Topological spaces after forcing

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I offer some notes on a general construction of topological spaces in forcing models.

I follow Kunen 80 in my treatment of forcing; in particular, for a forcing notion $\mathbb{P}$, terms in $V^{\mathbb{P}}$ are subsets of $V^{\mathbb{P}} \times \mathbb{P}$. For other unexplained notation it is worth checking in Fremlin 02, Fremlin 03 and Fremlin 08.

This is an abridged version with many proofs and comments omitted.

Contents

1 Universally Baire-property sets

1A Definition Let X be a topological space. I will say that a set $A \subseteq X$ is **universally Baire-property** if $f^{-1}[A]$ has the Baire property in Z whenever Z is a Čech-complete completely regular Hausdorff space and $f : Z \to X$ is a continuous function. Because the family $\widehat{\mathcal{B}}(Z)$ of subsets of Z with the Baire property is always a σ-algebra closed under Souslin's operation and including the Borel σ-algebra, the family $\mathcal{U}\widehat{\mathcal{B}}(X)$ of universally Baire-property subsets of X is a σ-algebra of subsets of X closed under Souslin's operation and including the Borel σ-algebra.

1B Elementary facts Let X be a topological space.

(a) If Y is another topological space, $h : X \to Y$ is continuous and $A \in \mathcal{U}\widehat{\mathcal{B}}(Y)$ then $h^{-1}[A] \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(b)(i) If $Y \subseteq X$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then $A \cap Y \in \mathcal{U}\widehat{\mathcal{B}}(Y)$.

(ii) If $F \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(F)$ then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(c) If $\langle X_i \rangle_{i \in I}$ is a countable family of topological spaces and $A_i \in \mathcal{U}\widehat{\mathcal{B}}(X_i)$ for every i, then $\prod_{i \in I} A_i \in \mathcal{U}\widehat{\mathcal{B}}(\prod_{i \in I} X_i)$.

(d) Suppose that $A \subseteq X$ and that $\mathcal{G}$ is a family of open subsets of X, covering A, such that $A \cap G \in \mathcal{U}\widehat{\mathcal{B}}(X)$ for every $G \in \mathcal{G}$. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(e) If X is Čech-complete, then $\mathcal{U}\widehat{\mathcal{B}}(X) \subseteq \widehat{\mathcal{B}}(X)$.

1C Proposition If X is a Hausdorff space and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then A is universally Radon-measurable in X in the sense of Fremlin 03, 434E.

1D Let X be a Hausdorff space such that every compact subset of X is scattered. Then $\mathcal{U}\widehat{\mathcal{B}}(X) = \mathcal{P}X$.

1E Theorem Let X be a compact Hausdorff space, and $A \subseteq X$. Then the following are equiveridical:

(i) $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$;

(ii) $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$ whenever W is a topological space and $f : W \to X$ is continuous;

(iii) $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is an extremally disconnected compact Hausdorff space and $f : Z \to X$ is continuous;

(iv) there are a compact Hausdorff space K and a continuous surjection $f : K \to X$ such that $f^{-1}[A] \in \mathcal{U}\widehat{\mathcal{B}}(K)$.

1F Corollary (a) Let X be a topological space which is homeomorphic to a universally Baire-property subset of some compact Hausdorff space, and W any topological space. Then any continuous function from W to X is $(\widehat{\mathcal{B}}(W),\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable.

(b) Let X be a locally compact Hausdorff space, and $A \subseteq X$ a set such that $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is an extremally disconnected compact Hausdorff space and $f : Z \to X$ is continuous. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

1G Proposition (a) Suppose that Z is a topological space, X is second-countable and $f : Z \to X$ is $\widehat{\mathcal{B}}(Z)$-measurable. Then there is a comeager $Z_1 \subseteq Z$ such that $f{\restriction}Z_1$ is continuous.

(b) Suppose that X is a topological space, Y is a second-countable space and $\phi : X \to Y$ is $\mathcal{U}\widehat{\mathcal{B}}(X)$-measurable. Then ϕ is $(\mathcal{U}\widehat{\mathcal{B}}(X),\mathcal{U}\widehat{\mathcal{B}}(Y))$-measurable.

1H Lemma If W is a non-empty topological space, κ a cardinal and $\pi(W) \le \kappa$, then $\kappa^{\mathbb{N}}$ (giving each copy of κ the discrete topology) and $W \times \kappa^{\mathbb{N}}$ have isomorphic regular open algebras.

1I Lemma Let X be a metrizable space, κ an infinite cardinal, W a Čech-complete space with regular open algebra isomorphic to that of $\kappa^{\mathbb{N}}$, and $f : W \to X$ a continuous function. Then there are a dense G_δ subset W' of W and continuous functions $g : W' \to \kappa^{\mathbb{N}}$ and $h : \kappa^{\mathbb{N}} \to X$ such that $hg = f{\restriction}W'$; moreover, we can choose g in such a way that it is surjective and $g[F]$ is not dense for any proper relatively closed set $F \subseteq W'$.

1J Lemma Let W be a topological space and Y a non-empty α-favourable topological space.

(a) If $A \subseteq W$ is such that $A \times Y$ is meager in $W \times Y$, then A is meager in W.

(a) If $A \subseteq W$ is such that $A \times Y \in \widehat{\mathcal{B}}(W \times Y)$, then $A \in \widehat{\mathcal{B}}(Y)$.

1K Theorem (see Feng Magidor & Woodin 92, Theorem 2.1) Let X be a metrizable space and $A \subseteq X$. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ iff whenever κ is a cardinal and $f : \kappa^{\mathbb{N}} \to X$ is continuous, then $f^{-1}[A] \in \widehat{\mathcal{B}}(\kappa^{\mathbb{N}})$.

2 Basic theory

2A Hausdorff spaces after forcing Let $(X,\mathfrak{T})$ be a Hausdorff space and $\mathbb{P}$ a forcing notion.

(a) Let Z be the Stone space of the regular open algebra $\mathrm{RO}(\mathbb{P})$ of $\mathbb{P}$; in this context I will interpret Boolean truth values $[\![\phi]\!]$ directly as open-and-closed sets in Z. For $p \in \mathbb{P}$ let $\widehat{p} \subseteq Z$ be the open-and-closed set corresponding to the regular open set $\{q :$ if r is stronger than q then r is compatible with $p\}$. For subsets S, T of Z I will say that $S \subseteq^* T$ if $S \setminus T$ is meager. Note that if S, $T \in \widehat{\mathcal{B}}(Z)$ and $S \not\subseteq^* T$, then there is a $p \in \mathbb{P}$ such that $\widehat{p} \subseteq^* S \setminus T$. Let $C^-(Z;X)$ be the space of continuous functions from dense G_δ subsets of Z to X.

