreeable. Let $(\mathfrak{C}, \bar\nu, G, \langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}}, \langle\pi_i\rangle_{i\in I})$ be the Furstenberg self-joining of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ over I. For $J \subseteq I$ set

[9] Later editions only.

$$\mathfrak{B}_J = \{a : a \in \mathfrak{C},\ g\bullet_i a = g\bullet_j a \text{ for all } i,\ j \in J \text{ and } g \in G\},$$

and let $\mathfrak{B}^*_J \subseteq \mathfrak{C}$ be the divaricate copy of $\mathfrak{B}_J$ (5C). Let $\mathcal{J}$, $\mathcal{K} \subseteq \mathcal{P}I$ be sets such that $J' \in \mathcal{J}$ whenever $J \in \mathcal{J}$ and $J \subseteq J' \subseteq I$ and $K' \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $K \subseteq K' \subseteq I$. Then $\bigvee_{J\in\mathcal{J}} \mathfrak{B}^*_J$ and $\bigvee_{K\in\mathcal{K}} \mathfrak{B}^*_K$ are relatively independent over $\bigvee_{L\in\mathcal{J}\cap\mathcal{K}} \mathfrak{B}^*_L$.

proof (a) Induce on $\#(\mathcal{K} \setminus \mathcal{J})$. If $\mathcal{K} \subseteq \mathcal{J}$ the result is trivial. So the rest of the argument will be the inductive step to $\#(\mathcal{K} \setminus \mathcal{J}) = n > 0$.

(b) Take a maximal member M of $\mathcal{K} \setminus \mathcal{J}$, and set $\mathcal{K}' = \mathcal{K} \setminus \{M\}$. If $M \subset J \subseteq I$ then $J \in \mathcal{K}$; thus M is maximal in $\mathcal{P}I \setminus \mathcal{K}'$. If $M \subseteq J \in \mathcal{J} \cup \mathcal{K}'$ then $J \in \mathcal{K}$ because $M \in \mathcal{K}$, while $J \neq M$, so $J \in \mathcal{K}'$. Thus M is also maximal in $\mathcal{P}I \setminus (\mathcal{J} \cup \mathcal{K}')$. Set

$$\mathfrak{D}_1 = \bigvee_{J\in\mathcal{J}} \mathfrak{B}^*_J, \quad \mathfrak{D}_2 = \bigvee_{K\in\mathcal{K}} \mathfrak{B}^*_K, \quad \mathfrak{E} = \bigvee_{L\in\mathcal{J}\cap\mathcal{K}} \mathfrak{B}^*_L = \bigvee_{L\in\mathcal{J}\cap\mathcal{K}'} \mathfrak{B}^*_L,$$

$$\mathfrak{D}'_2 = \bigvee_{K\in\mathcal{K}'} \mathfrak{B}^*_K, \quad \mathfrak{E}' = \bigvee_{M\subseteq J\in\mathcal{J}\cup\mathcal{K}'} \mathfrak{B}^*_J = \bigvee_{M\subseteq J\in\mathcal{K}'} \mathfrak{B}^*_J.$$

By the inductive hypothesis, $\mathfrak{D}_1$ and $\mathfrak{D}'_2$ are relatively independent over $\mathfrak{E}$.

If $c \in \mathfrak{B}^*_M$, then

$$Q_{\mathfrak{D}_1\vee\mathfrak{D}'_2}(\chi c) = Q_{\mathfrak{E}'}(\chi c)$$

(because $\mathfrak{D}_1 \vee \mathfrak{D}'_2$ and $\mathfrak{B}^*_M$ are relatively independent over $\mathfrak{E}'$, by 5D)

$$= Q_{\mathfrak{D}'_2}(\chi c)$$

because $\mathfrak{D}'_2$ and $\mathfrak{B}^*_M$ are relatively independent over $\mathfrak{E}'$, again by 5D. So if $c \in \mathfrak{B}^*_M$ and $d \in \mathfrak{D}'_2$,

$$\begin{aligned} Q_{\mathfrak{D}_1}(\chi c \times \chi d) &= Q_{\mathfrak{D}_1}(Q_{\mathfrak{D}_1\vee\mathfrak{D}'_2}(\chi c \times \chi d)) = Q_{\mathfrak{D}_1}(Q_{\mathfrak{D}_1\vee\mathfrak{D}'_2}(\chi c) \times \chi d) \\ &= Q_{\mathfrak{D}_1}(Q_{\mathfrak{D}'_2}(\chi c) \times \chi d) = Q_{\mathfrak{D}_1}(Q_{\mathfrak{D}'_2}(\chi c \times \chi d)) = Q_{\mathfrak{E}}(\chi c \times \chi d) \end{aligned}$$

because $\mathfrak{D}_1$ and $\mathfrak{D}'_2$ are relatively independent over $\mathfrak{E}$.

As c and d are arbitrary, $Q_{\mathfrak{D}_1}$ and $Q_{\mathfrak{E}}$ agree on $\mathfrak{B}^*_M \vee \mathfrak{D}'_2 = \mathfrak{D}_2$. Rearranging the notation, we have

$$\bar\nu(d_1 \cap d_2) = \int \chi d_1 \times Q_{\mathfrak{D}_1}(\chi d_2)\, d\bar\nu = \int \chi d_1 \times Q_{\mathfrak{E}}(\chi d_2)\, d\bar\nu = \int Q_{\mathfrak{E}}(\chi d_1) \times Q_{\mathfrak{E}}(\chi d_2)\, d\bar\nu$$

whenever $d_1 \in \mathfrak{D}_1$ and $d_2 \in \mathfrak{D}_2$, so $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are relatively independent over $\mathfrak{E}$.

