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Maharam algebras

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1 Foundations

1A Definitions (See Fremlin 04, §392.) Let $\mathfrak{A}$ be a Boolean algebra.

(a)(i) A **submeasure** on $\mathfrak{A}$ is a functional $\nu : \mathfrak{A} \to [0, \infty[$ such that

ν is **subadditive**, that is, $\nu(a \cup b) \le \nu a + \nu b$ for all a, $b \in \mathfrak{A}$,

$\nu 0_{\mathfrak{A}} = 0$, $\quad \nu a \le \nu b$ whenever $a \subseteq b \in \mathfrak{A}$.

(ii) Let ν be a submeasure on $\mathfrak{A}$. ν is **exhaustive** if $\lim_{n\to\infty} \nu a_n = 0$ for every disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. ν is **uniformly exhaustive** if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\inf_{a\in A} \nu a < \epsilon$ for every disjoint set $A \subseteq \mathfrak{A}$ of size greater than n. ν is **strictly positive** if $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$. ν is **countably subadditive** if $\nu(\sup_{n\in\mathbb{N}} a_n) \le \sum_{n=0}^{\infty} \nu a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ with a supremum in $\mathfrak{A}$. ν is a **Maharam submeasure** if $\lim_{n\to\infty} \nu a_n = 0$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0. ν is **atomless** if whenever $a \in \mathfrak{A}$ and $\nu a > 0$ there is a $b \subseteq a$ such that $\nu b > 0$ and $\nu(a \setminus b) > 0$. ν is **unital** if $\nu 1_{\mathfrak{A}} = 1$. ν is **additive** if $\nu(a \cup b) = \nu a + \nu b$ for all disjoint a, $b \subseteq \mathfrak{A}$. ν is **completely additive** if it is additive and $\inf_{a\in A} \nu a = 0$ whenever A is a non-empty downwards-directed set in $\mathfrak{A}$ with infimum 0 (see Fremlin 04, 326J). ν is **pathological** if it is non-zero and there is no non-zero additive functional μ on $\mathfrak{A}$ such that $0 \le \mu a \le \nu a$ for every $a \in \mathfrak{A}$. ν is a **Ramsey submeasure** (Zapletal p06) if $\inf_{m<n\in\mathbb{N}} \nu(a_m \cup a_n) \le \sup_{n\in\mathbb{N}} \nu a_n$ for every sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$. ν is **diffuse** (Farah) if for every $\epsilon > 0$ there is a finite partition D of the identity such that $\nu d \le \epsilon$ for every $d \in D$.

(iii) If μ and ν are two submeasures on $\mathfrak{A}$, I say that μ is **absolutely continuous** with respect to ν if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\mu a \le \epsilon$ whenever $\nu a \le \delta$.

(b) $\mathfrak{A}$ is a **Maharam algebra** (Veličković 05) if it is Dedekind complete and there is a strictly positive Maharam submeasure on $\mathfrak{A}$. $\mathfrak{A}$ is a **measurable algebra** (Fremlin 04, §391) if it is Dedekind complete and there is a strictly positive additive Maharam submeasure on $\mathfrak{A}$. (For an example of a Maharam algebra which is not measurable, see Talagrand 06 or Fremlin n06.) $\mathfrak{A}$ is **chargeable** if it has a strictly positive additive submeasure (Fremlin 04, 391X). If $\mathfrak{A}$ is Dedekind σ-complete, I will say that it is **nowhere measurable** if no non-zero principal ideal of $\mathfrak{A}$ is a measurable algebra.

(c) $\mathfrak{A}$ is **weakly (σ, ∞)-distributive** (Fremlin 04, §316) if for every sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of partitions of unity in $\mathfrak{A}$ there is a partition D of unity in $\mathfrak{A}$ such that $\{c : c \in C_n,\ c \cap d \ne 0\}$ is finite for every $n \in \mathbb{N}$ and every $d \in D$. $\mathfrak{A}$ is **weakly σ-distributive** if for every sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of countable partitions of unity in $\mathfrak{A}$ there is a partition D of unity in $\mathfrak{A}$ such that $\{c : c \in C_n,\ c \cap d \ne 0\}$ is finite for every $n \in \mathbb{N}$ and every $d \in D$. Note that every weakly (σ, ∞)-distributive algebra is weakly σ-distributive, and that a ccc weakly σ-distributive algebra is weakly (σ, ∞)-distributive.

If κ is any cardinal, $\mathfrak{A}$ is **weakly (κ, ∞)-distributive** if whenever $\langle C_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{A}$, there is a partition D of unity such that $\{c : c \in C_\xi,\ c \cap d \ne 0\}$ is finite for every $d \in D$ and $\xi < \kappa$. Now the **weak distributivity** $\operatorname{wdistr}(\mathfrak{A})$ of $\mathfrak{A}$ is the least cardinal κ such that $\mathfrak{A}$ is not weakly (κ, ∞)-distributive. (If there is no such cardinal, write $\operatorname{wdistr}(\mathfrak{A}) = \infty$.)

(d) $\mathfrak{A}$ is **σ-finite-cc** (condition (ii) of Horn & Tarski 48, Theorem 2.4) if there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets with union $\mathfrak{A}$ such that no infinite subset of any A_n is disjoint; it is **σ-bounded-cc** (condition (ii)$'$ of Horn & Tarski 48, p. 482) if there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets with union $\mathfrak{A}$ such that no A_n includes a disjoint set of size greater than n. For cardinals κ, λ and θ, say that $(\kappa, \lambda, {<}\theta)$ is a **precaliber triple** of $\mathfrak{A}$ if for every family $\langle a_\xi\rangle_{\xi<\kappa}$ in $\mathfrak{A}^+ = \mathfrak{A} \setminus \{0\}$ there is a $\Gamma \in [\kappa]^\lambda$ such that $\inf_{\xi\in I} a_\xi \ne 0$ for every $I \in [\Gamma]^{<\theta}$ (see Fremlin 08?, §511). I will say that $(\kappa, \lambda, \theta)$ is a precaliber triple of $\mathfrak{A}$ if $(\kappa, \lambda, {<}\theta^+)$ is a precaliber triple of $\mathfrak{A}$. [If (ω_1, ω_1, n) is a precaliber triple of $\mathfrak{A}$, $\mathfrak{A}$ is said to have **property $\mathbf{K}_n$**.]

I will examine a further chain condition on a Boolean algebra in §§2D and 6A:

(*) $\mathfrak{A}=\bigcup_{n\in\mathbb{N}}A_n$ where every infinite subset of every A_n has an infinite centered subset.

(e) A sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ **order*-converges** to $a\in\mathfrak{A}$ (Fremlin 04, §§367 and 392) if there is a partition C of unity in $\mathfrak{A}$ such that $\{n:c\cap(a_n\triangle a)\neq 0\}$ is finite for every $c\in C$. The **order-sequential topology** on $\mathfrak{A}$ (Fremlin 04, §392; compare Balcar Głowczyński & Jech 98) is the topology for which a set $F\subseteq\mathfrak{A}$ is closed iff $a\in F$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in F order*-converging to a.

1B Elementary remarks (a)(i) Any Maharam submeasure is sequentially order-continuous. **P** Let μ be a Maharam submeasure on a Boolean algebra $\mathfrak{A}$. (α) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and has supremum a, then $\langle a\setminus a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0; now

$$\mu a_n\le\mu a\le\mu a_n+\mu(a\setminus a_n)$$

for each n, so

$$\lim_{n\to\infty}|\mu a-\mu a_n|\le\lim_{n\to\infty}\mu(a\setminus a_n)=0.$$

(β) If $\langle a_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum a, then $\langle a_n\setminus a\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0; now

$$\lim_{n\to\infty}|\mu a-\mu a_n|\le\lim_{n\to\infty}\mu(a_n\setminus a)=0.\ \mathbf{Q}$$

(ii) A Maharam submeasure on a Dedekind σ-complete Boolean algebra is exhaustive (Fremlin 04, 392Hc).

(iii) Any Boolean algebra with a strictly positive exhaustive submeasure (in particular, any Maharam algebra) is σ-finite-cc therefore ccc.

(b) If $\mathfrak{A}$ is a Boolean algebra and ν is an exhaustive submeasure on $\mathfrak{A}$ which is sequentially order-continuous on the left (that is, $\nu a=\sup_{n\in\mathbb{N}}\nu a_n$ whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a) then ν is a Maharam submeasure. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{A}$ with infimum 0, then $\nu a_n=\lim_{i\to\infty}\nu(a_n\setminus a_i)$ for each n, so we can choose a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\nu(a_{n_k}\setminus a_{n_{k+1}})\ge\nu a_{n_k}-2^{-k}$ for each k; now

$$\lim_{n\to\infty}\nu a_n=\lim_{k\to\infty}\nu a_{n_k}=\lim_{k\to\infty}(\nu a_{n_k}\setminus a_{n_{k+1}})=0.\ \mathbf{Q}$$

(c) Let $\mathfrak{A}$ be a Boolean algebra. (i) If $\mathfrak{A}$ is σ-finite-cc then any subalgebra of $\mathfrak{A}$ is σ-finite-cc. (If $\langle A_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}$ is σ-finite-cc, and $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$, then $\langle A_n\cap\mathfrak{B}\rangle_{n\in\mathbb{N}}$ will witness that $\mathfrak{B}$ is σ-finite-cc.) (ii) If $\mathfrak{A}$ has an order-dense σ-finite-cc subalgebra $\mathfrak{B}$, then $\mathfrak{A}$ is σ-finite-cc. (If $\langle B_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{B}$ is σ-finite-cc, set $A_n=\{a:a\in\mathfrak{A},\ b\subseteq a$ for some $b\in B_n\}$ for each n; then $\langle A_n\rangle_{n\in\mathbb{N}}$ will witness that $\mathfrak{A}$ is σ-finite-cc.) (iii) If $\mathfrak{A}$ has an order-dense weakly (σ,∞)-distributive subalgebra $\mathfrak{B}$ then $\mathfrak{A}$ is weakly (σ,∞)-distributive. (If $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence of partitions of unity in $\mathfrak{A}$, then for each $n\in\mathbb{N}$ we can find a partition of unity C'_n in $\mathfrak{B}$ refining C_n. Now there is a partition D of unity in $\mathfrak{B}$ such that $\{c:c\in C'_n,\ c\cap d\neq 0\}$ is finite for every $n\in\mathbb{N}$ and $d\in D$; in this case, D is still a partition of unity in $\mathfrak{A}$ and $\{c:c\in C_n,\ c\cap d\neq 0\}$ is finite for every $n\in\mathbb{N}$ and $d\in D$.)

1C Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν an atomless Maharam submeasure on $\mathfrak{A}$.

(a) If $a\in\mathfrak{A}$ and $0\le\gamma\le\nu a$ there is a $b\in\mathfrak{A}$ such that $b\subseteq a$ and $\nu b=\gamma$.

(b) ν is diffuse.

proof (a)(i) Note first that if $\delta>0$, $c\in\mathfrak{A}$ and $\nu c>0$ then there is a $d\subseteq c$ such that $0<\nu d\le\delta$. **P** Choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively so that $c_0=c$, $c_{n+1}\subseteq c_n$, $\nu c_{n+1}>0$ and $\nu(c_n\setminus c_{n+1})>0$ for every n. By 1Ba, ν is exhaustive. So $\lim_{n\to\infty}\nu(c_n\setminus c_{n+1})=0$, and we can take $d=c_n\setminus c_{n+1}$ for an appropriate n. **Q**

(ii) Choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $b_0=0$. Given that $b_n\subseteq a$, set $\gamma_n=\sup\{\nu c:b_n\subseteq c\subseteq a,\ \nu c\le\gamma\}$ and choose b_{n+1} such that $b_n\subseteq b_{n+1}\subseteq a$, $\nu b_{n+1}\le\gamma$ and $\nu b_{n+1}\ge\gamma_n-2^{-n}$. Set $b=\sup_{n\in\mathbb{N}}b_n$; then $\langle b\setminus b_n\rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0, so $\lim_{n\to\infty}\nu(b\setminus b_n)=0$ and $\nu b=\lim_{n\to\infty}\nu b_n\le\gamma$.

If $b\subseteq b'\subseteq a$ and $\nu b'\le\gamma$, then $\nu b'=\nu b$. **P?** Otherwise, there is an $n\in\mathbb{N}$ such that $\nu b<\nu b'-2^{-n}$. But observe that $b_n\subseteq b$ and $\nu b\le\gamma$, so $\nu b_n\ge\nu b'-2^{-n}$. **XQ**

? Suppose, if possible, that $\nu b < \gamma$. Let D be a maximal disjoint family in $\mathfrak{A}$ such that $0 < \nu d \le \gamma - \nu b$ and $b \cap d = 0$ for every $d \in D$. Because ν is exhaustive, D must be countable; let $\langle d_n \rangle_{n\in\mathbb{N}}$ run over $D \cup \{0\}$. By the last remark, we can induce on n to see that $\nu(b \cup \sup_{i\le n} d_i) = \nu b$ for every $n \in \mathbb{N}$. Set $b^* = b \cup \sup_{i\in\mathbb{N}} d_i$; then

$$\nu b^* = \lim_{n\to\infty} \nu(b \cup \sup_{i\le n} b_i) = \nu b < \gamma,$$

and $\nu(a \setminus b^*) \ge \nu a - \nu b^* > 0$. By (a), there is a $d \subseteq a \setminus b^*$ such that $0 < \nu d \le \gamma - \nu b^*$. So we ought to have put d into D. **X**

Thus $\nu b = \gamma$, as required.

(b) Let $A_0 \subseteq \mathfrak{A}$ be a maximal disjoint set such that $\nu a = \epsilon$ for every $a \in A_0$. Because ν is exhaustive (1B(a-ii)), A_0 is finite. Set $c = 1 \setminus \sup A_0$; by (a), $\nu c < \epsilon$. So we can take $A = A_0 \cup \{c\}$.

1D Proposition Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive Boolean algebra and $\nu : \mathfrak{A} \to [0,\infty[$ a functional such that $\nu b \le \nu a$ whenever $b \subseteq a$. Set

$$\mu a = \inf\{\sup_{c\in C} \nu c : C \subseteq \mathfrak{A} \text{ is non-empty and upwards-directed and } \sup C = a\}.$$

(a) $\mu b \le \mu a$ whenever $b \subseteq a$ in $\mathfrak{A}$.

(b) If $\nu a > 0$ for every non-zero $a \in \mathfrak{A}$ then $\mu a > 0$ for every non-zero $a \in \mathfrak{A}$.

(c) μ is sequentially order-continuous on the left, that is, $\mu a = \sup_{n\in\mathbb{N}} a_n$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum a.

(d) If ν is subadditive, so is μ.

(e) If ν is an exhaustive submeasure, μ is a Maharam submeasure.

(f) If ν is a uniformly exhaustive submeasure, so is μ.

(g) If ν is additive, μ is countably additive.

proof (a) If $b \subseteq a$ and C is an upwards-directed set with supremum a, then $\{b \cap c : c \in C\}$ is an upwards-directed set with supremum b; so $\mu b \le \mu a$.

(b) If $\mu a = 0$, then for each $n \in \mathbb{N}$ we can find a non-empty upwards-directed set C_n such that $\sup C_n = a$ and $\sup_{b\in C_n} \nu b \le 2^{-n}$. Set

$$C = \{c : \text{ there is some } n \in \mathbb{N} \text{ such that for every } m \ge n \text{ there is a } b \in C_m \text{ such that } b \supseteq c\}.$$

Then C is upwards-directed and (because $\mathfrak{A}$ is weakly (σ,∞)-distributive) $\sup C = a$. But $\nu c = 0$ for every $c \in C$ so (because ν is strictly positive) $C = \{0\}$ and $a = 0$. Thus μ is strictly positive.

(c) Suppose that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{A}$ with supremum a, then of course $\mu a \ge \sup_{n\in\mathbb{N}} \mu a_n$. Now suppose that $\alpha > \sup_{n\in\mathbb{N}} \mu a_n$. For each $n \in \mathbb{N}$, we have a non-empty upwards-directed set B_n such that $\sup B_n = a_n$ and $\nu b \le \alpha$ for every $b \in B_n$. Set

$$C = \{c : \text{ there is some } n \in \mathbb{N} \text{ such that for every } m \ge n \text{ there is a } b \in B_m \text{ such that } b \supseteq c\}.$$

Then (as in (b)) C is upwards-directed and $\sup C = a$. So $\mu a \le \sup_{c\in C} \nu c \le \alpha$. As α is arbitrary, $\mu a = \sup_{n\in\mathbb{N}} \mu a_n$.

(d) If a, $a' \in \mathfrak{A}$, B is a non-empty upwards-directed set with supremum a, and B' is a non-empty upwards-directed set with supremum a', then $C = \{b \cup b' : b \in B,\, b' \in B'\}$ is a non-empty upwards-directed set with supremum $a \cup a'$. If ν is subadditive,

$$\mu(a \cup a') \le \sup_{c\in C} \nu c \le \mu a + \mu a';$$

thus μ is subadditive.

(e) If ν is an exhaustive submeasure, then μ is exhaustive, because $\mu \le \nu$. By 1Bb, μ is a Maharam submeasure.

(f) If ν is uniformly exhaustive, so is μ, because $\mu \le \nu$.

(g) If ν is additive and a, $a' \in \mathfrak{A}$ are disjoint, then $\mu(a \cup a') \ge \mu a + \mu a'$. **P** If C is non-empty, upwards directed and has supremum a, then $B = \{c \cap a : c \in C\}$ and $B' = \{c \cap a' : c \in C\}$ are upwards-directed with suprema a, a' respectively. So

$$\mu a + \mu a' \le \sup_{b \in B} \nu b + \sup_{b' \in B'} \nu b' = \sup_{b \in B, b' \in B'} \nu(b \cup b') \le \sup_{c \in C} \nu c.$$

because C is upwards-directed. As C is arbitrary, $\mu a + \mu a' \le \mu(a \cup a')$. **Q** But we know already that μ is subadditive, so it must be additive. Now it is actually countably additive because it is a Maharam submeasure.

1E Proposition Let $\mathfrak{A}$ be a Boolean algebra and μ a strictly positive exhaustive Maharam submeasure on $\mathfrak{A}$.

(a) μ is order-continuous.

(b) μ has a unique extension to a strictly positive Maharam submeasure $\hat{\mu}$ on the Dedekind completion $\widehat{\mathfrak{A}}$ of $\mathfrak{A}$, so $\widehat{\mathfrak{A}}$ is a Maharam algebra.

(c)(i) $\hat{\mu}$ is uniformly exhaustive iff μ is.

(ii) $\hat{\mu}$ is additive iff μ is.

proof (a) Because μ is strictly positive and exhaustive, $\mathfrak{A}$ is ccc (1Ba(iii)); because μ is sequentially order-continuous (1Ba(i)), μ is order-continuous (Fremlin 04, 316Fc).

(b) For $d \in \widehat{\mathfrak{A}}$, set $\hat{\mu} d = \inf\{\mu a : d \subseteq a \in \mathfrak{A}\}$. Then $\hat{\mu}$ extends μ, and $\hat{\mu} d \le \hat{\mu} d'$ whenever $d \subseteq d'$ in $\widehat{\mathfrak{A}}$. If d, $d' \in \widehat{\mathfrak{A}}$ then

$$\begin{aligned}\hat{\mu}(d \cup d') &= \inf\{\mu a : (d \cup d') \subseteq a \in \mathfrak{A}\} \le \inf\{\mu(a \cup a') : d \subseteq a \in \mathfrak{A},\ d' \subseteq a' \in \mathfrak{A}\} \\ &\le \inf\{\mu a + \mu a' : d \subseteq a \in \mathfrak{A},\ d' \subseteq a' \in \mathfrak{A}\} = \hat{\mu} d + \hat{\mu} d'.\end{aligned}$$

Thus $\hat{\mu}$ is a submeasure. If $d \in \widehat{\mathfrak{A}}$ is non-zero, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq d$, in which case $\hat{\mu} d \ge \mu a > 0$; so $\hat{\mu}$ is strictly positive. If $\langle d_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\widehat{\mathfrak{A}}$ with infimum 0, then $A = \{a : a \in \mathfrak{A},\ a \supseteq d_n \text{ for some } n \in \mathbb{N}\}$ is downwards-directed and has infimum 0 in $\widehat{\mathfrak{A}}$ and therefore in $\mathfrak{A}$. Because μ is order-continuous,

$$\inf_{n \in \mathbb{N}} \hat{\mu} d_n = \inf_{a \in A} \mu a = 0.$$

As $\langle d_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\hat{\mu}$ is a Maharam submeasure. By 1Ba(ii) (or otherwise), it is exhaustive.

(c)(i) If μ is uniformly exhaustive and $\epsilon > 0$, let $n \in \mathbb{N}$ be such that $\min_{i \le n} \mu a_i \le \epsilon$ whenever $a_0, \dots, a_n \in \mathfrak{A}$ are disjoint. If now $d_0, \dots, d_n \in \widehat{\mathfrak{A}}$ are disjoint and $\eta > 0$, we have $\hat{\mu} d_i = \sup\{\hat{\mu} a : a \in \mathfrak{A},\ a \subseteq d_i\}$ for each i, because $\hat{\mu}$ is order-continuous, by (a) here (or otherwise). Take $a_i \subseteq d_i$ such that $\hat{\mu} a_i \ge \hat{\mu} d_i - \eta$; then $a_0, \dots, a_n$ are disjoint, so

$$\min_{i \le n} \hat{\mu} d_i \le \eta + \min_{i \le n} \hat{\mu} a_i \le \eta + \min_{i \le n} \hat{\mu} a_i \le \eta + \epsilon.$$

As η and ϵ are arbitrary, $\hat{\mu}$ is uniformly exhaustive.

In the other direction, if $\hat{\mu}$ is uniformly exhaustive then $\mu = \hat{\mu} \restriction \mathfrak{A}$ must be uniformly exhaustive.

(ii) If μ is additive and d, $d' \in \mathfrak{A}$ are disjoint, set $A = \{a : a \in \mathfrak{A},\ a \subseteq d\}$ and $A' = \{a : a \in \mathfrak{A},\ a \subseteq d'\}$. Then A, A' and $B = \{a \cup a' : a \in A,\ a' \in A'\}$ are upwards-directed with suprema d, d' and $d \cup d'$ respectively. So

$$\hat{\mu}(d \cup d') = \sup_{b \in B} \mu b = \sup_{a \in A, a' \in A'} \mu(a \cup a') = \sup_{a \in A, a' \in A'} \mu a + \mu a' = \hat{\mu} d + \hat{\mu} d'.$$

As d and d' are arbitrary, $\hat{\mu}$ is additive.

In the other direction, if $\hat{\mu}$ is additive then $\mu = \hat{\mu} \restriction \mathfrak{A}$ must be additive.

1F Proposition (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Then it is nowhere measurable iff the only completely additive functional on $\mathfrak{A}$ is the zero functional.

(b) Let $\mathfrak{A}$ be a Maharam algebra, not $\{0\}$, and ν a strictly positive Maharam submeasure on $\mathfrak{A}$. Then ν is pathological iff $\mathfrak{A}$ is nowhere measurable.

proof (a) Suppose that $\mathfrak{A}$ is nowhere measurable, and that ν is a non-negative completely additive functional on $\mathfrak{A}$. By the Hahn decomposition theorem (Fremlin 04, 326O), there is an element $a = [\![\nu > 0]\!]$ of $\mathfrak{A}$ such that $\nu b > 0$ if $0 \ne b \subseteq a$ and $\nu b \le 0$ if $b \cap a = 0$. Now $\nu\restriction\mathfrak{A}_a$ witnesses that $\mathfrak{A}_a$ is measurable, so $a = 0$ and $\nu = 0$.

Conversely, if $\mathfrak{A}$ is not nowhere measurable, let $a \in \mathfrak{A}^+$ be such that $\mathfrak{A}_a$ is a measurable algebra. Let $\mu : \mathfrak{A}_a \to [0,1]$ be a strictly positive measure, and set $\nu b = \mu(a \cap b)$ for $b \in \mathfrak{A}$; then ν is a non-zero completely additive functional on $\mathfrak{A}$.

(b)(i) If $\mathfrak{A}$ is nowhere measurable and μ is an additive functional such that $0 \le \mu a \le \nu a$ for every $a \in \mathfrak{A}$, then μ must be completely additive. **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with infimum 0,

$$\lim_{n\to\infty} \mu a_n = \inf_{n\in\mathbb{N}} \mu a_n \le \inf_{n\in\mathbb{N}} \nu a_n = 0.$$

So μ is countably additive; because $\mathfrak{A}$ is ccc, μ is completely additive. **Q** By (a), $\mu = 0$; as μ is arbitrary, ν is pathological.

(ii) If $\mathfrak{A}$ is not nowhere measurable, let μ be a non-zero non-negative completely additive functional on $\mathfrak{A}$; re-scaling μ, we may suppose that $\mu 1 = \nu 1$. Set $C = \{c : \nu c < \mu c\}$, and let $D \subseteq C$ be a maximal disjoint set; set $b = \sup D$. Then either $b = 0$ or $\nu b \le \sum_{d\in D} \nu d < \sum_{d\in D} \mu d = \mu b$. So $b \ne 1$; setting $a = 1 \setminus b$, we have $\mu c \le \nu c$ for every $c \in \mathfrak{A}_a$. Now take $\mu' c = \mu(a \cap c)$ for every $c \in \mathfrak{A}$; then μ' is a non-zero non-negative additive functional and $\mu' \le \nu$, so ν is not pathological.

1G Lemma (Christensen 78) Let ν be a pathological unital submeasure on a Boolean algebra $\mathfrak{A}$. Then for every $\epsilon > 0$ there is a non-empty finite family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\nu b_i \le \epsilon$ for every $i \in I$ and $\sup_{i\in J} b_i = 1$ whenever $J \subseteq I$ and $\#(J) \ge \epsilon\#(I)$.

proof ? Suppose, if possible, otherwise. Set $C = \{1 \setminus b : \nu b \le \epsilon\}$. Then C has intersection number at least ϵ, so there is an additive functional $\mu : \mathfrak{A} \to [0,1]$ such that $\mu 1 = 1$ and $\mu c \ge \epsilon$ for every $c \in C$ (Fremlin 04, 391I).

