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The generalized McShane integral

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I develop an extension of the McShane integral and discuss its relationships with the Pettis, Talagrand and Bochner integrals.

Introduction A large number of different methods of integration of Banach-space-valued functions have been introduced, based on the various possible constructions of the Lebesgue integral. They commonly run fairly closely together when the range space is separable (or has w*-separable dual) and diverge more or less sharply for general range spaces. Here I describe a natural extension of the McShane integral to functions from any of a wide class of topological measure spaces to a Banach space, and give both positive and negative results concerning it and the other four integrals listed above.

1. The McShane integral I propose to use this name for a method of integrating vector-valued functions which is adapted from the integration process described in [McS83]. As I wish to make rather a large step (from real-valued functions defined on $\mathbb{R}^n$ or $\mathbb{R}^{\mathbb{N}}$ to vector-valued functions defined on σ-finite outer regular quasi-Radon measure spaces), I give a full list of the definitions and theorems in the elementary theory as I develop it, even though most of the proofs will not involve any new ideas.

1A Definitions Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a non-empty σ-finite quasi-Radon measure space which is **outer regular**, that is, such that $\mu E = \inf\{\mu G : E \subseteq G \in \mathfrak{T}\}$ for every $E \in \Sigma$. A **generalized McShane partition** of S is a sequence $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ such that $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint family of measurable sets of finite measure, $\mu(S \setminus \bigcup_{i\in\mathbb{N}} E_i) = 0$ and $t_i \in S$ for each i. A **gauge** on S is a function $\Delta : S \to \mathfrak{T}$ such that $s \in \Delta(s)$ for every $s \in S$. A generalized McShane partition $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ is **subordinate** to a gauge Δ if $E_i \subseteq \Delta(t_i)$ for every $i \in \mathbb{N}$.

Now let X be a Banach space. I will say that a function $\phi : S \to X$ is **McShane integrable**, with **McShane integral** w, if for every $\epsilon > 0$ there is a gauge $\Delta : S \to \mathfrak{T}$ such that

$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i.\phi(t_i)\| \le \epsilon$$

for every generalized McShane partition $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ of S subordinate to Δ.

1B Remarks (a) For the elementary theory of quasi-Radon measure spaces see [Fr74], [Frn82] and [Fr84]; the same idea, expressed in a more general context, underlies the 'Radon spaces of type $(\mathcal{H})$' of B.Rodriguez-Salinas ([RSJG79], [RS91]). The principal examples of σ-finite outer regular quasi-Radon measure spaces are

(i) all totally finite Radon and quasi-Radon measure spaces;
(ii) all Lindelöf Radon measure spaces (e.g., Lebesgue measure on $\mathbb{R}^n$);
(iii) all subspaces of such spaces (1L below);
(iv) finite products of such spaces ([Frn82], 4C, or [Fr84], A7Ea);
(v) all products of probability spaces of these types ([Frn82], 4F, or [Fr84], A7Eb).

(b) The essential facts I shall need here are that a quasi-Radon measure μ is inner regular for the closed sets (that is, $\mu E = \sup\{\mu F : F \subseteq E,\ F \text{ is closed}\}$ for every measurable E) and τ-smooth (that is, $\mu(\bigcup \mathcal{G}) = \sup_{G\in\mathcal{G}} \mu G$ for every non-empty upwards-directed family $\mathcal{G}$ of open sets).

(c) In addition, we shall need to know that an outer regular quasi-Radon measure is locally finite (that is, every point belongs to an open set of finite measure). Moreover, it has the following property, formally stronger than what is declared by the usual definition of 'outer regular': if E is any measurable set, and $\epsilon > 0$, there is an open set $G \supseteq E$ such that $\mu(G \setminus E) \le \epsilon$. Another elementary fact about outer regular measures is that if μ is an outer regular measure on S, and $f : S \to [0, \infty[$ is an integrable function, then for any $\epsilon > 0$ there is a lower semi-continuous function $h : S \to \mathbb{R}$ such that $f(t) < h(t)$ for every $t \in S$ and $\int h \le \epsilon + \int f$.

(d) I had better remark straight away that my version of the McShane integral is well-defined, in the sense that any given function has at most one value of the integral. Of course this is just because there are enough generalized McShane partitions: if $S \neq \emptyset$ and $\Delta : S \to \mathfrak{T}$ is any gauge, there is a generalized McShane partition subordinate to it. To see this, observe that

$$\mathcal{G} = \{G : G \in \mathfrak{T},\, \mu G < \infty,\, \exists\ s \in S,\, G \subseteq \Delta(s)\}$$

is an open cover of S, so that (because μ is τ-smooth) we have

$$\mu H = \sup\{\mu(H \cap \bigcup \mathcal{G}_0) : \mathcal{G}_0 \subseteq \mathcal{G} \text{ is finite}\}$$

for every open $H \subseteq S$; now, because μ is σ-finite, there is a sequence $\langle G_i \rangle_{i\in\mathbb{N}}$ in $\mathcal{G}$ such that $\mu(S \setminus \bigcup_{i\in\mathbb{N}} G_i) = 0$. If we choose for each i a $t_i \in S$ such that $G_i \subseteq \Delta(t_i)$, and write $E_i = G_i \setminus \bigcup_{j<i} G_j$ for $i \in \mathbb{N}$, we shall have a generalized McShane partition $\langle (E_i, t_i) \rangle_{i\in\mathbb{N}}$ subordinate to Δ.

Now because the family of gauges on S is directed downwards (if Δ_0 and Δ_1 are gauges, so is $s \mapsto \Delta_0(s) \cap \Delta_1(s)$) this shows that for any particular ϕ there will be at most one w satisfying the definition above.

(e) There is a technical fault in the definition of the McShane integral above. It ignores the case $S = \emptyset$. On the other hand, I certainly wish to count the empty set as a quasi-Radon measure space, and to accept the empty function as McShane integrable, with integral zero. Of course this is a triviality, and in the proofs below I shall systematically pass the case $S = \emptyset$ by, though I do wish it to be included in the statements of the results.

(f) It is in fact possible to define a McShane integral on outer regular quasi-Radon measure spaces which are not σ-finite. As however such a space must consist of a σ-finite part together with a family of closed sets, of strictly positive measure, on each of which the topology is indiscrete (see [GP84], §13), the McShane integral outside the σ-finite part corresponds just to unconditional summability of appropriate families in X; and the extra technical complications (we have to use uncountable families $\langle (E_i, t_i) \rangle_{i\in I}$ instead of sequences) seem more trouble than they're worth.

1C We are now ready for some elementary facts about the McShane integral. I give no proofs as the arguments are of a type familiar from [McS83].

Proposition Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon space and X a Banach space.

(a) If ϕ, $\psi : S \to X$ are McShane integrable functions with McShane integrals w, z respectively then $\phi + \psi$ is McShane integrable, with integral $w + z$.

(b) Let Y be another Banach space and $T : X \to Y$ a bounded linear operator. If $\phi : S \to X$ is McShane integrable, with McShane integral w, then $T\phi : S \to Y$ is McShane integrable, with McShane integral Tw.

(c) If $C \subseteq X$ is a closed cone and $\phi : S \to C$ is a McShane integrable function, then its McShane integral belongs to C.

Remark Of course the principal use of (b) is with $Y = \mathbb{R}$, and the principal use of (c) is with $X = \mathbb{R}$, $C = [0, \infty[$.

1D Readers familiar with [McS83] will already have observed that my definition of the McShane integral is significantly different from (and more complex than) the most natural generalisations of the work in [McS83]; a much simpler expression is used in [FMp91] and [Frp91]. The extra elaboration of my definition here is necessary to deal with the wider context in which I operate. However I must of course justify my terminology by showing that in the limited contexts considered in [McS83] and [Go90] my formulations agree with the simpler ones. The first point is that for compact spaces S there is no need to take infinite McShane partitions. Let us say that a **finite strict generalized McShane partition** of S is a family $\langle (E_i, t_i) \rangle_{i\leq n}$ such that $E_0, \dots, E_n$ is a finite disjoint cover of S by measurable sets (I find it convenient still to allow $E_i = \emptyset$ for some i) and $t_i \in S$ for each $i \leq n$. Now we have the following:

1E Proposition Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space and X a Banach space; let $\phi : S \to X$ be a function. Then ϕ is McShane integrable, with McShane integral w, if and only if for every $\epsilon > 0$ there is a gauge $\Delta : S \to \mathfrak{T}$ such that whenever $\langle (E_i, t_i) \rangle_{i\leq n}$ is a finite strict generalized McShane partition of S subordinate to Δ then $\|w - \sum_{i\leq n} \mu E_i \phi(t_i)\| \leq \epsilon$.