5F Lemma (Austin p08b, 7.1) Let G be an abelian group, I a finite set and $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system which is fully isotropized and fully agreeable. Let $(\mathfrak{C}, \bar\nu, G, \langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}}, \langle\pi_i\rangle_{i\in I})$ be the Furstenberg self-joining of $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ over I. For $\mathcal{J} \subseteq \mathcal{P}I$ set

$$\tilde{\mathfrak{B}}_{\mathcal{J}} = \bigvee_{J\in\mathcal{J}} \{a : a \in \mathfrak{A},\ g\bullet_i a = g\bullet_j a \text{ for all } i,\ j \in J \text{ and } g \in G\}$$

(interpreting $\tilde{\mathfrak{B}}_\emptyset$ as $\{0\}$, of course). Let $\mathbb{J} \subseteq I \times \mathcal{PP}I$ be such that if $(i, \mathcal{J}) \in \mathbb{J}$ then

$$I \in \mathcal{J}, \quad i \in J \text{ for every } J \in \mathcal{J}, \quad \text{if } J \in \mathcal{J} \text{ and } J \subseteq K \subseteq I \text{ then } K \in \mathcal{J}.$$

If $\langle a_{i\mathcal{J}}\rangle_{(i,\mathcal{J})\in\mathbb{J}}$ is a family in $\mathfrak{A}$ such that $a_{i\mathcal{J}} \in \tilde{\mathfrak{B}}_{\mathcal{J}}$ for all $(i, \mathcal{J}) \in \mathbb{J}$, and

$$\inf_{(i,\mathcal{J})\in\mathbb{J}} \pi_i(a_{i\mathcal{J}}) = 0,$$

then

$$\inf_{(i,J)\in\mathbb{J}} a_{i\mathcal{J}} = 0.$$

proof (a) Before starting on the main argument, it will be helpful to explain the way in which Lemma 5F will be applied. Import the notation of 5E, so that if $J \subseteq I$ then

$$\mathfrak{B}_J = \{a : g\bullet_i a = g\bullet_j a \text{ for all } i,\ j \in J \text{ and } g \in G\}, \quad \mathfrak{B}^*_J = \pi_i[\mathfrak{B}_J] \text{ whenever } i \in J,$$

(with $\mathfrak{B}^*_\emptyset = \mathfrak{C}$); then $\tilde{\mathfrak{B}}_{\mathcal{J}} = \bigvee_{J\in\mathcal{J}} \mathfrak{B}_J$ and $\pi_i[\tilde{\mathfrak{B}}_{\mathcal{J}}] = \bigvee_{J\in\mathcal{J}} \mathfrak{B}^*_J$ whenever $(i, \mathcal{J}) \in \mathbb{J}$. Take any $l_0 \in \mathbb{N}$ and for $\mathcal{J} \subseteq \mathcal{P}I$ set $\hat{\mathcal{J}} = \{J : J \in \mathcal{J},\ \#(J) > l_0\}$. Suppose that for each $J \subseteq I$ we are given a closed subalgebra $\mathfrak{G}_J$ of $\mathfrak{B}_J$, and for $\mathcal{J} \subseteq \mathcal{P}I$ set $\mathfrak{D}_{\mathcal{J}} = \bigvee_{J\in\hat{\mathcal{J}}} \mathfrak{B}_J \vee \bigvee_{J\in\mathcal{J}} \mathfrak{G}_J$. If $(l,\mathcal{L}) \in \mathbb{J}$ then $\mathfrak{E}_1 = \pi_l[\tilde{\mathfrak{B}}_{\mathcal{L}}]$ and $\mathfrak{E}_2 = \bigvee_{(i,\mathcal{J})\in\mathbb{J},(i,\mathcal{J})\ne(l,\mathcal{L})} \pi_i[\mathfrak{D}_{\mathcal{J}}]$ are relatively independent over $\mathfrak{E} = \pi_l[\mathfrak{D}_{\mathcal{L}}]$. **P** Set

$$\mathcal{K} = \bigcup_{(i,\mathcal{J})\in\mathbb{J}} \mathcal{J} \setminus (\mathcal{L} \setminus \hat{\mathcal{L}}).$$

Observe that if $K \in \mathcal{K}$ and $K \subseteq K' \subseteq I$ then $K' \in \mathcal{K}$. By 5F, $\mathfrak{E}_1 = \bigvee_{J\in\mathcal{L}} \mathfrak{B}^*_J$ and $\mathfrak{E}'_2 = \bigvee_{J\in\mathcal{K}} \mathfrak{B}^*_J$ are relatively independent over

$$\bigvee_{J\in\mathcal{K}\cap\mathcal{L}} \mathfrak{B}^*_J = \bigvee_{J\in\hat{\mathcal{L}}} \mathfrak{B}^*_J \subseteq \mathfrak{E} = \bigvee_{J\in\hat{\mathcal{L}}} \mathfrak{B}^*_J \vee \bigvee_{J\in\mathcal{L}} \pi_l[\mathfrak{G}_J] \subseteq \mathfrak{E}_1.$$

Consequently $\mathfrak{E}_1$ and $\mathfrak{E}'_2$ are relatively independent over $\mathfrak{E}$ (Fremlin 03, 458Ld[10]). It follows that $\mathfrak{E}_1$ and $\mathfrak{E}'_2 \vee \mathfrak{E}$ are relatively independent over $\mathfrak{E}$ (Fremlin 03, 458Ld again). But

$$\mathfrak{E}_2 \subseteq \mathfrak{E}'_2 \vee \bigvee_{J\in\mathcal{L}} \pi_l[\mathfrak{G}_J] \subseteq \mathfrak{E}'_2 \vee \mathfrak{E},$$

so $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are relatively independent over $\mathfrak{E}$. **Q**