Choose $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Given $\langle b_i\rangle_{i<n}$, set

$$\delta_n = \sup\{\mu b : b \cap b_i = 0 \text{ for every } i < n,\ \nu b \le \epsilon\mu b\},$$

and take b_n such that $b_n \cap b_i = 0$ for every $i < n$, $\nu b \le \epsilon\mu b$ and $\mu b_n \ge \frac{1}{2}\delta_n$. Note that $\langle b_n\rangle_{n\in\mathbb{N}}$ is disjoint; set $b'_n = \sup_{i<n} b_i$ for each n; then

$$\nu b'_n \le \sum_{i=0}^{n-1} \nu b_i \le \epsilon \sum_{i=0}^{n-1} \mu b_i = \epsilon\mu b'_n \le \epsilon$$

for every n, so $\mu(1 \setminus b'_n) \ge \epsilon$ for every n.

Set $\lambda a = \lim_{n\to\infty} \mu(a \setminus b'_n)$ for $a \in \mathfrak{A}$. Then λ is a finitely additive functional and $\lambda 1 \ge \epsilon$. Because ν is pathological, there is an $a \in \mathfrak{A}$ such that $\nu a < \epsilon\lambda a$. If $n \in \mathbb{N}$, then $a \setminus b'_n$ is disjoint from b_i for each $i < n$, while

$$\nu(a \setminus b'_n) \le \nu a \le \epsilon\lambda a \le \epsilon\mu(a \setminus b'_n).$$

So $\mu(a \setminus b'_n) \le \delta_n$ and

$$\lambda a \le \delta_n \le 2\mu b_n.$$

And this has to be true for every n, so $\sum_{n=0}^{\infty} \mu b_n = \infty$, which is impossible. **X**

1H Proposition A simple product of a countable family of Maharam algebras is a Maharam algebra.

proof Let $\langle \mathfrak{A}_i\rangle_{i\in I}$ be a countable family of Maharam algebras and $\mathfrak{A}$ its simple product. Then $\mathfrak{A}$ is Dedekind complete (Fremlin 04, 315De). For each $i \in I$, let ν_i be a strictly positive Maharam submeasure on $\mathfrak{A}_i$; let $\langle \epsilon_i\rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i < \infty$. Set $\nu(\langle a_i\rangle_{i\in I}) = \sum_{i\in I} \min(\epsilon_i, \nu_i a_i)$ for $\langle a_i\rangle_{i\in I} \in \mathfrak{A}$; it is easy to verify that ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, so that $\mathfrak{A}$ is a Maharam algebra.

1I The Loomis-Sikorski representation: Theorem (a) Let X be a set, Σ a σ-algebra of subsets of X, and μ a Maharam submeasure on Σ. Then $\mathfrak{A} = \Sigma/\mu^{-1}[\{0\}]$ is a Maharam algebra, with a strictly positive Maharam submeasure $\bar\mu$ defined by setting $\bar\mu E^\bullet = \mu E$ for every $E \in \Sigma$.

(b) Let $\mathfrak{A}$ be a Maharam algebra, and X its Stone space; write $\mathcal{B}\mathfrak{a}(X)$ for the Baire σ-algebra of X, and $\mathcal{M}(X)$ for the ideal of meager subsets of X. Then

(i) every member of $\mathcal{M}(X)$ is included in a nowhere dense zero set;

(ii) $\mathfrak{A} \cong \mathcal{B}\mathfrak{a}(X)/\mathcal{B}\mathfrak{a}(X) \cap \mathcal{M}(X)$;

(iii) there is a Maharam submeasure μ on $\mathcal{B}\mathfrak{a}(X)$ such that $\mu^{-1}[\{0\}] = \mathcal{B}\mathfrak{a}(X) \cap \mathcal{M}(X)$.

proof (a) Vér. fac.

(b) Because $\mathfrak{A}$ is weakly (σ, ∞)-distributive, every meager set in X is nowhere dense (FREMLIN 04, 316I). Because $\mathfrak{A}$ and X are ccc, every nowhere dense set in X is included in a nowhere dense zero set. **P** If E is nowhere dense, let $\mathcal{G}$ be a maximal disjoint family of cozero sets not meeting E; then $\mathcal{G}$ is countable so $\bigcup \mathcal{G}$ is cozero, and its complement is a nowhere dense zero set including E. **Q** Consequently $\mathfrak{A} \cong \mathcal{B}\mathfrak{a}(X)/\mathcal{B}\mathfrak{a}(X) \cap \mathcal{M}(X)$ (see the proof of 314L in FREMLIN 04).

Let $\pi : \mathcal{B}\mathfrak{a}(X) \to \mathfrak{A}$ be the corresponding Boolean homomorphism. Then π is sequentially order-continuous (FREMLIN 04, 313Pb). Let $\bar{\mu}$ be a strictly positive Maharam submeasure on $\mathfrak{A}$; then $\mu = \bar{\mu}\pi$ is a Maharam submeasure on $\mathcal{B}\mathfrak{a}(X)$ and $\mu^{-1}[\{0\}] = \mathcal{B}\mathfrak{a}(X) \cap \mathcal{M}(X)$.

1J Maharam-algebra topologies (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, μ a strictly positive countably subadditive submeasure on $\mathfrak{A}$ and ν a Maharam submeasure on $\mathfrak{A}$. Then ν is absolutely continuous with respect to μ. **P?** Otherwise, there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ and $\epsilon > 0$ such that $\mu a_n \le 2^{-n}$ and $\nu a_n \ge \epsilon$ for every n. Set $b_n = \sup_{m\ge n} a_m$; then $\mu b_n \le 2^{-n+1}$ for every $n \in \mathbb{N}$. Set $b = \inf_{n\in\mathbb{N}} b_n$; then $\mu b = 0$ so $b = 0$. As $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-increasing, $\lim_{n\to\infty} \nu b_n = 0$; but $\nu b_n \ge \nu a_n \ge \epsilon$ for every n. **XQ**

(b) If $\mathfrak{A}$ is a Boolean algebra and μ is a strictly positive submeasure on $\mathfrak{A}$, then we have a metric ρ on $\mathfrak{A}$ defined by setting $\rho(a, b) = \mu(a \triangle b)$ for all a, $b \in \mathfrak{A}$. If $\mathfrak{A}$ is a Maharam algebra and μ is a Maharam submeasure, the topology generated by ρ is the order-sequential topology of $\mathfrak{A}$. **P** (i) Suppose that $F \subseteq \mathfrak{A}$ is closed for the order-sequential topology and that $a \in \mathfrak{A}$ belongs to the ρ-closure of F. Then there is a sequence $\langle a_n \rangle_{n\in\mathbb{N}}$ in F such that $\mu(a_n \triangle a) \le 2^{-n}$ for every $n \in \mathbb{N}$. Set $b_n = \sup_{m\ge n} a_m \triangle a$ for each n; then $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0. So $\langle a_n \rangle_{n\in\mathbb{N}}$ order*-converges to a and $a \in F$. As a is arbitrary, F is ρ-closed. (ii) Suppose that F is ρ-closed and that $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in F which order*-converges to $a \in \mathfrak{A}$. Again set $b_n = \sup_{m\ge n} a_m \triangle a$ for each n; again, $\langle b_n \rangle_{n\in\mathbb{N}}$ is non-increasing and has infimum 0. So $\inf_{n\in\mathbb{N}} \rho(a_n, a) \le \inf_{n\in\mathbb{N}} \mu b_n = 0$ and $a \in F$. As $\langle a_n \rangle_{n\in\mathbb{N}}$ and a are arbitrary, F is closed for the order-sequential topology. **Q**

1K Modular functionals Recall that a real-valued functional f on a lattice P is called **supermodular** if $f(p \vee q) + f(p \wedge q) \ge f(p) + f(q)$ for all p, $q \in P$; **submodular** (also **strongly subadditive** when P is a Boolean algebra and f is non-negative) if $f(p \vee q) + f(p \wedge q) \le f(p) + f(q)$ for all p, $q \in P$; and **modular** if it is both supermodular and submodular. Now we have the following fact.

Proposition (a) A supermodular submeasure is uniformly exhaustive.

(b) A submodular exhaustive submeasure is uniformly exhaustive.

proof (a) Let $\mathfrak{A}$ be an algebra of sets and ν a supermodular submeasure on $\mathfrak{A}$. Identifying $\mathfrak{A}$ with the lattice of open-and-closed sets in its Stone space, Theorem 413P in FREMLIN 03 tells us that there is an additive $\mu : \mathfrak{A} \to [0, \infty[$ such that $\mu a \ge \nu a$ for every $a \in \mathfrak{A}$; now μ is uniformly exhaustive so ν also is.

(b)(i) If $\mathfrak{A}$ is a Boolean algebra and ν is a non-zero submodular submeasure on $\mathfrak{A}$, there is a non-zero additive $\mu : \mathfrak{A} \to [0, \infty[$ such that $\mu a \le \nu a$ for every $a \in \mathfrak{A}$. **P** Set $\nu' a = \nu 1 - \nu(1 \setminus a)$ for $a \in \mathfrak{A}$. It is easy to check that $\nu' : \mathfrak{A} \to [0, \infty[$ is order-preserving and supermodular, while $\nu' 0 = 0$. Again applying FREMLIN 03, 413P, in the Stone space of $\mathfrak{A}$, we have an additive functional $\mu : \mathfrak{A} \to [0, \infty[$ such that $\mu 1 = \nu' 1 = \nu 1$ and $\mu a \ge \nu' a$ for every $a \in \mathfrak{A}$. Now

$$\mu a = \mu 1 - \mu(1 \setminus a) \le \nu 1 - \nu'(1 \setminus a) = \nu a$$

for every $a \in \mathfrak{A}$.

(ii) If $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a strictly positive submodular Maharam submeasure, there is a non-zero $c \in \mathfrak{A}$ such that the principal ideal $\mathfrak{A}_c$ is a measurable algebra. **P** Let ν be a strictly positive submodular Maharam submeasure on $\mathfrak{A}$. By (i), there is a non-zero additive functional

μ on $\mathfrak{A}$ such that $\mu \le \nu$; it follows that μ is countably additive, therefore completely additive (since $\mathfrak{A}$ is ccc). Let c be the support of μ (Fremlin 02, 326O); then $\mu c > 0$ and $\mu\restriction\mathfrak{A}_c$ is strictly positive, so $\mathfrak{A}_c$ is measurable. **Q**

(iii) It follows immediately that if $\mathfrak{A}$ is a Dedekind complete Boolean algebra with a strictly positive submodular Maharam submeasure, it is itself a measurable algebra.

(iv) Now suppose only that $\mathfrak{A}$ is a Boolean algebra with a submodular exhaustive submeasure ν. Set $I = \{a : \nu a = 0\}$, $\mathfrak{C} = \mathfrak{A}/I$; then we have a submodular exhaustive submeasure $\bar\nu$ on $\mathfrak{C}$ defined by setting $\bar\nu a^\bullet = \nu a$ for every $a \in \mathfrak{A}$. Let $\widehat{\mathfrak{C}}$ be the metric completion of $\mathfrak{C}$ and $\hat\nu$ the continuous extension of $\bar\nu$ to $\widehat{\mathfrak{C}}$, as in Fremlin 02, 393B; then $\hat\nu$ is a strictly positive submodular Maharam submeasure on $\widehat{\mathfrak{C}}$, so $\widehat{\mathfrak{C}}$ is a measurable algebra and $\hat\nu$ is uniformly exhaustive. Accordingly $\bar\nu$ and ν are uniformly exhaustive.

1L Proposition (Zapletal p06, 4.3.12) Let ν be a Ramsey submeasure on a Boolean algebra $\mathfrak{A}$. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ and $\sup_{n\in\mathbb{N}} \nu a_n < \gamma$, there is an infinite set $I \subseteq \mathbb{N}$ such that $\nu(\sup_{i\in I\cap n} a_i) \le \gamma$ for every $n \in \mathbb{N}$.

proof Let $\langle\gamma_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence such that $\gamma_0 = \sup_{n\in\mathbb{N}} \nu a_n$ and $\gamma_n < \gamma$ for every n. Choose $\langle i_n\rangle_{n\in\mathbb{N}}$, $\langle c_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $J_0 = \mathbb{N}$, $c_0 = 0$. Given that $\nu(c_n \cup a_j) \le \gamma_n$ for every $j \in J_n$, then, because ν is a Ramsey submeasure, any infinite subset of J_n contains distinct i, j such that $\nu(c_n \cup a_i \cup a_j) \le \gamma_{n+1}$. By Ramsey's theorem, there is an infinite $J_{n+1} \subseteq J_n$ such that $\nu(c_n \cup a_i \cup a_j) \le \gamma_{n+1}$ for all i, $j \in J_n$. Take $i_n \in J_{n+1} \setminus n$ and set $c_{n+1} = c_n \cup a_{i_n}$; continue.

Now set $I = \{i_n : n \in \mathbb{N}\}$.

1M The lattice of submeasures Let $\mathfrak{A}$ be a Boolean algebra and M the set of submeasures on $\mathfrak{A}$.

(a) If $\langle\mu_i\rangle_{i\in I}$ is a family in M, then it is bounded above in M iff $\sup_{i\in I} \mu_i 1$ is finite, and in this case its supremum μ is given by $\mu a = \sup_{i\in I} \mu_i a$ for every $a \in \mathfrak{A}$ (counting $\sup\emptyset$ as 0).

Consequently M is a Dedekind complete lattice.

(b) If $\langle\mu_i\rangle_{i\in I}$ is a non-empty family in M, its infimum μ is given by

$$\mu a = \inf\{\textstyle\sum_{i\in J} \mu_i a_i : J \subseteq I \text{ is finite, } a \subseteq \sup_{i\in J} a_i\}$$

for every $a \in \mathfrak{A}$.

(c) If $\mathfrak{A}$ is Dedekind σ-complete and μ, ν are two Maharam submeasures on $\mathfrak{A}$ such that $\mu \wedge \nu = 0$, there is a $c \in \mathfrak{A}$ such that $\mu c = \nu(1 \setminus c) = 0$. **P** For each $n \in \mathbb{N}$ there is an $a_n \in \mathfrak{A}$ such that $\mu a_n + \nu(1 \setminus a_n) \le 2^{-n}$; set $c = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a_m$. **Q**

2 Sequences in Maharam algebras

2A Lemma Let $\mathfrak{A}$ be a ccc Boolean algebra, and $\langle a_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathfrak{A}$. Then

either there is an infinite $I \subseteq \mathbb{N}$ such that $\langle a_i\rangle_{i\in I}$ order*-converges to 0

or there are a non-zero $d \in \mathfrak{A}$ and an infinite $I \subseteq \mathbb{N}$ such that $\sup_{i\in J} d \cap a_i = d$ for every infinite $J \subseteq I$.

proof ? Suppose, if possible, otherwise. Choose inductively families $\langle I_\xi\rangle_{\xi<\omega_1}$ in $[\mathbb{N}]^\omega$ and $\langle c_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{A}^+$ as follows. $I_0 = \mathbb{N}$. Given $\langle I_\eta\rangle_{\eta\le\xi}$ such that $I_\eta \setminus I_\zeta$ is finite whenever $\zeta \le \eta \le \xi$, we are supposing that $\langle a_i\rangle_{i\in I_\xi}$ does not order*-converge to 0. Set $C_\xi = \{c : c \in \mathfrak{A}, \{i : i \in I_\xi, a_i \cap c \ne 0\} \text{ is finite}\}$. Then C_ξ does not include any partition of unity; as $c \in C_\xi$ whenever $c \subseteq c' \in C_\xi$, it follows that there is a $b \in \mathfrak{A}^+$ such that $b \cap c = 0$ for every $c \in C_\xi$. Now there must be an infinite $I_{\xi+1} \subseteq I_\xi$ such that b is not the supremum of $\{b \cap a_i : i \in I_{\xi+1}\}$; let $c_\xi \subseteq b$ be a non-zero element such that $c_\xi \cap a_i = 0$ for every $i \in I_{\xi+1}$. Note that now $I_\eta \setminus I_\zeta$ is finite whenever $\zeta \le \eta \le \xi + 1$, so that the induction continues. At non-zero countable limit ordinals ξ, let $I_\xi \in [\mathbb{N}]^\omega$ be such that $I_\xi \setminus I_\eta$ is finite for every $\eta < \xi$, and carry on.

Now observe that because $I_\xi \setminus I_\eta$ is finite, $C_\eta \subseteq C_\xi$ whenever $\eta \le \xi$. $I_{\eta+1}$ is constructed so that $c_\eta \in C_{\eta+1}$, and therefore $c_\eta \cap c_\xi = 0$ whenever $\eta < \xi$. But this means that we have an uncountable disjoint family $\langle c_\xi\rangle_{\xi<\omega_1}$ in $\mathfrak{A}^+$, and $\mathfrak{A}$ is not ccc. **X**

2B Theorem (Veličković 05, Theorem 2) If $\mathfrak{A}$ is an atomless Maharam algebra, there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I} a_n = 1$ and $\inf_{n\in I} a_n = 0$ for every infinite $I \subseteq \mathbb{N}$.

proof (a) Fix a strictly positive Maharam submeasure ν on $\mathfrak{A}$. Before embarking on the main argument, let me note a simple fact. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ order*-converging to 0, $\lim_{n\to\infty}\nu a_n=0$. **P** Let C be a partition of unity in $\mathfrak{A}$ such that $\{n:a_n\cap c\ne 0\}$ is finite for every $n\in\mathbb{N}$. Then C is countable; enumerate it as $\langle c_k\rangle_{k\in\mathbb{N}}$. Set $b_m=1\setminus\sup_{k\le m}c_k$ for each $m\in\mathbb{N}$; then $\langle b_m\rangle_{m\in\mathbb{N}}$ is non-increasing and has infimum 0, so $\lim_{m\to\infty}\nu b_m=0$. But each b_m includes all but finitely many of the a_n, so $\lim_{n\to\infty}\nu a_n=0$. **Q** Turning this round: if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\nu a_n>0$, it can have no subsequence order*-converging to 0, so by Lemma 2A there are a non-zero $d\in\mathfrak{A}$ and an infinite $I\subseteq\mathbb{N}$ such that $d=\sup_{i\in J}d\cap a_i$ for every infinite $J\subseteq I$.

(b) Let us say that a Boolean algebra $\mathfrak{A}$ **splits reals** if there is a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that $\sup_{n\in I}a_n=1$ and $\inf_{n\in I}a_n=0$ for every infinite $I\subseteq\mathbb{N}$. Now if $\mathfrak{A}$ is a Maharam algebra, the set of those $d\in\mathfrak{A}$ such that the principal ideal $\mathfrak{A}_d$ generated by d splits reals is order-dense in $\mathfrak{A}$. **P** Let $a\in\mathfrak{A}^+$.

(i) If $\nu\restriction\mathfrak{A}_a$ is uniformly exhaustive, then $\mathfrak{A}_a$ is measurable (Kalton & Roberts 83, or Fremlin 04, 392J). Let $\bar\mu$ be a probability measure on $\mathfrak{A}_a$; because $\mathfrak{A}_a$, like $\mathfrak{A}$, is atomless, there is a stochastically independent family $\langle a_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_a$ with $\bar\mu a_n=\frac{1}{2}$ for every n, and now $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}_a$ splits reals.

(ii) If $\nu\restriction\mathfrak{A}_a$ is not uniformly exhaustive, let $\langle b_{ni}\rangle_{i\le n\in\mathbb{N}}$ be a family of elements of $\mathfrak{A}_a$ such that $\langle b_{ni}\rangle_{i\le n}$ is disjoint for each n and $\epsilon=\inf_{i\le n\in\mathbb{N}}\nu b_{ni}$ is greater than 0. Let $\langle f_\xi\rangle_{\xi<\omega_1}$ be a family in $\prod_{n\in\mathbb{N}}\{0,\dots,n\}$ such that $\{n:f_\xi(n)=f_\eta(n)\}$ is finite whenever $\eta<\xi<\omega_1$. **?** If for every $\xi<\omega_1$ and $I\in[\mathbb{N}]^\omega$ there is a $J\in[I]^\omega$ such that $\inf_{i\in J}b_{i,f_\xi(i)}\ne 0$, choose $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively so that $I_\xi\in[\mathbb{N}]^\omega$, $I_\xi\setminus I_\eta$ is finite for every $\eta<\xi$, and $c_\xi=\inf_{i\in I_\xi}b_{i,f_\xi(i)}\ne 0$ for every $\xi<\omega_1$. Then whenever $\eta<\xi$ the set $I_\xi\cap I_\eta$ is infinite, so there is an $i\in I_\xi\cap I_\eta$ such that $f_\xi(i)\ne f_\eta(i)$; now $c_\xi\cap c_\eta\subseteq b_{i,f_\xi(i)}\cap b_{i,f_\eta(i)}=0$. But this means that we have an uncountable disjoint family in $\mathfrak{A}_a$, which is impossible, because every Maharam algebra is ccc (Fremlin 04, 392I). **X**

Thus we have a $\xi<\omega_1$ and an infinite $I\subseteq\mathbb{N}$ such that $\inf_{i\in J}d_i=0$ for every infinite $J\subseteq I$, where $d_i=b_{i,f_\xi(i)}$ for $i\in I$. Next, applying (a) to $\langle d_i\rangle_{i\in I}$, we have an infinite $K\subseteq I$ and a $d\ne 0$ such that $d=\sup_{i\in J}d_i$ for every infinite $J\subseteq K$. But this means that $\langle d\cap d_i\rangle_{i\in K}$ witnesses that $\mathfrak{A}_d$ splits reals; while $d\subseteq a$.

As a is arbitrary, we have the result. **Q**

(c) Let $D\subseteq\mathfrak{A}$ be a partition of unity such that $\mathfrak{A}_d$ splits reals for every $d\in D$; choose sequences $\langle a_{dn}\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_d$ witnessing this. Set $a_n=\sup_{d\in D}a_{dn}$ for each n. If $I\subseteq\mathbb{N}$ is infinite, then

$$\sup_{n\in I}a_n=\sup_{d\in D}\sup_{n\in I}a_{dn}=\sup D=1,$$

while

$$d\cap\inf_{n\in I}a_n=\inf_{n\in I}a_{dn}=0$$

for every $d\in D$, so $\inf_{n\in I}a_n=0$. Thus $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}$ splits reals, as claimed.

Remark More generally, a ccc Dedekind complete Boolean algebra splits reals iff no non-trivial principal ideal is sequentially compact in the order-sequential topology; see Balcar Jech & Pazák p04, §4.

2C Corollary (Zapletal p06, 4.3.23) If $\mathfrak{A}$ is a Boolean algebra and ν is a non-zero diffuse exhaustive submeasure on $\mathfrak{A}$, ν is not Ramsey.

proof (a) ? Suppose first that $\mathfrak{A}$ is a non-trivial Maharam algebra and that ν is a diffuse Ramsey strictly positive Maharam submeasure on $\mathfrak{A}$. Because ν is diffuse, $\mathfrak{A}$ can have no atom. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ as in 2B. Set $\gamma_n=(\frac{1}{2}+2^{-n-1})\nu 1$ for each n, and choose $\langle c_n\rangle_{n\in\mathbb{N}}$ and $\langle i_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0=1$. Given that $\nu c_n\ge\gamma_n$,

$$\sup_{m\ge n}\nu(c_n\cap\sup_{i\in I\cap m}a_i)=\nu(c_n\cap\sup_{i\in I}a_i)=\nu c_n\ge\gamma_n$$

for every infinite $I\subseteq\mathbb{N}\setminus n$, so Proposition 1L tells us that $\sup_{i\ge n}\nu(c_n\cap a_i)\ge\gamma_n$; take $i_n\ge n$ such that $\nu(c_n\cap a_{i_n})\ge\gamma_{n+1}$, and set $c_{n+1}=c_n\cap a_{i_n}$. Continue.

We now find that

$$c=\inf_{n\in\mathbb{N}}c_n\subseteq\inf_{n\in\mathbb{N}}a_{i_n}=0$$

while

$\nu c = \lim_{n\to\infty} \nu c_n = 0$. **X**

(b) Thus the result is true in the special case in which ν is a strictly positive Maharam submeasure on a Maharam algebra. Now suppose that ν is just a strictly positive diffuse exhaustive submeasure on a non-trivial Boolean algebra $\mathfrak{A}$. Let $\widehat{\mathfrak{A}}$ be the metric completion of $\mathfrak{A}$, and $\hat{\nu}$ the canonical extension of ν to $\widehat{\mathfrak{A}}$, as in Fremlin 02, 393B. Then $\hat{\nu}$ is a Maharam submeasure, and is still diffuse. By (a), it is not Ramsey; let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\widehat{\mathfrak{A}}$ such that

$$\hat{\nu}(a_m \cup a_n) \ge \gamma > \gamma' \ge \hat{\nu} a_n$$

for all distinct m, $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ we can find an $a'_n \in \mathfrak{A}$ such that $\hat{\nu}(a'_n \,\triangle\, a_n) \le \frac{1}{4}(\gamma - \gamma')$, and now $\langle a'_n\rangle_{n\in\mathbb{N}}$ witnesses that ν is not Ramsey.

(c) Finally, for the case in which ν is not strictly positive, let I be the ideal $\{a : \nu a = 0\}$, $\mathfrak{B}$ the quotient $\mathfrak{A}/I$ and ν' the submeasure on $\mathfrak{B}$ defined by setting $\nu' a^\bullet = \nu a$ for every $a \in \mathfrak{A}$. Then ν' is diffuse, exhaustive and strictly positive, so is not Ramsey. If $\langle a_n\rangle_{n\in\mathbb{N}}$ is such that $\langle a^\bullet_n\rangle_{n\in\mathbb{N}}$ witnesses that ν' is not Ramsey, $\langle a_n\rangle_{n\in\mathbb{N}}$ witnesses that ν is not Ramsey, as required.

