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Topological spaces after forcing

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I offer some notes on a general construction of topological spaces in forcing models.

I follow Kunen 80 in my treatment of forcing; in particular, for a forcing notion $\mathbb{P}$, terms in $V^{\mathbb{P}}$ are subsets of $V^{\mathbb{P}} \times \mathbb{P}$. For other unexplained notation it is worth checking in Fremlin 02, Fremlin 03 and Fremlin 08.

Contents

1 Universally Baire-property sets

1A Definition Let X be a topological space. I will say that a set $A \subseteq X$ is **universally Baire-property** if $f^{-1}[A]$ has the Baire property in Z whenever Z is a Čech-complete completely regular Hausdorff space and $f : Z \to X$ is a continuous function. Because the family $\widehat{\mathcal{B}}(Z)$ of subsets of Z with the Baire property is always a σ-algebra closed under Souslin's operation and including the Borel σ-algebra, the family $\mathcal{U}\widehat{\mathcal{B}}(X)$ of universally Baire-property subsets of X is a σ-algebra of subsets of X closed under Souslin's operation and including the Borel σ-algebra.

1B Elementary facts Let X be a topological space.

(a) If Y is another topological space, $h : X \to Y$ is continuous and $A \in \mathcal{U}\widehat{\mathcal{B}}(Y)$ then $h^{-1}[A] \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(b)(i) If $Y \subseteq X$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then $A \cap Y \in \mathcal{U}\widehat{\mathcal{B}}(Y)$.

(ii) If $F \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(F)$ then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. **P** Let Z be a Čech-complete space and $f : Z \to X$ a continuous function. Then there is an G_δ set $W \subseteq f^{-1}[F]$ such that $f^{-1}[F] \setminus W$ is meager. Set $g = f{\restriction}W : W \to F$; then $g^{-1}[A]$ has the Baire property in W, so is the intersection of W with a set which has the Baire property in Z, and $g^{-1}[A]$ has the Baire property in Z. As $f^{-1}[A] \triangle g^{-1}[A] \subseteq f^{-1}[F] \setminus W$ is meager in Z, $f^{-1}[A]$ has the Baire property in Z. **Q**

(c) If $\langle X_i \rangle_{i \in I}$ is a countable family of topological spaces and $A_i \in \mathcal{U}\widehat{\mathcal{B}}(X_i)$ for every i, then $\prod_{i \in I} A_i \in \mathcal{U}\widehat{\mathcal{B}}(\prod_{i \in I} X_i)$.

(d) Suppose that $A \subseteq X$ and that $\mathcal{G}$ is a family of open subsets of X, covering A, such that $A \cap G \in \mathcal{U}\widehat{\mathcal{B}}(X)$ for every $G \in \mathcal{G}$. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. **P** Let Z be a Čech-complete space and $f : Z \to \bigcup \mathcal{G}$ a continuous function. For each $G \in \mathcal{G}$, $A \cap G \in \mathcal{U}\widehat{\mathcal{B}}(\bigcup \mathcal{G})$ (by (b-i)), so $f^{-1}[A] \cap f^{-1}[G] = f^{-1}[A \cap G]$ belongs to $\widehat{\mathcal{B}}(Z)$; as $\{f^{-1}[G] : G \in \mathcal{G}\}$ is an open cover of Z, $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$. As Z and f are arbitrary, $A \in \mathcal{U}\widehat{\mathcal{B}}(\bigcup \mathcal{G})$; by (b-ii), $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. **Q**

(e) If X is Čech-complete, then $\mathcal{U}\widehat{\mathcal{B}}(X) \subseteq \widehat{B}(X)$.

1C Proposition If X is a Hausdorff space and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then A is universally Radon-measurable in X in the sense of Fremlin 03, 434E.

proof Let μ be a Radon probability measure on X and (Z, ν) the Stone space of the measure algebra of μ. Let f be the canonical inverse-measure-preserving map from a conegligible open subset W of Z to X (Fremlin 03, 416V). Then $\mu = \nu_W f^{-1}$, where ν_W is the subspace measure on W, and $f^{-1}[A]$ has the Baire property in W and Z, therefore is measured by ν.

1D Let X be a Hausdorff space such that every compact subset of X is scattered. Then $\mathcal{U}\widehat{\mathcal{B}}(X) = \mathcal{P}X$.

proof Take any $A \subseteq X$. Let Z be a Čech-complete space and $f : Z \to X$ a continuous function. Set $W = \bigcup_{x \in X} \operatorname{int} f^{-1}[\{x\}]$. **?** If W is not dense in Z, express Z as $\bigcap_{n \in \mathbb{N}} H_n$ where $\langle H_n \rangle_{n \in \mathbb{N}}$ is a sequence of dense open sets in a compact Hausdorff space $\hat{Z}$. Set $V = \hat{Z} \setminus \overline{W}$. Choose $\langle V_\sigma \rangle_{\sigma \in S_2^*}$ and $\langle G_\sigma \rangle_{\sigma \in S_2^*}$ as follows, where $S_2^* = \bigcup_{n \in \mathbb{N}} \{0,1\}^n$. $V_\emptyset = V$ and $G_\emptyset = X$. Given that V_σ is a non-empty open subset of V and $f[Z \cap V_\sigma] \subseteq G_\sigma$, then $V_\sigma \cap Z \not\subseteq W$, so $f[V_\sigma \cap Z]$ has more than one element; because X is Hausdorff, there must be non-empty open subsets $G_{\sigma^\frown<0>}$, $G_{\sigma^\frown<1>}$ of G_σ both meeting $f[Z \cap V_\sigma$. Choose non-empty open sets $V_{\sigma^\frown<0>}$, $V_{\sigma^\frown<1>} \subseteq V_\sigma$ such that $\overline{V}_{\sigma^\frown<i>} \subseteq V_\sigma \cap H_{\#(\sigma)}$ and $f[V_{\sigma^\frown<i>} \subseteq G_{\sigma^\frown<i>}$ for both i.

At the end of the construction, set

$$V' = \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{0,1\}^n} V_\sigma = \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \{0,1\}^n} \overline{V}_\sigma.$$

Then V' is a compact subset of Z, and we have a continuous surjection $h : f[V'] \to \{0,1\}^{\mathbb{N}}$ defined by saying that $\sigma \subseteq h(x)$ whenever $\sigma \in S_2^*$ and $x \in G_\sigma$. So $f[V'] \subseteq X$ is not scattered. **X**

So W is dense in Z, and if we set $U = \bigcup_{x \in A} \operatorname{int} f^{-1}[\{x\}]$ then $U \triangle f^{-1}[A]$ is nowhere dense, so $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$. As f and Z are arbitrary, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

1E Theorem Let X be a compact Hausdorff space, and $A \subseteq X$. Then the following are equiveridical:

(i) $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$;

(ii) $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$ whenever W is a topological space and $f : W \to X$ is continuous;

(iii) $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is an extremally disconnected compact Hausdorff space and $f : Z \to X$ is continuous;

(iv) there are a compact Hausdorff space K and a continuous surjection $f : K \to X$ such that $f^{-1}[A] \in \mathcal{U}\widehat{\mathcal{B}}(K)$.

proof (ii)⇒(i)⇒(iii) are immediate from the definition of $\mathcal{U}\widehat{\mathcal{B}}(X)$.

(iii)⇒(ii)($\boldsymbol{\alpha}$) Let Z be the Stone space of the regular open algebra $\mathrm{RO}(W)$ of W. Let $\mathcal{E}$ be the family of sets $U \subseteq W$ with nowhere dense boundaries; for $U \in \mathcal{U}$ write U^* for the open-and-closed subset of Z corresponding to $\operatorname{int}\overline{U} \in \mathrm{RO}(W)$. Note that $U \mapsto U^* : \mathcal{E} \to \mathcal{P}Z$ is a Boolean homomorphism with range the algebra of open-and-closed subsets of Z, and that $\mathcal{E}$ contains all open subsets of W; also $U^* \ne \emptyset$ if U is a non-empty open subset of W, and $\mathcal{E} \subseteq \widehat{\mathcal{B}}(W)$. Set

$$\begin{aligned} g &= \bigcap_{F\subseteq X \text{ is closed}} (Z \times F) \cup ((Z \setminus (f^{-1}[F])^*) \times X) \\ &= \bigcap_{G\subseteq X \text{ is open}} (Z \times (X \setminus G)) \cup ((f^{-1}[G])^* \times X), \end{aligned}$$

so that g is a closed subset of $Z \times X$. Now for any $z \in Z$ there is a unique $x \in X$ such that $(z, x) \in g$. **P** The set $\{F : F \subseteq X \text{ is closed}, z \in (f^{-1}[F])^*\}$ is a downwards-directed family of closed subsets of X so has non-empty intersection, and if x belongs to this intersection then $(z, x) \in g$. If x_0, $x_1 \in X$ are distinct and $z \in Z$, there are closed sets F_0, $F_1 \subseteq X$ such that $x_0 \notin F_0$, $x_1 \notin F_1$ and $F_0 \cup F_1 = X$. In this case $Z = (f^{-1}[F_0])^* \cup (f^{-1}[F_1])^*$. If $z \in (f^{-1}[F_i])^*$ then $(z, x_i) \notin g$. **Q**

($\boldsymbol{\beta}$) Thus g is a continuous function from Z to X. Now Z is an extremally disconnected compact Hausdorff space, so $g^{-1}[A] \in \widehat{\mathcal{B}}(Z)$; let $E \in \mathcal{E}$ be such that $E^*\triangle g^{-1}[A]$ is disjoint from $\bigcap_{n\in\mathbb{N}} Z_n$, where each Z_n is a dense open subset of Z. For each $n \in \mathbb{N}$ set $V_n = \bigcup\{U : U \subseteq W \text{ is open}, U^* \subseteq Z_n\}$. Then V_n is a dense open subset of W. **P** If $U \subseteq W \setminus Z_n$ is open, then $U^* \cap Z_n = \emptyset$ so U^* and U are empty. **Q** Now $\operatorname{int} E \setminus f^{-1}[A]$ is disjoint from $\bigcap_{n\in\mathbb{N}} V_n$. **P?** If $x \in \operatorname{int} E \cap \bigcap_{n\in\mathbb{N}} V_n \setminus f^{-1}[A]$, then consider $\{U^* : U \subseteq W \text{ is open}, x \in U\}$. This is a downwards-directed family of non-empty closed subsets of Z so there is a point z in the intersection. In this case, $z \in E^* \cap \bigcap_{n\in\mathbb{N}} Z_n$ so $g(z) \in A$ and $f(x) \ne g(z)$. Let G, H be disjoint open subsets of X containing $f(x)$, $g(z)$ respectively; then $z \in (f^{-1}[G])^* \cap (f^{-1}[H])^*$, which is impossible. **XQ**

Similarly, $\operatorname{int}(W \setminus E) \setminus f^{-1}[X \setminus A]$ is disjoint from $\bigcap_{n\in\mathbb{N}} V_n$, so

$$E\triangle f^{-1}[A] \subseteq (\overline{E} \setminus \operatorname{int} E) \cup (W \setminus \textstyle\bigcap_{n\in\mathbb{N}} V_n)$$

is meager, and $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$, as required.

(i)⇒(iv) is trivial.

(iv)⇒(iii) Suppose that (iv) is true, that Z is a compact Hausdorff space and that $g : Z \to X$ is continuous. Set

$$Q = \{(x, z) : x \in X, z \in Z, f(x) = g(z)\}.$$

Then Q is a compact subset of $X \times Z$. Writing $\pi_1 : Q \to X$ and $\pi_2 : Q \to Z$ for the coordinate maps, we have a continuous function $h = f\pi_1 = g\pi_2$ from Q to X. Note that $\pi_2 : Q \to Z$ is surjective because f is.

Let $L \subseteq Q$ be a compact set such that $\pi_2\restriction L$ is an irreducible surjection (Fremlin 03, 4A2Gi). Set $B = L \cap h^{-1}[A] = L \cap \pi_1^{-1}[f^{-1}[A]]$; then $B \in \widehat{\mathcal{B}}(L)$, by (i)⇒(ii) here applied to $\pi_1\restriction L$. Express B as $F\triangle M$ where $F \subseteq L$ is closed and $M \subseteq L$ is meager in L. Now $\pi_2[M]$ is meager in Z. **P?** If $C \subseteq L$ is closed and nowhere dense in L, but $\pi_2[C]$ is not nowhere dense in Z, there is a non-empty open $H \subseteq \pi_2[C]$. In this case, $L \cap \pi_2^{-1}[H]$ is relatively open and not empty, so cannot be included in C, and $L' = L \setminus (\pi_2^{-1}[H] \setminus C)$ is a proper closed subset of L; but $\pi_2[L'] = Z$. **X** Thus $\pi_2[C]$ is nowhere dense in Z for any closed $C \subseteq L$ which is nowhere dense in L; it follows at once that $\pi_2[M]$ is meager. **Q**

Accordingly

$$\pi_2[B]\triangle\pi_2[F] = \pi_2[F\triangle M]\triangle\pi_2[F] \subseteq \pi_2[(F\triangle M)\triangle F] = \pi_2[M]$$

is meager, and $\pi_2[B] \in \widehat{\mathcal{B}}(Z)$. But $\pi_2[B] = g^{-1}[A]$. **P** If $z \in g^{-1}[A]$, there is an $x \in X$ such that $(x, z) \in L$; now $h(x, z) = g(z)$ belongs to A, so $(x, z) \in B$ and $z \in \pi_2[B]$. On the other hand, if $z \in \pi_2[B]$, then there is an x such that $(x, z) \in B$, and $g(z) = h(x, z) \in A$. **Q**

So $g^{-1}[A] \in \widehat{\mathcal{B}}(Z)$. As Z and g are arbitrary, (iii) is true.

1F Corollary (a) Let X be a topological space which is homeomorphic to a universally Baire-property subset of some compact Hausdorff space, and W any topological space. Then any continuous function from W to X is $(\widehat{\mathcal{B}}(W),\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable.

(b) Let X be a locally compact Hausdorff space, and $A \subseteq X$ a set such that $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is an extremally disconnected compact Hausdorff space and $f : Z \to X$ is continuous. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

proof (a) Suppose that $X \in \mathcal{U}\widehat{\mathcal{B}}(\hat{X})$ where $\hat{X}$ is a compact Hausdorff space. Let $f : W \to X$ be continuous, and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. Then f can be regarded as a continuous function from W to $\hat{X}$, and $A \in \mathcal{U}\widehat{\mathcal{B}}(\hat{X})$, by 1B(b-ii). So 1E(i)⇒(ii) tells us that $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$; as A is arbitrary, f is $(\widehat{\mathcal{B}}(W),\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable.

(b) If $G \subseteq X$ is a relatively compact open set, then $A \cap \overline{G} \in \mathcal{U}\widehat{\mathcal{B}}(\overline{G})$ by 1E(iii)⇒(i). So $A \cap G \in \mathcal{U}\widehat{\mathcal{B}}(G)$ (1B(b-i)); as G is arbitrary, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ (1Bd).

Remark Compare Jech 03, 32.21-32.24.

1G Proposition (a) Suppose that Z is a topological space, X is second-countable and $f : Z \to X$ is $\widehat{\mathcal{B}}(Z)$-measurable. Then there is a comeager $Z_1 \subseteq Z$ such that $f\restriction Z_1$ is continuous.

(b) Suppose that X is a topological space, Y is a second-countable space and $\phi : X \to Y$ is $\mathcal{U}\widehat{\mathcal{B}}(X)$-measurable. Then ϕ is $(\mathcal{U}\widehat{\mathcal{B}}(X),\mathcal{U}\widehat{\mathcal{B}}(Y))$-measurable.

proof (a) Kuratowski 66 32.II.

(b) Let $A \in \mathcal{U}\widehat{\mathcal{B}}(Y)$. Let Z be a Čech-complete space and $f : Z \to X$ a continuous function. Then f is $(\widehat{\mathcal{B}}(Z),\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable, by the definition of $\mathcal{U}\widehat{\mathcal{B}}(X)$, so $\phi f : Z \to Y$ is $(\widehat{\mathcal{B}}(Z)$-measurable. By (i), there is a commeager $Z_1 \subseteq Z$ such that $\phi f\restriction Z_1$ is continuous; we may suppose that Z_1 is a G_δ set, so that Z_1 is Čech-complete. In this case,

$$Z_1 \cap f^{-1}[\phi^{-1}[A]] = (\phi f\restriction Z_1)^{-1}[A] \in \widehat{\mathcal{B}}(Z_1)$$

and $f^{-1}[\phi^{-1}[A]] \in \widehat{\mathcal{B}}(Z)$. As Z and f are arbitrary, $\phi^{-1}[A] \in \mathcal{U}\widehat{\mathcal{B}}(X)$. As A is arbitrary, ϕ is $(\mathcal{U}\widehat{\mathcal{B}}(X),\mathcal{U}\widehat{\mathcal{B}}(Y))$-measurable.

1H Lemma If W is a non-empty topological space, κ a cardinal and $\pi(W) \le \kappa$, then $\kappa^{\mathbb{N}}$ (giving each copy of κ the discrete topology) and $W \times \kappa^{\mathbb{N}}$ have isomorphic regular open algebras.

proof (a) To begin with (down to the end of (g)), suppose that $\mathrm{RO}(W)$ is atomless. Let $\langle U_\xi\rangle_{\xi<\kappa}$ run over a π-base $\mathcal{U}$ for the topology of W. Let P be the partially ordered set $S^*_\kappa \times S^*_2$, where $S^*_I = \bigcup_{n\in\mathbb{N}} I^n$; let $\mathfrak{T}$ be the topology of $W \times \kappa^{\mathbb{N}}$. Given $\sigma \in S^*_\kappa$ define $Q_\sigma \subseteq S^*_2$ and $\langle H_{\sigma\tau}\rangle_{\tau\in Q_\sigma}$ by saying that

$\emptyset \in Q_\sigma$ and $H_{\sigma\emptyset} = W$;

if $\tau \in Q_\sigma$ and there is an $i < n$ such that neither $H_{\sigma\tau} \cap U_{\sigma(i)}$ nor $H_{\sigma\tau} \setminus \overline{U}_{\sigma(i)}$ is empty, take the first such i; put both $\tau^\frown{<}0{>}$ and $\tau^\frown{<}1{>}$ into Q_σ; set $H_{\sigma,\tau^\frown<1>} = H_\tau \cap U_{\sigma(i)}$ and $H_{\sigma,\tau^\frown<0>} = H_\tau \setminus \overline{U}_{\sigma(i)}$;

if there is no such i then no proper extension of τ belongs to Q_σ.

Now set $Q = \{(\sigma,\tau) : \sigma \in S^*_\kappa,\ \tau \in Q_\sigma\}$ and $f(\sigma,\tau) = H_{\sigma\tau} \times \{x : \sigma \subseteq x \in \kappa^{\mathbb{N}}\}$ for $(\sigma,\tau) \in Q$.

(b) Every Q_σ is finite; in fact $\#(\tau) \le \#(\sigma)$ whenever $\tau \in Q_\sigma$. If σ, $\sigma' \in S^*_\kappa$ and $\sigma \subseteq \sigma'$ then $Q_\sigma \subseteq Q_{\sigma'}$ and $H_{\sigma\tau} = H_{\sigma'\tau}$ for every $\tau \in Q_\sigma$ (induce on $\#(\tau)$). If $\sigma \in S^*_\kappa$ then $\bigcup\{H_{\sigma\tau} : \tau \in Q_\sigma$ is maximal$\}$ is dense (induce on $\#(\sigma)$).

(c) Q is cofinal with P. **P** Suppose that $\sigma \in \kappa^n$ and $\tau \in \{0,1\}^m$ and $(\sigma,\tau) \notin Q$, Let τ' be the longest initial segment of τ such that $\tau' \in Q_\sigma$; set $l = \#(\tau) - \#(\tau')$. Because $\mathrm{RO}(W)$ is atomless, we can find $V_0,\dots,V_l$, $U'_0,\dots,U'_{l-1}$ such that

$V_0 = H_{\sigma,\tau'}$;

given that $j < l$ and $V_j \subseteq W$ is open and not empty, $U'_j \in \mathcal{U}$ and neither $V_j \cap U'_j$ nor $V_j \setminus \overline{U'_j}$ is empty;

if $\tau(m-l+j) = 1$ then $V_{j+1} = V_j \cap U'_j$;

if $\tau(m-l+j) = 0$ then $V_{j+1} = V_j \setminus \overline{U'_j}$.

Let σ' be an extension of σ to a member of κ^{n+l} such that $U_{\sigma'(n+j)} = U'_j$ for $j < l$. Then $(\sigma', \tau) \in Q$, with $H_{\sigma'\tau} = V_l$. **Q**

(d) If (σ, τ), $(\sigma', \tau') \in Q$ and $(\sigma, \tau) \le (\sigma', \tau')$ then

$$f(\sigma, \tau) = H_{\sigma\tau} \times \{x : \sigma \subseteq x \in \kappa^{\mathbb{N}}\} \supseteq H_{\sigma'\tau} \times \{x : \sigma' \subseteq x \in \kappa^{\mathbb{N}}\}$$
$$\supseteq H_{\sigma'\tau'} \times \{x : \sigma' \subseteq x \in \kappa^{\mathbb{N}}\} = f(\sigma', \tau').$$

So if (σ, τ), (σ', τ') are upwards-compatible in Q, $f(\sigma, \tau)$ and $f(\sigma', \tau')$ are downwards-compatible in $\mathfrak{T} \setminus \{\emptyset\}$,

(e) If (σ, τ), (σ', τ') are upwards-incompatible in Q, $f(\sigma, \tau)$ and $f(\sigma', \tau')$ are downwards-incompatible in $\mathfrak{T} \setminus \{\emptyset\}$, **P** Because Q is cofinal with P, (σ, τ) and (σ', τ') are upwards-incompatible in P. If σ, σ' are upwards-incompatible in S^*_κ, then

$$f(\sigma, \tau) \cap f(\sigma', \tau') \subseteq W \times (\{x : \sigma \subseteq x \text{ and } \sigma' \subseteq x\}) = \emptyset.$$

If $\sigma \subseteq \sigma'$ then τ and τ' must be incompatible in S^*_2; let j be the least integer such that $\tau(j) \ne \tau'(j)$. Then there must be an $i < \#(\sigma)$ such that one of $H_{\sigma\tau}$, $H_{\sigma\tau'} = H_{\sigma'\tau'}$ is included in $U_{\sigma(i)}$ and the other is disjoint from $U_{\sigma(i)}$. So $H_{\sigma\tau} \cap H_{\sigma'\tau'} = \emptyset$ and again $f(\sigma, \tau)$ and $f(\sigma', \tau')$ are disjoint. **Q**

(f) $f[Q]$ is coinitial with $\mathfrak{T} \setminus \{\emptyset\}$. **P** If $U \subseteq W$ is open and not empty and $\sigma \in S^*_\kappa$, let τ be a maximal member of Q_σ such that $H_{\sigma\tau} \cap U$ is not empty. Let U' be a member of $\mathcal{U}$ included in $H_{\sigma\tau} \cap U$ such that U' is not dense in $H_{\sigma\tau}$. Let $\xi < \kappa$ be such that $U' = U_\xi$. Set $\sigma' = \sigma^\frown\xi$ and $\tau' = \tau^\frown{<}1{>}$. Then $(\sigma', \tau') \in Q$ and

$$f(\sigma, \tau) = U' \times \{x : \sigma' \subseteq x\} \subseteq U \times \{x : \sigma \subseteq x\}. \text{ \textbf{Q}}$$

(g) By Fremlin 08, 514R,

$$\mathrm{RO}(\kappa^{\mathbb{N}}) \cong \mathrm{RO}(\kappa^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \cong \mathrm{RO}^{\uparrow}(P) \cong \mathrm{RO}^{\uparrow}(Q) \cong \mathrm{RO}^{\downarrow}(\mathfrak{T} \setminus \{\emptyset\}) \cong \mathrm{RO}(W \times \kappa^{\mathbb{N}}).$$

(h) All this has been on the assumption that $\mathrm{RO}(W)$ is atomless. For the general case, let $\mathcal{V}$ be the set of atoms in $\mathrm{RO}(W)$, and set $W' = W \setminus \overline{\bigcup \mathcal{V}}$. Then $\mathrm{RO}(W')$ is atomless, so (a)-(g) tell us that if W' is not empty, $\mathrm{RO}(W' \times \kappa^{\mathbb{N}})$ is isomorphic to $\mathrm{RO}(\kappa^{\mathbb{N}})$, and

$$\mathrm{RO}(W \times \kappa^{\mathbb{N}}) \cong \mathrm{RO}(W' \times \kappa^{\mathbb{N}}) \times \prod_{V \in \mathcal{V}} \mathrm{RO}(V \times \kappa^{\mathbb{N}})$$

(taking the simple product of the Boolean algebras)

$$\cong \mathrm{RO}(W' \times \kappa^{\mathbb{N}}) \times \prod_{V \in \mathcal{V}} \mathrm{RO}(\kappa^{\mathbb{N}}) \cong \mathrm{RO}(\kappa^{\mathbb{N}})^\lambda$$

(where $\lambda = \#(\mathcal{V})$ if $W' \ne \emptyset$, $\#(\mathcal{V} \cup \{W'\})$ otherwise)

$$\cong \mathrm{RO}(\lambda \times \kappa^{\mathbb{N}}) \cong \mathrm{RO}(\kappa^{\mathbb{N}})$$

because $\lambda \le c(W) \le \pi(W) \le \kappa$. So we have the general result.

1I Lemma Let X be a metrizable space, κ an infinite cardinal, W a Čech-complete space with regular open algebra isomorphic to that of $\kappa^{\mathbb{N}}$, and $f : W \to X$ a continuous function. Then there are a dense G_δ subset W' of W and continuous functions $g : W' \to \kappa^{\mathbb{N}}$ and $h : \kappa^{\mathbb{N}} \to X$ such that $hg = f\restriction W'$; moreover, we can choose g in such a way that it is surjective and $g[F]$ is not dense for any proper relatively closed set $F \subseteq W'$.

proof Express W as $\bigcap_{n\in\mathbb{N}} H_n$ where $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open sets in a compact Hausdorff space Z. Then W is dense in Z, so $\mathrm{RO}(Z) \cong \mathrm{RO}(W) \cong \mathrm{RO}(\kappa^{\mathbb{N}})$; set $S^*_\kappa = \bigcup_{n\in\mathbb{N}} \kappa^n$ and let $\langle V_\sigma\rangle_{\sigma\in S^*_\kappa}$ be a family in $\mathrm{RO}(Z)$ corresponding to the family $\langle\{x : \sigma \subseteq x\}\rangle_{\sigma\in S^*_\kappa}$ in $\mathrm{RO}(\kappa^{\mathbb{N}}$, so that $\{V_\sigma : \sigma \in S^*_\kappa\}$ is order-dense in $\mathrm{RO}(Z)$ and $\langle V_{\sigma^\frown\xi}\rangle_{\xi<\kappa}$ is always a disjoint family of subsets of V_σ with union dense in V_σ. Because the topology of Z is regular, $\{V_\sigma : \sigma \in S^*_\kappa\}$ is a π-base for it. In particular, every non-empty open subset of Z has saturation exactly κ^+. Give X a metric ρ inducing its topology. Now, for each $n \in \mathbb{N}$, choose a family

$\mathcal{V}_n$ of open subsets of Z, as follows. $\mathcal{V}_0 = \{Z\} = \{V_\emptyset\}$. Given that $\mathcal{V}_n$ is a disjoint family of non-empty open subsets of Z with dense union, let $\mathcal{V}'_n$ be the family of all non-empty open subsets V of Z such that (α) there is a $V' \in \mathcal{V}_n$ such that $\overline{V} \subseteq V' \cap H_n$ (β) $\operatorname{diam} f[W \cap V_n] \le 2^{-n}$. $\mathcal{V}'_n$ is a π-base for the topology of Z, so we can find a disjoint family $\mathcal{V}_{n+1} \subseteq \mathcal{V}'_n$ such that (i) $\{V : V \in \mathcal{V}_{n+1},\ V \subseteq V'\}$ has cardinal κ for every $V' \in \mathcal{V}_n$ (ii) $\bigcup \mathcal{V}_{n+1}$ is dense in Z. Continue.

Of course we can now index each $\mathcal{V}_n$ as $\langle V'_\sigma\rangle_{\sigma\in\kappa^n}$ in such a way that $\langle V'_{\sigma^\frown\xi}\rangle_{\xi<\kappa}$ enumerates $\{V : V \in \mathcal{V}_{n+1},$ $V \subseteq V'_\sigma\}$ whenever $\sigma \in \kappa^n$. Since every member of $\mathcal{V}_n$ is included in V_σ for some $\sigma \in \kappa^n$, $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is a π-base for the topology of Z. If $\alpha \in \kappa^{\mathbb{N}}$, then $\overline{V'_{\alpha\restriction n+1}} \subseteq V'_{\alpha\restriction n} \cap H_n$ for every n, so $K_\alpha = \bigcap_{n\in\mathbb{N}} V'_{\alpha\restriction n}$ is a non-empty compact subset of W; and

$$W' = \bigcup_{\alpha\in\kappa^{\mathbb{N}}} K_\alpha = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$$

is a dense G_δ subset of W. Define $g : W' \to \kappa^{\mathbb{N}}$ by setting $g(z) = \alpha$ whenever $z \in K_\alpha$. Then g is surjective, and it is continuous because $g(z)\restriction n$ is constant on each member of $\mathcal{V}_n$. If $F \subset W'$ is a proper relatively closed set, there is a $\sigma \in S^*_\kappa$ such that $F \cap V'_\sigma = \emptyset$, in which case $g[F]$ does not meet $\{\alpha : \sigma \subseteq \alpha \in \kappa^{\mathbb{N}}\}$ and is not dense.

If $\alpha \in \kappa^{\mathbb{N}}$ and z, $z' \in K_\alpha$, then the distance between $f(z)$ and $f(z')$ must be zero, so we can define $h : \kappa^{\mathbb{N}} \to X$ by saying that $h(\alpha) = f(z)$ whenever $z \in K_\alpha$. If $\alpha\restriction n = \beta\restriction n = \sigma$, $z \in K_\alpha$ and $z' \in K_\beta$, then both z and z' belong to V_σ, so (if $n \ge 1$)

$$\rho(h(\alpha), h(\beta)) = \rho(f(z), f(z')) \le 2^{-n+1}.$$

This shows that h is continuous. And of course $f\restriction W' = hg$.

1J Lemma Let W be a topological space and Y a non-empty α-favourable topological space.

(a) If $A \subseteq W$ is such that $A \times Y$ is meager in $W \times Y$, then A is meager in W.

(a) If $A \subseteq W$ is such that $A \times Y \in \widehat{\mathcal{B}}(W \times Y)$, then $A \in \widehat{\mathcal{B}}(Y)$.

proof (a) Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence of closed nowhere dense subsets of $W \times Y$ covering $A \times Y$. Let σ be a winning strategy for the second player in the Banach-Mazur game on Y. Choose $\langle \mathcal{G}_n\rangle_{n\in\mathbb{N}}$ and $\langle\langle H_{nG}\rangle_{G\in\mathcal{G}_n}\rangle_{n\in\mathbb{N}}$ inductively, as follows. The inductive hypothesis will be that

$\mathcal{G}_n$ is a disjoint family of open subsets of W with dense union;

$\mathcal{G}_{n+1}$ refines $\mathcal{G}_n$;

H_{nG} is always a non-empty open subset of Y;

if $G_0 \in \mathcal{G}_0, \dots, G_{n+1} \in \mathcal{G}_{n+1}$ and $G_0 \supseteq \dots \supseteq G_{n+1}$, then $H_{n+1,G_{n+1}} \subseteq \sigma(H_{0G_0}, \dots, H_{nG_n})$.

Start by setting $\mathcal{G}_0 = \{W\}$ and $H_{0W} = Y$. For the inductive step, given $G \in \mathcal{G}_n$, take G_i to be the unique member of $\mathcal{G}_i$ including G for each $i \le n$, and set $H^*_G = \sigma(H_{0G_0}, \dots, H_{nG_n})$, so that H^*_G is a non-empty open subset of Y. Let $\mathcal{U}_G$ be

$$\{U : \emptyset \ne U \subseteq G \text{ and there is a non-empty open } V \subseteq H^*_G \text{ such that } (U \times V) \cap F_n = \emptyset\}.$$

Then $\bigcup \mathcal{U}_G$ is dense in G so we have a disjoint family $\mathcal{U}'_G$ in $\mathcal{U}_G$ with union dense in G. Set $\mathcal{G}_{n+1} = \bigcup_{G\in\mathcal{G}_n} \mathcal{U}'_G$; for $U \in \mathcal{G}_{n+1}$ choose $H_{n+1,U}$ to be a non-empty open set in Y such that $U \times H_{n+1,U}$ is disjoint from F_n and $H_{n+1,U} \subseteq H^*_G$ where G is the member of $\mathcal{G}_n$ including U. Continue.

At the end of the induction, $W' = \bigcap_{n\in\mathbb{N}} \bigcup \mathcal{G}_n$ is comeager in W. Now W' is disjoint from A. **P?** Otherwise, take $z \in W' \cap A$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be such that $z \in G_n \in \mathcal{G}_n$ for each n. Then $G_0 \supseteq G_1 \supseteq \dots$ so $H_{n+1,G_{n+1}} \subseteq \sigma(H_{0G_0}, \dots, H_{nG_n})$ for each n. Because σ is a winning strategy, there is a point $y \in \bigcap_{n\in\mathbb{N}} H_{nG_n}$. But now $(z, y) \in G_{n+1} \cap H_{n+1,G_{n+1}}$ so $(z, y) \notin F_n$ for each n, and $(z, y) \in (A \times Y) \setminus \bigcup_{n\in\mathbb{N}} F_n$; which is impossible. **XQ**

So A must be meager.

(b) Let $\mathcal{G}$ be the family of those open subsets G of W such that for some non-empty $H \subseteq Y$, $(A \times Y) \cap (G \times H)$ is meager. Then (a) tells us that $A \cap G$ is meager for every $G \in \mathcal{G}$, so $A \cap \bigcup \mathcal{G}$ is meager. Similarly, if $\mathcal{G}'$ is the family of open $G \subseteq W$ such that $((W \setminus A) \times Y) \cap (G \times H)$ is meager for some non-empty open $H \subseteq Y$, $\bigcup \mathcal{G}' \setminus A$ is meager. But as $A \times Y$ has the Baire property, $\bigcup \mathcal{G} \cup \bigcup \mathcal{G}'$ is dense, so A has the Baire property.

1K Theorem (see Feng Magidor & Woodin 92, Theorem 2.1) Let X be a metrizable space and $A \subseteq X$. Then $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ iff whenever κ is a cardinal and $f : \kappa^{\mathbb{N}} \to X$ is continuous, then $f^{-1}[A] \in \widehat{\mathcal{B}}(\kappa^{\mathbb{N}})$.

proof Of course all the spaces $\kappa^{\mathbb{N}}$ are Čech-complete, so only one direction needs proof. Suppose that $f^{-1}[A]$ has the Baire property whenever κ is a cardinal and $f : \kappa^{\mathbb{N}} \to X$ is continuous.

(a) If W is a Čech-complete space with regular open algebra isomorphic to $\mathrm{RO}(\kappa^{\mathbb{N}})$, where κ is an infinite cardinal, and $f : W \to X$ is continuous, then $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$. **P** By 1I, there are a dense G_δ subset W' of W and continuous functions $g : W' \to \kappa^{\mathbb{N}}$ and $h : \kappa^{\mathbb{N}} \to X$ such that $hg = f\restriction W'$, g is surjective and $g[F]$ is not dense for any proper relatively closed set $F \subseteq W'$. Now $W' \cap f^{-1}[A] = g^{-1}[h^{-1}[A]]$. By hypothesis, $h^{-1}[A]$ has the Baire property in $\kappa^{\mathbb{N}}$. Now if $H \subseteq \kappa^{\mathbb{N}}$ is a dense open set, and $G \subseteq W'$ is a non-empty relatively open set, $g[W' \setminus G]$ is not dense and $H \setminus g[W' \setminus G] \neq \emptyset$, that is, $G \cap g^{-1}[H]$ is non-empty (as g is surjective). As G is arbitrary, $g^{-1}[H]$ is dense. It follows that $g^{-1}[E]$ is comeager whenever $E \subseteq \kappa^{\mathbb{N}}$ is comeager, and $g^{-1}[E]$ is meager whenever E is meager; consequently $g^{-1}[h^{-1}[A]] \in \widehat{\mathcal{B}}(W') \subseteq \widehat{\mathcal{B}}(W)$. So $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$. **Q**

(b) If W is any Čech-complete space and $f : W \to X$ is continuous, then $f^{-1}[A] \in \widehat{\mathcal{B}}(W)$. **P** Let $\kappa \geq \pi(W)$ be an infinite cardinal. Then $\mathrm{RO}(W \times \kappa^{\mathbb{N}}) \cong \mathrm{RO}(\kappa^{\mathbb{N}})$, by 1H, while $W \times \mathrm{RO}(\kappa^{\mathbb{N}})$ is Čech-complete. Set $g(z, \alpha) = f(z)$ for $z \in W$, $\alpha \in \kappa^{\mathbb{N}}$. Then $g^{-1}[A]$ has the Baire property, by (a). But $g^{-1}[A] = f^{-1}[A] \times \kappa^{\mathbb{N}}$, so $f^{-1}[A]$ has the Baire property, by 1J. **Q**

So $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, as claimed.

2 Basic theory

2A Hausdorff spaces after forcing Let $(X, \mathfrak{T})$ be a Hausdorff space and $\mathbb{P}$ a forcing notion.

(a) Let Z be the Stone space of the regular open algebra $\mathrm{RO}(\mathbb{P})$ of $\mathbb{P}$; in this context I will interpret Boolean truth values $[\![\phi]\!]$ directly as open-and-closed sets in Z. For $p \in \mathbb{P}$ let $\widehat{p} \subseteq Z$ be the open-and-closed set corresponding to the regular open set $\{q :$ if r is stronger than q then r is compatible with $p\}$. For subsets S, T of Z I will say that $S \subseteq^* T$ if $S \setminus T$ is meager. Note that if $S, T \in \widehat{\mathcal{B}}(Z)$ and $S \not\subseteq^* T$, then there is a $p \in \mathbb{P}$ such that $\widehat{p} \subseteq^* S \setminus T$. Let $C^-(Z; X)$ be the space of continuous functions from dense G_δ subsets of Z to X.

For a function $f \subseteq Z \times X$ let $\vec{f}$ be the $\mathbb{P}$-name

$$\{(\check{g}, p) : g \in C^-(Z; X),\ p \in \mathbb{P},\ \widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}\};$$

for $A \subseteq X$ let $\tilde{A}$ be the $\mathbb{P}$-name

$$\{(\vec{f}, p) : f \in C^-(Z; X),\ p \in \mathbb{P},\ \widehat{p} \subseteq^* f^{-1}[A]\}.$$

Remark Note that the definitions of $\vec{f}$ and $\tilde{A}$ involve the whole set X as well as the pair $(\mathbb{P}, Z)$ and the sets f and A themselves.

It will be some time before I will discuss $\vec{f}$ for anything but functions in $C^-(Z; X)$ but I slip the general formulation in here for future reference.

(b)(i) If $f \subseteq Z \times X$ is a function, $g \in C^-(Z; X)$ and $p \in \mathbb{P}$ then $p \Vdash_{\mathbb{P}} \check{g} \in \vec{f}$ iff $(\check{g}, p) \in \vec{f}$. **P** If $(\check{g}, p) \in \vec{f}$ then of course $p \Vdash_{\mathbb{P}} \check{g} \in \vec{f}$. If $p \Vdash_{\mathbb{P}} \check{g} \in \vec{f}$ and q is stronger than p, then there are r, q', h such that

$$h \in C^-(Z; X),\ (\check{h}, q') \in \vec{f},\ r \text{ is stronger than both } q \text{ and } q',\ r \Vdash_{\mathbb{P}} \check{h} = \check{g};$$

that is, q' is compatible with q, $h = g$ and $\widehat{q'} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$. But this means that, setting $D = \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$, every non-empty open subset of $\widehat{p}$ includes a non-empty open set meeting $Z \setminus D$ in a meager set, so $\widehat{p} \setminus D$ is meager and $(\check{g}, p) \in \vec{f}$. **Q**

(ii) If $f \subseteq Z \times X$ is a function, $g \in C^-(Z; X)$ and $p \in \mathbb{P}$ then $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$ iff $\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,$ $f(z) = g(z)\}$. **P** (α) If $\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$, $(\check{h}, q') \in \vec{f}$ and q is stronger than both p and q', then

$$\widehat{q} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\} \cap \{z : z \in \operatorname{dom} g \cap \operatorname{dom} h,\ f(z) = h(z)\}$$
$$\subseteq \{z : z \in \operatorname{dom} g \cap \operatorname{dom} h,\ h(z) = g(z)\},$$

so $(\check{h}, q) \in \vec{g}$; the same applies with f and g exchanged so $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$. (β) If $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$ and q is stronger than p, then $(\hat{g}, q) \in \vec{g}$ and $q \Vdash_{\mathbb{P}} \check{g} \in \vec{g} = \vec{f}$ so there are an r stronger than q such that $r \Vdash_{\mathbb{P}} \check{g} \in \vec{f}$. By (i), $(\check{g}, r) \in \vec{f}$ and $\widehat{r} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$. As q is arbitrary, $\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\}$. **Q**

(iii) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $f \in C^-(Z;X)$ and $p \in \mathbb{P}$, then $p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}$ iff $(\vec{f}, p) \in \tilde{A}$. **P** If $(\vec{f}, p) \in \tilde{A}$ then of course $p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}$. If $(\vec{f}, p) \notin \tilde{A}$ then $\widehat{p} \not\subseteq^* f^{-1}[A]$. So there is a q, stronger than p, such that $\widehat{q} \cap f^{-1}[A]$ is meager. If $(\vec{g}, q') \in \tilde{A}$ and r is stronger than both q' and q then $\widehat{r} \cap f^{-1}[A]$ and $\widehat{r} \setminus g^{-1}[A]$ are both meager, so $\{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) \neq g(z)\}$ is dense in $\widehat{r}$ and $r \not\Vdash_{\mathbb{P}} \vec{f} = \vec{g}$. So $p \not\Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}$. **Q**

(iv) Suppose that $*$ is one of the four Boolean operations $\cup$, $\cap$, $\setminus$ and $\triangle$. If A, B, $C \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $A * B = C$ then $\Vdash_{\mathbb{P}} \tilde{A} * \tilde{B} = \tilde{C}$. **P** If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A} \cup \tilde{B} \cup \tilde{C}$, then there are a q stronger than p and an $f \in C^-(Z;X)$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; now

$$\begin{aligned} q \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A} * \tilde{B} &\iff q \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A} * \tilde{B} \\ &\iff \widehat{q} \subseteq [\![\vec{f} \in \tilde{A} * \tilde{B}]\!] = [\![\vec{f} \in \tilde{A}]\!] * [\![\vec{f} \in \tilde{B}]\!] \\ &\iff \widehat{q} \subseteq^* (f^{-1}[A] * f^{-1}[B]) \\ &\iff \widehat{q} \subseteq^* f^{-1}[A * B] \\ &\iff \widehat{q} \subseteq^* f^{-1}[C] \\ &\iff q \Vdash_{\mathbb{P}} \vec{f} \in \tilde{C} \iff q \Vdash_{\mathbb{P}} \dot{x} \in \tilde{C}. \end{aligned}$$

As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \tilde{A} * \tilde{B} = \tilde{C}$. **Q**

(v) Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{U}\widehat{\mathcal{B}}(X)$ with union A. Then $\Vdash_{\mathbb{P}} \tilde{A} = \bigcup_{n\in\mathbb{N}} \tilde{A}_n$. **P** (α) If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$, then there are a q stronger than p and an $f \in C^-(Z;X)$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; now $\widehat{q} \subseteq^* f^{-1}[A]$, while $\langle f^{-1}[A_n] \rangle_{n\in\mathbb{N}}$ is a sequence in the Baire-property algebra of Z with union A; so there are an open subset H of Z and an $n \in \mathbb{N}$ such that $H \subseteq \widehat{q}$ and $H \subseteq^* f^{-1}[A_n]$. Let r be such that $\widehat{r} \subseteq H$; then

$$r \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{A}_n.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{A} \subseteq \textstyle\bigcup_{n\in\mathbb{N}} \tilde{A}_n.$$

(β) In the other direction, (iv) tells us that $\Vdash_{\mathbb{P}} \tilde{A}_n \cap \tilde{A} = \tilde{A}_n$ for every $n \in \mathbb{N}$, so that $\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \tilde{A}_n \subseteq \tilde{A}$. **Q**

(vi) Let $\langle G_i \rangle_{i\in I}$ be a family in $\mathfrak{T}$ with union G. Then

$$\Vdash_{\mathbb{P}} \tilde{G} = \textstyle\bigcup_{i\in\check{I}} \tilde{G}_i.$$[1]

P As in (v), we can use (iv) to see that

$$\Vdash_{\mathbb{P}} \textstyle\bigcup_{i\in\check{I}} \tilde{G}_i \subseteq \tilde{G}.$$

In the other direction, if $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G}$, let $q \in \mathbb{P}$ and $f \in C^-(Z;X)$ be such that q is stronger than p and $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Then $\widehat{q} \setminus f^{-1}[G]$ is nowhere dense, so there is an $i \in I$ such that $\widehat{q}$ meets $f^{-1}[G_i]$. As $\widehat{q} \cap f^{-1}[G_i]$ is relatively open in the comeager set $\operatorname{dom} f$, there is an r stronger than q such that $\widehat{r} \subseteq^* f^{-1}[G_i]$ and $r \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G}_{\check{\imath}}$. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{G} \subseteq \textstyle\bigcup_{i\in\check{I}} \tilde{G}_i.$$ **Q**

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[1] There is something of an abuse of notation here. Strictly speaking, $\langle G_i \rangle_{i\in I}$ is a subset of $I \times \mathfrak{T}$; now $\langle \tilde{G}_i \rangle_{i\in\check{I}}$ is to be a suitable $\mathbb{P}$-name for a corresponding subset of $\check{I} \times \tilde{\mathfrak{T}}$, e.g., $\{((\check{\imath}, \tilde{G}_i), \mathbb{1}) : i \in I\}$. See Fremlin 08, 5A3Eb.

(vii) Suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$. Then there is an $f \in C^-(Z;A)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. **P** Note first that there are surely a p_0 stronger than p and an $f_0 \in C^-(Z;X)$ such that

$$p_0 \Vdash_{\mathbb{P}} \dot{x} = \vec{f_0} \in \tilde{A},$$

so that $\widehat{p}_0 \subseteq^* f^{-1}[A]$ and $A \ne \emptyset$. Fix $x_0 \in A$. Next, for every q stronger than p there are an r stronger than q and an $f \in C^-(Z;X)$ such that $r \Vdash_{\mathbb{P}} \vec{f} = \dot{x} \in \tilde{A}$, so that $\widehat{r} \subseteq^* f^{-1}[A]$. We therefore have a maximal antichain $Q \subseteq \mathbb{P}$ such that for every $q \in Q$

— either q is stronger than p and we have a $g_q \in C^-(Z;X)$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{g}_q$

— or q is incompatible with p, in which case take g_q to be the constant function with domain Z and value x_0.

Now $\langle \widehat{q} \rangle_{q \in Q}$ is a disjoint family of open subsets of Z with dense union. For $q \in Q$, $\widehat{q} \subseteq^* g_q^{-1}[A]$; let E_q be a dense G_δ-subset of $\widehat{q}$ included in $g_q^{-1}[A]$. Set $E = \bigcup_{q \in Q} E_q$; then E is a dense G_δ set in Z. Define $f : E \to A$ by setting $f(z) = g_q(z)$ if $z \in E_q$; then $f \in C^-(Z;X)$ and $\Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}$. Also

$$q \Vdash_{\mathbb{P}} \vec{f} = \vec{g}_q = \dot{x}$$

whenever $q \in Q$ and q is stronger than p, so $p \Vdash_{\mathbb{P}} \vec{f} = \dot{x}$, as required. **Q**

(viii) If, in (vii), the set A is compact, then every member of $C^-(Z;A)$ will have a (unique) extension to a member of $C(Z;A)$, because Z is extremally disconnected; so we find that whenever $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$, then there is an $f \in C(Z;A)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$.

(c) Now set

$$\tilde{\mathfrak{T}} = \{(\tilde{G}, \mathbb{1}) : G \in \mathfrak{T}\}.$$

Then

$$\Vdash_{\mathbb{P}} \tilde{\mathfrak{T}} \text{ is a topology base on } \tilde{X} \text{ and generates a Hausdorff topology on } \tilde{X}.$$

P This is a first-order property so survives translation into the forcing language. More explicitly: suppose that $\dot{G}$ and $\dot{H}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \dot{G},\ \dot{H} \in \tilde{\mathfrak{T}}.$$

Then there are G, $H \in \mathfrak{T}$ and q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{G} = \tilde{G} \text{ and } \dot{H} = \tilde{H}.$$

In this case

$$q \Vdash_{\mathbb{P}} \tilde{G} \cap \tilde{H} = (G \cap H)^{\sim} \in \tilde{\mathfrak{T}}$$

by (b-iv). As p, $\dot{G}$ and $\dot{H}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{\mathfrak{T}} \text{ is closed under } \cap.$$

Of course

$$\Vdash_{\mathbb{P}} \tilde{X} \in \tilde{\mathfrak{T}} \text{ and } G \subseteq \tilde{X} \text{ for every } G \in \tilde{\mathfrak{T}},$$

so

$$\Vdash_{\mathbb{P}} \tilde{\mathfrak{T}} \text{ is a topology base on } \tilde{X}.$$

To see that we have a Hausdorff topology in $V^{\mathbb{P}}$, suppose that $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names and that $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \tilde{X},\ \dot{x} \ne \dot{y}.$$

By (b-vii), we have f, $g \in C^-(Z;X)$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f},\ \dot{y} = \vec{g}.$$

Now $\widehat{p}\cap\{z : z\in\operatorname{dom} f\cap\operatorname{dom} g,\ f(z)=g(z)\}$ has the Baire property and does not essentially include any open set, so

$$\bigcup_{G,H\in\mathfrak{T},G\cap H=\emptyset}\widehat{p}\cap f^{-1}[G]\cap f^{-1}[H]$$

is relatively comeager in the non-meager G_δ set $\widehat{p}\cap\operatorname{dom} f\cap\operatorname{dom} g$, and there are disjoint G, $H\in\mathfrak{T}$ such that $\widehat{p}\cap f^{-1}[G]\cap f^{-1}[H]$ is non-meager. Let q, stronger than p, be such that $\widehat{q}\subseteq^*(f^{-1}[G]\cap f^{-1}[H])$; then

$$q\Vdash_{\mathbb{P}}\dot{x}\in\tilde{G}\in\tilde{\mathfrak{T}},\ \dot{y}\in\tilde{H}\in\tilde{\mathfrak{T}}\text{ and }\tilde{G}\cap\tilde{H}=\emptyset.$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}}\tilde{\mathfrak{T}}\text{ generates a Hausdorff topology on }\tilde{X}.\ \mathbf{Q}$$

(d)(i) It is perhaps worth noting explicitly that we can use any base for $\mathfrak{T}$ to define the topology on $\tilde{X}$ in $V^{\mathbb{P}}$. If $\mathcal{U}$ is a base for $\mathfrak{T}$, set $\tilde{\mathcal{U}}=\{(\tilde{U},\mathbb{1}) : U\in\mathcal{U}\}$. Then

$$\Vdash_{\mathbb{P}}\tilde{\mathcal{U}}\text{ is a topology base on }\tilde{X}\text{ and generates the same topology as }\tilde{\mathfrak{T}}.$$

P Suppose that $p\in\mathbb{P}$ and that $\dot{U}$ and $\dot{V}$ are such that

$$p\Vdash_{\mathbb{P}}\dot{U},\ \dot{V}\in\tilde{\mathcal{U}}.$$

Then there are a q stronger than p and U, $V\in\mathcal{U}$ such that

$$q\Vdash_{\mathbb{P}}\dot{U}=\tilde{U}\text{ and }\dot{V}=\tilde{V}.$$

Set $\mathcal{W}=\{W : W\in\mathcal{U},\ W\subseteq U\cap V\}$; then $U\cap V=\bigcup_{W\in\mathcal{W}}W$, so (b-iv) and (b-vi) tell us that

$$\Vdash_{\mathbb{P}}\tilde{U}\cap\tilde{V}=(U\cap V)^{\sim}=\bigcup_{W\in\check{\mathcal{W}}}\tilde{W},$$

and accordingly that

$$q\Vdash_{\mathbb{P}}\dot{U}\cap\dot{V}\text{ is a union of members of }\tilde{\mathcal{U}}.$$

It is easy to check that

$$\Vdash_{\mathbb{P}}\bigcup\tilde{\mathcal{U}}=\tilde{X},$$

so

$$\Vdash_{\mathbb{P}}\tilde{\mathcal{U}}\text{ is a topology base on }\tilde{X}.$$

To check that we get the right topology, we surely have

$$\Vdash_{\mathbb{P}}\tilde{\mathcal{U}}\subseteq\tilde{\mathfrak{T}}.$$

If $p\in\mathbb{P}$ and $\dot{G}$ is a $\mathbb{P}$-name such that $p\Vdash_{\mathbb{P}}\dot{G}\in\tilde{\mathfrak{T}}$, there are a $G\in\mathfrak{T}$ and a q stronger than p such that $q\Vdash_{\mathbb{P}}\dot{G}=\tilde{G}$. Setting $\mathcal{W}=\{U : U\in\mathcal{U},\ U\subseteq G\}$ we now have

$$q\Vdash_{\mathbb{P}}\dot{G}=\tilde{G}=\bigcup_{W\in\check{\mathcal{W}}}\tilde{W},$$

so

$$q\Vdash_{\mathbb{P}}\dot{G}\text{ belongs to the topology generated by }\tilde{\mathcal{U}}.$$

As p and $\dot{G}$ are arbitrary,

$q\Vdash_{\mathbb{P}}$ the topology generated by $\tilde{\mathfrak{T}}$ is coarser than the topology generated by $\tilde{\mathcal{U}}$, so the two are equal.