Remark I follow [Fr74] in taking a Radon measure space to be a Hausdorff locally finite quasi-Radon measure space in which the measure is inner regular for the compact sets.

proof Evidently any McShane integrable function $\phi : S \to X$ must satisfy the condition offered, as this merely restricts the class of partitions considered (of course a finite McShane partition can be extended to an infinite one by adding empty E_i.) For the reverse implication, suppose that ϕ, w satisfy the condition. Let $\epsilon > 0$ and let $\Delta : S \to \mathfrak{T}$ be a gauge such that $\|w - \sum_{i\le n} \mu E_i \phi(t_i)\| \le \epsilon$ for every finite strict generalized McShane partition $\langle (E_i, t_i)\rangle_{i\le n}$ subordinate to Δ. Now let $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ be an infinite generalized McShane partition subordinate to Δ. Because S is compact, we can find a finite cover of it by sets of the form $\Delta(t)$; accordingly, adding finitely many negligible sets E_i to the beginning of the sequence if necessary, we may take it that $S = \bigcup_{i\in I} E_i$. For each $i \in \mathbb{N}$ choose an open set G_i such that $E_i \subseteq G_i \subseteq \Delta(t_i)$ and $\mu(G_i \setminus E_i)\|\phi(t_i)\| \le 2^{-i}\epsilon$.

There is a finite $k \in \mathbb{N}$ such that $S = \bigcup_{i\le k} G_i$. Now if $n \ge k$, we have $S = \bigcup_{i\le n} G_i$, so there is a disjoint family $\langle E'_i\rangle_{i\le n}$ of measurable sets such that $E_i \subseteq E'_i \subseteq G_i$ for every $i \le n$ and $S = \bigcup_{i\le n} E'_i$. But in this case $\langle (E'_i, t_i)\rangle_{i\le n}$ is a finite strict generalized McShane partition of S subordinate to Δ, so we must have

$$\|w - \textstyle\sum_{i\le n} \mu E'_i.\phi(t_i)\| \le \epsilon.$$

On the other hand, we also have

$$\|\textstyle\sum_{i\le n} \mu E'_i \phi(t_i) - \sum_{i\le n} \mu E_i \phi(t_i)\| \le \sum_{i\le n} (\mu E'_i - \mu E_i)\|\phi(t_i)\| \le \sum_{i\le n} 2^{-i}\epsilon \le 2\epsilon.$$

So

$$\|w - \textstyle\sum_{i\le n} \mu E_i \phi(t_i)\| \le 3\epsilon$$

for all $n \ge k$; as ϵ is arbitrary, ϕ is McShane integrable with integral w.

1F The definitions of [McS83] do not as a rule refer to partitions into arbitrary measurable sets; instead they use various types of 'interval' for the E_i – e.g., half-open intervals in $\mathbb{R}$. I can give a general criterion for the applicability of such methods, as follows.

Proposition Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space and X a Banach space. Let $\mathcal{A} \subseteq \Sigma$ be a subalgebra of Σ such that whenever $F \subseteq G \subseteq S$, F is closed and G is open there is an $A \in \mathcal{A}$ such that $F \subseteq A \subseteq G$; let $\mathcal{C} \subseteq \mathcal{A}$ be such that every member of $\mathcal{A}$ is a finite disjoint union of members of $\mathcal{C}$. Then a function $\phi : S \to X$ is McShane integrable, with McShane integral w, iff for every $\epsilon > 0$ there is a gauge $\Delta : S \to \mathfrak{T}$ such that $\|w - \sum_{i\le n} \mu C_i \phi(t_i)\| \le \epsilon$ for every finite strict generalized McShane partition $\langle (C_i, t_i)\rangle_{i\le n}$ of S, subordinate to Δ, such that $C_i \in \mathcal{C}$ for every $i \le n$.

proof (a) Of course a McShane integrable function (as I have defined it) must satisfy the condition.

(b) For the converse, I use the following facts.

(i) If $E \in \Sigma$ and $E \subseteq G \in \mathfrak{T}$ and $\eta > 0$ there is an $A \in \mathcal{A}$ such that $A \subseteq G$ and $\mu(E \triangle A) \le \eta$. For take any closed set $F \subseteq E$, open set $H \supseteq E$ such that $\mu(H \setminus F) \le \eta$, and take A such that $F \subseteq A \subseteq G \cap H$.

(ii) Suppose that $\Delta : S \to \mathfrak{T}$ is a gauge and that $\langle (E_i, t_i)\rangle_{i\le n}$ is a strict finite generalized McShane partition of S subordinate to Δ. Then for any $\epsilon > 0$ there are $A_0, \dots, A_n \in \mathcal{A}$ such that $\langle (A_i, t_i)\rangle_{i\le n}$ is a strict finite generalized McShane partition of S, subordinate to Δ, and $\sum_{i\le n} \mu(A_i \triangle E_i)\|\phi(t_i)\| < \epsilon$. To see this, take $\eta > 0$ so small that $2(n+1)^2\eta \sum_{i\le n} \|\phi(t_i)\| \le \epsilon$. Now for each $i \le n$ take $A'_i \in \mathcal{A}$ such that $A'_i \subseteq \Delta(t_i)$ and $\mu(E_i \triangle A'_i) \le \eta$. Set $A = S \setminus \bigcup_{i\le n} A'_i \in \mathcal{A}$. Because S is compact and Hausdorff and $S = \bigcup_{i\le n} \Delta(t_i)$, the set $\{G : G \in \mathfrak{T}, \exists\ i \le n, \overline{G} \subseteq \Delta(t_i)\}$ is an open cover of S and has a finite subcover, and there are closed sets $F_0, \dots, F_n$ such that $F_i \subseteq \Delta(t_i)$ for each i and $\bigcup_{i\le n} F_i = S$; consequently there are $A''_0, \dots, A''_n \in \mathcal{A}$ such that $A''_i \subseteq \Delta(t_i)$ for each i and $\bigcup_{i\le n} A''_i = S$ (take A''_i such that $F_i \subseteq A''_i \subseteq G_i$ for each i). Now set

$$A_i = (A'_i \cup (A \cap A''_i)) \setminus \textstyle\bigcup_{j<i} A_j$$

for each $i \le n$. Evidently $A_0, \dots, A_n$ are disjoint, belong to $\mathcal{A}$ and cover S, and $A_i \subseteq \Delta(t_i)$ for each i. Also

$$\mu(E_i \triangle A_i) \le \mu(E_i \triangle A'_i) + \mu A + \textstyle\sum_{j<i} \mu(E_i \cap A'_j) \le \eta + (n+1)\eta + i\eta \le (2n+2)\eta$$

for each i. So

$$\textstyle\sum_{i\le n} \mu(E_i \triangle A_i)\|\phi(t_i)\| \le 2(n+1)^2\eta \sum_{i\le n} \|\phi(t_i)\| \le \epsilon,$$

as required.

(c) Now suppose that ϕ satisfies the condition. Let $\epsilon > 0$ and let $\Delta : S \to \mathfrak{T}$ be a gauge such that $\|w - \sum_{i\le n} \mu C_i \phi(t_i)\| \le \epsilon$ whenever $\langle (C_i, t_i)\rangle_{i\le n}$ is a strict finite generalized McShane cover of S by members of $\mathcal{C}$ subordinate to Δ. Let $\langle (E_i, t_i)\rangle_{i\le n}$ be any strict finite generalized McShane cover of S subordinate to Δ. By (b), there are disjoint $A_0, \dots, A_n \in \mathcal{A}$ such that $\bigcup_{i\le n} A_i = S$, $A_i \subseteq \Delta(t_i)$ for each i and

$\sum_{i\le n}\mu(E_i\triangle A_i)\|\phi(t_i)\|\le\epsilon$. By the hypothesis on $\mathcal{C}$, we can express each non-empty A_i as a disjoint union $C_{i0}\cup\ldots\cup C_{i,k(i)}$ of members of $\mathcal{C}$. Now write $t_{ij}=t_i$ for each $j\le k(i)$; we see that $\langle(C_{ij},t_{ij})\rangle_{i\le n,j\le k(i)}$ is a strict finite generalized McShane cover of S subordinate to Δ, so

$$\begin{aligned}\|w-\sum_{i\le n}\mu E_i\phi(t_i)\| &\le \|w-\sum_{i\le n}\mu A_i\phi(t_i)\|+\sum_{i\le n}|\mu E_i-\mu A_i|\|\phi(t_i)\|\\ &\le \|w-\sum_{i\le n,j\le k(i)}\mu C_{ij}\phi(t_i)\|+\epsilon\\ &\le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, the criterion of 1E shows that ϕ is McShane integrable.