2D Lemma Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}} \nu a_n > 0$. Then there is an infinite $I \subseteq \mathbb{N}$ such that $\{a_n : n \in I\}$ is centered.

first proof Set $I = \{a : \nu a = 0\}$. Then $I \lhd \mathfrak{A}$. On the quotient algebra $\mathfrak{A}/I$ we have an exhaustive submeasure $\bar{\nu}$ defined by saying that $\bar{\nu} a^\bullet = \nu a$ for every $a \in \mathfrak{A}$ (see Fremlin 04, 392Xd). $\bar{\nu}$ is strictly positive. We can therefore embed $(\mathfrak{A}/I, \bar{\nu})$ in $(\mathfrak{B}, \bar{\nu})$ where $\mathfrak{B}$ is a Dedekind complete Boolean algebra and $\bar{\nu}$ is a strictly positive Maharam submeasure on $\mathfrak{B}$ (Fremlin 04, 393B). Working in $\mathfrak{B}$, $\inf_{n\in\mathbb{N}} \bar{\nu} a^\bullet_n > 0$, so $b = \inf_{n\in\mathbb{N}} \sup_{m\ge n} a^\bullet_m \ne 0$; now take $I \subseteq \mathbb{N}$ to be maximal such that $b \cap \inf_{i\in I\cap n} a^\bullet_i \ne 0$ for every n. In this case $\langle a_i\rangle^\bullet_{i\in I}$ is centered in $\mathfrak{B}$ so $\{a_i : i \in I\}$ is centered in $\mathfrak{A}$.

second proof For any $m \in \mathbb{N}$ and $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\nu(\sup_{n\le i<k} a_i \setminus \sup_{m\le i<n} a_i) \le \epsilon$ for every $k \in \mathbb{N}$. **P?** Otherwise, choose $\langle n_k\rangle_{k\in\mathbb{N}}$ so that $n_0 = m$ and $\nu c_k > \epsilon$ where $c_k = \sup_{n_k\le i<n_{k+1}} a_i \setminus \sup_{m\le i<n_k} a_i$ for every k. Then $\langle c_k\rangle_{k\in\mathbb{N}}$ is disjoint, so ν is not exhaustive. **XQ**

Set $\delta = \frac{1}{2}\inf_{n\in\mathbb{N}} \nu a_n$. Choose a strictly increasing sequence $\langle m_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$, a non-increasing sequence $\langle c_k\rangle_{k\in\mathbb{N}}$ in $\mathfrak{A}$, and a_{ki}, for i, $k \in \mathbb{N}$, as follows. $m_0 = 0$ and $a_{0i} = a_i$ for every i. Given that $\nu a_{kn} \ge (1+2^{-k})\delta$ for every $n \ge m_k$, let m_{k+1} be such that $\nu(\sup_{m_{k+1}\le i<l} a_{ki} \setminus \sup_{m_k\le i<m_{k+1}} a_{ki}) \le 2^{-k-1}\delta$ for every l. Set $c_k = \sup_{m_k\le i<m_{k+1}} a_{ki}$ and $a_{k+1,i} = a_{ki} \cap c_k$ for $i \ge m_{k+1}$. Then $\nu a_{k+1,i} \ge \nu a_{ki} - \nu(a_{ki} \setminus c_k) \ge (1+2^{-k-1})\delta$ for every $i \ge m_{k+1}$, so the induction continues.

Now $\langle c_k\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence of non-zero elements, so is centered; and $c_k \subseteq \sup_{m_k\le i<m_{k+1}} a_i$ for every k. Taking a maximal centered family C containing every c_k, the set $I = \{i : a_i \in C\}$ must meet $[m_k, m_{k+1}[$ for every k, so is infinite; and $\{a_i : i \in I\}$ is centered.

Remark Thus any Boolean algebra with a strictly positive exhaustive submeasure has the property (*) of 1Ad. Compare 2E, 2H below.

2E Proposition Let $\mathfrak{A}$ be a Boolean algebra, ν an exhaustive submeasure on $\mathfrak{A}$, and $\langle a_i\rangle_{i\in\mathbb{N}}$ a sequence in $\mathfrak{A}$ such that $\inf_{i\in\mathbb{N}} \nu a_i > 0$. Then for every $k \in \mathbb{N}$ there are an $I \in [\mathbb{N}]^\omega$ and a $\delta > 0$ such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [I]^k$.

proof Induce on k. The cases $k = 0$, $k = 1$ are trivial. For the inductive step to $k+1$, let $M \in [\mathbb{N}]^\omega$ and $\delta > 0$ be such that $\nu(\inf_{i\in J} a_i) \ge \delta$ for every $J \in [M]^k$. **?** Suppose, if possible, that for every $I \in [M]^\omega$ and $\gamma > 0$ there is a $J \in [I]^{k+1}$ such that $\nu(\inf_{i\in J} a_i) < \gamma$. Using Ramsey's theorem repeatedly, we can find $\langle I_n\rangle_{n\in\mathbb{N}}$ such that $I_0 = M$, $I_{n+1} \in [I_n]^\omega$, $r_n = \min I_n \notin I_{n+1}$ and $\nu(\inf_{i\in J} a_i) \le 2^{-n-2}\delta$ for every $J \in [I_n]^{k+1}$. Set $I = \{r_n : n \in \mathbb{N}\}$. If $J \in [I]^k$ and $\min J = r_n$, then $J \cup \{r_m\} \in [I_m]^{k+1}$, so $\nu(\inf_{i\in J} a_i \cap a_{r_m}) \le 2^{-m-2}\delta$, for every $m < n$. It follows that $\nu(\inf_{i\in J} a_i \cap \sup_{m<n} a_{r_m}) \le \frac{1}{2}\delta$ and $\nu(\inf_{i\in J} a_i \setminus \sup_{m<n} a_{r_m}) \ge \frac{1}{2}\delta$. But this means that $\nu c_n \ge \frac{1}{2}\delta$ where $c_n = a_{r_n} \setminus \sup_{m<n} a_{r_m}$ for each n. As $\langle c_n\rangle_{n\in\mathbb{N}}$ is disjoint, this is impossible. **X**

Thus we can find $\gamma > 0$ and $I \in [M]^\omega$ such that $\nu(\inf_{i\in J} a_i) \ge \gamma$ for every $J \in [I]^{k+1}$, and the induction continues.

2F Proposition Let κ be a regular uncountable cardinal, and ν an exhaustive submeasure on a Boolean algebra $\mathfrak{A}$. Suppose that $\langle a_\xi\rangle_{\xi<\kappa}$ is a family in $\mathfrak{A}$ such that $\inf_{\xi<\kappa}\nu a_\xi>0$. Then for every $n\in\mathbb{N}$ there are a stationary set $S\subseteq\kappa$ and a $\delta>0$ such that $\nu(\inf_{i\in J}a_i)\ge\delta$ for every $J\in[S]^n$.

proof Induce on n. The cases $n=0$, $n=1$ are trivial. For the inductive step to $n+1\ge 2$, write $c_J=\inf_{i\in J}a_i$ for $J\in[\kappa]^{<\omega}$. We know from the inductive hypothesis that there are a stationary set $S\subseteq\kappa$ and a $\delta>0$ such that $\nu c_J\ge 3\delta$ for every $J\in[S]^n$. For each $\xi\in S$, choose $m(\xi)\in\mathbb{N}$ and $\langle J_{\xi i}\rangle_{i<m(\xi)}$ as follows. Given $\langle J_{\xi i}\rangle_{i<j}$, where $j\in\mathbb{N}$, choose, if possible, $J_{\xi j}\in[S\cap\xi]^n$ such that $\nu(c_{J_{\xi j}}\cap c_{J_{\xi i}})\le 2^{-i}\delta$ for every $i<j$ and $\nu(a_\xi\cap c_{J_{\xi j}})\le 2^{-j}\delta$; if this is not possible, set $m(\xi)=j$ and stop. Now the point is that we always do have to stop. **P?** Otherwise, set $d_i=c_{J_{\xi i}}$ for each $i\in\mathbb{N}$. Because $J_{\xi i}\in[S]^n$, $\nu d_i\ge 3\delta$ for each i; also $\nu(d_i\cap d_j)\le 2^{-i}\delta$ for $i<j$; so $\nu d'_j\ge\delta$, where $d'_j=d_j\setminus\sup_{i<j}d_i$ for each j. But now $\langle d'_j\rangle_{j\in\mathbb{N}}$ is disjoint and ν is not exhaustive. **XQ**

At the end of the process, we have $m(\xi)$ and $\langle J_{\xi i}\rangle_{i<m(\xi)}$ for each $\xi\in S$. By the Pressing-Down Lemma, there are $\tilde m$ and $\langle\tilde J_i\rangle_{i<\tilde m}$ such that $S'=\{\xi:\xi\in S,\ m(\xi)=\tilde m,\ J_{\xi i}=\tilde J_i$ for every $i<\tilde m\}$ is stationary in κ. **?** Suppose, if possible, that $I\in[S']^{n+1}$ and $\nu c_I\le 2^{-\tilde m}\delta$. Set $\xi=\max I$, $J=I\setminus\{\xi\}$, $\eta=\min I\in J$. Then $J\in[S\cap\xi]^n$. For each $i<\tilde m=m(\xi)$,

$$\nu(c_J\cap c_{J_{\xi i}})\le\nu(a_\eta\cap c_{J_{\xi i}})=\nu(a_\eta\cap c_{J_{\eta i}})\le 2^{-i}\delta,$$

while

$$\nu(a_\xi\cap c_J)=\nu c_I\le 2^{-\tilde m}\delta.$$

But this means that we could have extended the sequence $\langle J_{\xi i}\rangle_{i<\tilde m}$ by setting $J_{\xi\tilde m}=J$. **X**

So S' and $2^{-\tilde m}\delta$ provide the next step in the induction.

2G Corollary If $\mathfrak{A}$ is a Boolean algebra with a strictly positive exhaustive submeasure, then (κ,κ,n) is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ and every $n\in\mathbb{N}$.

2H Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and ν a Maharam submeasure on $\mathfrak{A}$. Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}$ and $\delta=\inf_{n\in\mathbb{N}}\nu a_n$. Then for any $\delta'<\delta$ there is a strictly increasing sequence $\langle m_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\nu(\inf_{k\in\mathbb{N}}\sup_{m_k\le n<m_{k+1}}a_n)\ge\delta'$.

proof If $\delta'\le 0$ this is trivial; suppose that $0<\delta'<\delta$. Repeat the argument of the 'second proof' of Lemma 2C, but this time requiring $\nu a_{kn}\ge\delta_k$ for every $n\ge m_k$, where $\langle\delta_k\rangle_{k\in\mathbb{N}}$ is a strictly decreasing sequence in $[\delta',\delta]$. Then $\nu c_k\ge\delta_k$ for every k, so $\nu(\inf_{k\in\mathbb{N}}c_k)\ge\delta'$.

3 The theorems of Balcar-Główczyński-Jech, Balcar-Jech-Pazák and Todorčević

3A Lemma (Balcar Główcyński & Jech 98) Let $\mathfrak{A}$ be a ccc Dedekind complete weakly (σ,∞)-distributive Boolean algebra, endowed with its order-sequential topology. For $A\subseteq\mathfrak{A}$, set $\bigvee_0(A)=\{0\}$ and $\bigvee_{n+1}(A)=\{a\cup b:a\in A,\ b\in\bigvee_n(A)\}$ for $n\in\mathbb{N}$. Then for every open set G containing 0 there is an open set H containing 0 such that $\bigvee_3(H)\subseteq\bigvee_2(G)$.

proof ? Otherwise, choose H_n, a_n, b_n and c_n inductively, as follows. $H_0\subseteq G$ is to be an open neighbourhood of 0 such that $[0,a]\subseteq H_0$ whenever $a\in H_0$ (Fremlin 04, 392Mc). Given that H_n is an open set containing 0, we are supposing that $\bigvee_3(H_n)\not\subseteq\bigvee_2(G)$; choose a_n, b_n, $c_n\in H$ such that $a_n\cup b_n\cup c_n\notin\bigvee_2(G)$, and set

$$H'_n=\{a:a,\ a\triangle a_n,\ a\triangle b_n\text{ and }a\triangle c_n\text{ all belong to }H_n\},$$

so that H'_n is an open set containing 0. Let H_{n+1} be an open neighbourhood of 0, included in H'_n, such that $[0,a]\subseteq H_{n+1}$ for every $a\in H_{n+1}$. Continue.

Set $F=\bigcap_{n\in\mathbb{N}}\overline{H}_n$ and $a^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}a_i$. Then $a^*\cup c\in F$ for every $c\in F$. **P** For $m\le n\in\mathbb{N}$, $\sup_{m\le i\le n}a_i\cup b\in H_m$ for every $b\in H_{n+1}$ (induce downwards on m). So $\sup_{m\le i\le n}a_i\cup c\in\overline{H}_m$ for every $c\in F$. Letting $n\to\infty$, $c\cup\sup_{m\le i}a_i\in\overline{H}_m$ for every $c\in F$, $m\in\mathbb{N}$. Next, for any $b\in\mathfrak{A}$, $\{a:a\cap b\in\overline{H}_m\}$ is a closed set including H_m, so $a\cap b\in\overline{H}_m$ for every $a\in\overline{H}_m$; that is, $[0,a]\subseteq\overline{H}_m$ for every $a\in\overline{H}_m$. As $a^*\subseteq\sup_{i\ge m}a_i$. $c\cup a^*\in\overline{H}_m$ for every $c\in F$. As m is arbitrary, $c\cup a^*\in F$ for every $c\in F$. **Q**

Similarly, setting $b^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}b_i$ and $c^*=\inf_{n\in\mathbb{N}}\sup_{i\ge n}c_i$, $c\cup b^*$ and $c\cup c^*$ belong to F for every $c\in F$. So $d=a^*\cup b^*\cup c^*$ belongs to F. For each $n\in\mathbb{N}$, $a_n\cup b_n\cup c_n\notin\bigvee_2(H_0)$; but $[0,a]\subseteq\bigvee_2(H_0)$

for every $a \in \bigvee_2(H_0)$, so $\sup_{i\ge n} a_i \cup b_i \cup c_i \notin \bigvee_2(H_0)$. Accordingly $d = \inf_{n\in\mathbb{N}} \sup_{i\ge n} a_i \cup b_i \cup c_i$ does not belong to $\operatorname{int}(\bigvee_2(H_0))$. But $\bigvee_2(H_0) = \{a \triangle b : a, b \in H_0\}$ is an open set including $\overline{H}_0$, so $d \in F \setminus \overline{H}_0$; which is impossible. **X**

3B Theorem (Balcar Główczyński & Jech 98) Let $\mathfrak{A}$ be a Dedekind complete ccc Boolean algebra in which the order-sequential topology is Hausdorff. Then $\mathfrak{A}$ is a Maharam algebra.

proof (a) $\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of maximal antichains in $\mathfrak{A}$, and set

$$D = \{d : d \in \mathfrak{A},\ \{a : a \in A_n,\ a \cap d \ne 0\} \text{ is finite for every } n \in \mathbb{N}\}.$$

Take any $c \in \mathfrak{A}^+$. Let G, H be disjoint open sets containing 0, c respectively. Choose $\langle c_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $c_0 = c$. Given $c_n \in H$, let $\langle a_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence running over A_n, and set $c_{nj} = \sup_{i\le j} c_n \cap a_{ni}$; then $\langle c_{nj}\rangle_{j\in\mathbb{N}}$ order*-converges to c_n, so there is a j_n such that $c_{nj_n} \in H$; set $c_{n+1} = c_{nj_n}$, and continue.

This gives us a non-increasing sequence $\langle c_n\rangle_{n\in\mathbb{N}}$ in H. Set $d = \inf_{n\in\mathbb{N}} c_n$; then $d \notin G$ so $d \ne 0$, while $d \subseteq \sup_{i\le j_n} a_{ni}$ for each n, so $d \in D$.

As c is arbitrary, D is order-dense in $\mathfrak{A}$ and includes a maximal antichain. As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

(b) For any $a \in \mathfrak{A}^+$ there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $a \not\subseteq \sup(\bigcap_{n\in\mathbb{N}} H_n)$. **P** For $A \subseteq \mathfrak{A}$ and $n \in \mathbb{N}$, define $\bigvee_n(A)$ as in 3A. Let H_0 be a neighbourhood of 0 such that H_0 and $\{a \triangle b : b \in H_0\}$ are disjoint; by Fremlin 04, 392Mc again, we may suppose that $[0, b] \subseteq H_0$ for every $b \in H_0$, in which case $[0, b] \subseteq \bigvee_2(H_0)$ for every $b \in \bigvee_2(H_0)$, while $a \notin \bigvee_2(H_0)$. By Lemma 3A, we can choose neighbourhoods H_n of 0, for $n \ge 1$, such that $H_{n+1} \subseteq H_n$ and $\bigvee_3(H_{n+1}) \subseteq \bigvee_2(H_n)$ for every n. But this will ensure that $\bigvee_4(H_{n+2}) \subseteq \bigvee_2(H_n)$ for every n, so that $\bigvee_{2^k}(H_{2k}) \subseteq \bigvee_2(H_2)$ for every $k \ge 1$. Set $F = \bigcap_{n\in\mathbb{N}} H_n$. Then

$$\bigvee_{2^k}(F) \subseteq \bigvee_{2^k}(H_{2k}) \subseteq \bigvee_2(H_2)$$

for every $k \ge 1$. Since $\sup F$ is the limit of a sequence in $\bigcup_{k\ge1} \bigvee_{2^k}(F)$,

$$\sup F \in \overline{\bigvee_2(H_2)} \subseteq \bigvee_3(H_2) \subseteq \bigvee_2(H_0)$$

and cannot include a. **Q**

(c) Now consider the set D of those $d \in \mathfrak{A}$ such that there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d \cap \sup(\bigcap_{n\in\mathbb{N}} H_n) = 0$. By (b), D is order-dense, so includes a maximal antichain A. Now A is countable, so there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of 0 such that $d \cap \sup(\bigcap_{n\in\mathbb{N}} H_n) = 0$ for every $d \in A$; but this means that $\bigcap_{n\in\mathbb{N}} H_n = \{0\}$. By Fremlin 04, 392O, $\mathfrak{A}$ is a Maharam algebra.

3C Theorem (Todorčević p04) Let $\mathfrak{A}$ be a σ-finite-cc weakly (σ,∞)-distributive Dedekind complete Boolean algebra. Then $\mathfrak{A}$ is a Maharam algebra.

proof (Balcar n04) **(a)(i)** Suppose that $\mathfrak{A} \ne \{0\}$. Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets, with union $\mathfrak{A}^+$, such that no A_n includes any infinite disjoint set. For each n, set $B_n = \bigcup_{m\le n} \bigcup_{a\in A_m} [a, 1]$, so that B_n includes no infinite disjoint subset. Now there is an n such that 1 is in the interior of B_n for the order-sequential topology. **P?** Otherwise, there is for each $n \in \mathbb{N}$ a sequence $\langle b_{ni}\rangle_{i\in\mathbb{N}}$ in $\mathfrak{A} \setminus B_n$ which is order*-convergent to 1 (Fremlin 04, 392Mb). By Fremlin 04, 392Ma, there is a sequence $\langle k(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\langle b_{n,k(n)}\rangle_{n\in\mathbb{N}}$ order*-converges to 1. As $1 \ne 0$, there must be an $m \in \mathbb{N}$ such that $c = \inf_{i\ge m} b_{i,k(i)} \ne 0$. There is an n such that $c \in A_n$, in which case $b_{i,k(i)} \in B_m \subseteq B_i$ for every $i \ge \max(m, n)$. **XQ**

(ii) Set $H = \operatorname{int} B_n$. Then there is a $c \in H$ such that for every $d \in \mathfrak{A}$ one of $c \cap d$, $c \setminus d \notin H$. **P ?** Otherwise, we can choose a sequence $\langle c_i\rangle_{i\in\mathbb{N}}$ in H such that $c_0 = 1$ and, for each $i \in \mathbb{N}$, $c_{i+1} \subseteq c_i$ and $c_i \setminus c_{i+1} \in H$. But in this case $\langle c_i \setminus c_{i+1}\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in B_n, which is impossible. **XQ**

(iii) 0 and 1 can be separated by open sets. **P** Take H and c from (ii). Then $G_0 = \{d : c \setminus d \in H\}$ and $G_1 = \{d : c \cap d \in H\}$ are disjoint open sets containing 0 and 1 respectively. **Q**

(b) It follows that $\mathfrak{A}$ is actually Hausdorff in the order-sequential topology. **P** Let a_0, $a_1 \in \mathfrak{A}$ be such that $b = a_1 \setminus a_0$ is non-zero. Consider the principal ideal $\mathfrak{A}_b$. Like $\mathfrak{A}$, this is σ-finite-cc, weakly (σ,∞)-distributive and Dedekind complete. By (a), there are disjoint subsets U, V of $\mathfrak{A}_b$, open for the order-sequential topology of $\mathfrak{A}_b$, such that $0 \in U$ and $b \in V$. Now the function $a \mapsto a \cap b : \mathfrak{A} \to \mathfrak{A}_b$ is continuous for the order-sequential topologies (use Fremlin 04, 3A3Pb), so $G = \{a : a \cap b \in U\}$ and $H = \{a : a \cap b \in V\}$ are open. Now G and H are open sets in $\mathfrak{A}$ containing a_0, a_1 respectively. As a_0 and a_1 are arbitrary, $\mathfrak{A}$ is Hausdorff. **Q**

By Theorem 3B, $\mathfrak{A}$ is a Maharam algebra.

3D Lemma (Quickert 02) Let $\mathfrak{A}$ be a Boolean algebra, and $\mathcal{I}$ be the family of countable subsets I of $\mathfrak{A}^+$ for which there is a partition C of unity such that $\{a : a \in I,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(a) $\mathcal{I}$ is an ideal of $\mathcal{P}\mathfrak{A}$ including $[\mathfrak{A}]^{<\omega}$.

(b) If $A \subseteq \mathfrak{A}^+$ is such that $A \cap I$ is finite for every $I \in \mathcal{I}$, and $B = \{b : b \supseteq a \text{ for some } a \in A\}$, then $B \cap I$ is finite for every $I \in \mathcal{I}$.

(c) If $\mathfrak{A}$ is ccc, then there is no uncountable $B \subseteq \mathfrak{A}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$.

(d) If $\mathfrak{A}$ is ccc and weakly (σ,∞)-distributive, $\mathcal{I}$ is a **P-ideal**, that is, if $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \mathbb{N}$.

proof (a) Of course every finite subset of $\mathfrak{A}$ belongs to $\mathcal{I}$. If I_0, $I_1 \in \mathcal{I}$ and $J \subseteq I_0 \cup I_1$, then $J \in [\mathfrak{A}]^{\le\omega}$. For each j, we have a partition C_j of unity in $\mathfrak{A}$ such that $\{a : a \in I_j,\, a \cap c \neq 0\}$ is finite for every $c \in C_j$. Set $C = \{c_0 \cap c_1 : c_0 \in C_0,\, c_1 \in C_1\}$; then C is a partition of unity in $\mathfrak{A}$ and $\{a : a \in J,\, a \cap c \neq 0\}$ is finite for every $c \in C$.

(b) ? Otherwise, set $J = B \cap I \in \mathcal{I}$. For each $b \in J$, let $a_b \in A$ be such that $a_b \subseteq b$. Let C be a partition of unity such that $\{b : b \in J,\, b \cap c \neq 0\}$ is finite for every $c \in C$; then $\{a_b : b \in J,\, a_b \cap c \neq 0\}$ is finite for every $c \in C$, so $\{a_b : b \in J\}$ belongs to $\mathcal{I}$ and must be finite. There is therefore an $a \in A$ such that $K = \{b : b \in J,\, a = a_b\}$ is infinite; but in this case there is a $c \in C$ such that $a \cap c \neq 0$ and $b \cap c \neq 0$ for every $b \in K$. **X**

(c) Let $\widehat{\mathfrak{A}}$ be the Dedekind completion of $\mathfrak{A}$ (Fremlin 04, 314T). Let $\langle b_\xi\rangle_{\xi<\omega_1}$ be a family of distinct elements of B and set $d = \inf_{\xi<\omega_1}\sup_{\xi\le\eta<\omega_1} b_\eta$, taken in $\widehat{\mathfrak{A}}$. Then (because $\widehat{\mathfrak{A}}$ is ccc, see Fremlin 04, 316Xf) $d = \sup_{\xi\le\eta<\omega_1} b_\eta$ for some ξ; in particular, $d \neq 0$. Next, we can find a strictly increasing sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ in ω_1 such that $d \subseteq \sup_{\xi_n\le\eta<\xi_{n+1}} b_\eta$ for every $n \in \mathbb{N}$. Set $I = \{b_\eta : \eta < \sup_{n\in\mathbb{N}}\xi_n\} \in [B]^{\le\omega}$. If C is any partition of unity in $\mathfrak{A}$, there must be some $c \in C$ such that $c \cap d \neq 0$, and now $\{a : a \in I,\, a \cap c \neq 0\}$ is infinite. So $I \notin \mathcal{I}$. **Q**

(d) For each $n \in \mathbb{N}$, let C_n be a partition of unity such that $\{a : a \in I_n,\, a \cap c \neq 0\}$ is finite for every $c \in C_n$. Let D be a partition of unity such that $\{c : c \in C_n,\, c \cap d \neq 0\}$ is finite for every $d \in D$ and $n \in \mathbb{N}$. Then

$$\{a : a \in I_n,\, a \cap d \neq 0\} \subseteq \bigcup_{c\in C_n, c\cap d\neq 0}\{a : a \in I_n,\, a \cap c \neq 0\}$$

is finite for every $d \in D$ and $n \in \mathbb{N}$. Let $\langle d_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $D \cup \{\emptyset\}$ and set $I = \bigcup_{n\in\mathbb{N}}\{a : a \in I_n,\, a \cap d_i = 0 \text{ for every } i \le n\}$. Then

$$I_n \setminus I \subseteq \bigcup_{i\le n}\{a : a \in I_n,\, a \cap d_i \neq \emptyset\}$$

is finite for each n. Also

$$\{a : a \in I,\, a \cap d_n \neq 0\} \subseteq \bigcup_{i<n}\{a : a \in I_i,\, a \cap d_n \neq 0\}$$

is finite for each n, so $I \in \mathcal{I}$.

Remark In this context, $\mathcal{I}$ is called **Quickert's ideal**.

3E Lemma (Balcar Jech & Pazák p03) Let $\mathfrak{A}$ be a weakly (σ,∞)-distributive ccc Dedekind complete Boolean algebra, and suppose that $\mathfrak{A}^+$ is expressible as $\bigcup_{k\in\mathbb{N}} D_k$ where no infinite subset of any D_k belongs to Quickert's ideal $\mathcal{I}$. Then $\mathfrak{A}$ is a Maharam algebra.

proof The point is that if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ which order*-converges to 0, then $\{a_n : n \in \mathbb{N}\} \in \mathcal{I}$ (Fremlin 04, 392La). So no sequence in any D_k can order*-converge to 0. Because $\mathfrak{A}$ is weakly (σ,∞)-distributive and ccc, 0 does not belong to the closure $\overline{D}_k$ of D_k for the order-sequential topology on $\mathfrak{A}$ (Fremlin 04, 392Mb). So $\mathfrak{A}^+ = \bigcup_{k\in\mathbb{N}}\overline{D}_k$ is F_σ and $\{0\}$ is G_δ for the order-sequential topology. By Fremlin 04, 392O, $\mathfrak{A}$ is a Maharam algebra.