Q

(ii) Similarly, if $\mathcal{U}$ is any subbase for $\mathfrak{T}$, and we set $\tilde{\mathcal{U}}=\{(\tilde{U},\mathbb{1}) : U\in\mathcal{U}\}$, then

$$\Vdash_{\mathbb{P}}\tilde{U}\text{ generates the same topology as }\tilde{\mathfrak{T}}.$$

P Set $\mathcal{V}=\{X\}\cup\{U_0\cap\ldots\cap U_n : U_0,\ldots,U_n\in\mathcal{U}\}$. Then $\mathcal{V}$ is a base for $\mathfrak{T}$ so

$$\Vdash_{\mathbb{P}}\tilde{\mathcal{V}}\text{ defines the right topology.}$$

But it is easy to check that

$\Vdash_{\mathbb{P}} \tilde{\mathcal{V}} = \{\tilde{X}\} \cup \{U_0 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \tilde{\mathcal{U}}\}$, so $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{U}}$ define the same topology. **Q**

(e) Using (b-iv), we see that

$$\Vdash_{\mathbb{P}} \tilde{F} \text{ is closed in } \tilde{X}$$

whenever $F \subseteq X$ is closed. Using (b-v), we see that

$$\Vdash_{\mathbb{P}} \tilde{E} \text{ is Borel in } \tilde{X}$$

whenever $E \subseteq X$ is Borel. We also find that

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is nowhere dense in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ is nowhere dense in X. **P** Suppose that $p \in \mathbb{P}$ and $\dot{G}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is a non-empty open subset of } \tilde{X}.$$

Then there are a $q \in \mathbb{P}$, stronger than p, and a $U \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \tilde{U} \text{ is not empty and included in } \dot{G}.$$

Let $V \in \mathfrak{T}$ be not empty and included in $U \setminus A$; then

$$q \Vdash_{\mathbb{P}} \emptyset \ne \tilde{V} \subseteq \dot{G} \setminus \tilde{A}.$$

As p and $\dot{G}$ are arbitrary,

$$\begin{aligned} \Vdash_{\mathbb{P}} &\text{ if } G \subseteq \tilde{X} \text{ is a non-empty open set} \\ &\qquad\qquad \text{then there is a non-empty open subset of } G \setminus \tilde{A}, \end{aligned}$$

that is,

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is nowhere dense. } \mathbf{Q}$$

It follows that

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is meager in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ is meager in X (note that the meagerness of A can be witnessed by a sequence of nowhere dense sets which are relatively closed in A, therefore universally Baire-property in X), and that

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ has the Baire property in } \tilde{X}$$

whenever $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ has the Baire property in X.

(f)(i) For $x \in X$, let $e_x \in C^-(Z;X)$ be the constant function with domain Z and value x, and write $\tilde{x}$ for the $\mathbb{P}$-name $\vec{e}_x$. Set

$$\dot{\varphi} = \{((\check{x}, \tilde{x}), \mathbb{1}) : x \in X\},$$

so that

$$\Vdash_{\mathbb{P}} \dot{\varphi} \text{ is a function from } \check{X} \text{ to } \tilde{X}.$$

Since $\Vdash_{\mathbb{P}} \tilde{x} \ne \tilde{y}$ whenever x, $y \in X$ are distinct, $\Vdash_{\mathbb{P}} \dot{\phi}$ is injective.

(ii) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then

$$\Vdash_{\mathbb{P}} \check{A} = \dot{\varphi}^{-1}[\tilde{A}].$$

P If $\dot{x}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \check{X}$, then there are a q stronger than p and an $x \in X$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \check{x}$. In this case

$$\begin{aligned} q \Vdash_{\mathbb{P}} \dot{x} \in \check{A} &\iff q \Vdash_{\mathbb{P}} \check{x} \in \check{A} \\ &\iff x \in A \\ &\iff q \Vdash_{\mathbb{P}} \tilde{x} \in \tilde{A} \end{aligned}$$

(because $e_x^{-1}[A] = Z$ if $x \in A$, $\emptyset$ otherwise)

$$\iff q \Vdash_{\mathbb{P}} \dot\varphi(\check{x}) \in \tilde{A} \iff q \Vdash_{\mathbb{P}} \dot\varphi(\dot{x}) \in \tilde{A}$$

.

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{for every } x \in \check{X},\ x \in \check{A} \text{ iff } \dot\phi(x) \in \tilde{A}. \quad \mathbf{Q}$$

(iii) Next, if $D \subseteq X$ is dense,

$$\Vdash_{\mathbb{P}} \dot\varphi[\check{D}] \text{ is dense in } \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{G}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is a non-empty open set in } \tilde{X}.$$

Then there are a $q \in \mathbb{P}$, stronger than p, and $f \in C^-(Z;X)$, $U \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \vec{f} \in \tilde{U} \subseteq \dot{G}.$$

As $f^{-1}[U]$ is a dense G_δ set in the non-empty open-and-closed set $\widehat{q}$, $U \neq \emptyset$; take any $x \in U \cap D$. Then $\Vdash_{\mathbb{P}} \tilde{x} \in \tilde{U}$, so

$$q \Vdash_{\mathbb{P}} \dot\varphi(\check{x}) \in \tilde{U} \subseteq \dot{G}. \quad \mathbf{Q}$$

(g)(i) Suppose that every compact subset of X is scattered. Then, in the language of (f),

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot\varphi[\check{X}].$$

P? Otherwise, there are a $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{x}$ such that

$$\Vdash_{\mathbb{P}} \dot{x} \in \tilde{X} \setminus \dot\varphi[\check{X}].$$

Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Let $\langle W_n \rangle_{n\in\mathbb{N}}$ be a sequence of dense open sets in Z such that $\operatorname{dom} f = \bigcap_{n\in\mathbb{N}} W_n$. Set $S_2^* = \bigcup_{n\in\mathbb{N}} \{0,1\}^n$ and choose $\langle p_\sigma \rangle_{\sigma\in S_2^*}$ and $\langle G_\sigma \rangle_{\sigma\in S_2^*}$ inductively, as follows. $p_\emptyset = p$ and $G_\emptyset = X$. Given that $\sigma \in S_2^*$, $p_\sigma \in \mathbb{P}$ is stronger than p and G_σ is an open subset of X including $f[\widehat{p}_\sigma]$, consider $f{\restriction}\widehat{p}_\sigma$. Because

$$p_\sigma \Vdash_{\mathbb{P}} \vec{f} \notin \dot\varphi[\check{X}],$$

there must be at least two points in $f[\widehat{p}_\sigma]$, and we can find disjoint open subsets $G_{\sigma^\frown<0>}$ and $G_{\sigma^\frown<1>}$ of G_σ included in G_σ and both meeting $f[\widehat{p}_\sigma]$. Now, for each i, $\widehat{p}_\sigma \cap f^{-1}[G_{\sigma^\frown<i>}]$ is a non-empty relatively open subset of $\operatorname{dom} f$, so we can find $p_{\sigma^\frown<i>}$, stronger than p_σ, such that $\operatorname{dom} f \cap \widehat{p}_{\sigma^\frown<i>} \subseteq f^{-1}[G_{\sigma^\frown<i>}]$ and $\widehat{p}_{\sigma^\frown<i>} \subseteq W_n$, where $n = \#(\sigma)$. Continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} \bigcup_{\sigma\in\{0,1\}^n} \widehat{p}_\sigma$, so that K is a compact subset of $\operatorname{dom} f$ and $f[K]$ is a compact subset of X. For $\sigma \in S_2^*$, set $K_\sigma = K \cap \widehat{p}_\sigma$, so that $f[K_\sigma] = f[K] \cap G_\sigma$; in particular, $f[K_\sigma] \cap f[K_\tau] = \emptyset$ if $n \in \mathbb{N}$ and σ, $\tau \in \{0,1\}^n$ are distinct. We therefore have a function $h : f[K] \to \{0,1\}^{\mathbb{N}}$ defined by saying that $h(x){\restriction}n = \sigma$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^n$ and $x \in f[K_\sigma]$, and h is a continuous surjection from $f[K]$ onto $\{0,1\}^{\mathbb{N}}$, because $hf(z) = y$ whenever $y \in \{0,1\}^{\mathbb{N}}$ and $z \in \bigcap_{n\in\mathbb{N}} \widehat{p}_{y{\restriction}n}$. So $f[K]$ is a non-scattered compact subset of X. **XQ**

(ii) In particular, if $\#(X) < \mathfrak{c}$ or X is discrete,

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot\varphi[\check{X}].$$

(iii) In fact, if X is discrete, then

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot\varphi[\check{X}] \text{ is discrete.}$$

P Set $\mathcal{U} = \{\{x\} : x \in X\}$. If $p \in \mathbb{P}$ and $\dot{x}$, $\dot{U}$ are $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}} \dot{U} \in \tilde{\mathcal{U}}$, then there are $f \in C^-(Z;X)$, q stronger than p and $U \in \mathcal{U}$ such that $q \Vdash_{\mathbb{P}} \vec{f} = \dot{x} \in \dot{U} = \tilde{U}$. In this case,

$$\widehat{q} \subseteq^* f^{-1}[U] = \{z : z \in \operatorname{dom} f,\ f(z) = e_x(z)\},$$

so

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} = \tilde{x} = \dot{\varphi}(\check{x}).$$

As p, $\dot{U}$ and $\dot{x}$ are arbitrary, we have

$$\Vdash_{\mathbb{P}} \text{every non-empty member of } \tilde{\mathcal{U}} \text{ is of the form } \{y\} \text{ for some } y \in \dot{\varphi}[\check{X}].$$

Since $\mathcal{U}$ is a base for the topology of X we also have

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a base for the topology of } \tilde{X};$$

putting these together, we have the result. **Q**

2B Closures and interiors In the context of 2A, suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. Then

$$\Vdash_{\mathbb{P}} \operatorname{int} \tilde{A} = (\operatorname{int} A)^{\sim},\ \overline{\tilde{A}} = \overline{\dot{\varphi}[\check{A}]} = \tilde{\overline{A}} \text{ and } \partial \tilde{A} = (\partial A)^{\sim},$$

where ∂A is the topological boundary of A. **P** From 2Ab and 2Ac we know that

$$\Vdash_{\mathbb{P}} (\operatorname{int} A)^{\sim} \text{ is an open subset of } \tilde{A}, \text{ so } (\operatorname{int} A)^{\sim} \subseteq \operatorname{int} \tilde{A}.$$

Now suppose that $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \operatorname{int} \tilde{A}.$$

Then there are a q stronger than p, an $f \in C^{-}(Z;X)$ and an open set $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{U} \subseteq \tilde{A}.$$

In this case $U \subseteq A$ so $U \subseteq \operatorname{int} A$ and

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in (\operatorname{int} A)^{\sim}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{every member of } \operatorname{int} \tilde{A} \text{ belongs to } (\operatorname{int} A)^{\sim}, \text{ so } \operatorname{int} \tilde{A} = (\operatorname{int} A)^{\sim}.$$

Applying this to $X \setminus A$, and using 2A(b-iv), we have

$$\Vdash_{\mathbb{P}} \overline{\tilde{A}} = \tilde{\overline{A}},$$

$$\Vdash_{\mathbb{P}} \partial \tilde{A} = \overline{\tilde{A}} \setminus \operatorname{int} \tilde{A} = \tilde{\overline{A}} \setminus (\operatorname{int} A)^{\sim} = (\overline{A} \setminus \operatorname{int} A)^{\sim} = (\partial A)^{\sim}.$$

As for $\dot{\varphi}[\check{A}]$, we have only to repeat the argument of 2A(f-iii). Suppose that $p \in \mathbb{P}$ and that $\dot{G}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is an open set meeting } \tilde{A}.$$

Then there are a q stronger than p and an open set $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \tilde{G} \subseteq \dot{G} \text{ and } \tilde{G} \cap \tilde{A} \neq \emptyset.$$

So there must be an $x \in G \cap A$, in which case

$$q \Vdash_{\mathbb{P}} \dot{\varphi}(\check{x}) \in \dot{G}, \text{ so } \dot{G} \text{ meets } \dot{\varphi}[\check{A}].$$

As p and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{A} \subseteq \overline{\dot{\varphi}[\check{A}]} \text{ and } \overline{\dot{\varphi}[\check{A}]} = \overline{\tilde{A}}. \textbf{ Q}$$

2C Continuous functions, among others Let $\mathbb{P}$ be a forcing notion, Z the Stone space of its regular open algebra, $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ Hausdorff spaces, and $\tilde{X}$, $\tilde{\mathfrak{T}}$, $\tilde{Y}$ and $\tilde{\mathfrak{S}}$ the $\mathbb{P}$-names as defined in 2A. Let $\phi \subseteq X \times Y$ be a function.

(a) Let $\tilde{\phi}$ be the $\mathbb{P}$-name

$$\{((\vec{f}, \vec{g}), p) : f \in C^{-}(Z;X),\ g \in C^{-}(Z;Y),\ p \in \mathbb{P},\ \widehat{p} \subseteq^{*} \operatorname{dom}(g \cap \phi f)\}.$$

(Here $\mathrm{dom}(g \cap \phi f) = \{z : z \in \mathrm{dom}\, f \cap \mathrm{dom}\, g,\ g(z) = \phi f(z)\}$.[2]) Then

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is a function from a subset of } \tilde{X} \text{ to } \tilde{Y}.$$

P Of course

$$\Vdash_{\mathbb{P}} \tilde{\phi} \subseteq \tilde{X} \times \tilde{Y}.$$

To see that $\tilde{\phi}$ is a name for a function, suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{y}_0$ and $\dot{y}_1$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} (\dot{x}, \dot{y}_0) \text{ and } (\dot{x}, \dot{y}_1) \text{ belong to } \tilde{\phi}.$$

Then there are a q stronger than p, f_0, $f_1 \in C^-(Z;X)$ and g_0, $g_1 \in C^-(Z;Y)$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}_0 = \vec{f}_1,\ \dot{y}_0 = \vec{g}_0,\ \dot{y}_1 = \vec{g}_1$$

and

$$\widehat{q} \subseteq^* \mathrm{dom}(g_0 \cap \phi f_0) \cap \mathrm{dom}(g_1 \cap \phi f_1).$$

In this case,

$$\widehat{q} \subseteq^* \mathrm{dom}(f_0 \cap \phi f_1) \subseteq \mathrm{dom}(\phi f_0 \cap \phi f_1),$$

so $\widehat{q} \subseteq^* \mathrm{dom}(g_0 \cap g_1)$ and

$$q \Vdash_{\mathbb{P}} \dot{y}_0 = \vec{g}_0 = \vec{g}_1 = \dot{y}_1.$$

As p, $\dot{x}$, $\dot{y}_0$ and $\dot{y}_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is a function. } \mathbf{Q}$$

(b) Corresponding to 2A(b-vii), we have the following.

(i) If $p \in \mathbb{P}$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that $p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \dot{y}$, then there are $f \in C^-(Z;X)$ and $g \in C^-(Z;Y)$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \dot{y} = \vec{g},$$

$$\widehat{p} \subseteq \mathrm{dom}(g \cap \phi f).$$

P Argue as in 2A(b-vii), but in place of pairs q, g_q take triplets q, f_q, g_q such that, if q is stronger than p, then

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}_q \text{ and } \dot{y} = \vec{g}_q,$$

$$\widehat{q} \subseteq^* \mathrm{dom}(g_q \cap \phi f_q). \ \mathbf{Q}$$

(ii) In fact, if $p \in \mathbb{P}$ and $f \in C^-(Z;X)$ and $g \in C^-(Z;Y)$, then $p \Vdash_{\mathbb{P}} \tilde{\phi}(\vec{f}) = \vec{g}$ iff $\widehat{p} \subseteq^* \mathrm{dom}(g \cap \phi f)$. **P** (α) If $\widehat{p} \subseteq^* \mathrm{dom}(g \cap \phi f)$ then $((\vec{f}, \vec{g}), p) \in \tilde{\phi}$ so surely $p \Vdash_{\mathbb{P}} (\vec{f}, \vec{g}) \in \tilde{\phi}$ and $p \Vdash_{\mathbb{P}} \tilde{\phi}(\vec{f}) = \vec{g}$. (β) If $p \Vdash_{\mathbb{P}} \tilde{\phi}(\vec{f}) = \vec{g}$ then (i) tells us that there are $f_1 \in C^-(Z;X)$, $g_1 \in C^-(Z;Y)$ such that

$$p \Vdash_{\mathbb{P}} \vec{f} = \vec{f}_1 \text{ and } \vec{g} = \vec{g}_1,$$

$$\widehat{p} \subseteq^* \mathrm{dom}(g_1 \cap \phi f_1.$$

But in this case

$$\widehat{p} \subseteq^* \mathrm{dom}(g_1 \cap \phi f_1) \cap \mathrm{dom}(g \cap g_1) \cap \mathrm{dom}(f \cap f_1) \subseteq \mathrm{dom}(g \cap \phi f),$$

as required. **Q**

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[2] There is an abuse of notation in the displayed formula. The subformula $(\vec{f}, \vec{g})$ must be interpreted in the forcing language, so that instead of being the simple ordered pair $\{\{\vec{f}\}, \{\vec{f}, \vec{g}\}\}$ it is $\{(\{(\vec{f}, \mathbb{1})\}, \mathbb{1}), (\{(\vec{f}, \mathbb{1}), (\vec{g}, \mathbb{1})\}, \mathbb{1})\}$, or (to make myself quite clear) $\{\{\{\{\{\{\vec{f}\}, \{\vec{f}, \mathbb{1}\}\}\}\}, \{\{\{\{\vec{f}\}, \{\vec{f}, \mathbb{1}\}\}\}, \mathbb{1}\}\}, \{\{\{\{\{\vec{f}\}, \{\vec{f}, \mathbb{1}\}\}, \{\{\vec{g}\}, \{\vec{g}, \mathbb{1}\}\}\}\}, \{\{\{\{\vec{f}\}, \{\vec{f}, \mathbb{1}\}\}, \{\{\vec{g}\}, \{\vec{g}, \mathbb{1}\}\}\}, \mathbb{1}\}\}\}$. See the remark following 5A3H in Fremlin 08.

(c) Next, suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $A \subseteq \operatorname{dom}\phi$, $\phi\restriction A$ is continuous and $B \in \mathcal{U}\widehat{\mathcal{B}}(Y)$. Then $A \cap \phi^{-1}[B] \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and

$$\Vdash_{\mathbb{P}} \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{B}] = (A \cap \phi^{-1}[B])^{\sim}.$$

(In particular, $\Vdash_{\mathbb{P}} \tilde{A} \subseteq \operatorname{dom}\tilde{\phi}$.) **P** ($\alpha$) By 1B,

$$A \cap \phi^{-1}[B] = (\phi\restriction A)^{-1}[B] \in \mathcal{U}\widehat{\mathcal{B}}(A) \subseteq \mathcal{U}\widehat{\mathcal{B}}(X).$$

(β) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{B}]$. Then there is a $\mathbb{P}$-name $\dot{y}$ such that $p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \dot{y} \in \tilde{B}$; let $f \in C^-(Z;X)$ and $g \in C^-(Z;Y)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \dot{y} = \vec{g}$$

and $\widehat{p} \subseteq^* \operatorname{dom}(g \cap \phi f)$. Then

$$\widehat{p} \setminus \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\, f(z) \in A,\, g(z) \in B,\, g(z) = \phi(f(z))\}$$

is meager, so $\widehat{p} \subseteq^* f^{-1}[A \cap \phi^{-1}[B]]$ and $p \Vdash_{\mathbb{P}} \dot{x} \in (A \cap \phi^{-1}[B])^{\sim}$. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{B}] \subseteq (A \cap \phi^{-1}[B])^{\sim}.$$

(γ) In the other direction, suppose that $p \in \mathbb{P}$ and $\dot{x}$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in (A \cap \phi^{-1}[B])^{\sim}$. Let $f \in C^{-1}(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; then $\widehat{p} \subseteq^* f^{-1}[A \cap \phi^{-1}[B]]$. Let Z_0 be a G_δ subset of $\widehat{p} \cap f^{-1}[A]$ such that $\widehat{p} \subseteq^* Z_0$, and let $g : Z_0 \cup (Z \setminus \widehat{p}) \to Y$ be such that $g(z) = \phi f(z)$ for $z \in Z_0$ and g is constant on $Z \setminus \widehat{p}$. Then $g \in C^-(Z;Y)$ and $g(z) \in B$ whenever $z \in Z_0 \cap f^{-1}[\phi^{-1}[B]]$ so

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\vec{f}) = \vec{g} \in \tilde{B}$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{B}].$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{B}] \subseteq (A \cap \phi^{-1}[B])^{\sim}.$$

(δ) Applying this with $B = Y$, we see that

$$\Vdash_{\mathbb{P}} \operatorname{dom}\tilde{\phi} = \tilde{\phi}^{-1}[\tilde{Y}] \supseteq \tilde{A}. \ \mathbf{Q}$$

(d) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, $A \subseteq \operatorname{dom}\phi$ and $\phi\restriction A$ is continuous, then

$$\Vdash_{\mathbb{P}} \tilde{\phi}\restriction\tilde{A} \text{ is continuous.}$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{H}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{H} \in \tilde{\mathfrak{S}}$. Then there are a q stronger than p and an $H \in \mathfrak{S}$ such that $q \Vdash_{\mathbb{P}} \dot{H} = \tilde{H}$. Let $G \in \mathfrak{T}$ be such that $A \cap \phi^{-1}[H] = A \cap G$. Then (c) tells us that

$$q \Vdash_{\mathbb{P}} \tilde{A} \cap \tilde{\phi}^{-1}[\dot{H}] = \tilde{A} \cap \tilde{\phi}^{-1}[\tilde{H}] = (A \cap \phi^{-1}[H])^{\sim} = (A \cap G)^{\sim} = \tilde{A} \cap \tilde{G} \text{ is relatively open in } \tilde{A}.$$

As p and $\dot{H}$ are arbitrary,

$\Vdash_{\mathbb{P}} \tilde{A} \cap \tilde{\phi}^{-1}[H]$ is relatively open in $\tilde{A}$ for every $H \in \tilde{\mathfrak{S}}$, while $\tilde{\mathfrak{S}}$ is a base for the topology of $\tilde{Y}$, so $\tilde{\phi}\restriction\tilde{A}$ is continuous. **Q**

(e) If X_0, X_1, X_2 are Hausdorff spaces and $\phi : X_0 \to X_1$, $\psi : X_1 \to X_2$ are continuous functions, then

$$\Vdash_{\mathbb{P}} (\psi\phi)^{\sim} = \tilde{\psi}\tilde{\phi}.$$

P If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}_0$, then let $f \in C^-(Z;X_0)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \tilde{f}$. Then

$$\Vdash_{\mathbb{P}} \tilde{\phi}(\tilde{f}) = (\phi f)^{\sim},\ \tilde{\psi}(\tilde{\phi}(\tilde{f})) = (\psi\phi f)^{\sim} = (\psi\phi)^{\sim}(\tilde{f}),$$

so

$$p \Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{\phi}(\dot{x})) = (\psi\phi)^{\sim}(\dot{x}).$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} (\psi\phi)^{\sim} = \tilde{\psi}\tilde{\phi}. \quad \mathbf{Q}$$

(f) If ϕ is injective, then

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is injective.}$$

P Suppose that $p \in \mathbb{P}$ and $\dot{x}_0$, $\dot{x}_1$ are such that

$$p \Vdash_{\mathbb{P}} \dot{x}_0,\ \dot{x}_1 \in \operatorname{dom}\tilde{\phi} \text{ and } \dot{x}_0 \neq \dot{x}_1.$$

Let f_0, $f_1 \in C^-(Z;X)$ and g_0, $g_1 \in C^-(Z;Y)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x}_0 = \vec{f}_0 \text{ and } \dot{x}_1 = \vec{f}_1$$

and

$$\widehat{p} \subseteq^* \operatorname{dom}(g_0 \cap \phi f_0) \cap \operatorname{dom}(g_1 \cap \phi f_1).$$

Then the G_δ set $\widehat{p} \cap \operatorname{dom}(f_0 \cap f_1)$ cannot essentially include any non-empty open set, so must be meager, and

$$\widehat{p} \subseteq^* \operatorname{dom}(g_0 \cap \phi f_0) \cap \operatorname{dom}(g_1 \cap \phi f_1) \setminus \operatorname{dom}(f_0 \cap f_1) \subseteq (Z \setminus \operatorname{dom}(g_0 \cap g_1))$$

because ϕ is injective. Thus

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}_0) = \tilde{\phi}(\vec{f}_0) = \vec{g}_0 \neq \vec{g}_1 = \tilde{\phi}(\dot{x}_1).$$

As p, $\dot{x}_0$ and $\dot{x}_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is injective.} \quad \mathbf{Q}$$

(g) If ϕ is a homeomorphism between X and a set $B \in \mathcal{U}\widehat{\mathcal{B}}(Y)$, then

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is a homeomorphism between } \tilde{X} \text{ and } \tilde{B}.$$

P From (e), with $A = X$, together with (c) and (d), we know that

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is an injective continuous function from } \tilde{X} \text{ to } \tilde{Y}.$$

Let $G \subseteq X$ be any open set. Then G is expressible as $\phi^{-1}[H]$ for some open $H \subseteq Y$, so

$$\Vdash_{\mathbb{P}} \tilde{G} = \tilde{\phi}^{-1}[\tilde{H}] \text{ is the inverse image of an open set.}$$

If $p \in \mathbb{P}$ and $\dot{G}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{G} \in \tilde{\mathfrak{T}}$, and q stronger than p and $G \in \mathfrak{T}$ are such that $q \Vdash_{\mathbb{P}} \dot{G} = \tilde{G}$, then

$$q \Vdash_{\mathbb{P}} \dot{G} \text{ is the inverse image of an open set.}$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is the inverse image of an open set.}$$

As p and $\dot{G}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ every member of $\tilde{\mathfrak{T}}$ is the inverse image of an open set, so that $\tilde{X}$ is homeomorphic to its image in $\tilde{Y}$.

I still have to check that

$$\Vdash_{\mathbb{P}} \tilde{\phi}[\tilde{X}] = \tilde{B}.$$

But if $p \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{y} \in \tilde{B}$, there is a $g \in C^-(Z;B)$ such that $p \Vdash_{\mathbb{P}} \dot{y} = \vec{g}$; now $f = \phi^{-1}g$ belongs to $C^-(Z;X)$ and

$$\Vdash_{\mathbb{P}} \dot{y} = \vec{g} = \tilde{\phi}(\vec{f}) \in \tilde{\phi}[\tilde{X}].$$

As p and $\dot{y}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{\phi}[\tilde{X}] = \tilde{B}. \quad \mathbf{Q}$$

Remark The point of (e) here is that we can discuss subspace topologies without inhibitions, at least on universally Baire-property sets. If we have a topological space Y and a set $X \in \mathcal{U}\widehat{\mathcal{B}}(Y)$, then the formula of 2Aa forces a formal distinction between the $\mathbb{P}$-name

$$\tilde{X} = \{(\vec{f}, p) : f \in C^-(Z;Y),\, p \in \mathbb{P},\, \widehat{p} \subseteq^* f^{-1}[A]\}$$

when X is regarded as a subset of Y, and the $\mathbb{P}$-name

$$\tilde{X} = \{(\vec{f}, p) : f \in C^-(Z;X),\, p \in \mathbb{P}\}$$

when X is regarded as a topological space in its own right; indeed the subformula $\vec{f}$ demands different interpretations in the two cases. But the result just proved shows that for ordinary purposes we can expect any theorem concerning the topology of $\tilde{X}$ to be indifferent to which interpretation is being used.

2D Lemma Suppose, in the context of 2C, that $X = Y$ and we have a set $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that $\phi(x) = x$ for every $x \in A$. Then

$$\Vdash_{\mathbb{P}} \tilde{\phi}(x) = x \text{ for every } x \in \tilde{A}.$$

proof Let $p \in \mathbb{P}$ and $\dot{x}$ be a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$. Let q stronger than p and f, $g \in C^-(Z;X)$ be such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \tilde{\phi}(\dot{x}) = \vec{g}$$

and $\widehat{q} \subseteq \mathrm{dom}(g \cap \phi f)$. Then

$$\widehat{q} \subseteq^* \mathrm{dom}(g \cap \phi f) \cap f^{-1}[A] \subseteq \mathrm{dom}(g \cap f),$$

that is,

$$q \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \vec{g} = \vec{f} = \dot{x}.$$

This works for a set of q which covers p, so

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \dot{x};$$

as p and $\dot{x}$ are arbitrary, we have the result.

2E Alternative description of Borel sets Let $\mathbb{P}$, Z and $(X, \mathfrak{T})$ be as in §2A.

(a) If $\dot{G}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X},$$

consider the open set

$$W = \bigcup_{G \in \mathfrak{T}} [\![\tilde{G} \subseteq \dot{G}]\!] \times G \subseteq Z \times X.$$

If $\dot{E}$, $\dot{G}$ and $\dot{H}$ are $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ and } \dot{H} \text{ are open subsets of } \tilde{X} \text{ and } \dot{E} = \dot{G} \cap \dot{H},$$

and $W_{\dot{E}}$, $W_{\dot{G}}$ and $W_{\dot{H}}$ are the corresponding open subsets of $Z \times X$, then $W_{\dot{E}} = W_{\dot{G}} \cap W_{\dot{H}}$. **P** We have $W_{\dot{E}} \subseteq W_{\dot{G}}$ just because $[\![\tilde{G} \subseteq \dot{E}]\!] \subseteq [\![\tilde{G} \subseteq \dot{G}]\!]$ for every open $G \subseteq X$. Similarly, $W_{\dot{E}} \subseteq W_{\dot{G}}$. Now suppose that $(z, x) \in W_{\dot{G}} \cap W_{\dot{H}}$. Then there are open G, $H \subseteq X$ such that $x \in G \cap H$ and

$$z \in [\![\tilde{G} \subseteq \dot{G}]\!] \cap [\![\tilde{H} \subseteq \dot{H}]\!] \subseteq [\![(\tilde{G} \cap \tilde{H}) \subseteq \dot{G} \cap \dot{H}]\!] = [\![(G \cap H)^{\sim} \subseteq \dot{E}]\!],$$

so $(z, x) \in W_{\dot{E}}$. **Q**

In particular, $\Vdash_{\mathbb{P}} \dot{G} \cap \dot{H} = \emptyset$ iff $W_{\dot{G}}$ and $W_{\dot{H}}$ are disjoint.

(b) For any $W \subseteq Z \times X$, let $\vec{W}$ be the $\mathbb{P}$-name

$$\{(\vec{f}, p) : f \in C^-(Z;X),\, p \in \mathbb{P},\, \widehat{p} \subseteq^* \{z : (z, f(z)) \in W\}\}.$$

(i) If $\dot{G}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X},$$

$W_{\dot{G}}$ is the corresponding open subset of $Z \times X$, $p \in \mathbb{P}$ and $f \in C^-(Z;X)$, then $p \Vdash_{\mathbb{P}} \vec{f} \in \dot{G}$ iff $(\vec{f}, p) \in \vec{W}_{\dot{G}}$. **P** (i) If $p \Vdash_{\mathbb{P}} \vec{f} \in \dot{G}$, then for every q stronger than p there are an r stronger than q and a $G \in \mathfrak{T}$ such that $r \Vdash_{\mathbb{P}} \vec{f} \in \tilde{G} \subseteq \dot{G}$. In this case $\widehat{r} \times G \subseteq W_{\dot{G}}$ and

$$\{z : z \in \widehat{r} \cap \operatorname{dom} f,\ (z, f(z)) \notin W\} \subseteq \widehat{r} \setminus f^{-1}[G]$$

is meager. As q is arbitrary, $\{z : z \in \widehat{p} \cap \operatorname{dom} f,\ (z, f(z)) \notin W_{\dot{G}}\}$ is meager and $\widehat{p} \subseteq^* \{z : (z, f(z)) \in W_{\dot{G}}\}$. (ii) If $\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f,\ (z, f(z)) \in W_{\dot{G}}\}$, then for every q stronger than p there is a $G \in \mathfrak{T}$ such that $\widehat{q} \cap [\![\tilde{G} \subseteq \dot{G}]\!] \cap f^{-1}[G]$ is non-meager (because the function $z \mapsto (z, f(z)) : \operatorname{dom} f \to Z \times X$ is continuous). In this case there is an r stronger than q such that $\widehat{r} \subseteq [\![\tilde{G} \subseteq \dot{G}]\!]$ and $\widehat{r} \subseteq^* f^{-1}[G]$, so that

$$r \Vdash_{\mathbb{P}} \vec{f} \in \tilde{G} \subseteq \dot{G}.$$

As q is arbitrary, $p \Vdash_{\mathbb{P}} \vec{f} \in \dot{G}$. **Q**

(ii) Consequently

$$\Vdash_{\mathbb{P}} \vec{W}_{\dot{G}} = \dot{G}.$$

P (α) If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{G}$, then there is an $f \in C^-(Z;X)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; now (i) tells us that $(\vec{f}, p) \in \vec{W}_{\dot{G}}$, so of course $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \vec{W}_{\dot{G}}$. As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \dot{G} \subseteq \vec{W}_{\dot{G}}$. ($\beta$) If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}_{\dot{G}}$, then there are a q stronger than p and an $f \in C^-(Z;X)$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$ and $(\vec{f}, q) \in \vec{W}_{\dot{g}}$. Now (i) tells us that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \dot{G}$. As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \vec{W}_{\dot{G}} \subseteq \dot{G}$. **Q**

(iii) Note that $W_{\tilde{X}} = Z \times X$ and

$$\Vdash_{\mathbb{P}} \tilde{X} = (Z \times X)^{\rightarrow}.$$

(iv) Next, observe that if $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ and $f \in C^-(Z;X)$, then

$$[\![\vec{f} \in \vec{W}]\!] \triangle \{z : (z, f(z)) \in W\} \text{ is meager.}$$

P Because $z \mapsto (z, f(z)) : \operatorname{dom} f \to Z \times X$ is a continuous function from a Čech-complete space to a Hausdorff space, $\{z : (z, f(z)) \in W\} \in \widehat{B}(\operatorname{dom} f) \subseteq \widehat{B}(Z)$. Now, for $p \in \mathbb{P}$,

$$\begin{aligned}
\widehat{p} \subseteq^* [\![\vec{f} \in \vec{W}]\!] &\iff p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W} \\
&\iff \text{for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\
&\qquad\qquad \text{such that } (\vec{f}, r) \in \vec{W} \\
&\iff \text{for every } q \text{ stronger than } p \text{ there is an } r \text{ stronger than } q \\
&\qquad\qquad \text{such that } \widehat{r} \subseteq^* \{z : (z, f(z)) \in W\} \\
&\iff \widehat{p} \subseteq^* \{z : (z, f(z)) \in W\}.
\end{aligned}$$

As $[\![\vec{f} \in \vec{W}]\!]$ and $\{z : (z, f(z)) \in W\}$ both have the Baire property, and $\{\widehat{p} : p \in \mathbb{P}\}$ is a π-base for the topology of Z, this is enough. **Q**

(c)(i) If $p \in \mathbb{P}$, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $\widehat{p} \times A \subseteq W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$, then $p \Vdash_{\mathbb{P}} \tilde{A} \subseteq \vec{W}$. **P** If q is stronger than r and $\dot{x}$ is a $\mathbb{P}$-name such that $q \Vdash_{\mathbb{P}} \dot{x} \in \tilde{A}$, there is an $f \in C^-(Z;X)$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; now

$$\widehat{q} \subseteq^* \widehat{p} \cap f^{-1}[A] \subseteq \{z : (z, f(z)) \in W\},$$

so $q \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. As q and $\dot{x}$ are arbitrary, $p \Vdash_{\mathbb{P}} \tilde{A} \subseteq \vec{W}$. **Q**

(ii) If $W \subseteq Z \times X$ is open, then

$$\Vdash_{\mathbb{P}} \vec{W} \text{ is open.}$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, so that $\widehat{p} \subseteq^* \{z : (z, f(z)) \in W\}$. Take any $z_0 \in \widehat{p} \cap \operatorname{dom} f$ such that $(z_0, f(z_0)) \in W$. Because

W is open, there are a q stronger than p and an open set $G \subseteq X$ such that $(z_0, f(z_0)) \in \widehat{q} \times G \subseteq W$. Now $\widehat{q} \cap f^{-1}[G]$ is non-empty and relatively open in the dense G_δ set $\operatorname{dom} f$, so there is an r stronger than q such that $\widehat{r} \subseteq^* f^{-1}[G]$, that is,

$$r \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{G}.$$

Also $\widehat{r} \times G \subseteq W$, so $r \Vdash_{\mathbb{P}} \tilde{G} \subseteq \vec{W}$, by (i). Now

$$r \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G} \subseteq \vec{W} \text{ and } \dot{x} \in \operatorname{int} \vec{W}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{W} \subseteq \operatorname{int} \vec{W} \text{ and } \vec{W} \text{ is open. } \mathbf{Q}$$

(iii) If $V \subseteq Z$ is open-and-closed, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $W = V \times A$, then

$$V = [\![\vec{W} = \tilde{A}]\!], \quad Z \setminus V = [\![\vec{W} = \emptyset]\!].$$

$\mathbf{P}$ By (i), $p \Vdash_{\mathbb{P}} \tilde{A} \subseteq \vec{W}$ whenever $\widehat{p} \subseteq V$; similarly, if $\widehat{p} \subseteq V$, then

$$p \Vdash_{\mathbb{P}} (X \setminus A)\check{} \subseteq ((Z \times X) \setminus W)^{\rightarrow}.$$

But it is easy to see (cf. (d) below) that

$$\Vdash_{\mathbb{P}} ((Z \times X) \setminus W)^{\rightarrow} = \tilde{X} \setminus \vec{W},$$

while of course

$$\Vdash_{\mathbb{P}} (X \setminus A)^{\sim} = \tilde{X} \setminus \tilde{A},$$

so $p \Vdash_{\mathbb{P}} \vec{W} = \tilde{A}$ whenever $\widehat{p} \subseteq V$. And of course $p \Vdash_{\mathbb{P}} \vec{W} = \emptyset$ whenever $\widehat{p} \cap V = \emptyset$, since then $\widehat{p}$ is disjoint from $\{z : (z, f(z)) \in W\}$ for every $f \in C^-(Z; X)$. So we have the result. $\mathbf{Q}$

(d) (Compare 2A(b-iv).) If V_1, $V_2 \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$, $*$ is any of the Boolean operations $\cup$, $\cap$, $\setminus$ and $\triangle$ and $W = V_1 * V_2$, then

$$\Vdash_{\mathbb{P}} \vec{W} = \vec{V}_1 * \vec{V}_2.$$

$\mathbf{P}$ If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W} \cup \vec{V}_1 \cup \vec{V}_2$, then there is an $f \in C^-(Z; X)$ such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; now

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{x} \in \vec{V}_1 * \vec{V}_2 &\iff p \Vdash_{\mathbb{P}} \vec{f} \in \vec{V}_1 * \vec{V}_2 \\ &\iff \widehat{p} \subseteq [\![\vec{f} \in \vec{V}_1 * \vec{V}_2]\!] = [\![\vec{f} \in \vec{V}_1]\!] * [\![\vec{f} \in \vec{V}_2]\!] \\ &\iff \widehat{p} \subseteq^* \{z : (z, f(z)) \in V_1\} * \{z : (z, f(z)) \in V_2\} \\ &\iff \widehat{p} \subseteq^* \{z : (z, f(z)) \in W\}\} \iff p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}. \mathbf{Q} \end{aligned}$$

(d) If $\langle V_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ with union W, then $\Vdash_{\mathbb{P}} \vec{W} = \bigcup_{n\in\mathbb{N}} \vec{W}_n$. $\mathbf{P}$ As in 2A(b-v). $\mathbf{Q}$

(f) If $\langle W_i \rangle_{i\in I}$ is a family of open subsets of $Z \times X$ with union W, then $\Vdash_{\mathbb{P}} \vec{W} = \bigcup_{i\in\check{I}} \vec{W}_i$. $\mathbf{P}$ As in 2A(b-vi). $\mathbf{Q}$

(g) It follows that if $W \subseteq Z \times X$ is a Borel set, then $\Vdash_{\mathbb{P}} \vec{W}$ is a Borel set in $\tilde{X}$. (Induce on the Borel class of W.[3])

(h)(i) Now suppose that $p \in \mathbb{P}$, $\alpha < \omega_1$ and that $\dot{E}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel subset of } \tilde{X} \text{ of class } \alpha.$$

[3]I am not sure that there is a standard definition of Borel classes in general topological spaces. One I like is in Holický & Spurný 03, starting from $\mathcal{B}_0(X)$ the algebra of subsets of X generated by the open sets. But you can pick your own for the results here.

Then there is a Borel set $W \subseteq Z \times X$ of class α such that $p \Vdash_{\mathbb{P}} \dot{E} = \vec{W}$. **P** Induce on α. **Q**

(ii) If $p \in \mathbb{P}$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel subset of } \tilde{X},$$

then there is a $W \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that $p \Vdash_{\mathbb{P}} \dot{E} = \vec{W}$. **P** Let $Q \subseteq \mathbb{P}$ be a maximal antichain such that for each $q \in Q$ there is a Borel set $W_q \subseteq Z \times X$ such that $q \Vdash_{\mathbb{P}} \dot{E} = \vec{W}_q$. By (i), Q is dense subject to p, so $W = \bigcup_{q \in Q} W_q \cap (\widehat{q} \times X)$ will work. **Q**

(iii) If $\mathbb{P}$ is ccc, $p \in \mathbb{P}$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{E} \text{ is a Borel set in } \tilde{X},$$

then there is a Borel set $W \subseteq Z \times X$ such that $p \Vdash_{\mathbb{P}} \dot{E} = \vec{W}$. **P** As (ii), but noting that Q is countable so W is Borel. **Q**

(i) If $W \subseteq Z \times X$ is open then

$$\Vdash_{\mathbb{P}} \overline{\vec{W}} = \vec{\overline{W}}.$$

P Because $W \subseteq \overline{W}$ and $\overline{W}$ is closed,

$$\Vdash_{\mathbb{P}} \vec{W} \subseteq \vec{\overline{W}} \text{ and } \vec{\overline{W}} \text{ is closed,}$$

so

$$\Vdash_{\mathbb{P}} \overline{\vec{W}} \subseteq \vec{\overline{W}}.$$

In the other direction, suppose that $p \in \mathbb{P}$ and that $\dot{G}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is an open set meeting } \vec{\overline{W}}.$$

Then there are a q stronger than p and an open $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \tilde{G} \subseteq \dot{G} \text{ and } \tilde{G} \cap \vec{\overline{W}} \neq \emptyset.$$

Now

$$q \Vdash_{\mathbb{P}} \tilde{G} = (\widehat{q} \times G)^{\rightarrow},$$

so

$$q \Vdash_{\mathbb{P}} (\overline{W} \cap (\widehat{q} \times G))^{\rightarrow} = \vec{\overline{W}} \cap \tilde{G} \neq \emptyset,$$

and $\overline{W}$ meets the open set $\widehat{q} \times G$. So W also meets this, and there are r stronger than q and an $x \in G$ such that $\widehat{r} \times \{x\} \subseteq W$. But now

$$r \Vdash_{\mathbb{P}} \tilde{x} \in \vec{W} \cap \tilde{G} \subseteq \vec{W} \cap \dot{G}, \text{ so } \dot{G} \text{ meets } \vec{W}.$$

As p and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ every open set meeting } \vec{\overline{W}} \text{ meets } \vec{W}, \text{ so } \vec{\overline{W}} \subseteq \overline{\vec{W}}. \textbf{ Q}$$

2F Convergent sequences: Lemma Suppose that $\mathbb{P}$ is a forcing notion, Z the Stone space of its regular open algebra, and X a Hausdorff space. Suppose that $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence in $C^-(Z;X)$ and $f \in C^-(Z;X)$, $p \in \mathbb{P}$ are such that

$$\widehat{p} \subseteq^* \{z : f(z) = \lim_{n \to \infty} f_n(z) \text{ in } X\}.$$

Then

$$p \Vdash_{\mathbb{P}} \vec{f} = \lim_{n \to \infty} \vec{f}_n \text{ in } \tilde{X}.$$

proof Suppose that q is stronger than p and that $\dot{G}$ is a $\mathbb{P}$-name such that

$$q \Vdash_{\mathbb{P}} \dot{G} \text{ is an open subset of } \tilde{X} \text{ containing } \vec{f}.$$

Then there are r stronger than q and an open $G \subseteq X$ such that

$$r \Vdash_{\mathbb{P}} \vec{f} \in \tilde{G} \subseteq \dot{G}.$$

Set

$$W = \{z : z \in \operatorname{dom} f \cap \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n,\ \lim_{n\to\infty} f_n(z) = f(z) \in G\};$$

then $\widehat{r} \subseteq^* W$. Now

$$W \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{i\ge n} f_i^{-1}[G],$$

so there is an $n \in \mathbb{N}$ such that $\widehat{r} \cap \bigcap_{i\ge n} f_i^{-1}[G]$ is non-meager and there is an s stronger than r such that $\widehat{s} \subseteq^* \bigcap_{i\ge n} f_i^{-1}[G]$, that is,

$$s \Vdash_{\mathbb{P}} \vec{f}_i \in \tilde{G} \subseteq \dot{G} \text{ for every } i \ge n.$$

As q and $\dot{G}$ are arbitrary,

$$p \Vdash_{\mathbb{P}} \lim_{n\to\infty} \vec{f}_n = \vec{f}.$$

2G Supplementing the descriptions of open and closed sets in 2E, we have the following description of at least some names for compact sets.

Theorem Let X be a Hausdorff space and $\mathbb{P}$ a forcing notion, with Stone space Z. If $Z_0 \subseteq Z$ is comeager and $V \subseteq Z_0 \times X$ is an usco-compact relation in $Z_0 \times X$, then, in the language of 2E,

$$\Vdash_{\mathbb{P}} \vec{V} \text{ is compact in } \tilde{X}.$$

proof Let $\dot{\mathcal{F}}$ be a $\mathbb{P}$-name and $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \dot{\mathcal{F}} \text{ is an ultrafilter on } \tilde{X} \text{ containing } \vec{V}.$$

(a) Set

$$W = \bigcup_{G\subseteq X \text{ is open}} [\![\tilde{G} \notin \dot{\mathcal{F}}]\!] \times G.$$

Then $W \subseteq Z \times X$ is open. If $z \in Z_0 \cap \widehat{p}$ then $V[\{z\}] \not\subseteq W[\{z\}]$. **P?** Otherwise, because $V[\{z\}]$ is compact, there are open sets $G_0, \dots, G_n \subseteq X$ such that $z \in [\![\tilde{G}_i \notin \dot{\mathcal{F}}]\!]$ for each i and $V[\{z\}] \subseteq \bigcup_{i\le n} G_i$. Set $G = \bigcup_{i\le n} G_i$ and $U = \widehat{p} \cap [\![\tilde{G} \notin \dot{\mathcal{F}}]\!]$; then

$$\Vdash_{\mathbb{P}} \tilde{G} = \bigcup_{i\le n} \tilde{G}_i,$$

so

$$U = \widehat{p} \cap \inf_{i\le n} [\![\tilde{G}_i \notin \dot{\mathcal{F}}]\!] = \widehat{p} \cap \bigcap_{i\le n} [\![\tilde{G}_i \notin \dot{\mathcal{F}}]\!]$$

contains z. Now V is usco-compact, so $Z_0 \cap \widehat{p} \cap U \cap \{z' : V[\{z'\}] \subseteq G\}$ is an open neighbourhood of z in Z_0 and includes $\widehat{q} \cap Z_0$ for some q stronger than p. Now

$$q \Vdash_{\mathbb{P}} \vec{V} \subseteq \tilde{G} \notin \dot{\mathcal{F}},$$

which is impossible. **XQ**

(b) If $z \in Z_0 \cap \widehat{p}$ there is exactly one $f_0(z)$ such that $(z, f_0(z)) \in V \setminus W$. **P** By (a), there is at least one such point. **?** If (z, x) and (z, y) belong to $V \setminus W$ and $x \ne y$, let G, $H \subseteq X$ be open sets containing x, y respectively. Then $\Vdash_{\mathbb{P}} \tilde{G} \cap \tilde{H} = \emptyset$ so

$$[\![\tilde{G} \notin \dot{\mathcal{F}}]\!] \cup [\![\tilde{H} \notin \dot{\mathcal{F}}]\!] \supseteq \widehat{p}.$$

But z cannot belong to either of these sets. **XQ**

(c) $f_0 : Z_0 \cap \widehat{p} \to X$ is continuous. **P** The graph of f_0 is a closed subset of V so is itself an usco-compact relation. **Q**

(d) Let $f \in C^-(Z;X)$ be a function extending f_0. Since $(z, f(z)) \in V$ for every $z \in \widehat{p} \cap Z_0$, $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{V}$. Also

$$p \Vdash_{\mathbb{P}} \dot{\mathcal{F}} \to \vec{f}.$$

P Let q stronger than p and a $\mathbb{P}$-name $\dot{G}$ be such that

$$q \Vdash_{\mathbb{P}} \dot{G} \text{ is an open set containing } \vec{f}.$$

Then there are an r stronger than q and an open $G \subseteq X$ such that

$$r \Vdash_{\mathbb{P}} \vec{f} \in \tilde{G} \subseteq \dot{G}.$$

Now $r \subseteq^* f^{-1}[G]$, so there is an s stronger than r such that $Z_0 \cap \widehat{s} \subseteq f^{-1}[G]$; so $f(z) \in G$ for every $z \in Z_0 \cap \widehat{s}$ and $[\![\tilde{G} \notin \dot{\mathcal{F}}]\!]$ does not meet $\widehat{s}$, that is, $s \Vdash_{\mathbb{P}} \tilde{G} \in \dot{\mathcal{F}}$. But this means that

$$s \Vdash_{\mathbb{P}} \dot{G} \in \dot{\mathcal{F}}.$$

As q and $\dot{G}$ are arbitrary,

$$p \Vdash_{\mathbb{P}} \text{every open set containing } \vec{f} \text{ belongs to } \dot{\mathcal{F}},$$

that is,

$$p \Vdash_{\mathbb{P}} \vec{f} = \lim \dot{\mathcal{F}}. \quad \mathbf{Q}$$

(e) As p and $\dot{\mathcal{F}}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{every ultrafilter on } \tilde{X} \text{ containing } \vec{V} \text{ has a limit in } \vec{V}, \text{ and } \vec{V} \text{ is compact.}$$

2H Theorem Let X be a Hausdorff space, $\mathbb{P}$ a forcing notion and Z its Stone space. Set $S = \bigcup_{n \geq 1} \mathbb{N}^n$ and let $\langle W_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme in $\mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ with kernel W. Then

$$\Vdash_{\mathbb{P}} \vec{W} \text{ is the kernel of the Souslin scheme } \langle \vec{W}_\sigma \rangle_{\sigma \in S}.$$

proof (a) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \vec{f} = \dot{x}$. Then $\widehat{p} \subseteq^* \{z : (z, f(z)) \in W\}$.