1G Examples Examples relevant to the work of [McS83] are (i) $S=[a,b]$, $\mathcal{C}=\{[c,d[:\ a\le c<d\le b\}\cup\{\{b\}\}$ (ii) $S=\prod_{i\le n}[a_i,b_i]$, $\mathcal{C}=\{\prod_{i\le n}C_i:C_i\in\mathcal{C}_i\ \forall\ i\le n\}$ where $\mathcal{C}_i$ consists of intervals, as in (i). For infinite products, if each S_i (in a countable or uncountable product) is a compact Radon probability space with an associated family $\mathcal{C}_i$, then the corresponding cylinder sets in $S=\prod_i S_i$, of the form $\prod_i C_i$ where each C_i belongs to $\mathcal{C}_i\cup\{S_i\}$ and $\{i:C_i\ne S_i\}$ is finite, do the same for S.

Of course [McS83] uses gauge functions of the form $\delta:S\to\ ]0,\infty[$ rather than of the form $\Delta:S\to\mathfrak{T}$; but the translation from one to the other, in a metric space (S,ρ), is trivial, if we match $\delta(s)$ to the open set $\Delta(s)=\{t:\rho(t,s)<\delta(s)\}$.

In [Go90], [FMp91] and [Frp91] 'partitions' into non-overlapping closed intervals are used systematically; but of course these could be read throughout as half-open intervals without it making any difference.

1H The next step is to show that my version of the McShane integral agrees with the ordinary integral in the case $X=\mathbb{R}$. For the case $S=[0,1]$, this is already covered by 1F and the results of [Go90]; for other S we still have some work to do. In fact I show a more general result in one direction: for any Banach space X, if $\phi:S\to X$ is Bochner integrable, with Bochner integral w, then it is McShane integrable, with McShane integral w. (For the definition and elementary properties of the Bochner integral, see [DS58]).

We need two fairly straightforward lemmas.

1I Lemma Let $(S,\mathfrak{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $E\subseteq X$ be a set of finite measure and $x\in X$; let $\phi:S\to X$ be the function which takes the value x on E, 0 elsewhere. Then ϕ is McShane integrable, with integral $w=\mu E.x$.

proof Let $\epsilon>0$. Let F be a closed set and G an open set such that $F\subseteq E\subseteq G$ and $\mu(G\setminus F)\le\epsilon$. Set $\Delta(s)=G$ if $s\in F$, $G\setminus F$ if $s\in G\setminus F$, $S\setminus F$ if $s\in S\setminus G$. Then an easy calculation shows that $\lim_{n\to\infty}\|w-\sum_{i\le n}\mu E_i\phi(t_i)\|\le\epsilon\|x\|$ whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ.

1J Lemma Let $(S,\mathfrak{T},\Sigma,\mu)$ be an outer regular quasi-Radon measure space and X a Banach space. Let $\phi:S\to X$ be a function and suppose that $h:S\to\mathbb{R}$ is a lower semi-continuous function such that $\|\phi(s)\|<h(s)$ for every $s\in S$. Then there is a gauge $\Delta:S\to\mathfrak{T}$ such that $\sum_{i\in\mathbb{N}}\mu E_i\|\phi(t_i)\|\le\int h$ for every n whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to ϕ.

proof Set $\Delta(s)=\{t:h(t)>\|\phi(s)\|\}$ for each s; this works.

1K Theorem Let $(S,\mathfrak{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\phi:S\to X$ be a Bochner integrable function with Bochner integral w. Then ϕ is McShane integrable with McShane integral w.

proof Let $\epsilon>0$. Then there is a 'simple' function $\psi:S\to X$, of the form

$$\psi(s)=x_i\text{ when }s\in F_i,\ 0\text{ if }s\notin\textstyle\bigcup_{i\le n}F_i,$$

where $F_0,\ldots,F_n$ are disjoint sets of finite measure and each $x_i\in X$, such that $\int\|\phi(s)-\psi(s)\|\mu(ds)\le\epsilon$. Set $w_0=\sum_{i\le n}\mu F_ix_i$; then $\|w-w_0\|\le\epsilon$. As remarked in 1Bc above, there must be a lower semi-continuous function $h:S\to\mathbb{R}$ such that $\|\phi(s)-\psi(s)\|<h(s)$ for each $s\in S$ and $\int h\le 2\epsilon$. Now Lemma 1I tells us that ψ is McShane integrable, with McShane integral w_0; let Δ_0 be a gauge such that

$$\limsup_{n\to\infty}\|w_0-\textstyle\sum_{i\le n}\mu E_i\psi(t_i)\|\le\epsilon$$

whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ_0. Also Lemma 1J tells us that there is a gauge Δ_1 on S such that

$$\sum_{i\in\mathbb{N}}\mu E_i\|\phi(t_i)-\psi(t_i)\|\le 2\epsilon$$

whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ_1.

If we now take $\Delta(s)=\Delta_0(s)\cap\Delta_1(s)$ for each $s\in S$, we see that Δ is a gauge on S and that

$$\limsup_{n\to\infty}\|w-\sum_{i\le n}\mu E_i\phi(t_i)\|\le 4\epsilon$$

for every generalized McShane partition $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ of S subordinate to Δ. As ϵ is arbitrary, ϕ is McShane integrable with McShane integral w.

1L My next objective is to prove a result in the opposite direction: if $\phi:S\to\mathbb{R}$ is McShane integrable, it is integrable in the usual sense. This will lead directly to a more general result: if $\phi:S\to X$ is McShane integrable, it is Pettis integrable. My route to this takes us past some further useful facts.

Recall that if $(S,\mathfrak{T},\Sigma,\mu)$ is any quasi-Radon space, and $A\subseteq S$ is any set (not necessarily measurable), then $(A,\mathfrak{T}_A,\Sigma_A,\mu_A)$ is a quasi-Radon measure space, where $\mathfrak{T}_A$ is the induced topology on A, $\Sigma_A=\{E\cap A: E\in\Sigma\}$, and $\mu_A(B)=\min\{\mu E: B=A\cap E\}$ for $B\in\Sigma_A$. (See [Frn82], 5B and [Fr84], A7D.) It is easy to see that if $(S,\mathfrak{T},\Sigma,\mu)$ is σ-finite or outer regular, so is $(A,\mathfrak{T}_A,\Sigma_A,\mu_A)$. Accordingly, if X is a Banach space and $\phi:S\to X$ is a function, we may discuss the McShane integrability of $\phi\restriction A:A\to X$. Now we have the following results. The first is an elementary lemma.

1M Lemma Let $(S,\mathfrak{T},\Sigma,\mu)$ be a non-empty σ-finite outer regular quasi-Radon measure space and X a Banach space. Suppose that $\phi:S\to X$ has the property that for every $\epsilon>0$ there is a gauge $\Delta:S\to\mathfrak{T}$ such that

$$\limsup_{n\to\infty}\|\sum_{i\le n}\mu E_i\phi(t_i)-\sum_{i\le n}\mu F_i\phi(u_i)\|\le\epsilon$$

whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ and $\langle(F_i,u_i)\rangle_{i\in\mathbb{N}}$ are generalized McShane partitions of X subordinate to Δ. Then ϕ is McShane integrable.

proof Take ϵ, Δ as above. The point is that if $\langle(F_i,u_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ, and $\pi:\mathbb{N}\to\mathbb{N}$ is any bijection, than $\langle(F_{\pi(i)},u_{\pi(i)})\rangle_{i\in\mathbb{N}}$ is also a generalized McShane partition of S subordinate to Δ, so that

$$\limsup_{n\to\infty}\|\sum_{i\le n}\mu F_i\phi(u_i)-\sum_{i\le n}\mu F_{\pi(i)}\phi(u_{\pi(i)})\|\le\epsilon.$$

It follows at once that there is some $k\in\mathbb{N}$ such that

$$\sup_{n\ge k}\|w-\sum_{i\le n}\mu F_i\phi(u_i)\|\le 2\epsilon,$$

where $w=\sum_{i\le k}\mu F_i\phi(t_i)$. Now

$$\limsup_{n\to\infty}\|w-\sum_{i\le n}\mu E_i\phi(t_i)\|\le 3\epsilon$$

whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ.

If for each $\epsilon>0$ we use the method above to find a gauge Δ_ϵ and a vector w_ϵ, we see that $\|w_\epsilon-w_\eta\|\le 3(\epsilon+\eta)$ for all ϵ, $\eta>0$; so that $w=\lim_{\epsilon\downarrow 0}w_\epsilon$ is defined in X (this is one of the few points where we need X to be complete), and of course w will be the McShane integral of ϕ.

1N Theorem Let $(S,\mathfrak{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If $\phi:S\to X$ is McShane integrable, then $\phi\restriction A$ is McShane integrable for every $A\subseteq S$.

proof Let w be the McShane integral of ϕ, and $\epsilon>0$. Let $\Delta:S\to\mathfrak{T}$ be a gauge such that $\limsup_{n\to\infty}\|w-\sum_{i\le n}\mu E_i\phi(t_i)\|\le\epsilon$ whenever $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ.