3F Todorčević's P-ideal dichotomy This is the statement

whenever X is a set and $\mathcal{I}$ is a P-ideal of countable subsets of X, then

either there is a $B \in [X]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$

or X is expressible as $\bigcup_{n\in\mathbb{N}} X_n$ where $\mathcal{I} \cap \mathcal{P}X_n \subseteq [X_n]^{<\omega}$ for every $n \in \mathbb{N}$.

This is a consequence of the Proper Forcing Axiom, and is also relatively consistent with the generalized continuum hypothesis (Todorčević 00).

3G Theorem (Balcar Jech & Pazák p03) If Todorčević's P-ideal dichotomy is true, then every Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra.

proof Let $\mathfrak{A}$ be a Dedekind complete ccc weakly (σ,∞)-distributive Boolean algebra. Let $\mathcal{I}$ be Quickert's ideal on $\mathfrak{A}$; then $\mathcal{I}$ is a P-ideal (3Dd). By 3Dc, there is no $B \in [\mathfrak{A}]^{\omega_1}$ such that $[B]^{\le\omega} \subseteq \mathcal{I}$. We are assuming that Todorčević's P-ideal dichotomy is true; so $\mathfrak{A}$ must be expressible as $\bigcup_{n\in\mathbb{N}} D_n$ where no infinite subset of any D_n belongs to $\mathcal{I}$. By 3E, $\mathfrak{A}$ is a Maharam algebra.

3H Theorem (Jech l04) Let $\mathfrak{A}$ be a Boolean algebra. Then the following are equiveridical:

(i) the Dedekind completion of $\mathfrak{A}$ is a Maharam algebra;

(ii) there is a family S of sequences in $\mathfrak{A}$ such that

(α) $\langle a_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 0 for every $\langle a_n\rangle_{n\in\mathbb{N}} \in S$;

(β) if $\langle\langle a_{nk}\rangle_{k\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ is a sequence in S then $\langle a_{nn}\rangle_{n\in\mathbb{N}} \in S$;

(γ) every sequence which order*-converges to 0 has a subsequence in S.

proof (i)⇒(ii) If the Dedekind completion of $\mathfrak{A}$ is a Maharam algebra, then $\mathfrak{A}$ itself has a strictly positive Maharam submeasure ν. Let S be the set of all sequences $\langle a_n\rangle_{n\in\mathbb{N}}$ such that $\nu a_n \le 2^{-n}$ for every n; then S satisfies the conditions of (ii). **P** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ which is *not* order*-convergent to 0, there is a non-zero $c \in \mathfrak{A}$ such that $c = \sup_{i\ge n} c \cap a_i$ for every n. In this case,

$$0 < \nu c = \sup_{m\ge n} \nu(c \cap \sup_{n\le i\le m} a_i) \le \sum_{i=n}^{\infty} \nu a_i$$

for every n, and $\sum_{i=0}^{\infty} \nu a_i = \infty$, so $\langle a_n\rangle_{n\in\mathbb{N}} \notin S$. This shows that S satisfies (α). The others are elementary. **Q**

(ii)⇒(i) Given $S \subseteq \mathfrak{A}^{\mathbb{N}}$ satisfying the conditions in (ii), let A_n be the set $\{a_n : \langle a_k\rangle_{k\in\mathbb{N}} \in S\}$ for each n.

$\bigcap_{n\in\mathbb{N}} A_n = \{0\}$. **P** By ($\gamma$), the constant sequence with value 0 belongs to S, so $0 \in A_n$ for every n. If $a \in A_n$ for every n, then for each $n \in \mathbb{N}$ we have a sequence $\langle a_{nk}\rangle_{k\in\mathbb{N}} \in S$ such that $a = a_{nn}$; now the constant sequence $\langle a_{nn}\rangle_{n\in\mathbb{N}}$ belongs to S, by (β), so is order*-convergent to 0, by (α), and $a = 0$. **Q**

$\mathfrak{A}$ is σ-finite-cc. **P?** If $\langle a_k\rangle_{k\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A} \setminus A_n$, then it is order*-convergent to 0, so has a subsequence belonging to S which must enter A_n. **X** So $\langle\mathfrak{A} \setminus A_n\rangle_{n\in\mathbb{N}}$ witnesses that $\mathfrak{A}$ is σ-finite-cc. **Q**

$\mathfrak{A}$ is weakly (σ,∞)-distributive. **P** Let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence of partitions of unity in $\mathfrak{A}$. Set $C'_n = \{\inf_{i\le n} c_i : c_i \in C_i \text{ for } i \le n\}$, so that C'_n is a partition of unity refining C_n, and C'_{n+1} refines C'_n for each n. Let $\langle c_{nk}\rangle_{k\in\mathbb{N}}$ be a sequence running over $C'_n \cup \{0\}$. Set $c'_{nm} = 1 \setminus \sup_{k<m} c_{nk}$, so that $\langle c'_{nm}\rangle_{m\in\mathbb{N}}$ is non-increasing and has infimum 0. As $\langle c'_{nm}\rangle_{m\in\mathbb{N}}$ is order*-convergent to 0, it has a subsequence $\langle c'_{n,m(n,i)}\rangle_{i\in\mathbb{N}}$ belonging to S. Consider the sequence $\langle c'_{n,m(n,n)}\rangle_{n\in\mathbb{N}} \in S$. This is order*-convergent to 0 so there is a partition D of unity such that $\{n : d \cap c'_{n,m(n,n)} \ne 0\}$ is finite for each $d \in D$. So, given $d \in D$ and $j \in \mathbb{N}$, there is an $n \ge j$ such that $d \cap c'_{n,m(n,n)} = 0$, in which case $d \subseteq \sup_{i<m(n,n)} c_{ni}$ and

$$\begin{aligned}\{c : c \in C_j,\, d \cap c \ne 0\} &\subseteq \bigcup_{i<m(n,n)} \{c : c \in C_j,\, c \cap c_{ni} \ne 0\}\\ &= \bigcup_{i<m(n,n)} \{c : c \in C_j,\, 0 \ne c_{ni} \subseteq c\}\end{aligned}$$

is finite. As $\langle C_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathfrak{A}$ is weakly (σ,∞)-distributive. **Q**

Now the Dedekind completion of $\mathfrak{A}$ is still weakly (σ,∞)-distributive (1B(c-iii)) and σ-finite-cc (1B(c-ii)), so is a Maharam algebra by Todorčević's theorem 3C.

4 Products of submeasures

4A Construction There seems to be no satisfactory general method of constructing products of submeasures. However the following method may turn out to be useful.

(a) Let $\mathfrak{A}$ and $\mathfrak{B}$ be Boolean algebras with submeasures μ, ν respectively. On the free product $\mathfrak{A}\otimes\mathfrak{B}$ (Fremlin 04, §315), we have a functional λ defined by saying that whenever $c\in\mathfrak{A}\otimes\mathfrak{B}$ is of the form $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$, then

$$\begin{aligned}\lambda c&=\min_{J\subseteq I}\max(\{\mu(\sup_{i\in J}a_i)\}\cup\{\nu b_i:i\in I\setminus J\})\\&=\min\{\epsilon:\epsilon\ge 0,\ \mu(\sup\{a_i:i\in I,\ \nu b_i>\epsilon\})\le\epsilon\}.\end{aligned}$$

P Every $c\in\mathfrak{A}\otimes\mathfrak{B}$ can be expressed in this form (Fremlin 04, 315Na). Of course this can be done in many different ways. But if $c=\sup_{j\in J}a'_j\otimes b'_j$ is another expression of the same kind, then $b_i=b'_j$ whenever $a_i\cap a'_j\ne 0$. So

$$\begin{aligned}\sup\{a_i:i\in I,\ \nu b_i>\epsilon\}&=\sup\{a_i\cap a'_j:i\in I,\ j\in J,\ a_i\cap a'_j\ne 0,\ \nu b_i>\epsilon\}\\&=\sup\{a_i\cap a'_j:i\in I,\ j\in J,\ a_i\cap a'_j\ne 0,\ \nu b'_j>\epsilon\}\\&=\sup\{a'_j:j\in J,\ \nu b'_j>\epsilon\}\end{aligned}$$

for any ϵ, and the two calculations for λ give the same result. **Q**

Note that $\lambda(a\otimes b)=\min(\mu a,\nu b)$ for all $a\in\mathfrak{A}$ and $b\in\mathfrak{B}$.

(b) In the context of (a), λ is a submeasure.

P By definition, $\lambda c\ge 0$ for every $c\in\mathfrak{A}\otimes\mathfrak{B}$; and if $c=0$ then it is $1\otimes 0$ and $\lambda c=0$.

If c, c' are two members of $\mathfrak{A}\otimes\mathfrak{B}$, express them in the forms $c=\sup_{i\in I}a_i\otimes b_i$ and $c'=\sup_{j\in J}a'_j\otimes b'_j$ where $\langle a_i\rangle_{i\in I}$ and $\langle a'_j\rangle_{j\in J}$ are partitions of unity in $\mathfrak{A}$. Set $K=\{(i,j):a_i\cap a'_j\ne 0\}\subseteq I\times J$, $a''_{ij}=a_i\cap a'_j$ for $(i,j)\in K$; then $\langle a''_{ij}\rangle_{(i,j)\in K}$ is a partition of unity in $\mathfrak{A}$, $c=\sup_{(i,j)\in K}a''_{ij}\otimes b_i$ and $c'=\sup_{(i,j)\in K}a''_{ij}\otimes b'_j$. Set $\alpha=\lambda c$, $\beta=\lambda c'$, $L=\{(i,j):(i,j)\in K,\ \nu b_i>\alpha\}$, $L'=\{(i,j):(i,j)\in K,\ \nu b'_j>\beta\}$, $e=\sup\{a_{ij}:(i,j)\in L\}$ and $e'=\sup\{a_{ij}:(i,j)\in L'\}$; then $\mu e\le\alpha$ and $\mu e'\le\beta$. So $\mu(e\cup e')\le\alpha+\beta$; but $e\cup e'=\sup_{(i,j)\in L\cup L'}a''_{ij}$ and

$$\nu(b_i\cup b'_j)\le\nu b_i+\nu b'_j\le\alpha+\beta$$

for all $(i,j)\in K\setminus(L\cup L')$. So $\lambda(c\cup c')\le\alpha+\beta$.

If $c\subseteq c'$, then $b_i\subseteq b'_j$ for every $(i,j)\in K$. So $\nu b_i\le\beta$ for every $(i,j)\in K\setminus L'$ and $\lambda c\le\beta$.

Thus λ is subadditive and order-preserving and is a submeasure. **Q**

(c) In this context, I will write $\lambda=\mu\ltimes\nu$. I note that only in exceptional, and usually trivial, cases will $\mu\ltimes\nu$ be matched with $\nu\ltimes\mu$ by the canonical isomorphism between $\mathfrak{A}\otimes\mathfrak{B}$ and $\mathfrak{B}\otimes\mathfrak{A}$; this product is not 'commutative'. It is however 'associative', in the following sense. Let $(\mathfrak{A}_1,\mu_1)$, $(\mathfrak{A}_2,\mu_2)$, $\mathfrak{A}_3,\mu_3)$ be Boolean algebras endowed with submeasures. Set

$$\lambda_{12}=\mu_1\ltimes\mu_2,\quad\lambda_{(12)3}=\lambda_{12}\ltimes\mu_3,\quad\lambda_{23}=\mu_2\ltimes\mu_3,\quad\lambda_{1(23)}=\mu_1\ltimes\lambda_{23}.$$

Then the canonical isomorphisms between $(\mathfrak{A}_1\otimes\mathfrak{A}_2)\otimes\mathfrak{A}_3$, $\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$ and $(\mathfrak{A}_1\otimes(\mathfrak{A}_2\otimes\mathfrak{A}_3)$ (Fremlin 04, 315K) identify $\lambda_{(12)3}$ with $\lambda_{1(23)}$.

P Take any $d\in\mathfrak{A}_1\otimes\mathfrak{A}_2\otimes\mathfrak{A}_3$. Express d as $\sup_{i\in I}a_i\otimes e_i$ where $\langle a_i\rangle_{i\in I}$ is a partition of unity in $\mathfrak{A}_1$ and $e_i\in\mathfrak{A}_2\otimes\mathfrak{A}_3$ for each i; express each e_i as $\sup_{j\in J_i}b_{ij}\otimes c_{ij}$ where $\langle b_{ij}\rangle_{j\in J_i}$ is a partition of unity in $\mathfrak{A}_2$ and $c_{ij}\in\mathfrak{A}_3$ for $i\in I$, $j\in J_i$. In this case, $\langle a_i\otimes b_{ij}\rangle_{i\in I,j\in J_i}$ is a partition of unity in $\mathfrak{A}_1\otimes\mathfrak{A}_2$ and $d=\sup_{i\in I,j\in J_i}a_i\otimes b_{ij}\otimes c_{ij}$.

Let $\epsilon>0$. For $i\in I$, set $J'_i=\{j:j\in J_i,\ \mu_3c_{ij}>\epsilon\}$, $e'_i=\sup_{j\in J'_i}b_{ij}$. Then $\lambda_{23}(\sup_{j\in J_i}b_{ij}\otimes c_{ij})\le\epsilon$ iff $\mu_2e'_i\le\epsilon$. Set $I'=\{i:\mu_2e'_i>\epsilon\}$; then $\lambda_{1(23)}d\le\epsilon$ iff $\mu_1(\sup_{i\in I'}a_i)\le\epsilon$. From the other direction, set $f=\sup\{a_i\otimes b_{ij}:i\in I,\ j\in J'_i\}$; then $\lambda_{(12)3}d\le\epsilon$ iff $\lambda_{12}f\le\epsilon$. But $f=\sup_{i\in I}a_i\otimes e'_i$, so $\lambda_{12}f\le\epsilon$ iff $\mu_1(\sup_{i\in I'}a_i)\le\epsilon$.

As ϵ and d are arbitrary, $\lambda_{(12)3}=\lambda_{1(23)}$, as claimed. **Q**

(d) Returning to the notation of (a)-(b): if μ and ν are exhaustive, so is λ. **P** Let $\langle c_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{A}\otimes\mathfrak{B}$ such that $\lambda c_n>\epsilon>0$ for every n. For each n, express c_n as $\sup_{i\in I_n}a_{ni}\otimes b_{ni}$ where $\langle a_{ni}\rangle_{i\in I_n}$ is a partition of unity; set $I'_n=\{i:i\in I_n,\ \nu b_{ni}>\epsilon\}$, $a_n=\sup_{i\in I'_n}a_{ni}$; then $\mu a_n>\epsilon$. By Lemma 2D, there is an infinite $J \subseteq \mathbb{N}$ such that $\{a_n : n \in J\}$ is centered. Let $D \subseteq \mathfrak{A}$ be a maximal centered family including $\{a_n : n \in J\}$; then for every $n \in J$ there is an $i_n \in I'_n$ such that $a_{n,i_n} \in D$. But now observe that $\nu b_{n,i_n} > \epsilon$ for every $n \in J$, so there must be distinct m, $n \in J$ such that $b_{m,i_m} \cap b_{n,i_n} \neq 0$; as $a_{m,i_m} \cap a_{n,i_n}$ is also non-zero, $c_m \cap c_n \neq 0$. As $\langle c_n \rangle_{n \in \mathbb{N}}$ is arbitrary, λ is exhaustive. **Q**

(e) We can extend the construction to infinite products, as follows. Let I be a totally ordered set and $\langle (\mathfrak{A}_i, \mu_i) \rangle_{i \in I}$ a family of Boolean algebras endowed with unital submeasures. For a finite set $J = \{i_0, \ldots, i_n\}$ where $i_0 < \ldots < i_n$ in I, let λ_J be the product submeasure $(.(\mu_{i_0} \ltimes \mu_{i_1}) \ltimes \ldots) \ltimes \mu_{i_n}$ on $\mathfrak{C}_J = \bigotimes_{j \in J} \mathfrak{A}_j$; for definiteness, on $\mathfrak{C}_\emptyset = \{0, 1\}$ take $\lambda_\emptyset$ to be the unital submeasure. Using (c) repeatedly, we see that if J, $K \in [I]^{<\omega}$ and $j < k$ for every $j \in J$, $k \in K$, then the identification of $\mathfrak{C}_{J \cup K}$ with $\mathfrak{C}_J \otimes \mathfrak{C}_K$ (Fremlin 04, 315K) matches $\lambda_{J \cup K}$ with $\lambda_J \ltimes \lambda_K$. Moreover, if $K \in [I]^{<\omega}$ and J is any subset of K (not necessarily an initial segment) and $\varepsilon_{JK} : \mathfrak{C}_J \to \mathfrak{C}_K$ is the canonical embedding corresponding to the identification of $\mathfrak{C}_K$ with $\mathfrak{C}_J \otimes \mathfrak{C}_{K \setminus J}$, then $\lambda_J = \lambda_K \varepsilon_{JK}$; this is also an easy induction on $\#(K)$. What this means is that for any subset M of I we have a submeasure λ_M on $\mathfrak{C}_M = \bigcup \{\varepsilon_{JM} \mathfrak{C}_J : J \in [M]^{<\omega}\}$, being the unique functional such that $\lambda_M \varepsilon_{JM} = \lambda_J$ for every $J \in [M]^{<\omega}$. Finally, if L, M are subsets of I with $l < m$ for every $l \in L$ and $m \in M$, then $\lambda_{L \cup M}$ can be identified with $\lambda_L \ltimes \lambda_M$.

Unhappily it is not clear that we can get new exhaustive submeasures this way. If I is any infinite totally ordered set, and for each $i \in I$ we set $\mathfrak{A}_i = \mathcal{P}\{0, 1\}$ with $\nu_i\{0\} = \nu_i\{1\} = \nu_i\{0, 1\} = 1$, then $\bigotimes_{i \in I} \mathfrak{A}_i$ can be identified with the algebra $\mathcal{E}$ of open-and-closed subsets of $\{0, 1\}^I$, and λ_I with the submeasure on $\mathcal{E}$ which gives every non-empty set the submeasure 1; which is about as far from exhaustive as it could well be.

(f) Turning now to products of Maharam algebras, it is easy to see, in (a), that if μ and ν are strictly positive so is $\mu \ltimes \nu$. At this point it is worth observing that if μ, μ' are submeasures on $\mathfrak{A}$, ν and ν' are submeasures on $\mathfrak{B}$, μ is absolutely continuous with respect to μ' and ν is absolutely continuous with respect to ν', then $\mu \ltimes \nu$ is absolutely continuous with respect to $\mu' \otimes \nu'$. **P** For any $\epsilon > 0$ there is a $\delta > 0$ such that $\mu a \leq \epsilon$ whenever $\mu' a \leq \delta$ and $\nu b \leq \epsilon$ whenever $\nu' b \leq \delta$. If now $c \in \mathfrak{A} \otimes \mathfrak{B}$ and $(\mu' \ltimes \nu')(c) \leq \delta$, we have $c = \sup_{i \in I} a_i \otimes b_i$ and $J \subseteq I$ such that $\langle a_i \rangle_{i \in I}$ is a partition unity, $\mu'(\sup_{i \in J} a_i) \leq \delta$ and $\nu' b_i \leq \delta$ for every $i \in I \setminus J$; so $\mu(\sup_{i \in J} a_i) \leq \epsilon$ and $\nu b_i \leq \epsilon$ for every $i \in I \setminus J$ and $(\mu \ltimes \nu)(c) \leq \epsilon$. **Q**

Now suppose that $\langle \mathfrak{A}_i \rangle_{i \in I}$ is a family of non-trivial Maharam algebras, where I is a finite totally ordered set. Then we can take a strictly positive unital Maharam submeasure μ_i on each $\mathfrak{A}_i$, form an exhaustive submeasure λ on $\mathfrak{C}_I = \bigotimes_{i \in I} \mathfrak{A}_i$, and use λ to construct a metric completion $\widehat{\mathfrak{C}}_I$ which is a Maharam algebra, as in Fremlin 04, 393B. If we change each μ_i to μ'_i, where μ'_i is another strictly positive Maharam submeasure on $\mathfrak{A}_i$, then every μ'_i is absolutely continuous with respect to μ_i (Fremlin 04, 393E), so the corresponding λ' will be absolutely continuous with respect to λ, and vice versa; in which case the metrics on $\mathfrak{C}_I$ are uniformly equivalent and we get the same completion $\widehat{\mathfrak{C}}_I$ up to Boolean algebra isomorphism. We can therefore think of $\widehat{\mathfrak{C}}_I$ as 'the' Maharam algebra free product of the family $\langle \mathfrak{A}_i \rangle_{i \in I}$ of Boolean algebras; as before, we shall have an isomorphism between $\widehat{\mathfrak{C}}_J \widehat{\otimes} \widehat{\mathfrak{C}}_K$ and $\widehat{\mathfrak{C}}_{J \cup K}$ whenever J, $K \subseteq I$ and $j < k$ for every $j \in J$, $k \in K$.

(g) I should perhaps have remarked already that if μ and ν, in (a), are additive and unital, then we have an additive function λ' on $\mathfrak{A} \otimes \mathfrak{B}$ such that $\lambda'(a \otimes b) = \mu a \cdot \nu b$ for every $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$ (Fremlin 04, 326Q). Now if we take λ as constructed in (a), each of λ, λ' is absolutely continuous with respect to the other. **P** If $c \in \mathfrak{A} \otimes \mathfrak{B}$, express c as $\sup_{i \in I} a_i \otimes b_i$ where $\langle a_i \rangle_{i \in I}$ is a finite partition of unity. Then$\mu(\sup\{a_i : \nu b_i > \lambda c\}) \leq \lambda c$, so $\lambda' c = \sum_{i \in I} \mu a_i \cdot \nu b_i$ is at most $2\lambda c$. On the other hand, $\mu(\sup\{a_i : \nu b_i > \sqrt{\lambda' c}\}) \leq \sqrt{\lambda' c}$, so $\lambda c \leq \sqrt{\lambda' c}$. **Q**

What this means is that if $(\mathfrak{A}, \mu)$ and $(\mathfrak{B}, \nu)$ are probability algebras, then their Maharam algebra free product, regarded as a Boolean algebra, is identical to their probability algebra free product as defined in Fremlin 04, §326. Now this extends to finite products, as in (f) here.

4B Representing products of Maharam algebras: Theorem Let X and Y be sets, with σ-algebras Σ and T and Maharam submeasures μ and ν defined on Σ, T respectively. Set $\mathcal{I} = \mu^{-1}[\{0\}]$, $\mathcal{J} = \nu^{-1}[\{0\}]$, $\mathfrak{A} = \Sigma/\mathcal{I}$ and $\mathfrak{B} = \mathrm{T}/\mathcal{J}$, and write $\bar{\mu}$, $\bar{\nu}$ for the strictly positive Maharam submeasures on $\mathfrak{A}$ and $\mathfrak{B}$ induced by μ and ν as in 1I above. Let $\Sigma \widehat{\otimes} \mathrm{T}$ be the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma$, $F \in \mathrm{T}\}$.

(a) (Compare FREMLIN 03, 418T.) Give $\mathfrak{B}$ the topology induced by the metric $(b, b') \mapsto \bar{\nu}(b \triangle b')$[1]. If $W \in \Sigma\widehat{\otimes}\mathrm{T}$ then $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and the function $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Consequently $x \mapsto \nu W[\{x\}] : X \to [0, \infty[$ is Σ-measurable.

(b) For $W \in \Sigma\widehat{\otimes}\mathrm{T}$ set

$$\lambda W = \inf\{\epsilon : \epsilon > 0,\, \mu\{x : \nu W[\{x\}] > \epsilon\} \le \epsilon\}.$$

Then λ is a Maharam submeasure on $\Sigma\widehat{\otimes}\mathrm{T}$, and

$$\lambda^{-1}[\{0\}] = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\, \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}\}.$$

(c) $\mathfrak{C} = \Sigma\widehat{\otimes}\mathrm{T}/\lambda^{-1}[\{0\}]$ is a Maharam algebra with a strictly positive Maharam submeasure $\bar{\lambda}$ induced by λ.

(d) $\mathfrak{A} \otimes \mathfrak{B}$ can be embedded in $\mathfrak{C}$ by mapping $E^\bullet \otimes F^\bullet$ to $(E \times F)^\bullet$ for all $E \in \Sigma$, $F \in \mathrm{T}$.

(e) This embedding identifies $(\mathfrak{C}, \bar{\lambda})$ with the metric completion $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$ of $(\mathfrak{A} \otimes \mathfrak{B}, \bar{\mu} \ltimes \bar{\nu})$ as described in 4Af.

proof (a) Write $\mathcal{W}$ for the set of those $W \subseteq X \times Y$ such that $W[\{x\}] \in \mathrm{T}$ for every $x \in X$ and $x \mapsto W[\{x\}]^\bullet : X \to \mathfrak{B}$ is Σ-measurable and has separable range. Then $\Sigma \otimes \mathrm{T}$ (identified with the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma,\, F \in \mathrm{T}\}$) is included in $\mathcal{W}$.

If $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$ with union W, then $W \in \mathcal{W}$. **P** Of course $W[\{x\}] = \bigcup_{n\in\mathbb{N}} W_n[\{x\}]$ belongs to T for every $x \in X$. Set $f_n(x) = W_n[\{x\}]^\bullet$ for $n \in \mathbb{N}$ and $x \in X$. For each $x \in X$, $W[\{x\}] \setminus W_n[\{x\}]$ is a non-increasing sequence with empty intersection, so $\lim_{n\to\infty} \nu(W[\{x\}] \setminus W_n[\{x\}]) = 0$ and $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ converges to $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. By FREMLIN 03, 418B, f is measurable. Also $D = \overline{\{f_n(x) : x \in X,\, n \in \mathbb{N}\}}$ is a separable subspace of $\mathfrak{B}$ including $f[X]$. So $W \in \mathcal{W}$. **Q**

Similarly, $\bigcap_{n\in\mathbb{N}} W_n \in \mathcal{W}$ for any non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}$. $\mathcal{W}$ therefore includes the σ-algebra generated by $\Sigma \otimes \mathrm{T}$ (FREMLIN 00, 136G), which is $\Sigma\widehat{\otimes}\mathrm{T}$.

Now $x \mapsto \nu W[\{x\}] = \bar{\nu} W[\{x\}]^\bullet$ is measurable because $\bar{\nu} : \mathfrak{B} \to \mathbb{R}$ is continuous.