For $\sigma \in S^* = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ set

$$W'_\sigma = \bigcup_{\sigma \subseteq \alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \geq 1} W_{\alpha \restriction n} \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X).$$

Choose open-and-closed sets V_σ in Z so that

$$V_\emptyset = Z,$$

and for every $\sigma \in S^*$

$$\langle V_{\sigma^\frown <i>} \rangle_{i \in \mathbb{N}} \text{ is a disjoint sequence of subsets of } V_\sigma \text{ with union dense in } V_\sigma,$$

$$\widehat{p} \cap V_\sigma \subseteq^* \{z : (z, f(z)) \in W'_\sigma\}.$$

Then we have a $\mathbb{P}$-name $\dot{\alpha}$ such that

$$\Vdash_{\mathbb{P}} \dot{\alpha} \in \mathbb{N}^{\mathbb{N}} \text{ and } [\![\sigma \subseteq \dot{\alpha}]\!] = V_\sigma \text{ for every } \sigma \in S^*.$$

Now, for every $n \in \mathbb{N}$,

$$\begin{aligned} [\![\vec{f} \in \vec{W}_{\dot{\alpha} \restriction n}]\!] &= \sup_{\sigma \in \mathbb{N}^n} [\![\sigma \subseteq \dot{\alpha}]\!] \cap [\![\vec{f} \in \vec{W}_\sigma]\!] \\ &\supseteq \sup_{\sigma \in \mathbb{N}^n} V_\sigma \cap \widehat{p} \cap V_\sigma = \widehat{p}, \end{aligned}$$

so

$$\Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \bigcap_{n \in \mathbb{N}} \vec{W}_{\dot{\alpha} \restriction n} \text{ and } \dot{x} \text{ belongs to the kernel of } \langle \vec{W}_\sigma \rangle_{\sigma \in S}.$$

(b) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \text{ is in the kernel of the Souslin scheme } \langle \vec{W}_\sigma \rangle_{\sigma \in S}.$$

Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, and for $\sigma \in S^*$ set

$$V_\sigma = [\![\vec{f} \in \vec{W}_\sigma]\!] \cap [\![\vec{f} \text{ belongs to the kernel of the Souslin scheme } \langle \vec{W}_{\sigma^\frown \tau} \rangle_{\tau \in S}]\!].$$

Then $\widehat{p} \subseteq V_\emptyset$. Set

$$M = \bigcup_{\sigma \in S^*} (V_\sigma \setminus \bigcup_{n \in \mathbb{N}} V_{\sigma^\frown n}) \cup \bigcup_{\sigma \in S} (V_\sigma \setminus \{z : (z, f(z)) \in W_\sigma\}),$$

so that M is meager. For $z \in \widehat{p} \setminus M$, there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $z \in V_{\alpha \restriction n}$ for every n, and now $(z, f(z)) \in W_{\alpha \restriction n}$ for every n, so $(z, f(z)) \in W$. But this means that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \vec{W}$.

2I Corollary If $\langle A_\sigma \rangle_{\sigma \in S}$ is a Souslin scheme in $\mathcal{U}\widehat{\mathcal{B}}(X)$ with kernel A, then

$$\Vdash_{\mathbb{P}} \tilde{A} \text{ is the kernel of } \langle \tilde{A}_\sigma \rangle_{\sigma \in S}.$$

proof Apply 2H with $W_\sigma = Z \times A_\sigma$.

2J Finding the Boolean value $[\![\vec{W} \ne \emptyset]\!]$ Let X be a Hausdorff space, $\mathbb{P}$ a forcing notion and Z its Stone space. We have straightforward formulae for $[\![\tilde{f} = \tilde{g}]\!]$ and $[\![\tilde{f} \in \vec{W}]\!]$ when f, $g \in C^-(Z;X)$ and $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$. We do not have such elementary methods for finding $[\![\vec{W} = \vec{V}]\!] = [\![(W \triangle V)^{\rightarrow} = \emptyset]\!]$. Here I give a handful of partial results.

(a)(i) If $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ then

$$[\![\vec{W} \ne \emptyset]\!] \subseteq^* W^{-1}[X].$$

P Suppose that $p \in \mathbb{P}$ and $\widehat{p} \subseteq [\![\vec{W} \ne \emptyset]\!]$, that is, $p \Vdash_{\mathbb{P}} \vec{W} \ne \emptyset$, that is, there is a $\mathbb{P}$-name $\dot{x}$ such that $p \Vdash_{\mathbb{P}} \dot{x} \in \vec{W}$. Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$; Then

$$\widehat{p} \subseteq^* \{z : (z, f(z)) \in W\} \subseteq W^{-1}[X].$$

The union of such sets $\widehat{p}$ is open and dense in $[\![\vec{W} \ne \emptyset]\!]$ so $[\![\vec{W} \ne \emptyset]\!] \subseteq^* W^{-1}[X]$. **Q**

(ii) If V, $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ then

$$\{z : V[\{z\}] \subseteq W[\{z\}]\} \subseteq^* [\![\vec{V} \subseteq \vec{W}]\!].$$

P Apply (i) to $V \setminus W$. **Q**

(iii) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ and $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$ then

$$\{z : A \subseteq W[\{z\}]\} \subseteq^* [\![\tilde{A} \subseteq \vec{W}]\!].$$

P By 2E(c-iii) or otherwise, $\tilde{A} = (Z \times A)^{\rightarrow}$. **Q**

(b) If $Z_0 \subseteq Z$ is comeager and $W \subseteq Z_0 \times X$ is usco-compact, then $[\![\vec{W} \ne \emptyset]\!] \triangle W^{-1}[X]$ is meager. **P** Start by observing that $W^{-1}[X]$ is relatively closed in Z_0; express it as $Z_0 \cap F$ where $F \subseteq Z$ is closed; set $Z' = \operatorname{int} F$, so that Z' is open-and-closed in Z, and $Z' \cap Z_0$ is comeager in Z'. Set $W' = W \cap (Z' \times X)$, so that $W' \subseteq (Z_0 \cap Z') \times X$ is usco-compact.

Let $V \subseteq W'$ be a minimal relatively closed set such that $V^{-1}[X] = Z' \cap W^{-1}[X]$. Because $Z_0 \cap Z'$ is dense in the extremally disconnected space Z', and $V \subseteq (Z_0 \cap Z') \times X$ is usco-compact, V is the graph of a function $f_0 : Z' \cap Z_0 \to X$, which must be continuous, so there is an $f \in C^-(Z;X)$ agreeing with f_0 on $Z' \cap \operatorname{dom} f$. Now $[\![\vec{W} \ne \emptyset]\!] \supseteq [\![\vec{f} \in \vec{W}]\!]$ and $[\![\vec{f} \in \vec{W}]\!] \triangle \{z : (z, f(z)) \in W\}$ is meager, so $W^{-1}[X] \subseteq^* [\![\vec{W} \ne \emptyset]\!]$ is meager. With (a-i) this gives the result. **Q**

(c) If $W \subseteq Z \times X$ is K-analytic, then $[\![\vec{W} \ne \emptyset]\!] \triangle W^{-1}[X]$ is meager. **P** Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times (Z \times X)$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = W$. Then

$$R' = \{(\alpha, z) : (\alpha, z, x) \in R\} \subseteq \mathbb{N}^{\mathbb{N}} \times Z$$

is usco-compact, therefore closed in $\mathbb{N}^{\mathbb{N}} \times Z$ (use Fremlin 03, 422Dc and 422Df). So $W^{-1}[X] = R'[\mathbb{N}^{\mathbb{N}}]$ has the Baire property in Z and there is a $\widehat{\mathcal{B}}(Z)$-measurable function $h : W^{-1}[X] \to \mathbb{N}^{\mathbb{N}}$ which is a selector for R' (Fremlin 03, 423N). Because $\mathbb{N}^{\mathbb{N}}$ is second-countable, there is a $g \in C^{-}(Z;\mathbb{N}^{\mathbb{N}})$ included in h. Set $V = \{(z,x) : z \in \operatorname{dom} g,\ (g(z),z,x) \in R\}$; then $V \subseteq \operatorname{dom} g \times X$ is an usco-compact relation (use Fremlin 03, 422Df again). So (b) tells us that

$$W^{-1}[X] \subseteq^* V^{-1}[X] \subseteq^* [\![\vec{V} \ne \emptyset]\!] \subseteq [\![\vec{W} \ne \emptyset]\!]$$

and with (a-i) again we have the result. **Q**

(d) If $W \subseteq Z \times X$ is open then $[\![\vec{W} \ne \emptyset]\!] \triangle W^{-1}[X]$ is meager. **P** This time, if $z \in W^{-1}[X]$, there are an $x \in X$ and an open neighbourhood H of z such that $H \times \{x\} \subseteq W$, Now $H \subseteq [\![\tilde{x} \in \vec{W}]\!]$. As z is arbitrary, $W^{-1}[X] \subseteq [\![\vec{W} \ne \emptyset]\!]$, which is more than we need. **Q**

2K Examples (a) Let $\mathbb{P}$ be a forcing notion and Z its Stone space. Suppose that Z is expressible as the union of κ nowhere dense zero sets. Set $X = [0,1[^{\kappa}$. Then there is a closed set $W \subseteq Z \times X$ such that $W^{-1}[X] = Z$ but $\Vdash_{\mathbb{P}} \vec{W} = \emptyset$. **P** Let $\langle Z_\xi \rangle_{\xi<\kappa}$ be a family of nowhere dense zero sets covering Z, and for each $\xi < \kappa$ let $f_\xi \in C(Z;[0,1])$ be such that $Z_\xi = f_\xi^{-1}[\{1\}]$. Set $W = \{(z,x) : z \in Z,\ x \in X,\ x(\xi) - f_\xi(z) \in \mathbb{Z}$ for every $\xi\}$; then W is closed and $W^{-1}[X] = Z$. **?** If $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}} \vec{W} \ne \emptyset$, there is a $g \in C^{-}(Z;X)$ such that $\widehat{p} \subseteq^* \{z : (z,g(z)) \in W\}$. Take any $z \in \widehat{p} \cap \operatorname{dom} g$ such that $(z,g(z)) \in W$ and let $\xi < \kappa$ be such that $z \in Z_\xi$. Then $g(z')(\xi) = f_\xi(z')$ for every $z' \in \operatorname{dom} g \setminus Z_\xi$, which is dense in $\operatorname{dom} g$; so $g(z)(\xi) = f_\xi(z) = 1$, which is impossible. **X** So we must have $\Vdash_{\mathbb{P}} \vec{W} = \emptyset$. **Q**

(b) Suppose that $A \subseteq [0,1]$ is a coanalytic set with no perfect subset and that $\mathbb{P}$ is a forcing notion such that the Stone space Z of $\mathbb{P}$ can be covered by ω_1 nowhere dense sets. Then there is a set $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times [0,1])$ such that $W^{-1}[/,[0,1]/,] = Z$ but $\Vdash_{\mathbb{P}} \vec{W} = \emptyset$. **P** Express Z as $\bigcup_{x\in A} Z_x$ where every Z_x is closed and nowhere dense. Set $W = \bigcup_{x\in A} Z_x \times \{x\}$; then $W^{-1}[\,[0,1]\,] = Z$.

If Y is a Čech-complete space and $h : Y \to Z \times [0,1]$ is continuous then, because A is coanalytic, $Y_0 = h^{-1}[Z \times A] \in \widehat{\mathcal{B}}(Y)$; let $Y_1 \subseteq Y_0$ be a G_δ set such that $Y_0 \setminus Y_1$ is meager. If Y_1 is empty, then $h^{-1}[W] \subseteq Y_1$ is meager and has the Baire property. Otherwise, $\pi_2 h \restriction Y_1$ is a continuous function from the Čech-complete space Y_1 to A. As A has no perfect subset, there is an $x \in A$ such that $\{y : \pi_2 h(y) = x\}$ is non-meager and has non-empty relative interior $H_x \subseteq Y_1$. In this case, $H_x \cap h^{-1}[W] = \{y : y \in H_x,\ \pi_1 h(y) \in Z_x\}$ is relatively closed in H_x and has the Baire property in Y_1 and Y. The same argument applies to any non-empty relatively open subset of Y_1, so $Y_2 = \bigcup_{x\in A} H_x$ is dense in Y_1, while $Y_2 \cap h^{-1}[W]$ has the Baire property in Y; but $h^{-1}[W] \setminus Y_2$ is meager, so $h^{-1}[W] \in \widehat{\mathcal{B}}(Y)$. As Y and h are arbitrary, $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$.

? If $p \in \mathbb{P}$ and $f \in C^{-}(Z;[0,1])$ are such that $p \Vdash_{\mathbb{P}} \vec{f} \in \vec{W}$, then there is a non-meager G_δ set $V \subseteq \widehat{p} \cap \{z : (z,f(z)) \in W\}$. Now $f \restriction V : V \to A$ is continuous, so there is an $x \in A$ such that $V \cap f^{-1}[\{x\}]$ is not meager. But $V \cap f^{-1}[\{x\}] \subseteq Z_x$ is nowhere dense. **X**

Thus $\Vdash_{\mathbb{P}} \vec{W} = \emptyset$. **Q**

3 Identifying the new spaces

The most pressing problem is to find ways of getting a clear picture of the 'new' spaces as topological spaces. For actual examples it will be easiest to wait for §4 below. Here I put together a handful of basic techniques.

3A Theorem Let $\langle X_i \rangle_{i\in I}$ be a family of Hausdorff spaces with product X, and $\mathbb{P}$ a forcing notion. Suppose that $J = \{i : i \in I,\ X_i$ is not compact$\}$ is countable. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with } \textstyle\prod_{i\in\check{I}} \tilde{X}_i.$$

proof (a) For $i \in I$, let $\pi_i : X \to X_i$ be the canonical map. For any $f \in C^{-}(Z;X)$, let $f^{\#}$ be the $\mathbb{P}$-name

$$\{(\check{\imath}, (\pi_i f)^{\rightarrow}), \mathbb{1}) : i \in I\},$$

where in this formula each $(\pi_i f)^{\rightharpoonup}$ is to be the $\mathbb{P}$-name corresponding to $\pi_i f$ regarded as a member of $C^-(Z;X_i)$. Then

$$\Vdash_{\mathbb{P}} f^{\#} \in \prod_{i\in\check I} \tilde X_i$$

because $\Vdash_{\mathbb{P}} \check\imath \ne \check\jmath$ whenever i, $j \in I$ are distinct. (I hope it is clear that the formula '$\prod_{i\in\check I} \tilde X_i$' refers to the $\mathbb{P}$-name $\{((\check\imath, \tilde X_i), \mathbb{1}) : i \in I\}$ for a family of topological spaces.)

Let $\dot\psi$ be the $\mathbb{P}$-name

$$\{((\vec f, f^{\#}), \mathbb{1}) : f \in C^-(Z;X)\}.$$

I claim that $\dot\psi$ is a name for a homeomorphism between $\tilde X$ and $\prod_{i\in\check I} \tilde X_i$. We surely have

$$\Vdash_{\mathbb{P}} \dot\psi \subseteq \tilde X \times \prod_{i\in\check I} \tilde X_i.$$

(b) Suppose that $p \in \mathbb{P}$ and that $\dot x_0$, $\dot x_1$, $\dot y_0$, $\dot y_1$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} (\dot x_0, \dot y_0) \text{ and } (\dot x_1, \dot y_1) \text{ belong to } \dot\psi.$$

Let q, stronger than p, and f_0, $f_1 \in C^-(Z;X)$ be such that

$$q \Vdash_{\mathbb{P}} \dot x_0 = \vec f_0,\ \dot y_0 = f_0^{\#},\ \dot x_1 = \vec f_1,\ \dot y_1 = f_1^{\#}.$$

Then

$$\begin{aligned}
q \Vdash_{\mathbb{P}} \dot x_0 = \dot x_1 &\iff q \Vdash_{\mathbb{P}} \vec f_0 = \vec f_1\\
&\iff f_0(z) = f_1(z) \text{ for every } z \in \widehat q \cap \operatorname{dom} f_0 \cap \operatorname{dom} f_1\\
&\iff \text{ for every } i \in I,\ \pi_i f_0(z) = \pi_i f_1(z)\\
&\qquad\qquad \text{for every } z \in \widehat q \cap \operatorname{dom} f_0 \cap \operatorname{dom} f_1\\
&\iff \text{ for every } i \in I,\ \pi_i f_0(z) = \pi_i f_1(z)\\
&\qquad\qquad \text{for every } z \in \widehat q \cap \operatorname{dom} \pi_i f_0 \cap \operatorname{dom} \pi_i f_1\\
&\iff \text{ for every } i \in I,\ q \Vdash_{\mathbb{P}} (\pi_i f_0)^{\rightharpoonup} = (\pi_i f_1)^{\rightharpoonup}\\
&\iff \text{ for every } i \in I,\ q \Vdash_{\mathbb{P}} f_0^{\#}(\check\imath) = f_1^{\#}(\check\imath)\\
&\iff q \Vdash_{\mathbb{P}} f_0^{\#}(i) = f_1^{\#}(i) \text{ for every } i \in \check I\\
&\iff q \Vdash_{\mathbb{P}} f_0^{\#} = f_1^{\#} \iff q \Vdash_{\mathbb{P}} \dot y_0 = \dot y_1.
\end{aligned}$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot x_0 = \dot x_1 \text{ iff } \dot y_0 = \dot y_1.$$

As p, $\dot x_0$, $\dot y_0$, $\dot x_1$ and $\dot y_1$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\psi \text{ is an injective function.}$$

(c) Since $\Vdash_{\mathbb{P}} \dot\psi(\vec f) = f^{\#}$ for every $f \in C^-(Z;X)$,

$$\Vdash_{\mathbb{P}} \text{the domain of } \dot\psi \text{ is } \tilde X.$$

In the other direction, suppose that $p \in \mathbb{P}$ and $\dot y$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot y \in \prod_{i\in\check I} \tilde X_i$. Then, for each $i \in I$,

$$p \Vdash_{\mathbb{P}} \dot y(\check\imath) \in \tilde X_i,$$

so there is an $f_i \in C^-(Z;X_i)$ such that

$$p \Vdash_{\mathbb{P}} \dot y(\check\imath) = \vec f_i;$$

moreover, 2A(b-viii) tells us that we can arrange that $\operatorname{dom} f_i = Z$ for every $i \in I \setminus J$. Set $Z_0 = \bigcap_{i\in J} \operatorname{dom} f_i$; because J is countable, Z_0 is a dense G_δ set in Z. Set $f(z) = \langle f_i(z)\rangle_{i\in I}$ for $z \in Z_0$; then $f \in C^-(Z;X)$, and

$$p \Vdash_{\mathbb{P}} f^{\#}(\check{\imath}) = (\pi_i f)^{\rightarrow} = (f_i \restriction Z_0)^{\rightarrow} = \vec{f_i} = \dot{y}(\check{\imath})$$

for every $i \in I$, so

$$p \Vdash_{\mathbb{P}} f^{\#}(i) = \dot{y}(i) \text{ for every } i \in \check{I},$$

and

$$p \Vdash_{\mathbb{P}} \dot{y} = f^{\#} \text{ is a value of } \dot{\psi}.$$

As p and $\dot{y}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ the set of values of $\dot{\psi}$ is $\prod_{i \in \check{I}} \tilde{X}_i$, and $\dot{\psi} : \tilde{X} \to \prod_{i \in \check{I}} \tilde{X}_i$ is a bijection.

(d) Suppose now that $\langle G_i \rangle_{i \in I}$ is a family such that G_i is an open set in X_i for every $i \in I$, and $G_i = X_i$ for all but finitely many i; set $G = \prod_{i \in I} G_i$, so that G is an open set in X. Then, for any $f \in C^{-}(Z; X)$ and $p \in \mathbb{P}$,

$$p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{G} \iff \widehat{p} \subseteq^* f^{-1}[G] = \bigcap_{i \in I} (\pi_i f)^{-1}[G_i]$$

$$\iff \text{ for every } i \in I,\ \widehat{p} \subseteq^* (\pi_i f)^{-1}[G_i]$$

(because $(\pi_i^f)^{-1}[G_i] = \operatorname{dom} f$ for all but finitely many i)

$$\iff \text{ for every } i \in I,\ p \Vdash_{\mathbb{P}} f^{\#}(\check{\imath}) \in \tilde{G}_i$$

$$\iff p \Vdash_{\mathbb{P}} f^{\#}(i) \in \tilde{G}_i \text{ for every } i \in \check{I}$$

$$\iff p \Vdash_{\mathbb{P}} f^{\#} \in \prod_{i \in \check{I}} \tilde{G}_i \iff p \Vdash_{\mathbb{P}} \dot{\psi}(\vec{f}) \in \prod_{i \in \check{I}} \tilde{G}_i.$$

As p and f are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{G} = \dot{\psi}^{-1}[\textstyle\prod_{i \in \check{I}} \tilde{G}_i].$$

(e) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{W}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X},\ \dot{W} \subseteq \textstyle\prod_{i \in \check{I}} \tilde{X}_i \text{ is open and } \dot{\psi}(\dot{x}) \in \dot{W}.$$

Then

$p \Vdash_{\mathbb{P}}$ there is an open cylinder set in $\prod_{i \in \check{I}} \tilde{X}_i$, determined by coordinates in a finite subset of $\check{I}$, containing $\dot{\psi}(\dot{x})$ and included in $\dot{W}$.

We therefore have a q stronger than p, a finite set $K \subseteq I$ and a $\mathbb{P}$-name $\dot{V}$ such that

$q \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x}) \in \dot{V} \subseteq \dot{W}$ and $\dot{V}$ is an open cylinder set in $\prod_{i \in \check{I}} \tilde{X}_i$, determined by coordinates in $\check{K}$.

Accordingly there is a family $\langle \dot{G}_i \rangle_{i \in I}$ of $\mathbb{P}$-names such that $\dot{G}_i = \tilde{X}_i$ for $i \in I \setminus K$ and

$q \Vdash_{\mathbb{P}} \dot{G}_i$ is an open subset of $\tilde{X}_i$ containing $\psi(\dot{x})(i)$ for every $i \in \check{K}$, and $\prod_{i \in \check{I}} \dot{G}_i \subseteq \dot{W}$.

Now there must be an r stronger than q and a family $\langle G_i \rangle_{i \in J}$ such that G_i is an open set in X_i and

$$r \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x})(\check{\imath}) \in \tilde{G}_i \subseteq \dot{G}_i$$

for every $i \in K$. Setting $G_i = X_i$ for $i \in I \setminus K$,

$$r \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x})(\check{\imath}) \in \tilde{G}_i \subseteq \dot{G}_i$$

for every $i \in I$; so

$$r \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x})(i) \in \tilde{G}_i \subseteq \dot{G}_i \text{ for every } i \in \check{I},$$

that is,

$$r \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x}) \in \textstyle\prod_{i \in I} \tilde{G}_i \subseteq \dot{V} \subseteq \dot{W}.$$

Setting $G = \prod_{i \in I} G_i$, (d) above tells us that

$$r \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G} \subseteq \dot{\psi}^{-1}[\dot{W}],$$

and therefore that

$$r \Vdash_{\mathbb{P}} \dot{x} \in \operatorname{int} \dot{\psi}^{-1}[\dot{W}].$$

As p, $\dot{x}$ and $\dot{W}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is continuous.}$$

(f) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{G}$ are $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open subset of } \tilde{X} \text{ and } \dot{x} \in \dot{G}.$$

Let $f \in C^{-}(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Taking $\mathcal{U}$ to be the family of open cylinder sets in X, we know that

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a base for the topology of } \tilde{X},$$

so there must be a $G \in \mathcal{U}$ and a q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{G} \subseteq \dot{G}.$$

Now (d) tells us that

$$\Vdash_{\mathbb{P}} \dot{\psi}[\tilde{G}] \text{ is open in } \prod_{i \in \check{I}} \tilde{X}_i,$$

so we get

$$q \Vdash_{\mathbb{P}} \dot{\psi}(\dot{x}) \in \operatorname{int} \dot{\psi}[\dot{G}].$$

As p, $\dot{x}$ and $\dot{G}$ are arbitrary (and $\Vdash_{\mathbb{P}} \dot{\psi}$ is a bijection),

$$\Vdash_{\mathbb{P}} \dot{\psi}^{-1} \text{ is continuous,}$$

which completes the proof.

3B Regular open algebras Let $\mathbb{P}$, $(X, \mathfrak{T})$ and $\tilde{X}$ be as in §2A.

(a) If $G \subseteq X$ is a regular open set in X, then

$$\Vdash_{\mathbb{P}} \tilde{G} \text{ is a regular open set in } \tilde{X}.$$

P Of course

$$\Vdash_{\mathbb{P}} \tilde{G} \text{ is an open set in } \tilde{X}.$$

Now suppose that $p \in \mathbb{P}$ and that $\dot{V}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{V} \text{ is an open set in } \tilde{X} \text{ not included in } \tilde{G}.$$

Then

$$p \Vdash_{\mathbb{P}} \text{ there is a } V \in \tilde{\mathfrak{T}} \text{ such that } V \subseteq \dot{V} \text{ but } V \not\subseteq \tilde{G},$$

so there are a q stronger than p and an open set $V \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \tilde{V} \subseteq \dot{V} \text{ and } \tilde{V} \not\subseteq \tilde{G}.$$

Accordingly $V \not\subseteq G$. But G is supposed to be a regular open set, so $W = V \setminus \overline{G}$ is non-empty. Now

$$q \Vdash_{\mathbb{P}} \tilde{W} \text{ is a non-empty open subset of } \dot{V} \setminus \tilde{G}, \text{ so } \dot{V} \not\subseteq \overline{\tilde{G}}.$$

As p and $\dot{V}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ every open subset of } \overline{\tilde{G}} \text{ is included in } \tilde{G}, \text{ so } \tilde{G} \text{ is regular. } \mathbf{Q}$$

(b) Let $\mathrm{RO}(X)$ be the regular open algebra of X. Then Write $\dot{\vartheta}$ for the $\mathbb{P}$-name $\{((\check{G}, \tilde{G}), \mathbb{1}) : G \in \mathrm{RO}(X)\}$. By (a),

$$\Vdash_{\mathbb{P}} \dot{\vartheta} \text{ is a function from } \mathrm{RO}(X)\check{} \text{ to } \mathrm{RO}(\tilde{X}).$$

Now

$$\Vdash_{\mathbb{P}} \dot\vartheta \text{ is a Boolean homomorphism.}$$

P (i) Let $p \in \mathbb{P}$ and $\dot a$ be a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot a = 0 \text{ in } \mathrm{RO}(X)\check{}.$$

Then

$$p \Vdash_{\mathbb{P}} \dot a = \check\emptyset \text{ and } \dot\vartheta(\dot a) = \tilde\emptyset = 0 \text{ in } \mathrm{RO}(\tilde X).$$

As p and $\dot a$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\vartheta(0) = 0.$$

(ii) Suppose that $p \in \mathbb{P}$ and that $\dot a$, $\dot b$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot a, \dot b \in \mathrm{RO}(X)\check{}.$$

Then there are a q stronger than p and $G, H \in \mathrm{RO}(X)$ such that

$$q \Vdash_{\mathbb{P}} \dot a = \check G \text{ and } \dot b = \check H.$$

In this case,

$$q \Vdash_{\mathbb{P}} \dot a \cap \dot b = (G \cap H)\check{},$$

so

$$q \Vdash_{\mathbb{P}} \dot\vartheta(\dot a \cap \dot b) = (G \cap H)\tilde{} = \tilde G \cap \tilde H = \dot\vartheta(\dot a) \cap \dot\vartheta(\dot b) \text{ in } \mathrm{RO}(\tilde X).$$

As p, $\dot a$ and $\dot b$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\vartheta \text{ preserves the Boolean operation } \cap.$$

(iii) Suppose that $p \in \mathbb{P}$ and that $\dot a$, $\dot b$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot a \text{ and } \dot b \text{ are complementary elements of } \mathrm{RO}(X)\check{}.$$

Then there are a q stronger than p and $G, H \in \mathrm{RO}(X)$ such that

$q \Vdash_{\mathbb{P}} \dot a = \check G$ and $\dot b = \check H$, so $\check G$ and $\check H$ are complementary members of $\mathrm{RO}(X)\check{}$, that is, they are disjoint and no non-zero member of $\mathrm{RO}(X)\check{}$ can be disjoint from both.

But this means that G and H are disjoint and have union dense in X, so that

$\Vdash_{\mathbb{P}} \tilde G$ and $\tilde H$ are disjoint and have union dense in $\tilde X$, that is, they are complementary in $\mathrm{RO}(\tilde X)$.

So

$$q \Vdash_{\mathbb{P}} \dot\vartheta(\dot a) = \tilde G \text{ and } \dot\vartheta(\dot b) = \tilde H \text{ are complementary in } \mathrm{RO}(\tilde X).$$

As p, $\dot a$ and $\dot b$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\vartheta \text{ preserves complements.}$$

(iv) Putting (i)-(iii) together,

$$\Vdash_{\mathbb{P}} \dot\vartheta \text{ is a Boolean homomorphism. } \mathbf{Q}$$

(c) $\Vdash_{\mathbb{P}} \dot\vartheta$ is injective. **P** Suppose that $p \in \mathbb{P}$ and that $\dot a$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot a \text{ is a non-zero member of } \mathrm{RO}(X)\check{}.$$

Then there are a q stronger than p and a $G \in \mathrm{RO}(X)$ such that

$$q \Vdash_{\mathbb{P}} \dot a = \check G \neq 0 \text{ in } \mathrm{RO}(X)\check{},$$

so that $G \neq \emptyset$ and

$$q \Vdash_{\mathbb{P}} \dot\vartheta(\dot a) = \tilde G \neq 0 \text{ in } \mathrm{RO}(\tilde X).$$

As p and $\dot a$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot\vartheta(a) \ne 0 \text{ for every non-zero } a \in \mathrm{RO}(X)\check{} \text{ and } \dot\vartheta \text{ is injective. } \mathbf{Q}$$

(d) $\Vdash_{\mathbb{P}} \dot\vartheta[\mathrm{RO}(X)\check{}]$ is order-dense in $\mathrm{RO}(\tilde X)$. **P** Suppose that $p \in \mathbb{P}$ and that $\dot G$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot G \text{ is a non-empty regular open set in } \tilde X.$$

Then there are a q stronger than p and a non-empty open set $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \tilde G \subseteq \dot G.$$

Next, there is a non-empty $H \in \mathrm{RO}(X)$ such that $H \subseteq G$, in which case

$$q \Vdash_{\mathbb{P}} \dot\vartheta(\check H) = \tilde H \text{ is a non-empty member of } \dot\vartheta[\mathrm{RO}(X)\check{}] \text{ included in } \dot G.$$

As p and $\dot G$ are arbitrary, we have the result. **Q**

3C Corollary For any topological space X,

$$\Vdash_{\mathbb{P}} \mathrm{RO}(\tilde X) \text{ can be identified with the Dedekind completion of } \mathrm{RO}(X)\check{}.$$

3D Normal bases and the finite-cover uniformity (a) Let X be a set. I will say that a topology base $\mathcal{U}$ on X is **normal** if

(i) $U \cup V$ and $U \cap V$ belong to $\mathcal{U}$ for all U, $V \in \mathcal{U}$,

(ii) whenever $x \in U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $U \cup V = X$ and $x \notin V$,

(iii) whenever U, $V \in \mathcal{U}$ and $U \cup V = X$ then there are disjoint U', $V' \in \mathcal{U}$ such that $U \cup V' = U' \cup V = X$.

(b) Let $\mathcal{U}$ be a normal topology base on X.

(i) If $\mathcal{V} \subseteq \mathcal{U}$ is a finite cover of X, there is a finite $\mathcal{V}^* \subseteq \mathcal{U}$, a cover of X, which is a star-refinement of $\mathcal{V}$. **P** Induce on $n = \#(\mathcal{V})$. If $n \le 1$ we can take $\mathcal{V}^* = \mathcal{V}$. For the inductive step to $n+1$, fix $V_0 \in \mathcal{V}$ and set $\mathcal{V}_1 = \{V_0 \cup V : V \in \mathcal{V} \setminus \{V_0\}\}$. Then $\mathcal{V}_1$ is a subset of $\mathcal{U}$, covers X and has at most n members, so there is a finite star-refinement $\mathcal{V}_1^*$ of $\mathcal{V}_1$ included in $\mathcal{U}$ and covering X. For each $W \in \mathcal{V}_1^*$, set

$$W' = W \cap \bigcap\{V : V \in \mathcal{V} \setminus \{V_0\},\, W \subseteq V \cup V_0\};$$

then $W' \supseteq W \setminus V_0$. Accordingly $U = \bigcup\{W' : W \in \mathcal{V}_1^*\}$ includes $X \setminus V_0$. Let U_1, U_2 be disjoint members of $\mathcal{U}$ such that $U_1 \cup U = U_2 \cup V_0 = X$. Now consider

$$\mathcal{V}^* = \{U_1\} \cup \{W' \cap V_0 : W \in \mathcal{V}_1^*\} \cup \{W' \cap U_2 : W' \in \mathcal{V}_1^*\}.$$

Then $\mathcal{V}^* \subseteq \mathcal{U}$ is finite and

$$\bigcup \mathcal{V}^* = U_1 \cup \bigcup\{W' : W \in \mathcal{V}_1^*\} = X.$$

We have to check that $\mathcal{V}^*$ is a star-refinement of $\mathcal{V}$. If $x \in U_1$, then $x \notin U_2$, so $\bigcup\{W : x \in W \in \mathcal{V}^*\} \subseteq V_0$. If $x \in X \setminus U_1$, then there is a $V_1 \in \mathcal{V} \setminus \{V_0\}$ such that $\bigcup\{W : x \in W \in \mathcal{V}_1^*\} \subseteq V_1 \cup V_0$. Now

$$\begin{aligned}\bigcup\{W : x \in W \in \mathcal{V}^*\} &\subseteq \bigcup\{W' : x \in W \in \mathcal{V}_1^*\}\\ &\subseteq \bigcup\{W' : W \in \mathcal{V}_1^*,\, W \subseteq V_1 \cup V_0\} \subseteq V_1.\end{aligned}$$

So we have what we need. **Q**

(ii) We have a uniformity $\mathcal{W}$ on X defined by saying that a subset W of $X \times X$ belongs to $\mathcal{W}$ iff there is a finite subset $\mathcal{V}$ of $\mathcal{U}$, covering X, such that $W_{\mathcal{V}} \subseteq W$, where $W_{\mathcal{V}} = \bigcup_{V \in \mathcal{V}} V \times V$. **P** (α) If $\mathcal{V}_1$, $\mathcal{V}_2 \subseteq \mathcal{U}$ are finite covers of X, then $\mathcal{V} = \{V_1 \cap V_2 : V_1 \in \mathcal{V}_1, V_2 \in \mathcal{V}_2\}$ covers X and $W_{\mathcal{V}} \subseteq W_{\mathcal{V}_1} \cap W_{\mathcal{V}_2}$. So (if X is not empty) $\mathcal{W}$ is a filter on $X \times X$. (β) If $\mathcal{V} \subseteq \mathcal{U}$ is a finite cover of X, then $W_{\mathcal{V}}^{-1} = W_{\mathcal{V}}$, so $W^{-1} \in \mathcal{W}$ for every $W \in \mathcal{W}$. (γ) If $\mathcal{V} \subseteq \mathcal{U}$ is a finite cover of X, there is a finite $\mathcal{V}' \subseteq \mathcal{U}$ which covers X and is a star-refinement of $\mathcal{V}$; now $W_{\mathcal{V}'} \circ W_{\mathcal{V}'} \subseteq W_{\mathcal{V}}$. So for any $W \in \mathcal{W}$ there is a $W' \in \mathcal{W}$ such that $W' \circ W' \subseteq W$. **Q**

(iii) The topologies $\mathfrak{T}_{\mathcal{U}}$, $\mathfrak{T}_{\mathcal{W}}$ induced on X by $\mathcal{U}$, $\mathcal{W}$ respectively are equal. **P** If $x \in X$ and $\mathcal{V} \subseteq \mathcal{U}$ is a finite cover of X, then $W_{\mathcal{V}}[\{x\}] = \bigcup\{V : x \in V \in \mathcal{V}\}$ is open for the topology induced by $\mathcal{U}$; so $\mathfrak{T}_{\mathcal{U}} \supseteq \mathfrak{T}_{\mathcal{W}}$. If $x \in G \in \mathfrak{T}_{\mathcal{U}}$ there is a $U \in \mathcal{U}$ such that $x \in U \subseteq G$; now there is a $V \in \mathcal{U}$ such that $x \notin V$ and $U \cup V = X$; and in this case $W_{\{U,V\}}[\{x\}] = U \subseteq G$. So $\mathfrak{T}_{\mathcal{U}} \subseteq \mathfrak{T}_{\mathcal{W}}$. **Q**

(iv) I will call $\mathcal{W}$ the **finite-cover uniformity** derived from $\mathcal{U}$.

(c) The definition in (b-ii) makes it plain that X is totally bounded for the finite-cover uniformity.

(d) Let X be a compact Hausdorff space.

(i) If $\mathcal{U}$ is a base for the topology of X closed under $\cup$ and $\cap$, then $\mathcal{U}$ is a normal topology base.

(ii) If $Y \subseteq X$ is dense, $\mathcal{U}$ is a base for the topology of X and $\mathcal{U}_Y = \{Y \cap U : U \in \mathcal{U}\}$ is a normal topology base on Y, then X can be identified with the completion of Y for the finite-cover uniformity induced by $\mathcal{U}_Y$.

3E Descriptions of $\tilde{X}$ The most important spaces of analysis come to us not as abstract sets but defined by some more or less straightforward construction, and we shall be very much happier if we can relate the space $\tilde{X}$, as defined above, to the construction leading to the space X. One reasonably general method leads through 'normal topology bases' as just defined.

Proposition Let $\mathbb{P}$ be a forcing notion, X a compact Hausdorff space and $\mathcal{U}$ a normal base for the topology of X. Let Z, $\tilde{X}$, $\dot{\varphi} : \check{X} \to \tilde{X}$ be as in §2.

(a)

$$\Vdash_{\mathbb{P}} \check{\mathcal{U}} \text{ is a normal topology base on } \check{X}.$$

(b)

$\Vdash_{\mathbb{P}}$ the embedding $\dot{\varphi} : \check{X} \to \tilde{X}$ identifies $\tilde{X}$, with the unique uniformity compatible with its topology, with the completion of $\check{X}$ with the finite-cover uniformity on $\check{X}$ generated by $\check{\mathcal{U}}$.

proof (a) As in 2Ac, we are dealing with a first-order property. In detail: suppose that $\dot{U}$ and $\dot{V}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{U}, \dot{V} \in \check{\mathcal{U}}$. Then there are U, $V \in \mathcal{U}$ and q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{U} = \check{U} \text{ and } \dot{V} = \check{V}.$$

In this case, $U \cup V$ and $U \cap V$ belong to $\mathcal{U}$ and

$$\Vdash_{\mathbb{P}} (\check{U} \cup \check{V}) = (U \cup V)\check{}, \ (\check{U} \cap \check{V}) = (U \cap V)\check{} \text{ belong to } \check{\mathcal{U}},$$

so

$$q \Vdash_{\mathbb{P}} \dot{U} \cup \dot{V}, \ \dot{U} \cap \dot{V} \text{ belong to } \check{\mathcal{U}}.$$

As p, $\dot{U}$ and $\dot{V}$ are arbitrary,

$$\Vdash_{\mathbb{P}} U \cup V \text{ and } U \cap V \text{ belong to } \check{\mathcal{U}} \text{ for all } U, V \in \check{\mathcal{U}}.$$

If $\dot{x}$, $\dot{U}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{U} \in \check{\mathcal{U}}$, then there are $q \in \mathbb{P}$, $x \in X$ and $G \in \mathcal{U}$ such that

$$q \Vdash_{\mathbb{P}} \check{x} = \dot{x} \in \dot{U} = \check{G},$$

so that $x \in G$. Set $V = X \setminus \{x\}$; then

$$q \Vdash_{\mathbb{P}} \check{V} \in \check{\mathcal{U}}, \ \dot{x} \notin \check{V}, \ \dot{U} \cup \check{V} = \tilde{X}.$$

As p, $\dot{x}$ and $\dot{U}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ if } x \in U \in \check{\mathcal{U}} \text{ there is a } V \in \check{\mathcal{U}} \text{ such that } x \notin V \text{ and } U \cup V = \check{X}.$$

Finally, if $\dot{U}$, $\dot{V}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \dot{U} \text{ and } \dot{V} \text{ belong to } \check{\mathcal{U}} \text{ and their union is } \check{X},$$

then there are a $q \in \mathbb{P}$, stronger than p, and U, $V \in \mathcal{U}$ such that

$$q \Vdash_{\mathbb{P}} \dot{U} = \check{U} \text{ and } \dot{V} = \check{V}.$$

In this case,

$$q \Vdash_{\mathbb{P}} (U \cup V)\check{} = \check{U} \cup \check{V} = \check{X},$$

so $U \cup V = X$. Because $\mathcal{U}$ is a normal topology base, there are disjoint U_1, $V_1 \in \mathcal{U}$ such that $U_1 \cup V = U \cup V_1 = X$. In this case

$$q \Vdash_{\mathbb{P}} \check{U}_1 \in \check{\mathcal{U}},\ \check{V}_1 \in \check{\mathcal{U}},\ \check{U}_1 \cap \check{V}_1 = \emptyset,\ \check{U}_1 \cup \dot{V} = \dot{U} \cup \check{V}_2 = \check{X}.$$

So the normality condition 3D(a-iii) is satisfied.

(b) This follows from 2Ad and 3D(d-ii).

3F For certain classes of topological space, we have an alternative route to $\tilde{X}$, as follows.

Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$, which I think of as the algebra of open-and-closed sets in Z; let X be a non-empty Hausdorff space.

(a)(i) For every $f \in C^-(Z;X)$ we have a sequentially order-continuous Boolean homomorphism $\pi_f : \mathcal{U}\widehat{\mathcal{B}}(X) \to \mathrm{RO}(\mathbb{P})$ defined by saying that $\pi_f(A)\triangle f^{-1}[A]$ is meager for every $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(ii) $\pi_f(A) = [\![\vec{f} \in \tilde{A}]\!]$ for any $f \in C^-(Z;X)$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(iii) π_f is **τ-additive** in the sense that if $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, then $\pi_f H = \sup_{G\in\mathcal{G}} \pi_f G$ in $\mathrm{RO}(\mathbb{P})$.

(iv) If f, $g \in C^-(Z;X)$ and $p \in \mathbb{P}$, then the following are equiveridical:

(α) f and g agree on $\widehat{p} \cap \operatorname{dom} f \cap \operatorname{dom} g$;

(β) $\widehat{p} \subseteq^* \operatorname{dom}(f \cap g)$;

(γ) for any t and for any q stronger than p, $(t,q) \in \vec{f}$ iff $(t,q) \in \vec{g}$;

(δ) $p \Vdash_{\mathbb{P}} \vec{f} = \vec{g}$;

(ϵ) $\widehat{p} \cap \pi_f A = \widehat{p} \cap \pi_g A$ for every $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$;

(ζ) there is a base $\mathcal{U}$ for the topology of X such that $\widehat{p} \cap \pi_f G = \widehat{p} \cap \pi_g G$ for every $G \in \mathcal{U}$.

(b)(i) Suppose that X is Čech-complete and that $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ is a sequentially order-continuous Boolean homomorphism which is τ-additive in the sense that $\pi(\bigcup\mathcal{G}) = \sup_{G\in\mathcal{G}} \pi G$ whenever $\mathcal{G} \subseteq \mathcal{B}\mathfrak{a}(X)$ is a family of open sets with union in $\mathcal{B}\mathfrak{a}(X)$. Then there is an $f \in C^-(Z;X)$ such that π_f extends π.

(ii) If X is compact, then for every sequentially order-continuous $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ there is an $f \in C(Z;X)$ such that π_f extends π.

(iii) If X is Polish, then for every sequentially order-continuous $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ there is an $f \in C^-(Z;X)$ such that π_f extends π.

(c) Suppose that X is Čech-complete and that $\pi : \mathcal{B}(X) \to \mathrm{RO}(\mathbb{P})$ is a τ-additive sequentially order-continuous Boolean homomorphism. Then there is an $f \in C^-(Z;X)$ such that π_f extends π.

proof (a)(i) If $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ then $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ so there is a unique open-and-closed set $\pi_f A \subseteq Z$ such that $\pi_f A \triangle f^{-1}[A]$ is meager.

Now π_f is sequentially order-continuous because it corresponds to the composition of the sequentially order-continuous maps $A \mapsto f^{-1}[A] : \mathcal{U}\widehat{\mathcal{B}}(X) \to \widehat{\mathcal{B}}(Z)$ and the canonical map from $\widehat{\mathcal{B}}(Z)$ to the category algebra of Z.

(ii) For $p \in \mathbb{P}$,

$$\widehat{p} \subseteq \pi_f A \iff \widehat{p} \subseteq^* f^{-1}[A] \iff p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A} \iff \widehat{p} \subseteq [\![\vec{f} \in \tilde{A}]\!];$$

as $\{\widehat{p} : p \in \mathbb{P}\}$ is order-dense in $\mathrm{RO}(\mathbb{P})$, this gives the result.

(iii) $\bigcup_{G\in\mathcal{G}} \pi_f G \supseteq \bigcup_{G\in\mathcal{G}} f^{-1}[G] = f^{-1}[H]$ is dense in $\pi_f H$.

(iv)(α)$\Longrightarrow$(β) because $\operatorname{dom} f \cap \operatorname{dom} g$ is comeager.

(β)$\Longrightarrow$(γ) If $h \in C^-(Z;X)$ and q is stronger than p,

$$(\check{h}, q) \in \vec{f} \iff \widehat{q} \subseteq^* \operatorname{dom}(f \cap h) \iff \widehat{q} \subseteq^* \operatorname{dom}(g \cap h)$$

(because $\widehat{q} \setminus \operatorname{dom}(f \cap g)$ is meager)

$$\iff (\check{h}, q) \in \vec{g}.$$

(γ)⟹(δ) Use 2A(b-i).

(δ)⟹(ϵ) follows from (ii).

(ϵ)⟹(ζ) is trivial.

(¬α)⟹(¬ζ) Let $z \in \widehat{p} \cap \operatorname{dom} f \cap \operatorname{dom} g$ be such that $f(z) \neq g(z)$; let G, $H \in \mathcal{U}$ be disjoint sets containing $f(z)$, $g(z)$ respectively. Then $\widehat{p} \cap f^{-1}[G] \cap g^{-1}[H]$ is not empty and is the intersection of the dense Gϵ set $\operatorname{dom} f \cap \operatorname{dom} g$ with an open set, so is non-meager, and $\widehat{p} \cap \pi_f G \cap \pi_g H$ is non-empty; but $\pi_f G \cap \pi_f H = \emptyset$, so $\widehat{p} \cap \pi_f H \neq \widehat{p} \cap \pi_g H$.

(b)(i) Set

$$g = \bigcap_{G \subseteq X \text{ is a cozero set}} (\pi G \times X) \cup (Z \times (X \setminus G)).$$

(α) g is a function. **P?** If (z, x) and (z, y) both belong to g, where $x \neq y$, let G, $H \subseteq X$ be disjoint cozero sets containing x, y respectively. Then $(z, x) \in (\pi G \times X) \cup (Z \times (X \setminus G))$ so $z \in \pi G$; similarly, $z \in \pi H$; but $\pi G \cap \pi H = \pi(G \cap H)$ is empty. **XQ**

(β) If $G \subseteq X$ is a cozero set then $g^{-1}[G] = \pi G \cap \operatorname{dom} g$. **P** By the definition of g, $z \in \pi G$ whenever $x \in G$ and $(z, x) \in g$, that is, $\pi G \cap \operatorname{dom} g \subseteq g^{-1}[G]$. In the other direction, if $g(z) \in G$, there are disjoint cozero sets H, H' such that $g(z) \in H$ and $X \setminus G \subseteq H'$. Now $x \notin H'$ so $z \notin \pi H'$ and $z \notin \pi(X \setminus G)$ and $z \in \pi G$. As z is arbitrary, $g^{-1}[G] \subseteq \pi G \cap \operatorname{dom} g$. **Q**

It follows that g is continuous.

(γ) Express X as $\bigcap_{n \in \mathbb{N}} H_n$ where $\langle H_n \rangle_{n \in \mathbb{N}}$ is a sequence of open sets in a compact Hausdorff space Y. For each $n \in \mathbb{N}$, let $\mathcal{G}_n$ be the family of those cozero subsets G of X for which there is a zero set $F \subseteq Y$ such that $G \subseteq F \subseteq H_n$; then $\bigcup \mathcal{G}_n = X$, so $V_n = \bigcup_{G \in \mathcal{G}} \pi G$ is dense in Z. (This is where I use the hypothesis that π is τ-additive.) Set $V = \bigcap_{n \in \mathbb{N}} V_n$, so that V is a dense G$_\delta$ set in Z. Now $V \subseteq \operatorname{dom} g$. **P** Take $z \in V$ and consider the family $\mathcal{E}$ of zero sets $F \subseteq Y$ such that $z \in \pi(F \cap X)$. $\mathcal{E}$ is downwards-directed so there is a $y \in \bigcap \mathcal{E}$. For each $n \in \mathbb{N}$ there is a $G \in \mathcal{G}_n$ such that $z \in \pi G$ so there is an $F \in \mathcal{E}$ such that $F \subseteq H_n$ and $y \in \mathcal{H}_n$; accordingly $y \in X$. **?** If $(z, y) \notin g$, let $G \subseteq X$ be a cozero set such that $z \notin \pi G$ and $y \notin X \setminus G$. Then there is a cozero set $H \subseteq Y$ such that $y \in H$ and $H \cap X \subseteq G$. Now $z \in \pi(X \setminus G)$ (recall that Z is extremally disconnected, so RO($\mathbb{P}$) is just the algebra of open-and-closed subsets of Z, and its Boolean operations agree with those of $\mathcal{P}Z$, so $z \in \pi(X \setminus H)$ and $Y \setminus H \in \mathcal{E}$; but this is impossible. **X** Thus $g(z) = y$ and $z \in \operatorname{dom} g$. **Q**

(δ) Thus g is a continuous function with a comeager domain, and there is an $f \in C^-(Z; X)$ such that $f \subseteq g$. By (β), $f^{-1}[G] = \pi G \cap \operatorname{dom} f$ for every cozero $G \subseteq X$, so that $\pi_f G = \pi G$ for cozero sets G. By the Monotone Class Theorem, $\pi_f E = \pi E$ for every $E \in \mathcal{B}\mathfrak{a}(X)$.

(ii) If X is compact, then every open set in $\mathcal{B}\mathfrak{a}(X)$ is actually a cozero set (Fremlin 03, 4A3Xc) so is σ-compact, therefore Lindelöf. What this means is that if $\mathcal{G}$ is a family of cozero sets and $\bigcup \mathcal{G}$ is a cozero set, then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union $\bigcup \mathcal{G}$; as π is sequentially order-continuous,

$$\pi(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}_0} \pi G = \sup_{G \in \mathcal{G}} \pi G.$$

So π is τ-additive and there is an $f \in C^-(Z; X)$ such that $\pi = \pi_f \restriction \mathcal{B}\mathfrak{a}(X)$, Because X is compact, f extends to a member of $C(Z; X)$ with the same property.

(iii) This time, X is hereditarily Lindelöf so we can again apply (i).

(c) By (b), there is an $f \in C^-(Z; X)$ such that π_f extends $\pi \restriction \mathcal{B}\mathfrak{a}(X)$. Because π_f and π are both τ-additive and agree on a base for the topology of X, they agree on the open sets in X and therefore on $\mathcal{B}(X)$.