Let $\Delta_A(s)=A\cap\Delta(s)$ for $s\in A$; then Δ_A is a gauge on A. Let $\langle(E_i,t_i)\rangle_{i\in\mathbb{N}}$ and $\langle(F_i,u_i)\rangle_{i\in\mathbb{N}}$ be generalized McShane partitions of A subordinate to Δ_A. For each $i\in\mathbb{N}$ choose $\tilde{E}_i$, $\tilde{F}_i\in\Sigma$ such that $E_i=\tilde{E}_i\cap A$, $\mu_A E_i=\mu\tilde{E}_i$, $F_i=\tilde{F}_i\cap A$ and $\mu_A F_i=\mu\tilde{F}_i$. Set

$$H=\bigcup_{i\in\mathbb{N}}(\tilde{E}_i\cap\Delta(t_i))\cap\bigcup_{i\in\mathbb{N}}(\tilde{F}_i\cap\Delta(u_i)).$$

For $i\in\mathbb{N}$ set

$$E_i^*=H\cap\tilde{E}_i\cap\Delta(t_i)\setminus\bigcup_{j<i}E_j^*,$$
$$F_i^*=H\cap\tilde{F}_i\cap\Delta(u_i)\setminus\bigcup_{j<i}F_j^*.$$

Then $\bigcup_{i\in\mathbb{N}}E_i^*=\bigcup_{i\in\mathbb{N}}F_i^*=H$.

Fix any generalized McShane partition $\langle(H_i,v_i)\rangle_{i\in\mathbb{N}}$ of S subordinate to Δ. Define H_i', v_i', H_i'', v_i'' by writing

$$H'_{2i}=E_i^*,\ v'_{2i}=t_i,\ H'_{2i+1}=H_i\setminus H,\ v'_{2i+1}=v_i,$$
$$H''_{2i}=F_i^*,\ v''_{2i}=u_i,\ H'_{2i+1}=H_i\setminus H,\ v'_{2i+1}=v_i$$

for each $i \in \mathbb{N}$. Then $\langle (H'_i, v'_i)\rangle_{i\in\mathbb{N}}$ and $\langle (H''_i, v''_i)\rangle_{i\in\mathbb{N}}$ are both generalized McShane partitions of S subordinate to Δ. So

$$\limsup_{n\to\infty} \| \sum_{i\le n} \mu H'_i \phi(v'_i) - \sum_{i\le n} \mu H''_i \phi(v''_i)\| \le 2\epsilon.$$

But on translating this through the definitions above, we see that

$$\limsup_{n\to\infty} \| \sum_{i\le n} \mu_A E_i \phi(t_i) - \sum_{i\le n} \mu_A F_i \phi(u_i)\| \le 2\epsilon.$$

So the criterion of Lemma 1M is satisfied, and $\phi\restriction A$ is McShane integrable.

Remark I give this as a theorem about arbitrary subspaces, because of course this is one of the most important constructions of quasi-Radon measure spaces (see [Frn82], 6G). However the applications below will be to measurable A, in which case the argument can be slightly simplified.

See also 2B below.

1O Theorem Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and $h : S \to \mathbb{R}$ a function. Then h is McShane integrable iff it is integrable in the ordinary sense, and the two integrals are equal.

proof If h is integrable in the ordinary sense, it is Bochner integrable, and therefore McShane integrable, by 1K. If h is McShane integrable, it is measurable; this is a special case of 3Ea below, so I omit the argument here; the argument of 3Ea can be substantially simplified for this case. If we set $E = \{s : h(s) \ge 0\}$, then by Theorem 1N we have a McShane integral (McS)$\int_E h$. Now if $g : E \to [0,\infty[$ is any function which is integrable in the ordinary sense, and dominated by h, we must have $\int_E g =$ (McS) $\int_E g \le$ (McS) $\int_E h$; because h is measurable, it follows that $\int_E h$ is defined. Similarly, $\int_{S\setminus E} h$ is defined, so that h is integrable.

2. Convergence theorems

Since Lebesgue's time, the search for 'convergence theorems' has been central to the study of integration theories. Here I show that the integral I have defined performs well in this direction. I begin with a result of a technical type (Proposition 2B), showing that we can form integrals of the type $\int_E \phi$ without problems.

2A Lemma Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space, X a Banach space and $\phi : S \to X$ a McShane integrable function. Let $\mathcal{F}$ be an upwards-directed family of measurable subsets of S such that $\mu E = \sup\{\mu(E \cap F) : F \in \mathcal{F}\}$ for every $E \in \Sigma$. Then for every $\epsilon > 0$ there are an $F \in \mathcal{F}$ and a gauge $\Delta : S \to \mathfrak{T}$ such that $\| \sum_{i\le n} \mu(E_i)\phi(t_i)\| \le \epsilon$ whenever $E_0, \dots , E_n$ are measurable sets of finite measure disjoint from each other and from F, and $t_i \in S$ is such that $E_i \subseteq \Delta(t_i)$ for each $i \le n$.

proof Let w be the McShane integral of ϕ, and let Δ be a gauge on S such that

$$\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \phi(t_i)\| \le \tfrac{\epsilon}{3}$$

for every generalized McShane partition $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ of S subordinate to Δ.

Suppose, if possible, that there is no $F \in \mathcal{F}$ witnessing the truth of the lemma. Then we may choose F_k, $\langle (E_{ki}, t_{ki})\rangle_{i\le n(k)}$ inductively, as follows. Take F_0 to be any member of $\mathcal{F}$. Given $F_k \in \mathcal{F}$, choose $\langle (E_{ki}, t_{ki})\rangle_{i\le n(k)}$ such that the E_{ki} are measurable sets disjoint from each other and from F_k and included in $\Delta(t_{ki})$, and $\| \sum_{i\le n(k)} \mu E_{ki}\phi(t_{ki})\| > \epsilon$. Now take $F_{k+1} \in \mathcal{F}$ such that $F_{k+1} \supseteq F_k$ and $\| \sum_{i\le n(k)} \mu(E_{ki} \cap F_{k+1})\phi(t_{ki})\| \ge \epsilon$. Continue. At the end of the induction write $E = \bigcup_{k\in\mathbb{N}} \bigcup_{i\le n(k)} E_{ki} \cap F_{k+1}$.

Let $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ be any generalized McShane partition of S subordinate to Δ. Set $m(k) = \sum_{j<k}(n(j)+2)$ for each k, and define $\langle (H_i, u_i)\rangle_{i\in\mathbb{N}}$ as follows. For $k \in \mathbb{N}$, $i \le n(k)$ set

$$H_{m(k)+i} = E_{ki} \cap F_{k+1},\ u_{m(k)+i} = t_{ki};$$

now take $H_{m(k)+n(k)+1} = E_k \setminus E$, $u_{m(k)+n(k)+1} = t_k$ for each k.

Evidently $\langle (H_i, u_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ. So there must be some $N \in \mathbb{N}$ such that $\|w - \sum_{i\le r} \mu H_i\phi(u_i)\| < \frac{\epsilon}{2}$ for every $r \ge N$. On the other hand,

$$\| \sum_{m(k)\le i\le m(k)+n(k)} \mu H_i\phi(u_i)\| = \| \sum_{i\le n(k)} \mu(E_{ki} \cap F_{k+1})\phi(t_{ki})\| \ge \epsilon$$

for every k; which is impossible if $k \ge N$.

This contradiction proves the lemma.

2B Proposition Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $E \in \Sigma$ and let $\phi : S \to X$ be a function which is zero on $S \setminus E$. Then ϕ is McShane integrable iff $\phi\restriction E$ is McShane integrable, and in this case the integrals are equal.

proof The case $E = \emptyset$ is trivial and as usual I will ignore it. We have already seen in Theorem 1N that if ϕ is McShane integrable then $\phi\restriction E$ is McShane integrable. Now suppose that $\phi\restriction E$ is McShane integrable with integral w. Let $\epsilon > 0$, and let $\Delta_0 : E \to \mathfrak{T}_E$ be a gauge such that $\limsup_{n\to\infty} \|w - \sum_{i\le n} \mu E_i \phi(t_i)\| \le \epsilon$ whenever $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of E subordinate to Δ_0. Let $\mathcal{F}$ be the family of subsets of E which are closed in S; then (because μ is inner regular for the closed sets) we have $\mu_E H = \sup_{F\in\mathcal{F}} \mu_E(H \cap F)$ for every $H \in \Sigma_E$, so by Lemma 2A we can find an $F \in \mathcal{F}$ and a gauge Δ_1 on E such that $\|\sum_{i\le n} \mu E_i \phi(t_i)\| \le \epsilon$ whenever $t_0, \dots, t_n \in E$, $E_0, \dots, E_n$ are measurable subsets of E disjoint from each other and from F, and $E_i \subseteq \Delta_1(t_i)$ for every i.