For a function $f \subseteq Z \times X$ let $\vec{f}$ be the $\mathbb{P}$-name

$$\{(\check{g},p) : g \in C^-(Z;X),\, p \in \mathbb{P},\, \widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\, f(z) = g(z)\}\};$$

for $A \subseteq X$ let $\tilde{A}$ be the $\mathbb{P}$-name

$$\{(\vec{f}, p) : f \in C^{-}(Z; X),\ p \in \mathbb{P},\ \widehat{p} \subseteq^{*} f^{-1}[A]\}.$$

(b)(i) If $f \subseteq Z \times X$ is a function, $g \in C^{-}(Z; X)$ and $p \in \mathbb{P}$ then $p \Vdash_{\mathbb{P}} \breve{g} \in \vec{f}$ iff $(\breve{g}, p) \in \vec{f}$.

(ii) If $f \subseteq Z \times X$ is a function, $g \in C^{-}(Z; X)$ and $p \in \mathbb{P}$ then $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$ iff $\widehat{p} \subseteq^{*} \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$.

(iii) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $f \in C^{-}(Z; X)$ and $p \in \mathbb{P}$, then $p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}$ iff $(\vec{f}, p) \in \tilde{A}$.

(iv) Suppose that $*$ is one of the four Boolean operations $\cup$, $\cap$, $\setminus$ and $\triangle$. If A, B, $C \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $A * B = C$ then $\Vdash_{\mathbb{P}} \tilde{A} * \tilde{B} = \tilde{C}$.

(v) Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{U}\widehat{\mathcal{B}}(X)$ with union A. Then $\Vdash_{\mathbb{P}} \tilde{A} = \bigcup_{n \in \mathbb{N}} \tilde{A}_n$.

(vi) Let $\langle G_i \rangle_{i \in I}$ be a family in $\mathfrak{T}$ with union G. Then

$$\Vdash_{\mathbb{P}} \tilde{G} = \bigcup_{i \in \check{I}} \tilde{G}_i.$$

(vii) Suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$. Then there is an $f \in C^{-}(Z; A)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$.

(viii) If, in (vii), the set A is compact, then every member of $C^{-}(Z; A)$ will have a (unique) extension to a member of $C(Z; A)$, because Z is extremally disconnected; so we find that whenever $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$, then there is an $f \in C(Z; A)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$.

(c) Now set

$$\tilde{\mathfrak{T}} = \{(\tilde{G}, \mathbb{1}) : G \in \mathfrak{T}\}.$$

Then

$$\Vdash_{\mathbb{P}} \tilde{\mathfrak{T}} \text{ is a topology base on } \tilde{X} \text{ and generates a Hausdorff topology on } \tilde{X}.$$

(d)(i) It is perhaps worth noting explicitly that we can use any base for $\mathfrak{T}$ to define the topology on $\tilde{X}$ in $V^{\mathbb{P}}$. If $\mathcal{U}$ is a base for $\mathfrak{T}$, set $\tilde{\mathcal{U}} = \{(\tilde{U}, \mathbb{1}) : U \in \mathcal{U}\}$. Then

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a topology base on } \tilde{X} \text{ and generates the same topology as } \tilde{\mathfrak{T}}.$$

(ii) Similarly, if $\mathcal{U}$ is any subbase for $\mathfrak{T}$, and we set $\tilde{\mathcal{U}} = \{(\tilde{U}, \mathbb{1}) : U \in \mathcal{U}\}$, then

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ generates the same topology as } \tilde{\mathfrak{T}}.$$

(e)

$$\Vdash_{\mathbb{P}} \tilde{F} \text{ is closed in } \tilde{X}$$

whenever $F \subseteq X$ is closed.

$$\Vdash_{\mathbb{P}} \tilde{E} \text{ is Borel in } \tilde{X}$$

whenever $E \subseteq X$ is Borel.

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is nowhere dense in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ is nowhere dense in X.

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is meager in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ is meager in X, and

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ has the Baire property in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ has the Baire property in X.

(f)(i) For $x \in X$, let $e_x \in C^-(Z;X)$ be the constant function with domain Z and value x, and write $\tilde{x}$ for the $\mathbb{P}$-name $\vec{e}_x$. Set

$$\dot{\varphi} = \{((\check{x}, \tilde{x}), \mathbb{1}) : x \in X\},$$

so that

$$\Vdash_{\mathbb{P}} \dot{\varphi} \text{ is a function from } \check{X} \text{ to } \tilde{X}.$$

$\Vdash_{\mathbb{P}} \dot{\phi}$ is injective.

(ii) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then

$$\Vdash_{\mathbb{P}} \check{A} = \dot{\varphi}^{-1}[\tilde{A}].$$

(iii) Next, if $D \subseteq X$ is dense,

$$\Vdash_{\mathbb{P}} \dot{\varphi}[\check{D}] \text{ is dense in } \tilde{X}.$$

(g)(i) Suppose that every compact subset of X is scattered. Then

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot{\varphi}[\check{X}].$$

(ii) In particular, if $\#(X) < \mathfrak{c}$ or X is discrete,

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot{\varphi}[\check{X}].$$

(iii) In fact, if X is discrete, then

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot{\varphi}[\check{X}] \text{ is discrete.}$$

2B Closures and interiors In the context of 2A, suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. Then

$$\Vdash_{\mathbb{P}} \operatorname{int} \tilde{A} = (\operatorname{int} A)\tilde{}, \ \overline{\tilde{A}} = \overline{\dot{\varphi}[\check{A}]} = \tilde{\overline{A}} \text{ and } \partial\tilde{A} = (\partial A)\tilde{},$$

where ∂A is the topological boundary of A.

2C Continuous functions, among others Let $\mathbb{P}$ be a forcing notion, Z the Stone space of its regular open algebra, $(X,\mathfrak{T})$ and $(Y,\mathfrak{S})$ Hausdorff spaces, and $\tilde{X}$, $\tilde{\mathfrak{T}}$, $\tilde{Y}$ and $\tilde{\mathfrak{S}}$ the $\mathbb{P}$-names as defined in 2A. Let $\phi\subseteq X\times Y$ be a function.

(a) Let $\tilde{\phi}$ be the $\mathbb{P}$-name

$$\{((\vec{f},\vec{g}),p): f\in C^-(Z;X),\ g\in C^-(Z;Y),\ p\in\mathbb{P},\ \widehat{p}\subseteq^*\operatorname{dom}(g\cap\phi f)\}.$$

Then

$$\Vdash_{\mathbb{P}}\tilde{\phi}\text{ is a function from a subset of }\tilde{X}\text{ to }\tilde{Y}.$$

(b)(i) If $p\in\mathbb{P}$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that $p\Vdash_{\mathbb{P}}\tilde{\phi}(\dot{x})=\dot{y}$, then there are $f\in C^-(Z;X)$ and $g\in C^-(Z;Y)$ such that

$$p\Vdash_{\mathbb{P}}\dot{x}=\vec{f}\text{ and }\dot{y}=\vec{g},$$

$$\widehat{p}\subseteq\operatorname{dom}(g\cap\phi f).$$

(ii) In fact, if $p\in\mathbb{P}$ and $f\in C^-(Z;X)$ and $g\in C^-(Z;Y)$, then $p\Vdash_{\mathbb{P}}\tilde{\phi}(\vec{f})=\vec{g}$ iff $\widehat{p}\subseteq^*\operatorname{dom}(g\cap\phi f)$.