3G Notation Suppose that X is either compact or Polish, $\mathbb{P}$ is a forcing notion and $\pi : \mathcal{B}\mathfrak{a}(X) \to \text{RO}(\mathbb{P})$ is a sequentially order-continuous Boolean homomorphism. Then 3Fb tells us that we have a $\mathbb{P}$-name $\breve{\pi}$ defined by saying that $\breve{\pi} = \vec{f}$ whenever $f \in C^-(Z; X)$ and $\pi \subseteq \pi_f$. Now, of course, $\Vdash_{\mathbb{P}} \breve{\pi} \in \tilde{X}$; moreover, $[\![\breve{\pi} \in \tilde{F}]\!] = \pi F$ for every Baire set $F \subseteq X$. The following fact will be useful.

3H Proposition Suppose that X is either compact or Polish, $\mathbb{P}$ is a forcing notion and π, $\phi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ are sequentially order-continuous Boolean homomorphisms. Then, for any $p \in \mathbb{P}$, the following are equiveridical:

(i) $p \Vdash_{\mathbb{P}} \check{\pi} = \check{\phi}$;

(ii) $\widehat{p} \cap \pi E = \widehat{p} \cap \phi E$ for every $E \in \mathcal{B}\mathfrak{a}(X)$;

(iii) there is a base $\mathcal{U}$ for the topology of X, consisting of cozero sets, such that $\widehat{p} \cap \pi U = \widehat{p} \cap \phi U$ for every $U \in \mathcal{U}$.

proof Use 3F(a-iv).

4 Preservation of topological properties

4A Theorem Let $\mathbb{P}$, $(X, \mathfrak{T})$ and $\tilde{X}$ be as in §2A.

(a) If X is regular, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is regular.}$$

(b) If X is completely regular, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is completely regular.}$$

(c) If X is compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is compact.}$$

(d) If X is separable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is separable.}$$

(e) If X is metrizable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is metrizable.}$$

(f) If X is Čech-complete, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Čech-complete.}$$

(g) If X is Polish, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Polish.}$$

(h) If X is locally compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is locally compact.}$$

(i) If $\operatorname{ind} X \le n \in \mathbb{N}$, where $\operatorname{ind} X$ is the small inductive dimension of X, then

$$\Vdash_{\mathbb{P}} \operatorname{ind} \tilde{X} \le n.$$

(In particular, if X is zero-dimensional then $\Vdash_{\mathbb{P}} \tilde{X}$ is zero-dimensional.)

(j) If X is chargeable, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is chargeable.}$$

proof As in §2A, let Z be the Stone space of $\mathrm{RO}(\mathbb{P})$ and $C^-(Z; X)$ the set of continuous functions from dense G_δ subsets of Z to X.

(a) Let $\dot{x}$, $\dot{G}$ be $\mathbb{P}$-names and $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X} \text{ and } \dot{x} \in \dot{G}.$$

Then there are q stronger than p, $f \in C^-(Z; X)$ and $U \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{U} \subseteq \dot{G}.$$

Now $U = \bigcup\{V : V \in \mathfrak{T}, \overline{V} \subseteq U\}$, while $\widehat{q} \subseteq^* f^{-1}[U]$, so there are an open set V such that $\overline{V} \subseteq U$ and an r stronger than q such that $\widehat{r} \subseteq^* f^{-1}[V]$. Set $W = X \setminus \overline{V}$; then

$$r \Vdash_{\mathbb{P}} \dot{x} \in \tilde{V},\ \tilde{V} \cap \tilde{W} = \emptyset,\ \dot{G} \cup \tilde{W} = \tilde{X},$$

so that

$$r \Vdash_{\mathbb{P}} \dot{x} \text{ belongs to an open set with closure included in } \dot{G}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \text{ belongs to an open set with closure included in } \dot{G};$$

as p, $\dot{x}$ and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{the topology of } \tilde{X} \text{ is regular.}$$

(b)(i) Let $\mathcal{W}$ be a uniformity on X defining its topology, and $\mathcal{W}_0$ the family of those members of $\mathcal{W}$ which are open for the product topology of $X \times X$. For $W \in \mathcal{W}_0$ let $\tilde{W}$ be the $\mathbb{P}$-name

$$\{((\vec{f}, \vec{g}), p) : p \in \mathbb{P},\ f,\ g \in C^-(Z; X),$$
$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ (f(z), g(z)) \in W\}\}.$$

Then we have to check the following:

$$\Vdash_{\mathbb{P}} \tilde{W} \subseteq \tilde{X} \times \tilde{X}$$

for every $W \in \mathcal{W}_0$;

$$\Vdash_{\mathbb{P}} \tilde{W}_0 \cap \tilde{W}_1 = (W_0 \cap W_1)^{\sim}$$

whenever W_0, $W_1 \in \mathcal{W}_0$;

$$\Vdash_{\mathbb{P}} (\tilde{W})^{-1} = (W^{-1})^{\sim}$$

for every $W \in \mathcal{W}_0$;

$$\Vdash_{\mathbb{P}} \tilde{W}_0 \circ \tilde{W}_0 \subseteq \tilde{W}$$

whenever W_0, $W \in \mathcal{W}_0$ and $W_0 \circ W_0 \subseteq W$. These are all easy. Now, setting

$$\tilde{\mathcal{W}} = \{(\tilde{W}, \mathbb{1}) : W \in \mathcal{W}_0\},$$

we have

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{W}} \text{ is a filter base on } \tilde{X} \times \tilde{X}, \text{ and the filter it generates is a uniformity on } \tilde{X}.$$

(ii) Now

$$\Vdash_{\mathbb{P}} \text{the uniformity generated by } \tilde{\mathcal{W}} \text{ is finer than the given topology on } \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{G}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is open in } \tilde{X} \text{ and } \dot{x} \in \dot{G}.$$

Let q, stronger than p, and $f \in C^-(Z; X)$, $G \in \mathfrak{T}$ be such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \in \tilde{G} \subseteq \dot{G}.$$

Now

$$G = \bigcup \{H : W \in \mathcal{W}_0,\ H \in \mathfrak{T},\ W[H] \subseteq G\},$$

so we have r stronger than q, $W \in \mathcal{W}_0$ and $H \in \mathfrak{T}$ such that $W[H] \subseteq G$ and $\widehat{r} \subseteq^* f^{-1}[H]$. Suppose now that we have s stronger than r and a $\mathbb{P}$-name $\dot{y}$ such that $s \Vdash_{\mathbb{P}} \dot{y} \in \tilde{W}[\vec{f}]$. Then we have a t stronger than s and a $g \in C^-(Z; X)$ such that $t \Vdash_{\mathbb{P}} \dot{y} = \vec{g}$ and $\widehat{t} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ (f(z), g(z)) \in W\}$. Now

$$\widehat{t} \setminus g^{-1}[G] \supseteq \widehat{t} \setminus (\{z : (f(z), g(z) \in W\} \cap f^{-1}[H])$$

is meager, so $t \Vdash_{\mathbb{P}} \dot{y} \in \dot{G}$. As s and $\dot{y}$ are arbitrary,

$$r \Vdash_{\mathbb{P}} \tilde{W}[\dot{x}] \subseteq \dot{G}.$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \text{there is a } W \text{ in the uniformity such that } W[\dot{x}] \subseteq \dot{G}.$$

As p, $\dot{x}$ and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{the topology defined by the uniformity is finer than the given topology on } \tilde{X}. \quad \mathbf{Q}$$

(iii) Next, if $W \in \mathcal{W}_0$ and $f \in C^-(Z;X)$ then

$$\Vdash_{\mathbb{P}} \tilde{W}[\{\vec{f}\}] \text{ is open in } \tilde{X}.$$

$\mathbf{P}$ Let $p \in \mathbb{P}$ and $\dot{y}$ be such that

$$p \Vdash_{\mathbb{P}} \dot{y} \in \tilde{W}[\{\vec{f}\}].$$

Then there are q stronger than p and $g \in C^-(Z;X)$ such that $\widehat{q} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g, (f(z), g(z)) \in W\}$ and $q \Vdash_{\mathbb{P}} \dot{y} = \vec{g}$. Now W_0 is open in $X \times X$, so whenever $(f(z), g(z)) \in W_0$ there are H_0, $H_1 \in \mathfrak{T}$ such that $(f(z), g(z)) \in H_0 \times H_1 \subseteq W$. We can therefore find H_0, $H_1 \in \mathfrak{T}$ such that $H_0 \times H_1 \subseteq W$, and r stronger than q, such that $\widehat{r} \subseteq^* (f^{-1}[H_0] \cap g^{-1}[H_1])$. In this case

$$r \Vdash_{\mathbb{P}} \dot{y} \in \tilde{H}_1,\ \tilde{H}_1 \text{ is open and } (\vec{f}, \dot{z}) \in \tilde{W} \text{ whenever } \dot{z} \in \tilde{H}_1,$$

so that

$$r \Vdash_{\mathbb{P}} \dot{y} \in \operatorname{int} \tilde{W}[\vec{f}].$$

As q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{y} \in \operatorname{int} \tilde{W}[\vec{f}];$$

as p and $\dot{y}$ are arbitrary,

$$p \Vdash_{\mathbb{P}} \tilde{W}[\vec{f}] \text{ is open.} \quad \mathbf{Q}$$

(iv) It follows that

$$\Vdash_{\mathbb{P}} \text{the topology generated by the uniformity is coarser than the given topology on } \tilde{X}.$$

$\mathbf{P}$ This time, take $p \in \mathbb{P}$ and $\mathbb{P}$-names $\dot{x}$, $\dot{G}$ such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is open for the topology generated by the uniformity and } \dot{x} \in \dot{G}.$$

Let q stronger than p and $f \in C^-(Z;X)$, $W \in \mathcal{W}_0$ be such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \tilde{W}[\{\vec{f}\}] \subseteq \dot{G}.$$

Then (iii) tells us that

$$q \Vdash_{\mathbb{P}} \dot{x} \in \operatorname{int} \dot{G};$$

as q is arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{x} \in \operatorname{int} \dot{G};$$

as p, $\dot{G}$ and $\dot{x}$ are arbitrary, we have the result. $\mathbf{Q}$

(v) Thus

$$\Vdash_{\mathbb{P}} \text{the topology of } \tilde{X} \text{ is generated by a uniformity and is completely regular.}$$

(c) This follows at once from 2G, since if X is compact then $Z \times X$ is usco-compact.

(d) This follows at once from 2A(f-iii), because if D is countable then $\Vdash_{\mathbb{P}} \check{D}$ is countable.

(e) Use the ideas of (b) to show that

$$\Vdash_{\mathbb{P}} \text{the topology of } \tilde{X} \text{ is generated by a uniformity with a countable base.}$$

(f) If X is homeomorphic to a G_δ subset B of a compact Hausdorff space Y, then the results above tell us that

$\Vdash_{\mathbb{P}} \tilde{Y}$ is compact, $\tilde{B}$ is a G_δ subset of $\tilde{Y}$ and $\tilde{X}$ is homeomorphic to $\tilde{B}$, therefore Čech-complete.

(g) A topological space is Polish iff it is separable, metrizable and Čech-complete.

(h) Let $\mathcal{U}$ be a base for the topology of X consisting of relatively compact open sets. Then 2Ad tells us that

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is an open cover of } \tilde{X}.$$

But we also have

$$\Vdash_{\mathbb{P}} \text{every member of } \tilde{\mathcal{U}} \text{ is relatively compact in } \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{U}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{U} \in \tilde{\mathcal{U}}$. Then there are a q stronger than p and a $U \in \mathcal{U}$ such that $q \Vdash_{\mathbb{P}} \dot{U} = \tilde{U}$. Consider $K = \overline{U}$ and the identity embedding $\phi : K \to X$. I think I need to distinguish for a moment between K, thought of as a topological space standing alone, from itself thought of as a subspace of X; I write L for the latter incarnation. We know from (c) that

$$\Vdash_{\mathbb{P}} \tilde{K} \text{ is compact},$$

from 2Ab that

$$\Vdash_{\mathbb{P}} \tilde{U} \subseteq \tilde{L},$$

and from 2Cg that

$$\Vdash_{\mathbb{P}} \tilde{K} \text{ is homeomorphic to } \tilde{L} \text{ with its subspace topology.}$$

So

$$q \Vdash_{\mathbb{P}} \dot{U} \subseteq \tilde{L} \text{ is relatively compact.}$$

As p and $\dot{U}$ are arbitrary, we have the result. **Q**

So

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is covered by a family of relatively compact open sets and is locally compact.}$$

(i) Induce on n. If $\operatorname{ind} X \le -1$ then $X = \emptyset$ so

$$\Vdash_{\mathbb{P}} \tilde{X} = \emptyset \text{ and } \operatorname{ind} \tilde{X} = -1.$$

For the inductive step to $n \ge 0$, suppose that $\operatorname{ind} X \le n$. Then there is a base $\mathcal{U}$ for $\mathfrak{T}$ such that $\operatorname{ind}(\partial U) < n$ for every $U \in \mathcal{U}$. Now

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ is a base for the topology of } \tilde{X}.$$

If $p \in \mathbb{P}$ and $\dot{U}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{U} \in \tilde{U}$, then there are a q stronger than p and a $U \in \mathcal{U}$ such that $q \Vdash_{\mathbb{P}} \dot{U} = \tilde{U}$. So

$$q \Vdash_{\mathbb{P}} \operatorname{ind}(\partial\dot{U}) = \operatorname{ind}(\partial\tilde{U}) = \operatorname{ind}(\partial U)^{\sim} < n$$

by 2B, the inductive hypothesis and 2Cg. As p and $\dot{U}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \operatorname{ind}(\partial U) < n \text{ for every } U \in \tilde{U}, \text{ so } \operatorname{ind} \tilde{X} \le n,$$

and the induction continues.

(j) Recall that X is 'chargeable' if there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu G > 0$ for every non-empty open $G \subseteq X$. It is easy to check (using Kelley's theorem, Fremlin 02, 391J) that X is chargeable iff there is a base $\mathcal{U}$ for its topology which is expressible as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where the intersection number of each $\mathcal{U}_n$ is at least 2^{-n}. In this case, writing $\tilde{\mathcal{U}}_n = \{(\tilde{U}, \mathbb{1}) : U \in \mathcal{U}_n\}$ for each n,

$$\Vdash_{\mathbb{P}} \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{U}}_n \text{ is a base for the topology of } \tilde{X}.$$

But we also have, for any $n \in \mathbb{N}$,

$$\Vdash_{\mathbb{P}} \text{the intersection number of } \tilde{\mathcal{U}}_n \text{ is at least } 2^{-n}.$$

P Suppose that $p \in \mathbb{P}$, $m \in \mathbb{N}$ and $\dot{U}_0, \ldots, \dot{U}_{m-1}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{U}_i \in \tilde{\mathcal{U}}_n \text{ for every } i < m.$$

Then there are a q stronger than p and $U_0, \ldots, U_{m-1} \in \mathcal{U}_n$ such that

$$q \Vdash_{\mathbb{P}} \dot{U}_i = \tilde{U}_i \text{ for every } i < m.$$

Now there is a $J \subseteq m$ with $\#(J) \geq 2^{-n}m$ such that $\bigcap_{i\in J} U_i \neq \emptyset$, in which case

$$q \Vdash_{\mathbb{P}} \check{J} \subseteq m,\ \#(\check{J}) \geq 2^{-n}m \text{ and } \bigcap_{i\in\check{J}} \dot{U}_i \neq \emptyset.$$

As p and $\dot{U}_0, \ldots, \dot{U}_{m-1}$ are arbitrary, we have the result. **Q**

So

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is chargeable.}$$

4B Corollary Let X be a zero-dimensional compact Hausdorff space, and $\mathcal{E}$ the algebra of open-and-closed sets in X. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with the Stone space of the Boolean algebra } \check{\mathcal{E}}.$$

proof Note that $\mathcal{E}$ is a normal base for the topology of X. By 3Ea and 2Ad,

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{E}} \text{ is a normal base for the topology of } \tilde{X},$$

and of course

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{E}} \text{ is an algebra of subsets of } \tilde{X} \text{ and } \tilde{X} \text{ is compact and Hausdorff}$$

by 2A(b-iv) and 4c. It follows at once that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is zero-dimensional and } \tilde{\mathcal{E}} \text{ is its algebra of open-and-closed sets.}$$

Since also

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{E}} \text{ is isomorphic, as Boolean algebra, to } \check{\mathcal{E}},$$

we have

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with the Stone space of } \check{\mathcal{E}}.$$

4C Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$; let X be a topological group.

(a) We have a $\mathbb{P}$-name for a group operation on $\tilde{X}$, defined by saying that

$$\Vdash_{\mathbb{P}} \vec{f} \cdot \vec{g} = \vec{h}$$

whenever f, g, $h \in C^-(Z; X)$ and $h(z) = f(z)g(z)$ for every $z \in \operatorname{dom} f \cap \operatorname{dom} g$; and now

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is a topological group with identity } \tilde{e}$$

where e is the identity of X.

(b)(i) For any $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{A}^{-1} = (A^{-1})^{\sim}.$$

(ii) For any $a \in X$ and $B \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{a} \cdot \tilde{B} = (aB)^{\sim},\ \tilde{B} \cdot \tilde{a} = (Ba)^{\sim}.$$

(iii) For any open set $G \subseteq X$ and $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$,

$$\Vdash_{\mathbb{P}} \tilde{G} \cdot \tilde{A} = (GA)^{\sim},\ \tilde{A} \cdot \tilde{G} = (AG)^{\sim}.$$

proof a) Let $\phi : X \times X \to X$ and $\psi : X \to X$ be the operations of multiplication and inversion. These are continuous, so we have corresponding names $\tilde{\phi}$, $\tilde{\psi}$ such that

$$\Vdash_{\mathbb{P}} \tilde{\phi} : (X \times X)^{\sim} \to \tilde{X},\ \tilde{\psi} : \tilde{X} \to \tilde{X} \text{ are continuous.}$$

Now the identification

$$\Vdash_{\mathbb{P}} (X \times X)^{\sim} \equiv \tilde{X}$$

and the definition of $\tilde{\phi}$ make it plain that

$$\Vdash_{\mathbb{P}} \tilde{\phi}(\vec{f}, \vec{g}) = \vec{h}$$

iff $\{z : z \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h,\ f(z)g(z) = h(z)\}$ is comeager. It is now elementary to check that

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ acts as a group operation on } \tilde{X}, \text{ with inversion function } \tilde{\psi} \text{ and identity } \tilde{e}.$$

(b)(i) Because inversion is a homeomorphism, $\tilde{A} \in \mathcal{U}\widehat{\mathcal{B}}(X)$. The point is just that if $f \in C^{-}(Z; X)$ and $g(z) = f(z)^{-1}$ for $z \in \operatorname{dom} f$, then

$$\Vdash_{\mathbb{P}} (\vec{f})^{-1} = \vec{g},$$

so that, for any $p \in \mathbb{P}$,

$$\begin{aligned} p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{A}^{-1} &\iff p \Vdash_{\mathbb{P}} \vec{g} \in \tilde{A} \\ &\iff \widehat{p} \subseteq^{*} g^{-1}[A] = f^{-1}[A^{-1}] \iff p \Vdash_{\mathbb{P}} \vec{f} \in (A^{-1})^{\sim}. \end{aligned}$$

(ii) Note that as $x \mapsto ax$ is a homeomorphism, aB certainly belongs to $\mathcal{U}\widehat{\mathcal{B}}(X)$. If $\dot{x}$ is a $\mathbb{P}$-name and $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$, let $f \in C^{-}(Z; X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Set $g(z) = a^{-1} f(z)$ for $z \in \operatorname{dom} f$. Then

$$p \Vdash_{\mathbb{P}} \vec{g} = \tilde{a}^{-1} \cdot \vec{f}.$$

So

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{a} \cdot \tilde{B} &\iff p \Vdash_{\mathbb{P}} \vec{g} = \tilde{a}^{-1} \cdot \vec{f} \in \tilde{B} \\ &\iff \widehat{p} \subseteq^{*} \{z : g(z) \in B\} = \{z : f(z) \in aB\} \\ &\iff p \Vdash_{\mathbb{P}} \dot{x} \in (aB)^{\sim}. \end{aligned}$$

As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \tilde{a} \cdot \tilde{B} = (aB)^{\sim}$. Similarly, $\Vdash_{\mathbb{P}} \tilde{B} \cdot \tilde{a} = (Ba)^{\sim}$.

(iii)(α) Suppose that $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G} \cdot \tilde{A}$. Then there must be $\mathbb{P}$-names $\dot{y}_1$, $\dot{y}_2$ such that

$$p \Vdash_{\mathbb{P}} \dot{y}_1 \in \tilde{G},\ \dot{y}_2 \in \tilde{A} \text{ and } \dot{y}_1 \dot{y}_2 = \dot{x}.$$

Let g_1, $g_2 \in C^{-}(Z; X)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{y}_1 = \vec{g}_1 \text{ and } \dot{y}_2 = \vec{g}_2,$$

and set $f(z) = g_1(z) g_2(z)$ for $z \in \operatorname{dom} g_1 \cap \operatorname{dom} g_2$. Then $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$ and

$$\widehat{p} \subseteq^{*} g_1^{-1}[G] \cap g_2^{-1}[A] \subseteq f^{-1}[GA],$$

so $p \Vdash_{\mathbb{P}} \dot{x} \in (GA)^{\sim}$. As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{G} \cdot \tilde{A} \subseteq (GA)^{\sim}.$$

(β) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in (GA)^{\sim}$. Let $f \in C^{-}(Z; X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Then $\widehat{p} \subseteq^{*} f^{-1}[GA]$. Take any $z_0 \in \widehat{p} \cap f^{-1}[GA]$. Then we can express $f(z_0)$ as $y_1 y_2$ where $y_1 \in G$ and $y_2 \in A$. Set $g(z) = f(z) y_2^{-1}$ for $z \in \operatorname{dom} f$, so that $g \in C^{-}(Z; X)$. Because g is continuous and G is open, there is a neighbourhood V of z_0 such that $g(z) \in G$ whenever $z \in V \cap \operatorname{dom} g$. Let q stronger than p be such that $\widehat{q} \subseteq V$. Then

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} = \vec{g} \cdot \tilde{y}_2 \in \tilde{G} \cdot \tilde{A}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \tilde{G} \cdot \tilde{A} \supseteq (GA)^{\sim}.$$

(γ) Similarly,

$$\Vdash_{\mathbb{P}} \tilde{A} \cdot \tilde{G} = (AG)^{\sim}.$$

4D Examples Let $\mathbb{P}$ be a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$.

(a) Suppose that X is a totally ordered set with its order topology. Let $\tilde{\le}$ be the $\mathbb{P}$-name

$$\{((\vec{f}, \vec{g}), p) : f, g \in C^{-}(Z; X),\ p \in \mathbb{P},$$
$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) \le g(z)\}\}.$$

(i) $\tilde{\le}$ is a $\mathbb{P}$-name for a total ordering of $\tilde{X}$. **P** Let $\dot{x}$, $\dot{y}$ and $\dot{z}$ be $\mathbb{P}$-names, and $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y},\ \dot{z} \in \tilde{X}.$$

Then there are f, g, $h \in C^{-}(Z; X)$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f},\ \dot{y} = \vec{g} \text{ and } \dot{z} = \vec{h}.$$

(α)

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \,\tilde{\le}\, \vec{f} = \dot{x}.$$

(β) If

$$p \Vdash_{\mathbb{P}} \dot{x} \,\tilde{\le}\, \dot{y} \text{ and } \dot{y} \,\tilde{\le}\, \dot{z},$$

then

$$\widehat{p} \setminus (\{z : z \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h,\ f(z) \le g(z) \le h(z)\}$$

is the union of two meager sets and is meager, so

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \,\tilde{\le}\, \vec{h} = \dot{z}.$$

(γ) If

$$p \Vdash_{\mathbb{P}} \dot{x} \,\tilde{\le}\, \dot{y} \text{ and } \dot{y} \,\tilde{\le}\, \dot{x},$$

then

$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) \le g(z),\ g(z) \le f(z)\}),$$

so

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} = \vec{g} = \dot{y}.$$

(δ) At least one of the Borel sets $\widehat{p} \cap \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) \le g(z)\}$, $\widehat{p} \cap \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ g(z) \le f(z)\}$ is non-meager, suppose the former; then it essentially includes $\widehat{q}$ for some q stronger than p, and in this case

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \,\tilde{\le}\, \vec{g} = \dot{y}.$$

As p, $\dot{x}$, $\dot{y}$ and $\dot{z}$ are arbitrary, we have the result. **Q**

(ii) Now

$$\Vdash_{\mathbb{P}} \text{the order topology defined by } \tilde{\le} \text{ is the topology on } \tilde{X} \text{ generated by } \tilde{\mathfrak{T}}.$$

P (α) Suppose that $\dot{x}$, $\dot{U}$ are $\mathbb{P}$-names and that $p \in \mathbb{P}$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{U} \in \tilde{\mathfrak{T}}$. Then there are q stronger than p and $f \in C^{-}(Z; X)$, $U \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \vec{f} = \dot{x} \in \dot{U} = \tilde{U}.$$

The open set $f^{-1}[U]$ is the union of sets of the form $f^{-1}[\,]u,v[\,]$ where u, $v\in X\cup\{-\infty,\infty\}$ and $]u,v[\,\subseteq U$, so there are such u, v such that $\widehat{q}\cap f^{-1}[\,]u,v[\,]$ is non-meager, that is, there is an r stronger than q such that $f[\widehat{r}]\subseteq\,]u,v[$. Now

$$r\Vdash_{\mathbb{P}}\dot{x}=\vec{f}\in\,]\tilde{u},\tilde{v}[\,\subseteq\tilde{U}=\dot{U}$$

(interpreting $\{x:\tilde{u}<x\}$ as $\tilde{X}$ if $u=-\infty$, of course). As p, $\dot{x}$ and $\dot{G}$ are arbitrary,

> $\Vdash_{\mathbb{P}} G$ is the union of the open intervals it includes, for every G in the topology generated by $\tilde{\mathfrak{T}}$,

and

$$\Vdash_{\mathbb{P}}\text{ the topology generated by }\tilde{\mathfrak{T}}\text{ is coarser than the order topology.}$$

In the other direction, if $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names and

$$p\Vdash_{\mathbb{P}}\dot{x},\,\dot{y}\in\tilde{X},\,\dot{x}\mathrel{\dot{<}}\dot{y},$$

take f, $g\in C^-(Z;X)$ such that

$$p\Vdash_{\mathbb{P}}\vec{f}=\dot{x}\mathrel{\dot{<}}\dot{y}=\vec{g}.$$

Then there is a $u\in X$ such that $\widehat{p}\cap\{z:z\in\operatorname{dom}f\cap\operatorname{dom}g,\ f(z)<u<g(z)\}$ is not meager, so essentially includes $\widehat{q}$ for some q stronger than p. In this case, setting $G=\,]u,\infty[\,\in\mathfrak{T}$,

$$q\Vdash_{\mathbb{P}}\dot{y}=\vec{g}\in\tilde{G}\subseteq\Big]\vec{f},\infty\Big[=\,]\dot{x},\infty[,$$

where of course we interpret $\Big]\vec{f},\infty\Big[$ and $]\dot{x},\infty[$ in $V^{\mathbb{P}}$. As p, $\dot{x}$ and $\dot{y}$ are arbitrary,

> $\Vdash_{\mathbb{P}}$ for every $x\in\tilde{X}$, $]x,\infty[$ is the union of the members of the topology generated by $\tilde{\mathfrak{T}}$ which it includes.

Similarly,

> $\Vdash_{\mathbb{P}}$ for every $x\in\tilde{X}$, $]-\infty,x[$ is the union of the members of the topology generated by $\tilde{\mathfrak{T}}$ which it includes,

and

$$\Vdash_{\mathbb{P}}\text{ the topology generated by }\tilde{\mathfrak{T}}\text{ is finer than the order topology. }\mathbf{Q}$$

(iii) For any f, $g\in C^{-1}(Z;X)$, $f(z)\le g(z)$ for every $z\in\operatorname{dom}f\cap\operatorname{dom}g\cap[\![\vec{f}\mathrel{\tilde{\le}}\vec{g}]\!]$. **P?** Otherwise, there is a non-empty open set $U\subseteq[\![\vec{f}\mathrel{\tilde{\le}}\vec{g}]\!]$ such that $g(z)<f(z)$ for every $z\in U\cap\operatorname{dom}f\cap\operatorname{dom}g$. Let $p\in\mathbb{P}$ be such that $\widehat{p}\subseteq U$. Then $p\Vdash_{\mathbb{P}}\vec{f}\mathrel{\tilde{\le}}\vec{g}$ so there must be a q compatible with p and f_1, $g_1\in C^-(Z;X)$ such that

$$((\vec{f}_1,\vec{g}_1),q)\in\tilde{\le},\quad q\Vdash_{\mathbb{P}}\vec{f}_1=\vec{f}\text{ and }\vec{g}_1=\vec{g}.$$

But now $\widehat{p}\cap\widehat{q}$ is a non-empty open set included in

$$\{z:f_1(z)=f(z)\}\cap\{z:g_1(z)=g(z)\}\cap\{z:f_1(z)\le g_1(z)\}\cap\{z:g(z)<f(z)\}$$

which is meager. **XQ**

(iv) In the language of 2Af,

$$\Vdash_{\mathbb{P}}\dot{\varphi}[\check{X}]\text{ is cofinal and coinitial with }\tilde{X}.$$

P Suppose that $p\in\mathbb{P}$ and a $\mathbb{P}$-name $\dot{x}$ are such that $p\Vdash_{\mathbb{P}}\dot{x}\in\tilde{X}$. Let $f\in C^-(Z;X)$ be such that $p\Vdash_{\mathbb{P}}\dot{x}=\tilde{f}$. Take any $z_0\in\widehat{p}\cap\operatorname{dom}f$. If $f(z_0)$ is the greatest element of X, then, writing e_{z_0} for the constant function with value z_0,

$$\Vdash_{\mathbb{P}}\tilde{f}\mathrel{\tilde{\le}}\tilde{e}_{z_0}=\dot{\varphi}\check{z}_0$$

and $p\Vdash_{\mathbb{P}}\dot{x}\mathrel{\tilde{\le}}\dot{\varphi}\check{z}_0$. Otherwise, take any $y>f(z_0)$; then $\{z:z\in\widehat{p}\cap\operatorname{dom}f,\ f(z)<y\}$ is a non-empty relatively open set in $\operatorname{dom}f$, so includes $\widehat{q}\cap\operatorname{dom}f$ for some q stronger than p, and

$$q\Vdash_{\mathbb{P}}\dot{x}=\tilde{f}\le\dot{\varphi}\check{y}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\varphi}[\check{X}] \text{ is cofinal with } \tilde{X}.$$

The argument for coinitiality is the same, upside down. **Q**

(v) If X is Dedekind complete, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Dedekind complete.}$$

The point is that a totally ordered set is Dedekind complete iff there is a cofinal-and-coinitial set A such that $[a,b]$ is compact in the order topology whenever a, $b \in A$ and $a \le b$. So 4Ac and (iv) above, with a little care over subspace topologies and identification of intervals, give the result. **Q**

(b)(i) If $X = [0,1]$ with its usual topology, then

$$\Vdash_{\mathbb{P}} \tilde{X}, \text{ with the topology generated by } \tilde{\mathfrak{T}}, \text{ can be identified with the unit interval.}$$

P By (a) and 4Ac we know that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is compact in its order topology,}$$

and therefore that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is Dedekind complete and has greatest and least elements.}$$

So all we need to check is that

$$\Vdash_{\mathbb{P}} \text{ if } x,\, y \in \tilde{X} \text{ and } x \mathrel{\dot{<}} y, \text{ there is a } q \in \check{\mathbb{Q}} \text{ such that } x \mathrel{\dot{<}} \dot{\varphi}(q) \mathrel{\dot{<}} y,$$

where $\dot{\varphi}$ is the map of 2Ae above, and this is a trifling refinement of one of the steps in the proof of (a-ii). **Q**

(ii) If $X = \mathbb{R}$ with its usual topology, then

$$\Vdash_{\mathbb{P}} \tilde{X}, \text{ with the topology generated by } \tilde{\mathfrak{T}}, \text{ can be identified with the real line.}$$

P This time, we can use 4Ah to see that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is locally compact in its order topology,}$$

and as above we know that

$$\Vdash_{\mathbb{P}} \check{\mathbb{Q}} \text{ is dense in } \tilde{X}.$$

Modifying the argument in 4Ah by taking $\mathcal{U}$ to be the set of open intervals with rational endpoints, we see in fact that

$$\Vdash_{\mathbb{P}} \mathbb{Q} \text{ is cofinal and coinitial with } \tilde{X} \text{ and closed intervals in } \tilde{X} \text{ with rational endpoints are compact.}$$

This is plenty. **Q**

(c) Let I be any set, and $X = \{0,1\}^I$. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified, as topological space, with } \{0,1\}^{\check{I}}.$$

P Put 2A(g-iii) and 3A together. **Q**

(d) If $X = \mathbb{N}^{\mathbb{N}}$ then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with } \check{\mathbb{N}}^{\check{\mathbb{N}}}.$$

P Put 2A(g-iii) and 3A together. **Q**

(e) If X is an n-dimensional manifold, where $n \ge 1$, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is an } n\text{-dimensional manifold.}$$

P Follow the argument of 4Ah, but this time taking $\mathcal{U}$ to be the family of open subsets of X which are homeomorphic to $\mathbb{R}^n$. This time we need to use (c-ii) here and Theorem 3A to see that

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ is homeomorphic to } n\text{-dimensional Euclidean space}$$

for every $U \in \mathcal{U}$, so that

$$\Vdash_{\mathbb{P}} \text{every member of } \tilde{\mathcal{U}} \text{ is homeomorphic to } n\text{-dimensional Euclidean space}$$

and therefore that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ has a base consisting of sets homeomorphic to } n\text{-dimensional Euclidean space.}$$ **Q**

4E Zero sets: Proposition If X is a topological space and $F \subseteq X$ is a zero set, then

$$\Vdash_{\mathbb{P}} \tilde{F} \text{ is a zero set in } \tilde{X}.$$

proof Let $\phi : X \to \mathbb{R}$ be a continuous function such that $F = \phi^{-1}[\{0\}]$. Let $\tilde{\phi}$ be the $\mathbb{P}$-name as defined in 2C, so that

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is a continuous function from } \tilde{X} \text{ to } \tilde{\mathbb{R}}$$

(2Cd). Now

$$\Vdash_{\mathbb{P}} \tilde{F} = \tilde{\phi}^{-1}[\{0\}].$$

P Suppose that $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names and $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \dot{y}.$$

Let $f \in C^{-}(Z; X)$ and $g \in C^{-}(Z; \mathbb{R})$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \dot{y} = \vec{g}$$

and $\widehat{p} \subseteq^* \operatorname{dom}(g \cap \phi f)$. Then

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{F} &\iff \widehat{p} \subseteq^* f^{-1}[F] \iff \widehat{p} \subseteq^* (\phi f)^{-1}[\{0\}] \\ &\iff \widehat{p} \subseteq^* g^{-1}[\{0\}] \iff p \Vdash_{\mathbb{P}} \vec{g} = 0 \iff p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = 0. \end{aligned}$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary, we have the result. **Q**

Since $\tilde{\mathbb{R}}$ is a $\mathbb{P}$-name for the real line (4D(b-ii)), we see that

$$\Vdash_{\mathbb{P}} \tilde{F} \text{ is a zero set.}$$

4F Proposition Let X be a connected Hausdorff space and $\mathbb{P}$ a forcing notion. Then

(a) If X is compact,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

(b) If X is analytic,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

proof (a) ? Otherwise, by Theorem 4Ac, there are $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{G}$ such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is a compact open set in } \tilde{X}, \text{ and is neither } \emptyset \text{ nor } \tilde{X}.$$

Now

$$p \Vdash_{\mathbb{P}} \text{there is a finite subset of } \tilde{\mathfrak{T}} \text{ with union } \dot{G},$$

so there are a q stronger than p, an $n \in \mathbb{N}$ and $G_0, \dots, G_n \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \dot{G} = \tilde{G}_0 \cup \dots \tilde{G}_n;$$

setting $G = \bigcup_{i \le n} G_i$,

$$q \Vdash_{\mathbb{P}} \dot{G} = \tilde{G}.$$

Similarly, there are an r stronger than q and an open set $H \subseteq X$ such that

$$r \Vdash_{\mathbb{P}} \tilde{X} \setminus \dot{G} = \tilde{H}.$$

But now G and H must be disjoint non-empty open sets in X. **X**

(b) ? Otherwise, there are $p \in \mathbb{P}$ and $\mathbb{P}$-names $\dot{G}$, $\dot{H}$ such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ and } \dot{H} \text{ are disjoint non-empty open sets with union } \tilde{X}.$$

Adjusting the names $\dot{G}$, $\dot{H}$ if necessary, we can suppose that

$$Z \setminus \widehat{p} = [\![\dot{G} = \tilde{X}]\!] = [\![\dot{H} = \emptyset]\!],$$

so that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ and } \dot{H} \text{ are disjoint open sets with union } \tilde{X}.$$

Let $W_{\dot{G}}$, $W_{\dot{H}}$ be the corresponding open subsets of $Z \times X$ as described in §2E, so that they are disjoint (2Eb); set $F = (Z \times X) \setminus (W_{\dot{G}} \cup W_{\dot{H}})$. The projections $\pi_1[W_{\dot{G}}]$, $\pi_1[W_{\dot{H}}]$ are open subsets of Z both dense in $\widehat{p}$, and (because X is connected) their intersection includes $\pi_1[F]$; so $\operatorname{int}\pi_1[F]$ is dense in $\widehat{p}$. By the von Neumann-Jankow selection theorem (Fremlin 03, 423N), $\pi_1[F] \in \widehat{\mathcal{B}}(Z)$ and there is a selector $h_0 : \pi_1[F] \to X$ for F which is $\widehat{\mathcal{B}}(Z)$-measurable. Extending h_0 to a function h which is constant on $Z \setminus \pi_1[F]$, $h : Z \to X$ is $\widehat{\mathcal{B}}(Z)$-measurable.

Because X has a countable network consisting of Souslin-F sets, there is a dense G_δ set $E \subseteq Z$ such that $f = h{\restriction}E$ is continuous and belongs to $C^-(Z;X)$. Now consider $[\![\vec{f} \in \dot{G}]\!]$. If $q \in \mathbb{P}$ is such that $q \Vdash_{\mathbb{P}} \vec{f} \in \dot{G}$, then

$$\widehat{q} \subseteq^* \{z : z \in E,\ (z, f(z)) \in W_{\dot{G}}\} \subseteq E \setminus \operatorname{int}\pi_1[F] \subseteq^* Z \setminus \widehat{p}.$$

So $p \Vdash_{\mathbb{P}} \vec{f} \notin \dot{G}$. Similarly,

$$p \Vdash_{\mathbb{P}} \vec{f} \notin \dot{H} \text{ and } \dot{G} \cup \dot{H} \neq \tilde{X},$$

which is absurd. **X**

4G Corollary Let X be a Hausdorff space such that for any two points x, $y \in X$ there is a connected compact set containing both. (For instance, X might be path-connected.) Then for any forcing notion $\mathbb{P}$,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is connected.}$$

proof Let $\dot{\varphi}$ be the $\mathbb{P}$-name described in 2Af. Then

$$\Vdash_{\mathbb{P}} \text{any two points of } \dot{\varphi}[\check{X}] \text{ belong to the same component of } \tilde{X}.$$

P Let $\dot{x}$, $\dot{y}$ be $\mathbb{P}$-names and $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \text{ belong to } \dot{\varphi}[\check{X}].$$

Then there are a q stronger than p and x, $y \in X$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \tilde{x} \text{ and } \dot{y} = \tilde{y}.$$

Let K be a connected compact subset of X containing both x and y. Putting 2Cg and 4Fa together, we see that

$$\Vdash_{\mathbb{P}} \tilde{K} \text{ is a connected subset of } \tilde{X},$$

while

$$q \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \tilde{K},$$

so

$$q \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \text{ belong to the same component of } \tilde{X}.$$

As p, $\dot{x}$ and $\dot{y}$ are arbitrary, we have the result. **Q**

Now 2A(f-iii) tells us that

$$\Vdash_{\mathbb{P}} \dot{\phi}[\check{X}] \text{ is dense in } \tilde{X},$$

so

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ has a dense component and is connected.}$$

Remark Of course the same idea works if we start from an adequate collection of connected Polish subspaces.

4H For completeness, I set out two elementary remarks.

(a) If X is not connected then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not connected.}$$

(For if U is a non-trivial open-and-closed subset of X, then

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ is a non-trivial open-and-closed subset of } \tilde{X}.)$$

(b) If X is not compact, then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not compact.}$$

P Let $\langle G_i\rangle_{i\in I}$ be an open cover of X with no finite subcover. Then 2A(b-vi) tells us that

$$\Vdash_{\mathbb{P}} \langle\tilde{G}_i\rangle_{i\in\check{I}} \text{ is an open cover of } \tilde{X}.$$

? If $p\in\mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \tilde{X} \text{ is compact,}$$

then

$$p \Vdash_{\mathbb{P}} \text{there is a finite set } J\subseteq\check{I} \text{ such that } \tilde{X}=\bigcup_{i\in J}\tilde{G}_i.$$

Now there are a q stronger than p and an $n\in\mathbb{N}$ such that

$$q \Vdash_{\mathbb{P}} \text{there is a } J\in[\check{I}]^n \text{ such that } \tilde{X}=\bigcup_{i\in J}\tilde{G}_i;$$

in which case there must be an r stronger than q and a $J\in[I]^{\le n}$ such that

$$r \Vdash_{\mathbb{P}} \tilde{X}=\bigcup_{i\in\check{J}}\tilde{G}_i.$$

But this means that $X=\bigcup_{i\in J}G_i$, contrary to the choice of $\langle G_i\rangle_{i\in I}$. **X**

So we must have

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not compact.}$$

4I Metric spaces: Theorem Let (X,ρ) be a metric space.

(a) There is a $\mathbb{P}$-name $\tilde{\rho}$ such that

$\Vdash_{\mathbb{P}} \tilde{\rho}$ is a metric on $\tilde{X}$ defining its topology, and $\dot{\varphi}:\check{X}\to\tilde{X}$ is an isometry for $\check{\rho}$ and $\dot{\rho}$.

(b) If (X,ρ) is complete, then

$$\Vdash_{\mathbb{P}} (\tilde{X},\tilde{\rho}) \text{ is complete.}$$

proof (a)(i) For f, $g\in C^-(Z;X)$ define $a_{fg}\in C^-(Z;[0,\infty[)$ by setting

$$a_{fg}(z)=\rho(f(z),g(z)) \text{ for } z\in\operatorname{dom}f\cap\operatorname{dom}g.$$

Let $\tilde{\rho}$ be the $\mathbb{P}$-name

$$\{(((\vec{f},\vec{g}),\tilde{a}_{fg}),\mathbb{1}) : f,\, g\in C^-(Z;X)\}.$$

Then

$$\Vdash_{\mathbb{P}} \tilde{\rho} \text{ is a function from } \tilde{X}\times\tilde{X} \text{ to the non-negative reals.}$$

P (α) Suppose that $p\in\mathbb{P}$ and that $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} (\dot{x},\dot{y})\in\tilde{X}\times\tilde{X}.$$

Then there are f, $g \in C^-(Z;X)$ such that

$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$ and $\dot{y} = \vec{g}$, so $((\dot{x},\dot{y}),\tilde{a}_{fg}) \in \tilde{\rho}$, while $\tilde{a}_{fg}$ is a non-negative real number (using 4D(b-ii)). (β) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{y}$, $\dot{a}$ and $\dot{a}'$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} ((\dot{x},\dot{y}),\dot{a}) \text{ and } ((\dot{x},\dot{y}),\dot{a}') \text{ both belong to } \tilde{\rho}.$$

Then there are a q stronger than p and f, g, f', $g' \in C^-(Z;X)$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} = \vec{f}',\ \dot{y} = \vec{g} = \vec{g}',\ \dot{a} = \tilde{a}_{fg} \text{ and } \dot{a}' = \tilde{a}_{f'g'}.$$

In this case,

$$\begin{aligned}\widehat{q} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} f',\ f(z) = f'(z)\}\\ \cap \{z : z \in \operatorname{dom} g \cap \operatorname{dom} g',\ g(z) = g'(z)\}\\ \subseteq \{z : z \in \operatorname{dom} a_{fg} \cap \operatorname{dom} a_{f'g'},\ a_{fg}(z) = a_{f'g'}(z)\}\end{aligned}$$

and

$$q \Vdash_{\mathbb{P}} \dot{a} = \dot{a}'. \ \mathbf{Q}$$

(ii) If f, g, $h \in C^-(Z;X)$, then

$$a_{fh}(z) \le a_{fg}(z) + a_{gh}(z) \text{ for every } z \in \operatorname{dom} f \cap \operatorname{dom} g \cap \operatorname{dom} h;$$

it follows at once that

$$\Vdash_{\mathbb{P}} \tilde{\rho}(x,w) \le \tilde{\rho}(x,y) + \tilde{\rho}(y,w) \text{ for all } x,\ y,\ w \in \tilde{X}.$$

So we have the triangle inequality.

(iii) If $p \in \mathbb{P}$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \dot{y} \in \tilde{X},$$

let f, $g \in C^-(Z;X)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \dot{y} = \vec{g}.$$

Then

$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\} = \{z : z \in \operatorname{dom} a_{fg},\ a_{fg}(z) = 0\},$$

so

$$p \Vdash_{\mathbb{P}} \tilde{\rho}(\dot{x},\dot{y}) = \tilde{a}_{fg} = 0.$$

(iv) In the other direction, if $p \in \mathbb{P}$ and $\dot{x}$, $\dot{y}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{x},\ \dot{y} \in \tilde{X} \text{ and } \tilde{\rho}(\dot{x},\dot{y}) = 0,$$

let f, $g \in C^-(Z;X)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \dot{y} = \vec{g}.$$

Then

$$\widehat{p} \subseteq^* \{z : z \in \operatorname{dom} a_{fg},\ a_{fg}(z) = 0\} = \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(z) = g(z)\},$$

so

$$p \Vdash_{\mathbb{P}} \dot{x} = \dot{y}.$$

Putting this together with (ii) and (iii),

$$\Vdash_{\mathbb{P}} \text{ for all } x,\ y \in \tilde{X},\ \tilde{\rho}(x,y) = 0 \text{ iff } x = y,$$

and

$$\Vdash_{\mathbb{P}} \tilde{\rho} \text{ is a metric on } \tilde{X}.$$

(v) Now suppose that $p \in \mathbb{P}$, $\epsilon > 0$ is a rational and $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x}_0 \in \tilde{X}.$$

Then there is an $f \in C^-(Z;X)$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}.$$

Let $U \subseteq X$ be an open set of diameter at most ϵ such that $f^{-1}[U]$ is not empty; then there is a q stronger than p such that $\widehat{q} \subseteq^* f^{-1}[U]$. Now

$$q \Vdash_{\mathbb{P}} \tilde{U} \text{ is an open set containing } \dot{x}.$$

Suppose that r is stronger than q and $\dot{y}$ is a $\mathbb{P}$-name such that

$$r \Vdash_{\mathbb{P}} \dot{y} \in \tilde{U}.$$

Let $\vec{g}$ be such that

$$r \Vdash_{\mathbb{P}} \dot{y} = \vec{g};$$

then

$$\begin{aligned}\widehat{r} &\subseteq^* f^{-1}[U] \cap g^{-1}[U]\\ &\subseteq \{z : a_{fg}(z) \text{ is defined and at most } \epsilon\}.\end{aligned}$$

So

$$r \Vdash_{\mathbb{P}} \tilde{\rho}(\dot{x},\dot{y}) \le \epsilon.$$

As p and $\dot{x}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ if $x \in \tilde{X}$ there is an open set U containing x such that $\tilde{\rho}(x,y) \le \epsilon$ for every $y \in U$;

as ϵ is arbitrary,

$$\Vdash_{\mathbb{P}} \text{ the topology induced by } \tilde{\rho} \text{ is coarser than the standard topology on } \tilde{X}.$$

(vi) In the other direction, suppose that $p \in \mathbb{P}$ and $\dot{x}$, $\dot{G}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{G} \subseteq \tilde{X} \text{ is open and } \dot{x} \in \dot{G}.$$

Then there are a q stronger than p and an open $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} \in \tilde{G} \subseteq \dot{G};$$

let $f \in C^-(Z;X)$ be such that $q \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$, so that $\widehat{q} \subseteq^* f^{-1}[G]$. Let $\epsilon > 0$ be a rational such that $\widehat{q}$ meets $f^{-1}[H_\epsilon]$, where $H_\epsilon = \{x : \rho(x, X \setminus G) > \epsilon\}$. Let r stronger than q be such that $\widehat{r} \subseteq^* f^{-1}[H_\epsilon]$. Now suppose that s is stronger than r and that $\dot{y}$ is a $\mathbb{P}$-name such that

$$s \Vdash_{\mathbb{P}} \dot{y} \in \tilde{X} \text{ and } \tilde{\rho}(\dot{x},\dot{y}) < \epsilon.$$

Let $g \in C^-(Z;X)$ be such that $s \Vdash_{\mathbb{P}} \dot{y} = \vec{g}$. Then

$$\widehat{s} \subseteq^* \{z : z \in \operatorname{dom} f \cap \operatorname{dom} g,\ \rho(f(z),g(z)) < \epsilon\} \cap f^{-1}[H_\epsilon] \subseteq g^{-1}[G],$$

so

$$s \Vdash_{\mathbb{P}} \dot{y} \in \tilde{G} \subseteq \dot{G}.$$

As p, $\dot{x}$ and $\dot{G}$ are arbitrary, we see that

$\Vdash_{\mathbb{P}}$ whenever $G \subseteq \tilde{X}$ is open and $x \in G$, there is an $\epsilon > 0$ such that $y \in G$ whenever $y \in \tilde{X}$ and $\tilde{\rho}(x,y) < \epsilon$;

so, with (v), we have

$$\Vdash_{\mathbb{P}} \text{ the topology defined by } \tilde{\rho} \text{ is the standard topology on } \tilde{X}.$$

(vii) I still have to check the assertion that $\dot{\varphi}$ is a name for an isometry. But suppose that x, $y \in X$. Then (in the notation of 2Af) a_{e_x,e_y} is the constant function with value $\rho(x,y)$, so

$$\Vdash_{\mathbb{P}} \tilde\rho(\dot\varphi(\check x),\dot\varphi(\check y))=\tilde\rho(\vec e_x,\vec e_y)=\tilde a_{e_x,e_y}=\rho(x,y)\check{}=\check\rho(\check x,\check y).$$

So

$$\Vdash_{\mathbb{P}} \dot\varphi:\check X\to\tilde X \text{ is an isometry.}$$

(b) Now suppose that (X,ρ) is complete. Take $p\in\mathbb{P}$, and let $\dot{\boldsymbol g}$ be a $\mathbb{P}$-name such that

$$p\Vdash_{\mathbb{P}}\dot{\boldsymbol g} \text{ is a sequence in } \tilde X \text{ such that } \tilde\rho(\dot{\boldsymbol g}(n+1),\dot{\boldsymbol g}(n))<2^{-n} \text{ for every } n\in\mathbb{N}.$$

For each $n\in\mathbb{N}$ let $f_n\in C^-(Z;X)$ be such that

$$p\Vdash_{\mathbb{P}}\dot{\boldsymbol g}(n)=\vec f_n.$$

Then

$$\widehat p\subseteq^*\bigcap_{n\in\mathbb{N}}\{z:z\in\operatorname{dom}a_{f_n,f_{n+1}},\ a_{f_{n+1},f_n}(z)<2^{-n}\}=E$$

say. Set $E'=E\cap\operatorname{int}\overline{E}$, so that $E\setminus E'$ is nowhere dense, and E', like E, is a G_δ set. For $z\in E'$, we have $\rho(f_{n+1}(z),f_n(z))<2^{-n}$ for every n, so $f(z)=\lim_{n\to\infty}f_n(z)$ is defined in X; for $z\in Z\setminus\overline{E}$ take $f(z)$ to be any point of X. (I am passing over the trivial case $X=\emptyset$.) Because $\langle f_n\restriction E'\rangle_{n\in\mathbb{N}}$ is uniformly convergent to $f\restriction E'$, $f\restriction E'$ and f are continuous, and $f\in C^-(Z;X)$. Now, for any $n\in\mathbb{N}$, $\rho(f(z),f_n(z))<2^{-n+1}$ for every $z\in E'$, so $\widehat p\subseteq^*\{z:z\in\operatorname{dom}a_{f,f_n},\ a_{f,f_n}(z)<2^{-n+1}\}$ and

$$p\Vdash_{\mathbb{P}}\tilde\rho(\dot{\boldsymbol g}(n),\vec f)<2^{-n+1}.$$

Accordingly

$$p\Vdash_{\mathbb{P}}\vec f=\lim_{n\to\infty}\dot{\boldsymbol g}(n);$$

as p and $\dot{\boldsymbol g}$ are arbitrary,

$$\Vdash_{\mathbb{P}}(\tilde X,\tilde\rho) \text{ is complete.}$$

4J When studying random and Cohen forcing, among others, it is often useful to know when a name for a Borel set in $\tilde X$ can be represented, in the manner of 2E, by a set $W\subseteq Z\times X$ which factors through a continuous function from Z to $\{0,1\}^{\mathbb{N}}$. Here I collect some simple cases in which this can be done, in preparation for §8 below.