For each $n \in \mathbb{N}$ choose an open set $G_n \supseteq E$ such that $\mu(G_n \setminus E) \le 2^{-n}\epsilon/(n+1)$. Now let $\Delta : S \to \mathfrak{T}$ be a gauge such that

(i) if $s \in E$ then $\Delta(s) \cap E \subseteq \Delta_0(s) \cap \Delta_1(s)$;

(ii) if $s \in E$ and $n \le \|\phi(s)\| < n+1$ then $\Delta(s) \subseteq G_n$;

(iii) if $s \in S \setminus E$ then $\Delta(s) \subseteq S \setminus F$.

Let $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ be a generalized McShane partition of S subordinate to Δ. Write $I = \{i : t_i \in E\}$, $H = \bigcup_{i\in I} E_i$; observe that $F \subseteq H$. Let $\langle (F_i, u_i)\rangle_{i\in\mathbb{N}}$ be any generalized McShane partition of E subordinate to Δ_0. Fix some $u \in E$. Define $\langle (F'_i, u'_i)\rangle_{i\in\mathbb{N}}$ by setting

$F'_{2i} = E_i \cap E$, $u'_{2i} = t_i$ if $i \in I$;

$F'_{2i} = \emptyset$, $t'_{2i} = u$ if $i \in \mathbb{N} \setminus I$;

$F'_{2i+1} = F_i \setminus H$, $u'_{2i+1} = u_i$ for all $i \in \mathbb{N}$.

Then $\langle (F'_i, u'_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of E subordinate to Δ_0, so

$$\limsup_{n\to\infty} \|w - \textstyle\sum_{i\le n} \mu F'_i \phi(u'_i)\| \le \epsilon.$$

On the other hand, if $n \in \mathbb{N}$,

$$\begin{aligned}\|\sum_{i<n} \mu E_i \phi(t_i) - \sum_{i<2n} \mu F'_i \phi(u'_i)\| &= \|\sum_{i<n, i\in I} \mu(E_i \setminus E)\phi(t_i) + \sum_{i<n} \mu(F_i \setminus H)\phi(u_i)\| \\ &\le \sum_{k\in\mathbb{N}} \sum_{k\le\|\phi(t_i)\|<k+1} (k+1)\mu(E_i \setminus E) + \epsilon \\ &\le \sum_{k\in\mathbb{N}} (k+1)\mu(G_k \setminus E) + \epsilon \\ &\le \sum_{k\in\mathbb{N}} 2^{-k}\epsilon + \epsilon \le 3\epsilon\end{aligned}$$

because $E_i \subseteq G_k$ if $i \in I$ and $k \le \|\phi(t_i)\| < k+1$, while $\langle F_i \setminus H\rangle_{i<n}$ is a disjoint family of subsets of $E \setminus F$ with $F_i \setminus H \subseteq \Delta_1(u_i)$ for each i. So

$$\limsup_{n\to\infty} \|w - \textstyle\sum_{i\le n} \mu E_i \phi(t_i)\| \le 4\epsilon.$$

As ϵ is arbitrary, w is the McShane integral of ϕ.

2C For the principal theorem of this section, we need to recall some well-known facts concerning vector measures. Suppose that Σ is a σ-algebra of sets and X a Banach space.

(i) Let us say that a function $\nu : \Sigma \to X$ is 'weakly countably additive' if $f(\nu(\bigcup_{i\in\mathbb{N}} E_i)) = \sum_{i\in\mathbb{N}} f(\nu E_i)$ for every disjoint sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ in Σ and every $f \in X^*$. The first fact is that in this case ν is countably additive, that is, $\sum_{i\in\mathbb{N}} \nu E_i$ is unconditionally summable to $\nu(\bigcup_{i\in\mathbb{N}} E_i)$ for the norm topology whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence of measurable sets with union E ([Ta84], 2-6-1; [DU77], p. 22, Corollary 4).

(ii) If now μ is a measure with domain Σ such that $\nu E = 0$ whenever $\mu E = 0$, then for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\nu E\| \le \epsilon$ whenever $\mu E \le \delta$.

(iii) Thirdly, suppose that $\langle \nu_n\rangle_{n\in\mathbb{N}}$ is a sequence of countably additive functions from Σ to X such that $\nu E = \lim_{n\to\infty} \nu_n E$ exists in X, for the weak topology of X, for every $E \in \Sigma$; then ν is countably additive. (Use Nikodým's theorem ([Di84], p. 90) to see that ν is weakly countably additive.)

If $(S, \mathfrak{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space and $\phi : S \to X$ is a McShane integrable function, then by Theorem 1N we have an indefinite integral $\nu : \Sigma \to X$ given by $\nu E = \int \phi\restriction E$; now Theorem 1O shows us that ν is weakly countably additive, and accordingly countably additive.

2D Lemma Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If $\phi : S \to X$ is McShane integrable with McShane integral w, then

$$\|w\| \le \underline{\int} \|\phi(t)\| \mu(dt).$$

proof Take any f in the unit ball of X^*. By 1C, $f(w)$ is the McShane integral of $f\phi : S \to \mathbb{R}$, and by Theorem 1O this is the ordinary integral of $f\phi$. So we have

$$|f(w)| = |\int f\phi| \le \int |f\phi| \le \underline{\int}\|\phi\|.$$

As f is arbitrary, $\|w\| \le \underline{\int}\|\phi\|$.

2E Lemma Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space, X a Banach space and $\phi : S \to X$ a McShane integrable function. Then for any $\epsilon > 0$ there is a gauge $\Delta : S \to \mathfrak{T}$ such that

$$\limsup_{n\to\infty} \|\sum_{i\le n} \mu E_i \phi(t_i) - \int_E \phi\| \le \epsilon$$

whenever $E_0, \dots$ are disjoint sets of finite measure with union E and $t_0, \dots \in S$ are such that $E_i \subseteq \Delta(t_i)$ for each i.

proof Recall that by Theorem 1N we can speak of $\int_E \phi = \int \phi{\restriction}E$, and by Proposition 2B we can identify this with the integral of $\chi_E \times \phi$, so that $\int \phi = \int_E \phi + \int_{S\setminus E} \phi$ for every $E \in \Sigma$. Let $\Delta : S \to \mathfrak{T}$ be a gauge such that $\limsup_{n\to\infty} \|\int \phi - \sum_{i\le n} \mu E_i \phi(t_i)\| \le \frac{1}{2}\epsilon$ whenever $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ. Now let $E_0, \dots$ be disjoint sets of finite measure with union E, and $t_0, \dots \in S$ such that $E_i \subseteq \Delta(t_i)$ for each i. Let $\langle (F_i, u_i)\rangle_{i\in\mathbb{N}}$ be a generalized McShane partition of $S \setminus E$, subordinate to Δ, such that $\limsup_{n\to\infty} \|\int_{S\setminus E} \phi - \sum_{i\le n} \mu F_i \phi(u_i)\| \le \frac{1}{2}\epsilon$. (Readers will have no difficulty in dealing separately with the case $E = S$.)

If we set

$$E'_{2i} = E_i,\ t'_{2i} = t_i,\ E'_{2i+1} = F_i,\ t'_{2i+1} = u_i$$

for $i \in \mathbb{N}$, then $\langle (E'_i, t'_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ. So

$$\begin{aligned}\limsup_{n\to\infty} &\|\int_E \phi - \sum_{i<n} \mu E_i \phi(t_i)\| \\ &\le \limsup_{n\to\infty} (\|\int \phi - \sum_{i<2n} \mu E'_i \phi(t'_i)\| + \|\int_{S\setminus E} \phi - \sum_{i<n} \mu F_i \phi(u_i)\|) \\ &\le \epsilon,\end{aligned}$$

as required.

2F Theorem Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. Let $\langle \phi_n\rangle_{n\in\mathbb{N}}$ be a sequence of McShane integrable functions from S to X, and suppose that $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists in X for every $t \in S$. If moreover the limit

$$\nu E = \lim_{n\to\infty} \int_E \phi_n$$

exists in X, for the weak topology, for every $E \in \Sigma$, ϕ is McShane integrable and $\int \phi = \nu S$.

proof Fix $\epsilon > 0$.