(c) Next, suppose that $A\in\mathcal{U}\widehat{\mathcal{B}}(X)$, $A\subseteq\operatorname{dom}\phi$, $\phi\restriction A$ is continuous and $B\in\mathcal{U}\widehat{\mathcal{B}}(Y)$. Then $A\cap\phi^{-1}[B]\in\mathcal{U}\widehat{\mathcal{B}}(X)$ and

$$\Vdash_{\mathbb{P}}\tilde{A}\cap\tilde{\phi}^{-1}[\tilde{B}]=(A\cap\phi^{-1}[B])^{\sim}.$$

(In particular, $\Vdash_{\mathbb{P}}\tilde{A}\subseteq\operatorname{dom}\tilde{\phi}$.)

(d) If $A\in\mathcal{U}\widehat{\mathcal{B}}(X)$, $A\subseteq\operatorname{dom}\phi$ and $\phi\restriction A$ is continuous, then

$$\Vdash_{\mathbb{P}}\tilde{\phi}\restriction\tilde{A}\text{ is continuous.}$$

(e) If X_0, X_1, X_2 are Hausdorff spaces and $\phi:X_0\to X_1$, $\psi:X_1\to X_2$ are continuous functions, then

$$\Vdash_{\mathbb{P}}(\psi\phi)^{\sim}=\tilde{\psi}\tilde{\phi}.$$

(f) If ϕ is injective, then

$$\Vdash_{\mathbb{P}}\tilde{\phi}\text{ is injective.}$$

(g) If ϕ is a homeomorphism between X and a set $B\in\mathcal{U}\widehat{\mathcal{B}}(Y)$, then

$$\Vdash_{\mathbb{P}}\tilde{\phi}\text{ is a homeomorphism between }\tilde{X}\text{ and }\tilde{B}.$$

2D Lemma Suppose, in the context of 2C, that $X=Y$ and we have a set $A\in\mathcal{U}\widehat{\mathcal{B}}(X)$ such that $\phi(x)=x$ for every $x\in A$. Then

$$\Vdash_{\mathbb{P}}\tilde{\phi}(x)=x\text{ for every }x\in\tilde{A}.$$

2E Alternative description of Borel sets Let $\mathbb{P}$, Z and $(X,\mathfrak{T})$ be as in §2A.

(a) If $\dot{G}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X},$$

consider the open set

$$W = \bigcup_{G\in\mathfrak{T}} [\![\tilde{G} \subseteq \dot{G}]\!] \times G \subseteq Z \times X.$$

If $\dot{E}$, $\dot{G}$ and $\dot{H}$ are $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ and } \dot{H} \text{ are open subsets of } \tilde{X} \text{ and } \dot{E} = \dot{G} \cap \dot{H},$$

and $W_{\dot{E}}$, $W_{\dot{G}}$ and $W_{\dot{H}}$ are the corresponding open subsets of $Z \times X$, then $W_{\dot{E}} = W_{\dot{G}} \cap W_{\dot{H}}$.

In particular, $\Vdash_{\mathbb{P}} \dot{G} \cap \dot{H} = \emptyset$ iff $W_{\dot{G}}$ and $W_{\dot{H}}$ are disjoint.

(b) For any $W \subseteq Z \times X$, let $\vec{W}$ be the $\mathbb{P}$-name

$$\{(\vec{f}, p) : f \in C^{-}(Z;X),\, p \in \mathbb{P},\, \widehat{p} \subseteq^{*} \{z : (z, f(z)) \in W\}\}.$$

(i) If $\dot{G}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X},$$

$W_{\dot{G}}$ is the corresponding open subset of $Z \times X$, $p \in \mathbb{P}$ and $f \in C^{-}(Z;X)$, then $p \Vdash_{\mathbb{P}} \vec{f} \in \dot{G}$ iff $(\vec{f}, p) \in \vec{W}_{\dot{G}}$.

(ii)

$$\Vdash_{\mathbb{P}} \vec{W}_{\dot{G}} = \dot{G}.$$

(iii) $W_{\tilde{X}} = Z \times X$ and

$$\Vdash_{\mathbb{P}} \tilde{X} = (Z \times X)^{\rightarrow}.$$

(iv) Next, observe that if $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ and $f \in C^{-}(Z;X)$, then

$$[\![\vec{f} \in \vec{W}]\!] \triangle \{z : (z, f(z)) \in W\} \text{ is meager.}$$

(c)(i) If $p \in \mathbb{P}$, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $\widehat{p} \times A \subseteq W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$, then $p \Vdash_{\mathbb{P}} \tilde{A} \subseteq \vec{W}$.

(ii) If $W \subseteq Z \times X$ is open, then

$$\Vdash_{\mathbb{P}} \vec{W} \text{ is open.}$$

(iii) If $V \subseteq Z$ is open-and-closed, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $W = V \times A$, then

$$V = [\![\vec{W} = \tilde{A}]\!], \quad Z \setminus V = [\![\vec{W} = \emptyset]\!].$$

(d) If V_1, $V_2 \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$, $*$ is any of the Boolean operations $\cup$, $\cap$, $\setminus$ and $\triangle$ and $W = V_1 * V_2$, then

$$\Vdash_{\mathbb{P}} \vec{W} = \vec{V}_1 * \vec{V}_2.$$

(d) If $\langle V_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}\widehat{\mathcal{B}}(Z\times X)$ with union W, then $\Vdash_{\mathbb{P}} \vec{W} = \bigcup_{n\in\mathbb{N}}\vec{W}_n$.

(f) If $\langle W_i\rangle_{i\in I}$ is a family of open subsets of $Z\times X$ with union W, then $\Vdash_{\mathbb{P}} \vec{W} = \bigcup_{i\in\check{I}}\vec{W}_i$.

(g) It follows that if $W\subseteq Z\times X$ is a Borel set, then $\Vdash_{\mathbb{P}} \vec{W}$ is a Borel set in $\tilde{X}$.

(h)(i) Now suppose that $p\in\mathbb{P}$, $\alpha<\omega_1$ and that $\dot{E}$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel subset of } \tilde{X} \text{ of class } \alpha.$$

Then there is a Borel set $W\subseteq Z\times X$ of class α such that $p\Vdash_{\mathbb{P}} \dot{E}=\vec{W}$.

(ii) If $p\in\mathbb{P}$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel subset of } \tilde{X},$$

then there is a $W\in\mathcal{U}\widehat{\mathcal{B}}(X)$ such that $p\Vdash_{\mathbb{P}} \dot{E}=\vec{W}$.

(iii) If $\mathbb{P}$ is ccc, $p\in\mathbb{P}$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel set in } \tilde{X},$$

then there is a Borel set $W\subseteq Z\times X$ such that $p\Vdash_{\mathbb{P}} \dot{E}=\vec{W}$.