(b) Of course $\lambda\emptyset = 0$ and $\lambda W \le \lambda W'$ if W, $W' \in \Sigma\widehat{\otimes}\mathrm{T}$ and $W \subseteq W'$. If W_1, $W_2 \in \Sigma\widehat{\otimes}\mathrm{T}$ have union W, $\lambda W_1 = \alpha_1$ and $\lambda W_2 = \alpha_2$, then

$$\{x : \nu W[\{x\}] > \alpha_1 + \alpha_2\} \subseteq \{x : \nu W_1[\{x\}] > \alpha_1\} \cup \{x : \nu W_2[\{x\}] > \alpha_2\},$$

so, setting $\alpha = \alpha_1 + \alpha_2$,

$$\mu\{x : \nu W[\{x\}] > \alpha\} \le \mu\{x : \nu W_1[\{x\}] > \alpha_1\} + \mu\{x : \nu W_2[\{x\}] > \alpha_2\} \le \alpha_1 + \alpha_2 = \alpha,$$

and $\lambda W \le \alpha$. Thus λ is monotonic and subadditive.

If now $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\Sigma\widehat{\otimes}\mathrm{T}$ with empty intersection, and $\epsilon > 0$, set $E_n = \{x : \nu W_n[\{x\}] \ge \epsilon\}$ for each n. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ is non-increasing; moreover, for any $x \in X$, $\langle W_n[\{x\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T with empty intersection, so $\lim_{n\to\infty} \nu W_n[\{x\}] = 0$ and $x \notin \bigcap_{n\in\mathbb{N}} E_n$. There is therefore an n such that $\mu E_n \le \epsilon$ and $\lambda W_n \le \epsilon$. As $\langle W_n\rangle_{n\in\mathbb{N}}$ and ϵ are arbitrary, λ is a Maharam submeasure.

Finally, for $W \in \Sigma\widehat{\otimes}\mathrm{T}$,

$$\begin{aligned}
\lambda W = 0 &\iff \mu\{x : \nu W[\{x\}] \ge 2^{-n}\} \le 2^{-n} \text{ for every } n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] \ge 2^{-m}\} \le 2^{-n} \text{ for every } m,\, n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] > 0\} \le 2^{-n} \text{ for every } n \in \mathbb{N} \\
&\iff \mu\{x : \nu W[\{x\}] > 0\} = 0 \iff \{x : W[\{x\}] \notin \mathcal{J}\} \in \mathcal{I}.
\end{aligned}$$

(c) Put (b) together with Theorem 1I.

(d) If either $\mathfrak{A}$ or $\mathfrak{B}$ is $\{0\}$, this is trivial. Otherwise, we have a Boolean homomorphism $E \mapsto (E \times Y)^\bullet : \Sigma \to \mathfrak{C}$ with kernel $\mathcal{I}$, so there is a corresponding Boolean homomorphism $E^\bullet \mapsto (E \times Y)^\bullet : \mathfrak{A} \to \mathfrak{C}$. Similarly we have a Boolean homomorphism $F^\bullet \mapsto (X \times F)^\bullet : \mathfrak{B} \to \mathfrak{C}$. Accordingly we have a Boolean homomorphism $\phi : \mathfrak{A} \otimes \mathfrak{B} \to \mathfrak{C}$ defined by saying that

[1] that is, its order-sequential topology (1Jb).

$$\phi(E^\bullet\otimes F^\bullet)=(E\times Y)^\bullet\cap(X\times F)^\bullet=(E\times F)^\bullet$$

for $E\in\Sigma$ and $F\in\mathrm{T}$. Now ϕ is injective. **P** If $e\in\mathfrak{A}\otimes\mathfrak{B}$ is non-zero, there are $E\in\Sigma$, $F\in\mathrm{T}$ such that $0\ne E^\bullet\otimes F^\bullet\subseteq e$. In this case, $E\notin\mathcal{I}$ and $F\notin\mathcal{J}$ so $\lambda(E\times F)>0$ and

$$\phi e\supseteq\phi(E^\bullet\otimes F^\bullet)=(E\times F)^\bullet\ne 0.\ \mathbf{Q}$$

(e) $\bar\lambda(\phi e)=(\mu\ltimes\nu)(e)$ for every $e\in\mathfrak{A}\otimes\mathfrak{B}$. **P** Express e as $\sup_{i\in I}a_i\otimes b_i$ where $\langle a_i\rangle_{i\in I}$ is a finite partition of unity in $\mathfrak{A}$ and $b_i\in\mathfrak{B}$ for each i. For each i, we can express a_i, b_i as $E_i^\bullet$, $F_i^\bullet$ where $E_i\in\Sigma$ and $F_i\in\mathrm{T}$; moreover, we can do this in such a way that $\langle E_i\rangle_{i\in I}$ is a partition of X. In this case, $\phi e=W^\bullet$ where $W=\bigcup_{i\in I}E_i\times F_i$, so that, for $\epsilon>0$,

$$\mu\{x:\nu W[\{x\}]>\epsilon\}=\mu(\bigcup\{E_i:i\in I,\,\nu F_i>\epsilon\})=\bar\mu(\sup\{a_i:i\in I,\,\bar\nu b_i>\epsilon\}).$$

Accordingly

$$\begin{aligned}(\mu\ltimes\nu)(e)&=\inf\{\epsilon:\bar\mu(\sup\{a_i:i\in I,\,\bar\nu b_i>\epsilon\})\le\epsilon\}\\&=\inf\{\epsilon:\mu\{x:\nu W[\{x\}]>\epsilon\}\le\epsilon\}=\lambda W=\bar\lambda W^\bullet=\bar\lambda(\phi e).\ \mathbf{Q}\end{aligned}$$

Next, $\phi[\mathfrak{A}\otimes\mathfrak{B}]$ is dense in $\mathfrak{C}$ for the metric induced by $\bar\lambda$. **P** Let $\mathfrak{D}$ be the metric closure of $\phi[\mathfrak{A}\otimes\mathfrak{B}]$ and set $\mathcal{V}=\{V:V\in\Sigma\otimes\mathrm{T},\,V^\bullet\in\mathfrak{D}\}$. Then $\mathcal{V}$ includes $\Sigma\otimes\mathrm{T}$ and is closed under unions and intersections of monotonic sequences, so is the whole of $\Sigma\widehat{\otimes}\mathrm{T}$, and $\mathfrak{D}=\mathfrak{C}$, as required. **Q** But this means that we can identify $\mathfrak{C}$ with the metric completion of $\phi[\mathfrak{A}\otimes\mathfrak{B}]$ and with $\mathfrak{A}\widehat{\otimes}\mathfrak{B}$.

4C The robust σ-bounded-cc (a) Let $\mathfrak{A}$ be a Boolean algebra and μ a strictly positive submeasure on $\mathfrak{A}$. I will say that $(\mathfrak{A},\mu)$ is **robustly σ-bounded-cc** if $\mathfrak{A}^+$ can be expressed as $\bigcup_{n\in\mathbb{N}}A_n$ where for each $n\in\mathbb{N}$ there are $m\in\mathbb{N}$, $\delta>0$ such that whenever $a_0,\dots,a_m\in A_n$ then there are distinct i, $j<m$ such that $\mu(a_i\cap a_j)\ge\delta$.

(b) Observe that if $\mathfrak{A}$ is a Boolean algebra and μ is a strictly positive additive functional on $\mathfrak{A}$, then $(\mathfrak{A},\mu)$ is robustly σ-finite-cc. **P** Set $A_n=\{a:\mu a\ge\frac{\mu 1}{n+1}\}$ for each $n\in\mathbb{N}$. If $a_0,\dots,a_{n+1}\in A_n$, then

$$\tfrac{n+2}{n+1}\mu 1\le\sum_{i=0}^{n+1}\mu a_i\le\mu 1+\mu(\sup_{i<j\le n+1}a_i\cap a_j),$$

so there must be distinct i, $j\le n+1$ such that $\mu(a_i\cap a_j)\ge\frac{2\mu 1}{(n+2)(n+1)^2}$. **Q**

(c) If μ, ν are two strictly positive submeasures on $\mathfrak{A}$, each absolutely continuous with respect to the other, then $(\mathfrak{A},\mu)$ is robustly σ-bounded-cc iff $(\mathfrak{A},\nu)$ is.

4D Proposition Let $\mathfrak{A}$ be a σ-bounded-cc Maharam algebra, and μ a strictly positive Maharam submeasure on $\mathfrak{A}$. Then $(\mathfrak{A},\mu)$ is robustly σ-bounded-cc.

proof Let $\langle A_n\rangle_{n\in\mathbb{N}}$, $\langle m_n\rangle_{n\in\mathbb{N}}$ be such that $\mathfrak{A}^+=\bigcup_{n\in\mathbb{N}}A_n$ and no A_n includes any disjoint set of size greater than m_n. For $n\in\mathbb{N}$ set $A'_n=\bigcup\{[a,1]:a\in A_n\}$; then A'_n includes no disjoint set of size greater than m_n. For n, $k\in\mathbb{N}$ set

$$B_{nk}=\{a:a\in\mathfrak{A},\,a\setminus b\in A'_n\text{ whenever }\mu b\le 2^{-k}\}.$$

Then $\bigcup_{n,k\in\mathbb{N}}B_{nk}=\mathfrak{A}^+$. **P?** Otherwise, there is an $a\in\mathfrak{A}^+$ such that for every $n\in\mathbb{N}$ there is a b_n such that $\mu b_n\le 2^{-n-2}\mu a$ and $a\setminus b_n\notin A'_n$. Set $a'=a\setminus\sup_{n\in\mathbb{N}}b_n$; then $\mu a'>0$ but $a'\notin\bigcup_{n\in\mathbb{N}}A_n$. **XQ**

Set $\delta_{nk}=\frac{1}{2^k(m_n+1)}$ for m, $n\in\mathbb{N}$. If n, $k\in\mathbb{N}$ and $a_0,\dots,a_{m_n}\in B_{nk}$, then there are distinct i, $j\le m_n$ such that $\mu(a_i\cap a_j)\ge\delta_{nk}$. **P?** Otherwise, set $b_i=\sup_{j\le m_n,j\ne i}a_i\cap a_j$ for each i. Then $\mu b_i\le 2^{-k}$ so $a_i\setminus b_i\in A'_n$ for each $i\le m_n$. But $\langle a_i\setminus b_i\rangle_{i\le m_n}$ is disjoint. **XQ**

So $\langle B_{nk}\rangle_{n,k\in\mathbb{N}}$ witnesses that $(\mathfrak{A},\mu)$ is robustly σ-bounded-cc.

4E Proposition Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are σ-bounded-cc Maharam algebras. Then their Maharam algebra free product $\mathfrak{C}$ is σ-bounded-cc.

proof (a) We may suppose throughout that neither $\mathfrak{A}$ nor $\mathfrak{B}$ is $\{0\}$. Express $\mathfrak{A}$ and $\mathfrak{B}$ as quotients of (X,Σ,μ) and (Y,T,ν) as in 1Ib. Then we can identify $\mathfrak{C}$ with the quotient of $(X\times Y,\Sigma\otimes\mathrm{T},\lambda)$ where λ is defined as in Theorem 4B. Let $\langle A_n\rangle_{n\in\mathbb{N}}$, $\langle B_n\rangle_{n\in\mathbb{N}}$ witness that $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{B},\bar\nu)$ are robustly σ-bounded-cc (Proposition 4D); for each $n\in\mathbb{N}$ let m_n, $m'_n\in\mathbb{N}$ and δ_n, $\delta'_n>0$ be appropriate parameters as required in the definition 4Ca. Set

$$\mathcal{E}_n=\{E:E\in\Sigma,\ E^\bullet\in A_n\},\quad \mathcal{F}_n=\{F:F\in\mathrm{T},\ F^\bullet\in B_n\}$$

for each n. Then $\bigcup_{n\in\mathbb{N}}\mathcal{E}_n=\{E:\mu E>0\}$ and $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n=\{F:\nu F>0\}$.

(b) If $W\in\Sigma\widehat{\otimes}\mathrm{T}$ and $\lambda W>0$, there is an $F\in\mathrm{T}$ such that $\nu F>0$ and $\mu\{x:\nu(F\setminus W[\{x\}])<\eta\}>0$ for every $\eta>0$. **P** By Theorem 4Ba, $x\mapsto W[\{x\}]^\bullet$ is measurable and has separable range. Set $E=\{x:W[\{x\}]^\bullet\ne 0\}$, $D=\{W[\{x\}]^\bullet:x\in E\}$; then $\mu E>0$ and D is separable. As D is Lindelöf, there is a $b\in D$ such that $\mu\{x:W[\{x\}]^\bullet\in U\}>0$ for every open neighbourhood U of b. Take $F\in\mathrm{T}$ such that $F^\bullet=b$; then $\nu F>0$. If $\eta>0$, then $U=\{b':\bar\nu(b\setminus b')<\eta\}$ is a neighbourhood of b, so

$$\mu\{x:\nu(F\setminus W[\{x\}])<\eta\}=\mu\{x:W[\{x\}]^\bullet\in U\}>0.\ \mathbf{Q}$$

(c) For k, $l\in\mathbb{N}$, let $\mathcal{W}_{kl}$ be the set of those $W\in\Sigma\widehat{\otimes}\mathrm{T}$ for which there are $E\in\mathcal{E}_k$, $F\in\mathcal{F}_l$ such that $\nu(F\setminus W[\{x\}])\le\frac{1}{3}\delta'_l$ for every $x\in E$. By (b), every $W\in\Sigma\widehat{\otimes}\mathrm{T}$ such that $\lambda W>0$ belongs to W_{kl} for some k, l.

(d) Take k, $l\in\mathbb{N}$. Let $m\ge 1$ be so large that whenever $S\subseteq[m+1]^2$ there is either an $I\in[m+1]^{m_k+1}$ such that $[I]^2\subseteq S$ or a $J\in[m+1]^{m_l+1}$ such that $[J]^2\cap S=\emptyset$. Then if $W_0,\dots,W_m\in\mathcal{W}_{kl}$ there are distinct i, $j\le m$ such that $\lambda(W_i\cap W_j)>0$. **P** For each $i\le m$ choose $E_i\in\mathcal{E}_k$ and $F_i\in\mathcal{F}_l$ such that $\nu(F_i\setminus W_i[\{x\}])\le\frac{1}{3}\delta'_l$ for every $x\in E_i$. Consider $S=\{\{i,j\}:i<j\le m,\ \mu(E_i\cap E_j)<\delta_k\}$. If $I\subseteq m+1$ and $\#(I)=m_k+1$ there must be distinct i, $j\in I$ such that $\mu(E_i\cap E_j)\ge\delta_k$, so that $\{i,j\}\notin S$. Accordingly there is a set $J\subseteq m+1$ such that $\#(J)=m_l+1$ and $[J]^2\cap S=\emptyset$. Let i, j be distinct members of J such that $\nu(F_i\cap F_j)\ge\delta'_l$. Then

$$\nu(W_i\cap W_j)[\{x\}]=\nu(W_i[\{x\}]\cap W_j[\{x\}])\ge\nu(F_i\cap F_j)-\frac{2}{3}\delta'_l\ge\frac{1}{2}\delta'_l$$

for every $x\in E_i\cap E_j$. So

$$\lambda(W_i\cap W_j)\ge\min(\mu(E_i\cap E_j),\frac{1}{3}\delta'_l)>0.\ \mathbf{Q}$$

Accordingly, setting $C_{kl}=\{W^\bullet:W\in\mathcal{W}_{kl}\}$ for k, $l\in\mathbb{N}$, $\langle C_{kl}\rangle_{k,l\in\mathbb{N}}$ witnesses that $(\mathfrak{C},\bar\lambda)$ is σ-bounded-cc.

4F Definitions (Fremlin 08?, §527) Suppose that $\mathcal{I}\lhd\mathcal{P}X$ and $\mathcal{J}\lhd\mathcal{P}Y$ are ideals of subsets of sets X, Y respectively.

(a) I will write $\mathcal{I}\ltimes\mathcal{J}$ for their **skew product** $\{W:W\subseteq X\times Y,\ \{x:W[\{x\}]\notin\mathcal{J}\}\in\mathcal{I}\}$. and $\mathcal{I}\rtimes\mathcal{J}$ for $\{W:W\subseteq X\times Y,\ \{y:W^{-1}[\{y\}]\notin\mathcal{I}\}\in\mathcal{J}\}$; these are ideals of subsets of $X\times Y$.

(b) If Λ is a family of subsets of $X\times Y$, write $\mathcal{I}\ltimes_\Lambda\mathcal{J}$, $\mathcal{I}\rtimes_\Lambda\mathcal{J}$ for the ideals generated by $(\mathcal{I}\ltimes\mathcal{J})\cap\Lambda$, $(\mathcal{I}\rtimes\mathcal{J})\cap\Lambda$ respectively.

4G Proposition Let X be a set, Σ a σ-algebra of subsets of X, $\mu:\Sigma\to[0,\infty[$ a Maharam submeasure, and $\mathcal{N}(\mu)$ the null ideal of μ, that is, the ideal of subsets of X generated by $\mu^{-1}[\{0\}]$. Let μ_L be Lebesgue measure on $[0,1]$.

(a) If $\mathcal{N}(\mu)\ltimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)\subseteq\mathcal{N}(\mu)\rtimes\mathcal{N}(\mu_L)$ then μ is uniformly exhaustive.

(b) If μ is uniformly exhaustive then $\mathcal{N}(\mu)\ltimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)=\mathcal{N}(\mu)\rtimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)$.

proof (a) ? Otherwise, there are an $\epsilon>0$ and a family $\langle E_{ij}\rangle_{i\in\mathbb{N},j<2^i}$ in Σ such that $\mu E_{ij}\ge\epsilon$ for all i and j and $\langle E_{ij}\rangle_{j<2^i}$ is disjoint for each i. Let $\langle F_{ij}\rangle_{i\in\mathbb{N},j<2^i}$ be a family in Σ_L such that $\mu F_{ij}=2^{-i}$ for all i and j and $\bigcup_{j<2^i}F_{ij}=[0,1]$ for each i. Set

$$W=\bigcap_{k\in\mathbb{N}}\bigcup_{i\ge k,j<2^i}E_{ij}\times F_{ij}\in\Sigma\widehat{\otimes}\Sigma_L.$$

For any $x\in X$, set $K_x=\{i:x\in\bigcup_{j<2^n}E_{ij}\}$, and for $i\in K_x$ define $f(x,i)$ by saying that $x\in E_{i,f(x,i)}$; then

$$W[\{x\}]=\bigcap_{k\in\mathbb{N}}\bigcup_{i\in K_x\setminus k}F_{i,f(x,i)}\in\mathcal{N}(\mu_L)$$

because $\sum_{i\in K_x}\mu F_{i,f(x,i)}$ is finite. So $W\in\mathcal{N}(\mu)\ltimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)$. For any $t\in[0,1]$, $i\in\mathbb{N}$ choose $g(t,i)$ such that $t\in F_{i,g(t,i)}$; then

$$\mu W^{-1}[\{t\}]\ge\mu(\bigcap_{k\in\mathbb{N}}\bigcup_{i\ge k}E_{i,g(t,i)})=\inf_{k\in\mathbb{N}}\mu(\bigcup_{i\ge k}E_{i,g(t,i)})\ge\epsilon.$$

So $W\notin\mathcal{N}(\mu)\rtimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)$. **X**

(b) The quotient $\mathfrak{A}=\Sigma/\mathcal{N}(\mu)$ has a strictly positive uniformly exhaustive submeasure, so is a measurable algebra; there is therefore a totally finite measure ν with domain Σ and the same null ideal as μ. Now we can use Fubini's theorem to see that $\mathcal{N}(\nu)\ltimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)$ and $\mathcal{N}(\nu)\rtimes_{\Sigma\widehat{\otimes}\Sigma_L}\mathcal{N}(\mu_L)$ are both the null ideal of the product measure $\nu\times\mu_L$.

5 Forcing

5A Proposition Suppose that $\mathfrak{A}$ and $\mathfrak{C}$ are Boolean algebras such that
(i) $\mathfrak{A}$ is weakly (σ,∞)-distributive, has a strictly positive exhaustive submeasure and is not $\{0\}$;
(ii) $\Vdash_{\mathfrak{A}}$ '$\check{\mathfrak{C}}$ has a strictly positive exhaustive submeasure'.
Then $\mathfrak{C}$ has a strictly positive exhaustive submeasure.

proof Replacing $\mathfrak{A}$ by its completion, if necessary, we may suppose that $\mathfrak{A}$ is a Maharam algebra (Prop. 1E), with a strictly positive Maharam submeasure ν. Let $\dot\mu$ be an $\mathfrak{A}$-name for a strictly positive exhaustive submeasure on $\mathfrak{C}$. For $c\in\mathfrak{C}$, set

$$\lambda c=\inf\{\epsilon:\epsilon\in\mathbb{Q},\,\epsilon\ge 0,\,\nu([\![\dot\mu c>\epsilon]\!])\le\epsilon\}.$$

Now λ is a submeasure. **P** If $c=0$ then $[\![\dot\mu c>0]\!]=0$ and $\lambda c=0$. If $c\subseteq c'$ then $[\![\dot\mu c>\epsilon]\!]\subseteq[\![\dot\mu c'>\epsilon]\!]$ for every $\epsilon>0$ and $\lambda c\le\lambda c'$. If c, $c'\in\mathfrak{C}$ and $\delta>0$, there are ϵ, $\epsilon'\in\mathbb{Q}$ such that

$$\epsilon\le\lambda c+\delta,\quad \nu([\![\dot\mu c>\epsilon]\!])\le\epsilon,\quad \epsilon'\le\lambda c'+\delta,\quad \nu([\![\dot\mu c'>\epsilon']\!])\le\epsilon'.$$

Now if $a=1\setminus([\![\dot\mu c>\epsilon]\!]\cup[\![\dot\mu c'>\epsilon']\!])$,

$$a\Vdash \text{`}\dot\mu c\le\epsilon\,\&\,\dot\mu c'\le\epsilon'\text{'},$$

so $a\Vdash$ '$\dot\mu(c\cup c')\le\epsilon+\epsilon'$', that is,

$$[\![\dot\mu(c\cup c')>\epsilon+\epsilon']\!]\subseteq[\![\dot\mu c>\epsilon]\!]\cup[\![\dot\mu c'>\epsilon']\!];$$

consequently $\lambda(c\cup c')\le\lambda c+\lambda c'+2\delta$; as δ, c and c' are arbitrary, λ is subadditive. **Q**

λ is strictly positive. **P** If $c\in\mathfrak{C}^+$, then $1_{\mathfrak{A}}=[\![\dot\mu c>0]\!]=\sup_{n\in\mathbb{N}}[\![\dot\mu c>2^{-n}]\!]$, so there must be some $n\in\mathbb{N}$ such that $\nu([\![\dot\mu c>2^{-n}]\!])>2^{-n}$ and $\lambda c\ge 2^{-n}$. **Q**

Finally, λ is exhaustive. **P** Suppose that $\langle c_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{C}$ such that $\lambda c_n>\epsilon$ for every n, where $\epsilon>0$ is rational. Set $a_n=[\![\dot\mu c_n\ge\epsilon]\!]$; then $\nu a_n\ge\epsilon$ for every n. Set $a=\inf_{n\in\mathbb{N}}\sup_{m\ge n}a_m$; then $\nu a\ge\epsilon$ so $a\ne 0$. Now

$$a\Vdash \text{`for every } n\in\mathbb{N} \text{ there is an } m\ge n \text{ such that } \dot\mu c_m\ge\epsilon\text{'};$$

since $\dot\mu$ is a name for an exhaustive submeasure,

$$a\Vdash \text{`there are distinct } m,\,n\in\mathbb{N} \text{ such that } c_m\cap c_n\ne 0\text{'}.$$

So there are distinct m, $n\in\mathbb{N}$ and a non-zero $a'\subseteq a$ such that

$$a'\Vdash \text{`}c_m\cap c_n\ne 0\text{'}.$$

But since the objects c_m, c_n are in the ground model, $c_m\cap c_n\ne 0$ in the real world, and $\langle c_n\rangle_{n\in\mathbb{N}}$ is not disjoint. **Q**

5B Corollary Suppose that $\mathfrak{A}$ is a non-zero Maharam algebra and $\mathfrak{C}$ is a Dedekind complete Boolean algebra such that

$$\Vdash_{\mathfrak{A}} \text{`the Dedekind completion of } \check{\mathfrak{C}} \text{ is a Maharam algebra'}.$$

Then $\mathfrak{C}$ is a Maharam algebra.

proof Since

$$\Vdash_{\mathfrak{A}} \text{ ‘the Dedekind completion of } \check{\mathfrak{C}} \text{ has a strictly positive exhaustive submeasure’},$$

we surely have

$$\Vdash_{\mathfrak{A}} \text{ ‘}\check{\mathfrak{C}} \text{ has a strictly positive exhaustive submeasure’}.$$

By Proposition 5A, $\mathfrak{C}$ has a strictly positive exhaustive submeasure; in particular, it is ccc. Also $\mathfrak{A}$, being weakly (σ,∞)-distributive, is weakly σ-distributive, and

$$\Vdash_{\mathfrak{A}} \text{ ‘}\check{\mathfrak{C}} \text{ is weakly } \sigma\text{-distributive’},$$

so $\mathfrak{C}$ is weakly σ-distributive; as it is ccc, $\mathfrak{C}$ is weakly (σ,∞)-distributive. Now Proposition 1E tells us that $\mathfrak{C}$ is a Maharam algebra.

5C Pre-ordered sets (In the following paragraphs, all pre-ordered sets will be active upwards; that is to say, $p \le q$ will mean that q is stronger than p. In the language of FREMLIN 08?, this would be represented by adding the word ‘upwards’ to each definition.) Let P be a pre-ordered set (‘p.o.set’ in KUNEN 80), that is, a set with a reflexive transitive relation $\le$. I will say that P is ‘Maharam’, or ‘measurable’, or ‘chargeable’, or ‘weakly σ-distributive’, or ‘σ-finite-cc’, or ‘σ-bounded-cc’, or ‘weakly (σ,∞)-distributive’, if its regular open algebra is. The last three have reasonably simple translations:

P is σ-finite-cc iff it is expressible as $\bigcup_{n\in\mathbb{N}} A_n$ where no A_n includes any infinite antichain;

P is σ-bounded-cc iff it is expressible as $\bigcup_{n\in\mathbb{N}} A_n$ where no A_n includes any antichain with more than n members;

P is weakly (σ,∞)-distributive iff whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of maximal antichains in P, then there is a maximal antichain B such that $\{a : a \in A_n,\ a \text{ is compatible with } b\}$ is finite for every $n \in \mathbb{N}$.