(a) For $t \in S$, $n \in \mathbb{N}$ set $q_n(t) = \sup_{j\ge i\ge n} \|\phi_j(t) - \phi_i(t)\|$. Let $h : S \to \mathbb{R}$ be a strictly positive function such that $\int h \le \epsilon$. For each t, write $r(t) = \min\{n : q_n(t) \le h(t), \|\phi(t)\| \le n\}$; set $A_k = \{t : r(t) = k\}$ for each k. For each $k \in \mathbb{N}$, let $W_k \supseteq A_k$ be a measurable set with $\mu_*(W_k \setminus A_k) = 0$; set $V_k = W_k \setminus \bigcup_{j<k} W_j$ for each k, so that $\langle V_k\rangle_{k\in\mathbb{N}}$ is a disjoint cover of S by measurable sets, and $A_k \subseteq \bigcup_{j\le k} V_j$ and $\mu_*(V_k \setminus A_k) = 0$ for each k. For each k, write $V_k^* = \bigcup_{j\le k} V_j = \bigcup_{j\le k} W_j$; take $\delta_k > 0$ such that $\|\nu E\| \le 2^{-k}\epsilon$ whenever $\mu E \le \delta_k$ (see (ii) of 2C above); let $G_k \supseteq V_k^*$ be an open set such that $\mu(G_k \setminus V_k^*) \le \min(\delta_k, 2^{-k}\epsilon)$.

(b) If $k \in \mathbb{N}$ and $E \subseteq V_k^*$ is measurable, then $\|\nu E - \int_E \phi_k\| \le \int_E h$. To see this, it is enough to consider the case $E \subseteq V_j$ where $j \le k$. In this case, observe that

$$\|\nu E - \int_E \phi_k\| \le \limsup_{n\to\infty} \|\int_E \phi_n - \int_E \phi_k\| \le \sup_{n\ge k} \underline{\int}_E \|\phi_n(t) - \phi_k(t)\|\mu(dt)$$

by Lemma 2D. Now $\mu_*(E \setminus A_j) = 0$ and for $t \in A_j$ we have $\|\phi_n(t) - \phi_k(t)\| \le q_j(t) \le h(t)$ for every $n \ge k$, so

$$\underline{\int}_E \|\phi_n(t) - \phi_k(t)\|\mu(dt) \le \int_E h$$

for every $n \ge k$, giving the result.

(c) Let $\Delta' : S \to \mathfrak{T}$ be a gauge such that $\limsup_{n\to\infty} |\int h - \sum_{i\le n} \mu E_i.h(t_i)| \le \epsilon$ whenever $\langle (E_i, t_i)\rangle_{i\in\mathbb{N}}$ is a generalized McShane partition of S subordinate to Δ'; then $\sum_{i\in\mathbb{N}} \mu E_i.h(t_i) \le 2\epsilon$ for any such partition. For each $k \in \mathbb{N}$ let $\Delta_k : S \to \mathfrak{T}$ be a gauge such that

$$\| \int_E \phi_k - \sum_{i\le n} \mu E_i \phi_k(t_i)\| \le 2^{-k}\epsilon$$

whenever $E_0, \dots , E_n$ are disjoint measurable sets with union E and $t_0, \dots , t_n \in S$ are such that $E_i \subseteq \Delta_k(t_i)$ for each i; such a gauge exists by Lemma 2E. Define $\Delta : S \to \mathfrak{T}$ by setting $\Delta(t) = \Delta_k(t) \cap \Delta'(t) \cap G_k$ for $t \in A_k$.

(d) Let $\langle(E_i, t_i)\rangle_{i\in\mathbb{N}}$ be a generalized McShane partition of S subordinate to Δ. I seek to estimate $\nu S - x_n$, where $x_n = \sum_{i\le n} \mu E_i \phi(t_i)$. Fix n for the moment.

Set $I_k = \{i : i \le n,\ t_i \in A_k\}$ for each k; of course all but finitely many of the I_k are empty. For $i \in I_k$, set $E'_i = E_i \cap V^*_k$. We have $E_i \subseteq \Delta(t_i) \subseteq G_k$, so $\sum_{i\in I_k} \mu(E_i \setminus E'_i) \le 2^{-k}\epsilon$, and $\sum_{i\in I_k} \mu(E_i \setminus E'_i)\|\phi(t_i)\| \le 2^{-k}k\epsilon$, because $\|\phi(t)\| \le k$ for $t \in A_k$. Consequently, if we write

$$y_0 = \sum_{i\le n} \mu E'_i \phi(t_i),$$

we shall have $\|x_n - y_0\| \le \sum_{k\in\mathbb{N}} 2^{-k}k\epsilon = 2\epsilon$.

For each $i \le n$, let $k(i)$ be such that $t_i \in A_{k(i)}$. Then we have $\|\phi(t_i) - \phi_{k(i)}(t_i)\| \le h(t_i)$ for each i. So

$$\sum_{i\le n} \mu E'_i \|\phi(t_i) - \phi_{k(i)}(t_i)\| \le \sum_{i\in\mathbb{N}} \mu E_i h(t_i) \le 2\epsilon,$$

because $\langle(E_i, t_i)\rangle_{i\in\mathbb{N}}$ is subordinate to Δ'. Accordingly, writing

$$y_1 = \sum_{i\le n} \mu E'_i \phi_{k(i)}(t_i),$$

we have $\|x_n - y_1\| \le 4\epsilon$.

Set $H_k = \bigcup\{E'_i : i \in I_k\}$ for each k. Because $E'_i \subseteq \Delta_k(t_i)$ for each $i \in I_k$, we have

$$\| \sum_{i\in I_k} \mu E'_i \phi_k(t_i) - \int_{H_k} \phi_k\| \le 2^{-k}\epsilon.$$

Consequently, writing

$$y_2 = \sum_{k\in\mathbb{N}} \int_{H_k} \phi_k,$$

we have $\|x_n - y_2\| \le 6\epsilon$.

Next, for any k, $H_k \subseteq V^*_k$, so we have

$$\|\nu H_k - \int_{H_k} \phi_k\| \le \int_{H_k} h,$$

by (b) above. So writing $y_3 = \sum_{k\in\mathbb{N}} \nu H_k$ we have $\|y_2 - y_3\| \le \int h$ and $\|x_n - y_3\| \le 7\epsilon$.

If we set $H'_k = \bigcup\{E_i : i \in I_k\}$, then $\mu(H'_k \setminus H_k) \le \delta_k$, so that $\|\nu H'_k - \nu H_k\| \le 2^{-k}\epsilon$, for each k. Accordingly $\|x_n - y_4\| \le 9\epsilon$, where

$$y_4 = \sum_{k\in\mathbb{N}} \nu H'_k = \nu(\bigcup_{k\in\mathbb{N}} H'_k) = \nu(\bigcup_{i\le n} E_i).$$

Thus

$$\|\nu(\bigcup_{i\le n} E_i) - \sum_{i\le n} \mu E_i \phi(t_i)\| \le 9\epsilon.$$

Because ν is countably additive,

$$\limsup_{n\to\infty} \|\nu S - \sum_{i\le n} \mu E_i \phi(t_i)\| \le 9\epsilon.$$

This shows that ϕ is McShane integrable, with integral νS.

Remark This generalizes Theorem 2I of [FMp91].

2G Corollary Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space.

(a) Let $\langle\phi_n\rangle_{n\in\mathbb{N}}$ be a sequence of McShane integrable functions from S to X such that $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists in X for every $t \in S$. If

$$C = \{f\phi_n : f \in X^*,\ \|f\| \le 1,\ n \in \mathbb{N}\}$$

is uniformly integrable, then ϕ is McShane integrable.

(b) Let $\phi : S \to X$ be a Pettis integrable function and $\langle E_i\rangle_{i\in\mathbb{N}}$ a cover of S by measurable sets. Suppose that $\phi\restriction E_i$ is McShane integrable for each i. Then ϕ is McShane integrable.

proof (a) The point is that ϕ_n, ϕ satisfy the conditions of Theorem 2F. To see this, take $E \in \Sigma$ and $\epsilon > 0$. Because C is uniformly integrable, there are a set F of finite measure and a $\delta > 0$ such that $\int_H |g| \le \epsilon$ whenever $g \in C$ and $\mu(H \cap F) \le \delta$; consequently $\| \int_{E\setminus G} \phi_n\| \le \epsilon$ for all $n \in \mathbb{N}$ whenever $G \in \Sigma$ and $\mu(F \setminus G) \le \delta$. Now set

$$A_n = \{t : \|\phi_i(t) - \phi_j(t)\|\mu F \le \epsilon\ \forall\ i, j \ge n\};$$

then $\langle A_n\rangle_{n\in\mathbb{N}}$ is an increasing sequence with union S, so there is an n such that $\mu^*(F \cap A_n) \ge \mu F - \delta$. Let $G \in \Sigma$ be such that $A_n \cap F \subseteq G \subseteq F$ and $\mu G = \mu^*(A_n \cap F)$. Then whenever $i, j \ge n$ we have

$$\| \int_{E\cap G} \phi_i - \int_{E\cap G} \phi_j\| \le \underline{\int}_{E\cap G} \|\phi_i(t) - \phi_j(t)\|\mu(dt) \le \mu G \sup_{t\in A_n} \|\phi_i(t) - \phi_j(t)\| \le \epsilon.$$

Also $\|\int_{E\setminus G}\phi_i\|$ and $\|\int_{E\setminus G}\phi_j\|$ are both less than or equal to ϵ, so $\|\int_E \phi_i - \int_E \phi_j\| \le 3\epsilon$. This shows that $\langle \int_E \phi_i\rangle_{i\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent, for every $E\in\Sigma$. Accordingly the conditions of 2F are satisfied and ϕ is McShane integrable.