(i) If $W\subseteq Z\times X$ is open then

$$\Vdash_{\mathbb{P}} \overline{\vec{W}} = \vec{\overline{W}}.$$

2F Convergent sequences: Lemma Suppose that $\mathbb{P}$ is a forcing notion, Z the Stone space of its regular open algebra, and X a Hausdorff space. Suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in $C^-(Z;X)$ and $f\in C^-(Z;X)$, $p\in\mathbb{P}$ are such that

$$\widehat{p}\subseteq^* \{z : f(z)=\lim_{n\to\infty} f_n(z) \text{ in } X\}.$$

Then

$$p\Vdash_{\mathbb{P}} \vec{f}=\lim_{n\to\infty}\vec{f}_n \text{ in } \tilde{X}.$$

2G Theorem Let X be a Hausdorff space and $\mathbb{P}$ a forcing notion, with Stone space Z. If $Z_0\subseteq Z$ is comeager and $V\subseteq Z_0\times X$ is an usco-compact relation in $Z_0\times X$, then, in the language of 2E,

$$\Vdash_{\mathbb{P}} \vec{V} \text{ is compact in } \tilde{X}.$$

2H Theorem Let X be a Hausdorff space, $\mathbb{P}$ a forcing notion and Z its Stone space. Set $S=\bigcup_{n\geq1}\mathbb{N}^n$ and let $\langle W_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{U}\widehat{\mathcal{B}}(Z\times X)$ with kernel W. Then

$$\Vdash_{\mathbb{P}} \vec{W} \text{ is the kernel of the Souslin scheme } \langle\vec{W}_\sigma\rangle_{\sigma\in S}.$$

2I Corollary If $\langle A_\sigma\rangle_{\sigma\in S}$ is a Souslin scheme in $\mathcal{U}\widehat{\mathcal{B}}(X)$ with kernel A, then

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is the kernel of } \langle\tilde{A}_\sigma\rangle_{\sigma\in S}.$$

2J Finding the Boolean value $[\![\vec{W}\neq\emptyset]\!]$ Let X be a Hausdorff space, $\mathbb{P}$ a forcing notion and Z its Stone space. If $W\in\mathcal{U}\widehat{\mathcal{B}}(Z\times X)$ then

$$[\![\vec{W}\neq\emptyset]\!]\subseteq^* W^{-1}[X].$$

(ii) If V, $W\in\mathcal{U}\widehat{\mathcal{B}}(Z\times X)$ then

$$\{z:V[\{z\}]\subseteq W[\{z\}]\}\subseteq^* [\![\vec{V}\subseteq\vec{W}]\!].$$

(iii) If $A\in\mathcal{U}\widehat{\mathcal{B}}(X)$ and $W\in\mathcal{U}\widehat{\mathcal{B}}(Z\times X)$ then

$$\{z:A\subseteq W[\{z\}]\}\subseteq^* [\![\tilde{A}\subseteq\vec{W}]\!].$$

(b) If $Z_0\subseteq Z$ is comeager and $W\subseteq Z_0\times X$ is usco-compact, then $[\![\vec{W}\neq\emptyset]\!]\triangle W^{-1}[X]$ is meager.

(c) If $W\subseteq Z\times X$ is K-analytic, then $[\![\vec{W}\neq\emptyset]\!]\triangle W^{-1}[X]$ is meager.

(d) If $W\subseteq Z\times X$ is open then $[\![\vec{W}\neq\emptyset]\!]\triangle W^{-1}[X]$ is meager.

2K Examples (a) Let $\mathbb{P}$ be a forcing notion and Z its Stone space. Suppose that Z is expressible as the union of κ nowhere dense zero sets. Set $X=[0,1[^\kappa$. Then there is a closed set $W\subseteq Z\times X$ such that $W^{-1}[X]=Z$ but $\Vdash_{\mathbb{P}}\vec{W}=\emptyset$.

(b) Suppose that $A\subseteq[0,1]$ is a coanalytic set with no perfect subset and that $\mathbb{P}$ is a forcing notion such that the Stone space Z of $\mathbb{P}$ can be covered by ω_1 nowhere dense sets. Then there is a set $W\in\mathcal{U}\widehat{\mathcal{B}}(Z\times[0,1])$ such that $W^{-1}[/,[0,1]/,]=Z$ but $\Vdash_{\mathbb{P}}\vec{W}=\emptyset$.

3 Identifying the new spaces

The most pressing problem is to find ways of getting a clear picture of the 'new' spaces as topological spaces. For actual examples it will be easiest to wait for §4 below. Here I put together a handful of basic techniques.

3A Theorem Let $\langle X_i\rangle_{i\in I}$ be a family of Hausdorff spaces with product X, and $\mathbb{P}$ a forcing notion. Suppose that $J=\{i:i\in I,\ X_i\text{ is not compact}\}$ is countable. Then

$$\Vdash_{\mathbb{P}}\tilde{X}\text{ can be identified with }\textstyle\prod_{i\in\check{I}}\tilde{X}_i.$$

3B Regular open algebras Let $\mathbb{P}$, $(X,\mathfrak{T})$ and $\tilde{X}$ be as in §2A.

(a) If $G\subseteq X$ is a regular open set in X, then

$$\Vdash_{\mathbb{P}}\tilde{G}\text{ is a regular open set in }\tilde{X}.$$

(b) Let $\mathrm{RO}(X)$ be the regular open algebra of X. Then Write $\dot\vartheta$ for the $\mathbb{P}$-name $\{((\check G,\tilde G),\mathbb{1}):G\in\mathrm{RO}(X)\}$. By (a),

$$\Vdash_{\mathbb{P}}\dot\vartheta\text{ is a function from }\mathrm{RO}(X)\check{}\text{ to }\mathrm{RO}(\tilde X).$$

Now

$$\Vdash_{\mathbb{P}}\dot\vartheta\text{ is a Boolean homomorphism.}$$

(c) $\Vdash_{\mathbb{P}}\dot\vartheta$ is injective.

(d) $\Vdash_{\mathbb{P}}\dot\vartheta[\mathrm{RO}(X)\check{}\,]$ is order-dense in $\mathrm{RO}(\tilde X)$.

3C Corollary For any topological space X,

$\Vdash_{\mathbb{P}}\mathrm{RO}(\tilde X)$ can be identified with the Dedekind completion of $\mathrm{RO}(X)\check{}$.

3D Normal bases and the finite-cover uniformity (a) Let X be a set. I will say that a topology base $\mathcal{U}$ on X is **normal** if

(i) $U\cup V$ and $U\cap V$ belong to $\mathcal{U}$ for all U, $V\in\mathcal{U}$,

(ii) whenever $x\in U\in\mathcal{U}$ there is a $V\in\mathcal{U}$ such that $U\cup V=X$ and $x\notin V$,

(iii) whenever U, $V\in\mathcal{U}$ and $U\cup V=X$ then there are disjoint U', $V'\in\mathcal{U}$ such that $U\cup V'=U'\cup V=X$.

(b) Let $\mathcal{U}$ be a normal topology base on X.