Theorems 3C and 3G tell us that P is Maharam iff it is weakly (σ,∞)-distributive and σ-finite-cc, and that if Todorčević’s P-ideal dichotomy is true, then P is Maharam iff it is weakly (σ,∞)-distributive and ccc. I note that P is measurable iff it is weakly (σ,∞)-distributive and chargeable (FREMLIN 04, 391D). We can translate Kelley’s criterion (FREMLIN 04, 391J) as follows:

P is chargeable iff it is expressible as $\bigcup_{n\in\mathbb{N}} A_n$ where for every $n \in \mathbb{N}$ and every non-empty finite indexed family $\langle p_i\rangle_{i\in I}$ in A_n, there is a $J \subseteq I$ such that $\#(J) \ge 2^{-n}\#(I)$ and $\{p_i : i \in J\}$ has an upper bound in P.

Now we have the following result.

5D Theorem Let P be a pre-ordered set and $\dot{Q}$ a P-name for a pre-ordered set.

(a) If P is weakly σ-distributive and $\Vdash_P$ ‘$\dot{Q}$ is weakly σ-distributive’, then $P * \dot{Q}$ is weakly σ-distributive.

(b) (I.Farah) If P is Maharam and $\Vdash_P$ ‘$\dot{Q}$ is Maharam’, then $P * \dot{Q}$ is Maharam.

proof (a) The point is that P is weakly σ-distributive iff it is ω^ω-bounding, so we can use (for instance) Theorem 6.3.5 of BARTOSZYŃSKI & JUDAH 95.

(b) For $p \in P$, let $\widehat{p} = \operatorname{int}\overline{[p,\infty[}$ be the corresponding element of the regular open algebra $\mathrm{RO}(P)$. By Proposition 2H, we can express P as a union $\bigcup_{n\in\mathbb{N}} A_n$ where for any sequence $\langle p_j\rangle_{j\in\mathbb{N}}$ in any A_n there are a strictly increasing sequence $\langle k_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a $p \in P$ such that $\widehat{p} \subseteq \sup_{k_i\le j<k_{i+1}} \widehat{p}_j$ for every i. At the same time,

$$\Vdash_P \text{ ‘}\dot{Q} \text{ is } \sigma\text{-finite-cc’},$$

so there is a sequence $\langle \dot{B}_n\rangle_{n\in\mathbb{N}}$ of P-names for subsets of $\dot{Q}$ such that

$$\Vdash_P \text{ ‘}\textstyle\bigcup_{n\in\mathbb{N}} \dot{B}_n = \dot{Q} \text{ and there is no infinite antichain in } \dot{B}_n\text{’}$$

for every n. Set

$$C_{mn} = \{(p,\dot{q}) : p \in A_m,\ p \Vdash \text{‘}\dot{q} \in \dot{B}_n\text{’}$$

for m, $n \in \mathbb{N}$. Then $\bigcup_{m,n\in\mathbb{N}} C_{mn}$ is cofinal with $P * \dot{Q}$. Also no C_{mn} includes an infinite antichain. **P** Let $\langle (p_i,\dot{q}_i)\rangle_{i\in\mathbb{N}}$ be a sequence in C_{mn}. Because $p_i \in A_n$ for every i, we have a $p \in P$ and a strictly increasing sequence $\langle k_j\rangle_{j\in\mathbb{N}}$ such that $\widehat{p}\subseteq\sup_{k_j\le i<k_{j+1}}\widehat{p}_i$ for every j. We can therefore find maximal antichains A'_j, for $j\in\mathbb{N}$, such that if $p'\in A'_j$ either p' is incompatible with p or $p\le p'$ and there is an $i\in k_{j+1}\setminus k_j$ with $p_i\le p'$. Let $\dot q'_j$ be a P-name for a member of $\dot Q$ such that whenever $p'\in A'_j$ and $p\le p'$ there is an $i\in k_{j+1}\setminus k_j$ such that $p_i\le p'$ and

$$p'\Vdash `\dot q'_j=\dot q_i\text{'},$$

so that

$$p\Vdash `\dot q'_j\in\dot B_n\text{'}.$$

There must therefore be distinct j, j' such that

$$p\Vdash `\dot q'_j\text{ and }\dot q'_{j'}\text{ are compatible'}.$$

But now there must be a $p'\ge p$ and $i\in k_{j+1}\setminus k_j$, $i'\in k_{j'+1}\setminus k_{j'}$ such that $p\ge p_i$, $p'\ge p_{i'}$ and

$$p'\Vdash `\dot q'_j=\dot q_i\text{ and }\dot q'_{j'}=\dot q_{i'}\text{'};$$

in which case $i\ne i'$ and $(p_i,\dot q'_i)$ and $(p_{i'},\dot q'_{i'})$ are compatible. **Q**

So $P*\dot Q$ is σ-finite-cc; by Theorem 3C and (a) above, it is Maharam.

Remark Of course there is an alternative proof working with the regular open algebras RO(P) and RO($\dot Q$) and Maharam submeasures and using Proposition 1E.

5E The Tukey ordering If P and Q are pre-ordered sets, a function $\phi:P\to Q$ is a **Tukey function** if $\{p:f(p)\le q\}$ is bounded above in P for every $q\in Q$. If there is a Tukey function from P to Q, I write $P\preccurlyeq_{\mathrm T}Q$. (See Fremlin 08?, §513.)

5F Proposition Let P and Q be pre-ordered sets such that $P\preccurlyeq_{\mathrm T}Q$. If Q is chargeable, so is P.

proof Let $\phi:P\to Q$ be a Tukey function, and express Q as $\bigcup_{n\in\mathbb{N}}B_n$ where for every $n\in\mathbb{N}$ and every finite indexed family $\langle q_i\rangle_{i\in I}$ in B_n, there is a $J\subseteq I$ such that $\#(J)\ge 2^{-n}\#(I)$ and $\{q_i:i\in J\}$ has an upper bound in Q. Set $A_n=\phi^{-1}[B_n]$ for each n; then $P=\bigcup_{n\in\mathbb{N}}A_n$, and for every $n\in\mathbb{N}$ and every finite indexed family $\langle p_i\rangle_{i\in I}$ in A_n, there is a $J\subseteq I$ such that $\#(J)\ge 2^{-n}\#(I)$ and $\{\phi(p_i):i\in J\}$ has an upper bound in Q, so $\{p_i:i\in J\}$ has an upper bound in P.

6 Examples

6A Proposition (S.Todorčević) Let RO(X) be the regular open algebra of the space X described in Fremlin 04, 391N (‘Gaifman's example'; see Gaifman 64). Then RO(X) has the property (*) defined in 1Ad.

proof I recall the definition of X from Fremlin 04. Enumerate as $\langle I_n\rangle_{n\in\mathbb{N}}$ the set of half-open intervals $[q,q'[$ in $\mathbb{R}$ with q, $q'\in\mathbb{Q}$ and $q<q'$. For each $n\in\mathbb{N}$ let $\mathcal{J}_n$ be a disjoint family of non-trivial subintervals of I_n. Let X be the set of those $x\in\{0,1\}^{\mathbb{R}}$ such that for each n the set $\{J:J\in\mathcal{J}_n,\ x(t)=1\text{ for some }t\in J\}$ has at most $n+1$ members, with its compact Hausdorff zero-dimensional topology inherited from $\{0,1\}^{\mathbb{R}}$.

For each $n\in\mathbb{N}$ let $\mathcal{G}_n$ be the set of those regular open subsets G of X for which there are K, $L\in[\mathbb{R}]^{<\omega}$ such that (i) taking $\mathcal{E}_n$ to be the finite subalgebra of subsets of $\mathbb{R}$ generated by $\{I_i:i<n\}$, any two distinct points t, u of $K\cup L$ belong to different atoms of $\mathcal{E}_n$ (ii) $\{x:x\in X,\ x(t)=1\text{ for every }t\in K,\ x(t)=0\text{ for every }t\in L\}$ is non-empty and included in G. Then every non-empty regular open subset of X belongs to some $\mathcal{G}_n$. Now suppose that $n\in\mathbb{N}$ and we are given a sequence $\langle G_k\rangle_{k\in\mathbb{N}}$ in $\mathcal{G}_n$. For each $k\in\mathbb{N}$ let K_k, L_k be finite sets witnessing that $G_k\in\mathcal{G}_n$. Let $\langle k_r\rangle_{r\in\mathbb{N}}$ be a strictly increasing sequence such that

for every $r\in\mathbb{N}$ and $E\in\mathcal{E}_n$, $K_{k_r}\cap E\ne\emptyset$ iff $K_{k_0}\cap E\ne\emptyset$,

for every $r\in\mathbb{N}$ and $E\in\mathcal{E}_n$, $L_{k_r}\cap E\ne\emptyset$ iff $L_{k_0}\cap E\ne\emptyset$,

whenever $m\in\mathbb{N}$ and $r\ge\lfloor\frac{m}{n}\rfloor-1$ and $J\in\mathcal{J}_m$ then $K_{k_{r+1}}\cap J\ne\emptyset$ iff $K_{k_r}\cap J=\emptyset$.

(At each stage we have to choose k_r belonging to an infinite set belonging to a given finite partition of the previous infinite set.) Now set $x(t)=1$ if $t\in\bigcup_{r\in\mathbb{N}}K_{k_r}$, 0 otherwise. For $m\in\mathbb{N}$, $r\in\mathbb{N}$ set $\mathcal{J}_{mr}=\{J:J\in\mathcal{J}_m,\ J\cap K_{k_r}\ne\emptyset\}$; for $m\in\mathbb{N}$, set $\mathcal{J}'_m=\bigcup_{r\in\mathbb{N}}\mathcal{J}_{mr}$. Then $\mathcal{J}_{m,r+1}=\mathcal{J}_{mr}$ if $r\ge\lfloor\frac{m}{n}\rfloor-1$, while

$\#(\mathcal{J}_{mr}) \le \#(K_{k_r}) \le n$ for all m and r. In particular, if $m < 2n$, $\mathcal{J}_{mr} = \mathcal{J}_{m0}$ for every r; as $\{x : x(t) = 1$ for $t \in K_{k_0}\}$ meets X, $\#(\mathcal{J}'_m) \le m$ for such m. If $(l+1)n \le m < (l+2)n$, where $l \ge 1$, then $\mathcal{J}_{mr} = \mathcal{J}_{ml}$ for every $r \ge l$, so

$$\#(\mathcal{J}'_m) = \#(\bigcup_{r \le l} \mathcal{J}_{mr}) \le (l+1)n \le m.$$

What this means is that $\#(\{J : J \in \mathcal{J}_m,\ x(t) = 1 \text{ for some } t \in J\}) \le m$ for every $m \in \mathbb{N}$, and $x \in X$. I have still to confirm that $x \in G_{k_r}$ for every r. But, given r, then if $t \in K_{k_r}$ we certainly have $x(t) = 1$; while if $u \in L_{k_r}$ then there is an atom E of $\mathcal{E}_n$ containing u, E must contain a point of L_{k_0}, E cannot contain any point of K_{k_0} and therefore does not contain any point of $\bigcup_{s \in \mathbb{N}} K_{k_s}$, so $x(u) = 0$. Thus $x \in G_{k_r}$ for every r, and $\{G_{k_r} : r \in \mathbb{N}\}$ is centered in $\mathrm{RO}(X)$.

Remark Recall that $\mathrm{RO}(X)$ is σ-n-linked for every n (Fremlin 04, 391Yh); in particular, it is σ-bounded-cc.

6B Remark Głowczyński 91 presents the following example. Starting from a two-valued-measurable cardinal κ we can find a ccc forcing to give us a model in which $\kappa < \mathfrak{c} = \mathfrak{m}$. This gives us an ω_1-saturated σ-ideal $\mathcal{I}$ of $\mathcal{P}\kappa$ such that the quotient $\mathfrak{A} = \mathcal{P}\kappa/\mathcal{I}$ is ccc, Dedekind complete, weakly (σ,∞)-distributive, has Maharam type ω and is not a Maharam algebra. Since Martin's axiom is true, $\mathfrak{A}$ satisfies Knaster's condition; by Theorem 3B, or otherwise, it is not σ-finite-cc.

7 Rank functions for exhaustive submeasures

7A Definitions Suppose that $\mathfrak{A}$ is a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$. For $\epsilon > 0$, say that $a \prec_\epsilon b$ if $a \subseteq b$ and $\nu(b \setminus a) > \epsilon$. Then $\prec_\epsilon$ is a well-founded relation on $\mathfrak{A}$; for $a \in \mathfrak{A}$, write $r_\epsilon(a)$ for the height of the relation restricted to the principal ideal $\mathfrak{A}_a$ generated by a, that is, $r_\epsilon(a) = \sup_{b \prec_\epsilon a}(r_\epsilon(b) + 1)$.

7B Elementary facts Let $\mathfrak{A}$ is a Boolean algebra with an exhaustive submeasure ν and associated rank functions r_ϵ for $\epsilon > 0$.

(a)

$$r_\delta(a) \le r_\epsilon(b) \text{ whenever } \nu(a \setminus b) \le \delta - \epsilon.$$

P Induce on $r_\epsilon(b)$. If $r_\epsilon(b) = 0$, then $\nu b \le \epsilon$ so $\nu a \le \delta$ and $r_\delta(a) = 0$. For the inductive step to $r_\epsilon(b) = \xi$, if $c \subseteq a$ and $\nu(a \setminus c) > \delta$ then $\nu(b \setminus c) > \epsilon$ and $r_\epsilon(b \cap c) < \xi$. Also $\nu(c \setminus b) \le \delta - \epsilon$ so, by the inductive hypothesis, $r_\delta(c) \neq r_\delta(b \cap c) < \xi$; as c is arbitrary, $r_\delta(a) \le \xi$ and the induction continues. **Q** In particular,

$$r_\epsilon(a) \le r_\epsilon(b) \text{ if } a \subseteq b, \quad r_\delta(a) \le r_\epsilon(a) \text{ if } \epsilon \le \delta.$$

(b) For $a \in \mathfrak{A}$ let $T_\epsilon^{(a)}$ be the set of all decreasing strings $\tau = (a_0, a_1, \dots, a_n)$ where $a_0 = a$ and $\nu(a_i \setminus a_{i+1}) > \epsilon$ for $i < n$; for such τ, set $s_\epsilon(\tau) = r_\epsilon(a_n)$. Then $T_\epsilon^{(a)}$ is a tree with no infinite branches. If $\sigma \in T_\epsilon^{(a)}$ then

$$s_\epsilon(\sigma) = \sup\{s_\epsilon(\tau) + 1 : \tau \in T_\epsilon^{(a)} \text{ properly extends } \sigma\}$$

(induce on $s_\epsilon(\sigma)$).

(c) If a, $b \in \mathfrak{A}$ are disjoint and $\epsilon > 0$, then $r_\epsilon(a \cup b) \ge r_\epsilon(a) + r_\epsilon(b)$, the latter being the ordinal sum. **P** Induce on $r_\epsilon(b)$. If $r_\epsilon(b) = 0$, the result is immediate from (a) above. For the inductive step to $r_\epsilon(b) = \xi$, we have for any $\eta < \xi$ a $c \subseteq b$ such that $\nu(b \setminus c) > \epsilon$ and $\eta \le r_\epsilon(c) < \xi$. Now $r_\epsilon(a \cup c) \ge r_\epsilon(a) + \eta$, by the inductive hypothesis, and $\nu((a \cup b) \setminus (a \cup c)) > \epsilon$, so $r_\epsilon(a \cup b) > r_\epsilon(a) + \eta$; as η is arbitrary, $r_\epsilon(a \cup b) \ge r_\epsilon(a) + \xi$ and the induction continues. **Q**

(d) If ν' is another exhaustive submeasure on $\mathfrak{A}$ with rank functions r'_ϵ, and $\nu a \le \alpha \nu' a$ for every $a \in \mathfrak{A}$, where $\alpha > 0$, then $r_{\alpha\epsilon}(a) \ge r'_\epsilon(a)$ for every $a \in \mathfrak{A}$ and $\epsilon > 0$ (induce on $r'_\epsilon(a)$, as usual).

7C Proposition Let $\mathfrak{A}$ be a Boolean algebra with a strictly positive exhaustive submeasure ν, and $\widehat{\mathfrak{A}}$ the metric completion of $\mathfrak{A}$ under the metric $(a, b) \mapsto \nu(a \triangle b)$ (Fremlin 04, 393B), so that ν extends naturally to a Maharam submeasure $\hat{\nu}$ on $\widehat{\mathfrak{A}}$. For $\epsilon > 0$ let $r_\epsilon : \mathfrak{A} \to \mathrm{On}$ and $\hat{r}_\epsilon : \widehat{\mathfrak{A}} \to \mathrm{On}$ be the rank functions associated with ν and $\hat{\nu}$ respectively. Then whenever $a \in \mathfrak{A}$ and $0 < \epsilon < \delta$,

$$r_\delta(a) \le \hat r_\delta(a) \le r_\epsilon(a) \le \hat r_\epsilon(a).$$

proof (a) To see that $r_\epsilon(a) \le \hat r_\epsilon(a)$, induce on $\hat r_\epsilon(a)$. If $\hat r_\epsilon(a) = 0$ then $\nu a = \hat\nu a \le \epsilon$ and $r_\epsilon(a) = 0$. For the inductive step to $\hat r_\epsilon(a) = \xi$, if $b \in \mathfrak{A}$ and $b \subseteq a$ and $\nu(a \setminus b) > \epsilon$, then $\hat\nu(a \setminus b) > \epsilon$ so $\hat r_\epsilon(b) < \xi$; by the inductive hypothesis, $r_\epsilon(b) < \xi$; as b is arbitrary, $r_\epsilon(a) \le \xi$ and the induction proceeds. **Q** Similarly, $r_\delta(a) \le \hat r_\delta(a)$.

(b) For the middle inequality, let $T_\epsilon^{(a)} \subseteq \bigcup_{n\ge1} \mathfrak{A}^n$ and $\hat T_\delta^{(a)} \subseteq \bigcup_{n\ge1} \widehat{\mathfrak{A}}^n$ be the trees constructed by the method in §7B. For each $c \in \widehat{\mathfrak{A}}$ choose $a_i(c) \in \mathfrak{A}$, for $i \in \mathbb{N}$, such that $\hat\nu(c \triangle a_i(c)) \le 2^{-i-2}(\delta - \epsilon)$ (and $a_i(c) = c$ if $c \in \mathfrak{A}$). For $\tau = (c_0, \dots, c_n) \in \hat T_\delta^{(a)}$, set $\tau' = (b_0, \dots, b_n) \in \mathfrak{A}^{n+1}$ where $b_j = \inf_{i\le j} a_i(c_i)$ for each $j \le n$. Then $b_{j+1} \subseteq b_j$ for $j < n$; moreover, $b_0 = c_0 = a$ and

$$\hat\nu(b_j \triangle c_j) \le \sum_{i=0}^{j} \hat\nu(c_i \triangle a_i(c_i)) \le \frac{1}{2}(\delta - \epsilon)$$

for $j \le n$, so

$$\nu(b_j \setminus b_{j+1}) \ge \hat\nu(c_j \setminus c_{j+1}) - (\delta - \epsilon) > \epsilon$$

for $j < n$, and $\tau' \in T_\epsilon^{(a)}$. The construction ensures that if σ, $\tau \in \hat T_\delta^{(a)}$ and τ extends σ, then τ' extends σ'. It follows at once that, defining $s_\epsilon : T_\epsilon^{(a)} \to \mathrm{On}$ and $\hat s_\delta : \hat T_\delta^{(a)} \to \mathrm{On}$ as in §7A, $\hat s_\delta(\tau) \le s_\epsilon(\tau')$ for every $\tau \in \hat T_\delta^{(a)}$ (induce on $s_\epsilon(\tau')$, as usual). In particular,

$$\hat r_\delta(a) = \hat s_\delta(\langle a\rangle) \le s_\epsilon(\langle a\rangle) = r_\epsilon(a),$$

as required.

7D Corollary If, in §7A, we set $r^*_\epsilon(a) = \sup_{\delta>\epsilon} r_\delta(a)$ for $a \in \mathfrak{A}$ and $\epsilon \ge 0$, then we shall still have the results

$$r^*_\delta(a) \le r^*_\epsilon(b) \text{ whenever } \nu(a \setminus b) \le \delta - \epsilon,$$

$$r^*_\epsilon(a \cup b) \ge r^*_\epsilon(a) + r_\epsilon(b) \text{ whenever } a \cap b = 0,$$

and moreover, in the context of §7C, $r^*_\delta(a)$ is the same, for $a \in \mathfrak{A}$, whether calculated in $\mathfrak{A}$ or in the metric completion $\widehat{\mathfrak{A}}$.

7E The rank of a Maharam algebra Note that the rank function r_ϵ associated with an exhaustive submeasure ν depends only on the set $\{a : \nu a > \epsilon\}$. In particular, if μ and ν are exhaustive submeasures on a Boolean algebra $\mathfrak{A}$ and $\mu a \le \epsilon$ whenever $\nu a \le \delta$, then $r_\epsilon^{(\mu)}(a) \le r_\delta^{(\nu)}(a)$ for every $a \in \mathfrak{A}$. If $\mathfrak{A}$ is a Maharam algebra, then any two Maharam submeasures on $\mathfrak{A}$ are mutually absolutely continuous, so we get the same value for $r^*_0(1)$ from either; I will call this the **Maharam submeasure rank** of $\mathfrak{A}$, $\mathrm{Mhsm}(\mathfrak{A})$. Note that if $a \in \mathfrak{A}$ then $\mathrm{Mhsm}(\mathfrak{A}_a) \le \mathrm{Mhsm}(\mathfrak{A})$.

If $\mathfrak{A}$ is a measurable algebra, $\mathrm{Mhsm}(\mathfrak{A}) \le \omega$, because if μ is a unital additive functional and $\epsilon > 0$, then $r_\epsilon^{(\mu)}(1) < \frac{1}{\epsilon}$. More generally, for any uniformly exhaustive submeasure ν and any $\epsilon > 0$, $r_\epsilon^{(\nu)}(1)$ is the maximal size of any disjoint set consisting of elements of submeasure greater than ϵ.

7F Reductions of submeasures Let $\mathfrak{A}$ be a Boolean algebra, and $\nu : \mathfrak{A} \to [0, \infty[$ a submeasure.

(a) For $a \in \mathfrak{A}$, set

$$\check\nu a = \inf_{n\in\mathbb{N}} \sup\{\min_{i\le n} \nu a_i : a_0, \dots, a_n \subseteq a \text{ are disjoint}\}.$$

Then $\check\nu$ is a submeasure. **P** Of course $\check\nu 0 = 0$ and $\check\nu a \le \check\nu b$ whenever $a \subseteq b$. If a, $b \in \mathfrak{A}$ and $\epsilon > 0$, then there are n_0, $n_1 \in \mathbb{N}$ such that whenever $\langle c_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$, then $\#(\{i : \nu(c_i \cap a) \ge \check\nu a + \epsilon\}) \le n_0$ and $\#(\{i : \nu(c_i \cap b) \ge \check\nu b + \epsilon\}) \le n_1$. So

$$\#(\{i : \nu(c_i \cap (a \cup b)) \ge \check\nu a + \check\nu b + 2\epsilon\}) \le n_0 + n_1.$$

It follows that $\check\nu(a \cup b) \le \check\nu a + \check\nu b + 2\epsilon$; as ϵ, a and b are arbitrary, $\check\nu$ is a submeasure. **Q**

(b) Of course $\check\nu a \le \nu a$ for every $a \in \mathfrak{A}$; in particular, $\check\nu$ is exhaustive, or Maharam, if ν is. Observe that $\check\nu a = 0$ iff $\nu\restriction\mathfrak{A}_a$ is uniformly exhaustive. So if $\mathfrak{A}$ is a Maharam algebra which is nowhere measurable and ν is a strictly positive Maharam submeasure on $\mathfrak{A}$, then $\check\nu$ is also strictly positive.

(c) In this context I will call $\check\nu$ the **reduction** of ν.

7G Proposition (Fremlin & Kupka n90) Let $\mathfrak{A}$ be a Boolean algebra and ν an exhaustive submeasure on $\mathfrak{A}$ with reduction $\check\nu$. Let r_ϵ, $\check r_\epsilon$ be the associated rank functions. Then

$$r_\epsilon(a) \ge \omega \cdot \check r_\epsilon(a)$$

for every $a \in \mathfrak{A}$, $\epsilon > 0$.

proof Induce on $\check r_\epsilon(a)$. If $\check r_\epsilon(a) = 0$, the result is trivial. For the inductive step to $\check r_\epsilon(a) = \xi + 1$, take $b \subseteq a$ such that $\check\nu b > \epsilon$ and $\check r_\epsilon(a \setminus b) = \xi$. Then for every $n \in \mathbb{N}$ there are disjoint $b_0, \dots, b_n \subseteq b$ such that $\nu b_i > \epsilon$ for every i, and $r_\epsilon(b) \ge \omega$; by the inductive hypothesis, $r_\epsilon(a \setminus b) \ge \omega \cdot \xi$; by 7Bc, $r_\epsilon(a) \ge \omega\cdot\xi + \omega = \omega\cdot(\xi+1)$, and the induction proceeds. The inductive step to non-zero limit ξ is elementary.

7H Theorem (J.Kupka) Let ν be a pathological submeasure on a Boolean algebra $\mathfrak{A}$, with reduction $\check\nu$. Then $\check\nu a \ge \frac{1}{3}\nu a$ for every $a \in \mathfrak{A}$.

proof (a) Since $\nu\restriction\mathfrak{A}_a$ is also a pathological submeasure, and $\check\nu\restriction\mathfrak{A}_a$ is the reduction of $\nu\restriction\mathfrak{A}_a$, it is enough to consider the case $a = 1$; and since the operation of reduction commutes with scalar multiplication of the submeasures, it is enough to consider the case $\nu 1 = 1$.

(b) ? Suppose, if possible, that $\check\nu 1 < \frac{1}{3}$. Take γ such that $\check\nu 1 < \gamma < \frac{1}{3}$. Let $n \ge 1$ be such that there is no disjoint family $\langle a_i\rangle_{i\le n}$ in $\mathfrak{A}$ with $\nu a_i \ge \gamma$ for every $i \le n$. Then we see that

$$\textstyle\sum_{i\in I} \nu(a_i) \le n + \gamma\#(I)$$

for every disjoint family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$.

Set

$$\epsilon = \frac{1-3\gamma}{n+3} > 0, \quad \delta = \min(\frac{\epsilon^2}{18}, \frac{\epsilon}{n}) > 0.$$

By 1G, there is a non-empty finite family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that $\nu b_i \le \delta$ for every $i \in I$ and $\sup_{i\in J} b_i = 1$ whenever $J \subseteq I$ and $\#(J) \ge \delta\#(I)$. Note that we can repeat copies of $\langle b_i\rangle_{i\in I}$ if necessary, so that we can assume that $\#(I) = m$ is at least $\frac{3}{\delta}$. We must have $\sup_{i\in I} b_i = 1$ so

$$\frac{m\epsilon}{n} \ge m\delta \ge \textstyle\sum_{i\in I} \nu b_i \ge 1$$

and $n \le \epsilon m$.