(b) Apply 2F with $\phi_n(t)=\phi(t)$ for $t\in\bigcup_{i\le n}E_i$, 0 elsewhere.

3. Relations with other integrals

I come now to a discussion of the relationship between the McShane integral, as I have defined it, and other integrals of vector-valued functions. I have already observed that any Bochner integrable function is McShane integrable (1K). Complementing this we have Theorem 3B: every McShane integrable function is Pettis integrable.

3A Definitions Let (S,Σ,μ) be a probability space and X a Banach space, with dual X^*.

(a) A function $\phi : S\to X$ is **Pettis integrable** if for every $E\in\Sigma$ there is a $w_E\in X$ such that $\int_E f(\phi(x))\mu(dx)$ exists and is equal to $f(w_E)$ for every $f\in X^*$; in this case w_S is the **Pettis integral** of ϕ, and the map $E\to w_E:\Sigma\to X$ is the **indefinite Pettis integral** of ϕ.

(b) A function $\phi : S\to X$ is **Talagrand integrable**, with **Talagrand integral** w, if $w=\lim_{n\to\infty}\frac{1}{n}\sum_{i<n}\phi(s_i)$ for almost all sequences $\langle s_i\rangle_{i\in\mathbb{N}}\in S^{\mathbb{N}}$, where $S^{\mathbb{N}}$ is given its product probability. (See [Ta87], Theorem 8.)

3B Theorem Let $(S,\mathfrak{T},\Sigma,\mu)$ be a σ-finite outer regular quasi-Radon measure space and X a Banach space. If $\phi : S\to X$ is McShane integrable, with McShane integral w, then it is Pettis integrable, with Pettis integral w.

proof For every $E\in\Sigma$ we have a McShane integral w_E of $\phi\restriction E$, by 1N. If $g\in X^*$ then $g\phi\restriction E : E\to\mathbb{R}$ is McShane integrable, with integral $g(w_E)$, by Proposition 1C. But we have seen in 1O that this means that $\int_E g\phi$ exists and is $g(w_E)$. As g is arbitrary, ϕ is Pettis integrable, with indefinite Pettis integral $E\mapsto w_E$; and the Pettis integral of ϕ is $w_S=w$.

Remark This generalises Theorem 2C of [FMp91].

3C I come now to a result connecting the McShane and Talagrand integrals. Recall that if (S,Σ,μ) is a probability space, a set A of real-valued functions is **stable** (in Talagrand's terminology) if for every $E\in\Sigma$, with $\mu E>0$, and all real numbers $\alpha<\beta$, there are m, $n\ge 1$ such that $\mu_{m+n}Z(A,E,m,n,\alpha,\beta)<(\mu E)^{m+n}$, where throughout the rest of paper I write $Z(A,E,I,J,\alpha,\beta)$ for

$$\{(t,u) : t\in E^I,\ u\in E^J,\ \exists\ f\in A,\ f(t(i))\le\alpha\ \forall\ i\in I,\ f(u(j))\ge\beta\ \forall\ j\in J\},$$

and μ^*_{m+n} is the ordinary product outer measure on $S^m\times S^n$. Now if X is a Banach space, a function $\phi : S\to X$ is **properly measurable** if $\{h\phi : h\in X^*,\ \|h\|\le 1\}$ is stable. Talagrand proved ([Ta87], Theorem 8) that ϕ is Talagrand integrable iff it is properly measurable and the upper integral $\overline{\int}\|\phi(t)\|\mu(dt)$ is finite.

The next proposition requires a lemma about gauges in quasi-Radon spaces.

3D Lemma Let $(S,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon probability space and $\Delta : S\to\mathfrak{T}$ a gauge. Then
(a) $\{x : x\in S^{\mathbb{N}},\ \mu(\bigcup_{i\in\mathbb{N}}\Delta(x(i)))=1\}$ has outer measure 1 in $S^{\mathbb{N}}$;
(b) writing μ_n for the quasi-Radon product measure on S^n,

$$\lim_{n\to\infty}\overline{\int}\mu(\textstyle\bigcup_{i<n}\Delta(u(i)))\mu_n(du)=1.$$

Remark The definition and properties of product quasi-Radon measures are sketched in [Fr84], A7E and discussed in detail in [Frn82]. For the purposes of this paper it would be enough to prove the lemma with μ_n the ordinary product measure of S^n. The crucial fact is that both product measures satisfy Fubini's theorem in the sense that if I, J are disjoint sets and μ_I, μ_J, $\mu_{I\cup J}$ the measures of S^I, etc., then for any $\mu_{I\cup J}$-measurable set $W\subseteq S^{I\cup J}$ we have almost every section $W_u=\{v : u^\frown v\in W\}$ measurable, and $\int\mu_J(W_u)\mu_I(du)=\mu_{I\cup J}W$.

proof (a) Suppose, if possible, otherwise.

(i) Set $h(x)=\mu(\bigcup_{i\in\mathbb{N}}\Delta(x(i)))$ for each $x\in S^{\mathbb{N}}$. For any set I let μ_I be the product quasi-Radon measure on S^I.

There is a closed set $W\subseteq S^{\mathbb{N}}$ such that $\mu_{\mathbb{N}}W>0$ and $h(x)<1$ for every $x\in W$. Set

$$T=\bigcup_{n\in\mathbb{N}}\{u : u\in S^n,\ \mu_{\mathbb{N}\setminus n}\{v : v\in S^{\mathbb{N}\setminus n},\ u^\frown v\in W\}>0\}.$$

For $u \in T$ set $g(u) = \mu(\bigcup_{i<\mathrm{dom}(u)} \Delta(u(i)))$. Choose a sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in T as follows. u_0 is to be the empty sequence. Given $u_n \in T$, choose $u_{n+1} \in T$ such that u_{n+1} properly extends u_n and $g(u_{n+1}) \geq \sup\{g(u) : u_n \subset u \in T\} - 2^{-n}$. Now we see that if $u_n \in S^{k(n)}$ for each n, $\langle k(n)\rangle_{n\in\mathbb{N}}$ is strictly increasing, so $x = \bigcup_{n\in\mathbb{N}} u_n \in S^{\mathbb{N}}$; also, for each $n \in \mathbb{N}$,

$$\{v : v \in S^{\mathbb{N}\setminus k(n)},\ (x{\restriction}k(n))^\frown v \in W\} \neq \emptyset,$$

so $x \in W$ because W is closed. Consequently $h(x) < 1$.

Let $F \subseteq S \setminus \bigcup_{i\in\mathbb{N}} \Delta(x(i))$ be a non-empty self-supporting closed set, so that $\mu(F \cap G) > 0$ for every open set G meeting F. Then, in particular, $\mu(F \cap \Delta(t)) > 0$ for every $t \in F$, so there is a $\delta > 0$ such that $\mu^* D > 0$, where

$$D = \{t : t \in F,\ \mu(F \cap \Delta(t)) \geq \delta\}.$$

(ii) Because $\langle g(u_n)\rangle_{n\in\mathbb{N}}$ is a bounded non-decreasing sequence, there is an $n \in \mathbb{N}$ such that $g(u_{n+1}) - g(u_n) + 2^{-n} < \delta$. We have

$$\mu_{\mathbb{N}\setminus k(n)}\{v : u_n^\frown v \in W\} > 0,$$

while

$$\mu^*_{\mathbb{N}\setminus k(n)}\{v : \exists\ i \geq k(n),\ v(i) \in D\} = 1,$$

so there is some $i \geq k(n)$ such that

$$\mu^*_{\mathbb{N}\setminus k(n)}\{v : u_n^\frown v \in W,\ v(i) \in D\} > 0.$$

Set $m = i + 1$,

$$E = \{w : w \in S^{m\setminus k(n)},\ \mu_{\mathbb{N}\setminus m}\{y : u_n^\frown w^\frown y \in W\} > 0\};$$

then E is $\mu_{m\setminus k(n)}$-measurable and

$$\mu_{\mathbb{N}\setminus k(n)}\{v : u_n^\frown v \in W,\ v{\restriction}m\setminus k(n) \notin E\} = 0.$$

Consequently there is a $v \in S^{\mathbb{N}\setminus k(n)}$ such that $v{\restriction}m\setminus k(n) \in E$ and $v(i) \in D$. But now consider

$$u = u_n^\frown(v{\restriction}m\setminus k(n)).$$

We see that $u \in T$ and $u_n \subset u$, so

$$g(u) \leq g(u_{n+1}) + 2^{-n}.$$

On the other hand, $u(i) \in D$, so

$$g(u) - g(u_n) \geq \mu(\Delta(u(i)) \setminus \bigcup_{j<k(n)} \Delta(u(j)))$$

$$\geq \mu(\Delta(u(i)) \cap F) \geq \delta.$$

Thus

$$g(u_{n+1}) \geq g(u) - 2^{-n} \geq g(u_n) - 2^{-n} + \delta,$$

contrary to the choice of n.