(i) If $\mathcal{V}\subseteq\mathcal{U}$ is a finite cover of X, there is a finite $\mathcal{V}^*\subseteq\mathcal{U}$, a cover of X, which is a star-refinement of $\mathcal{V}$.

(ii) We have a uniformity $\mathcal{W}$ on X defined by saying that a subset W of $X\times X$ belongs to $\mathcal{W}$ iff there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, covering X, such that $W_{\mathcal{V}}\subseteq W$, where $W_{\mathcal{V}}=\bigcup_{V\in\mathcal{V}}V\times V$.

(iii) The topologies $\mathfrak{T}_{\mathcal{U}}$, $\mathfrak{T}_{\mathcal{W}}$ induced on X by $\mathcal{U}$, $\mathcal{W}$ respectively are equal.

(iv) I will call $\mathcal{W}$ the **finite-cover uniformity** derived from $\mathcal{U}$.

(c) The definition in (b-ii) makes it plain that X is totally bounded for the finite-cover uniformity.

(d) Let X be a compact Hausdorff space.

(i) If $\mathcal{U}$ is a base for the topology of X closed under $\cup$ and $\cap$, then $\mathcal{U}$ is a normal topology base.

(ii) If $Y\subseteq X$ is dense, $\mathcal{U}$ is a base for the topology of X and $\mathcal{U}_Y=\{Y\cap U:U\in\mathcal{U}\}$ is a normal topology base on Y, then X can be identified with the completion of Y for the finite-cover uniformity induced by $\mathcal{U}_Y$.

3E Descriptions of $\tilde X$: Proposition Let $\mathbb{P}$ be a forcing notion, X a compact Hausdorff space and $\mathcal{U}$ a normal base for the topology of X. Let Z, $\tilde X$, $\dot\varphi:\check X\to\tilde X$ be as in §2.

(a)

$$\Vdash_{\mathbb{P}}\check{\mathcal{U}}\text{ is a normal topology base on }\check X.$$

(b)

$\Vdash_{\mathbb{P}}$ the embedding $\dot{\varphi} : \check{X} \to \tilde{X}$ identifies $\tilde{X}$, with the unique uniformity compatible with its topology, with the completion of $\check{X}$ with the finite-cover uniformity on $\check{X}$ generated by $\check{\mathcal{U}}$.

3F Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of RO($\mathbb{P}$), which I think of as the algebra of open-and-closed sets in Z; let X be a non-empty Hausdorff space.

(a)(i) For every $f \in C^-(Z;X)$ we have a sequentially order-continuous Boolean homomorphism $\pi_f : \mathcal{U}\widehat{\mathcal{B}}(X) \to \mathrm{RO}(\mathbb{P})$ defined by saying that $\pi_f(A) \triangle f^{-1}[A]$ is meager for every $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(ii) $\pi_f(A) = [\![\vec{f} \in \tilde{A}]\!]$ for any $f \in C^-(Z;X)$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(iii) π_f is **τ-additive** in the sense that if $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, then $\pi_f H = \sup_{G \in \mathcal{G}} \pi_f G$ in RO($\mathbb{P}$).

(iv) If f, $g \in C^-(Z;X)$ and $p \in \mathbb{P}$, then the following are equiveridical:

(α) f and g agree on $\widehat{p} \cap \mathrm{dom}\, f \cap \mathrm{dom}\, g$;

(β) $\widehat{p} \subseteq^* \mathrm{dom}(f \cap g)$;

(γ) for any t and for any q stronger than p, $(t,q) \in \vec{f}$ iff $(t,q) \in \vec{g}$;

(δ) $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$;

(ϵ) $\widehat{p} \cap \pi_f A = \widehat{p} \cap \pi_g A$ for every $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$;

(ζ) there is a base $\mathcal{U}$ for the topology of X such that $\widehat{p} \cap \pi_f G = \widehat{p} \cap \pi_g G$ for every $G \in \mathcal{U}$.

(b)(i) Suppose that X is Čech-complete and that $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ is a sequentially order-continuous Boolean homomorphism which is τ-additive in the sense that $\pi(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \pi G$ whenever $\mathcal{G} \subseteq \mathcal{B}\mathfrak{a}(X)$ is a family of open sets with union in $\mathcal{B}\mathfrak{a}(X)$. Then there is an $f \in C^-(Z;X)$ such that π_f extends π.

(ii) If X is compact, then for every sequentially order-continuous $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ there is an $f \in C(Z;X)$ such that π_f extends π.

(iii) If X is Polish, then for every sequentially order-continuous $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ there is an $f \in C^-(Z;X)$ such that π_f extends π.

(c) Suppose that X is Čech-complete and that $\pi : \mathcal{B}(X) \to \mathrm{RO}(\mathbb{P})$ is a τ-additive sequentially order-continuous Boolean homomorphism. Then there is an $f \in C^-(Z;X)$ such that π_f extends π.

3G Notation Suppose that X is either compact or Polish, $\mathbb{P}$ is a forcing notion and $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ is a sequentially order-continuous Boolean homomorphism. Then 3Fb tells us that we have a $\mathbb{P}$-name $\breve{\pi}$ defined by saying that $\breve{\pi} = \vec{f}$ whenever $f \in C^-(Z;X)$ and $\pi \subseteq \pi_f$. $\Vdash_{\mathbb{P}} \breve{\pi} \in \tilde{X}$; $[\![\breve{\pi} \in \tilde{F}]\!] = \pi F$ for every Baire set $F \subseteq X$.

3H Proposition Suppose that X is either compact or Polish, $\mathbb{P}$ is a forcing notion and π, $\phi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ are sequentially order-continuous Boolean homomorphisms. Then, for any $p \in \mathbb{P}$, the following are equiveridical:

(i) $p \Vdash_{\mathbb{P}} \breve{\pi} = \breve{\phi}$;

(ii) $\widehat{p} \cap \pi E = \widehat{p} \cap \phi E$ for every $E \in \mathcal{B}\mathfrak{a}(X)$;

(iii) there is a base $\mathcal{U}$ for the topology of X, consisting of cozero sets, such that $\widehat{p} \cap \pi U = \widehat{p} \cap \phi U$ for every $U \in \mathcal{U}$.

4 Preservation of topological properties

4A Theorem Let $\mathbb{P}$, $(X, \mathfrak{T})$ and $\tilde{X}$ be as in §2A.
(a) If X is regular, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is regular.}$$

(b) If X is completely regular, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is completely regular.}$$

(c) If X is compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is compact.}$$

(d) If X is separable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is separable.}$$

(e) If X is metrizable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is metrizable.}$$

(f) If X is Čech-complete, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Čech-complete.}$$

(g) If X is Polish, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Polish.}$$

(h) If X is locally compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is locally compact.}$$

(i) If $\operatorname{ind} X \le n \in \mathbb{N}$, where $\operatorname{ind} X$ is the small inductive dimension of X, then

$$\Vdash_{\mathbb{P}} \operatorname{ind} \tilde{X} \le n.$$

(In particular, if X is zero-dimensional then $\Vdash_{\mathbb{P}} \tilde{X}$ is zero-dimensional.)
(j) If X is chargeable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is chargeable.}$$

4B Corollary Let X be a zero-dimensional compact Hausdorff space, and $\mathcal{E}$ the algebra of open-and-closed sets in X. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with the Stone space of the Boolean algebra } \check{\mathcal{E}}.$$

4C Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$; let X be a topological group.
(a) We have a $\mathbb{P}$-name for a group operation on $\tilde{X}$, defined by saying that

$$\Vdash_{\mathbb{P}} \vec{f} \cdot \vec{g} = \vec{h}$$

whenever f, g, $h \in C^-(Z; X)$ and $h(z) = f(z)g(z)$ for every $z \in \operatorname{dom} f \cap \operatorname{dom} g$; and now

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is a topological group with identity } \tilde{e}$$

where e is the identity of X.