Proposition Let $\mathbb{P}$ be a forcing notion and Z the Stone space of its regular open algebra. Write $\mathcal{B}\mathfrak{a}(Z)$ for the Baire σ-algebra of Z. Let X be a Hausdorff space and Σ a σ-algebra of subsets of X including a base for the topology of X. I will say that a $\mathbb{P}$-name $\dot E$ is **$(\mathcal{B}\mathfrak{a},\Sigma)$-representable** if there is a $W\in\mathcal{B}\mathfrak{a}(Z)\widehat\otimes\Sigma$ such that

$$\Vdash_{\mathbb{P}}\dot E=\vec W,$$

defining $\vec W$ as in 2E.

(a) Suppose that X is second-countable and that

$$\Vdash_{\mathbb{P}}\dot E \text{ is a Borel subset of } \tilde X.$$

If *either* $\mathbb{P}$ is ccc *or* there is an $\alpha<\omega_1$ such that

$$\Vdash_{\mathbb{P}}\dot E \text{ is of Borel class at most } \alpha,$$

then $\dot E$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

(b) Suppose that $\mathbb{P}$ is ccc.

(i) If

$$\Vdash_{\mathbb{P}}\dot E \text{ is a compact } \mathrm{G}_\delta \text{ set}$$

then $\dot E$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

(ii) If X is compact and

$$\Vdash_{\mathbb{P}}\dot E\in\mathcal{B}\mathfrak{a}(\tilde X),$$

then $\dot E$ is $(\mathcal{B}\mathfrak{a},\Sigma)$-representable.

proof (a)(i) Suppose that

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is open.}$$

Let $\mathcal{U} \subseteq \Sigma$ be a countable base for the topology of X and set

$$W = \bigcup_{U \in \mathcal{U}} [\![\tilde{U} \subseteq \dot{E}]\!] \times U \in \mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\Sigma;$$

then $\Vdash_{\mathbb{P}} \dot{E} = \vec{W}$ so $\dot{E}$ is $(\mathcal{B}\mathfrak{a}, \Sigma)$-representable.

(ii) Inducing on α, we see that if

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is of class at most } \alpha,$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a}, \Sigma)$-representable.

(iii) If $\mathbb{P}$ is ccc then we can apply (ii).

(b)(i) Let $\langle \dot{G}_n \rangle_{n\in\mathbb{N}}$ be a sequence of $\mathbb{P}$-names such that

$$\Vdash_{\mathbb{P}} \dot{G}_n \text{ is open for every } n \text{ and } \dot{K} = \bigcap_{n\in\mathbb{N}} G_n.$$

Let $\mathcal{U} \subseteq \Sigma$ be a base for the topology of X closed under finite unions. Then, in the language of 2A,

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a base for the topology of } \tilde{X} \text{ closed under finite unions.}$$

Fix $n \in \mathbb{N}$ for the moment. Then

$$\Vdash_{\mathbb{P}} \text{ there is a } U \in \tilde{\mathcal{U}} \text{ such that } \dot{E} \subseteq U \subseteq \dot{G}_n,$$

so there are a maximal antichain $Q_n \subseteq \mathbb{P}$ and a family $\langle U_{nq} \rangle_{q\in Q_n}$ in $\mathcal{U}$ such that

$$q \Vdash_{\mathbb{P}} \dot{E} \subseteq \tilde{U}_{nq} \subseteq \dot{G}_n$$

for every $q \in Q_n$. Set $W_n = \bigcup_{q\in Q_n} \widehat{q} \times U_{nq}$; then

$$q \Vdash_{\mathbb{P}} \dot{E} \subseteq \tilde{U}_{nq} = \vec{W}_n \subseteq \dot{G}_n$$

for every $q \in Q_n$, and

$$\Vdash_{\mathbb{P}} \dot{E} \subseteq \vec{W}_n \subseteq \dot{G}_n.$$

So if we now set $W = \bigcap_{n\in\mathbb{N}} W_n$, we shall have

$$\Vdash_{\mathbb{P}} \dot{E} \subseteq \vec{W} \subseteq \bigcap_{n\in\mathbb{N}} \dot{G}_n \text{ and } \dot{E} = \vec{W}.$$

But also, because $\mathbb{P}$ is ccc, every Q_n is countable, so every W_n belongs to $\mathcal{B}\mathfrak{a}(Z)\widehat{\otimes}\Sigma$ and W also does.

(ii) From 4Ac and (i), we see that if

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is a zero set in } \tilde{X}, \text{ therefore a compact } \mathrm{G}_\delta \text{ set}$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a}, \Sigma)$-representable. Now, writing $\mathcal{B}\mathfrak{a}_\alpha$, for $1 \le \alpha < \omega_1$, for the additive classes in the Baire hierarchy[4], then we see by induction on α that if

$$\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}_\alpha(\tilde{X})$$

then $\dot{E}$ is $(\mathcal{B}\mathfrak{a}, \Sigma)$-representable. Finally, if

$$\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}(\tilde{X})$$

then, because $\mathbb{P}$ is ccc, there is an $\alpha < \omega_1$ such that $\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}_\alpha(\tilde{X})$ and $\dot{E}$ is therefore $(\mathcal{B}\mathfrak{a}, \Sigma)$-representable.

[4] For any topological space Y, start with $\mathcal{B}\mathfrak{a}_1(Y)$ the family of cozero sets, $\mathcal{B}\mathfrak{a}_{\alpha+1}(Y) = \{\bigcup_{n\in\mathbb{N}}(Y \setminus E_n) : E_n \in \mathcal{B}\mathfrak{a}_\alpha(Y)$ for every $n\}$, $\mathcal{B}\mathfrak{a}_\alpha(Y) = \bigcup_{1\le\beta<\alpha} \mathcal{B}\mathfrak{a}_\beta(Y)$ for non-zero limit ordinals α.

5 Cardinal functions

5A Theorem Let $\mathbb{P}$, $(X,\mathfrak{T})$ and $\tilde{X}$ be as in §2A, and θ a cardinal.
(a) If the weight $w(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} w(\tilde{X}) \le \#(\check{\theta}).^5$$

(b) If the π-weight $\pi(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} \pi(\tilde{X}) \le \#(\check{\theta}).$$

(c) If the density $d(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} d(\tilde{X}) \le \#(\check{\theta}).$$

(d) If the saturation $\operatorname{sat}(X)$ of X is θ then

$$\Vdash_{\mathbb{P}} \operatorname{sat}(\tilde{X}) \ge \#(\check{\theta}).$$

proof (a) Apply 2Ad with a base $\mathcal{U}$ of cardinal θ.

(b) Let $\langle U_\xi\rangle_{\xi<\theta}$ enumerate a π-base for the topology of X. Consider the $\mathbb{P}$-name

$$\dot{\psi} = \{((\check{\xi},\tilde{U}_\xi),\mathbb{1}) : \xi < \theta\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\psi} \text{ is a function from } \check{\theta} \text{ to } \tilde{\mathfrak{T}}.$$

Now

$$\Vdash_{\mathbb{P}} \{\dot{\psi}(\xi) : \xi < \check{\theta}\} \text{ is a } \pi\text{-base for the topology of } \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{G}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is a non-empty open subset of } \tilde{X}.$$

Then

$$p \Vdash_{\mathbb{P}} \text{there is a } G \in \tilde{\mathfrak{T}} \text{ such that } \emptyset \ne G \subseteq \dot{G},$$

so there are a q stronger than p and a $G \in \mathfrak{T}$ such that

$$q \Vdash_{\mathbb{P}} \emptyset \ne \tilde{G} \subseteq \dot{G}.$$

In this case, $G \ne \emptyset$ so there is a $\xi < \theta$ such that $\emptyset \ne U_\xi \subseteq G$, in which case

$$q \Vdash_{\mathbb{P}} \dot{\psi}(\check{\xi}) = \tilde{U}_\xi \text{ is non-empty and included in } \dot{G}.$$

As p and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{every non-empty open subset of } \tilde{X} \text{ includes a non-empty value of } \dot{\psi},$$

which is what we need to know. **Q**
Now the result follows at once.

(c) Use 2A(f-iii) with a set D of cardinal θ.

(d) **?** Otherwise, there are a $p \in \mathbb{P}$ and an ordinal κ such that

$$p \Vdash_{\mathbb{P}} \operatorname{sat}(\tilde{X}) = \check{\kappa} < \#(\check{\theta}).$$

Now $\kappa < \operatorname{sat}(X)$, so there is a disjoint family $\langle G_\xi\rangle_{\xi<\kappa}$ of non-empty open sets in X. But now
$\Vdash_{\mathbb{P}} \langle\tilde{G}_\xi\rangle_{\xi<\check{\kappa}}$ is a disjoint family of non-empty open subsets of $\tilde{X}$, so $\#(\check{\kappa}) < \operatorname{sat}(\tilde{X})$. **X**

5B Theorem Let $\mathbb{P}$, Z, $(X,\mathfrak{T})$ and $\tilde{X}$ be as in §2, and θ a cardinal.
(a) If X is compact and $w(X) = \theta$, then

[5] Recall that $\Vdash_{\mathbb{P}} \check{\theta}$ is an ordinal, but that in many cases $\Vdash_{\mathbb{P}} \check{\theta}$ is not a cardinal.

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

(b) If X is metrizable and $w(X) = \theta$, then

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

proof (a) Suppose that $p \in \mathbb{P}$ and that $\dot{\lambda}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{\lambda} \le \check{\theta} \text{ is a regular cardinal.}$$

Then there are an ordinal λ and a q stronger than p such that

$$q \Vdash_{\mathbb{P}} \dot{\lambda} = \check{\lambda},$$

and λ must be a regular cardinal in the ground model, less than or equal to θ. Write I for the unit interval $[0,1]$. Because $\lambda \le w(X)$, there is a continuous function $\phi : X \to I^{\lambda}$ such that whenever $\xi < \lambda$ there are x, $y \in X$ such that $\phi(x)\restriction\alpha = \phi(y)\restriction\alpha$ but $\phi(x)(\alpha) \ne \phi(y)(\alpha)$. Let $\tilde{\phi}$ be the corresponding $\mathbb{P}$-name defined from ϕ by the construction of 2C. By Theorem 3A,

$$\Vdash_{\mathbb{P}} (I^{\lambda})^{\sim} \text{ can be identified with } \tilde{I}^{\check{\lambda}};$$

working through the identifications in §§2C and 3A, we have a $\mathbb{P}$-name $\tilde{\psi}$ such that

$$\Vdash_{\mathbb{P}} \tilde{\psi} : \tilde{X} \to \tilde{I}^{\check{\lambda}} \text{ is a continuous function,}$$

and whenever $f \in C(Z;X)$ then

$$\Vdash_{\mathbb{P}} \tilde{\psi}(\vec{f}) = (\phi f)^{\#} = (\langle \pi_{\xi}\phi f\rangle_{\xi<\lambda})^{\rightarrow}.$$

Now

$$q \Vdash_{\mathbb{P}} \text{if } \alpha < \check{\lambda} \text{ there are } x,\ y \in \tilde{X} \text{ such that } \tilde{\psi}(x)\restriction\alpha = \tilde{\psi}(y)\restriction\alpha \text{ but } \tilde{\psi}(x) \ne \tilde{\psi}(y).$$

P Let r be stronger than q and $\dot{\alpha}$ a $\mathbb{P}$-name such that $r \Vdash_{\mathbb{P}} \dot{\alpha} < \check{\lambda}$. Then there are an s stronger than r and an ordinal $\alpha < \lambda$ such that $s \Vdash_{\mathbb{P}} \dot{\alpha} = \check{\alpha}$. Now we have x, $y \in X$ such that $\phi(x)\restriction\alpha = \phi(y)\restriction\alpha$ but $\phi(x)(\alpha) \ne \phi(y)(\alpha)$. Let e_x, $e_y \in C(Z;X)$ be the corresponding constant functions. Then

$$\Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{x}) = (\langle \pi_{\xi}\phi e_x\rangle_{\xi<\lambda})^{\rightarrow},$$

so

$$\Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{x})(\xi) = (\pi_{\xi}\phi e_x)^{\rightarrow} = \vec{f}_{\pi_{\xi}\phi(x)}$$

for every $\xi < \lambda$, where $f_{\pi_{\xi}\phi(x)} : Z \to I$ is now the constant function with value $\phi(x)(\xi)$. But this means that

$$\Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{x})(\xi) = \tilde{\psi}(\tilde{y})(\xi)$$

for every $\xi < \alpha$, while

$$\Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{x})(\check{\alpha}) \ne \tilde{\psi}(\tilde{y})(\check{\alpha}).$$

So

$$s \Vdash_{\mathbb{P}} \tilde{\psi}(\tilde{x})\restriction\dot{\alpha} = \tilde{\psi}(\tilde{y})\restriction\dot{\alpha} \text{ but } \tilde{\psi}(\tilde{x}) \ne \tilde{\psi}(\tilde{y}).$$

As r and $\dot{\alpha}$ are arbitrary, we have the result. **Q**

Because

$$q \Vdash_{\mathbb{P}} \check{\lambda} \text{ is a regular cardinal and } \tilde{I} \text{ is the unit interval}$$

(4D(b-i)),

$$q \Vdash_{\mathbb{P}} w(\tilde{X}) \ge \check{\lambda} = \dot{\lambda}.$$

As p and $\dot{\lambda}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{if } \lambda \le \#(\theta) \text{ is a regular cardinal, then } w(\tilde{X}) \ge \lambda,$$

so

$$\Vdash_{\mathbb{P}} w(\tilde{X}) \ge \#(\check{\theta}).$$

Putting this together with Theorem 5Aa,

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

(b) If ρ is a metric on X inducing its topology, then X has a σ-metrically-discrete base, so there is a sequence $\langle \mathcal{G}_n \rangle_{n\in\mathbb{N}}$ such that each $\mathcal{G}_n$ is a disjoint family of non-empty open sets and $\sup_{n\in\mathbb{N}} \theta_n = \theta$, where $\theta_n = \#(\mathcal{G}_n)$. Now, setting $\tilde{\mathcal{G}}_n = \{(\tilde{G}, \mathbb{1}) : G \in \mathcal{G}_n\}$ for each n,

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{G}}_n \text{ is a disjoint family of non-empty open sets, so } \#(\check{\theta}_n) = \#(\tilde{\mathcal{G}}_n) \le w(\tilde{X})$$

for each n. But as

$$\Vdash_{\mathbb{P}} \check{\theta} = \bigcup_{n\in\mathbb{N}} \check{\theta}_n,$$

we have

$$\Vdash_{\mathbb{P}} \#(\check{\theta}) = \sup_{n\in\mathbb{N}} \#(\check{\theta}_n) \le w(\tilde{X});$$

putting this together with 5Ac,

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \#(\check{\theta}).$$

5C Theorem (A.Dow) Suppose that GCH is true, and that $\mathbb{P}$ is any forcing notion.

(a) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra and set $\kappa = \pi(\mathfrak{A})$. Then

$$\Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) = \#(\check{\kappa}).$$

(b) Let X be a regular topological space and set $\kappa = \pi(X)$. Then

$$\Vdash_{\mathbb{P}} \pi(\tilde{X}) = \#(\check{\kappa}).$$

(c) Let $\mathfrak{A}$ be any Boolean algebra and set $\kappa = \pi(\mathfrak{A})$. Then

$$\Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) = \#(\check{\kappa}).$$

proof (a) It is easy to see that

$$\Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) \le \#(\check{\kappa}),$$

since if A is order-dense in $\mathfrak{A}$ then

$$\Vdash_{\mathbb{P}} \check{A} \text{ is order-dense in } \check{\mathfrak{A}}.$$

? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}} \#(\check{\kappa}) \le \pi(\check{\mathfrak{A}}).$$

Then there must be a $p \in \mathbb{P}$ and ordinals λ_1, λ_2 such that

$$p \Vdash_{\mathbb{P}} \pi(\check{\mathfrak{A}}) = \check{\lambda}_1 < \check{\lambda}_2 \le \#(\check{\kappa}) \text{ and } \check{\lambda}_2 \text{ is a regular cardinal.}$$

Of course λ_1 is a cardinal and λ_2 is a regular cardinal. Note that $\operatorname{sat}(\mathfrak{A}) \le \lambda_2$. **P** If $A \subseteq \mathfrak{A} \setminus \{0\}$ is a disjoint set of size λ then

$$\Vdash_{\mathbb{P}} \check{A} \text{ is a disjoint family in } \check{\mathfrak{A}} \setminus \{0\} \text{ so } \#(\check{\lambda}) = \#(\check{A}) \le \pi(\check{\mathfrak{A}})$$

and

$$p \Vdash_{\mathbb{P}} \#(\check{\lambda}) < \#(\check{\lambda}_2)$$

so $\lambda < \lambda_2$. **Q**

It follows that if $B \subseteq \mathfrak{A}$ is any set of cardinal at most λ_2 there is an order-closed subalgebra of $\mathfrak{A}$, including $\mathfrak{B}$, of cardinal at most λ_2. (This is where we need the continuum hypothesis, to see that $2^\lambda \le \lambda_2$ for every $\lambda < \lambda_2$.) At the same time, $\lambda_2 \le \kappa = \pi(\mathfrak{A})$. We can therefore find families $\langle \mathfrak{B}_\xi \rangle_{\xi<\lambda_2}$ and $\langle b_\xi \rangle_{\xi<\lambda_2}$ such that

$$\langle \mathfrak{B}_\xi \rangle_{\xi<\lambda_2} \text{ is a non-decreasing family of subalgebras of } \mathfrak{A}, \text{ all of cardinal less than } \lambda_2,$$

$$\text{for each } \xi < \lambda_2,\ b_\xi \in \mathfrak{B}_{\xi+1} \setminus \{0\} \text{ and } b_\xi \not\supseteq a \text{ for any } a \in \mathfrak{B}_\xi,$$

$$\mathfrak{B} = \bigcup_{\xi<\lambda_2} \mathfrak{B}_\xi \text{ is an order-closed subalgebra of } \mathfrak{A}.$$

For $a \in \mathfrak{A}$ set $f(a) = \operatorname{upr}(a, \mathfrak{B})$, the smallest element of $\mathfrak{B}$ including a.

Let $\dot{A}$ be a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A} \text{ is an order-dense subset of } \check{\mathfrak{A}} \text{ and } \#(\dot{A}) \le \check{\lambda}_1.$$

Let $\dot{g}$ be the $\mathbb{P}$-name

$$\{((\check{a},\xi),q) : a \in \mathfrak{A},\ f(a) \in \mathfrak{B}_\xi \setminus \bigcup_{\eta<\xi} \mathfrak{B}_\eta,\ q \Vdash_{\mathbb{P}} a \in \dot{A}\}.$$

Then

$$p \Vdash_{\mathbb{P}} \dot{g} \text{ is a function from } \dot{A} \text{ to } \check{\lambda}_2.$$

Since

$$p \Vdash_{\mathbb{P}} \check{\lambda}_2 \text{ is a regular cardinal greater than } \#(\dot{A}),$$

there must be a q stronger than p and a $\zeta < \lambda_2$ such that

$$q \Vdash_{\mathbb{P}} \dot{g}(a) \le \check{\zeta} \text{ for every } a \in \dot{A}.$$

However,

$$q \Vdash_{\mathbb{P}} \text{ there is some } a \in \dot{A} \text{ such that } 0 \ne a \subseteq \check{b}_{\check{\zeta}},$$

so there are an r stronger than q and an $a \in \mathfrak{A} \setminus \{0\}$ such that $a \subseteq b_\zeta$ and $r \Vdash_{\mathbb{P}} \check{a} \in \dot{A}$. As $r \Vdash_{\mathbb{P}} \dot{g}(\check{a}) \le \check{\zeta}$, there are an s stronger than r and a $\zeta' \le \zeta$ such that $s \Vdash_{\mathbb{P}} \dot{g}(\check{a}) = \check{\zeta}'$; but this means that $f(a) \in \mathfrak{B}_{\zeta'}$ and b_ζ includes a non-zero element of $\mathfrak{B}_\zeta$. **X**

So (a) must be true.

(b) Apply (a) to $\mathrm{RO}(X)$; since $\pi(\mathrm{RO}(X)) = \pi(X) = \kappa$ and

$$\Vdash_{\mathbb{P}} \mathrm{RO}(X)\check{} \text{ is isomorphic to an order-dense subalgebra of } \mathrm{RO}(\tilde{X}),$$

we have (using the fact that $\Vdash_{\mathbb{P}} \tilde{X}$ is regular)

$$\Vdash_{\mathbb{P}} \pi(\tilde{X}) = \pi(\mathrm{RO}(\tilde{X})) = \pi(\mathrm{RO}(X)\check{}) = \#(\check{\kappa}),$$

as required.

(c) Apply (b) to the Stone space of $\mathfrak{A}$.

5D Proposition Let X be a ccc Hausdorff space, and $\mathbb{P}$ a productively ccc forcing notion. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is ccc.}$$

proof If Z is the Stone space of $\mathrm{RO}(\mathbb{P})$ then Z is productively ccc so $Z \times X$ is ccc.

? Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}} \tilde{X} \text{ is ccc.}$$

Then there is a $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \text{ there is an uncountable disjoint family of non-empty open sets in } \tilde{X};$$

let $\langle \dot{G}_\xi \rangle_{\xi<\omega_1}$ be a family of $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{G}_\xi \text{ is a non-empty open subset of } \tilde{X} \text{ and } \dot{G}_\xi \cap \dot{G}_\eta = \emptyset \text{ whenever } \xi < \eta < \omega_1.$$

By 2Eb we have for each $\xi < \omega_1$ an open set $W_\xi \subseteq Z \times X$ such that $p \Vdash_{\mathbb{P}} \dot{G}_\xi = \vec{W}_\xi$. Now $(\widehat{p} \times X) \cap W_\xi$ is never empty, so there are $\xi < \eta < \omega$ such that $(\widehat{p} \times X) \cap W_\xi \cap W_\eta \ne \emptyset$. So we have an r stronger than p and a non-empty open $H \subseteq X$ such that $(\widehat{r} \times H) \subseteq W_\xi \cap W_\eta$. But now

$$r \Vdash_{\mathbb{P}} \emptyset \ne \tilde{H} \subseteq \vec{W}_\xi \cap \vec{W}_\eta = \dot{G}_\xi \cap \dot{G}_\eta,$$

which is impossible. **X**

5E Proposition Suppose that X is a hereditarily ccc compact Hausdorff space and that $\mathbb{P}$ is a forcing notion such that ω_1 is a precaliber of $\mathbb{P}$. Then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is hereditarily ccc.}$$

proof ? Otherwise, there are a $p \in \mathbb{P}$ and $\mathbb{P}$-names $\dot{x}_\xi$, $\dot{G}_\xi$ for $\xi < \omega_1$ such that

$$p \Vdash_{\mathbb{P}} \dot{G}_\xi \subseteq \tilde{X} \text{ is open and } \dot{x}_\xi \in \dot{G}_\xi \setminus \dot{G}_\eta \text{ whenever } \xi,\, \eta < \omega_1 \text{ are distinct.}$$

Let Z be the Stone space of $\mathrm{RO}(\mathbb{P})$; for each $\xi < \omega_1$ let $f_\xi \in C(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x}_\xi = \vec{f}_\xi$. Then we can find an open set $G_\xi \subseteq X$ and a p_ξ stronger than p such that

$$p_\xi \Vdash_{\mathbb{P}} \vec{f}_\xi \subseteq \tilde{G}_\xi \subseteq \dot{G}_\xi.$$

Now if ξ, $\eta < \omega_1$ are distinct and r is stronger than both p_ξ and p_η,

$$r \Vdash_{\mathbb{P}} \vec{f}_\xi \notin \dot{G}_\eta \supseteq \tilde{G}_\eta$$

and $\widehat{r} \cap f_\xi^{-1}[G_\eta]$ must be empty. As r is arbitrary, $\widehat{p}_\xi \cap \widehat{p}_\eta \cap f_\xi^{-1}[G_\eta]$ is empty.

Because ω_1 is a precaliber of $\mathbb{P}$, there is a $z \in Z$ such that $D = \{\xi : \xi < \omega_1,\, z \in \widehat{p}_\xi\}$ is uncountable. But now $f_\xi(z) \in G_\xi \setminus G_\eta$ for all distinct ξ, $\eta \in D$, so $\{f_\xi(z) : \xi \in D\}$ is not ccc. **X**

6 Radon measures

6A Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $\mathbb{P}$ a forcing notion. Let $\tilde{\mu}$ be the $\mathbb{P}$-name

$$\{((\tilde{A}, (\mu A)\check{}\,), \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X)\}.$$

Then

$$\Vdash_{\mathbb{P}} \text{ there is a unique Radon measure on } \tilde{X} \text{ extending } \tilde{\mu}.$$

Remark Perhaps a note is in order on the interpretation of the formula $(\mu A)\check{}$. If we take a real number α to be the set of rational numbers less than or equal to α, then $\check{\alpha}$ becomes a $\mathbb{P}$-name for a real number. If, in this context, we interpret ∞ as the set of all rational numbers, then we can equally regard $\check{\infty} = \check{\mathbb{Q}}$ as a $\mathbb{P}$-name for the top point of the two-point compactification of the reals.

proof (a) By 1C, every member of $\mathcal{U}\widehat{\mathcal{B}}(X)$ is universally Radon-measurable, so the formula for $\tilde{\mu}$ makes sense. Since we know that

$$\Vdash_{\mathbb{P}} \tilde{A} \neq \tilde{B}$$

whenever A and B are distinct elements of $\mathcal{U}\widehat{\mathcal{B}}(X)$, we have

$$\Vdash_{\mathbb{P}} \tilde{\mu} \text{ is a function,}$$

and from 2A we see that

$$\Vdash_{\mathbb{P}} \tilde{\mu} \text{ is an additive function from an algebra of subsets of } \tilde{X} \text{ to } [0, \infty].$$

(b) Let $\dot{\mathrm{T}}$ be the $\mathbb{P}$-name

$$\{(\tilde{A}, \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X) \text{ is included in an open set of finite measure}\};$$

then

$$\Vdash_{\mathbb{P}} \dot{\mathrm{T}} \text{ is a ring of subsets of } \tilde{X}.$$

Let $\dot{\nu}$ be the $\mathbb{P}$-name

$$\{((\tilde{A}, (\mu A)\check{}\,), \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X) \text{ is included in an open set of finite measure}\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\nu} : \dot{\mathrm{T}} \to [0, \infty[\text{ is additive.}$$

Let $\mathcal{U}$ be the family of open subsets of X of finite measure, so that $\mathcal{U}$ is upwards-directed and covers X. Then

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \subseteq \dot{\mathrm{T}} \text{ is an upwards-directed family of open sets with union } \tilde{X}.$$

By Fremlin 03, 416K, there is a $\mathbb{P}$-name $\dot{\mu}$ such that

$\Vdash_{\mathbb{P}} \dot{\mu}$ is a Radon measure on $\tilde{X}$, $\dot{\mu}K \ge \dot{\nu}K$ whenever $K \in \dot{\mathrm{T}}$ is compact, and $\dot{\mu}G \le \dot{\nu}G$ whenever $G \in \dot{\mathrm{T}}$ is open.

(c) Suppose that $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ is included in an open set of finite measure, and γ_1, γ_2 are rationals such that $\gamma_1 < \mu A < \gamma_2$. Then there are a compact $K \subseteq A$ such that $\mu K \ge \gamma_1$ and an open set $G \supseteq A$ such that $\mu G \le \gamma_2$. In this case K and G are universally Baire-property sets included in open sets of finite measure, so

$\Vdash_{\mathbb{P}} \tilde{K} \in \dot{\mathrm{T}}$ is a compact subset of $\tilde{A}$ and $\tilde{G} \in \dot{\mathrm{T}}$ is an open set including $\tilde{A}$, so $\gamma_1 \le (\mu K)\check{} = \dot{\nu}\tilde{K} \le \dot{\mu}\tilde{K} \le \dot{\mu}_*\tilde{A} \le \dot{\mu}^*\tilde{A} \le \dot{\mu}\tilde{G} \le \dot{\nu}\tilde{G} = (\mu G)\check{} \le \gamma_2$.

As γ_1 and γ_2 are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mu}_*\tilde{A} = \dot{\mu}^*\tilde{A} = (\mu A)\check{},$$

so

$$\Vdash_{\mathbb{P}} \dot{\mu}\tilde{A} \text{ is defined and equal to } (\mu A)\check{} = \tilde{\mu}\tilde{A}.$$

(d) Now take any $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$.

(i)

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ measures } \tilde{A} \cap U \text{ for every } U \in \tilde{U}.$$

P Let $p \in \mathbb{P}$ and $\dot{U}$ be such that

$$p \Vdash_{\mathbb{P}} \dot{U} \in \tilde{U}.$$

Then there are a q stronger than p and a $U \in \mathcal{U}$ such that

$$q \Vdash_{\mathbb{P}} \dot{U} = \tilde{U},$$

in which case

$$q \Vdash_{\mathbb{P}} \tilde{A} \cap \dot{U} = (A \cap U)\tilde{} \in \operatorname{dom} \dot{\mu}. \quad \mathbf{Q}$$

Since

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is a Radon measure and } \tilde{U} \text{ is an open cover of } \tilde{X},$$

we have

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ measures } \tilde{A}.$$

(ii) If $\gamma < \mu A$ is rational, there is a compact set $K \subseteq A$ such that $\mu K \ge \gamma$, and now

$$\Vdash_{\mathbb{P}} \tilde{K} \subseteq \tilde{A} \text{ and } \dot{\mu}\tilde{K} \ge \gamma, \text{ so } \dot{\mu}\tilde{A} \ge \gamma.$$

As γ is arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mu}\tilde{A} \ge (\mu A)\check{} = \tilde{\mu}\tilde{A}.$$

(iii) If $U \in \mathcal{U}$, then

$$\Vdash_{\mathbb{P}} \dot{\mu}(\tilde{A} \cap \tilde{U}) = \dot{\mu}(A \cap U)\tilde{} = \tilde{\mu}(A \cap U)\tilde{} \le \tilde{\mu}\tilde{A}.$$

So

$$\Vdash_{\mathbb{P}} \dot{\mu}(\tilde{A} \cap U) \le \tilde{\mu}\tilde{A} \text{ for every } U \in \tilde{\mathcal{U}}.$$

Since

$\Vdash_{\mathbb{P}} \dot{\mu}$ is a Radon measure and $\tilde{\mathcal{U}}$ is an upwards-directed family of open sets with union $\tilde{X}$, we have

$$\Vdash_{\mathbb{P}} \dot{\mu}\tilde{A} \le \tilde{\mu}\tilde{A}.$$

(iv) Putting these together, we see that

$$\Vdash_{\mathbb{P}} \dot{\mu}\tilde{A} = \tilde{\mu}\tilde{A}$$

for every $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$, so that

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ extends } \tilde{\mu}.$$

6B Theorem Let $\mathbb{P}$ be a forcing notion. Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces such that $J = \{i : i \in I, X_i \text{ is not compact}\}$ is countable. Let μ be the product Radon measure on $X = \prod_{i\in I} X_i$. Let $\dot{\mu}$, $\dot{\mu}_i$, for $i \in I$, be $\mathbb{P}$-names for Radon measures on $\tilde{X}$, $\tilde{X}_i$ respectively, defined as in 6A. Then

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ can be identified with the Radon product of } \langle \dot{\mu}_i\rangle_{i\in\check{I}}.$$

proof We need to begin by checking that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ can be identified with } \textstyle\prod_{i\in\check{I}} \tilde{X}_i;$$

this is Theorem 3A. Next, consider the base $\mathcal{U}$ for the topology of X consisting of open cylinder sets, and the corresponding name $\tilde{\mathcal{U}}$, so that

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a base for the topology of } \tilde{X} \text{ closed under finite intersections.}$$

If $U \in \mathcal{U}$, then U can be expressed as $\prod_{i\in I} U_i$ where $U_i \subseteq X_i$ is open for every i and $K = \{i : U_i \ne X_i\}$ is finite. In this case,

$$\Vdash_{\mathbb{P}} \tilde{U} \text{ is matched with } \textstyle\prod_{i\in I} \tilde{U}_i.$$

Moreover,

$\Vdash_{\mathbb{P}} \dot{\mu}\tilde{U} = \tilde{\mu}\tilde{U} = (\mu U)\check{} = (\prod_{i\in K} \mu_i U_i)\check{} = \prod_{i\in K}(\mu_i U_i)\check{} = \prod_{i\in K} \tilde{\mu}_i\tilde{U}_i = \prod_{i\in I} \dot{\mu}_i\tilde{U}_i$, so the Radon measure $\dot{\mu}$ agrees with the Radon product measure $\dot{\mu}^{\#}$ on $\tilde{U}$.

As $\tilde{U}$ is arbitrary,

$\Vdash_{\mathbb{P}} \dot{\mu}$ agrees with $\dot{\mu}^{\#}$ on $\tilde{\mathcal{U}}$, and as these are both Radon measures they must coincide.

6C I extract a couple of simple facts about quasi-Radon measures for use in the next theorem.

Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra.

(a) For every $E \in \Sigma$ there is an $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that $A \subseteq E$ and $E \setminus A$ is negligible.

(b) If $\mathcal{U}$ is any base for $\mathfrak{T}$ closed under finite unions, then $\{U^\bullet : U \in \mathcal{U}\}$ is dense in $\mathfrak{A}$ for the measure-algebra topology.

proof (a) Let $\mathcal{G}$ be the set of open sets of finite measure, and G^* its union. Let $\langle F_i\rangle_{i\in I}$ be a maximal disjoint family of non-empty self-supporting sets all included in members of $\mathcal{G}$. Then $X \setminus \bigcup_{i\in I} F_i$ is negligible, and every member of $\mathcal{G}$ meets only countably many of the F_i. For each $i \in I$, let $E_i \subseteq E \cap F_i$ be a Borel set such that $(E \cap F_i) \setminus E_i$ is negligible, and set $A = \bigcup_{i\in I} E_i$. By 1Bd, $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$. Of course $A \subseteq E$, and as $(E \setminus A) \cap F_i$ is negligible for every i, $E \setminus A$ is negligible.

(b) Suppose that $H \in \Sigma$ has finite measure, $E \in \Sigma$ and $\epsilon > 0$. Then there is an open set G of finite measure such that $\mu(E \setminus G) \le \epsilon$. There is a closed set $F \subseteq G \setminus H$ such that $\mu F \ge \mu(G \setminus H) - \epsilon$, so that $\mu((E \triangle (G \setminus F)) \cap H) \le 2\epsilon$. Because $\mathcal{U}$ is a base for $\mathfrak{T}$ closed under finite unions, there is a $U \in \mathcal{U}$ such that $U \subseteq G \setminus F$ and $\mu U \ge \mu(G \setminus F) - \epsilon$, so that $\mu((E \triangle U) \cap H) \le 3\epsilon$. As E, H and ϵ are arbitrary, we have the result.

6D Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $\mathbb{P}$ be a forcing notion, and $\dot{\mu}$ a $\mathbb{P}$-name for a Radon measure on $\tilde{X}$ as described in 6A; let $(\dot{\mathfrak{A}}, \dot{\bar{\mu}})$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} (\dot{\mathfrak{A}}, \dot{\bar{\mu}}) \text{ is the measure algebra of } \dot{\mu}.$$

Let $\dot{\varpi}$ be the $\mathbb{P}$-name

$$\{((A^\bullet)\check{}, \tilde{A}^\bullet), \mathbb{1}) : A \in \mathcal{U}\widehat{\mathcal{B}}(X)\}.$$

Then

$\Vdash_{\mathbb{P}} \dot{\varpi}$ is a measure-preserving Boolean homomorphism from $(\check{\mathfrak{A}}, \check{\bar{\mu}})$ to $(\dot{\mathfrak{A}}, \dot{\bar{\mu}})$, and $\dot{\varpi}[\check{\mathfrak{A}}]$ is dense in $\dot{\mathfrak{A}}$ for the measure-algebra topology.

proof (a) The first step is to check that

$$\Vdash_{\mathbb{P}} \dot{\varpi} \text{ is a function.}$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{a}$, $\dot{b}$, $\dot{c}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} (\dot{a}, \dot{b}) \text{ and } (\dot{a}, \dot{c}) \text{ belong to } \dot{\varpi}.$$

Then there are a q stronger than p and A, $B \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that

$$q \Vdash_{\mathbb{P}} \dot{a} = (A^\bullet)\check{} = (B^\bullet)\check{},\ \dot{b} = \tilde{A}^\bullet \text{ and } \dot{c} = \tilde{B}^\bullet.$$

In this case, $A^\bullet = B^\bullet$ in $\mathfrak{A}$, so $\mu(A \triangle B) = 0$ and

$$\Vdash_{\mathbb{P}} \dot{\mu}(\tilde{A} \triangle \tilde{B}) = \dot{\mu}(A \triangle B)\tilde{} = 0,$$

so

$$q \Vdash_{\mathbb{P}} \dot{b} = \dot{c}. \ \mathbf{Q}$$

(b) $\Vdash_{\mathbb{P}} \operatorname{dom} \dot{\varpi} = \check{\mathfrak{A}}$. **P** If $p \in \mathbb{P}$ and $\dot{a}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{a} \in \check{\mathfrak{A}}$, then there are a q stronger than p and an $a \in \mathfrak{A}$ such that $q \Vdash_{\mathbb{P}} \dot{a} = \check{a}$. Now 6C tells us that there is an $A \in \mathcal{U}\widehat{\mathcal{B}}(X)$ such that $A^\bullet = a$, so that

$$q \Vdash_{\mathbb{P}} (\dot{a}, \tilde{A}^\bullet) \in \dot{\varpi}. \ \mathbf{Q}$$

(c) It is now elementary to check that

$$\Vdash_{\mathbb{P}} \dot{\varpi} \text{ is a measure-preserving Boolean homomorphism.}$$

(d) As for the density of the range, use 6Cb. Let $\mathcal{U}$ be the family of open sets of finite measure in X, so that $\mathcal{U}$ is a base for $\mathfrak{T}$, and

$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}}$ is a base for the topology of $\tilde{X}$ closed under $\cup$, so $\{U^\bullet : U \in \tilde{\mathcal{U}}\}$ is dense in $\dot{\mathfrak{A}}$.

Since

$$\Vdash_{\mathbb{P}} \dot{\varpi}((U^\bullet)\check{}) = \tilde{U}^\bullet$$

for every $U \in \mathcal{U}$,

$$\Vdash_{\mathbb{P}} \dot{\varpi}[\check{\mathfrak{A}}] \supseteq \{U^\bullet : U \in \tilde{\mathcal{U}}\} \text{ is dense in } \dot{\mathfrak{A}}.$$

6E Proposition (see Džamonja & Kunen 95) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Maharam-type-homogeneous Radon measure space and κ its Maharam type. Let $\mathbb{P}$ be a forcing notion and $\dot{\mu}$ the $\mathbb{P}$-name for a Radon measure on $\tilde{X}$ derived from μ. Then

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is Maharam-type-homogeneous and its Maharam type is } \#(\kappa).$$

proof Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a stochastically independent generating set in the measure algebra $\mathfrak{A}$ of μ, all of measure $\frac{1}{2}$. Then, in the language of 6D,

$\Vdash_{\mathbb{P}} \langle \dot{\varpi}(\check{a}_\xi) \rangle_{\xi<\kappa}$ is a stochastically independent family of elements of measure $\frac{1}{2}$ in the measure algebra $\dot{\mathfrak{A}}$ of $\dot{\mu}$, and the algebra they generate is dense for the measure metric, so $\dot{\mathfrak{A}}$ is isomorphic to the measure algebra of the usual measure on $\{0,1\}^\kappa$.

6F Lemma Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, $E\in\mathcal{U}\widehat{\mathcal{B}}(X)$ and $\langle E_i\rangle_{i\in I}$ a family in $\mathcal{U}\widehat{\mathcal{B}}(X)$ such that $E^\bullet=\sup_{i\in I}E_i^\bullet$ in the measure algebra of μ. Then, for any forcing notion $\mathbb{P}$,

$$\Vdash_{\mathbb{P}}\tilde{E}^\bullet=\sup_{i\in\check{I}}\tilde{E}_i^\bullet\text{ in the measure algebra of }\dot{\mu}.$$

proof (a) For each $i\in I$, $\mu(E_i\setminus E)=0$ so (in the language of 6A)

$$\Vdash_{\mathbb{P}}\dot{\mu}(\tilde{E}_i\setminus\tilde{E})=\tilde{\mu}(E_i\setminus E)^{\sim}=0.$$

Accordingly

$$\Vdash_{\mathbb{P}}\sup_{i\in\check{I}}\tilde{E}_i^\bullet\subseteq\tilde{E}^\bullet.$$

(b) On the other side, writing $\mathcal{U}$ for the family of open subsets of X of finite measure, we have

for every $U\in\mathcal{U}$ and rational $\epsilon>0$ there is a finite $J\subseteq I$ such that $\mu(U\cap E)\le\mu(U\cap\bigcup_{i\in J}E_i)+\epsilon$.

So

$\Vdash_{\mathbb{P}}$ for every $U\in\tilde{\mathcal{U}}$ and rational $\epsilon>0$ there is a finite $J\subseteq\check{I}$ such that $\dot{\mu}(U\cap\tilde{E})\le\dot{\mu}(U\cap\bigcup_{i\in J}\tilde{E}_i)+\epsilon$.

But as

$$\Vdash_{\mathbb{P}}\tilde{U}\text{ is a cover of }\tilde{X}\text{ by open sets of finite measure,}$$

this is enough to show that

$$\Vdash_{\mathbb{P}}\tilde{E}^\bullet\subseteq\sup_{i\in I}\tilde{E}_i^\bullet\text{ in the measure algebra of }\dot{\mu}.$$

6G Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space, Y a Hausdorff space and $\phi:X\to Y$ an almost continuous function. Let $\mathbb{P}$ be a forcing notion. Then, defining $\tilde{\phi}$ as in 2C, and taking a $\mathbb{P}$-name $\dot{\mu}$ for a Radon measure on $\tilde{X}$ as in 6A,

$$\Vdash_{\mathbb{P}}\tilde{\phi}\text{ is a }\dot{\mu}\text{-almost continuous function from a conegligible subset of }\tilde{X}\text{ to }\tilde{Y}.$$

proof Let $\langle K_i\rangle_{i\in I}$ be a maximal disjoint family of non-empty self-supporting compact subsets of X such that $\phi\restriction K_i$ is continuous for every $i\in I$. Then 2C tells us that

$$\Vdash_{\mathbb{P}}\tilde{K}_i\subseteq\operatorname{dom}\tilde{\phi}\text{ and }\tilde{\phi}\restriction\tilde{K}_i\text{ is continuous for every }i\in\check{I}.$$

But 6F tells us that

$$\Vdash_{\mathbb{P}}\sup_{i\in\check{I}}\tilde{K}_i^\bullet=1\text{ in the measure algebra of }\dot{\mu},$$

and the result follows at once.

6H Theorem Let X be a locally compact Hausdorff group, and μ a left Haar measure on X. Let $\mathbb{P}$ be a forcing notion and $\dot{\mu}$ a $\mathbb{P}$-name for a Radon measure on $\tilde{X}$ as in 6A. Then

$$\Vdash_{\mathbb{P}}\dot{\mu}\text{ is a left Haar measure on }\tilde{X}$$

when $\tilde{X}$ is given its topological group structure as in 4C.

proof (a)

$$p\Vdash_{\mathbb{P}}\dot{\mu}(\dot{x}\cdot\dot{G})\ge\dot{\mu}\dot{G}$$

whenever $\dot{x}$ and $\dot{G}$ are $\mathbb{P}$-names and $p\in\mathbb{P}$ is such that

$$p\Vdash_{\mathbb{P}}\dot{x}\in\tilde{X}\text{ and }\dot{G}\text{ is an open subset of }\tilde{X}.$$

P Suppose that γ is rational, q is stronger than p and $q\Vdash_{\mathbb{P}}\gamma<\dot{\mu}\dot{G}$. Then there are an r stronger than q and an open $G\subseteq X$ such that

$$r\Vdash_{\mathbb{P}}\tilde{G}\subseteq\dot{G}\text{ and }\dot{\mu}\tilde{G}>\gamma.$$

In this case $\mu G > \gamma$, so there is a compact set $K \subseteq G$ such that $\mu K \ge \gamma$. Let H be an open neighbourhood of the identity such that $HK \subseteq G$. Because $\{xH^{-1} : x \in X\}$ is an open cover of X, there are an s stronger than r and an $x \in X$ such that

$$s \Vdash_{\mathbb{P}} \dot{x} \in (xH^{-1})\tilde{} = \tilde{x}\tilde{H}^{-1}$$

(4C(b-i) and (b-ii)). So

$$s \Vdash_{\mathbb{P}} (xK)\tilde{} = \tilde{x}\tilde{K} \subseteq \dot{x}\tilde{H}\tilde{K} = \dot{x}(HK)\tilde{} \subseteq \dot{x}\tilde{G}$$

(4C(b-iii)) and

$$s \Vdash_{\mathbb{P}} \dot{\mu}(\dot{x}\tilde{G}) \ge \dot{\mu}(xK)\tilde{} = \mu(xK)\check{} = (\mu K)\check{} \ge \gamma.$$

As q and γ are arbitrary,

$$p \Vdash_{\mathbb{P}} \dot{\mu}(\dot{x} \cdot \dot{G}) \ge \dot{\mu}\dot{G}. \quad \mathbf{Q}$$

(b) Since we already know that

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is a Radon measure on the Hausdorff topological group } \tilde{X},$$

this is enough to show that

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is a left Haar measure on } \tilde{X}.$$

6I Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite Radon measure space and $\dot{\mu}$ a $\mathbb{P}$-name for a Radon measure on $\tilde{X}$ defined from μ as in 6A. Suppose that W is a Borel set in $Z \times X$ and that $\vec{W}$ is the corresponding $\mathbb{P}$-name for a subset of $\tilde{X}$, as in 2E. Then there is an $f \in C^-(Z; [0, \infty])$ such that $f(z) = \mu W[\{z\}]$ for every $z \in \operatorname{dom} f$ and

$$\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = \vec{f}.$$

proof (a) Suppose that W is open. Then $g(z) = \mu W[\{z\}]$ is defined for every z, and g is lower semi-continuous, so there is an $f \in C^-(Z; [0, \infty])$ such that $f \subseteq g$. Also $\Vdash_{\mathbb{P}} \vec{W}$ is open (2E(c-ii)). (α) If γ is rational, $z \in \operatorname{dom} f$ and $f(z) > \gamma$, then there are an open set U containing z and an open set $G \subseteq X$ such that $\mu G > \gamma$ and $U \times G \subseteq W$. Now there is a q stronger than p such that $z \in \widehat{q} \subseteq U$, so that

$$q \Vdash_{\mathbb{P}} \tilde{G} \subseteq \vec{W} \text{ and } \dot{\mu}W \ge \dot{\mu}\tilde{G} = (\mu G)\check{} > \gamma.$$

Thus $z \in \widehat{q} \subseteq [\![\dot{\mu}\vec{W} > \gamma]\!]$. As z is arbitrary, $f^{-1}[\,]\gamma, \infty]\,] \subseteq [\![\dot{\mu}\vec{W} > \gamma]\!]$ and

$$\Vdash_{\mathbb{P}} \text{if } \vec{f} > \gamma \text{ then } \dot{\mu}\vec{W} > \gamma;$$

as γ is arbitrary,

$$\Vdash_{\mathbb{P}} \vec{f} \le \dot{\mu}\vec{W}.$$

(β) If γ is rational, $p \in \mathbb{P}$ and $p \Vdash_{\mathbb{P}} \dot{\mu}\vec{W} > \gamma$, then let $\langle (q_i, G_i) \rangle_{i \in I}$ run over the set

$$\{(q, G) : q \in \mathbb{P},\ \widehat{q} \times G \subseteq W\},$$

and set $W_i = \widehat{q_i} \times G_i$ for $i \in I$. Then $\bigcup_{i \in I} W_i = W$, so by 2Eg we have

$$\Vdash_{\mathbb{P}} \vec{W} = \bigcup_{i \in \check{I}} \vec{W}_i = (\bigcup_{i \in I} W_i)\vec{}.$$

There must therefore be a q stronger than p and a finite set $J \subseteq I$ such that

$$q \Vdash_{\mathbb{P}} \dot{\mu}(\bigcup_{i \in \check{J}} \vec{W}_i) > \gamma.$$

But now there is an r stronger than q such that, for every $i \in J$, either r is stronger than q_i or r is incompatible with q_i; setting K for the set $\{i : i \in J,\ r \text{ is stronger than } q_i\}$ and $G = \bigcup_{i \in K} G_i$, we have

$$r \Vdash_{\mathbb{P}} \bigcup_{i \in \check{J}} \vec{W}_i = \tilde{G},$$

so

$$r \Vdash_{\mathbb{P}} \dot{\mu}\tilde{G} > \gamma$$

and $\mu G > \gamma$. Also $\widehat{q} \times G \subseteq W$ so $f(z) > \gamma$ for every $z \in \widehat{q}$ and $q \Vdash_{\mathbb{P}} \vec{f} > \gamma$. As p and γ are arbitrary,

$$\Vdash_{\mathbb{P}} \vec{f} \ge \dot{\mu}\vec{W} \text{ and } \vec{f} = \dot{\mu}\vec{W}.$$

(b) Let $G \subseteq X$ be an open set of finite measure and consider

$$\mathcal{W}_G = \{W : W \subseteq Z \times G \text{ is Borel and satisfies the conclusion of the theorem}\}.$$

Then $\mathcal{W}_G$ is a Dynkin class of subsets of $Z \times G$. **P** (α) $\emptyset \in \mathcal{W}_G$, witnessed by f constant 0. (β) If $W \in \mathcal{W}_G$, witnessed by $f \in C^-(Z;[0,\infty])$, then $f(z) \le \mu G$ for every $z \in \operatorname{dom} f$. Set $g(z) = \mu G - f(z)$ for $z \in \operatorname{dom} f$; then g witnesses that $(Z \times G) \setminus W$ belongs to $\mathcal{W}_G$, because

$$\Vdash_{\mathbb{P}} ((Z \times G) \setminus W)^{\to} = (Z \times G)^{\to} \setminus \vec{W} = \tilde{G} \setminus \vec{W}$$

(2E(c-iii), 2Ed) and $\Vdash_{\mathbb{P}} \dot{\mu}\tilde{G} = (\mu G)^{\check{}}$. (γ) If $\langle W_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}_G$, witnessed by $\langle f_n \rangle_{n\in\mathbb{N}}$ in $C^-(Z;X)$, and $W = \bigcup_{n\in\mathbb{N}} W_n$, set $h(z) = \sup_{n\in\mathbb{N}} f_n(z)$ for $z \in \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n$. Then h is lower semi-continuous and $\operatorname{dom} h$ is comeager in the extremally disconnected set Z, so there is a comeager G_δ set $V \subseteq \operatorname{dom} h$ such that $g = h \restriction V$ is continuous. Now

$$\Vdash_{\mathbb{P}} \vec{g} = \sup_{n\in\mathbb{N}} \vec{f}_n = \sup_{n\in\mathbb{N}} \dot{\mu}\vec{W}_n = \dot{\mu}(\bigcup_{n\in\mathbb{N}} \vec{W}_n) = \dot{\mu}\vec{W}$$

and g witnesses that $W \in \mathcal{W}_G$. **Q**

With (a), this is enough to show that $W \cap (Z \times G) \in \mathcal{W}_G$ for every Borel set $W \subseteq Z \times X$.