(b) Now for the main line of the proof. The case $\mathbb{J} = \emptyset$ is trivial; suppose that $\mathbb{J}$ is non-empty. Induce on the triple $(\#(I) - l_0, l_1, l_2)$ where

$$l_0 = \min\{\#(J) : J \in \bigcup_{(i,\mathcal{J})\in\mathbb{J}} \mathcal{J}\},$$

$$l_1 = \#(\{(i,\mathcal{J}) : (i,\mathcal{J}) \in \mathbb{J},\ \min\{\#(J) : J \in \mathcal{J}\} = l_0,\ \mathcal{J} \text{ has no least element}\},$$

$$l_2 = \#(\{(i,\mathcal{J}) : (i,\mathcal{J}) \in \mathbb{J},\ \min\{\#(J) : J \in \mathcal{J}\} = l_0,\ \mathcal{J} \text{ has a least element}\}.$$

The case $l_1 = l_2 = 0$ is vacuous. Let M be $\{(i,\mathcal{J}) : (i,\mathcal{J}) \in \mathbb{J},\ \min\{\#(J) : J \in \mathcal{J}\} = l_0\}$.

(c) Suppose that there are an $\mathcal{L} \subseteq \mathcal{P}I$ and distinct j, $k \in I$ such that $(j,\mathcal{L})$ and $(k,\mathcal{L})$ both belong to M. In this case, every member of $\mathcal{L}$ must contain both j and k, so $\mathfrak{B}_J \subseteq \mathfrak{B}_{\{j,k\}}$ for every $J \in \mathcal{L}$, $\tilde{\mathfrak{B}}_{\mathcal{L}} \subseteq \mathfrak{B}_{\{j,k\}}$, $g\bullet_j a = g\bullet_k a$ whenever $a \in \tilde{\mathfrak{B}}_{\mathcal{L}}$ and $g \in G$, and π_j and π_k agree on $\tilde{\mathfrak{B}}_{\mathcal{L}}$, by 5Ba.

Set $\mathbb{J}' = \mathbb{J} \setminus \{(k,\mathcal{L})\}$. Then $\mathbb{J}'$ yields the triple $(\#(I) - l_0, l'_1, l'_2)$ where $l'_1 \le l_1$, $l'_2 \le l_2$ and $l'_1 + l'_2 < l_1 + l_2$, so has been previously dealt with. Set

$$\begin{aligned} a'_{i\mathcal{J}} &= a_{j\mathcal{L}} \cap a_{k\mathcal{L}} \text{ if } i = j \text{ and } \mathcal{J} = \mathcal{L}, \\ &= a_{i\mathcal{J}} \text{ if } (i,\mathcal{J}) \in \mathbb{J}' \text{ and } (i,\mathcal{J}) \ne (j,\mathcal{L}). \end{aligned}$$

Since $a_{k\mathcal{L}} \in \tilde{\mathfrak{B}}_{\mathcal{L}}$,

$$\begin{aligned} \inf_{(i,\mathcal{J})\in\mathbb{J}'} \pi_i a'_{i\mathcal{J}} &= \inf_{(i,\mathcal{J})\in\mathbb{J}'} \pi_i(a_{i\mathcal{J}}) \cap \pi_j(a_{k\mathcal{L}}) \\ &= \inf_{(i,\mathcal{J})\in\mathbb{J}'} \pi_i(a_{i\mathcal{J}}) \cap \pi_k(a_{k\mathcal{L}}) = \inf_{(i,\mathcal{J})\in\mathbb{J}} \pi_i(a_{i\mathcal{J}}) = 0. \end{aligned}$$

By the inductive hypothesis,

$$0 = \inf_{(i,\mathcal{J})\in\mathbb{J}'} a'_{i\mathcal{J}} = \inf_{(i,\mathcal{J})\in\mathbb{J}'} a_{i\mathcal{J}} \cap a_{k\mathcal{L}} = \inf_{(i,\mathcal{J})\in\mathbb{J}} a_{i\mathcal{J}}$$

and the induction proceeds.

We can therefore assume, for the rest of the argument, that there are no such $\mathcal{L}$, j and k.

(d) Inductive step to $(l_0, 0, l_2)$ when $l_2 > 0$: In this case, for every $(i,\mathcal{J}) \in M$, $\mathcal{J}$ has a least member.

(i) Take any $(l,\mathcal{L}) \in M$, and let L be the least member of $\mathcal{L}$. Set $\hat{\mathcal{L}} = \mathcal{L} \setminus \{L\}$,

$$\mathbb{J}' = (\mathbb{J} \setminus \{(l,\mathcal{L})\}) \cup \{(l,\hat{\mathcal{L}})\}.$$

Then $\mathbb{J}'$ yields a triple $(\#(I) - l'_0, l'_1, l'_2)$ where either $l'_0 > l_0$ (because $(l,\mathcal{L})$ was the only member of M) or $l'_0 = l_0$ and $l'_1 = 0$ and $l'_2 = l_2 - 1$; in either case, it has already been dealt with.