(b)(i) For any $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{A}^{-1} = (A^{-1})^{\sim}.$$

(ii) For any $a \in X$ and $B \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{a} \cdot \tilde{B} = (aB)^{\sim},\ \tilde{B} \cdot \tilde{a} = (Ba)^{\sim}.$$

(iii) For any open set $G \subseteq X$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{G} \cdot \tilde{A} = (GA)^{\sim},\ \tilde{A} \cdot \tilde{G} = (AG)^{\sim}.$$

4D Examples Let $\mathbb{P}$ be a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$.

(a) Suppose that X is a totally ordered set with its order topology. Let $\tilde{\le}$ be the $\mathbb{P}$-name

$$\{((\vec{f}, \vec{g}), p) : f, g \in C^{-}(Z; X),\ p \in \mathbb{P},$$
$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) \le g(z)\}\}.$$

(i) $\tilde{\le}$ is a $\mathbb{P}$-name for a total ordering of $\tilde{X}$.

(ii) Now

$\Vdash_{\mathbb{P}}$ the order topology defined by $\tilde{\le}$ is the topology on $\tilde{X}$ generated by $\tilde{\mathfrak{T}}$.

(iii) For any $f,\ g \in C^{-1}(Z; X)$, $f(z) \le g(z)$ for every $z \in \operatorname{dom} f \cap \operatorname{dom} g \cap [\![\vec{f} \,\tilde{\le}\, \vec{g}]\!]$.

(iv) In the language of 2Af,

$$\Vdash_{\mathbb{P}} \dot{\varphi}[\check{X}] \text{ is cofinal and coinitial with } \tilde{X}.$$

(v) If X is Dedekind complete, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Dedekind complete.}$$

(b)(i) If $X = [0, 1]$ with its usual topology, then

$\Vdash_{\mathbb{P}} \tilde{X}$, with the topology generated by $\tilde{\mathfrak{T}}$, can be identified with the unit interval.

(ii) If $X = \mathbb{R}$ with its usual topology, then

$\Vdash_{\mathbb{P}} \tilde{X}$, with the topology generated by $\tilde{\mathfrak{T}}$, can be identified with the real line.

(c) Let I be any set, and $X = \{0, 1\}^I$. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified, as topological space, with } \{0, 1\}^{\check{I}}.$$

(d) If $X = \mathbb{N}^{\mathbb{N}}$ then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with } \check{\mathbb{N}}^{\check{\mathbb{N}}}.$$

(e) If X is an n-dimensional manifold, where $n \geq 1$, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is an } n\text{-dimensional manifold.}$$

4E Zero sets: Proposition If X is a topological space and $F \subseteq X$ is a zero set, then

$$\Vdash_{\mathbb{P}} \tilde{F} \text{ is a zero set in } \tilde{X}.$$

4F Proposition Let X be a connected Hausdorff space and $\mathbb{P}$ a forcing notion. Then

(a) If X is compact,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

(b) If X is analytic,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

4G Corollary Let X be a Hausdorff space such that for any two points x, $y \in X$ there is a connected compact set containing both. (For instance, X might be path-connected.) Then for any forcing notion $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

4H For completeness, I set out two elementary remarks.

(a) If X is not connected then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not connected.}$$

(For if U is a non-trivial open-and-closed subset of X, then

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ is a non-trivial open-and-closed subset of } \tilde{X}.)$$

(b) If X is not compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not compact.}$$

4I Metric spaces: Theorem Let (X, ρ) be a metric space.

(a) There is a $\mathbb{P}$-name $\tilde{\rho}$ such that

$\Vdash_{\mathbb{P}} \tilde{\rho}$ is a metric on $\tilde{X}$ defining its topology, and $\dot{\varphi} : \check{X} \to \tilde{X}$ is an isometry for $\check{\rho}$ and $\dot{\rho}$.

(b) If (X, ρ) is complete, then

$$\Vdash_{\mathbb{P}} (\tilde{X}, \tilde{\rho}) \text{ is complete.}$$

4J When studying random and Cohen forcing, among others, it is often useful to know when a name for a Borel set in $\tilde{X}$ can be represented, in the manner of 2E, by a set $W \subseteq Z \times X$ which factors through a continuous function from Z to

$\{0,1\}^{\mathbb{N}}$. Here I collect some simple cases in which this can be done, in preparation for §8 below.

Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of its regular open algebra. Write $\mathcal{B}\mathfrak{a}(Z)$ for the Baire σ-algebra of Z. Let X be a Hausdorff space and Σ a σ-algebra of subsets of X including a base for the topology of X. I will say that a $\mathbb{P}$-name $\dot{E}$ is **$(\mathcal{B}\mathfrak{a},\Sigma)$-representable** if there is a $W \in \mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\Sigma$ such that

$$\Vdash_{\mathbb{P}} \dot{E} = \vec{W},$$

defining $\vec{W}$ as in 2E.

(a) Suppose that X is second-countable and that

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel subset of } \tilde{X}.$$

If *either* $\mathbb{P}$ is ccc *or* there is an $\alpha < \omega_1$ such that

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is of Borel class at most } \alpha,$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

(b) Suppose that $\mathbb{P}$ is ccc.

(i) If

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is a compact } \mathrm{G}_\delta \text{ set}$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

(ii) If X is compact and

$$\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}(\tilde{X}),$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

5 Cardinal functions

5A Theorem Let $\mathbb{P}$, $(X,\mathfrak{T})$ and $\tilde{X}$ be as in §2A, and θ a cardinal.

(a) If the weight $w(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} w(\tilde{X}) \le \#(\check{\theta}).$$[1]

(b) If the π-weight $\pi(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} \pi(\tilde{X}) \le \#(\check{\theta}).$$

(c) If the density $d(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} d(\tilde{X}) \le \#(\check{\theta}).$$

(d) If the saturation $\mathrm{sat}(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} \mathrm{sat}(\tilde{X}) \ge \#(\check{\theta}).$$

5B Theorem Let $\mathbb{P}$, Z, $(X,\mathfrak{T})$ and $\tilde{X}$ be as in §2, and θ a cardinal.