(c) Because μ is σ-finite, there is a non-decreasing sequence $\langle G_n \rangle_{n\in\mathbb{N}}$ of open sets of finite measure with conegligible union in X. Repeating the argument of (b-γ) just above, we see that if $W \subseteq Z \times X$ is Borel and $X_0 = \bigcup_{n\in\mathbb{N}} G_n$, then $W \cap (Z \times X_0)$ satisfies the conclusion of the theorem. But now

$$\Vdash_{\mathbb{P}} (W \setminus (Z \times X_0))^{\to} \subseteq (X \setminus X_0)^{\sim} \text{ is } \dot{\mu}\text{-negligible, so } \dot{\mu}\vec{W} = \dot{\mu}(W \cap (Z \times X_0))^{\to},$$

$$\mu W[\{z\}] = \mu(W \cap (Z \times X_0))[\{z\}] \text{ for every } z \in Z,$$

so W also does.

6J Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite Radon measure space, $\mathbb{P}$ a forcing notion and $\dot{\mu}$ a corresponding $\mathbb{P}$-name for a Radon measure on $\tilde{X}$. Suppose that $\dot{E}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{E} \text{ is a } \dot{\mu}\text{-negligible subset of } \tilde{X}.$$

Then there is a G_δ subset W of $Z \times X$ such that

$$\Vdash_{\mathbb{P}} \dot{E} \subseteq \vec{W},$$

$$W[\{z\}] \text{ is } \mu\text{-negligible for every } z \in Z.$$

proof Because

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is } \sigma\text{-finite}$$

(apply 6G to a suitable sequence $\langle E_n \rangle_{n\in\mathbb{N}}$), there is a $\mathbb{P}$-name $\dot{H}$ such that

$$\Vdash_{\mathbb{P}} \dot{H} \text{ is a } \dot{\mu}\text{-negligible } \mathrm{G}_\delta \text{ set including } \dot{E}.$$

By 2Eh, there is a G_δ set $V \subseteq Z \times X$ such that

$$\Vdash_{\mathbb{P}} \dot{H} = \vec{V}.$$

By 6I, there is an $h \in C^-(Z;X)$ such that $h(z) = \mu H[\{z\}]$ for $z \in \operatorname{dom} h$ and

$$\Vdash_{\mathbb{P}} \vec{h} = \dot{\mu}\dot{H}.$$

But this means that $h(z) = 0$ for every $z \in \operatorname{dom} h$. Set $W = V \cap (\operatorname{dom} h \times X)$; then $\vec{W} = \vec{V}$ (see the definition in 2Eb), so $\Vdash_{\mathbb{P}} \dot{E} \subseteq \vec{W}$, while W is G_δ and has negligible vertical sections.

6K Example We really do need 'σ-finite' in the last two results, as the following elementary example shows. Let $\mathbb{P}$ be any atomless forcing notion, so that Z has no isolated points. Let X be Z with its discrete topology, and μ counting measure on X. Consider $W = \{(z,z) : z \in Z\}$. Then $W \subseteq Z \times X$ is closed and $\mu W[\{z\}] = 1$ for every z. But $\vec{W} = \emptyset$ so $\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = 0$.

However there may be much more to be said along the lines of 6I; see 12D.

6L Theorem Let $\mathbb{P}$ be a forcing notion, $\langle X_i \rangle_{i\in I}$ a family of Hausdorff spaces, each either Polish or compact, and μ a Baire probability measure on $X = \prod_{i\in I} X_i$. Then there is a $\mathbb{P}$-name $\dot{\mu}$ such that

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is a Baire probability measure on } \prod_{i\in\check{I}} \tilde{X}_i$$

and whenever $J \subseteq I$ is finite and $G_i \subseteq X_i$ is cozero for each $i \in J$,

$$\Vdash_{\mathbb{P}} \dot{\mu}\{x : x(\check{\imath}) \in \tilde{G}_i \ \forall\, i \in J\} = (\mu\{x : x(i) \in G_i \ \forall\, i \in J\})\check{}.$$

proof For each finite $J \subseteq I$ we have a unique Radon probability measure μ_J on $X_J = \prod_{i\in J} X_i$ such that $\mu_J H = \mu\{x : x\restriction J \in H\}$ for every cozero $H \subseteq X_J$. By 6A (with 3A) we have a corresponding name $\dot{\mu}_J$ such that

$\Vdash_{\mathbb{P}} \dot{\mu}_J$ is a Radon probability measure on $\tilde{X}_J \cong \prod_{i\in J} \tilde{X}_i$ and $\dot{\mu}_J\tilde{H} = (\mu_J H)\check{}$ for every open $H \subseteq X_J$.

It follows at once that if $J \subseteq K \in [I]^{<\omega}$ then

$$\Vdash_{\mathbb{P}} \text{the natural map from } \tilde{X}_K \text{ to } \tilde{X}_J \text{ is inverse-measure-preserving.}$$

Now 4A tells us that

$$\Vdash_{\mathbb{P}} \text{for every } i \in \check{I},\ \tilde{X}_i \text{ is either Polish or compact,}$$

so

$$\Vdash_{\mathbb{P}} \mathcal{B}\mathfrak{a}(\textstyle\prod_{i\in\check{I}} \tilde{X}_i) = \widehat{\bigotimes}_{i\in\check{I}} \mathcal{B}\mathfrak{a}(\tilde{X}_i)$$

(Fremlin n05). Using Kolmogorov's theorem (Fremlin 03, 454D-454G) in $V^{\mathbb{P}}$, we see that there is a $\mathbb{P}$-name $\dot{\mu}$ such that

$\Vdash_{\mathbb{P}} \dot{\mu}$ is a Baire measure on $\prod_{i\in\check{I}} \tilde{X}_i$ such that the canonical map onto every $\tilde{X}_J$ is inverse-measure-preserving for $\dot{\mu}$ and $\dot{\mu}_J\restriction\mathcal{B}(\tilde{X}_J)$.

This $\dot{\mu}$ will serve.

6M Theorem Let X be a compact Hausdorff space and $\mathbb{P}$ a forcing notion, with Z the Stone space of its regular open algebra. Let $\dot{\mu}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{\mu} \text{ is a Radon probability measure on } \tilde{X}.$$

Then there is a family $\langle \mu_z \rangle_{z\in Z}$ of Radon probability measures on X such that whenever $W \subseteq Z \times X$ is a Borel set then

$$\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = \vec{h}_W,$$

where $h_W(z) = \mu_z W[\{z\}]$ for every $z \in Z$.

proof (a) For $E \in \mathcal{B}(X)$ let $f_E \in C(Z;[0,1])$ be such that

$$\Vdash_{\mathbb{P}} \vec{f}_E = \dot{\mu}\tilde{E}.$$

Note that f_E is uniquely determined by this, so that $E \mapsto f_E : \mathcal{B}(X) \to C(Z;[0,1])$ is additive. For $z \in Z$, $E \mapsto f_E(z) : \mathcal{B}(X) \to [0,1]$ is additive, so there is a Radon measure μ_z on X such that $f_K(z) \le \mu_z K$ for every compact $K \subseteq X$ and $f_G(z) \ge \mu_z G$ for every open $G \subseteq X$; of course $\mu_z X = 1$.

(b) If $W \subseteq Z \times X$ is open, let $g_W \in C(Z;[0,1])$ be such that $\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = \vec{g}_W$. Then $h_W(z) \le g_W(z)$ for every $z \in Z$. **P** Suppose that $\gamma < h_W(z)$. Consider

$$\mathcal{G} = \{G : G \subseteq X \text{ is open, } U \times G \subseteq W \text{ for some open } U \subseteq Z \text{ containing } z\}.$$

Then $\mathcal{G}$ is upwards-directed and has union $W[\{z\}]$, so there is a $G \in \mathcal{G}$ such that $\mu_z G \ge \gamma$; let U be an open neighbourhood of z such that $U \times G \subseteq W$. Then

$$U \subseteq [\![\tilde{G} \subseteq \vec{W}]\!] \subseteq [\![\dot{\mu}\tilde{G} \le \dot{\mu}\vec{W}]\!]$$

(2J(a-iii)) and $f_G(z) \le g_W(z)$ for every $z \in U$ (4D(b-iii)). But now

$$\gamma \le \mu_z G \le f_G(z) \le g_W(z).$$

As γ is arbitrary, $h_W(z) \le g_W(z)$. **Q**

(b) If $W \subseteq Z \times X$ is open, then $\{z : g_W(z) \le h_W(z)\}$ is comeager. **P** Let $n \in \mathbb{N}$. Let $\mathcal{V}$ be the family of finite unions of sets of the form $U \times G$ where $U \subseteq Z$ is open-and-closed, $G \subseteq X$ is open and $U \times \overline{G} \subseteq W$; let $\tilde{\mathcal{V}}$ be the $\mathbb{P}$-name $\{(\vec{V}, \mathbb{1}) : V \in \mathcal{V}\}$. Then $\mathcal{V}$ is an upwards-directed family of open sets with union W, so by 2Ef we have

$\Vdash_{\mathbb{P}} \tilde{\mathcal{V}}$ is an upwards-directed family of open sets with union $\vec{W}$, and $\dot{\mu}\vec{W} = \sup_{V \in \tilde{\mathcal{V}}} \dot{\mu}V\}$.

There are therefore a maximal antichain $D \subseteq \mathbb{P}$ and a family $\langle V_d \rangle_{d \in D}$ in $\mathcal{V}$ such that

$$d \Vdash_{\mathbb{P}} \dot{\mu}\vec{V}_d \ge \dot{\mu}\vec{W} - 2^{-n}$$

for every $d \in D$.

Take $d \in D$ and $z \in \widehat{d}$. Express V_d as $\bigcup_{i \le n} U_i \times G_i$ where $U_i \subseteq Z$ is open-and-closed and $G_i \subseteq X$ is open for each i; set $J = \{i : i \le n,\ z \in U_i\}$, $U = \widehat{d} \cap \bigcap_{i \in J} U_i$ and $G = \bigcup_{i \in J} G_i$, so that $z \in U$, $U \times \overline{G} \subseteq W$ and

$$h_W(z) \ge \mu_z \overline{G} \ge f_{\overline{G}}(z) \ge f_G(z).$$

Also

$$z \in U \subseteq [\![\vec{V}_d = \tilde{G}]\!] \subseteq [\![\dot{\mu}\vec{W} - 2^{-n} \le \dot{\mu}\vec{V}_d = \dot{\mu}\tilde{G}]\!]$$

and $g_W(z) - 2^{-n} \le f_G(z)$. Putting these together,

$$g_W(z) - 2^{-n} \le h_W(z).$$

This is true for every $z \in \bigcup_{d \in D} \widehat{d}$, which is a dense open subset of Z. So $\{z : g_W(z) - 2^{-n} \le h_W(z)\}$ has dense interior. As n is arbitrary, $\{z : g_W(z) \le h_W(z)\}$ is comeager. **Q**

(c) Thus $\{z : g_W(z) = h_W(z)\}$ is comeager and $\vec{g}_W = \vec{h}_W$. So

$$\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = \vec{g}_W = \vec{h}_W.$$

And this is true for every open $W \subseteq Z \times X$. Using the formulae of 2E and the Monotone Class Theorem, we have

$$\Vdash_{\mathbb{P}} \dot{\mu}\vec{W} = \vec{h}_W$$

for every Borel set $W \subseteq Z \times X$, as claimed.

7 Second-countable spaces and Borel functions

7A Theorem Let $\mathbb{P}$ be a forcing notion, X a Hausdorff space, Y a second-countable Hausdorff space and $\phi : X \to Y$ a $\mathcal{U}\widehat{\mathcal{B}}(X)$-measurable function.

(a) $\Vdash_{\mathbb{P}} \tilde{\phi}$ is a function from $\tilde{X}$ to $\tilde{Y}$.

(b) If $B \in \mathcal{U}\widehat{\mathcal{B}}(Y)$, then

$$\Vdash_{\mathbb{P}} \tilde{\phi}^{-1}[\tilde{B}] = (\phi^{-1}[B])^{\sim}.$$

(c) If ϕ is Borel measurable,

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ is Borel measurable.}$$

proof (a) Suppose that $p \in \mathbb{P}$ and $\dot{x}$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$. Let $f \in C^-(Z; X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Then f is $(\widehat{\mathcal{B}}(Z), \mathcal{U}\widehat{\mathcal{B}}(X))$-measurable (by the definition of $\mathcal{U}\widehat{\mathcal{B}}(X)$), so ϕf is $\widehat{\mathcal{B}}(Z)$-measurable and defined on a comeager subset of Z. Because Y is second-countable, there is a $g \in C^-(Z; Y)$ such that $\mathrm{dom}(g \cap \phi f)$ is comeager. So $((\vec{f}, \vec{g}), \mathbb{1}) \in \tilde{\phi}$ and

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \tilde{\phi}(\vec{f}) = \vec{g} \text{ is defined.}$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \operatorname{dom}\tilde{\phi} = \tilde{X}. \ \mathbf{Q}$$

(b) By 1G, $\phi^{-1}[B] \in \mathcal{U}\widehat{\mathcal{B}}(X)$, so $\phi^{-1}[B]^{\sim}$ is defined. If $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$, let $f \in C^{-}(Z;X)$ and $g \in C^{-}(Z;Y)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \tilde{\phi}(\dot{x}) = \vec{g}$$

and $\widehat{p} \subseteq^* \operatorname{dom}(g \cap \phi f)$. Then

$$\begin{aligned} p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{\phi}^{-1}[\tilde{B}] \iff p \Vdash_{\mathbb{P}} \vec{g} \in \tilde{B} \iff \widehat{p} \subseteq^* g^{-1}[B] \\ \iff \widehat{p} \subseteq^* f^{-1}[\phi^{-1}[B]] \iff p \Vdash_{\mathbb{P}} \dot{x} \in \phi^{-1}[B]^{\sim}. \end{aligned}$$

(c) Let $\mathcal{U}$ be a countable base for the topology of Y. Then

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ is a countable base for the topology of } \tilde{Y}.$$

Since

$$\Vdash_{\mathbb{P}} \tilde{\phi}^{-1}[\tilde{U}] = \phi^{-1}[U]^{\sim} \text{ is a Borel set in } \tilde{X}$$

for every $U \in \mathcal{U}$ (2Ae),

$$\Vdash_{\mathbb{P}} \tilde{\phi}^{-1}[U] \in \mathcal{B}(\tilde{X}) \text{ for every } U \in \tilde{\mathcal{U}}, \text{ and } \tilde{\phi} \text{ is Borel measurable.}$$

7B Proposition Let $\mathbb{P}$ be a forcing notion, X a Hausdorff space, Y a second-countable Hausdorff space and $\langle\phi_n\rangle_{n\in\mathbb{N}}$ a sequence of $\mathcal{U}\widehat{\mathcal{B}}(X)$-measurable functions such that $\phi(x) = \lim_{n\to\infty}\phi_n(x)$ is defined for every $x \in X$. Then

$$\Vdash_{\mathbb{P}} \lim_{n\to\infty}\tilde{\phi}_n(x) = \tilde{\phi}(x) \text{ for every } x \in \tilde{X}.$$

proof Let $p \in \mathbb{P}$ and $\dot{x}$ be such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$. For each $n \in \mathbb{N}$ let $f_n \in C^{-}(Z;X)$ and $g_n \in C^{-}(Z;Y)$ be such that

$$p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}_n \text{ and } \tilde{\phi}_n(\dot{x}) = \vec{g}_n$$

and

$$\widehat{p} \subseteq^* \operatorname{dom}(g_n \cap \phi_n f_n).$$

Set

$$W = \{z : z \in \textstyle\bigcap_{n\in\mathbb{N}} \operatorname{dom}(g_n \cap \phi_n f_n),\ f_m(z) = f_n(z) \text{ for all } m,\, n \in \mathbb{N}\},$$

so that $\widehat{p} \subseteq^* W$. For $z \in W$,

$$g(z) = \lim_{n\to\infty}\phi_n f_n(z) = \lim_{n\to\infty} g_n(z)$$

is defined, and $g : W \to Y$ is $\widehat{\mathcal{B}}(Z)$-measurable, so there is an $h \in C^{-}(Z;Y)$ such that $W \subseteq^* \operatorname{dom}(g \cap h)$. Now

$$\widehat{p} \subseteq^* \{z : \lim_{n\to\infty} g_n(z) = h(z)\},$$

By 2F,

$$p \Vdash_{\mathbb{P}} \vec{g} = \vec{h} = \lim_{n\to\infty}\vec{g}_n = \lim_{n\to\infty}\tilde{\phi}_n(\dot{x}).$$

At the same time, for $z \in W$,

$$g(z) = \lim_{n\to\infty}\phi_n f_0(z) = \phi f_0(z),$$

so

$$p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) = \tilde{\phi}(\vec{f}_0) = \vec{g} = \lim_{n\to\infty}\tilde{\phi}_n(\dot{x}).$$

As p and $\dot{x}$ are arbitrary, we have the result. $\mathbf{Q}$

7C Corollary Let $\mathbb{P}$ be a forcing notion, X a Hausdorff space and α a countable ordinal. If $\phi : X \to \mathbb{R}$ belongs to the αth Baire class, then

$$\Vdash_{\mathbb{P}} \tilde{\phi} \text{ belongs to the } \alpha\text{th Baire class.}$$

proof Induce on α.

7D Proposition Let X be a Hausdorff space and Φ a set of $\mathcal{U}\widehat{\mathcal{B}}(X)$-measurable real-valued functions such that $\{\phi(x) : \phi \in \Phi\}$ is bounded for every $x \in X$. Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$. Then there is a $\mathbb{P}$-name $\dot{\alpha}$ such that $p \Vdash_{\mathbb{P}} \dot{\alpha} \in \mathbb{N}$ and $p \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) \le \dot{\alpha}$ for every $\phi \in \Phi$.

proof ? Otherwise, let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. There is a q stronger than p such that whenever r is stronger than q and $n \in \mathbb{N}$ there are an s stronger than r and a $\phi \in \Phi$ such that $s \Vdash_{\mathbb{P}} \tilde{\phi}(\dot{x}) > n$, that is, $\widehat{s} \subseteq^* \{z : z \in \operatorname{dom} f,\ \phi f(z) > n\}$. So for each $n \in \mathbb{N}$ we have a maximal antichain R_n and a family $\langle \phi_{nr} \rangle_{r \in R_n}$ such that for each $r \in R_n$ either r is incompatible with q or $\phi_{nr} \in \Phi$ and $\widehat{r} \subseteq^* \{z : \phi_{nr} f(z) > n\}$. Now for almost every $z \in \widehat{q}$ we have

$$\text{for every } n \in \mathbb{N} \text{ there is an } r \in R_n \text{ such that } \phi_{nr} f(z) > n,$$

and $\{\phi f(z) : \phi \in \Phi\}$ is unbounded above. **X**

Remark Compare §A, Lemma 1 of TODORČEVIĆ 99.

7E Proposition Let X be an analytic Hausdorff space, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathbb{P}$. If $W \subseteq Z \times X$ is a Borel set then $[\![\vec{W} \ne \emptyset]\!] \triangle W^{-1}[X]$ is meager.

proof (a) If W is open there is a K-analytic $V \subseteq Z \times X$ such that $(W \triangle V)^{-1}[X]$ is meager. **P** Let $h : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. For each $\sigma \in S^* = \bigcup_{n \in \mathbb{N}} \mathbb{N}^{\mathbb{N}}$ set $X_\sigma = \{h(\alpha) : \sigma \subseteq \alpha \in \mathbb{N}^{\mathbb{N}}\}$, so that X_σ is analytic; set $H_\sigma = \operatorname{int}\{z : \{z\} \times X_\sigma \subseteq W\}$. Then $W = \bigcup_{\sigma \in S^*} H_\sigma \times X_\sigma$. Set $V = \bigcup_{\sigma \in S^*} \overline{H}_\sigma \times X_\sigma$; then V is K-analytic and $(W \triangle V)^{-1}[X] \subseteq \bigcup_{\sigma \in S^*} \overline{H}_\sigma \setminus H_\sigma$ is meager. **Q**

(b) Consider the family of those sets $W \subseteq Z \times X$ such that there are K-analytic V_1, $V_2 \subseteq Z \times X$ such that $(W \triangle V_1)^{-1}[X]$ and $(((Z \times X) \setminus W) \triangle V_2)^{-1}[X]$ are meager; then $\mathcal{W}$ is closed under complements and countable unions and contains all the open subsets of $Z \times X$, so includes $\mathcal{B}(Z \times X)$.

In particular, if $W \subseteq Z \times X$ is Borel, there is a K-analytic V such that $(W \triangle V)^{-1}[X]$ is meager, so that $[\![\vec{W} \ne \emptyset]\!] = [\![\vec{V} \ne \emptyset]\!]$ differs by a meager set from each of $V^{-1}[X]$ and $W^{-1}[X]$ (using 2Jc).

7F Proposition Let X be a separable metrizable space, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathbb{P}$. Suppose that $W \subseteq Z \times X$ is K-analytic, and set $W^+ = \{(z,x) : z \in Z,\ x \in \overline{W[\{z\}]}\}$. Then $\Vdash_{\mathbb{P}} \overline{\vec{W}} = \overrightarrow{W^+}$.

proof (a) Suppose that $p \in \mathbb{P}$, $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \overrightarrow{W^+}$ and $\dot{G}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{G}$ is an open neighbourhood of $\dot{x}$. Then there are a q stronger than p, an $f \in C^-(Z;X)$ and an open $G \subseteq X$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \vec{f} \text{ and } \vec{f} \in \tilde{G} \text{ and } \tilde{\overline{G}} \subseteq \dot{G}.$$

Now

$$\widehat{q} \subseteq^* \{z : f(z) \in \overline{W[\{z\}]} \cap G\} \subseteq (W \cap (Z \times \overline{G}))^{-1}[X] \subseteq^* [\![\vec{W} \cap \tilde{\overline{G}} \ne \emptyset]\!]$$

by 2Jc. So there is a $\mathbb{P}$-name $\dot{y}$ such that $q \Vdash_{\mathbb{P}} \dot{y} \in \vec{W} \cap \tilde{\overline{G}} \subseteq \vec{W} \cap \dot{G}$. As p, $\dot{x}$ and $\dot{G}$ are arbitrary, $\Vdash_{\mathbb{P}} \overrightarrow{W^+} \subseteq \overline{\vec{W}}$.

(b) Suppose that $p \in \mathbb{P}$ and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \overline{\vec{W}}$. Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. Let $\mathcal{G}$ be a countable base for the topology of X. Then

$$\{z : z \in \operatorname{dom} f,\ f(z) \notin \overline{W[\{z\}]}\} = \bigcup_{G \in \mathcal{G}} \{z : f(z) \in G,\ \overline{G} \cap W[\{z\}] = \emptyset\}.$$

For each $G \in \mathcal{G}$,

$$\widehat{p} \cap \{z : f(z) \in G,\ \overline{G} \cap W[\{z\}] = \emptyset\} \subseteq^* \widehat{p} \cap [\![\dot{x} \in \tilde{G},\ \tilde{\overline{G}} \cap \vec{W} = \emptyset]\!] = \emptyset,$$

so $\widehat{p} \cap \{z : z \in \operatorname{dom} f,\ f(z) \notin \overline{W[\{z\}]}\}$ is meager and $p \Vdash_{\mathbb{P}} \dot{x} \in \overrightarrow{W^+}$. As p and $\dot{x}$ are arbitrary, $\Vdash_{\mathbb{P}} \overline{\vec{W}} \subseteq \overrightarrow{W^+}$.

7G Corollary Let X be an analytic separable metrizable space, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathbb{P}$. Suppose that $W \subseteq Z \times X$ is Borel, and set $W^+ = \{(z,x) : z \in Z,\, x \in \overline{W[\{z\}]}\}$. Then $\Vdash_{\mathbb{P}} \vec{W} = \overrightarrow{W^+}$.

proof As in the proof of 7E, we can find a K-analytic set V to put in the place of W and apply 7F to V.

8 Forcing with quotient algebras

For random and Cohen reals, in the first place, but in other cases too, we have a forcing notion which is naturally representable as the non-zero elements of a Boolean algebra which comes to us as a quotient algebra $\Sigma/\mathcal{I}$ where Σ is a σ-algebra of subsets of a set Ω. In this case, it is often helpful to be able to represent names for members of $\tilde{X}$ by functions from Ω to X. I run through some simple cases in which we can do this.

8A Definitions A **measurable space with negligibles** is a triple $(\Omega,\Sigma,\mathcal{I})$ such that Ω is a set, Σ is a σ-algebra of subsets of Ω, and $\mathcal{I}$ is a σ-ideal of subsets of Ω generated by $\Sigma\cap\mathcal{I}$. It is **non-trivial** if $\Omega \notin \mathcal{I}$, **complete** if $\Sigma \subseteq \mathcal{I}$, ω_1**-saturated** if there is no uncountable disjoint family in $\Sigma\setminus\mathcal{I}$, that is, the quotient algebra $\Sigma/\Sigma\cap\mathcal{I}$ is ccc. Note that $\mathfrak{A}$ is always Dedekind σ-complete, so if (Ω,Σ,μ) is ω_1-saturated it is Dedekind complete. If $(\Omega,\Sigma,\mathcal{I})$ is non-trivial, the **associated forcing notion** is $\Sigma\setminus\mathcal{I}$, active downwards. Note that the regular open algebra of this forcing notion can be identified with the Dedekind completion of the quotient Boolean algebra $\mathfrak{A} = \Sigma/\Sigma\cap\mathcal{I}$, and that for $E \in \mathbb{P}$ the corresponding member $\widehat{E}$ of $\mathrm{RO}(\mathbb{P})$ is just the equivalence class $E^\bullet \in \mathfrak{A}$.

(For the general theory of measurable spaces with negligibles, see Fremlin 87.)

8B Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion; set $\mathfrak{A} = \Sigma/\Sigma\cap\mathcal{I}$.

(a) If X is either compact or Polish, and $f : \Omega \to X$ is $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable, then there is a $\mathbb{P}$-name $\dot{x}$ such that $\Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$ and, for every $F \in \mathcal{B}\mathfrak{a}(X)$,

$$[\![\dot{x} \in \tilde{F}]\!] = f^{-1}[F]^\bullet$$

in $\mathfrak{A} \subseteq \mathrm{RO}(\mathbb{P})$.

(b) Suppose that $\mathfrak{A}$ is Dedekind complete and that X is Polish. If $\dot{x}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$, then there is a Σ-measurable function $f : \Omega \to X$ such that

$$[\![\dot{x} \in \tilde{F}]\!] = f^{-1}[F]^\bullet$$

for every $F \in \mathcal{B}(X)$.

(c) Still supposing that $\mathfrak{A}$ is Dedekind complete, let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces with product X, and $\dot{x}$ a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{x} \in \prod_{i\in\check{I}} \tilde{X}_i$. Then there is a $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable function $f : \Omega \to X$ such that whenever $J \subseteq X$ is countable and $F \subseteq \prod_{i\in J} X_i$ is a Borel set, then

$$[\![\dot{x}\restriction\check{J} \in \tilde{F}]\!] = f^{-1}[\{x : x \in X,\, x\restriction J \in F\}]^\bullet.$$

proof (a) We have a sequentially order-continuous Boolean homomorphism $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathrm{RO}(\mathbb{P})$ defined by saying that $\pi F = f^{-1}[F]^\bullet$ for every $F \in \mathcal{B}\mathfrak{a}(X)$. Let Z be the Stone space of $\mathrm{RO}(\mathbb{P})$. By 3Fb, there is a $g \in C^-(Z;X)$ such that $[\![\vec{g} \in \tilde{F}]\!] = \pi F$ for every Baire set $F \subseteq X$. So we can take $\dot{x} = \vec{g}$.

(b) Let $\mathcal{U}$ be a countable base for the topology of X containing X. For each $U \in \mathcal{U}$, let $E_U \in \Sigma$ be such that $E_U^\bullet = [\![\dot{x} \in \tilde{U}]\!]$ in $\mathfrak{A}$; arrange that $E_X = \Omega$. (This is where we need to suppose that $\mathfrak{A}$ is Dedekind complete, so that we can identify it with $\mathrm{RO}(\mathbb{P})$.) As in (b-iv) of the proof of 3F, let ρ be a complete metric defining the topology of X and such that $\operatorname{diam} X$ is finite, and for $U \in \mathcal{U}$ let $\mathcal{V}_U$ be $\{V : V \in \mathcal{U},\, \overline{V} \subseteq U,\, \operatorname{diam} V \le \frac{1}{2}\operatorname{diam} U\}$. Set

$$\begin{aligned} B = (\bigcup_{U\in\mathcal{U}} E_U \setminus \bigcup_{V\in\mathcal{V}_U} E_V) &\cup \bigcup\{E_U \setminus E_V : U,\, V \in \mathcal{U},\, U \subseteq V\} \\ &\cup \bigcup\{E_U \cap E_V : U,\, V \in \mathcal{U},\, U\cap V = \emptyset\} \\ \in \mathcal{I}. \end{aligned}$$

Then for $\omega\in\Omega\setminus B$ there is a unique $x\in X$ such that $\omega\in E_U$ whenever $x\in U\in\mathcal{U}$. **P** Choose $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $\omega\in E_{U_n}$, $\overline{U}_{n+1}\subseteq U_n$ and $\operatorname{diam}U_n\le 2^{-n}\operatorname{diam}X$ for each n; let x be the member of $\bigcap_{n\in\mathbb{N}}U_n$. If $U\in\mathcal{U}$ and $x\in U$, there is an $n\in\mathbb{N}$ such that $U_n\subseteq U$, in which case $\omega\in E_U$. If $y\in X$ and $y\ne x$, there are $n\in\mathbb{N}$, $V\in\mathcal{U}$ such that $y\in V$ and $U_n\cap V=\emptyset$, in which case $\omega\notin E_V$. **Q**

So we can define $f_0:\Omega\setminus B\to X$ by saying that $\omega\in E_U$ whenever $f_0(\omega)\in U\in\mathcal{U}$, and any extension f of f_0 to a function which is constant on B will be measurable and have the property that $[\![\dot x\in\tilde U]\!]=f^{-1}[U]^\bullet$ for every $U\in\mathcal{U}$, and therefore for every Borel set $U\subseteq X$.

(c) Choose $\mathbb{P}$-names $\dot x_i$ such that $\Vdash_{\mathbb{P}}\dot x_i\in\tilde X_i$ for each $i\in I$ and $\Vdash_{\mathbb{P}}\dot x=\langle\dot x_i\rangle_{i\in\check I}$. By (b), we have a measurable function $f_i:\Omega\to X_i$ such that $f_i^{-1}[F]^\bullet=[\![\dot x_i\in\tilde F]\!]$ for every Baire set $F\subseteq X_i$. Setting $f(\omega)=\langle f_i(\omega)\rangle_{i\in I}$ for $\omega\in\Omega$, $f:\Omega\to X$ is $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable, because $\mathcal{B}\mathfrak{a}(X)=\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$. If $J\subseteq I$ is countable, then

$$[\![\dot x\restriction\check J\in\tilde F]\!]=f^{-1}[\{x:x\in X,\,x\restriction J\in F\}]^\bullet$$

whenever $F\subseteq\prod_{i\in J}X_i$ is of the form $\prod_{i\in J}F_i$ and every F_i is a Baire set; by the Monotone Class Theorem, the formula is valid for every Borel set $F\subseteq\prod_{i\in J}F_i$.

8C Notation Suppose that, as in 8B, we have a non-trivial measurable space with negligibles $(\Omega,\Sigma,\mathcal{I})$ with quotient algebra $\mathfrak{A}$ and associated forcing notion $\mathbb{P}$, and a topological space X which is either compact or Polish. If $f:\Omega\to X$ is $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable, then we can define $\overrightarrow{f}$ to be the $\mathbb{P}$-name $\breve\pi$, as defined in 3F-3G, where $\pi F=f^{-1}[F]^\bullet$ for $F\in\mathcal{B}\mathfrak{a}(X)$. In this case, $\Vdash_{\mathbb{P}}\overrightarrow{f}\in\tilde X$ and $[\![\overrightarrow{f}\in\tilde F]\!]=f^{-1}[F]^\bullet$ for every $F\in\mathcal{B}\mathfrak{a}(X)$.

When we have a family $\langle X_i\rangle_{i\in I}$ of spaces with product X, each of the X_i being either Polish or compact, and a $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable function $f:\Omega\to X$, write $\overrightarrow{\overrightarrow{f}}$ for the $\mathbb{P}$-name

$$\{(\langle\overrightarrow{f_i}\rangle_{i\in\check I},\mathbb{1})\},$$

where $f_i(\omega)=f(\omega)(i)$ for $\omega\in\Omega$ and $i\in I$; of course the subformula $\langle\overrightarrow{f_i}\rangle_{i\in\check I}$ must be interpreted in the forcing language, as noted in the footnote to 2A(b-vi).

The content of 8Bb is now expressible by saying that if X is a Polish space and $\mathfrak{A}$ is Dedekind complete, $\dot x$ is a $\mathbb{P}$-name and $E\in\mathbb{P}$ is such that $E\Vdash_{\mathbb{P}}\dot x\in\tilde X$, then there is a measurable $f:\Omega\to X$ such that $E\Vdash_{\mathbb{P}}\dot x=\overrightarrow{f}$.

Moving to 8Bc, we see that if $\dot x$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}\dot x\in\prod_{i\in\check I}\tilde X_i$, there is a $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable $f:\Omega\to X$ such that $\Vdash_{\mathbb{P}}\dot x=\overrightarrow{\overrightarrow{f}}$.

I should perhaps remark that if, in 8Ba, X is a non-metrizable compact space, then we can have $(\Sigma,\mathcal{B}\mathfrak{a}(X))$-measurable functions f, $g:\Omega\to X$ which are nowhere equal but for which $\overrightarrow{f}=\overrightarrow{g}$ because $f^{-1}[F]\triangle g^{-1}[F]\in\mathcal{I}$ for every Baire set $F\subseteq X$. What we do have, for both Polish and compact X, is: if $E\in\Sigma\setminus\mathcal{I}$ and $f,g:\Omega\to\mathcal{B}\mathfrak{a}(X)$ are $(\Sigma,\mathcal{B}\mathfrak{a})$-measurable, then $E\Vdash_{\mathbb{P}}\overrightarrow{f}=\overrightarrow{g}$ iff $E\setminus(f^{-1}[F]\triangle g^{-1}[F])\in\mathcal{I}$ for every Baire set $F\subseteq X$. **P** Let h_f, $h_g\in C(Z;X)$ be the functions defined from the Boolean homomorphisms π_f, $\pi_g:\mathcal{B}\mathfrak{a}(X)\to\mathfrak{A}$, so that $\overrightarrow{f}=\breve\pi_f=\vec h_f$ and $\overrightarrow{g}=\vec h_g$. Now, for $E\in\Sigma\setminus\mathcal{I}$,

$$\begin{aligned}E\Vdash_{\mathbb{P}}\overrightarrow{f}=\overrightarrow{g}&\iff E^\bullet\setminus(\pi_fF\triangle\pi_gF)=0\text{ for every }F\in\mathcal{B}\mathfrak{a}(X)\\ &\text{(3H)}\\ &\iff E\setminus(f^{-1}[F]\triangle g^{-1}[F])\in\mathcal{I}\text{ for every }F\in\mathcal{B}\mathfrak{a}(X).\ \textbf{Q}\end{aligned}$$

8D Representing names for sets Let $(\Omega,\Sigma,\mathcal{I})$ be a non-trivial measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion; let $\mathfrak{A}$ be the quotient $\Sigma/\Sigma\cap\mathcal{I}$. Let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces with product X, and $W\subseteq\Omega\times X$. Write $\overrightarrow{\overrightarrow{W}}$ for the $\mathbb{P}$-name

$$\{(\overrightarrow{f},E) : E \in \Sigma \setminus \mathcal{I},\ f : \Omega \to X \text{ is } (\Sigma, \mathcal{B}\mathfrak{a}(X))\text{-measurable}, \\ E \setminus \{\omega : (\omega, f(\omega)) \in W\} \in \mathcal{I}\},$$

defining the $\mathbb{P}$-name $\overrightarrow{f}$ as in 8C.

8E Theorem Let $(\Omega, \Sigma, \mathcal{I})$ be a non-trivial measurable space with negligibles, $\mathfrak{A}$ be the quotient $\Sigma/\Sigma \cap \mathcal{I}$, and $\mathbb{P}$ the associated forcing notion. Let $\langle X_i \rangle_{i \in I}$ be a family of Polish spaces with product X.

(a) If $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}(X)$, then

$$\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{W}} \in \mathcal{B}\mathfrak{a}(\textstyle\prod_{i \in \check{I}} \tilde{X}_i).$$

(b) Now suppose that $(\Omega.\Sigma, \mathcal{I})$ is ω_1-saturated. Let $\dot{W}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}(\textstyle\prod_{i \in \check{I}} \tilde{X}_i).$$

Then there is a $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}(X)$ such that $\Vdash_{\mathbb{P}} \dot{W} = \overrightarrow{\overrightarrow{W}}$.

proof (a) Let $\mathcal{W}$ be the family of those $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}(X)$ such that

(α) for every $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable function $f : \Omega \to X$, $\{\omega : (\omega, f(\omega)) \in W\} \in \Sigma$;

(β) $\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{W}} \in \mathcal{B}\mathfrak{a}(\prod_{i \in \check{I}} \tilde{X}_i)$.

(i) The key fact is the following: if $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}(X)$, $f : \Omega \to X$ is a $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable function and $E = \{\omega : (\omega, f(\omega)) \in W\} \in \Sigma$, then $E^\bullet = [\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}]\!]$. **P** Since (if $E \notin \mathcal{I}$) $(\overrightarrow{f}, E) \in \overrightarrow{\overrightarrow{W}}$, we surely have

$$E \Vdash_{\mathbb{P}} \overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}, \quad E^\bullet \subseteq [\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}]\!].$$

? If $[\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}]\!] \setminus E^\bullet$ is non-zero, then there must be an $E' \in \mathbb{P}$ and a $g : \Omega \to \mathbb{R}$ such that $E' \Vdash_{\mathbb{P}} \overrightarrow{g} = \overrightarrow{f}$, $(\overrightarrow{g}, E') \in \overrightarrow{\overrightarrow{W}}$ and $E' \setminus E \notin \mathcal{I}$. In this case $E' \setminus \{\omega : (\omega, g(\omega)) \in W\} \in \mathcal{I}$. Because $\mathcal{I}$ is generated by $\Sigma \setminus \mathcal{I}$, there is an $E'' \subseteq E' \setminus E$ such that $E'' \in \Sigma \setminus \mathcal{I}$ and E'' is disjoint from $\{\omega : (\omega, g(\omega)) \notin W\}$. Now, however, recall that as $W \in \Sigma \widehat{\otimes} \mathcal{B}\mathfrak{a}(X)$ there is a countable set $J \subseteq I$ such that $(\omega, x) \in W$ whenever $(\omega, y) \in W$, $x \in X$ and $x \restriction J = y \restriction J$. Take $i \in J$ and let f_i, $g_i : \Omega \to X_i$ be the corresponding coordinates of f, g respectively. Then

$$E'' \Vdash_{\mathbb{P}} \overrightarrow{f_i} = \overrightarrow{g_i},$$

so, for any open set $G \subseteq X_i$, $E'' \cap (f_i^{-1}[G] \triangle g_i^{-1}[G]) \in \mathcal{I}$. As X_i is second-countable, $E'' \cap \{\omega : f_i(\omega) \neq g_i(\omega)\} \in \mathcal{I}$. This is true for every $i \in J$, so

$$E'' = E'' \cap \{\omega : (\omega, f(\omega)) \notin W,\ (\omega, g(\omega)) \in W\} \subseteq E'' \cap \textstyle\bigcup_{i \in J} \{\omega : f_i(\omega) \neq g_i(\omega)\} \in \mathcal{I},$$

which is impossible. **X** So $p = 0$ and $E^\bullet = [\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}]\!]$. **Q**[6]

(ii) If $W \in \mathcal{W}$ then $W' = (\Omega \times X) \setminus W$ belongs to $\mathcal{W}$. **P** (α) If $f : \Omega \to X$ is $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable then

$$\{\omega : (\omega, f(\omega)) \in W'\} = \Omega \setminus \{\omega : (\omega, f(\omega)) \in W\} \in \Sigma.$$

(β) Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \prod_{i \in \check{I}} \tilde{X}_i$. For each $i \in I$ let $f_i : \Omega \to X_i$ be a measurable function such that $p \Vdash_{\mathbb{P}} \overrightarrow{f_i} = \dot{x}(\check{\imath})$ (8B-8C). Setting $f(\omega) = \langle f_i(\omega) \rangle_{i \in I}$ for $\omega \in \Omega$, f is $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable; set $E = \{\omega : (\omega, f(\omega)) \in W\}$. Then

$$[\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W'}}]\!] = (\Omega \setminus E)^\bullet = 1 \setminus [\![\overrightarrow{f} \in \overrightarrow{\overrightarrow{W}}]\!],$$

[6] I don't know if it is clear what is going on here. If $\Vdash_{\mathbb{P}} \overrightarrow{f} = \overrightarrow{g}$, then, for each $i \in I$, $\Vdash_{\mathbb{P}} \overrightarrow{f_i} = \overrightarrow{g_i}$; because X is second-countable and Hausdorff, $f_i = g_i$ almost everywhere (that is, except on a set belonging to $\mathcal{I}$). But it does not at all follow that $f = g$ almost everywhere. The point of this argument is that because W is determined by coordinates in a countable subset of I, we can ignore all the other coordinates.

so

$$\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{f}} \in \overrightarrow{\overrightarrow{W'}} \iff \overrightarrow{\overrightarrow{f}} \notin \overrightarrow{\overrightarrow{W}}$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} \in \overrightarrow{\overrightarrow{W'}} \iff \dot{x} \notin \overrightarrow{\overrightarrow{W}}.$$

As p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{W'}} = (\prod_{i\in\check{I}} \tilde{X}_i) \setminus \overrightarrow{\overrightarrow{W}} \in \mathcal{B}\mathfrak{a}(\prod_{i\in\check{I}} \tilde{X}_i). \quad \mathbf{Q}$$

(iii) If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{W}$ then $W = \bigcup_{n\in\mathbb{N}} W_n$ belongs to $\mathcal{W}$. **P** (α) If $f : \Omega \to X$ is $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable then

$$\{\omega : (\omega, f(\omega)) \in W\} = \bigcup_{n\in\mathbb{N}}\{\omega : (\omega, f(\omega)) \in W_n\} \in \Sigma.$$

(β) If $p \in \mathbb{P}$ and $\dot{x}$ is such that $p \Vdash_{\mathbb{P}} \dot{x} \in \prod_{i\in\check{I}} \tilde{X}_i$, then, as before, there is a $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable function f such that $p \Vdash_{\mathbb{P}} \dot{x} = \overrightarrow{\overrightarrow{f}}$. Now

$$\begin{aligned}[\![\overrightarrow{\overrightarrow{f}} \in \overrightarrow{\overrightarrow{W}}]\!] &= \{\omega : (\omega, f(\omega)) \in W\}^\bullet = \sup_{n\in\mathbb{N}}\{\omega : (\omega, f(\omega)) \in W_n\}^\bullet \\ &= \sup_{n\in\mathbb{N}} [\![\overrightarrow{\overrightarrow{f}} \in \overrightarrow{\overrightarrow{W_n}}]\!] = [\![\overrightarrow{\overrightarrow{f}} \in \textstyle\bigcup_{n\in\mathbb{N}} \overrightarrow{\overrightarrow{W_n}}]\!]\end{aligned}$$

and

$$p \Vdash_{\mathbb{P}} \dot{x} \in \overrightarrow{\overrightarrow{W}} \iff \dot{x} \in \bigcup_{n\in\mathbb{N}} \overrightarrow{\overrightarrow{W_n}};$$

as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{W}} = \bigcup_{n\in\mathbb{N}} \overrightarrow{\overrightarrow{W_n}} \in \mathcal{B}\mathfrak{a}(\prod_{i\in\check{I}} \tilde{X}_i). \quad \mathbf{Q}$$

(iv) Of course $\emptyset \in \mathcal{W}$, so $\mathcal{W}$ is a σ-subalgebra of $\Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}(X)$. If $E \in \Sigma$, $j \in I$ and $G \subseteq X_j$ is open, then $W = E \times \{x : x \in X,\, x(j) \in G\}$ belongs to $\mathcal{W}$. **P** (α) If $f : \Omega \to X$ is $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable then

$$\{\omega : (\omega, f(\omega)) \in W\} = E \cap f_j^{-1}[G_j] \in \Sigma,$$

writing $f_j(\omega) = f(\omega)(j)$ as usual. (β) Let $\dot{V}$ be a $\mathbb{P}$-name such that

$$[\![\dot{V} = \{x : x \in \prod_{i\in\check{I}} \tilde{X}_i,\, x(\check{\jmath}) \in \tilde{G}\}]\!] = E^\bullet,$$

$$[\![\dot{V} = \emptyset]\!] = (\Omega \setminus E)^\bullet.$$

We have

$$\Vdash_{\mathbb{P}} \tilde{X}_j \text{ is Polish, so } \tilde{G} \text{ is a cozero set in } \tilde{X}_j \text{ and } \dot{V} \text{ is a cozero set in } \prod_{i\in\check{I}} \tilde{X}_i.$$

If $f : \Omega \to X$ is $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable, then

$$[\![\overrightarrow{\overrightarrow{f}} \in \overrightarrow{\overrightarrow{W}}]\!] = (E \cap f_j^{-1}[G])^\bullet = E^\bullet \cap [\![\overrightarrow{\overrightarrow{f}}(\check{\jmath}) \in \tilde{G}]\!] = [\![\overrightarrow{\overrightarrow{f}} \in \dot{V}]\!].$$

If $p \in \mathbb{P}$ and $\dot{x}$ are such that $p \Vdash_{\mathbb{P}} \dot{x} \in \prod_{i\in\check{I}} \tilde{X}_i$, let f be a $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable function such that $p \Vdash_{\mathbb{P}} \dot{x} = \overrightarrow{\overrightarrow{f}}$. Then

$$\begin{aligned}p \Vdash_{\mathbb{P}} \dot{x} \in \overrightarrow{\overrightarrow{W}} &\iff p \subseteq [\![\overrightarrow{\overrightarrow{f}} \in \overrightarrow{\overrightarrow{W}}]\!] \\ &\iff p \subseteq [\![\overrightarrow{\overrightarrow{f}} \in \dot{V}]\!] \iff p \Vdash_{\mathbb{P}} \dot{x} \in \dot{V};\end{aligned}$$

as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \overrightarrow{\overrightarrow{W}} = \dot{V} \in \mathcal{B}\mathfrak{a}(\prod_{i\in\check{I}} \tilde{X}_i). \quad \mathbf{Q}$$

(v) So $\mathcal{W} = \mathcal{B}\mathfrak{a}(X)$, as required.

(b) Let us say that a $\mathbb{P}$-name $\dot{W}$ is 'representable' if there is some $W \in \Sigma\widehat{\otimes}\mathcal{B}\mathfrak{a}(X)$ such that $\Vdash_{\mathbb{P}} \dot{W} = \vec{\vec{W}}$.

(i)(*α*) If $\dot{W}$ is a representable $\mathbb{P}$-name and $\dot{V}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{V} = (\prod_{i \in \check{I}} \tilde{X}_i) \setminus \dot{W}$, then $\dot{V}$ is representable. **P** If $\Vdash_{\mathbb{P}} \dot{W} = \vec{\vec{W}}$ then $\Vdash_{\mathbb{P}} \dot{V} = ((\Omega \times X) \setminus W)^{\Rightarrow}$, as in (a-ii). **Q**

(*β*) If $\langle \dot{W}_n \rangle_{n \in \mathbb{N}}$ is a sequence of representable names, and $\dot{W}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{W} = \bigcup_{n \in \mathbb{N}} \dot{W}_n$, then $\dot{W}$ is representable. **P** See (a-iii). **Q**

(*γ*) If $\langle \dot{W}_n \rangle_{n \in \mathbb{N}}$ is a sequence of representable names, and $\dot{W}$ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{W} = \bigcap_{n \in \mathbb{N}} \dot{W}_n$, then $\dot{W}$ is representable. **P** Put (α) and (β) together. **Q**

(ii) Let $\dot{W}$ be a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \dot{W} \subseteq \prod_{i \in \check{I}} \tilde{X}_i$.

(*α*) Suppose there are $j \in I$, $a \in \mathfrak{A}$ and an open set $G \subseteq X_j$ such that

$$a = [\![\dot{W} = \{x : x(\check{\jmath}) \in \tilde{G}\}]\!], \quad 1 \setminus a = [\![\dot{W} = \emptyset]\!].$$

Then $\dot{W}$ is representable. **P** Let $F \in \Sigma$ be such that $F^\bullet = a$ and set $W = F \times \{x : x(j) \in G\}$. Then for any $(\Sigma, \mathcal{B}\mathfrak{a}(X))$-measurable $f : \Omega \to X$,

$$\begin{aligned}[\![\,\vec{\vec{f}} \in \vec{\vec{W}}]\!] &= (F \cap f_j^{-1}[G])^\bullet \\ &= a \cap [\![\vec{f_j} \in \tilde{G}]\!] = [\![\,\vec{\vec{f}} \in \dot{W}]\!].\end{aligned}$$

As f is arbitrary, 8Bc-8C show that $\Vdash_{\mathbb{P}} \dot{W} = \vec{\vec{W}}$ and $\dot{W}$ is representable. **Q**

(*β*) Suppose there are $j \in I$ and a $\mathbb{P}$-name $\dot{G}$ such that

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open set in } \tilde{X}_j \text{ and } \dot{W} = \{x : x(\check{\jmath}) \in \dot{G}\}.$$

Then $\dot{W}$ is representable. **P** Let $\langle U_n \rangle_{n \in \mathbb{N}}$ run over a base for the topology of X_j. For $n \in \mathbb{N}$ set $a_n = [\![\tilde{U}_n \subseteq \dot{G}]\!]$ and choose $F_n \in \Sigma$ such that $F_n^\bullet = a_n$. Set $W_n = F_n \times \{x : x(j) \in U_n\}$ and $W = \bigcup_{n \in \mathbb{N}} W_n$; then

$$\Vdash_{\mathbb{P}} \dot{G} = \bigcup\{\tilde{U}_n : n \in \mathbb{N}, \tilde{U}_n \subseteq \dot{G}\},$$

so

$$\Vdash_{\mathbb{P}} \dot{W} = \bigcup_{n \in \mathbb{N}, \tilde{U}_n \subseteq \dot{G}_n} \{x : x(\check{\jmath}) \in \tilde{U}_n\} = \bigcup_{n \in \mathbb{N}} \vec{\vec{W_n}} = \vec{\vec{W}}$$

and $\dot{W}$ is representable. **Q**

(*γ*) Suppose that

$$\Vdash_{\mathbb{P}} \dot{W} \text{ is a basic open cylinder in } \textstyle\prod_{i \in \check{I}} \tilde{X}_i.$$

Then $\dot{W}$ is representable. **P** Use (β) and (i-γ). **Q**

(*δ*) Suppose that

$$\Vdash_{\mathbb{P}} \dot{W} \text{ is a cozero set.}$$

Then $\dot{W}$ is representable. **P** By 4Ag,

$$\Vdash_{\mathbb{P}} \tilde{X}_i \text{ is Polish for every } i \in \check{I}.$$

So

$$\Vdash_{\mathbb{P}} \dot{W} \text{ is the union of a sequence of basic open sets,}$$

and we can use (i-β). **Q**

(*ϵ*) Suppose that there is an $\alpha < \omega_1$ such that

$$\Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_\alpha(\textstyle\prod_{i \in \check{I}} \tilde{X}_i).$$

Then $\dot{W}$ is representable. **P** Induce on α. **Q**

(iii) Finally, suppose that $\Vdash_{\mathbb{P}} \dot{W}$ is a Baire set. Because $\mathbb{P}$ is ccc, there is an $\alpha < \omega_1$ such that $\Vdash_{\mathbb{P}} \dot{W} \in \mathcal{B}\mathfrak{a}_\alpha(\prod_{i \in \check{I}} \tilde{X}_i)$, so $\dot{W}$ is representable, as required.