Set $l = \lceil \epsilon m\rceil$, $k = \lfloor \delta m\rfloor$. Then

$$3 \le k \le l \le m, \quad 18km \le \epsilon^2 m^2 \le l^2,$$

so there is an $R \subseteq I \times l$ such that $\#(R) = 3m$ (in fact, $\#(R[\{i\}]) = 3$ for every $i \in I$) and $\#(R[E]) \ge \#(E)$ for every $E \in [I]^{\le k}$ (Kalton & Roberts 83, or Fremlin 04, 392D). For $E \subseteq I$ set

$$c_E = \inf_{i\in E}(1 \setminus b_i) \cap \inf_{i\in I\setminus E} b_i;$$

observe that $c_E = 0$ when $\#(E) > k$, so that

$$\sup\{c_E : E \in [I]^{\le k}\} = 1.$$

For $E \in [I]^{\le k}$ take an injective function $f_E : E \to l$ such that $f_E \subseteq R$. Set

$$a_{ij} = \sup\{c_E : i \in E \in [I]^{\le k},\ f_E(i) = j\}$$

for $i \in I$, $j < l$. Then, for any particular $j < l$, $\langle a_{ij}\rangle_{i\in I}$ is disjoint (because every f_E is injective), so

$$\textstyle\sum_{i\in I} \nu a_{ij} \le n + \gamma\#(\{i : a_{ij} \ne 0\}) \le n + \gamma\#(R^{-1}[\{j\}]).$$

Accordingly

$$\sum_{i\in I,j<l}\nu(a_{ij})\le nl+\gamma\sum_{j<l}\#(R^{-1}[\{j\}])\le n(\epsilon m+1)+\gamma\#(R)$$
$$\le n\epsilon m+\epsilon m+3\gamma m=m(3\gamma+(n+1)\epsilon).$$

On the other hand, for each $i\in I$,

$$1\setminus b_i=\sup\{c_E:i\in E\subseteq I\}$$
$$=\sup\{c_E:i\in E\in[I]^{\le k}\}=\sup_{j<l}a_{ij},$$

so

$$1=\nu 1\le\nu(b_i)+\nu(1\setminus b_i)\le\delta+\textstyle\sum_{j<l}\nu a_{ij}.$$

Now, summing over $i\in I$,

$$m\le m\delta+\sum_{i\in I,j<l}\nu a_{ij}\le m(\delta+3\gamma+(n+1)\epsilon)$$
$$\le m(3\gamma+(n+2)\epsilon)<m(3\gamma+1-3\gamma)=m,$$

which is impossible. **X**

So we have the result.

Remark Of course this result includes the Kalton-Roberts theorem, since it shows that no uniformly exhaustive submeasure can be pathological.

7J Theorem Suppose that $\mathfrak{A}$ is a non-measurable Maharam algebra. Then $\mathrm{Mhsm}(\mathfrak{A})$ is at least the ordinal power ω^ω.

proof Let $a\in\mathfrak{A}^+$ be such that the principal ideal $\mathfrak{A}_a$ is nowhere measurable. Let ν be a strictly positive Maharam submeasure on $\mathfrak{A}_a$, $\check\nu$ its reduction, and r_ϵ, $\check r_\epsilon$ the associated rank functions. As observed in 7Fb, $\check\nu$ is strictly positive. If

$$\alpha<\mathrm{Mhsm}(\mathfrak{A}_a)=\sup\nolimits_{\epsilon>0}r_\epsilon(a)=\sup\nolimits_{\epsilon>0}\check r_\epsilon(a)$$

(as noted in 7E), then there is an $\epsilon>0$ such that $\check r_\epsilon(a)\ge\alpha$, in which case

$$\mathrm{Mhsm}(\mathfrak{A}_a)\ge r_\epsilon(a)\ge\omega\cdot\check r_\epsilon(a)\ge\omega\cdot\alpha$$

by 7G. Since $\mathrm{Mhsm}(\mathfrak{A}_a)$ is surely infinite, $\mathrm{Mhsm}(\mathfrak{A})\ge\mathrm{Mhsm}(\mathfrak{A}_a)\ge\omega^n$ for every n, and $\mathrm{Mhsm}(\mathfrak{A})\ge\omega^\omega$.

7K Proposition Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras with exhaustive submeasures μ, ν respectively, and that $\lambda=\mu\ltimes\nu$ as constructed in §4. Then $r_\epsilon(a\otimes b)$ is at least the ordinal product $r_\epsilon(b)\cdot r_\epsilon(a)$ for all $a\in\mathfrak{A}$, $b\in\mathfrak{B}$ and $\epsilon>0$.

proof (a) I show first that if $\mu a>\epsilon$ then $r_\epsilon(a\otimes b)\ge r_\epsilon(b)$. **P** Induce on $r_\epsilon(b)$. If $r_\epsilon(b)=0$, the result is trivial. For the inductive step to $r_\epsilon(b)=\xi>0$, for every $\eta<\xi$ there is a $b'\subseteq b$ such that $r_\epsilon(b')\ge\eta$ and $\nu(b\setminus b')>\epsilon$; now $r_\epsilon(a\otimes b')\ge\eta$, by the inductive hypothesis, and $\lambda(a\otimes(b\setminus b'))=\min(\mu a,\nu(b\setminus b'))>\epsilon$, so $r_\epsilon(a\otimes b)>\eta$; as η is arbitrary, $r_\epsilon(a\otimes b)\ge\xi$ and the induction proceeds. **Q**

(b) Now induce on $r_\epsilon(a)$. If $r_\epsilon(a)=0$ the result is trivial. For the inductive step to $r_\epsilon(a)=\xi>1$, observe that for every $\eta<\xi$ there is an $a'\subseteq a$ such that $r_\epsilon(a')\ge\eta$ and $\mu(a\setminus a')>\epsilon$. Now

$$r_\epsilon(a\otimes b)\ge r_\epsilon(a'\otimes b)+r_\epsilon((a\setminus a')\otimes b)$$

(7Bc)

$$\ge r_\epsilon(b)\cdot\eta+r_\epsilon(b)$$

(by the inductive hypothesis and (a) above)

$$=r_\epsilon(b)\cdot(\eta+1);$$

as η is arbitrary, $r_\epsilon(a\otimes b)\ge r_\epsilon(b)\cdot\xi$ and the induction continues.

8 Strategically weakly (σ,∞)-distributive algebras

8A Definitions Let $\mathfrak{A}$ be a Boolean algebra.

(a) Consider the following infinite game $\Gamma_{\text{wd}}(\mathfrak{A})$. (This is called '$\mathcal{G}_{\text{fin}}$' in Jech 84 and '$\mathcal{G}^{\omega}_{<\omega}$' in Dobrinen 03; see also Grey 82.) I plays $a_0\in\mathfrak{A}^+$ and a maximal antichain $A_0\subseteq\mathfrak{A}$. In the position $(a_0,A_0,\dots,a_n,A_n)$, II plays a non-zero $a_{n+1}\subseteq a_n$ meeting only finitely many members of A_n. In the position $(a_0,A_0,\dots,a_n,A_n,a_{n+1})$, I plays a maximal antichain A_{n+1}. I wins if $\inf_{n\in\mathbb{N}}a_n=0$; otherwise II wins. (If $\mathfrak{A}=\{0\}$, so that I has no first move, II wins.)

$\mathfrak{A}$ is **strategically weakly** (σ,∞)**-distributive** if II has a winning strategy in $\Gamma_{\text{wd}}(\mathfrak{A})$; $\mathfrak{A}$ is **tactically weakly** (σ,∞)**-distributive** if II has a winning tactic, that is, a winning strategy σ such that $\sigma(a_0,A_0,\dots,a_n,A_n)=\tau(a_n,A_n)$ for some function τ.

(b) A variant of the above game is $\Gamma^*_{\text{wd}}(\mathfrak{A})$, defined as follows. This time, I starts with an antichain $A_0\subseteq\mathfrak{A}$. In the position $(A_0,a_0,A_1,\dots,a_{n-1},A_n)$, II plays a_n meeting only finitely many members of A_n. In the position $(A_0,\dots,A_n,a_n)$, I plays an antichain A_{n+1}. II wins if $\langle a_n\rangle_{n\in\mathbb{N}}$ is order*-convergent to 1; otherwise I wins. $\mathfrak{A}$ is **strongly strategically weakly** (σ,∞)**-distributive** if II has a winning strategy in $\Gamma^*_{\text{wd}}(\mathfrak{A})$.

T.Jech has suggested the following variant of $\Gamma^*_{\text{wd}}(\mathfrak{A})$. In this game, I plays sequences order*-convergent to 0, and II must choose a term in each sequence as it appears; II wins if the sequence of his choices is again order*-convergent to 0. It is easy to see that for ccc algebras this game is equivalent to $\Gamma^*_{\text{wd}}(\mathfrak{A})$, in the sense that a winning strategy for either player in one game can be used to generate a winning strategy for the same player in the other game.

8B Proposition (a) A tactically weakly (σ,∞)-distributive Boolean algebra is strategically weakly (σ,∞)-distributive. A strategically weakly (σ,∞)-distributive Boolean algebra is weakly (σ,∞)-distributive. A strongly strategically weakly (σ,∞)-distributive Boolean algebra is strategically weakly (σ,∞)-distributive.

(b)(Jech 84) If $\mathfrak{A}$ is a ccc Boolean algebra, then $\mathfrak{A}$ is weakly (σ,∞)-distributive iff I has no winning strategy in $\Gamma_{\text{wd}}(\mathfrak{A})$ iff I has no winning tactic in $\Gamma_{\text{wd}}(\mathfrak{A})$.

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}$ an order-dense subalgebra of $\mathfrak{A}$. Then $\mathfrak{B}$ is strategically (resp. tactically, resp. strongly strategically) weakly (σ,∞)-distributive iff $\mathfrak{A}$ is.

(d) A principal ideal of a strategically (resp. tactically, resp. strongly strategically) weakly (σ,∞)-distributive Boolean algebra is again strategically (resp. tactically) weakly (σ,∞)-distributive.

(e) A regularly embedded subalgebra of a strategically (resp. tactically, resp. strongly strategically) weakly (σ,∞)-distributive Boolean algebra is again strategically (resp. tactically) weakly (σ,∞)-distributive.

proof (a) Trivial.

(b)(i) If $\mathfrak{A}$ is not weakly (σ,∞)-distributive, then I has a winning tactic. **P** There are a non-zero $a_0\in\mathfrak{A}$ and a sequence $\langle C_n\rangle_{n\in\mathbb{N}}$ of maximal antichains such that $\inf_{n\in\mathbb{N}}a_n=0$ whenever each a_n, for $n\ge1$, meets only finitely many elements of C_{n-1}. We may suppose that C_{n+1} refines C_n for each n. I starts with a_0. Given a_n, I plays $A_n=C_k$ where $k\in\mathbb{N}$ is minimal such that $\{c:c\in C_k,\,a_n\cap c\ne0\}$ is infinite; this must be possible if $a_n\subseteq a_0$ is non-zero. In any play of the game, we must have A_n refining C_n for each n, so I wins. **Q**

(ii) If I has a winning strategy, and $\mathfrak{A}$ is ccc, then $\mathfrak{A}$ is not weakly (σ,∞)-distributive. **P** Consider all the plays in $\Gamma_{\text{wd}}(\mathfrak{A})$ in which I follows his strategy and II always plays $a_{n+1}=a_n\cap\sup I_n$ for some finite $I_n\subseteq A_n$. There are only countably many such plays; let $\mathcal{C}$ be the countable set of maximal antichains occurring in any of them. If $J_C\in[C]^{<\omega}$ for each $C\in\mathcal{C}$, consider the play in which I follows his strategy and II plays $a_{n+1}=a_n\cap\sup J_{A_n}$ at each move. Then

$$0=\inf_{n\in\mathbb{N}}a_n\supseteq\inf_{C\in\mathcal{C}}a_0\cap\sup J_C;$$

as $\langle J_C\rangle_{C\in\mathcal{C}}$ is arbitrary, a_0 and $\mathcal{C}$ witness that $\mathfrak{A}$ is not weakly (σ,∞)-distributive. **Q**

(c)(i) Suppose that $\mathfrak{A}$ is strategically weakly (σ,∞)-distributive. Let σ be a winning strategy for II in $\Gamma_{\text{wd}}(\mathfrak{A})$. Then there is a winning strategy σ' for II in $\Gamma_{\text{wd}}(\mathfrak{A})$ such that $\sigma'(a_0,A_0,\dots,a_n,A_n)$ always belongs to the subalgebra generated by $A_0 \cup \ldots \cup A_n \cup \{a_0, \ldots, a_n\}$; just enlarge values of σ slightly if necessary. Now apply σ' directly to positions in $\Gamma_{\mathrm{wd}}(\mathfrak{B})$ to get a winning strategy in $\Gamma_{\mathrm{wd}}(\mathfrak{B})$.

(ii) Similarly, if σ is a winning strategy for II in $\Gamma_{\mathrm{wd}}(\mathfrak{B})$, then for each maximal antichain $A \subseteq \mathfrak{A}$ let $A' \subseteq \mathfrak{B}$ be a maximal antichain refining A, for each $a \in \mathfrak{A}^+$ let $a' \in \mathfrak{B}^+$ be such that $a' \subseteq a$, and set $\sigma'(a_0, A_0, \ldots, a_n, A_n) = \sigma(a_0', A_0', a_1, A_1', \ldots, a_n, A_n')$ whenever $a_1, \ldots, a_n$ all belong to $\mathfrak{B}$.

(iii) The same tricks work for tactically weakly (σ, ∞)-distributive and strongly strategically weakly (σ, ∞)-distributive algebras.

(d) Elementary.

(e) Use the argument of (c-i) above.

8C Proposition A Maharam algebra is tactically weakly (σ, ∞)-distributive and strongly strategically weakly (σ, ∞)-distributive.

proof Let ν be a strictly positive Maharam submeasure on $\mathfrak{A}$.

(a) Given $a \in \mathfrak{A}^+$ and a maximal antichain $A \subseteq \mathfrak{A}$, choose $\tau(a, A)$ such that $0 \neq \tau(a, A) \subseteq a$, $\tau(a, A)$ meets only finitely many members of A and $\nu(\tau(a, A)) > \frac{1}{n}$ where n is the least integer greater than $\frac{1}{\nu a}$. Then τ is a winning tactic for II in $\Gamma_{\mathrm{wd}}(\mathfrak{A})$.

(b) Given a position $(A_0, a_0, \ldots, A_n)$ in $\Gamma^*_{\mathrm{wd}}(\mathfrak{A})$, let $\sigma(A_0, a_0, \ldots, A_n)$ be an element c of $\mathfrak{A}$ such that $\{a : a \in A_n,\ a \cap c \neq 0\}$ is finite and $\nu(1 \setminus c) \leq 2^{-n}$. Then σ is a winning strategy for II in $\Gamma^*_{\mathrm{wd}}(\mathfrak{A})$.

Remark Note that in (b) the strategy for II is defined from n and A_n; so with a trifling adaptation (except in the trivial case of finite $\mathfrak{A}$, take a_n such that $\nu(1 \setminus a_n) \leq \frac{1}{2}\nu(1 \setminus a_{n-1})$) can be defined from a_{n-1} and A_n.

8D Proposition (Dobrinen 03) If Jensen's $\diamondsuit$ is true, there is a Souslin algebra which is not strategically weakly (σ, ∞)-distributive.

proof I use the construction of a Souslin tree $(\omega_1, \triangleleft)$ in Kunen 80, II.7.8. Start from a $\diamondsuit$-sequence $\langle A_\alpha \rangle_{\alpha < \omega_1}$. Set $I_\beta = \{(\omega \cdot \beta) + n : n \in \mathbb{N}\}$ for $\beta < \omega_1$. The new element in the construction is a bijection $h : \omega_1 \to [\omega_1]^{<\omega} \times [\mathbb{N}]^{<\omega}$. Let C be the set of those non-zero limit ordinals $\gamma < \omega_1$ such that $f_\gamma = h[\gamma]$ is a function from $[\gamma]^{<\omega}$ to $[\mathbb{N}]^{<\omega}$; then C is a closed cofinal subset of ω_1. **P** Of course C is closed, because the union of a non-decreasing sequence of functions is a function. To see that it is unbounded, note that if $f : [\omega_1]^{<\omega} \to [\mathbb{N}]^{<\omega}$ is any function then $f = h[A]$ for some $A \subseteq \omega_1$ and that $\{\gamma : f \restriction [\gamma]^{<\omega} = h[A \cap \gamma]\} \subseteq C$ is uncountable. **Q**

Let $C' \subseteq C$ be the set of those members of C which are the suprema of strictly increasing sequences of limit ordinals; for $\gamma \in C'$ choose a such a sequence $\langle \theta_{\gamma n} \rangle_{n \in \mathbb{N}}$ of limit ordinals with supremum γ. Set $K_{\gamma n} = \{\theta_{\gamma i} : i \leq n\}$, $L_{\gamma n} = \{\omega \cdot \theta_{\gamma n} + i : i \in f_\gamma(K_{\gamma n})\}$ for $n \in \mathbb{N}$. Now construct $\triangleleft$ inductively so that

(i) for each $\beta < \omega_1$, $\triangleleft_\beta = \triangleleft \cap (\omega \cdot \beta \times \omega \cdot \beta)$ is a tree ordering on $\omega \cdot \beta$;

(ii) for $\beta < \omega_1$ and $n \in \mathbb{N}$, $\omega \cdot \beta + n \triangleleft \omega \cdot (\beta + 1) + m$ iff $\lfloor m/2 \rfloor = n$;

(iii) if $\alpha < \omega_1$ is a limit ordinal and $\xi \in \omega \cdot \alpha$ then there is an $n \in \mathbb{N}$ such that $\xi \triangleleft \omega \cdot \alpha + n$;

(iv) if $\alpha < \omega_1$ is a limit ordinal and A_α is a maximal up-antichain for $\triangleleft_\alpha$ then for every $n \in \mathbb{N}$ there is a $\xi \in A_\alpha$ such that $\xi \triangleleft \omega \cdot \alpha + n$;

(v) (the new bit) if $\gamma \in C'$ and $\eta \in I_\gamma$ then there is an $n \in \mathbb{N}$ such that $\xi \ntriangleleft \eta$ for any $\xi \in L_{\gamma n}$.

To see that there is no obstacle to (v), note that when we come to $\gamma \in C'$, and need to choose a $\triangleleft_\gamma$-branch passing through a given $\xi < \omega \cdot \gamma$ to have a continuation, we first move to $\xi_1 \triangleright_\gamma \xi$ such that (if A_γ is a maximal up-antichain for $\triangleleft_\gamma$) there is a $\zeta \in A_\gamma$ such that $\zeta \triangleleft_\gamma \xi_1$. Next, taking m such that $\xi_1 \leq \omega \cdot \theta_{\gamma m}$, there must be infinitely many members of $I_{\theta_{\gamma, m+1}}$ above ξ_1, so we can find $\xi_2 \in I_{\theta_{\gamma, m+1}} \setminus L_{\gamma, m+1}$; assign any branch through ξ_2 for continuation.

As in Kunen 80, this process builds an ever-branching Souslin tree. Let $\mathfrak{A}$ be the corresponding regular open algebra (Fremlin 08?, §514), so that $\mathfrak{A}$ is ccc and weakly (σ, ∞)-distributive. Let σ be a strategy for II in $\Gamma_{\mathrm{wd}}(\mathfrak{A})$. For $\alpha < \omega_1$, let $D_\alpha \subseteq \mathfrak{A}$ be the maximal antichain $\{[\xi, \infty[: \xi \in I_\alpha\}$. (Because our tree is ever-branching, all the sets $[\xi, \infty[$ are regular open sets for the up-topology.) Define $f : [\omega_1]^{<\omega} \to [\mathbb{N}]^{<\omega}$ as follows.

$f(\emptyset)=\emptyset$. Given that $K\subseteq\omega_1$ is a non-empty finite set, express it as $\{\alpha_0,\dots,\alpha_m\}$ where $\alpha_0<\alpha_1<\ldots<\alpha_m$. Set $a_0=1$ and $a_{j+1}=\sigma(a_0,D_{\alpha_0},\dots,a_j,D_{\alpha_j})$ for $j\le m$. Set $f(K)=\{i:a_{m+1}\cap[\omega\cdot\alpha_m+i,\infty[\,\ne 0\}$.

Let $A\subseteq\omega_1$ be such that $f=h[A]$. Because $\langle A_\alpha\rangle_{\alpha<\omega_1}$ is a $\diamondsuit$-sequence and C' is a closed cofinal subset of ω_1, there is a $\gamma\in C'$ such that $A\cap\gamma=A_\gamma$. Now consider the play of the game $\Gamma_{\text{wd}}(\mathfrak{A})$ in which I plays $(1,D_{\theta_{\gamma 0}})$ for his first move and $D_{\theta_{\gamma 1}}$, $D_{\theta_{\gamma 2}}$, $\ldots$ thereafter; let a_1, $a_2\ldots$ be the responses of II following his strategy σ. Then $f_\gamma(K_{\gamma n})=f(K_{\gamma n})$, so $a_{n+1}\cap[\xi,\infty[\,=0$ whenever $\xi\in I_{\theta_{\gamma n}}\setminus L_{\gamma n}$. But the construction of $\triangleleft_{\gamma+1}$ ensured that for every $\eta\in I_\gamma$ there must be some n such that the predecessor of η in $I_{\theta_{\gamma n}}$ does not belong to $L_{\gamma n}$ and $a_{n+1}\cap[\eta,\infty[\,=0$. So $\inf_{n\in\mathbb{N}}a_n=0$, I wins and σ is not a winning strategy.

Thus $\mathfrak{A}$ is not strategically weakly (σ,∞)-distributive.

Remark See Problem 9K.

8E Example (Jech) Let $S\subseteq\omega_1$ be a stationary set such that $\omega_1\setminus S$ is also stationary, and let P be the set of subsets of S which are closed in the order topology of ω_1, ordered by end-extension (that is, for p, $q\in P$, $p\le q$ iff $p=q\cap\xi$ for some $\xi<\omega_1$). Let $\mathfrak{A}$ be the regular open algebra of P. Then $\mathfrak{A}$ is weakly (σ,∞)-distributive but not strategically weakly (σ,∞)-distributive.

proof (a) ? If $\mathfrak{A}$ is strategically weakly (σ,∞)-distributive then player II has a winning strategy in $\Gamma_{\text{wd}}(\mathfrak{A})$. For each $\alpha<\omega_1$, let Q_α be the cofinal subset $\{p:p\in P,\ \sup p\ge\alpha\}$ of P, and fix a maximal antichain $C_\alpha\subseteq Q_\alpha$; then $A_\alpha=\{[p,\infty[\,:p\in C_\alpha\}$ is a maximal antichain in $\mathfrak{A}$. (The partial order on P is separative, so $A_\alpha\subseteq\mathfrak{A}$.) Consider plays of the game $\Gamma_{\text{wd}}(\mathfrak{A})$ in which I starts with $a_0=P$ and plays only antichains of the form A_α, while II follows his strategy. For each such play $(P,A_{\alpha_0},a_1,A_{\alpha_1},\dots)$, set $D_n=\{p:p\in A_{\alpha_n}$, $a_{n+1}\cap[p,\infty[\,\ne\emptyset\}$ and $\gamma_n=\sup_{p\in D_n}\sup p$; note that γ_n is determined by $\alpha_0,\dots,\alpha_n$. So the set

$$Q=\{\gamma:\gamma_n(\alpha_0,\dots,\alpha_n)<\gamma\text{ whenever }n\in\mathbb{N}\text{ and }\alpha_0,\dots,\alpha_n<\gamma\}$$

is a closed cofinal set in ω_1 and there is a non-zero limit ordinal $\alpha\in Q\setminus S$. Let $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum α and consider the corresponding play of $\Gamma_{\text{wd}}(\mathfrak{A})$. For the corresponding sequence $\langle D_n\rangle_{n\in\mathbb{N}}$, we have $\alpha_n\le\sup p<\alpha$ for every $n\in\mathbb{N}$, $p\in D_n$. But now we are supposed to have a non-zero $a\in\mathfrak{A}$ such that $a\subseteq\overline{\bigcup_{p\in D_n}[p,\infty[}$ for every $n\in\mathbb{N}$. If $p^*\in P$ is such that $[p^*,\infty[\,\subseteq a$, then for each $n\in\mathbb{N}$ there is a $p\in D_n$ such that p^* and p are compatible in P, that is, one is included in the other. As every extension of p^* is compatible with some member of D_n, we cannot have $p^*\subset p$, and instead we have $p\subseteq p^*$, so that p^* meets $\alpha\setminus\alpha_n$. As p^* is closed, $\alpha\in p^*$; but p^* is supposed to be a subset of S. **X**

(b) ? If $\mathfrak{A}$ is not weakly (σ,∞)-distributive then player I has a winning strategy in $\Gamma_{\text{wd}}(\mathfrak{A})$. Let $\preccurlyeq$ be a well-ordering of P. This time, consider plays $(a_0,A_0,a_1,A_1,\dots)$ in $\Gamma_{\text{wd}}(\mathfrak{A})$ in which I follows his strategy and II always plays a move of the form $a_{n+1}=[p_n\cup\{\alpha_n\},\infty[$ where p_n is the $\preccurlyeq$-least member of P such that $[p_n,\infty[$ is included in some $a_n\cap a$ where $a\in A_n$, and $\alpha_n\in S$ is such that $\alpha_n>\sup p$. This time, let Q be the set of those $\alpha<\omega_1$ such that whenever $\langle\alpha_i\rangle_{i<n}$ are permitted selections by II when playing according to the recipe just described, then he will be able to continue with $\alpha_n<\alpha$. Again Q is a closed cofinal set, so there is an non-zero $\alpha\in Q\cap S$ such that $S\cap\alpha$ is cofinal with α. Let $\langle\beta_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in S with supremum α. Then II will be able to play by selecting α_n with $\beta_n\le\alpha_n<\alpha$ for each n. (At the nth move, given $\langle\alpha_i\rangle_{i<n}$, he will have the option of selecting some $\alpha'_n<\alpha$. Now he can amend this to $\alpha_n=\max(\alpha'_n,\beta_n)$.) But now, if we look at the corresponding p_n such that II's move a_{n+1} was $[p_n\cup\{\alpha_n\},\infty[$, we must have $p_n\cup\{\alpha_n\}\subseteq p_{n+1}$ for each n, so that $p^*=\bigcup_{n\in\mathbb{N}}p_n\cup\{\alpha\}$ belongs to P, and $[p^*,\infty[\,\subseteq a_n$ for every n; in which case II wins the play, which is supposed to be impossible. **X**

8F Theorem A Dedekind σ-complete strongly strategically weakly (σ,∞)-distributive Boolean algebra is a Maharam algebra.

proof Let $\mathfrak{A}$ be a Dedekind σ-complete strongly strategically weakly (σ,∞)-distributive Boolean algebra.