(a) If X is compact and $w(X) = \theta$, then

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

(b) If X is metrizable and $w(X) = \theta$, then

[1] Recall that $\Vdash_{\mathbb{P}} \check{\theta}$ is an ordinal, but that in many cases $\Vdash_{\mathbb{P}} \check{\theta}$ is not a cardinal.

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

5C Theorem Suppose that GCH is true, and that $\mathbb{P}$ is any forcing notion.
(a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and set $\kappa = \pi(\mathfrak{A})$. Then

$$\Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) = \#(\check{\kappa}).$$

(b) Let X be a regular topological space and set $\kappa = \pi(X)$. Then

$$\Vdash_{\mathbb{P}} \pi(\tilde{X}) = \#(\check{\kappa}).$$

(c) Let $\mathfrak{A}$ be any Boolean algebra and set $\kappa = \pi(\mathfrak{A})$. Then

$$\Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) = \#(\check{\kappa}).$$

5D Proposition Let X be a ccc Hausdorff space, and $\mathbb{P}$ a productively ccc forcing notion. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is ccc.}$$

5E Proposition Suppose that X is a hereditarily ccc compact Hausdorff space and that $\mathbb{P}$ is a forcing notion such that ω_1 is a precaliber of $\mathbb{P}$. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is hereditarily ccc.}$$

6 Radon measures

6A Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $\mathbb{P}$ a forcing notion. Let $\tilde{\mu}$ be the $\mathbb{P}$-name

$$\{((\tilde{A}, (\mu A)\check{}\,), \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X)\}.$$

Then

$$\Vdash_{\mathbb{P}} \text{there is a unique Radon measure on } \tilde{X} \text{ extending } \tilde{\mu}.$$

Remark Perhaps a note is in order on the interpretation of the formula $(\mu A)\check{}$. If we take a real number α to be the set of rational numbers less than or equal to α, then $\check{\alpha}$ becomes a $\mathbb{P}$-name for a real number. If, in this context, we interpret ∞ as the set of all rational numbers, then we can equally regard $\check{\infty} = \check{\mathbb{Q}}$ as a $\mathbb{P}$-name for the top point of the two-point compactification of the reals.

6B Theorem Let $\mathbb{P}$ be a forcing notion. Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of Radon probability spaces such that $J = \{i : i \in I,\ X_i \text{ is not compact}\}$ is countable. Let μ be the product Radon measure on $X = \prod_{i \in I} X_i$. Let $\dot{\mu}$, $\dot{\mu}_i$, for $i \in I$, be $\mathbb{P}$-names for Radon measures on $\tilde{X}$, $\tilde{X}_i$ respectively, defined as in 6A. Then

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ can be identified with the Radon product of } \langle \dot{\mu}_i \rangle_{i \in \check{I}}.$$

6C I extract a couple of simple facts about quasi-Radon measures for use in the next theorem.

Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra.

(a) For every $E \in \Sigma$ there is an $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that $A \subseteq E$ and $E \setminus A$ is negligible.

(b) If $\mathcal{U}$ is any base for $\mathfrak{T}$ closed under finite unions, then $\{U^\bullet : U \in \mathcal{U}\}$ is dense in $\mathfrak{A}$ for the measure-algebra topology.

6D Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $\mathbb{P}$ be a forcing notion, and $\dot{\mu}$ a $\mathbb{P}$-name for a Radon measure on $\tilde{X}$ as described in 6A; let $(\dot{\mathfrak{A}}, \dot{\bar{\mu}})$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ is the measure algebra of } \dot{\mu}.$$

Let $\dot{\varpi}$ be the $\mathbb{P}$-name

$$\{(((A^\bullet)\check{}, \tilde{A}^\bullet), \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X)\}.$$

Then

$\Vdash_{\mathbb{P}} \dot{\varpi}$ is a measure-preserving Boolean homomorphism from $(\check{\mathfrak{A}}, \check{\bar{\mu}})$ to $(\dot{\mathfrak{A}}, \dot{\bar{\mu}})$, and $\dot{\varpi}[\check{\mathfrak{A}}]$ is dense in $\dot{\mathfrak{A}}$ for the measure-algebra topology. ∎

6E Proposition Let $\mathbb{P}$ be a well-pruned Souslin tree, active upwards. Then there is a compact Hausdorff space X such that every Radon measure on X has metrizable support, but

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ has a subspace homeomorphic to } \{0,1\}^{\omega_1}.$$

$\mathrm{Mah}_{\mathrm{R}}(X) = \{0, \omega\}$ but

$$\Vdash_{\mathbb{P}} \mathrm{Mah}_{\mathrm{R}}(\tilde{X}) \neq \{0, \omega\}.$$

11 Possibilities

Here I collect some conjectures which look as if they might sometime be worth exploring.

11B Let X, Y be Hausdorff spaces, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$.

(a) If $Z_0 \subseteq Z$ is comeager and $h : Z_0 \times X \to Y$ is continuous, then

$$\Vdash_{\mathbb{P}} \vec{h} \text{ is a continuous function from } \tilde{X} \text{ to } \tilde{Y}.$$

11D Let X, Y be Hausdorff spaces and $\mathbb{P}$ a forcing notion.

(a) If $R \subseteq X \times Y$ is an usco-compact relation, then

$$\Vdash_{\mathbb{P}} \tilde{R} \subseteq \tilde{X} \times \tilde{Y} \text{ is usco-compact.}$$

(b) If X is K-analytic then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is K-analytic.}$$

(c) If X is analytic then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is analytic.}$$

11G Let $\mathbb{P}$ be a forcing notion and Z the Stone space of its regular open algebra.

(a) If X is a K-analytic Hausdorff space, Y is a compact metrizable space and $\dot{h}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{h} \text{ is a continuous function from } \tilde{X} \text{ to } \tilde{Y},$$

then there is a function $h : X \to Y$ such that

$$\Vdash_{\mathbb{P}} \dot{h} = \vec{h}.$$

(b) If X is a K-analytic Hausdorff space, $\alpha < \omega_1$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}_\alpha(\tilde{X}),$$

then there are a comeager set $Z_0 \subseteq Z$ and a $W \in \mathcal{B}\mathfrak{a}_\alpha(Z_0 \times X)$ such that

$$\Vdash_{\mathbb{P}} \dot{E} = \vec{W}.$$

12 Problems

12A Suppose that $\operatorname{add}\mathcal{N} = \kappa < \operatorname{add}\mathcal{M}$, where $\mathcal{N}$, $\mathcal{M}$ are the Lebesgue null ideal and the ideal of meager subsets of $\mathbb{R}$. Then there is a family $\langle E_\xi \rangle_{\xi<\kappa}$ of Borel subsets of $[0,1]$ such that $A = \bigcup_{\xi<\kappa} E_\xi$ is not Lebesgue measurable, therefore not universally Baire-property, by 1C. But if Z is any Polish space and $f : Z \to [0,1]$ is continuous, $f^{-1}[A]$ has the Baire property in Z (cf. Matheron Solecki & Zelený p05).