8F Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a measurable space with negligibles and $\mathbb{P}$ the associated forcing notion. Let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces and μ a Baire probability measure on $X=\prod_{i\in I}X_i$; let $\dot\mu$ be a corresponding $\mathbb{P}$-name for a Baire probability measure on $\prod_{i\in\check I}\tilde X_i$ as in 6L. Take $W\in \mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(X)$ and define $\overrightarrow{\overrightarrow{W}}$ as in 8D. Set $f(\omega)=\mu W[\{\omega\}]$ for $\omega\in\Omega$. Then $f:\Omega\to[0,1]$ is T-measurable, so $\Vdash_{\mathbb{P}}\overrightarrow{f}\in[0,1]$ (8C, 4Db). Now

$$\Vdash_{\mathbb{P}}\dot\mu\overrightarrow{\overrightarrow{W}}=\overrightarrow{f}.$$

proof Use the method of the proof of 8Ea.

8G Liftings Let $(\Omega,\Sigma,\mathcal{I})$ be a measurable space with negligibles. A **lifting** for $(\Omega,\Sigma,\mathcal{I})$ is a Boolean homomorphism $\theta:\mathfrak{A}\to\Sigma$, where $\mathfrak{A}=\Sigma/\Sigma\cap\mathcal{I}$, such that $a=(\theta a)^\bullet$ for every $a\in\mathfrak{A}$. Note that if (Ω,Σ,μ) is a complete σ-finite measure space and $\mathcal{N}(\mu)$ is the null ideal of μ, then $(\Omega,\Sigma,\mathcal{N}(\mu))$ has a lifting (Fremlin 02, §341); similarly, if W is a Baire topological space and $\mathcal{M}$ is the ideal of meager subsets of W, then $(W,\widehat{\mathcal{B}}(W),\mathcal{M})$ has a lifting.

If $\theta:\mathfrak{A}\to\Sigma$ is a lifting, and Z is the Stone space of $\mathfrak{A}$, we have a corresponding map $h:\Omega\to Z$ defined by saying that $h(\omega)(a)=\chi(\theta a)(\omega)$ for $\omega\in\Omega$ and $a\in\mathfrak{A}$; that is, writing $\widehat a$ for the open-and-closed subset of Z corresponding to $a\in\mathfrak{A}$, $h^{-1}[\widehat a]=\theta a$.

If $\theta:\mathfrak{A}\to\Sigma$ is a lifting, the **lifting topology** $\mathfrak{T}_\theta$ on Ω is the topology generated by the algebra $\theta[\mathfrak{A}]$.

A lifting $\theta:\mathfrak{A}\to\Sigma$ is **strong** if for every $E\in\Sigma\setminus\mathcal{I}$ there is a non-zero $a\in\mathfrak{A}$ such that $\theta a\subseteq E$. (See Fremlin 03, §453.)

8H Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a measurable space with negligibles and $\theta:\mathfrak{A}\to\Sigma$ a lifting, where $\mathfrak{A}=\Sigma/\mathcal{I}$. Let Z be the Stone space of $\mathfrak{A}$, $h:\Omega\to Z$ the function associated with θ and $\mathfrak{T}_\theta$ the lifting topology.

(a)(i) h is continuous for $\mathfrak{T}_\theta$ and the usual topology of Z.

(ii) If $W\subseteq Z$ is a dense open set, $h^{-1}[W]$ is dense for $\mathfrak{T}_\theta$.

(iii) If $M\subseteq Z$ is meager, $h^{-1}[M]$ is meager for $\mathfrak{T}_\theta$.

(iv) h is $(\widehat{\mathcal{B}}(\Omega),\widehat{\mathcal{B}}(Z))$-measurable.

(b) Suppose that $(\Omega,\Sigma,\mathcal{I})$ is complete and ω_1-saturated.

(i) $\mathfrak{T}_\theta\subseteq\Sigma$.

(ii) Every $\mathfrak{T}_\theta$-meager set belongs to $\mathcal{I}$ and $\widehat{\mathcal{B}}(\Omega)\subseteq\Sigma$. So h is $(\Sigma,\widehat{\mathcal{B}}(Z))$-measurable.

(c) Now suppose that $(\Omega,\Sigma,\mathcal{I})$ is complete and ω_1-saturated, and that θ is a strong lifting. Then $\mathcal{I}$ is the ideal of $\mathfrak{T}_\theta$-nowhere-dense sets, and $\Sigma=\widehat{\mathcal{B}}(\Omega)$.

proof (a)(i) This is immediate from the definition of $\mathfrak{T}_\theta$, since $\{\widehat a:a\in\mathfrak{A}\}$ is a base for the topology of Z.

(ii) If $G\in\mathfrak{T}_\theta$ is non-empty, there is a non-zero $a\in\mathfrak{A}$ such that $\widehat a\subseteq G$. Now $\widehat a$ is a non-empty open subset of Z, so there is a non-zero $b\in\mathfrak{A}$ such that $\widehat b\subseteq W\cap\widehat a$. In this case, θb is a non-empty subset of $W\cap G$. As G is arbitrary, $h^{-1}[W]$ is dense.

(iii)-(iv) follow immediately from (ii).

(b)(i) If $E\in\mathfrak{T}_\theta$, set $A=\{a:\theta a\subseteq E\}$. Then A is upwards-directed; because $\mathfrak{A}$ is ccc, there is a non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A such that $c=\sup_{n\in\mathbb{N}}a_n$ is also the supremum of A. Set $E'=\bigcup_{n\in\mathbb{N}}\theta a_n\in\Sigma$; then $E'\subseteq E=\bigcup_{a\in A}\theta a\subseteq\theta c$. But also $(\theta c\setminus E')^\bullet=c\setminus\sup_{n\in\mathbb{N}}a_n=0$, so $\theta c\setminus E'\in\mathcal{I}$; because $(\Omega,\Sigma,\mathcal{I})$ is complete, $E\setminus E'$ and E belong to Σ.

(ii) If $F\subseteq\Omega$ is a nowhere dense closed set, then it belongs to $\mathcal{I}$. **P** Set $a=F^\bullet$. If $b\in\mathfrak{A}$ and $\theta b\cap F=\emptyset$, then $a\cap b=F^\bullet\cap(\theta b)^\bullet=0$ and $\theta a\cap\theta b=\emptyset$; as F is closed, $\theta a\subseteq F$; as F is nowhere dense, $a=0$ and $F\in\mathcal{I}$. **Q**

As $\mathcal{I}$ is a σ-ideal, every meager set belongs to $\mathcal{I}$. Since $\mathcal{I}\subseteq\Sigma$ and $\mathfrak{T}_\theta\subseteq\Sigma$, $\widehat{\mathcal{B}}(\Omega)\subseteq\Sigma$. By (a-iv), h is $(\Sigma,\widehat{\mathcal{B}}(Z))$-measurable.

(c) If $E\in\mathcal{I}$, consider its closure $\overline{E}$ for $\mathfrak{T}_\theta$. There can be no non-zero $a\in\mathfrak{A}$ such that $\theta a\subseteq\overline{E}\setminus E$, so $\overline{E}\setminus E\in\mathcal{I}$ and $\overline{E}\in\mathcal{I}$; so there is no non-zero a such that $\theta a\subseteq\overline{E}$ and E is nowhere dense. If E is any member of Σ, set $F=\theta E^\bullet$; then $E\triangle F\in\mathcal{I}$ so $E\in\widehat{\mathcal{B}}(\Omega)$. With (b-ii) this shows that $\mathcal{I}$ is just the ideal of nowhere dense sets and $\Sigma=\widehat{\mathcal{B}}(\Omega)$.

8I Proposition Let $(\Omega,\Sigma,\mathcal{I})$ be a complete ω_1-saturated measurable space with negligibles, and $\mathbb{P}$ the associated forcing notion; suppose that $\theta:\mathfrak{A}\to\Sigma$ is a lifting, where $\mathfrak{A}=\Sigma/\mathcal{I}$. Let X be a Hausdorff space, and $\dot{x}$ a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}}\dot{x}\in\tilde{X}$. Then there is a $(\Sigma,\mathcal{U}\widehat{\mathcal{B}}(X))$-measurable function $g:\Omega\to X$ such that $[\![\dot{x}\in\tilde{F}]\!]=g^{-1}[F]^\bullet$ for every $F\in\mathcal{U}\widehat{\mathcal{B}}(X)$.

proof Let Z be the Stone space of $\mathrm{RO}(\mathbb{P})\cong\mathfrak{A}$, and $h:\Omega\to Z$ the function associated with θ as in 8G-8H. Let $f\in C^-(Z;X)$ be such that $\Vdash_{\mathbb{P}}\dot{x}=\vec{f}$. Then $h^{-1}[Z\setminus\operatorname{dom}f]\in\mathcal{I}$ (put 8H(a-iii) and 8H(b-ii) together). Let $g:\Omega\to X$ be any function extending fh. If $F\in\mathcal{U}\widehat{\mathcal{B}}(X)$ then $f^{-1}[F]\in\widehat{\mathcal{B}}(Z)$ so $h^{-1}[f^{-1}[F]]\in\Sigma$ (putting 8H(a-iv) and 8H(b-ii) together). Now there is an $a\in\mathfrak{A}$ such that $f^{-1}[F]\triangle\widehat{a}$ is meager, in which case $h^{-1}[f^{-1}[F]]\triangle\theta a\in\mathcal{I}$ and

$$g^{-1}[F]^\bullet=h^{-1}[f^{-1}[F]]^\bullet=a=[\![\vec{f}\in\tilde{F}]\!]=[\![\dot{x}\in\tilde{F}]\!],$$

as required.

9 Banach spaces

9A Theorem Let X be a normed space, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathbb{P}$.

(a)

$$\Vdash_{\mathbb{P}}\tilde{X},\text{ with its natural linear space structure and norm, is a normed space.}$$

(b) Write $X^*_{w^*}$ for the dual of X with its weak* topology. Then we have a $\mathbb{P}$-name for a bilinear duality between $\tilde{X}$ and $(X^*_{w^*})^\sim$ such that

$$\Vdash_{\mathbb{P}}(\tilde{g}|\tilde{f})=(g|f)^\sim$$

whenever $g\in C^-(Z;X^*_{w^*})$ and $f\in C^-(Z;X)$, writing $(g|f)(z)=g(z)(f(z))$ for $z\in\operatorname{dom}f\cap\operatorname{dom}g$.

(c) Now

$$\Vdash_{\mathbb{P}}\text{ this duality identifies }(X^*_{w^*})^\sim\text{ with the normed space dual of }\tilde{X}.$$

proof (a) The checks are straightforward; compare 4C. The algebraic operations are given by the formulae

$$\Vdash_{\mathbb{P}}\tilde{f}+\tilde{g}=(f+g)^\sim,\ \tilde{a}\tilde{f}=(af)^\sim$$

for f, $g\in C^-(Z;X)$ and $a\in C^-(Z;\mathbb{R})$, setting $(f+g)(z)=f(z)+g(z)$ for $z\in\operatorname{dom}f\cap\operatorname{dom}g$ and $(af)(z)=a(z)f(z)$ for $z\in\operatorname{dom}a\cap\operatorname{dom}f$. The norm is given by the formula

$$\Vdash_{\mathbb{P}}\|\tilde{f}\|=\|f\|^\sim,$$

where $\|f\|(z)=\|f(z)\|$ for $z\in\operatorname{dom}f$.

(b)(i) The first thing to check is that for any $f\in C^-(Z;X)$ and $g\in C^-(Z;X^*_{w^*})$ there is a dense G_δ set $Z_0\subseteq\operatorname{dom}f\cap\operatorname{dom}g$ such that $(g|f)\restriction Z_0$ is continuous. **P** For $n\in\mathbb{N}$, set $V_n=\{z:z\in\operatorname{dom}g,\ \|g(z)\|\le n\}$. Then V_n is relatively closed in $\operatorname{dom}g$ so $\overline{V}_n\setminus\operatorname{int}\overline{V_n}$ is nowhere dense in Z; set $Z_0=\operatorname{dom}f\cap\operatorname{dom}g\setminus\bigcup_{n\in\mathbb{N}}\overline{V}_n\setminus\operatorname{int}\overline{V}_n$. If $z_0\in Z_0$ and $\epsilon>0$, there is an $n\ge1$ such that $z_0\in V_n$ so $z_0\in\operatorname{int}\overline{V}_n$. Now

$$U=\{z:z\in Z_0,\ \|f(z)-f(z_0)\|\le\frac{\epsilon}{n},\ z\in\operatorname{int}\overline{V}_n,\ |g(z)(f(z_0))-g(z_0)(f(z_0))|\le\epsilon\}$$

is a neighbourhood of z_0, and if $z\in U$ then

$$\begin{aligned}|g(z)(f(z))-g(z_0)(f(z_0))|&\le|g(z)(f(z))-g(z)(f(z_0))|+|g(z)(f(z_0))-g(z_0)(f(z_0))|\\&\le\|g(z)\|\|f(z)-f(z_0)\|+\epsilon\le2\epsilon.\end{aligned}$$

As z_0 and ϵ are arbitrary, $(g|f)\restriction Z_0$ is continuous. **Q**

(ii) Consequently $\Vdash_{\mathbb{P}} (g|f)^{\sim} \in \mathbb{R}$ whenever $g \in C^-(Z;X^*_{w^*})$ and $f \in C^-(Z;X)$. It is now easy to check that the given formula defines a name for a bilinear duality between $\tilde{X}$ and $(X^*_{w^*})^{\sim}$.

(c)(i) Note first that we have a $\mathbb{P}$-name for a norm on $(X^*_{w^*})^{\sim}$ given by the formula

$$\Vdash_{\mathbb{P}} \|\tilde{g}\| = \|g\|^{\sim}$$

whenever $g \in C^-(Z;X^*_{w^*})$. **P** For $\gamma \in \mathbb{Q}$, set $V_\gamma = \{z : z \in \operatorname{dom} g,\ \|g(z)\| \le \gamma\}$. If $Z_0 = \operatorname{dom} g \setminus \bigcup_{\gamma\in\mathbb{Q}}(\overline{V}_\gamma \setminus \operatorname{int}\overline{V}_\gamma)$, Z_0 is a dense G_δ set in Z and $\|g\|{\restriction}Z_0$ is continuous, so $\Vdash_{\mathbb{P}} \|g|^{\sim} \in \mathbb{R}$. The algebraic checks required are now straightforward. **Q**

Similarly, it is now elementary that

$$\Vdash_{\mathbb{P}} |(\dot{y}|\dot{x})| \le \|\dot{y}\|\|\dot{x}\| \text{ whenever } \dot{x} \in \tilde{X} \text{ and } \dot{y} \in (X^*_{w^*})^{\sim}.$$

(ii) Next, let $K \subseteq X^*_{w^*}$ be the unit ball of $X^*_{w^*}$. Then

$$\Vdash_{\mathbb{P}} \tilde{K} \text{ is a balanced mid-convex set in the unit ball of } (X^*_{w^*})^{\sim}.$$

Also

$$\Vdash_{\mathbb{P}} \|x\| = \sup_{y\in\tilde{K}}(y|x) \text{ for every } x \in \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$, $\gamma \in \mathbb{Q}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X} \text{ and } \|\dot{x}\| > \gamma'.$$

Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \tilde{f}$. Take any $z_0 \in \widehat{p} \cap \operatorname{dom} f$. Then $\|f(z_0)\| > \gamma$, So there is a $y \in K$ such that $y(f(z_0)) > \gamma$. Let U be a neighbourhood of z_0 such that $y(f(z)) > \gamma$ for $z \in U \cap \operatorname{dom} f$, and let q stronger than p be such that $\widehat{q} \subseteq U$, Then

$$q \Vdash_{\mathbb{P}} \tilde{y} \in \tilde{K} \text{ and } (\tilde{y}|\dot{x}) > \gamma.$$

As p, $\dot{x}$ and γ are arbitrary, we have the result. **Q**

(iii) We can also identify the topology of $\tilde{K}$:

$$\Vdash_{\mathbb{P}} \text{the topology of } \tilde{K} \text{ corresponds to the weak topology induced by the duality.}$$

P(α) Let $\mathcal{U}$ be the family of sets of the form $\{y : y \in K,\ y(x) < \gamma\}$ where $x \in X$ and $\gamma \in \mathbb{Q}$. Then $\mathcal{U}$ generates the topology of K so

$$\Vdash_{\mathbb{P}} \tilde{\mathcal{U}} \text{ generates the topology of } \tilde{K}$$

(2A(d-ii)). But if $U = \{y : y \in K,\ y(x) < \gamma\}$ then

$$\Vdash_{\mathbb{P}} \tilde{U} = \{y : y \in \tilde{K},\ (y|\tilde{x}) < \gamma\},$$

so

$\Vdash_{\mathbb{P}}$ every member of $\tilde{U}$ is open for the weak duality topology on $\tilde{K}$, so the usual topology on $\tilde{K}$ is weaker than the duality topology.

(β) On the other hand, suppose that $p \in \mathbb{P}$ and $\dot{y}_0$, $\dot{G}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is an open subset of } \tilde{K} \text{ for the duality topology and } \dot{y}_0 \in \dot{G}.$$

Then there are a q stronger than p, an $n \in \mathbb{N}$, $\gamma_0, \dots, \gamma_n \in \mathbb{Q}$ and $\mathbb{P}$-names $\dot{x}_0, \dots, \dot{x}_n$ such that

$$q \Vdash_{\mathbb{P}} \dot{x}_i \in \tilde{X} \text{ for each } i \le n \text{ and } \dot{y}_0 \in \{y : y \in \tilde{K},\ (y|\dot{x}_i) < \gamma_i \text{ for every } i\} \subseteq \dot{G}.$$

Let $g_0 \in C^-(Z;X^*_{w^*})$ and $f_0, \dots, f_n \in C^-(Z;X)$ be such that

$$q \Vdash_{\mathbb{P}} \dot{x}_i = \tilde{f}_i \text{ for each } i \text{ and } \dot{y}_0 = \tilde{g}_0.$$

Then $\widehat{q} \subseteq^* \{z : g_0(z)(f_i(z)) \text{ is defined and less than } \gamma_i \text{ for each } i \le n\}$. We can therefore find an open set $U \subseteq K$ such that

$$\widehat{q} \cap \{z : g_0(z) \in U,\ y(f_i(z)) < \gamma_i \text{ for every } i \le n \text{ and } y \in U\}$$

is non-meager and essentially includes $\widehat{r}$ for some r stronger than q. In this case,

$$r \Vdash_{\mathbb{P}} \dot{y}_0 \in \tilde{U} \subseteq \dot{G}.$$

As p, $\dot{y}_0$ and $\dot{G}$ are arbitrary,

$\Vdash_{\mathbb{P}}$ every weakly open set in $\tilde{K}$ is open for the usual topology of $\tilde{K}$, so the two topologies coincide. **Q**

(iv) Since we know that

$$\tilde{K} \text{ is compact,}$$

we have

$\Vdash_{\mathbb{P}} \tilde{K}$ acts on $\tilde{X}$ as a mid-convex norming subset of the unit ball of the dual of $\tilde{X}$, and is compact in the corresponding weak* topology, so acts as the unit ball of the dual of $\tilde{X}$.

At the same time, $\Vdash_{\mathbb{P}} (X^*_{w^*})^{\sim} = \bigcup_{n\in\mathbb{N}} n\tilde{K}$. It follows easily that

$$\Vdash_{\mathbb{P}} (X^*_{w^*})^{\sim} \text{ acts on } \tilde{X} \text{ as the dual of } \tilde{X}.$$

9B Lemma Let X be a normed space, W a Čech-complete topological space, and $\phi : W \to X$ a function which is continuous for the weak topology of X. Then there is a comeager set $W' \subseteq W$ such that $\phi\restriction W'$ is continuous for the norm topology of X.

proof (a) For each $n \in \mathbb{N}$ let W_n be $\bigcup\{G : G \subseteq W$ is open, $\operatorname{diam}\phi[G] \le 2^{-n}\}$; set $W' = \bigcap_{n\in\mathbb{N}} W_n$. Then $\phi\restriction W'$ is norm-continuous.

(b) ? Suppose, if possible, that W' is not comeager in W. Then there is an $n \in \mathbb{N}$ such that W_n is not dense. Express W as $\bigcap_{m\in\mathbb{N}} H_m$ where $\langle H_m\rangle_{m\in\mathbb{N}}$ is a sequence of dense open sets in a compact Hausdorff space Z, and set $V = Z \setminus \overline{W}_n$, so that $V \subseteq Z$ is a non-empty open set and $W \cap V$ is dense in V. Choose $\langle V_\sigma\rangle_{\sigma\in S_2^*}$ inductively, where $S_2^* = \bigcup_{m\in\mathbb{N}}\{0,1\}^m$, as follows. $V_\emptyset = V$. Given that V_σ is a non-empty open subset of V, then $V_\sigma \cap W$ is a non-empty relatively open subset of W disjoint from W_n, so $\operatorname{diam}\phi[V_\sigma \cap W] > 2^{-n}$; let x_σ, x'_σ be points of $\phi[W \cap V_\sigma]$ such that $\|x_\sigma - x'_\sigma\| > 2^{-n}$; let $y_\sigma \in X^*$ be such that $\|y_\sigma\| = 1$ and $|y_\sigma(x_\sigma) - y_\sigma(x'_\sigma)| > 2^{-n}$; let $V_{\sigma^\frown<0>}$ and $V_{\sigma^\frown<1>}$ be open subsets of V_σ such that

$\overline{V}_{\sigma^\frown<i>} \subseteq V_\sigma \cap H_{\#(\sigma)}$ for both i,

$|y_\sigma(\phi(z)) - y_\sigma(\phi(z'))| \ge 2^{-n}$ for every $z \in W \cap V_{\sigma^\frown<0>}$ and $z' \in W \cap V_{\sigma^\frown<1>}$.

Note that this ensures that $\|\phi(z) - \phi(z')\| \ge 2^{-n}$ whenever $z \in W \cap V_{\sigma^\frown<0>}$ and $z' \in W \cap V_{\sigma^\frown<1>}$.

At the end of the construction, we have $\|\phi(z) - \phi(z')\| \ge 2^{-n}$ whenever σ, $\tau \in S_2^*$ are incomparable, $z \in W \cap V_\sigma$ and $z' \in W \cap V_\tau$. Set

$$K = \bigcap_{m\in\mathbb{N}} \bigcup_{\sigma\in\{0,1\}^m} V_\sigma,$$

so that $K \subseteq W$ is compact. All the $K \cap V_\sigma$ are compact and not empty, so we have a Radon probability measure μ_0 on W such that $\mu_0(K \cap V_\sigma) = 2^{-\#(\sigma)}$ for every $\sigma \in S_2^*$ (Fremlin 03, 416K). Then $\mu_1 = \mu_0\phi^{-1}$ is a Radon measure on X for the weak topology on X (Fremlin 03, 418I), and therefore also for the norm topology of X (Fremlin 03, 466A). Since $\mu_1\phi[K] = \mu_0\phi^{-1}[\phi[K]] = 1$, there is a norm-compact set $L \subseteq \phi[K]$ such that

$$0 < \mu_1 L = \mu_0\phi^{-1}[L] = \mu_0(K \cap \phi^{-1}[L]).$$

For $m \in \mathbb{N}$, set $A_m = \{\sigma : \sigma \in \{0,1\}^m,\ V_\sigma \cap K \cap \phi^{-1}[L] \ne \emptyset\}$. Then $\#(A_m) \ge 2^m\mu_1 L$. But note that for each $\sigma \in A_m$ there is an $a_\sigma \in \phi[K \cap V_\sigma \cap L]$, and that if σ, σ' are distinct members of A_m then $\|a_\sigma - a_{\sigma'}\| \ge 2^{-n}$. So L cannot be covered by fewer than $2^m\mu_1 L$ sets of diameter less than 2^{-n}. As this is true for every m, L is not totally bounded and cannot be norm-compact. **X**

(c) So W' is comeager, as required.

9C Theorem Let X be a normed space, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathbb{P}$. Write X_w for X endowed with its weak topology. Let $\phi : X \to X_w$ be the identity function, regarded as a continuous function form the norm and weak topologies of X. Then

$$\Vdash_{\mathbb{P}} \tilde\phi : \tilde X \to \tilde X_w \text{ is surjective,}$$

so

$$\Vdash_{\mathbb{P}} \tilde X_w \text{ can be identified with } \tilde X \text{ with its weak topology.}$$

proof (a) If $f \in C^-(Z;X_w)$, then applying 9B to the Čech-complete space $\operatorname{dom} f$ we have a dense G_δ set $Z_0 \subseteq \operatorname{dom} f$ such that $f\restriction Z_0$ belongs to $C^-(Z;X)$; now

$$\Vdash_{\mathbb{P}} \tilde f = ((f\restriction Z_0)\phi)^{\sim} = \tilde\phi((f\restriction Z_0)^{\sim}) \in \tilde\phi[\tilde X].$$

Thus $\Vdash_{\mathbb{P}} \tilde\phi$ is surjective.

(b) Now use the identification of $\tilde X^*$ in 9A.

(i) Suppose that $p \in \mathbb{P}$ and that $\dot x$, $\dot G$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot G \text{ is an open subset of } \tilde X \text{ in its weak topology and } \dot x \in \dot G.$$

Then there are a q stronger than p, an $n \in \mathbb{N}$, rational numbers $\gamma_0,\dots,\gamma_n,\gamma'_0,\dots,\gamma'_n$ and $\mathbb{P}$-names $\dot y_0,\dots,\dot y_n$ such that

$$p \Vdash_{\mathbb{P}} \dot y_0,\dots,\dot y_n \in \tilde X^* \text{ and } \dot x \in \{u : u \in \tilde X,\ \gamma_i < \dot y_i(u) < \gamma'_i\ \forall\, i \le n\} \subseteq \dot G.$$

Using 9A, we have $f \in C^-(Z;X)$ and $g_0,\dots,g_n \in C^-(Z;X^*_{w^*})$ such that

$$q \Vdash_{\mathbb{P}} \dot x = \tilde f \in \{u : u \in \tilde X,\ \gamma_i < (\tilde g_i|u) < \gamma'_i\ \forall\, i \le n\} \subseteq \dot G.$$

As in the proof of 9A, we can now use the norm-continuity of f and the weak*-continuity of the g_i to see that there are a weakly open subset G of X and an r stronger than q such that

$r \Vdash_{\mathbb{P}} \tilde f \in \tilde G \subseteq \{u : u \in \tilde X,\ \gamma_i < (\tilde g_i|u) < \gamma'_i\ \forall\, i \le n\}$, so $\dot x$ belongs to the interior of $\tilde G$ in the topology of $\tilde X_w$.

As p, $\dot x$ and $\dot G$ are arbitrary,

$$\Vdash_{\mathbb{P}} \text{ the weak topology of } \tilde X \text{ is coarser than the topology of } \tilde X_w.$$

(ii) In the other direction, suppose that $p \in \mathbb{P}$ and that $\dot x$, $\dot G$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot G \text{ is an open subset of } \tilde X_w \text{ and } \dot x \in \dot G.$$

Then we have a q stronger than p, a basic weakly open subset G of X and an $f \in C^-(Z;X)$ such that

$$q \Vdash_{\mathbb{P}} \dot x = \tilde f \in \tilde G \subseteq \dot G.$$

Now there are an $n \in \mathbb{N}$, rational numbers $\gamma_0,\dots,\gamma_n,\gamma'_0,\dots,\gamma'_n$ and $y_0,\dots,y_n \in X^*$ such that

$$G = \{x : x \in X,\ \gamma_i < y_i(x) < \gamma'_i\ \forall\, i \le n\}.$$

So

$q \Vdash_{\mathbb{P}} \dot x \in \{u : u \in \tilde X,\ \gamma_i < (\tilde y_i|u) < \gamma'_i\ \forall\, i \le n\} = \tilde G \subseteq \dot G$ and $\dot x$ belongs to the interior of $\dot G$ for the weak topology of $\tilde X$.

As before, this means that

$$\Vdash_{\mathbb{P}} \text{ the topology of } \tilde X_w \text{ is coarser than the weak topology of } \tilde X \text{, so the two coincide.}$$

10 Examples

10A Souslin lines and random reals: Proposition Let X be a Souslin line, that is, a ccc Dedekind complete totally ordered space such that every countable set is nowhere dense. Let $\mathbb{P}$ be a random forcing, that is, $\mathbb{P} = \Sigma \setminus \mathcal{N}(\mu)$, active downwards, for some semi-finite measure space (Ω,Σ,μ), writing $\mathcal{N}(\mu)$ for the null ideal of μ. Then

$$\Vdash_{\mathbb{P}} \tilde X \text{ is a Souslin line.}$$

proof (a) By 4Da,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is totally ordered and Dedekind complete and its topology is its order topology.}$$

By 5D,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is ccc.}$$

(b) As for separable subspaces of $\tilde{X}$, the fact we need is the following. If $p \in \mathbb{P}$ has finite measure, and $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$, there is a nowhere dense closed set $F \subseteq X$ such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{F}$. **P** Let $f \in C^-(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \vec{f}$. $\widehat{p}$ is the Stone space of the subspace measure on p so carries a non-zero totally finite Radon measure ν for which meager sets are negligible; in particular, $\nu(\widehat{p} \cap \operatorname{dom} f) > 0$. Consider the subspace measure ν_1 on $\widehat{p} \cap \operatorname{dom} f$; this is again a non-zero totally finite Radon measure. So the image measure $\nu_1(f{\restriction}\widehat{p})^{-1}$ is a Radon measure on X. But the support of any Radon measure on X is nowhere dense, so there is a nowhere dense closed set $F \subseteq X$ such that $\nu(\widehat{p} \setminus f^{-1}[F]) = 0$, that is, $\widehat{p} \subseteq^* f^{-1}[F]$ and $p \Vdash_{\mathbb{P}} \dot{x} \in F$. **Q**

Because every meager subset of X is nowhere dense, we see that whenever $p \in \mathbb{P}$ has finite measure and $\dot{A}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{A} \text{ is a countable subset of } \tilde{X},$$

there is a closed nowhere dense set $F \subseteq X$ such that

$$p \Vdash_{\mathbb{P}} \dot{A} \subseteq \tilde{F}.$$

But we also know that

$$p \Vdash_{\mathbb{P}} \tilde{F} \text{ is closed and nowhere dense}$$

(2B). As p and $\dot{A}$ are arbitrary (and μ is semi-finite, so the elements of finite measure are dense in $\mathbb{P}$),

$$\Vdash_{\mathbb{P}} \text{countable subsets of } \tilde{X} \text{ are nowhere dense, so } \tilde{X} \text{ is a Souslin line.}$$

10B Kunen's compact L-space In Kunen 81 there is an example of a non-separable hereditarily Lindelöf chargeable compact Hausdorff space X, constructed with the aid of the continuum hypothesis; it is not hard to show that the construction can be performed if we assume that the cofinality of the Lebesgue null ideal is ω_1. This space has the additional property that it is expressible as the union of a non-decreasing family $\langle X_\xi \rangle_{\xi<\omega_1}$ of compact metrizable subspaces. The following proposition shows that at least some aspects of the construction can be carried over into contexts in which the cofinality of the Lebesgue null ideal is large.

Proposition Let X be a hereditarily Lindelöf chargeable compact Hausdorff space, of density ω_1, in which every separable subspace is metrizable, and $\mathbb{P}$ a forcing notion such that ω_1 is a precaliber of $\mathbb{P}$. (For instance, $\mathbb{P}$ could be $\mathrm{Fn}_{<\omega}(I,\{0,1\})$ for any set I.) Then

$\Vdash_{\mathbb{P}} \tilde{X}$ is a hereditarily Lindelöf chargeable compact Hausdorff space, of uncountable density, in which every separable subspace is metrizable.

proof (a) By 4Ac, 4Aj and 5Ac,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is a chargeable compact Hausdorff space of density at most } \omega_1.$$

(b) Because $d(X) = \omega_1$ and X is first-countable, X is expressible as the union of a strictly increasing family $\langle X_\xi \rangle_{\xi<\omega_1}$ of closed separable subspaces, all of which are metrizable. Now

$$\Vdash_{\mathbb{P}} \langle \tilde{X}_\xi \rangle_{\xi<\omega_1} \text{ is a strictly increasing family of compact metrizable subspaces of } \tilde{X}$$

by 2Ad and 4Ae. The point here is that

$$\Vdash_{\mathbb{P}} \tilde{X} = \textstyle\bigcup_{\xi<\omega_1} \tilde{X}_\xi.$$

P? Otherwise, we have a $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{x}$ such that

$$p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X} \setminus \textstyle\bigcup_{\xi<\omega_1} \tilde{X}_\xi.$$

Let Z be the Stone space of $\mathrm{RO}(\mathbb{P})$ and let $f \in C(Z;X)$ be such that $p \Vdash_{\mathbb{P}} \dot{x} = \tilde{f}$. Then none of the closed sets $f^{-1}[X_\xi]$ can include $\widehat{p}$, so we have for each $\xi < \omega_1$ a p_ξ stronger than p such that $\widehat{p_\xi} \cap f^{-1}[X_\xi] = \emptyset$. But as ω_1 is a precaliber of $\mathbb{P}$ there is a $z \in Z$ such that $D = \{\xi : \xi < \omega_1,\ z \in \widehat{p_\xi}\}$ is uncountable, and now $f(z) \notin \bigcup_{\xi \in D} X_\xi = X$. **XQ**

(c) As $\mathbb{P}$ is ccc, we have

$$\Vdash_{\mathbb{P}} \omega_1 \text{ is uncountable;}$$

consequently

$\Vdash_{\mathbb{P}}$ every countable subset of $\tilde{X}$ is included in some $\tilde{X}_\xi$, every separable subspace of $\tilde{X}$ is metrizable, and $d(\tilde{X})$ is the first uncountable cardinal.

(d) Next,

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is hereditarily ccc}$$

by 5E. Now we have

$\Vdash_{\mathbb{P}} \tilde{X}$ is hereditarily ccc and every separable subspace of $\tilde{X}$ is hereditarily Lindelöf, so $\tilde{X}$ is hereditarily Lindelöf,

which completes the proof.

10C Example Let $X \subseteq [0,1]^2$ be a Bernstein set. Let $\mathbb{P}$ be the partially ordered set of non-negligible compact subsets of $[0,1]$, active downwards, so that its regular open algebra is isomorphic to the measure algebra of Lebesgue measure on $[0,1]$. Then X is connected but

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not connected.}$$

proof (a) If G and H are disjoint non-empty open subsets of X, then there are $x \in G$, $y \in H$ and $\delta > 0$ such that the open balls $U(x,\delta)$ and $U(y,\delta)$ are disjoint and included in $[0,1]^2$, and $X \cap U(x,\delta) \subseteq G$, $X \cap U(y,\delta) \subseteq H$. Express G, H as $G_0 \cap X$ and $H_0 \cap X$ where G_0, H_0 are disjoint open sets in $[0,1]^2$. Then $[0,1]^2 \setminus (G_0 \cup H_0)$ meets the line segment from $x+w$ to $y+w$ for every $w \in U(0,\frac{1}{2}\delta)$, so has cardinal $\mathfrak{c}$ and must meet X; thus $X \neq G \cup H$. As G and H are arbitrary, X is connected.

(b) Every compact subset of X is countable, so 2Ag tells us that

$$\Vdash_{\mathbb{P}} \tilde{X} = \dot{\varphi}[\check{X}].$$

(c) Consider the $\mathbb{P}$-names

$$\dot{G} = \{(\tilde{x}, p) : x \in X,\ p \in \mathbb{P},\ \pi_1(x) > \sup p\},$$

$$\dot{H} = \{(\tilde{x}, p) : x \in X,\ p \in \mathbb{P},\ \pi_1(x) < \inf p\},$$

where $\pi_1 : [0,1]^2 \to [0,1]$ is the first-coordinate map (recall that every member of $\mathbb{P}$ is actually a compact non-empty subset of $[0,1]$). Then

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is an open subset of } \tilde{X}.$$

P Note that if $(\tilde{x}, p) \in \dot{G}$ and q is stronger than p, then $(\tilde{x}, q) \in \dot{G}$. Of course $\Vdash_{\mathbb{P}} \dot{G} \subseteq \tilde{X}$. Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \dot{G}$. Then there are a q stronger than p and an $x \in X$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \tilde{x}$ and $\sup q < \pi_1(x)$. Set $U = \{y : y \in X,\ \sup q < \pi_1(y)\}$. Then

$$q \Vdash_{\mathbb{P}} \tilde{U} \text{ is an open neighbourhood of } \dot{x}.$$

Now suppose that r is stronger than q and that $\dot{y}$ is a $\mathbb{P}$-name such that $r \Vdash_{\mathbb{P}} \dot{y} \in \tilde{U}$. In this case (by (b)) there are an s stronger than r and a $y \in X$ such that $s \Vdash_{\mathbb{P}} \dot{y} = \tilde{y}$, in which case $y \in U$, $\pi_1(y) > \sup s$ and

$$s \Vdash_{\mathbb{P}} \dot{y} = \tilde{y} \in \dot{G}.$$

As r and $\dot{y}$ are arbitrary,

$$q \Vdash_{\mathbb{P}} \tilde{U} \subseteq \dot{G} \text{ so } \dot{x} \in \operatorname{int} \dot{G};$$

as p and $\dot{x}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{G} \subseteq \operatorname{int} \dot{G} \text{ and } \dot{G} \text{ is open. } \mathbf{Q}$$

Next,

$$\Vdash_{\mathbb{P}} \dot{G} \text{ is not empty.}$$

P If $p \in \mathbb{P}$, there is a q stronger than p such that $\sup q < 1$, so that if x is any member of $X \cap (\{1\} \times [0,1])$ then $q \Vdash_{\mathbb{P}} \tilde{x} \in \dot{G}$. **Q**

Similarly,

$$\Vdash_{\mathbb{P}} \dot{H} \text{ is open and not empty.}$$

(d) Of course

$$\Vdash_{\mathbb{P}} \dot{G} \cap \dot{H} = \emptyset.$$

Also

$$\Vdash_{\mathbb{P}} \dot{G} \cup \dot{H} = \tilde{X}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{x}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}} \dot{x} \in \tilde{X}$. Then there are a q stronger than p and an $x \in X$ such that $q \Vdash_{\mathbb{P}} \dot{x} = \tilde{x}$. Now one of $q \cap [0, \pi_1(x)[$, $q \cap]\pi_1(x), 1]$ is non-negligible and includes an $r \in \mathbb{P}$; in which case $(\tilde{x}, r)$ belongs to one of $\dot{G}$, $\dot{H}$ and

$$r \Vdash_{\mathbb{P}} \dot{x} = \tilde{x} \in \dot{G} \cup \dot{H}.$$

As p and $\dot{x}$ are arbitrary, we have the result. **Q**

So $\dot{G}$, $\dot{H}$ witness that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not connected.}$$

10D Example There are a path-connected separable metrizable space X and a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not path-connected.}$$

proof (a) I start with some general remarks about spaces of the type to be set up, so as to shorten the part of the argument which must be done in the forcing language. Consider the following situation. X will be a Hausdorff space expressed as $Y_0 \cup X_1 \cup Y_1 \cup X_2$ where Y_0, X_1, Y_1 and X_2 are disjoint and not empty; Y_0 and Y_1 are open; X_1, X_2 and $Y_0 \cup X_1$ are closed; and X_1 and X_2 are zero-dimensional. There will be a continuous function $\psi : X_1 \cup Y_1 \to X_1$ such that $\psi(x) = x$ for $x \in X_1$, and a continuous function $\phi : [0,1] \to X$ such that $\phi(0) \in Y_0$ and $\phi(1) \in X_2$. In this case, there will be a greatest $t_0 \in [0,1[$ such that $\phi(t_0) \in Y_0 \cup X_1$, and a least $t_1 \in]t_0, 1]$ such that $\phi(t_1) \in X_2$. For $t \in [t_0, t_1[$, $\phi(t) \in X_1 \cup Y_1$, so $\psi\phi\restriction [t_0, t_1[$ is continuous; as X_1 is totally disconnected, $\psi\phi$ is constant on $[t_0, t_1[$. Moreover, $\phi(t_1) = \phi(1)$. **P?** Otherwise, there are t_2, t_3 such that $t_1 \le t_2 < t_3 \le 1$, $\phi(t_2)$ and $\phi(t_3)$ are different points of X_2, and $\phi(t) \in Y_1$ for $t_2 < t < t_3$. In this case, $\psi\phi(t)$ is constant for $t \in]t_2, t_3[$; let x be the constant value, so that $\phi(t)$ belongs to the line segment from x to $\alpha(x)$ for every $t \in]t_2, t_3[$. But this means that $\phi(t_1) = \alpha(x) = \phi(t_2)$. **XQ**

(b) Now for the actual space X, which will be in the form considered above. X will be a subset of $\mathbb{R}^3$. X_2 will be a subset of the line segment K_2 from $(-1,0,2)$ to $(1,0,2)$ homeomorphic to the Cantor set. X_1 will be a Bernstein subset of the line segment K_1 from $(0,-1,1)$ to $(0,1,1)$. Take any bijection $\alpha : X_1 \to X_2$; Y_1 will be the union of the open line segments from x to $\alpha(x)$ as x runs over X_1. Now set $y_0 = (0,0,0)$, and let Y_0 be the set consisting of y_0 together with all the open line segments from y_0 to points of X_1. If K is the convex hull of $K_1 \cup K_2$, we have a continuous function $\psi_0 : K \setminus K_2 \to K_1$ given by saying that $\psi_0(x) = y$ whenever $y \in K_1$, $y' \in K_2$ and x lies on the line segment from y to y'. Set $\psi = \psi_0\restriction X_1 \cup Y_1$. Then it is easy to check that X, Y_0, X_1, Y_1, X_2 and ψ have the properties listed in (a). Evidently X is path-connected, because every point of X belongs to a path ending in y_0.

(c) This time, let μ be an atomless Radon measure on X_2, and let $\mathbb{P}$ be the partially ordered set of non-negligible compact subsets of X_2. Putting 2A, 2C, 2D and 4B together we see that

> $\Vdash_{\mathbb{P}} \tilde{X}$ is the disjoint union of $\tilde{Y}_0$, $\tilde{X}_1$, $\tilde{Y}_1$ and $\tilde{X}_2$; $\tilde{Y}_0$, $\tilde{Y}_1 \cup \tilde{X}_2$, $\tilde{Y}_1$, $\tilde{Y}_0 \cup \tilde{X}_1 \cup \tilde{Y}_1$, $Y_1 \cup X_2$ are open; $\tilde{X}_2$ is zero-dimensional; and $\tilde{\psi}$ is a continuous function from $\tilde{X}_1 \cup \tilde{Y}_1$ to $\tilde{X}_1$ such that $\tilde{\psi}_1(x) = x$ for every $x \in \tilde{X}_1$.

(d) Exactly as in 10C, we have

$$\Vdash_{\mathbb{P}} \dot{\varphi}[\check{X}_1] = \tilde{X}_1.$$

It follows that

$$\Vdash_{\mathbb{P}} \tilde{X}_1 \text{ is zero-dimensional.}$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{x}$, $\dot{U}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{U} \text{ is a relatively open set in } \tilde{X}_1 \text{ and } \dot{x} \in \dot{U}.$$

Then there must be a q stronger than p, an $x \in X_1$ and a relatively open set $U \subseteq X_1$ such that

$$q \Vdash_{\mathbb{P}} \dot{x} = \tilde{x} \in \tilde{U} \subseteq \dot{U}.$$

So $x \in U$. As X_1 is zero-dimensional, there is a partition (V_1, V_2) of X_1 into relatively open sets such that $x \in V_1 \subseteq U$. But now

$$q \Vdash_{\mathbb{P}} \dot{x} \in \tilde{V}_1 \subseteq \dot{U} \text{ and } \tilde{V}_1 \text{ is relatively open-and-closed in } \tilde{X}_1.$$

As p, $\dot{x}$ and $\dot{U}$ are arbitrary, we have the result. **Q**

(e) Let Z be the Stone space of the regular open algebra of $\mathbb{P}$ and $f : Z \to X_2$ the canonical map; then $\Vdash_{\mathbb{P}} \vec{f} \in \tilde{X}_2$. Note that if $p \in \mathbb{P}$ then p is actually a compact subset of X, and $p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{p}$. Let $y_0 = (0,0,0)$ be the apex of Y_0. **?** If $p \in \mathbb{P}$ is such that

$$p \Vdash_{\mathbb{P}} \tilde{X} \text{ is path-connected,}$$

then there is a $\mathbb{P}$-name $\dot{\phi}$ such that

> $p \Vdash_{\mathbb{P}} \dot{\phi}$ is a continuous function from the unit interval to $\tilde{X}$, $\dot{\phi}(0) = \tilde{y}_0$ and $\dot{\phi}(1) = \vec{f}$.

Now (a) tells us that

> $p \Vdash_{\mathbb{P}}$ there are real numbers $t_0 < t_1$ and an $x \in \tilde{X}_1$ such that $\tilde{\psi}\dot{\phi}(t) = x$ for $t_0 \le t < t_1$ and $\dot{\phi}(t_1) = \vec{f}$.

Let q stronger than p and $x \in X_1$ be such that

> $q \Vdash_{\mathbb{P}}$ there are real numbers $t_0 < t_1$ such that $\tilde{\psi}\dot{\phi}(t) = \tilde{x}$ for $t_0 \le t < t_1$ and $\dot{\phi}(t_1) = \vec{f}$.

Now consider $\alpha(x) \in X_2$. There is an r stronger than q such that $\alpha(x)$ does not belong to the convex hull of r, just because μ is atomless. So we have disjoint convex relatively open sets U_0, $V_0 \subseteq K_2$ such that $r \subseteq U_0$ and $\alpha(x) \in V_0$. In this case, the sets $U = \Gamma(U_0 \cup K_1) \setminus K_1$, $V = \Gamma(V_0 \cup K_1) \setminus K_1$ are disjoint relatively open subsets of K, and $U \cap X$, $V \cap X$ are disjoint open subsets of X including r and $\psi^{-1}[\{x\}] \setminus \{x\}$ respectively. So

$$\Vdash_{\mathbb{P}} \tilde{Y}_1 \cap \tilde{\psi}^{-1}[\{\tilde{x}\}] \subseteq (V \cap X)^{\sim}.$$

P Suppose that $p' \in \mathbb{P}$ and $\dot{y}$ is a $\mathbb{P}$-name such that

$$p' \Vdash_{\mathbb{P}} \dot{y} \in \tilde{Y}_1 \text{ and } \tilde{\psi}(\dot{y}) = \tilde{x}.$$

Then there is a $g \in C^{-}(Z, \tilde{Y}_1)$ such that $p' \Vdash_{\mathbb{P}} \dot{y} = \vec{g}$, so that

$$\widehat{p'} \subseteq^* \{z : z \in \operatorname{dom} g,\, \psi g(z) = x\} = g^{-1}[V \cap X]$$

and $p' \Vdash_{\mathbb{P}} \dot{y} \in (V \cap X)^{\sim}$. **Q**

We have

$$r \Vdash_{\mathbb{P}} \vec{f} \in \tilde{U}.$$

Now there must be rational numbers γ, γ' such that $\gamma < \gamma' \le 1$ and an s stronger than r such that

> $s \Vdash_{\mathbb{P}} \dot{\phi}(t) \in (U \cap X)^{\sim}$ for $\gamma \le t \le \gamma'$, while $\dot{\phi}(\gamma) \in \tilde{Y}_1$ and $\tilde{\psi}\dot{\phi}(\gamma) = \tilde{x} \in (V \cap X)^{\sim}$.

But this is impossible, because

$$\Vdash_{\mathbb{P}} (U \cap X)^{\sim} \text{ is disjoint from } (V \cap X)^{\sim}. \quad \mathbf{X}$$

So

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not path-connected,}$$

and we have the required example.

10E I do not know whether any other cardinal functions are preserved in the way that weight, π-weight and density are. 'Character' is not, as the following example (due to A.Dow and G.Gruenhage) shows.

Example There are a first-countable compact Hausdorff space X and a forcing notion $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not first-countable.}$$

proof (a) Let $S \subseteq \omega_1$ be a stationary set such that $\omega_1 \setminus S$ is also stationary. Let T be the tree consisting of subsets of S which are closed in the order topology of ω_1, ordered by end-extension, so that $p \leq q$ iff there is some α such that $p = q \cap \alpha$. Then $\#(T) = \mathfrak{c}$, T has no uncountable branches and every element of T has more than one immediate successor, so there is a first-countable compact Hausdorff space X with a π-base $\mathcal{V}$ isomorphic to T inverted (Todorčević 84, 9.13). Replacing each member of $\mathcal{V}$ by the interior of its closure, if necessary, we can suppose that every member of $\mathcal{V}$ is a regular open set. For $p \in \mathbb{P}$ let V_p be the corresponding member of $\mathcal{V}$. If we take the forcing notion $\mathbb{P}$ to be T itself, acting upwards, $\{[p, \infty[: p \in \mathbb{P}\}$ is an order-dense subset of the regular open algebra $\mathrm{RO}(\mathbb{P})$, because T is separative. Now we have an order-isomorphism between $\mathcal{V}$ and $\{[p, \infty[: p \in \mathbb{P}\}$ matching V_p with $[p, \infty[$ for every p, and this order-isomorphism extends to a Boolean isomorphism between the Dedekind complete Boolean algebras $\mathrm{RO}(X)$ and $\mathrm{RO}(\mathbb{P})$. We can therefore identify the Stone space Z of $\mathrm{RO}(\mathbb{P})$ with the projective cover (projective resolution, absolute, Gleason space) of X, in the sense of Mill 84; and under this identification the regular open subset V_p of X is matched with the open-and-closed subset $\widehat{p}$ of Z, for each $p \in \mathbb{P}$.

(b) We need a special property of the partially ordered set T or $\mathbb{P}$: if $\langle A_n \rangle_{n \in \mathbb{N}}$ is any sequence of maximal up-antichains in T, there is a maximal up-antichain C refining every A_n, in the sense that for every $p \in C$ and $n \in \mathbb{N}$ there is a $q \in A_n$ such that $q_n \leq p$. **P** For each $n \in \mathbb{N}$ set $T_n = \bigcup_{q \in A_n} [q, \infty[$, so that T_n is a cofinal up-open subset of T; set $Q = \bigcap_{n \in \mathbb{N}} T_n$. **?** If Q is not cofinal with T, take $p_0 \in T$ such that $Q \cap [p_0, \infty[= \emptyset$. Let S' be the set of non-zero limit ordinals belonging to S, so that S' is stationary, and for each $\alpha \in S'$ let $\langle \gamma_{\alpha n} \rangle_{n \in \mathbb{N}}$ be a sequence in α with supremum α. For $p \in T$, $\gamma < \omega_1$ and $n \in \mathbb{N}$, let $q(p, \gamma, n)$ be a member of T_n such that $p \leq q(p, \gamma, n)$ and $q(p, \gamma, n) \not\subseteq \gamma$. For each $\alpha \in S'$ define a sequence $\langle p_{\alpha n} \rangle_{n \in \mathbb{N}}$ by saying that $p_{\alpha 0} = p_0$ and $p_{\alpha, n+1} = q(p_{\alpha n}, \gamma_{\alpha n}, n)$ for each n. Consider $\alpha' = \sup(\bigcup_{n \in \mathbb{N}} p_{\alpha n})$. $\{\alpha'\} \cup \bigcup_{n \in \mathbb{N}} p_{\alpha n}$ is a closed subset of ω_1, but cannot belong to T, since if it did it would dominate every $p_{\alpha n}$, so belong to every T_n, and we are supposing that there is no such member of T greater than or equal to p_0. So $\alpha' \notin S$, and, in particular, $\alpha' \neq \alpha$. Since $\alpha' \geq \gamma_{\alpha n}$ for every n, $\alpha' > \alpha$, and there is a first $n_\alpha \in \mathbb{N}$ such that $p_{\alpha, n_\alpha + 1} \not\subseteq \alpha$.