Set $\mathfrak{D} = \tilde{\mathfrak{B}}_{\hat{\mathcal{L}}}$,

[10] Later editions only.

$$a'_{i\mathcal{J}} = \operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D}) \text{ if } i = l \text{ and } \mathcal{J} = \hat{\mathcal{L}} \text{ and } (l, \mathcal{J}) \notin \mathbb{J},$$

(recall that $\operatorname{upr}(a, \mathfrak{D}) = \inf\{d : a \subseteq d \in \mathfrak{D}\}$; see Fremlin, 313S[11])

$$= a_{l\mathcal{J}} \cap \operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D}) \text{ if } i = l \text{ and } \mathcal{J} = \hat{\mathcal{L}} \text{ and } (l, \mathcal{J}) \in \mathbb{J},$$
$$= a_{l\mathcal{J}} \text{ if } i = l \text{ and } (l, \mathcal{J}) \in \mathbb{J}' \text{ and } \mathcal{J} \neq \hat{\mathcal{L}},$$
$$= a_{i\mathcal{J}} \text{ if } i \in I \setminus \{l\} \text{ and } (i, \mathcal{J}) \in \mathbb{J}.$$

Then $a'_{i\mathcal{J}} \in \tilde{\mathfrak{B}}_{\mathcal{J}}$ whenever $(i, \mathcal{J}) \in \mathbb{J}'$. **P** If $i = l$ and $\mathcal{J} = \hat{\mathcal{L}}$, then

$$\operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D}) \in \mathfrak{D} = \tilde{\mathfrak{B}}_{\mathcal{J}}.$$

If $\mathcal{J} = \hat{\mathcal{L}}$ and $(l, \mathcal{J}) \notin \mathbb{J}$ then $a'_{i\mathcal{J}} = \operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D})$; if $\mathcal{J} = \hat{\mathcal{L}}$ and $(l, \mathcal{J}) \in \mathbb{J}$ then $a'_{i\mathcal{J}} = a_{i\mathcal{J}} \cap \operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D})$; in either case it belongs to $\tilde{\mathfrak{B}}_{\mathcal{J}}$. In all other cases, $a'_{i\mathcal{J}} = a_{i\mathcal{J}} \in \tilde{\mathfrak{B}}_{\mathcal{J}}$. **Q**

(ii) Write N for $\mathbb{J} \setminus \{(l, \mathcal{L})\}$. In (a), set $\mathfrak{G}_L = \{0\}$ and $\mathfrak{G}_J = \mathfrak{B}_J$ for other $J \subseteq I$. Then $\mathfrak{D}_{\mathcal{J}} = \tilde{\mathfrak{B}}_{\mathcal{J}}$ whenever $(i, \mathcal{J}) \in N$. **P** The point is that $L \notin \mathcal{J}$. For if $J \in \mathcal{J}$ then either $\#(J) > l_0$ or $\#(J) = l_0$ is the least member of $\mathcal{J}$; since $\mathcal{J} \neq \mathcal{L}$, as settled in (b) above, and $\mathcal{J}$ and $\mathcal{L}$ both have least members, their least members must be different, and $J \neq L$. So

$$\mathfrak{D}_{\mathcal{J}} = \bigvee_{J \in \hat{\mathcal{J}}} \mathfrak{B}_J \vee \bigvee_{J \in \mathcal{J}} \mathfrak{G}_J = \bigvee_{J \in \hat{\mathcal{J}}} \mathfrak{B}_J \vee \bigvee_{J \in \mathcal{J}} \mathfrak{B}_J = \tilde{\mathfrak{B}}_{\mathcal{J}}. \text{ } \mathbf{Q}$$

On the other hand,

$$\mathfrak{D}_{\mathcal{L}} = \bigvee_{J \in \hat{\mathcal{L}}} \mathfrak{B}_J = \mathfrak{D}$$

because $\mathcal{L} = \hat{\mathcal{L}} \cup \{L\}$ and $\mathfrak{G}_L = \{0\}$.

Now observe that, in the notation of (a),

$$\mathfrak{C}_1 = \pi_l[\tilde{\mathfrak{B}}_{\mathcal{L}}]$$

contains $\pi_l(a_{l\mathcal{L}})$,

$$\mathfrak{C}_2 = \bigvee_{(i,\mathcal{J}) \in N} \pi_i[\mathfrak{D}_{\mathcal{J}}] = \bigvee_{(i,\mathcal{J}) \in N} \pi_i[\tilde{\mathfrak{B}}_{\mathcal{J}}]$$

contains $\inf_{(i,\mathcal{J}) \in N} a_{i\mathcal{J}}$, and

$$\mathfrak{C} = \pi_l[\mathfrak{D}_{\mathcal{L}}] = \pi_l[\mathfrak{D}].$$

Since $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are relatively independent over $\mathfrak{C}$, by (a), and $\pi_l(a_{l\mathcal{L}}) \cap \inf_{(i,\mathcal{J}) \in N} \pi_i(a_{i\mathcal{J}}) = 0$, we also have

$$0 = \operatorname{upr}(\pi_l(a_{l\mathcal{L}}), \mathfrak{C}) \cap \inf_{(i,\mathcal{J}) \in N} \pi_i(a_{i\mathcal{J}})$$

(Fremlin 03, 458Lf[12])

$$= \pi_l(\operatorname{upr}(a_{l\mathcal{L}}), \mathfrak{D}) \cap \inf_{(i,\mathcal{J}) \in N} \pi_i(a_{i\mathcal{J}})$$

(Fremlin 02, 313Xs[12])

$$= \pi_l(\operatorname{upr}(a_{l\mathcal{L}}), \mathfrak{D}) \cap \inf_{(i,\mathcal{J}) \in \mathbb{J}} \pi_i(a_{i\mathcal{J}}) = \inf_{(i,\mathcal{J}) \in \mathbb{J}'} \pi_i(a'_{i\mathcal{J}}).$$

By the inductive hypothesis,

$$0 = \inf_{(i,\mathcal{J}) \in \mathbb{J}'} a'_{i\mathcal{J}} = \operatorname{upr}(a_{l\mathcal{L}}, \mathfrak{D}) \cap \inf_{(i,\mathcal{J}) \in N} a_{i\mathcal{J}} \supseteq \inf_{(i,\mathcal{J}) \in \mathbb{J}} a_{i\mathcal{J}}$$

and the induction proceeds in this case also.