This contradiction proves the first part of the lemma.

(b) The second part follows. For each $n \in \mathbb{N}$ define $h_n : S^{\mathbb{N}} \to [a, b]$ by setting

$$h_n(x) = \mu(\textstyle\bigcup_{i<n} \Delta(x(i)))\ \ \forall\ x \in S^{\mathbb{N}}.$$

Then $\lim_{n\to\infty} h_n(x) = h(x)$ for every x, so

$$1 = \overline{\int} h(x)\mu_{\mathbb{N}}(dx) = \lim_{n\to\infty} \overline{\int} h_n(x)\mu_{\mathbb{N}}(dx) = \lim_{n\to\infty} \overline{\int} \mu(\textstyle\bigcup_{i<n} \Delta(u(i))\mu_n(du),$$

as required.

3E Proposition Let X be a Banach space such that the unit ball B of X^* is w*-separable. If $(S, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon probability space and $\phi : S \to X$ is a McShane integrable function, then it is properly measurable.

proof (a) Let w be the McShane integral of ϕ. Set $A = \{h\phi : h \in X^*,\ \|h\| \leq 1\} \subseteq \mathbb{R}^S$; we have to show that A is stable. Note that because the unit ball of X^* is separable for the w*-topology on X^*, and the map $h \mapsto h\phi : X^* \to \mathbb{R}^S$ is continuous for the w*-topology on X^* and the topology of pointwise convergence on $\mathbb{R}^S$, A has a countable dense subset A_0. Take $E \in \Sigma$, with $\mu E > 0$, and $\alpha < \alpha' < \beta' < \beta$ in $\mathbb{R}$. For $m, n \geq 1$ set $H_{mn} = Z(A, E, m, n, \alpha, \beta)$, $H'_{mn} = Z(A_0, E, m, n, \alpha', \beta')$; then $H_{mn} \subseteq H'_{mn}$ and H'_{mn} is measurable for the usual (completed) product measure on $E^m \times E^n$. We seek an m with $\mu_{2m} H'_{mm} < (\mu E)^{2m}$.

Set $\epsilon = \frac{1}{6}(\beta' - \alpha')\mu E > 0$. Let $\Delta : S \to \mathfrak{T}$ be a gauge such that

$$\limsup_{J\subseteq I \text{ is finite}} \|w - \textstyle\sum_{i\in J} \mu E_i.\phi(t_i)\| \leq \epsilon$$

whenever $\langle(E_i, t_i)\rangle_{i\in I}$ is a generalized McShane partition of S subordinate to Δ.

The set E, with its induced topology and measure, is a quasi-Radon measure space. So we may apply Lemma 3D to E, with an appropriate normalization of its measure, to see that there is an $m \in \mathbb{N}$ such that $\mu_m^* D > 0$, where

$$D = \{t : t \in E^m,\ \mu(\bigcup_{i<m} E \cap \Delta(t(i))) \geq \tfrac{3}{4}\mu E\}.$$

Suppose, if possible, that $\mu_{2m} H'_{mm} = (\mu E)^{2m}$. Then H'_{mm} must meet D^2; take t, $u \in D$ such that $(t, u) \in H'_{mm}$. Set

$$H = \bigcup_{i<m} \Delta(t(i)) \cap \bigcup_{i<m} \Delta(u(i));$$

then $\mu H \geq \frac{1}{2}\mu E$. Let $\langle t(i)\rangle_{m \leq i \in \mathbb{N}}$ be any sequence in S such that $\mu(\bigcup_{i \geq m} \Delta(t(i))) = 1$ (see §3 above).

Choose disjoint covers $\langle E_i\rangle_{i<m}$, $\langle F_i\rangle_{i<m}$ of H by measurable sets such that $E_i \subseteq \Delta(t(i))$ and $F_i \subseteq \Delta(u(i))$ for each $i < m$. Choose a disjoint cover $\langle E_i\rangle_{i \geq m}$ of $\bigcup_{i \geq m} \Delta(t(i)) \setminus H$ by measurable sets such that $E_i \subseteq \Delta(t(i))$ for each $i \geq m$. Set $u(i) = t(i)$, $F_i = E_i$ for $i \geq m$. Then we see that $\langle(E_i, t(i))\rangle_{i \in \mathbb{N}}$ and $\langle(F_i, u(i))\rangle_{i \in \mathbb{N}}$ are both generalized McShane partitions of S subordinate to the gauge Δ. So we must have

$$\limsup_{n \to \infty} \| \sum_{i \leq n} \mu(E_i).\phi(t(i)) - \sum_{i \leq n} \mu(F_i).\phi(u(i))\| \leq 2\epsilon,$$

by the choice of Δ. But this says just that

$$\| \sum_{i<m} \mu E_i.\phi(t(i)) - \mu F_i.\phi(u(i))\| \leq 2\epsilon.$$

Now $(t, u) \in H'_{mm}$, so is there is an $f \in A$ such that $f(t(i)) \leq \alpha'$ and $f(u(i)) \geq \beta'$ for every $i < m$. f is of the form $h\phi$ for some h of norm at most 1, so

$$| \sum_{i<m} \mu E_i.f(t(i)) - \mu F_i.f(u(i))| \leq 2\epsilon.$$

However, $f(t(i)) \leq \alpha'$ for each i and $\sum_{i<m} \mu E_i = \mu H$, so

$$\sum_{i<m} \mu E_i.f(t(i)) \leq \alpha' \mu H;$$

similarly $\sum_{i<m} \mu F_i.f(u(i)) \geq \beta' \mu H$, and we get

$$2\epsilon \geq (\beta' - \alpha')\mu H \geq (\beta' - \alpha')\tfrac{1}{2}\mu E = 3\epsilon,$$

which is absurd.

This shows that A is indeed stable, so that ϕ is properly measurable.

3F Corollary Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and X a Banach space such that the unit ball of X^* is w*-separable. If $\phi : S \to X$ is McShane integrable and $\overline{\int}\|\phi(s)\|\mu(ds) < \infty$, then ϕ is Talagrand integrable.

Remark These generalise Proposition 2L and Corollary 2M of [FMp91].

3G Corollary Let $(S, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon probability space and X a Banach space. If $\phi : S \to X$ is McShane integrable then its indefinite Pettis integral has totally bounded range.

proof By 4-1-5 of [Ta84], it is enough to show that $C = \{f\phi : f \in X^*, \|f\| \leq 1\}$ is totally bounded for $\| \ \|_1$. This will be so iff every countable subset of C is totally bounded. But 3E shows that any countable subset of C is stable, and therefore totally bounded by [Ta84], 9-5-2.

3H I conclude with some questions left open by the work above.

Problems (a) Suppose that $(S, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon probability space, X is a Banach space, and $\phi : S \to X$ a Pettis integrable function. Does it follow that the indefinite integral of ϕ has totally bounded range?

(b) Suppose that (S, Σ, μ) is a σ-finite measure space, X is a Banach space, and $\psi : S \to X$ is a function. Suppose that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two topologies on S making it an outer regular quasi-Radon measure space. If ϕ is McShane integrable for $\mathfrak{T}_1$, must it be McShane integrable for $\mathfrak{T}_2$?

As a special case of this, take $S = [0, 1]$ with its usual measure, $\mathfrak{T}_1$ the topology associated with a strong lifting ([Frn82], 3G), and $\mathfrak{T}_2$ the usual topology.

If the unit ball of X^* is w*-separable, then the answer is 'yes'; see [Frn92].

(c) Suppose that $(S, \mathfrak{T}, \Sigma, \mu)$ is a quasi-Radon probability space, X a Banach space, and $\phi : S \to X$ a measurable Pettis integrable function. Must ϕ be McShane integrable? (If there is no real-valued-measurable cardinal, yes.)

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