By the Pressing-Down Lemma, there is a stationary set $S_1 \subseteq S'$ such that $n_\alpha = n_\beta = n^*$ say for every α, $\beta \in S_1$, and moreover $\gamma_{\alpha i} = \gamma_{\beta i} = \gamma_i^*$ say for every α, $\beta \in S_1$ and $i \leq n^*$. It follows that $p_{\alpha i} = p_{\beta i} = p_i^*$ say for every $i \leq n^*$ and α, $\beta \in S_1$. But this means that $q(p_{n^*}^*, \gamma_{n^*}^*, n^*) \not\subseteq \alpha$ for every $\alpha \in S_1$; which is absurd. **X**

So Q is cofinal with T, and includes a maximal antichain C, which will have the property required. **Q**

(c) Because Z can be identified with the projective cover of X, there is a canonical continuous surjection f from Z onto X, defined by saying that $f(z) \in \overline{G}$ whenever $z \in Z$, $G \subseteq X$ is a regular open set and z belongs to the open-and-closed subset of Z corresponding to G. In particular, if $p \in P$ and V_p is the corresponding member of $\mathcal{V}$, $f(z) \in \overline{V_p}$ for every $z \in \widehat{p}$. In the language of this note, f belongs to the space $\tilde{X}$ defined from X and $\mathbb{P}$ and $p \Vdash_{\mathbb{P}} \vec{f} \in (\overline{V_p})^{\sim}$ for every $p \in \mathbb{P}$. Now

$$\Vdash_{\mathbb{P}} \chi(\vec{f}, \tilde{X}) > \omega.$$

P? Suppose, if possible, otherwise. Then there are $p_0 \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\mathcal{W}}$ such that

$$p_0 \Vdash_{\mathbb{P}} \dot{\mathcal{W}} \subseteq \tilde{\mathfrak{T}} \text{ is a countable base of neighbourhoods of } \vec{f};$$

let $\langle \dot{U}_n \rangle_{n \in \mathbb{N}}$ be a sequence of $\mathbb{P}$-names such that

$$p_0 \Vdash_{\mathbb{P}} \dot{U}_n \in \mathfrak{T} \text{ for every } n \text{ and } \dot{\mathcal{W}} = \{\dot{U}_n : n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ we have a maximal antichain A_n in $\mathbb{P}$ and a family $\langle U_{np}\rangle_{p\in A_n}$ in $\mathfrak{T}$ such that, for each $p \in A_n$, either p is incompatible with p_0 or $p \ge p_0$ and $p \Vdash_{\mathbb{P}} \dot{U}_n = \tilde{U}_{np}$. Let C be a maximal antichain in $\mathbb{P}$ refining every A_n; take any $p \in C$ such that $p \ge p_0$, so that for each $n \in \mathbb{N}$ there is a $q_n \in A_n$ dominated by p, and $p \Vdash_{\mathbb{P}} \dot{U}_n = \tilde{U}_{q_n n}$ for every n.

Now take two incompatible extensions p_1, p_2 of P and consider the corresponding members V_{p_1}, V_{p_2} of $\mathcal{V}$. These are non-empty disjoint open sets, and $\mathcal{V}$ is a π-base for a compact Hausdorff topology, so there are V_1', $V_2' \in \mathcal{V}$ such that $\overline{V_j'} \subseteq V_{p_j}$ for both j; if $V_j' = V_{r_j}$, we have $r_j \ge p_j$ for both j. Now

$$r_1 \Vdash_{\mathbb{P}} \vec{f} \in (\overline{V_{r_1}})^{\sim} \subseteq \tilde{V}_{p_1} \in \tilde{\mathfrak{T}},$$

so there must be an $r_1' \ge r_1$ and an $m \in \mathbb{N}$ such that $r_1' \Vdash_{\mathbb{P}} \dot{U}_m \subseteq \tilde{V}_{p_j}$. But we have $r_1' \ge q_m$, so $r_1' \Vdash_{\mathbb{P}} \tilde{U}_{q_m m} \subseteq \tilde{V}_{p_1}$ and $U_{q_m m} \subseteq V_{p_1}$. Similarly, there is an $n \in \mathbb{N}$ such that $U_{q_n n} \subseteq V_{p_2}$, so that $U_{q_m m} \cap U_{q_n n} = \emptyset$ and

$$\Vdash_{\mathbb{P}} \tilde{U}_{q_m m} \text{ is disjoint from } \tilde{U}_{q_n n}.$$

But we also have

$$p \Vdash_{\mathbb{P}} \vec{f} \in \tilde{U}_{q_m m} \cap \tilde{U}_{q_n n},$$

so this is impossible. **XQ**

(d) It follows at once that

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is not first-countable},$$

and we have the required example.

Remark In view of 4Da above, it is perhaps worth noting that the space X here can be thought of as a totally ordered set with its order topology and is a Corson compact (see TODORČEVIĆ 84, 9.14). The same phenomenon occurs if we start from a Souslin tree in place of the tree T here, and in this case we have a ccc forcing notion $\mathbb{P}$, at the cost of moving outside ZFC.

10F Example (A.Dow) If $\mathfrak{b} = \mathfrak{d} = \omega_2$ there are a forcing notion $\mathbb{P}$ and
(a) a countable Hausdorff space X of weight ω_2 such that $\Vdash_{\mathbb{P}} w(\tilde{X}) < \#(\omega_2)$;
(b) a compact Hausdorff space Y of π-weight ω_2 such that $\Vdash_{\mathbb{P}} \pi(\tilde{Y}) < \#(\omega_2)$.

proof (a)(i) Set $X = [\mathbb{N}]^{<\omega}$ with the topology $\mathfrak{T}$ defined by saying that $U \subseteq X$ is open iff for every $I \in U$ there is an $n \in \mathbb{N}$ such that $I \cup \{m\} \in U$ for every $m \ge n$.

(ii) For $f \in \mathbb{N}^{\mathbb{N}}$, $n \in \mathbb{N}$ and $K \subseteq n$ set

$$U_{nKf} = \{I : I \in X,\, I \cap n = K \text{ and } f(i) \le j \text{ whenever } i \in I \setminus n,\, j \in I \text{ and } i < j\}.$$

Then U_{nKf} is open, and also closed. **P** In fact U_{nKf} is closed for the coarser topology on X induced by the usual topology of $\mathcal{P}\mathbb{N}$. **Q**

If $F \subseteq \mathbb{N}^{\mathbb{N}}$ is $\le^*$-cofinal with $\mathbb{N}^{\mathbb{N}}$, then $\{U_{nKf} : f \in F,\, K \subseteq n \in \mathbb{N}\}$ is a base for $\mathfrak{T}$. **P** Suppose that $I \in U \in \mathfrak{T}$. Let $g : [\mathbb{N}]^{\omega} \to \mathbb{N}$ be such that $J \cup \{m\} \in U$ whenever $J \in U$ and $m \ge g(J)$, and set $h(m) = \max\{g(J) : J \subseteq m+1\}$ for $m \in \mathbb{N}$. Let $f \in F$ be such that $h \le^* f$, and take n such that $I \subseteq n$, $g(I) \le n$ and $h(m) \le f(m)$ for every $m \ge n$. Then $I \in U_{nIf} \subseteq U$. **Q**

So X is zero-dimensional.

(iii) $w(X) = \pi(X) = \mathfrak{d}$. **P** By (a-ii), $w(X) \le \mathfrak{d}$. Now let $\mathcal{V}$ be a π-base for $\mathfrak{T}$ of size $\pi(X)$; we can suppose that $\mathcal{V} = \{U_{nKf} : K \subseteq n \in \mathbb{N},\, f \in F\}$ where $F \subseteq \mathbb{N}^{\mathbb{N}}$ and $\#(F) = \pi(X)$. For $f \in F$ say that $f'(i) = \max(f(i), i+1)$ for every i. Take any $g \in \mathbb{N}^{\mathbb{N}}$. Then there are $K_g \subseteq n_g \in \mathbb{N}$ and $f_g \in F$ such that $U_{0\emptyset g} \supseteq U_{n_g K_g f_g}$. If $i \ge n_g$, $i < j$ and $f_g(i) \le j$ then $K_g \cup \{i,j\} \in U_{n_g K_g f_g}$ so $g(i) \le j$. Thus $g(i) \le \max(i+1, f_g(i))$ for every $i \ge n_g$ and $g \le^* f_g'$. As g is arbitrary, $\{f' : f \in F\}$ is $\le^*$-cofinal with $\mathbb{N}^{\mathbb{N}}$ and $\mathfrak{d} \le \#(F) = \pi(X)$. **Q**

(iv) Now suppose that $\mathfrak{b} = \mathfrak{d} = \omega_2$ and that $\mathbb{P}$ is the ω_2-Namba forcing notion (A1 below). Then

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \pi(\tilde{X}) = \omega.$$

P Let $\langle f_\xi\rangle_{\xi<\omega_2}$ run over a cofinal family in $\mathbb{N}^{\mathbb{N}}$ such that $f_\xi \le^* f_\eta$ whenever $\xi \le \eta < \omega_2$. Set $\mathcal{U} = \{U_{nKf_\xi} : K \subseteq n \in \mathbb{N},\ \xi < \omega_2\}$, so that $\mathcal{U}$ is a base for $\mathfrak{T}$. Let $\dot{A}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{A} \text{ is a countable cofinal subset of } \omega_2$$

(Proposition A4). Consider the $\mathbb{P}$-name

$$\dot{\mathcal{V}} = \{(\tilde{U}_{nKf_\xi}, p) : K \subseteq n \in \mathbb{N},\ \xi < \omega_2,\ p \Vdash_{\mathbb{P}} \xi \in \dot{A}\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{V}} \text{ is a countable subset of } \tilde{\mathfrak{T}}.$$

Now suppose that $p \in \mathbb{P}$ and that $\dot{I}$, $\dot{G}$ are $\mathbb{P}$-names such that

$$p \Vdash_{\mathbb{P}} \dot{G} \text{ is an open subset of } \tilde{X} \text{ containing } \dot{I}.$$

Because X is countable, there are a q stronger than p, $I \in X$, $K \subseteq n \in \mathbb{N}$ and $\xi < \omega_2$ such that

$$q \Vdash_{\mathbb{P}} \dot{I} = \tilde{I} \subseteq \tilde{U}_{nKf_\xi} \subseteq \dot{G}$$

(I am using 2Ad). It follows that $I \in U_{nKf_\xi}$. Now there are $\eta < \omega_2$ and r stronger than q such that

$$r \Vdash_{\mathbb{P}} \xi \le \eta \in \dot{A}.$$

Let $m \ge n$ be such that $f_\xi(i) \le f_\eta(i)$ for every $i \ge m$; then $I \in U_{mIf_\eta} \subseteq U_{nKf_\xi}$, so

$$r \Vdash_{\mathbb{P}} \dot{I} = \tilde{I} \in \tilde{U}_{mIf_\eta} \in \dot{\mathcal{V}} \text{ and } \tilde{U}_{mIf_\eta} \subseteq \tilde{U}_{nKf_\xi} \subseteq \dot{G}.$$

As p, $\dot{I}$ and $\dot{G}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\mathcal{V}} \text{ is a base for the topology of } \tilde{X}, \text{ so } w(\tilde{X}) \le \omega.$$

Since of course

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is infinite and Hausdorff},$$

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \pi(\tilde{X}) = \omega. \ \mathbf{Q}$$

(v) So, in the circumstances of (iv), we have a Hausdorff space X and a forcing notion $\mathbb{P}$ such that $w(X) = \pi(X) = \omega_2$ and

$$\Vdash_{\mathbb{P}} w(\tilde{X}) = \pi(\tilde{X}) = \omega < \omega_1 = \#(\omega_2)$$

(using Corollary A3).

(b) Again suppose that $\mathfrak{b} = \mathfrak{d} = \omega_2$ and that $\mathbb{P}$ is the ω_2-Namba forcing notion.

(i) Let $\mathrm{RO}(X)$ be the regular open algebra of the topological space X described in (a). Then $\pi(\mathrm{RO}(X)) = \pi(X) = \omega_2$. But

$$\Vdash_{\mathbb{P}} \pi(\mathrm{RO}(X)\check{}) = \omega.$$

P By §3B,

$\Vdash_{\mathbb{P}} \mathrm{RO}(X)\check{}$ is isomorphic to an order-dense subalgebra of $\mathrm{RO}(\tilde{X})$, so $\pi(\mathrm{RO}(X)\check{}) = \pi(\tilde{X}) = \omega$

((a-iv) above). **Q**

(ii) Let Y be the Stone space of $\mathrm{RO}(X)$. Then Y is a compact Hausdorff space and $\pi(Y) = \pi(\mathrm{RO}(X)) = \omega_2$. But

$$\Vdash_{\mathbb{P}} \tilde{Y} \text{ is homeomorphic to the Stone space of } \mathrm{RO}(X)\check{}$$

(Corollary 4B), so

$$\Vdash_{\mathbb{P}} \pi(\tilde{Y}) = \pi(\mathrm{RO}(X)\check{}) = \omega.$$

Remark What these show are that in Theorem 5B, we cannot (at least, if we are looking for a theorem in ZFC) omit the hypothesis that X is compact; and moreover that π-weight need not be preserved in the way that weight is, even for compact spaces. In the next proposition, we shall see that the same thing happens for density. By Theorem 5C, on the other hand, we have a positive result if the generalized continuum hypothesis is true.

10G Proposition Suppose that there is a set $A \subseteq \mathbb{R}$ such that $\kappa = \#(A)$ is a regular cardinal greater than ω_1 and every Lebesgue negligible subset of $\mathbb{R}$ meets A in a set of size less than κ. (Such a set exists, for instance, if there is a Sierpiński set of size $\kappa = \omega_2$, or if $\kappa = \mathfrak{m} = \mathfrak{c} > \omega_1$.) Let $\mathbb{P}$ be the κ-Namba forcing notion (A1 below).

(a) If $\mathfrak{A}$ is the Lebesgue measure algebra, and λ is its centering number, then

$$\Vdash_{\mathbb{P}} d(\check{\mathfrak{A}}) = \omega < \#(\check{\lambda}).$$

(b) If X is the Stone space of $\mathfrak{A}$, then its density $d(X)$ is λ, but

$$\Vdash_{\mathbb{P}} d(\tilde{X}) < \#(\check{\lambda}).$$

proof (a) Enumerate A as $\langle t_\xi \rangle_{\xi<\kappa}$. Let $\theta : \mathfrak{A} \to \Sigma$ be a lifting, where Σ is the σ-algebra of Lebesgue measurable sets (Fremlin 02, §341). For $t \in \mathbb{R}$ set $C_t = \{a : a \in \mathfrak{A},\, t \in \theta(a)\}$, so that C_t is a centered subset of $\mathfrak{A}$. Of course $\lambda \geq \omega_1$ (Fremlin 08, 524Ne, or otherwise).

Let $\dot{B}$ be a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{B} \text{ is a countable cofinal subset of } \check{\kappa}.$$

Consider the $\mathbb{P}$-name

$$\dot{\mathcal{D}} = \{(\check{D}_{t_\xi+q}, p) : p \in \mathbb{P},\, \xi < \kappa,\, p \Vdash_{\mathbb{P}} \xi \in \dot{B},\, \gamma \in \mathbb{Q}\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{\mathcal{D}} \text{ is a countable family of centered subsets of } \check{\mathfrak{A}}.$$

Also

$$\Vdash_{\mathbb{P}} \bigcup \dot{\mathcal{D}} = \check{\mathfrak{A}} \setminus \{0\}.$$

P Suppose that $p \in \mathbb{P}$ and that $\dot{a}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{a} \text{ is a non-zero member of } \check{\mathfrak{A}}.$$

Then there are a q stronger than p and an $a \in \mathfrak{A} \setminus \{0\}$ such that $q \Vdash_{\mathbb{P}} \dot{a} = \check{a}$. Consider the non-negligible measurable set $\theta(a)$. The set $\theta(a) + \mathbb{Q}$ is a conegligible measurable set, so there is a $\xi_0 < \kappa$ such that $t_\xi \in \theta(a) + \mathbb{Q}$ for every $\xi \geq \xi_0$. Now there are r stronger than q and $\xi \geq \xi_0$ such that $r \Vdash_{\mathbb{P}} \xi \in \dot{B}$. Let $\gamma \in \mathbb{Q}$ be such that $t_\xi + \gamma \in \theta(a)$, that is, $a \in D_{t_\xi+\gamma}$. Then

$$r \Vdash_{\mathbb{P}} \dot{a} = \check{a} \in \check{D}_{t_\xi+\gamma} \in \dot{\mathcal{D}}.$$

As p and $\dot{a}$ are arbitrary, we have the result. **Q**

So

$$\Vdash_{\mathbb{P}} d(\check{\mathfrak{A}}) \leq \omega < \omega_1 = \#(\omega_1) \leq \#(\lambda).$$

(b) The Stone space of any Boolean algebra has density equal to the centering number of the algebra, so we have $d(X) = \lambda$. Now

$$\Vdash_{\mathbb{P}} d(\tilde{X}) = d(\check{\mathfrak{A}}) = \omega < \#(\lambda)$$

by Corollary 4B and (a) above.

10H Example Suppose that there is a weakly inaccessible cardinal θ. Let X be a topological space such that $c(X) = \operatorname{sat}(X) = \theta$. Let $\mathbb{P}$ be the Lévy collapsing order for θ (Kunen 80, VII.8.6). Then

$$\Vdash_{\mathbb{P}} c(\tilde{X}) = \omega < \#(\check{\theta}).$$

proof We know that

$$\Vdash_{\mathbb{P}} \check\theta \text{ is the first uncountable cardinal}$$

(Kunen 80, VII.8.8). **?** Suppose, if possible, that

$$\neg \Vdash_{\mathbb{P}} \tilde X \text{ is ccc.}$$

Then there are a $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{\boldsymbol{g}}$ such that

$p \Vdash_{\mathbb{P}} \dot{\boldsymbol{g}}$ is a function from $\check\theta$ to the family of non-empty open sets in $\tilde X$ and $\dot{\boldsymbol{g}}(\xi) \cap \dot{\boldsymbol{g}}(\eta) = \emptyset$ whenever $\xi < \eta < \check\theta$.

Now for each $\xi < \theta$ we have a $p_\xi \in \mathbb{P}$, stronger than p, and a non-empty open set $G_\xi \subseteq X$ such that

$$p_\xi \Vdash_{\mathbb{P}} \tilde G_\xi \subseteq \dot{\boldsymbol{g}}(\xi).$$

Let $I_\xi \subseteq \theta \times \mathbb{N}$ be the domain of p_ξ; by the Δ-system lemma, there is a $\Gamma \in [\theta]^\theta$ such that $\langle I_\xi \rangle_{\xi \in \Gamma}$ is a Δ-system with root I say. Now there is a function $q : I \to \theta$ such that $\Gamma' = \{\xi : \xi \in \Gamma,\ p_\xi \restriction I = q\}$ has cardinal θ. But now p_ξ and p_η are compatible for all ξ, $\eta \in \Gamma'$, so $\langle G_\xi \rangle_{\xi \in \Gamma'}$ is disjoint and $\mathrm{sat}(X) > \theta$. **X**

So

$$\Vdash_{\mathbb{P}} c(\tilde X) = \omega,$$

as required.

10I Examples (a) Let μ be Dieudonné's measure on $X = \omega_1$, and $\mathbb{P}$ a forcing which collapses ω_1 to ω, that is,

$$\Vdash_{\mathbb{P}} \check\omega_1 \text{ is countable.}$$

Note that as all the compact subsets of X are countable,

$$\Vdash_{\mathbb{P}} \tilde X = \dot\varphi[\check X]$$

(2Ag). We still have a $\mathbb{P}$-name $\tilde\mu$ as defined in 6A, but since

$$\Vdash_{\mathbb{P}} \tilde\mu \tilde X = 1,\ \tilde\mu\{\xi\} = 0 \text{ for every } \xi \in \tilde X$$

there is no way to turn it into the name of a countably additive measure.

(b) Let X be the right-facing Sorgenfrey line and μ Lebesgue measure, so that μ is a locally finite σ-finite quasi-Radon measure on X. Let $\mathbb{P}$ be simple Cohen forcing, expressed as the set of functions from finite subsets of $\mathbb{N}$ to $\mathbb{Z}$. Again define $\tilde\mu$ as in 6A. Again, every compact subset of X is countable, so

$$\Vdash_{\mathbb{P}} \tilde X = \dot\varphi[\check X].$$

Consider the $\mathbb{P}$-name

$$\dot{\boldsymbol{g}} = \{((n, [2^{-n}i, 2^{-n}(i+1)]), p) : p \in \mathbb{P},\ n \in \operatorname{dom} p,\ p(n) = i\},$$

where in the formula here the interval $[2^{-n}i, 2^{-n}(i+1)]$ is of course to be interpreted in $V^{\mathbb{P}}$. Then

$$\Vdash_{\mathbb{P}} \tilde\mu\dot{\boldsymbol{g}}(n) = 2^{-n} \text{ for every } n \in \mathbb{N}, \text{ while } \tilde X = \textstyle\bigcap_{n\in\mathbb{N}} \bigcup_{m \ge n} \dot{\boldsymbol{g}}(m).$$

So again we have

$$\Vdash_{\mathbb{P}} \tilde\mu \text{ has no extension to a measure on } \tilde X.$$

10J Example (J.Hart, K.Kunen) Let X be the long line $[0, \omega_1]$ with a top point added (that is, X is the one-point compactification of $\omega_1 \times [0, 1[$ when this is given the order topology defined from the lexicographic ordering). Then X is compact and Hausdorff and connected but not path-connected. If $\mathbb{P}$ is any forcing which collapses ω_1, then

$\Vdash_{\mathbb{P}} \tilde X$ is totally ordered, compact, connected and has countable weight, so is homeomorphic to the unit interval, and in particular is path-connected.

10K Example (Džamonja & Kunen 95) Let $\mathbb{P}$ be a well-pruned Souslin tree, active upwards. Then there is a compact Hausdorff space X such that every Radon measure on X has metrizable support, but

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ has a subspace homeomorphic to } \{0,1\}^{\omega_1}.$$

In the language of Fremlin 08, §533, $\mathrm{Mah}_{\mathrm{R}}(X) = \{0, \omega\}$ but

$$\Vdash_{\mathbb{P}} \mathrm{Mah}_{\mathrm{R}}(\tilde{X}) \neq \{0, \omega\}.$$

proof (a) I begin by noting that because $\mathbb{P}$ is ccc,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is uncountable.}$$

(b) Let X be the set of functions $x \subseteq \mathbb{P} \times \{0,1\}$ such that $\operatorname{dom} x$ is a totally ordered subset of $\mathbb{P}$ and $p \in \operatorname{dom} x$ whenever $q \in \operatorname{dom} x$ and q is stronger than p. Then X is a closed subset of $\mathcal{P}(\mathbb{P} \times \{0,1\})$ with its usual topology, so is a compact Hausdorff space. Now every Radon measure μ on X has metrizable support, so has countable Maharam type. **P** For $p \in \mathbb{P}$, set $G_p = \{x : x \in X, \, p \in \operatorname{dom} x\}$. Then G_p is an open-and-closed subset of X, and $G_p \subseteq G_q$ whenever p is stronger than q in $\mathbb{P}$, while $G_p \cap G_q = \emptyset$ if p and q are incompatible in $\mathbb{P}$. For $\epsilon > 0$, let A_ϵ be $\{p : p \in \mathbb{P}, \, \mu G_p \geq \epsilon\}$. Then A_ϵ is a subtree of $\mathbb{P}$, and every level of A_ϵ is finite; consequently A_ϵ is countable. It follows that $A = \bigcup_{\epsilon > 0} A_\epsilon$ is a countable subtree of $\mathbb{P}$. Set $F = X \setminus \bigcup_{p \in \mathbb{P} \setminus A} G_p$; then F is a closed conegligible subset of X, and includes the support of μ. However F is homeomorphic to a subset of $\mathcal{P}(A \times \{0,1\})$, so is metrizable. **Q**

Set

$$V = \bigcap_{p \in \mathbb{P}} ((Z \setminus \widehat{p}) \times X) \cup (Z \times \times \{x : x \in X, \, p \in \operatorname{dom} x\}.$$

Then V is a closed subset of $Z \times X$, so

$$\Vdash_{\mathbb{P}} \vec{V} \text{ is a closed subset of } \tilde{X}.$$

(c) Let $\mathrm{rank} : \mathbb{P} \to \omega_1$ be the rank function of $\mathbb{P}$, and Z the Stone space of $\mathrm{RO}(\mathbb{P})$. For $\xi < \omega_1$ let H_ξ be the open set

$$\{x : x \in X, \, \mathrm{otp}(\operatorname{dom} x) > \xi\} = \{x : \text{there is some } p \in \operatorname{dom} x \text{ such that } \operatorname{rank} p \geq \xi\},$$

and define a function $\phi_\xi : H_\xi \to \{0,1\}$ by saying that $\phi_\xi(x) = x(p)$ when $p \in \operatorname{dom} x$ and $\operatorname{rank} p = \xi$. Let $\tilde{\phi}_\xi$ be the corresponding $\mathbb{P}$-name for a function as defined in 2C. Because ϕ_ξ is continuous,

$$\Vdash_{\mathbb{P}} \tilde{\phi}_\xi : \tilde{H}_\xi \to \{0,1\} \text{ is continuous.}$$

Note that

$$\Vdash_{\mathbb{P}} \vec{V} \subseteq \tilde{H}_\xi.$$

P If $f \in C(Z; X)$, $p \in \mathbb{P}$ and $\widehat{p} \subseteq^* \{z : (z, f(z)) \in V\}$, let q stronger than p be such that $\operatorname{rank} q \geq \xi$ (remember that $\mathbb{P}$ is supposed to be well-pruned). If $z \in \widehat{q}$ and $(z, f(z)) \in V$, then $q \in \operatorname{dom} f(z)$ so $f(z) \in H_\xi$. Thus $q \Vdash_{\mathbb{P}} \vec{f} \in \tilde{H}_\xi$. **Q**

(d) We therefore have a $\mathbb{P}$-name $\dot{\phi}$ such that

$$\Vdash_{\mathbb{P}} \dot{\phi} : \vec{V} \to \{0,1\}^{\omega_1} \text{ is continuous and } \dot{\phi}(x)(\xi) = \tilde{\phi}_\xi(x) \text{ for every } x \in \vec{V} \text{ and } \xi < \omega_1.$$

Now

$$\Vdash_{\mathbb{P}} \dot{\phi} \text{ is surjective.}$$

P Suppose that $I \in [\omega_1]^{<\omega}$ and $w \in \{0,1\}^I$. Let $\xi < \omega_1$ be such that $I \subseteq \xi$, and set $U = \bigcup_{p \in \mathbb{P}, \operatorname{rank} p = \xi} \widehat{p}$, so that U is a dense open subset of Z. For $z \in U$ define $f(z) \in X$ by saying that

$$\operatorname{dom} f(z) = \{p : p \in \mathbb{P}, \, z \in \widehat{p}\},$$

$$\begin{aligned} f(z)(p) &= w(\eta) \text{ if } z \in \widehat{p} \text{ and } \operatorname{rank} p = \eta \in I, \\ &= 0 \text{ if } z \in \widehat{p} \text{ and } \operatorname{rank} p \notin I. \end{aligned}$$

Then $f : U \to X$ is continuous so $\Vdash_{\mathbb{P}} \vec{f} \in \vec{V}$, while

$$\{z : \phi_\eta(f(z)) = w(\eta) \text{ for every } \eta \in I\} \supseteq \bigcup_{p \in \mathbb{P}, \operatorname{rank} p \geq \xi} \widehat{p}$$

is dense, so

$$\Vdash_{\mathbb{P}} w \subseteq \dot{\phi}(\vec{f}).$$

As w is arbitrary,

$$\Vdash_{\mathbb{P}} \dot{\phi}[\vec{V}] \text{ is dense in } \{0,1\}^{\omega_1} \text{ and } \dot{\phi} \text{ is surjective. } \mathbf{Q}$$

(e) Also

$$\Vdash_{\mathbb{P}} \dot{\phi} \text{ is injective.}$$

P Suppose that f, $g \in C(Z,X)$ and that $p \in \mathbb{P}$ is such that

$$\widehat{p} \subseteq^* \{z : (z, f(z)) \in V,\ (z, g(z)) \in V \text{ and } f(z) \neq g(z)\}.$$

Then there must be a $(q_0, i_0) \in \mathbb{P} \times \{0,1\}$ and a q stronger than p such that $\widehat{q}$ is included in one of $\{z : (q_0, i_0) \in f(z)\}$, $\{z : (q_0, i_0) \in g(z)\}$, and is disjoint from the other; suppose it is included in the former. Because $\{z : (z, f(z)) \in V\}$ is closed and essentially includes $\widehat{q}$, we have $(z, f(z)) \in V$ for every $z \in \widehat{q}$, and q, $q_0 \in \operatorname{dom} f(z)$ for every $z \in \widehat{q}$; accordingly q_0 is weaker than q. Similarly, for any $z \in \widehat{q}$, $q \in \operatorname{dom} g(z)$, so $q_0 \in \operatorname{dom} g(z)$; as $(q_0, i_0) \notin g(z)$, $g(z)(q_0) \neq f(z)(q_0)$. But this means that if $\xi = \operatorname{rank} q_0$ then $\phi_\xi(f(z)) \neq \phi_\xi(g(z))$ for every $z \in \widehat{q}$ and

$$q \Vdash_{\mathbb{P}} \tilde{\phi}_\xi(\vec{f}) \neq \tilde{\phi}_\xi(\vec{g}) \text{ so } \dot{\phi}(\vec{f}) \neq \dot{\phi}(\vec{g}). \ \mathbf{Q}$$

Putting these together,

$$\Vdash_{\mathbb{P}} \vec{V} \text{ is a subspace of } \tilde{X} \text{ homeomorphic to } \{0,1\}^{\omega_1}.$$

Remark I have gone to some trouble to express the ideas of Džamonja & Kunen 95 in the language of this note. Readers may find that the original version gives hints as to how the formulations here can be related to other approaches to forcing; in particular, to models built from generic filters.

11 Possibilities

Here I collect some conjectures which look as if they might sometime be worth exploring.

11B Let X, Y be Hausdorff spaces, $\mathbb{P}$ a forcing notion and Z the Stone space of $\mathrm{RO}(\mathbb{P})$.

(a) If $Z_0 \subseteq Z$ is comeager and $h : Z_0 \times X \to Y$ is continuous, then

$$\Vdash_{\mathbb{P}} \vec{h} \text{ is a continuous function from } \tilde{X} \text{ to } \tilde{Y}.$$

11D Let X, Y be Hausdorff spaces and $\mathbb{P}$ a forcing notion.

(a) If $R \subseteq X \times Y$ is an usco-compact relation, then

$$\Vdash_{\mathbb{P}} \tilde{R} \subseteq \tilde{X} \times \tilde{Y} \text{ is usco-compact.}$$

(b) If X is K-analytic then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is K-analytic.}$$

(c) If X is analytic then

$$\Vdash_{\mathbb{P}} \tilde{X} \text{ is analytic.}$$

11G Let $\mathbb{P}$ be a forcing notion and Z the Stone space of its regular open algebra.

(a) If X is a K-analytic Hausdorff space, Y is a compact metrizable space and $\dot{h}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{h} \text{ is a continuous function from } \tilde{X} \text{ to } \tilde{Y},$$

then there is a function $h : X \to Y$ such that

$$\Vdash_{\mathbb{P}} \dot{h} = \vec{h}.$$

(b) If X is a K-analytic Hausdorff space, $\alpha < \omega_1$ and $\dot{E}$ is a $\mathbb{P}$-name such that

$$\Vdash_{\mathbb{P}} \dot{E} \in \mathcal{B}\mathfrak{a}_\alpha(\tilde{X}),$$

then there are a comeager set $Z_0 \subseteq Z$ and a $W \in \mathcal{B}\mathfrak{a}_\alpha(Z_0 \times X)$ such that

$$\Vdash_{\mathbb{P}} \dot{E} = \vec{W}.$$

12 Problems

12A Suppose that $\operatorname{add}\mathcal{N} = \kappa < \operatorname{add}\mathcal{M}$, where $\mathcal{N}$, $\mathcal{M}$ are the Lebesgue null ideal and the ideal of meager subsets of $\mathbb{R}$. Then there is a family $\langle E_\xi \rangle_{\xi<\kappa}$ of Borel subsets of $[0,1]$ such that $A = \bigcup_{\xi<\kappa} E_\xi$ is not Lebesgue measurable, therefore not universally Baire-property, by 1C. But if Z is any Polish space and $f : Z \to [0,1]$ is continuous, $f^{-1}[A]$ has the Baire property in Z (cf. Matheron Solecki & Zelený p05).

However, we can still ask: is there an example in ZFC of a Polish space X and a set $A \subseteq X$ such that $f^{-1}[A] \in \widehat{\mathcal{B}}(Z)$ whenever Z is Polish and $f : Z \to X$ is continuous, but $A \notin \mathcal{U}\widehat{\mathcal{B}}(X)$?

12B In Theorem 5C, is there a corresponding result for topological density, or for centering numbers of Boolean algebras?

12C In Corollary 7C, do we have a converse? that is, can $\tilde{\phi}$ belong to a Baire class lower than the first Baire class containing ϕ?

12D In Theorem 6I, what can can we do for non-Borel sets $W \subseteq Z \times X$? Maybe we can reach a class closed under Souslin's operation. What about arbitrary $W \in \mathcal{U}\widehat{\mathcal{B}}(Z \times X)$?

12E In Proposition 3F, are there any other natural classes of topological space for which 3Fb or 3Fc will be valid? What about analytic Hausdorff spaces?

12F In Theorem 2G, can we characterize those $V \subseteq Z \times X$ for which $\Vdash_{\mathbb{P}} \vec{V}$ is compact?

12G In Proposition 8I, can we characterize those $(\Sigma, \mathcal{U}\widehat{\mathcal{B}}(X))$-measurable functions g for which there is a $\mathbb{P}$-name $\dot{x}$ such that $[\![\dot{x} \in \tilde{F}]\!] = g^{-1}[F]^\bullet$ for every $F \in \mathcal{U}\widehat{\mathcal{B}}(X)$?

12H In Theorem 4A, can we add

if X is a Hausdorff k-space, then $\Vdash_{\mathbb{P}} \tilde{X}$ is a k-space,

if X is compact, Hausdorff and path-connected, then $\Vdash_{\mathbb{P}} \tilde{X}$ is path-connected?

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Appendix: Namba forcing

For Example 10F we need a classic forcing notion. It is discussed at length in Shelah 82. In Fremlin n86 I wrote out my own version of the following theorem, itself derived from notes taken by G.Gruenhage at a lecture by M.Magidor. As Fremlin n86 exists only as photocopies-of-photocopies-of-typescript I reproduce the argument here with a different set of typos.

A1 Let X be a set and $\mathcal{I}$ a proper ideal of subsets of X. Consider the forcing notion $\mathbb{P}$ defined by saying that $\mathbb{P}$ is the set of those $p \subseteq \bigcup_{n\in\mathbb{N}} X^n$ such that

$\sigma\restriction n \in p$ whenever $\sigma \in p$ and $n \in \mathbb{N}$

there is an element stem(p) of p such that for every $\sigma \in p$

either $\sigma \subseteq \text{stem}(p)$

or $\text{stem}(p) \subseteq \sigma$ and $\{x : \sigma^\frown{<}x{>} \in p\} \notin \mathcal{I}$,

where, for $\sigma \in X^n$ and $x \in X$, $\sigma^\frown{<}x{>} = \sigma \cup \{(n, x)\} \in X^{n+1}$; and that p is stronger than q if $p \subseteq q$. I will call this the $(X, \mathcal{I})$**-Namba forcing notion**; when $X = \kappa$ is an infinite cardinal and $\mathcal{I} = [\kappa]^{<\kappa}$ I will call it the κ**-Namba forcing notion**.

Note that if p is stronger than q then $\text{stem}(p) \supseteq \text{stem}(q)$.

A2 Theorem Let X be a set, $\mathcal{I}$ a proper ideal of subsets of X with additivity and saturation greater than ω_1, and $\mathbb{P}$ the $(X, \mathcal{I})$-Namba forcing notion. If $S \subseteq \omega_1$ is stationary then

$$\Vdash_{\mathbb{P}} \check{S} \text{ is stationary in } \check{\omega}_1.$$

Remark As for any forcing notion,

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a non-zero limit ordinal.}$$

We do not yet know that

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal}$$

(this will be considered in A3 below), so we need to say: if α is an ordinal, a subset A of α is 'stationary' if it meets every relatively closed subset of α which is cofinal with α. If α is a non-zero limit ordinal of countable cofinality, this can happen only if $\sup(\alpha \setminus A) < \alpha$, of course.

proof (a) For $\sigma \in \bigcup_{n\in\mathbb{N}} X^n$ set

$$I_\sigma = \{\tau : \tau \in \textstyle\bigcup_{n\in\mathbb{N}} X^n \text{ and either } \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}.$$

Then $I_\sigma \in \mathbb{P}$ and $p \cap I_\sigma \in \mathbb{P}$ whenever $\sigma \in p \in \mathbb{P}$.

It will be convenient to fix here on a ladder system on ω_1, that is, a family $\langle\theta(\zeta, n)\rangle_{\zeta\in\Omega, n\in\mathbb{N}}$, where Ω is the set of non-zero countable limit ordinals, such that $\langle\theta(\zeta, n)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with supremum ζ for each $\zeta \in \Omega$.

Suppose that $p \in \mathbb{P}$ and that $\dot{C}$ is a $\mathbb{P}$-name such that

$$p \Vdash_{\mathbb{P}} \dot{C} \text{ is a closed cofinal subset of } \check{\omega}_1.$$

For each $r \in \mathbb{P}$ stronger than p set $C_r = \{\beta : \beta < \omega_1,\ r \Vdash_{\mathbb{P}} \check{\beta} \in \dot{C}\}$. Note that (α) C_r is always a closed subset of ω_1 (β) $C_r \subseteq C_{r'}$ if r' is stronger than r (γ) if r is stronger than p and $\alpha < \omega_1$ then there is an r' stronger than r such that $C_{r'} \not\subseteq \alpha$.

(b) Whenever q is stronger than p and $\alpha < \omega_1$ there is an r stronger than q such that $\text{stem}(r) = \text{stem}(q)$ and $C_r \not\subseteq \alpha$. **P** Set

$$s = \{\text{stem}(r) : r \in \mathbb{P},\ r \subseteq q,\ C_r \not\subseteq \alpha\},$$

$$q^* = \{\sigma : \sigma \in q,\ \sigma\restriction n \notin s \text{ for every } n \in \mathbb{N}\}.$$

? If $q^* \in \mathbb{P}$ there is an r stronger than q^* such that $C_r \not\subseteq \alpha$; but in this case $\text{stem}(r) \in q^* \cap s$. **X** So $q^* \notin \mathbb{P}$. Next, if σ is a proper initial segment of stem(q), then σ cannot be equal to stem(r) for any r stronger than q, so $\sigma \in q^*$; and $\sigma\restriction n \in s$ whenever $\sigma \in s$ and $n \in \mathbb{N}$.

? If $\text{stem}(q) \notin s$ then $\text{stem}(q) \in q^*$. So there must be a $\sigma \in q^*$ such that $\text{stem}(q) \subseteq \sigma$ and $\{x : \sigma^\frown{<}x{>} \in q^*\} = \{x : \sigma^\frown{<}x{>} \notin s\}$ belongs to $\mathcal{I}$. In this case, $A = \{x : \sigma^\frown{<}x{>} \in q \setminus q^*\}$ does not belong to $\mathcal{I}$. For each $x \in A$ choose $q_x \in \mathbb{P}$ such that $\text{stem}(q_x) = \sigma^\frown{<}x{>}$ and $C_{q_x} \not\subseteq \alpha$. As $\text{add}\,\mathcal{I} > \omega_1$, there is a $\beta < \omega_1$ such that $B = \{x : x \in A,\ \beta \in C_{q_x} \setminus \alpha\} \notin \mathcal{I}$. Set $r = \bigcup_{x\in B} q_x$. Then $r \in \mathbb{P}$ and $\text{stem}(r) = \sigma$. If r' is stronger than r there is some $x \in B$ such that r' is compatible with q_x, so there is an r'' stronger than r' such that $r'' \Vdash_{\mathbb{P}} \check{\beta} \in \dot{C}$; accordingly $r \Vdash_{\mathbb{P}} \check{\beta} \in \dot{C}$, $\beta \in C_r \setminus \alpha$ and $\sigma \in s$. But σ is supposed to belong to q^* which is disjoint from s. **X**

So $\mathrm{stem}(q) \in s$, as claimed. **Q**

(c) Let R be the set of pairs (r, g) such that $r \in \mathbb{P}$ is stronger than p and $g : r \to \omega_1$ is such that

whenever $\mathrm{stem}(r) \subseteq \sigma \in r$ then $g(\sigma) \in C_{r\cap I_\sigma}$, $g(\sigma) < g(\tau)$ whenever $\tau \in r$ properly extends σ, and $\{x : \sigma^\frown{<}x{>} \in r,\ g(\sigma^\frown{<}x{>}) \ge \alpha\} \notin \mathcal{I}$ for every $\alpha < \omega_1$.

Then for any q stronger than p there is a pair $(r, g) \in R$ such that r is stronger than q and $\mathrm{stem}(r) = \mathrm{stem}(q)$. **P** Use (b) repeatedly, as follows. Set $k = \#(\mathrm{stem}(q))$ and take q_0 stronger than q such that $\mathrm{stem}(q_0) = \mathrm{stem}(q)$ and $C_{q_0} \ne \emptyset$; take $g(\sigma) \in C_{q_0}$ for initial segments σ of $\mathrm{stem}(q)$.

Given that $q_n \in P$ is stronger than q, $\mathrm{stem}(q_n) = \mathrm{stem}(q)$ and that $g(\sigma)$ has been defined when $\sigma \in q_n$ and $\#(\sigma) \le k + n$, then for each $\sigma \in q_n \cap X^{k+n}$ set

$$A_\sigma = \{x : \sigma^\frown{<}x{>} \in q_n \cap X^{k+n+1}\}.$$

Then $A_\sigma \notin \mathcal{I}$; let $f_\sigma : A_\sigma \to \omega_1$ be such that $f_\sigma(x) > g(\sigma)$ for every $\tau \in A_\sigma$ and $\{x : x \in A_\sigma,\ f_\sigma(x) \ge \alpha\} \notin \mathcal{I}$ for every $\alpha < \omega_1$. For each $\tau \in q_n \cap X^{k+n+1}$ use (b) to find $r_\tau \in \mathbb{P}$ and $g(\tau)$ such that $\mathrm{stem}(r_\tau) = \tau$ and $g(\tau) \in C_{r_\tau} \not\subseteq f_{\tau\restriction k+n}(\tau(k + n))$. Now set $q_{n+1} = \bigcup\{r_\tau : \tau \in q_n \cap X^{k+n+1}\}$. Then $q_{n+1} \in \mathbb{P}$, $q_{n+1} \subseteq q_n$, $q_{n+1} \cap X^{k+n+1} = q_n \cap X^{k+n+1}$, $\mathrm{stem}(q_{n+1}) = \mathrm{stem}(q)$, and $g(\tau) \in C_{r_\tau} = C_{q_{n+1}\cap I_\tau}$ and $g(\tau) > g(\tau\restriction k + n)$ whenever $\tau \in q_{n+1} \cap X^{k+n+1}$.

So if we set $r = \bigcap_{n\in\mathbb{N}} q_n$, we shall have $r \cap X^{k+n} = q_n \cap X^{k+n}$ for every n, $r \in \mathbb{P}$ is stronger than q, $\mathrm{stem}(r) = \mathrm{stem}(q)$ and $r = \mathrm{dom}\, g$. If $\mathrm{stem}(r) \subseteq \sigma \in r$, and $x \in A_\sigma$, then $g(\sigma^\frown{<}x{>}) \ge f_\sigma(x) > g(\sigma)$, so $\{x : \sigma^\frown{<}x{>} \in r,\ g(\sigma^\frown{<}x{>}) \ge \alpha\} \notin \mathcal{I}$ for every $\alpha < \omega_1$. Also $g(\tau) \in C_{q_0} \subseteq C_r$ if τ is an initial segment of $\mathrm{stem}(r) = \mathrm{stem}(q)$, while for any other member of r we have $g(\tau) \in C_{r_\tau} \subseteq C_{r\cap I_\tau}$. So we have a suitable r. **Q**

(d) Take $(r, g) \in R$, and set $k = \#(\mathrm{stem}(r))$. For $\zeta \in \Omega$ let $W_{rg\zeta}$ be the set of those $w \in X^{\mathbb{N}}$ such that $w\restriction n \in r$ and $\theta(\zeta, n) \le g(w\restriction k+n+1) < \zeta$ for every $n \in \mathbb{N}$. For $\sigma \in r$ let $\Omega_{rg\sigma}$ be the set of those $\zeta \in \Omega$ such that whenever $h : \bigcup_{n\in\mathbb{N}} X^n \to \mathcal{I}$ is a function there is a $w \in W_{rg\zeta}$ such that $\sigma \subseteq w$ and $w(n) \notin h(w\restriction n)$ for $n \ge \#(\sigma)$. Then

$$A = \{x : \sigma^\frown{<}x{>} \in r,\ \zeta \in \Omega_{r,g,\sigma^\frown{<}x{>}}\}$$

does not belong to $\mathcal{I}$ for any $\zeta \in \Omega_{rg\sigma}$. **P?** Otherwise, for each $x \in X \setminus A$ choose $h_x : \bigcup_{n\in\mathbb{N}} X^n \to \mathcal{I}$ such that there is no $w \in W_{rg\zeta}$ such that $\sigma^\frown{<}x{>} \subseteq w$ and $w(n) \notin h_x(w\restriction n)$ for $n \ge \#(\sigma) + 1$. Define $h : \bigcup_{n\in\mathbb{N}} X^n \to \mathcal{I}$ by setting

$$\begin{aligned} h(\tau) &= A \text{ if } \tau = \sigma,\\ &= h_x(\tau) \text{ if } \tau \text{ properly extends } \sigma \text{ and } x = \tau(\#(\sigma)),\\ &= \emptyset \text{ if } \sigma \not\subseteq \tau. \end{aligned}$$

As $\zeta \in \Omega_{rg\sigma}$, there is supposed to be a $w \in W_{rg\zeta}$ such that $\sigma \subseteq w$ and $w(n) \notin h(w\restriction n)$ for $n \ge \#(\sigma)$; but in this case, setting $x = w(n) \in X \setminus A$, $w \notin h_x(w\restriction n)$ for $n \ge \#(\sigma) + 1$. **XQ**

(e) Recall that we were given a stationary set S in the statement of the theorem. Take $(r, g) \in R$. Then S meets $\Omega_{r,g,\mathrm{stem}(r)}$. **P** Set $k = \#(\mathrm{stem}(r))$. Again because $\mathrm{add}\,\mathcal{I} > \omega_1$, there is an $h : \bigcup_{n\in\mathbb{N}} X^n \to \mathcal{I}$ such that whenever $\zeta \in \Omega \setminus \Omega_{r,g,\mathrm{stem}(r)}$ there is no $w \in W_{rg\zeta}$ such that $\mathrm{stem}(r) \subseteq w$ and $w \notin h(w\restriction n)$ for $n \ge k$. Choose $\langle\sigma_\upsilon\rangle_{\upsilon\in\bigcup_{n\in\mathbb{N}}\omega_1^n}$ so that

$\sigma_\emptyset = \mathrm{stem}(r)$,

whenever $n \in \mathbb{N}$, $\upsilon \in \omega_1^n$, $\sigma_\upsilon \in r \cap \omega_1^{k+n}$ and $\alpha < \omega_1$, then $\sigma_{\upsilon^\frown{<}\alpha{>}} = \sigma_\upsilon^\frown{<}x{>}$ for some x such that $x \notin h(\sigma_\upsilon)$ and $g(\sigma_\upsilon^\frown{<}x{>}) \ge \alpha$.

Because S is stationary, there is a $\zeta \in S \cap \Omega$ such that $g(\sigma_\upsilon) < \zeta$ for every $\upsilon \in \bigcup_{n\in\mathbb{N}} \zeta^n$. Set $\upsilon_n = \langle\theta(\zeta, i)\rangle_{i<n}$ for each n; then we have a $w \in X^{\mathbb{N}}$ such that $w\restriction k + n = \sigma_{\upsilon_n}$ for each n. Now

$$g(w\restriction k + n + 1) = g(\sigma_{\upsilon_{n+1}}) \in \zeta \setminus \upsilon_{n+1}(n) \subseteq \zeta \setminus \theta(\zeta, n)$$

for each n, so $w \in W_{rg\zeta}$, while $w(n) \notin h(w\restriction n)$ for $n \ge k$. By the choice of h, ζ must belong to $\Omega_{r,g,\mathrm{stem}(r)}$. **Q**

(f) If $(r, g) \in R$ and $S \subseteq \omega_1$ is stationary, there is an r', stronger than r, such that $C_{r'} \cap S$ is non-empty. **P** Take $\zeta \in S \cap \Omega_{r,g,\mathrm{stem}(r)}$. Set

$$r' = \{\sigma : \mathrm{stem}(r) \subseteq \sigma \in r,\, \zeta \in \Omega_{rg\sigma}\} \cup \{\mathrm{stem}(r)\restriction n : n \in \mathbb{N}\}.$$

Then (d)-(e) tell us that $r' \in \mathbb{P}$. Also, setting $k = \#(\mathrm{stem}(r))$ as usual,

$$g(\sigma) \in C_{r\cap I_\sigma}) \subseteq C_{r'\cap I_\sigma} \cap \zeta \setminus \theta(\zeta, n)$$

whenever $n \in \mathbb{N}$ and $\sigma \in r' \cap X^{k+n}$. Now if $n \in \mathbb{N}$ and r_1 is stronger than r', there is an r_2 stronger than r_1 such that C_{r_2} meets $\zeta \setminus \theta(\zeta, n)$, that is, $r_2 \Vdash_{\mathbb{P}} \dot{C}$ meets $(\zeta \setminus \theta(\zeta, n))^\vee$. So $r' \Vdash_{\mathbb{P}} \dot{C}$ meets $(\zeta \setminus \theta(\zeta, n))^\vee$. As this is true for every n, and $r' \Vdash_{\mathbb{P}} \dot{C}$ is closed, $r' \Vdash_{\mathbb{P}} \check{\zeta} \in \dot{C}$ and $\zeta \in C_{r'} \cap S$. **Q**

(g) We are nearly home. For any q stronger than p there are an $(r, g) \in R$ such that r is stronger than q and an r' stronger than r such that $C_{r'} \cap S$ is non-empty, so surely $r' \Vdash_{\mathbb{P}} \check{S} \cap \dot{C} \neq \emptyset$. But this means that $p \Vdash_{\mathbb{P}} \check{S} \cap \dot{C} \neq \emptyset$. As p and $\dot{C}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \check{S} \text{ is stationary,}$$

as required.

A3 Corollary If X is a set, $\mathcal{I}$ is a proper ideal of subsets of X which is ω_2-additive and not ω_1-saturated, and $\mathbb{P}$ is the $(X, \mathcal{I})$-Namba forcing notion, then

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is a cardinal.}$$

proof Take any stationary set $S \subseteq \omega_1$ such that $\omega_1 \setminus S$ is uncountable. Then

$\Vdash_{\mathbb{P}} \check{\omega}_1 \setminus \check{S}$ is cofinal with $\check{\omega}_1$ and for every cofinal subset A of $\check{\omega}_1$ there is a $\zeta \in \check{S}$ such that $\zeta = \sup(\zeta \cap A)$.

But this implies that

$$\Vdash_{\mathbb{P}} \mathrm{cf}\,\check{\omega}_1 \neq \omega,$$

so

$$\Vdash_{\mathbb{P}} \check{\omega}_1 \text{ is the first uncountable ordinal.}$$

A4 Proposition If κ is an infinite cardinal and $\mathbb{P}$ is the κ-Namba forcing notion,

$$\Vdash_{\mathbb{P}} \mathrm{cf}\,\check{\kappa} = \omega.$$

proof Let $\dot{f}$ be the $\mathbb{P}$-name

$$\{((n, \xi), p) : p \in \mathbb{P},\, n < \#(\mathrm{stem}(p)),\, \mathrm{stem}(p)(n) = \xi\}.$$

Then

$$\Vdash_{\mathbb{P}} \dot{f} : \omega \to \check{\kappa} \text{ is a function and } \dot{f}[\omega] \text{ is cofinal with } \check{\kappa}\}.$$

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