(e) Inductive step to (l_0, l_1, l_2) **when** $l_1 > 0$: For $\mathcal{J} \subseteq \mathcal{P}I$, set $\hat{\mathcal{J}} = \{J : J \in \mathcal{J}, \#(J) > l_0\}$. Note that $\tilde{\mathfrak{B}}_{\mathcal{J}} = \tilde{\mathfrak{B}}_{\hat{\mathcal{J}}} \vee \bigvee_{J \in \mathcal{J}, \#(J) = l_0} \mathfrak{B}_J$ whenever $(i, \mathcal{J}) \in \mathbb{J}$.

[11] Formerly 314V.
[12] Later editions only.

case 1 Suppose there is a pair $(l,\mathcal{L})\in M$ such that $a_{l\mathcal{L}}$ is of the form $b\cap\inf_{L\in\mathcal{L},\#(L)=l_0}b_L$ where $b\in\tilde{\mathfrak{B}}_{\hat{\mathcal{L}}}$ and $b_L\in\mathfrak{B}_L$ for each $L\in\mathcal{L}\setminus\hat{\mathcal{L}}$.

Set $\mathcal{K}_L=\{J:L\subseteq J\subseteq I\}$ for $L\subseteq I$, so that $\tilde{\mathfrak{B}}_{\mathcal{K}_L}=\mathfrak{B}_L$ for each L, and

$$\mathbb{J}'=(\mathbb{J}\setminus\{(l,\mathcal{L})\})\cup\{(l,\mathcal{K}_L):L\in\mathcal{L}\setminus\hat{\mathcal{L}}\}\cup\{(l,\hat{\mathcal{L}})\}.$$

Then $\mathbb{J}'$ yields a triple $(\#(I)-l_0,l_1-1,l_2')$, because every $\mathcal{K}_L$ has a least element of size l_0, while $\hat{\mathcal{L}}$ contains no set of size l_0; so $\mathbb{J}'$ has been previously dealt with. Set

$$\begin{aligned}a'_{i\mathcal{J}}&=b\text{ if }i=l,\ \mathcal{J}=\hat{\mathcal{L}}\text{ and }(l,\mathcal{J})\notin\mathbb{J},\\&=b\cap a_{l\mathcal{L}}\text{ if }i=l,\ \mathcal{J}=\hat{\mathcal{L}}\text{ and }(l,\mathcal{J})\in\mathbb{J},\\&=b_L\text{ if }i=l,\ L\in\mathcal{L}\setminus\hat{\mathcal{L}},\ \mathcal{J}=\mathcal{K}_L\text{ and }(l,\mathcal{J})\notin\mathbb{J},\\&=b_L\cap a_{l\mathcal{J}}\text{ if }i=l,\ L\in\mathcal{L}\setminus\hat{\mathcal{L}},\ \mathcal{J}=\mathcal{K}_L\text{ and }(l,\mathcal{J})\in\mathbb{J},\\&=a_{l\mathcal{J}}\text{ if }i=l,\ (l,\mathcal{J})\in\mathbb{J}\text{ and }\mathcal{J}\notin\{\mathcal{L},\hat{\mathcal{L}}\}\cup\{\mathcal{K}_L:L\in\mathcal{L}\setminus\hat{\mathcal{L}}\},\\&=a_{i\mathcal{J}}\text{ if }i\in I\setminus\{l\}\text{ and }(i,\mathcal{J})\in\mathbb{J}.\end{aligned}$$

Then

$$\begin{aligned}\inf_{(i,\mathcal{J})\in\mathbb{J}'}\pi_i(a'_{i\mathcal{J}})&=\pi_l(b\cap\inf_{L\in\mathcal{L}\setminus\hat{\mathcal{L}}}b_L)\cap\inf_{\substack{(i,\mathcal{J})\in\mathbb{J}\\(i,\mathcal{J})\neq(l,\mathcal{L})}}\pi_i(a_{i\mathcal{J}})\\&=\inf_{(i,\mathcal{J})\in\mathbb{J}}\pi_i(a_{i\mathcal{J}})=0.\end{aligned}$$

By the inductive hypothesis,

$$0=\inf_{(i,\mathcal{J})\in\mathbb{J}'}a'_{i\mathcal{J}}=b\cap\inf_{L\in\mathcal{L}\setminus\hat{\mathcal{L}}}b_L\cap\inf_{\substack{(i,\mathcal{J})\in\mathbb{J}\\(i,\mathcal{J})\neq(l,\mathcal{L})}}a_{i\mathcal{J}}=\inf_{(i,\mathcal{J})\in\mathbb{J}}a_{i\mathcal{J}}$$

and again we can move forward.

case 2 Suppose there is a pair $(l,\mathcal{L})\in M$ such that $a_{l\mathcal{L}}$ belongs to the *subalgebra* of $\mathfrak{A}$ generated by $\tilde{\mathfrak{B}}_{\hat{\mathcal{L}}}\cup\bigcup\{\mathfrak{B}_J:J\in\mathcal{L}\}$. Then it is a finite supremum of elements of the form considered in case 1 and, applying the argument above to each of these, we again find that $\inf_{(i,\mathcal{J})\in\mathbb{J}}a_{i\mathcal{J}}=0$.