However, we can still ask: is there an example in ZFC of a Polish space X and a set $A \subseteq X$ such that $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is Polish and $f : Z \to X$ is continuous, but $A \notin \mathcal{U}\widehat{\mathcal{B}}(X)$?

12B In Theorem 5C, is there a corresponding result for topological density, or for centering numbers of Boolean algebras?

12C In Corollary 7C, do we have a converse? that is, can $\tilde{\phi}$ belong to a Baire class lower than the first Baire class containing ϕ?

12D In Theorem 6I, what can can we do for non-Borel sets $W \subseteq Z \times X$? Maybe we can reach a class closed under Souslin's operation. What about arbitrary $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$?

12E In Proposition 3F, are there any other natural classes of topological space for which 3Fb or 3Fc will be valid? What about analytic Hausdorff spaces?

12F In Theorem 2G, can we characterize those $V \subseteq Z \times X$ for which $\Vdash_{\mathbb{P}} \vec{V}$ is compact?

12G In Proposition 8I, can we characterize those $(\Sigma,\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable functions g for which there is a $\mathbb{P}$-name $\dot{x}$ such that $[\![\dot{x} \in \tilde{F}]\!] = g^{-1}[F]^\bullet$ for every $F \in \mathcal{U}\widehat{\mathcal{B}}(X)$?

12H In Theorem 4A, can we add

if X is a Hausdorff k-space, then $\Vdash_{\mathbb{P}} \tilde{X}$ is a k-space,

if X is compact, Hausdorff and path-connected, then $\Vdash_{\mathbb{P}} \tilde{X}$ is path-connected?

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Appendix: Namba forcing

A1 Let X be a set and $\mathcal{I}$ a proper ideal of subsets of X. Consider the forcing notion $\mathbb{P}$ defined by saying that $\mathbb{P}$ is the set of those $p \subseteq \bigcup_{n\in\mathbb{N}} X^n$ such that

$\sigma{\restriction}n \in p$ whenever $\sigma \in p$ and $n \in \mathbb{N}$

there is an element stem(p) of p such that for every $\sigma \in p$

either $\sigma \subseteq \text{stem}(p)$

or $\text{stem}(p) \subseteq \sigma$ and $\{x : \sigma^\frown{<}x{>} \in p\} \notin \mathcal{I}$,

where, for $\sigma \in X^n$ and $x \in X$, $\sigma^\frown{<}x{>} = \sigma \cup \{(n,x)\} \in X^{n+1}$; and that p is stronger than q if $p \subseteq q$. I will call this the **$(X,\mathcal{I})$-Namba forcing notion**; when $X = \kappa$ is an infinite cardinal and $\mathcal{I} = [\kappa]^{<\kappa}$ I will call it the **κ-Namba forcing notion**.

Note that if p is stronger than q then $\text{stem}(p) \supseteq \text{stem}(q)$.

A2 Theorem Let X be a set, $\mathcal{I}$ a proper ideal of subsets of X with additivity and saturation greater than ω_1, and $\mathbb{P}$ the $(X,\mathcal{I})$-Namba forcing notion. If $S \subseteq \omega_1$ is stationary then

$$\Vdash_{\mathbb{P}} \check{S} \text{ is stationary in } \check{\omega}_1.$$

Remark As for any forcing notion,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a non-zero limit ordinal.}$$

We do not yet know that

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal}$$

(this will be considered in A3 below), so we need to say: if α is an ordinal, a subset A of α is 'stationary' if it meets every relatively closed subset of α which is cofinal with α. If α is a non-zero limit ordinal of countable cofinality, this can happen only if $\sup(\alpha \setminus A) < \alpha$, of course.

A3 Corollary If X is a set, $\mathcal{I}$ is a proper ideal of subsets of X which is ω_2-additive and not ω_1-saturated, and $\mathbb{P}$ is the $(X,\mathcal{I})$-Namba forcing notion, then

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal.}$$

A4 Proposition If κ is an infinite cardinal and $\mathbb{P}$ is the κ-Namba forcing notion,

$$\Vdash_{\mathbb{P}} \operatorname{cf}\check{\kappa} = \omega.$$

References

Balogh Z. & Gruenhage G. [05] 'Two more perfectly normal non-metrizable manifolds', Topology and its Appl. 151 (2005) 260-272.

Dow A. [02] 'Recent results in set-theoretic topology', pp. 131-152 in Hušek & Mill 02.

Džamonja M. & Kunen K. [95] 'Properties of the class of measure separable compact spaces', Fundamenta Math. 147 (1995) 261-277.

Feng Q., Magidor M. & Woodin H. [92] 'Universally Baire sets of reals', pp. 203-242 in Judah Just & Woodin 92.

Foreman M., Magidor M. & Shelah S. [88] 'Martin's maximum, saturated ideals, and non-regular ultrafilters', Annals of Math. (2) 127 (1988) 1-47.

Fremlin D.H. [87] *Measure-additive coverings and measurable selectors.* Dissertationes Math. 260 (1987).

Fremlin D.H. [02] *Measure Theory, Vol. 3: Measure Algebras.* Torres Fremlin, 2002 (`http://www.lulu.com/content/8005793`).

Fremlin D.H. [03] *Measure Theory, Vol. 4: Topological Measure Spaces.* Torres Fremlin, 2003.

Fremlin D.H. [08] *Measure Theory, Vol. 5: Set-theoretic Measure Theory.* Torres Fremlin, 2008 (`http://www.lulu.com/content/3365665`, `http://www.lulu.com/content/4745305`).

Fremlin D.H. [n86] 'Consequences of Martin's Maximum', note of 31.7.86

Fremlin D.H. [n05] 'Baire σ-algebras in product spaces', note of 6.9.05 (`http://www.essex.ac.uk/maths/people/fremlin/preprints.htm`).

Holický P. & Spurný J. [03] 'Perfect images of absolute Souslin and absolute Borel Tychonoff spaces', Topology and its Appl. 131 (2003) 281-294.

Hušek M. & Mill J.van [02] (eds.) *Recent Progress in General Topology.* Elsevier, 2002.

Jech T. [03] *Set Theory.* Springer, 2003.

Judah H., Just W. & Woodin H. [92] (eds.) *Set Theory of the Continuum.* Springer, 1992.

Kunen K. [80] *Set Theory.* North-Holland, 1980.

Kunen K. [81] 'A compact L-space under CH', Topology and its Appl. 12 (1981) 283-287.

Kunen K. & Vaughan J.E. [84] (eds.) *Handbook of Set-Theoretic Topology.* North-Holland, 1984.

Kuratowski K. [66] *Topology,* vol. I. Academic, 1966.

Matheron É., Solecki S. & Zelenyý M. [p05] 'Trichotomies for ideals of compact sets', preprint, 2005.

Mill J.van [84] 'An introduction to $\beta\omega$', pp. 503-567 in Kunen & Vaughan 84.

Todorčević S. [84] 'Trees and linearly ordered sets', pp. 235-295 in Kunen & Vaughan 84.

Todorčević S. [99] 'Compact subsets of the first Baire class', J. Amer. Math. Soc. 12 (1999) 1179-1212.