case 3 Now for the case of general $a_{i\mathcal{J}}$. Take any $\epsilon\in\,]0,1]$. Set $\delta=\epsilon/2\#(\mathbb{J})$. For each $(i,\mathcal{J})\in\mathbb{J}$,

$$\begin{aligned}a_{i\mathcal{J}}\in\tilde{\mathfrak{B}}_{\mathcal{J}}&=\tilde{\mathfrak{B}}_{\hat{\mathcal{J}}}\vee\bigvee_{J\in\mathcal{J}\setminus\hat{\mathcal{J}}}\mathfrak{B}_J\\&=\overline{\bigcup\{\tilde{\mathfrak{B}}_{\hat{\mathcal{J}}}\vee\bigvee_{J\in\mathcal{J}}\mathfrak{G}_J:\mathfrak{G}_J\text{ is a finite subalgebra of }\mathfrak{B}_J\text{ for every }J\in\mathcal{J}\}}.\end{aligned}$$

We can therefore find families $\langle\mathfrak{G}_J\rangle_{J\subseteq I}$ and $\langle b_{i\mathcal{J}}\rangle_{(i,\mathcal{J})\in\mathbb{J}}$ such that $\mathfrak{G}_J$ is a finite subalgebra of $\mathfrak{B}_J$ for every J, $b_{i\mathcal{J}}\in\tilde{\mathfrak{B}}_{\hat{\mathcal{J}}}\vee\bigvee_{J\in\mathcal{J}}\mathfrak{G}_J$ for every $(i,\mathcal{J})\in\mathbb{J}$, and $\bar\mu(a_{i\mathcal{J}}\triangle b_{i\mathcal{J}})\le\delta^2$ for every $(i,\mathcal{J})\in\mathcal{J}$. As in (a), set $\mathfrak{D}_{\mathcal{J}}=\tilde{\mathfrak{B}}_{\hat{\mathcal{J}}}\vee\bigvee_{J\in\mathcal{J}}\mathfrak{G}_J$ for $\mathcal{J}\subseteq\mathcal{P}I$. For $(i,\mathcal{J})\in\mathbb{J}$, set $d_{i\mathcal{J}}=[\![Q_{\mathfrak{D}_{\mathcal{J}}}(\chi a_{i\mathcal{J}})>1-\delta]\!]$. Then

$$\begin{aligned}Q_{\mathfrak{D}_{\mathcal{J}}}\chi(d_{i\mathcal{J}}\setminus a_{i\mathcal{J}})&=Q_{\mathfrak{D}_{\mathcal{J}}}\bigl(\chi(d_{i\mathcal{J}})-\chi(d_{i\mathcal{J}})\times\chi(a_{i\mathcal{J}})\bigr)\\&=\chi d_{i\mathcal{J}}-\chi(d_{i\mathcal{J}})\times Q_{\mathfrak{D}_{\mathcal{J}}}\chi(a_{i\mathcal{J}})\le\delta\chi d_{i\mathcal{J}};\end{aligned}$$

on the other hand,

$$\begin{aligned}\delta\bar\mu(a_{i\mathcal{J}}\setminus d_{i\mathcal{J}})&\le\int_{a\setminus d}\chi a_{i\mathcal{J}}-Q_{\mathfrak{D}_{\mathcal{J}}}(\chi a_{i\mathcal{J}})\,d\bar\mu\le\|\chi a_{i\mathcal{J}}-Q_{\mathfrak{D}_{\mathcal{J}}}(\chi a_{i\mathcal{J}})\|_1\\&\le\|\chi a_{i\mathcal{J}}-\chi b_{i\mathcal{J}}\|_1+\|\chi b_{i\mathcal{J}}-Q_{\mathfrak{D}_{\mathcal{J}}}(\chi b_{i\mathcal{J}})\|_1+\|Q_{\mathfrak{D}_{\mathcal{J}}}(\chi b_{i\mathcal{J}}-\chi a_{i\mathcal{J}})\|_1\\&\le2\|\chi a_{i\mathcal{J}}-\chi b_{i\mathcal{J}}\|_1\le2\delta^2,\end{aligned}$$

so $\bar\mu(a_{i\mathcal{J}}\setminus d_{i\mathcal{J}})\le 2\delta$.

Consider $c=\inf_{(i,\mathcal{J})\in\mathbb{J}}\pi_i(d_{i\mathcal{J}})$. For $(l,\mathcal{L})\in\mathbb{J}$, we know from (a) that $\mathfrak{E}_1=\pi_l[\tilde{\mathfrak{B}}_{\mathcal{L}}]$ and $\mathfrak{E}_2=\bigvee_{(i,\mathcal{J})\in\mathbb{J},(i,\mathcal{J})\ne(l,\mathcal{L})}\pi_i[\mathfrak{D}_{\mathcal{J}}]$ are relatively independent over $\mathfrak{E}=\pi_l[\mathfrak{D}_{\mathcal{L}}]$. Since $\pi_l(d_{l\mathcal{L}}\setminus a_{l\mathcal{L}})\in\mathfrak{E}_1$ and $e=\inf_{(i,\mathcal{J})\in\mathbb{J},\,(i,\mathcal{J})\ne(l,\mathcal{L})}\pi_i(d_{i\mathcal{J}})$ belongs to $\mathfrak{E}_2$,

$$\bar\nu(c\setminus\pi_l(a_{l\mathcal{L}}))=\int\chi\pi_l(d_{l\mathcal{L}}\setminus a_{l\mathcal{L}})\times\chi e\,d\bar\nu=\int Q_{\mathfrak{E}}(\chi\pi_l(d_{l\mathcal{L}}\setminus a_{l\mathcal{L}}))\times\chi e\,d\bar\nu$$
$$=\int R_lQ_{\mathfrak{D}_{\mathcal{L}}}\chi(d_{l\mathcal{L}}\setminus a_{l\mathcal{L}})\times\chi e\,d\bar\nu$$

(where $R_l:L^0(\mathfrak{A})\to L^0(\mathfrak{C})$ corresponds to $\pi_l:\mathfrak{A}\to\mathfrak{C}$, as usual)

$$\le\delta\int R_l\chi(d_{l\mathcal{L}})\times\chi e\,d\bar\nu=\delta\int\chi(\inf_{(i,\mathcal{J})\in\mathbb{J}}\pi_i(d_{i\mathcal{J}}))\,d\bar\nu=\delta\nu c.$$

Summing over $(l,\mathcal{L})\in\mathbb{J}$,

$$\bar\nu c=\bar\nu(c\setminus\inf_{(l,\mathcal{L})\in\mathbb{J}}\pi_la_{l\mathcal{L}})$$

(because $\inf_{(l,\mathcal{L})\in\mathbb{J}}\pi_l(a_{l\mathcal{L}})=0$)

$$\le\sum_{(l,\mathcal{L})\in\mathbb{J}}\bar\nu(c\setminus\pi_la_{l\mathcal{L}})\le\delta\#(\mathbb{J})\bar\nu c\le\frac{1}{2}\bar\nu c,$$

and $\bar\nu c=0$, that is, $c=0$.

Now observe that, because every $\mathfrak{G}_J$ is finite, the subalgebra of $\mathfrak{A}$ generated by $\tilde{\mathfrak{B}}_{\hat{\mathcal{J}}}\cup\bigcup_{J\in\mathcal{J}}\mathfrak{G}_J$ is closed, and is equal to $\mathfrak{D}_{\mathcal{J}}$, for every $\mathcal{J}\subseteq\mathcal{P}I$. Applying case 2 to the family $\langle d_{i\mathcal{J}}\rangle_{(i,\mathcal{J})\in\mathbb{J}}$ and any $(l,\mathcal{L})\in M$, we see that $\inf_{(i,\mathcal{J})\in\mathbb{J}}d_{i\mathcal{J}}=0$. But this means that

$$\bar\mu(\inf_{(i,\mathcal{J})\in\mathbb{J}}a_{i\mathcal{J}})\le\sum_{(i,\mathcal{J})\in\mathbb{J}}\bar\mu(a_{i\mathcal{J}}\setminus d_{i\mathcal{J}})\le 2\delta\#(\mathbb{J})\le\epsilon.$$

As ϵ is arbitrary, $\inf_{(i,\mathcal{J})\in\mathbb{J}}a_{i\mathcal{J}}=0$ and the induction proceeds in this case also.

This completes the proof.

5G Theorem Let G be an abelian group, I a finite set and $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ a commuting measure-automorphism action system. Then

$$\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_ia)>0$$

for every non-zero $a\in\mathfrak{A}$.

proof (a)(i) If $I=\emptyset$ we have to interpret the infimum of the empty set in $\mathfrak{A}$, but this is 1, so we get $\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_ia)=1$ for every $a\in\mathfrak{A}$.

(ii) If $I=\{j\}$ is a singleton, then

$$\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_ia)=\mathrm{WDL}_{g\to G}\,\bar\mu(g\bullet_ja)=\bar\mu a>0$$

for every non-zero a. So henceforth we can assume that $\#(I)\ge 2$.

(iii) It may make you more comfortable if I remind you that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is measure-averaging, by Theorem 3G, so

$$\mathrm{WDL}_{g\to G}\,\chi(\inf_{i\in I}g\bullet_ia)=\mathrm{WDL}_{g\to G}\prod_{i\in I}g\bullet_i\chi a$$

is defined in $L^1(\mathfrak{A},\bar\mu)$ for every $a\in\mathfrak{A}$, and $\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_ia)$ is always defined.

(b) Suppose that $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is fully isotropized and fully agreeable. Let $(\mathfrak{C},\bar\nu,G,\langle\tilde\bullet_i\rangle_{i\in I\cup\{\infty\}},\langle\pi_i\rangle_{i\in I})$ be the Furstenberg self-joining of $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ over I.

Take $a\in\mathfrak{A}$ such that $\mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}g\bullet_ia)=0$. Because $(\mathfrak{A},\bar\mu,G,\langle\bullet_i\rangle_{i\in I})$ is (I,j)-agreeable for every $j\in I$,

$$0 = \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I} g\bullet_i a) = \bar\nu(\inf_{i\in I}\pi_i a) = \int\prod_{i\in I} R_i P_i \chi a\,d\bar\nu$$

where $R_i : L^0(\mathfrak{A}) \to L^0(\mathfrak{C})$ is the Riesz homomorphism corresponding to $\pi_i : \mathfrak{A} \to \mathfrak{C}$, and P_i is the conditional expectation operator corresponding to the closed subalgebra $\bigvee_{j\in I\setminus\{i\}}\{a : g\bullet_j a = g\bullet_i a \text{ for every } g \in G\} \subseteq \mathfrak{A}$. Set $a_i = [\![P_i\chi a > 0]\!]$ for each i; then $\pi_i a_i = [\![R_i P_i \chi a]\!]$ for each i, so $\inf_{i\in I}\pi_i a_i = 0$. Applying 5F with $\mathcal{J}_i = \{J : i \in J \subseteq I,\ \#(J) \ge 2\}$, $\mathbb{J} = \{(i, \mathcal{J}_i) : i \in I\}$, we see that

$$a_i \in \bigvee_{j\in I\setminus\{i\}}\{a : g\bullet_j a = g\bullet_i a \text{ for every } g \in G\} = \tilde{\mathfrak{B}}_{\mathcal{J}_i}$$

for each i, so $\inf_{i\in I} a_i = 0$. But $a \subseteq a_i$ for each i, so $a = 0$.

(c) In general, $(\mathfrak{A}, \bar\mu, G, \langle\bullet_i\rangle_{i\in I})$ has a fully isotropized and fully agreeable extension $(\mathfrak{A}', \bar\mu', G, \langle\bullet_i'\rangle_{i\in I}, \phi)$, by Proposition 4G. If $a \in \mathfrak{A}\setminus\{0\}$, then $\phi a \ne 0$ so

$$\begin{aligned}0 < \mathrm{WDL}_{g\to G}\,\bar\mu'(\inf_{i\in I} g\bullet_i'\phi a) &= \mathrm{WDL}_{g\to G}\,\bar\mu'(\inf_{i\in I}\phi(g\bullet_i a))\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu'(\phi(\inf_{i\in I} g\bullet_i a)) = \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I} g\bullet_i a),\end{aligned}$$

as required.

Remark The special case of this theorem in which $G = \mathbb{Z}$ is the Multiple Recurrence Theorem (Furstenberg & Katznelson 78).

5H Corollary Let G be an infinite abelian group, I a finite set and $(X, G, \langle\bullet_i\rangle_{i\in I})$ a commuting action system. Suppose that there is a finitely additive functional $\mu : \mathcal{P}X \to [0,\infty[$ which is G-invariant, that is, $\mu(g\hat\bullet_i A) = \mu A$ whenever $A \subseteq X$, $i \in I$ and $g \in G$, writing $g\hat\bullet_i A$ for $\{g\bullet_i x : x \in A\}$. If $A \subseteq X$ and $\mu A > 0$, there are a $g \in G$, not the identity, and an $x \in X$ such that $g\bullet_i x \in A$ for every $i \in I$.

proof If $\mu X = 0$ this is vacuous; otherwise, taking a scalar multiple of μ if necessary, we can assume that $\mu X = 1$. Of course we can take it that I is non-empty. Applying 2B to the system $(\mathcal{P}X, G, \langle\hat\bullet_i\rangle_{i\in I})$, we get a commuting measure-preserving action system $(\mathfrak{A}, \bar\mu, G, \langle\tilde\bullet_i\rangle_{i\in I})$ together with a Boolean homomorphism $\phi : \mathcal{P}X \to \mathfrak{A}$ such that $\bar\mu\phi(A) = \mu A$ for every $A \subseteq X$ and $g\tilde\bullet_i\phi(A) = \phi(g\hat\bullet_i A)$ whenever $A \subseteq X$, $i \in I$ and $g \in G$. If $\mu A > 0$, then $\bar\mu\phi(A) > 0$ so

$$\begin{aligned}\mathrm{WDL}_{g\to G}\,\mu(\bigcap_{i\in I} g\hat\bullet_i A) = \mathrm{WDL}_{g\to G}\,\bar\mu(\phi(\bigcap_{i\in I} g\hat\bullet_i A)) &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I}\phi(g\hat\bullet_i A))\\ &= \mathrm{WDL}_{g\to G}\,\bar\mu(\inf_{i\in I} g\tilde\bullet_i\phi(A)) > 0\end{aligned}$$

by Theorem 5G. In particular, there is a $g \in G$, other than the identity, such that $\mu(\bigcap_{i\in I} g\hat\bullet_i A) > 0$ (1Hd); in which case, there is surely an $x \in \bigcap_{i\in I} g\hat\bullet_i A$. Now $g^{-1}\bullet_i x \in A$ for every $i \in I$.

5J Corollary Let R be an infinite ring and X an R-module. Suppose that $I \subseteq X$ is a finite set and that $A \subseteq X$ has $\overline{\mathrm{WDL}}_{x\to X}\chi A(x) > 0$, where $\overline{\mathrm{WDL}}_{x\to X}$ is defined with respect to the additive group $(X, +)$. Then there is a similar copy $x + rI$ of I included in A, where $x \in X$ and $r \in R \setminus \{0\}$.

proof By 1Hc, there is a translation-invariant finitely additive functional $\mu : \mathcal{P}X \to [0,1]$ such that $\mu A > 0$. For $i \in I$, $r \in R$ and $x \in X$, set $r\bullet_i x = x + ri$. It is easy to check that $(X, R, \langle\bullet_i\rangle_{i\in I})$ is a commuting action system when R is given its additive group structure. Because μ is translation-invariant, it is R-invariant. By 5I, there are an $x \in X$ and an $r \in R\setminus\{0\}$ such that $x + ri = r\bullet_i x \in A$ for every $i \in I$.

References

Austin T. [p08a] 'On the norm convergence of nonconventional ergodic averages', preprint, 2008 (`arXiv:0805.0320`).

Austin T. [p08b] 'Deducing the multidimensional Szemerédi theorem from an infinitary removal lemma', preprint, 2008 (`arXiv:0808.2267`).

Bergelson V., McCutcheon R. & Zhang Q. [97] 'A Roth theorem for amenable groups', Amer. J. Math. 119 (1997) 1173-1211.

Clarkson J.A. [36] 'Uniformly convex spaces', Trans. Amer. Math. Soc. 40 (1936) 396-414.

Fremlin D.H. [01] *Measure Theory, Vol. 2: Broad Foundations.* Torres Fremlin, 2001.

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002.

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.

Fremlin D.H. [n08] 'Well-distributed sequences and Banach density', note of 22.10.08 (`http://www.essex.ac.uk/maths/staff/fremlin/preprints.htm`).

Furstenberg H. & Katznelson Y. [78] 'An ergodic Szemerédi theorem for commuting transformations', J. d'Analyse Math. 34 (1978) 275-291.

Furstenberg H. [81] *Recurrence in Ergodic Theory and Combinatorial Number Theory.* Princeton U.P., 1981.

Szemerédi E. [75] 'On sets of integers containing no k elements in arithmetic progression', Acta Arithmetica 27 (1975) 199-245.