(a) I begin by checking that $\mathfrak{A}$ is ccc. **P** Let A be an antichain in $\mathfrak{A}$, and consider the play of $\Gamma^*_{\text{wd}}(\mathfrak{A})$ in which I plays A at every move. If II plays a_0, $a_1,\dots$ then $\sup_{n\in\mathbb{N}}a_n=1$, while each a_n meets only finitely many members of A; so A is countable. As A is arbitrary, $\mathfrak{A}$ is ccc. **Q**

(b) If $\mathfrak{A}\ne\{0\}$, then 0 and 1 can be separated by open sets. **P** Let σ be a winning strategy for II in $\Gamma^*_{\text{wd}}(\mathfrak{A})$, regarded as a function on finite strings of antichains in $\mathfrak{A}$. Choose antichains A_0, A'_0, A_1, $A'_1,\dots$ as follows. $A_0=A'_0=\{1\}$. Given A_i and A'_i for $i\le n$, set

$$D_n = \{d : \sigma(A_0, \dots , A_n, A) \subseteq d \text{ for some antichain } A\},$$

$$D'_n = \{d : \sigma(A'_0, \dots , A'_n, A) \subseteq d \text{ for some antichain} A\}.$$

If there is an element of D_n with a complement in D'_n, choose such a d_n and antichains A_{n+1}, A'_{n+1} such that $\sigma(A_0, \dots , A_n, A_{n+1}) \subseteq d_n$ and $\sigma(A'_0, \dots , A'_n, A'_{n+1}) \subseteq 1 \setminus d_n$; otherwise stop.

? If the process here continued indefinitely, we should have a sequence $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}$ such that both $\langle d_n\rangle_{n\in\mathbb{N}}$ and $\langle 1 \setminus d_n\rangle_{n\in\mathbb{N}}$ are order*-convergent to 1; but in this case $\langle d_n\rangle_{n\in\mathbb{N}}$ is also order*-convergent to 0 and $0 = 1$, contrary to hypothesis. **X** So the process terminates at some stage with $1 \setminus d \notin D'_n$ for every $d \in D_n$.

? If 1 does not belong to the interior of D_n for the order-sequential topology of $\mathfrak{A}$, then (because $\mathfrak{A}$ is certainly weakly (σ, ∞)-distributive, and we have just seen that it is ccc) there is a sequence $\langle b_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{A} \setminus D_n$ which is order*-convergent to 1. Let A be a maximal antichain in $\mathfrak{A}$ such that $\{a : a \in A,\ a \setminus b_i \neq 0\}$ is finite for every $a \in A$. Then $b_i \supseteq \sigma(A_0, \dots , A_n, A)$ for all but finitely many i, that is, $b_i \in D_n$ for all but finitely many i, which is absurd. **X**

Thus $1 \in \operatorname{int} D_n$; similarly, $1 \in \operatorname{int} D'_n$ and $0 \in \operatorname{int}\{1 \setminus d : d \in D'_n\}$. But we stopped at a point which made these sets disjoint. **Q**

(c) Applying (b) to principal ideals of $\mathfrak{A}$, as in the proof of Theorem 3C, we see that the order-sequential topology of $\mathfrak{A}$ is Hausdorff, so that $\mathfrak{A}$ is Maharam.

9 Cardinal Functions

9A Galois-Tukey connections (see Fremlin 08?, §512)

(a) A **supported relation** is a triple (A, R, B) where A and B are sets and R is a relation.

If R is a relation I write R' for the relation $\{(a, I) : a \in R^{-1}[I]\}$. (If you don't like proper classes, interpret each occasion of this notation by cutting it down to a suitable set.)

(b) If (A, R, B) is a supported relation then $\operatorname{cov}(A, R, B)$ is the least cardinal of any $I \subseteq B$ such that $A \subseteq R^{-1}[I]$ (taken as ∞ if $A \not\subseteq R^{-1}[B]$). $\operatorname{add}(A, R, B)$ is the smallest cardinal of any $I \subseteq A$ such that $I \not\subseteq R^{-1}[\{b\}]$ for any $b \in B$ (or ∞ if there is no such I).

(c) If (A, R, B) and (C, S, D) are supported relations a **Galois-Tukey connection** from (A, R, B) to (C, S, D) is a pair (ϕ, ψ) where $\phi : A \to C$ and $\psi : D \to B$ are functions and $(a, \psi(d)) \in R$ whenever $a \in A$, $d \in D$ and $(\phi(a), d) \in S$. I will write $(A, R, B) \preccurlyeq_{\mathrm{GT}} (S, S, D)$ if there is a Galois-Tukey connection from (A, R, B) to (C, S, D).

(d) If $(A, R, B) \preccurlyeq_{\mathrm{GT}} (C, S, D)$ then $\operatorname{cov}(A, R, B) \leq \operatorname{cov}(C, S, D)$ and $\operatorname{add}(C, S, D) \leq \operatorname{add}(A, R, B)$ (Fremlin 08?, 512D).

9B Proposition Let $\mathfrak{A}$ be a Maharam algebra, $\tau(\mathfrak{A})$ its Maharam type and $d(\mathfrak{A})$ its topological density in its order-sequential topology. Then $\tau(\mathfrak{A}) \leq d(\mathfrak{A}) \leq \max(\omega, \tau(\mathfrak{A}))$.

proof If $D \subseteq \mathfrak{A}$ is topologically dense, then every element of $\mathfrak{A}$ is expressible as $\inf_{n\in\mathbb{N}} \sup_{m\geq n} a_m$ for some sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in D, so D τ-generates $\mathfrak{A}$ and $\tau(\mathfrak{A}) \leq \#(D)$; accordingly $\tau(\mathfrak{A}) \leq d(\mathfrak{A})$. If $D \subseteq \mathfrak{A}$ τ-generates $\mathfrak{A}$, let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by D and $\overline{\mathfrak{B}}$ its topological closure. Then $\overline{\mathfrak{B}}$ is order-closed (because $\mathfrak{A}$ is ccc), so is the whole of $\mathfrak{A}$, and $d(\mathfrak{A}) \leq \#(\mathfrak{B}) \leq \max(\omega, \#(D))$; accordingly $d(\mathfrak{A}) \leq \max(\omega, \tau(\mathfrak{A}))$.

9C The localization relation (Fremlin 08?, §521) Let $\mathcal{S}$ be the family of sets $S \subseteq \mathbb{N} \times \mathbb{N}$ such that $\#(S[\{n\}]) \leq 2^n$ for every $n \in \mathbb{N}$. For $f \in \mathbb{N}^{\mathbb{N}}$, $S \in \mathcal{S}$ say that $f \subseteq^* S$ if $\{n : f(n) \notin S[\{n\}]\}$ is finite. Now $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$ is the **localization relation**.

9D Theorem (compare Fremlin 08?, 523J) Let $\mathfrak{A}$ be a Maharam algebra with countable Maharam type, not $\{0\}$. Then $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\leq\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$.

proof (a) Fix a strictly positive Maharam submeasure μ on $\mathfrak{A}$ such that $\mu 1 = 1$, and a countable subalgebra $D \subseteq \mathfrak{A}$ which is dense for the order-sequential topology; let $\langle a_n\rangle_{n\in\mathbb{N}}$ run over D.

(b) For $S \in \mathcal{S}$, $d \in D \setminus \{0\}$, $n \in \mathbb{N}$ set

$$\psi_{dn}(S) = d \setminus \sup_{m\ge n} \sup\{a_i : (m,i) \in S,\ \mu a_i \le 2^{-2m-2}\nu d\}.$$

Then

$$\mu\psi_{dn}(S) \ge \mu d - \sum_{m=n}^{\infty} 2^{-m-2}\mu d > 0,$$

so $\psi_{dn}(S) \ne 0$; set $\psi(S) = \{\psi_{dn}(S) : d \in D \setminus \{0\},\ n \in \mathbb{N}\} \in [\mathfrak{A}^+]^{\le\omega}$.

(c) For $a \in \mathfrak{A}^+$ choose $\phi(a) \in \mathbb{N}^{\mathbb{N}}$ as follows. Start by taking $d_m \in D$, for $m \in \mathbb{N}$, such that $\mu(d_m \triangle a) \le 2^{-2m-4}\mu a$ for every j; then certainly $\mu d_m \ge \frac{1}{2}\mu a$ for every m, so that if $m \ge n$ then

$$\mu(d_m \setminus d_{m+1}) \le 2^{-2m-3}\mu a \le 2^{-2m-2}\mu d_n.$$

Take $\phi(a)$ so that $a_{\phi(a)(i)} = d_i \setminus d_{i+1}$ for every i.

(d) (ϕ, ψ) is a Galois-Tukey correspondence from $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\omega})$ to $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$. **P** Suppose that $a \in \mathfrak{A}^+$ and $S \in \mathcal{S}$ are such that $\phi(a) \subseteq^* S$; let $n \in \mathbb{N}$ be such that $\phi(a)(m) \in S[\{m\}]$ for $m \ge n$. Let $\langle d_m\rangle_{m\in\mathbb{N}}$ be the sequence constructed in the definition of $\phi(a)$ as described in (c), and set $d = d_n$. Then

$$\psi_{dn}(S) \subseteq d \setminus \sup_{m\ge n}(d_m \setminus d_{m+1}) \subseteq \inf_{m\ge n} d_m \subseteq a.$$

So $a \supseteq' \psi(S)$. **Q**

Accordingly $(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\omega}) \preccurlyeq_{\mathrm{GT}} (\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S})$.

9E Corollary Let $\mathfrak{A}$ be a Maharam algebra with countable Maharam type, and $\mathcal{N}$ the Lebesgue null ideal. Then $\pi(\mathfrak{A}) \le \mathrm{cf}\,\mathcal{N}$.

proof

$$\pi(\mathfrak{A}) = \mathrm{cov}(\mathfrak{A}^+, \supseteq, \mathfrak{A}^+) \le \max(\omega, \mathrm{cov}(\mathfrak{A}^+, \supseteq', [\mathfrak{A}^+]^{\le\omega})) \le \max(\omega, \mathrm{cov}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}))$$

(putting 9Ad and 9D together)

$$= \mathrm{cov}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) = \mathrm{cf}\,\mathcal{N}$$

by Fremlin 08?, 521M.

9F Theorem Let $\mathfrak{A}$ be a Maharam algebra with countable Maharam type, and $\mathcal{N}$ the Lebesgue null ideal. Then $\mathrm{wdistr}(\mathfrak{A}) \ge \mathrm{add}\,\mathcal{N}$.

proof Fix a strictly positive Maharam submeasure μ on A, a countable topologically dense subalgebra $D \subseteq \mathfrak{A}$ and a sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ running over D. For any partition of unity $C \subseteq \mathfrak{A}$ choose $f_C \in \mathbb{N}^{\mathbb{N}}$ as follows. Let $C' = \{c : 1 \setminus c$ meets only finitely many members of $C\}$. Choose $c_n \in C'$ such that $\mu c_n < 8^{-n}$ for every n; for each n, choose a sequence $\langle d_{ni}\rangle_{i\in\mathbb{N}}$ in D such that $c_n \subseteq \sup_{i\ge n} d_{ni}$ and $\mu d_{ni} \le 4^{-i} \cdot 2^{-n-1}$ for every i. Set $c'_i = \sup_{n\le i} d_{ni}$ for each i, so that $c'_i \in D$ and $\mu c'_i \le 4^{-i}$, while $\sup_{i\ge n} c'_i \supseteq c_n$ belongs to C' for every n. Now choose $f_C(i)$ so that $c'_i = a_{f_C(i)}$ for each i.

If $\kappa < \mathrm{add}\,\mathcal{N}$ and $\langle C_\xi\rangle_{\xi<\kappa}$ is a family of partitions of unity in $\mathfrak{A}$, then there is an $S \in \mathcal{S}$ such that $f_{C_\xi} \subseteq^* S$ for every ξ, because $\mathrm{add}(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}) = \mathrm{add}\,\mathcal{N}$ (Fremlin 08?, 521M). Set $b_0 = 1$,

$$b_{n+1} = \sup_{m\ge n} \sup\{a_i : (m,i) \in S,\ \mu a_i \le 4^{-m}\}$$

for each n; then $\mu b_{n+1} \le \sum_{m=n}^{\infty} 2^{-m} = 2^{-n+1}$ for every n, so $B = \{b_n \setminus b_{n+1} : n \in \mathbb{N}\}$ is a partition of unity. Also, given $\xi < \kappa$, there is an $n \in \mathbb{N}$ such that $f_{C_\xi}(m) \in S[\{m\}]$ for every $m \ge n$. Since $\mu(a_{f_{C_\xi}}(m)) \le 4^{-m}$ for every m, it follows that if $m \ge n$ then $\sup_{i\ge m} a_{f_{C_\xi}}(i) \subseteq b_{m+1}$ and $b_{m+1} \in C'_\xi$. Thus every member of B meets only finitely many members of C_ξ; and this is true for every $\xi < \kappa$. As $\langle C_\xi\rangle_{\xi<\kappa}$ is arbitrary, $\mathrm{wdistr}(\mathfrak{A}) \ge \mathrm{add}\,\mathcal{N}$.

9G Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with a non-zero atomless Maharam submeasure μ. Then $d(\mathfrak{A}) \ge \mathfrak{m}_{\mathrm{countable}} = \mathrm{cov}\,\mathcal{M}$, where $\mathcal{M}$ is the ideal of meager subsets of $\mathbb{R}$.

proof We can suppose that $\mu 1 = 1$. Then for each $n \in \mathbb{N}$ we have a finite partition A_n of unity in $\mathfrak{A}$ such that $\mu a \le 2^{-n-2}$ for every $a \in A_n$. Enumerate A_n as $\langle a_{ni} \rangle_{i<k(n)}$.

Suppose that $\kappa < \operatorname{cov} \mathcal{M}$ and $\langle C_\xi \rangle_{\xi<\kappa}$ is a family of maximal centered subsets of $\mathfrak{A}$. Then $a \in C_\xi$ whenever $\xi < \kappa$, $c \in C_\xi$ and $c \subseteq a$. For $\xi < \kappa$ and $n \in \mathbb{N}$, $C_\xi \cap A_n \ne \emptyset$; let $f_\xi(n) < k(n)$ be such that $a_{n,f_\xi(n)} \in C_\xi$. Because $\kappa < \mathfrak{m}_{\text{countable}}$, there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $f \cap f_\xi \ne \emptyset$ for every $\xi < \kappa$ (Fremlin 08?, 521Rb); we may suppose that $f(n) < k(n)$ for every n; set $a = \sup_{n\in\mathbb{N}} a_{n,f(n)}$. Then $a \in C_\xi$ for every $\xi < \kappa$ and $\mu a < 1$. So $1 \setminus a \in \mathfrak{A}^+ \setminus \bigcup_{\xi<\kappa} C_\xi$.

As $\langle C_\xi \rangle_{\xi<\kappa}$ is arbitrary, $d(\mathfrak{A}) \ge \operatorname{cov} \mathcal{M}$.

10 Topological submeasures

10A Definitions (a) Let μ be a submeasure defined on an algebra Σ of subsets of a set X, and $\mathcal{K}$ a family of sets. I say that μ is **inner regular with respect to** $\mathcal{K}$ if whenever $E \in \Sigma$ and $\epsilon > 0$ there is a $K \in \mathcal{K} \cup \{\emptyset\}$ such that $K \in \Sigma$, $K \subseteq E$ and $\mu(E \setminus K) \le \epsilon$.

(b) A submeasure μ defined on an algebra of sets is **(countably) compact** if it is inner regular with respect to some (countably) compact family of sets.

(c) Now suppose that X is a Hausdorff space. Then a submeasure μ defined on a σ-algebra Σ of subsets of X is a **Radon submeasure** if (i) Σ contains every open set (ii) whenever $E \subseteq F \in \Sigma$ and $\mu F = 0$ then $E \in \Sigma$ (iii) μ is inner regular with respect to the compact sets.

10B Remarks These definitions are of course based on the corresponding notions for measures; see Fremlin 03, §§412, 416 and 451. But watch out for the translations; thus the definition of 'inner regular' for submeasures matches the definition for totally finite measures, but not the definition for general measures, which of course need not be exhaustive.

10C Proposition (a) Suppose that μ is an exhaustive submeasure defined on an algebra Σ of sets, and that $\mathcal{K}$ is a family of sets such that $K \cup L \in \mathcal{K}$ whenever K, $L \in \mathcal{K}$ are disjoint and $\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma, K \subseteq E\}$ for every $E \in \Sigma$. Then μ is inner regular with respect to $\mathcal{K}$.

(b) Suppose that μ is a countably compact submeasure defined on a σ-algebra Σ of sets. Then μ is a Maharam submeasure.

(c) Any Radon submeasure is a Maharam submeasure.

proof (a) ? Otherwise, there are $E \in \Sigma$ and $\epsilon > 0$ such that $\mu(E \setminus K) > \epsilon$ whenever $K \in \Sigma \cap \mathcal{K} \cap \mathcal{P}E$. Choose $\langle K_n \rangle_{n\in\mathbb{N}}$ inductively so that $K_n \in \Sigma \cap \mathcal{K}$, $K_n \subseteq E \setminus \bigcup_{i<n} K_i$ and $\mu K_n \ge \mu(E \setminus \bigcup_{i<n} K_i) - \frac{1}{2}\epsilon$ for every n. Then $\bigcup_{i<n} K_i \in \mathcal{K}$ so $\mu K_n \ge \frac{1}{2}\epsilon$ for every n; but μ was supposed to be exhaustive. **X**

(b) Let $\mathcal{K} \subseteq \Sigma$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$. Let $\langle E_n \rangle_{n\in\mathbb{N}}$ be a non-increasing sequence in Σ with infimum $\emptyset$ in Σ; since Σ is a σ-algebra, $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. **?** If $\inf_{n\in\mathbb{N}} \mu E_n = \gamma > 0$, then for each $n \in \mathbb{N}$ choose $K_n \in \Sigma \cap \mathcal{K}$ such that $K_n \subseteq E_n$ and $\mu(E_n \setminus K_n) \le 2^{-n-1}\gamma$. Then

$$\mu(\bigcap_{i\le n} K_i) \ge \mu E_n - \sum_{i=0}^{n} \mu(E_n \setminus K_i) \ge \mu E_n - \sum_{i=0}^{n} \mu(E_i \setminus K_i)$$

$$\ge \gamma - \sum_{i=0}^{n} 2^{-i-1}\gamma > 0$$

and $\bigcap_{i\le n} K_i \ne \emptyset$ for every n. But $\bigcap_{n\in\mathbb{N}} K_n \subseteq \bigcap_{n\in\mathbb{N}} E_n$ is empty and $\mathcal{K}$ is supposed to be countably compact. **X**

(c) Immediate from the definitions and (b).

10D Theorem Let X be a Hausdorff space and $\mathcal{K}$ the family of compact subsets of X. Let $\phi : \mathcal{K} \to [0, \infty[$ be a bounded functional such that

(α) $\phi\emptyset = 0$ and $\phi K \le \phi(K \cup L) \le \phi K + \phi L$ for all K, $L \in \mathcal{K}$;

(β) whenever $K \in \mathcal{K}$ and $\epsilon > 0$ there is an $L \in \mathcal{K}$ such that $L \subseteq X \setminus K$ and $\phi K' \leq \epsilon$ whenever $K' \in \mathcal{K}$ is disjoint from $K \cup L$;

(γ) whenever $K, L \in \mathcal{K}$ and $K \subseteq L$ then $\phi L \leq \phi K + \sup\{\phi K' : K' \in \mathcal{K}, K' \subseteq L \setminus K\}$.

Then there is a unique Radon submeasure defined on an algebra of subsets of X and extending ϕ.

proof (a) For $A \subseteq X$ write $\phi_* A = \sup\{\phi K : K \subseteq A$ is compact$\}$. Then ϕ_* extends ϕ. Also $\phi_*(\bigcup_{n\in\mathbb{N}} G_n) \leq \sum_{n=0}^{\infty} \phi_* G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open subsets of X. **P** If $K \subseteq \bigcup_{n\in\mathbb{N}} G_n$ is compact, it is expressible as $\bigcup_{i\leq n} K_i$ where $n \in \mathbb{N}$ and $K_i \subseteq G_i$ is compact for every $i \leq n$. **Q**

(b) Let Σ be the family of subsets E of X such that for every $\epsilon > 0$ there is a $K \subseteq X$ such that $K \cap E$ and $K \setminus E$ are both compact and $\phi_*(X \setminus K) \leq \epsilon$. Then Σ is an algebra of subsets of X including $\mathcal{K}$. **P** (i) Of course $X \setminus E \in \Sigma$ whenever $E \in \Sigma$. (ii) If $E, F \in \Sigma$ and $\epsilon > 0$, let $K, L \subseteq X$ be such that $K \cap E$, $K \setminus E$, $L \cap F$ and $L \setminus F$ are all compact and $\phi_*(X \setminus K)$, $\phi_*(X \setminus L)$ are both at most $\frac{1}{2}\epsilon$. Then $(K \cap L) \cap (E \cup F)$ and $(K \cap L) \setminus (E \cup F)$ are both compact, and $\phi_*(X \setminus (K \cap L)) \leq \epsilon$. As ϵ is arbitrary, $E \cup F \in \Sigma$. (iii) By hypothesis (β), $\mathcal{K} \subseteq \Sigma$. **Q**

(c) Σ is a σ-algebra of subsets of X. **P** Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in Σ with intersection E, and $\epsilon > 0$. For each $n \in \mathbb{N}$ let $K_n \subseteq X$ be such that $K_n \cap E_n$ and $K_n \setminus E_n$ are compact and $\phi_*(X \setminus K_n) \leq 2^{-n}\epsilon$; set $K = \bigcap_{n\in\mathbb{N}} K_n$. Set $L = \bigcap_{n\in\mathbb{N}} K_n \cap E_n$, so that $L \subseteq E$ is compact, and let $L' \subseteq X \setminus L$ be a compact set such that $\phi_*(X \setminus (L \cup L')) \leq \epsilon$; set $K' = K \cap (L \cup L')$. Then $\phi_*(X \setminus K') \leq 3\epsilon$. As $L' \cap L = \emptyset$ there is an $n \in \mathbb{N}$ such that $L' \cap \bigcap_{i\leq n} K_i \cap E_i$ is empty. Now

$$K \cap L' \subseteq \textstyle\bigcup_{i\leq n}(X \setminus (K_i \cap E_i)) \cap \bigcap_{i\leq n} K_i \subseteq \bigcup_{i\leq n} X \setminus E_i \subseteq X \setminus E,$$

so $K' \cap E = K \cap L$ and $K' \setminus E = K \cap L'$ are compact. As ϵ is arbitrary, $E \in \Sigma$. **Q**

(d) Set $\mu = \phi_* \restriction \Sigma$. Then μ is subadditive. **P** Suppose that $E, F \in \Sigma$ and $K \subseteq E \cup F$ is compact. Let $\epsilon > 0$. Then there are $L_1, L_2 \in \mathcal{K}$ such that $L_1 \cap E$, $L_1 \setminus E$, $L_2 \cap F$ and $L_2 \setminus F$ are all compact, while $\phi_*(X \setminus L_1)$ and $\phi_*(X \setminus L_2)$ are both at most ϵ. Set $K_1 = L_1 \cap E$ and $K_2 = L_2 \cap F$, so that

$$\phi K \leq \phi(K \cup K_1 \cup K_2) \leq \phi(K_1 \cup K_2) + \phi_*(K \setminus (K_1 \cup K_2))$$

(by hypothesis (γ))

$$\leq \phi K_1 + \phi K_2 + \phi_*(X \setminus (L_1 \cap L_2)) \leq \phi_* E + \phi_* F + 2\epsilon.$$

As ϵ and K are arbitrary, $\phi_*(E \cup F) \leq \phi_* E + \phi_* F$. **Q**

(e) If $E \subseteq F \in \Sigma$ and $\mu F = 0$ then $E \in \Sigma$. **P** Let $\epsilon > 0$. Let $K \subseteq X$ be such that $K \cap F$ and $K \setminus F$ are both compact and $\phi_*(X \setminus K) \leq \epsilon$. If $L \in \mathcal{K}$ and $L \cap K \subseteq F$ then $\phi_*(L \setminus K) \leq \epsilon$ so

$$\phi(L \cup (K \cap F)) \leq \epsilon + \phi(K \cap F) = \epsilon.$$

Accordingly $\phi_*(X \setminus (K \setminus F)) \leq \epsilon$. But $(K \setminus F) \cap E$ and $(K \setminus F) \setminus E$ are both compact. As ϵ is arbitrary, $E \in \Sigma$. **Q**

(f) μ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \Sigma$ and $\epsilon > 0$, let $K \subseteq X$ be such that $K \cap E$ and $K \setminus E$ are both compact and $\phi_*(X \setminus K) \leq \epsilon$. If $L \in \mathcal{K}$ and $L \subseteq E \setminus K$ then $\phi L \leq \phi_*(X \setminus K) \leq \epsilon$; so $\mu(E \setminus K) \leq \epsilon$. **Q**

(g) Every open set belongs to Σ. **P** Let $G \subseteq X$ be open, and $\epsilon > 0$. Applying (β) with $K = \emptyset$ we have an $L \in \mathcal{K}$ such that $\phi_*(X \setminus L) \leq \epsilon$. Next, there is an $L' \in \mathcal{K}$, disjoint from $L \setminus G$, such that $\phi_*(X \setminus ((L \setminus G) \cup L')) \leq \epsilon$. Set $L'' = L \cap ((L \setminus G) \cup L')$. Then $L'' \cap G = L \cap L'$ and $L'' \setminus G = L \setminus G$ are compact and $\phi_*(X \setminus L'') \leq 2\epsilon$. **Q**

(h) So μ is a Radon submeasure. To see that it is unique, let μ' be another Radon submeasure with the same properties, and Σ' its domain. If $E \in \Sigma$ there are sequences $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ of compact sets such that $K_n \subseteq E$, $L_n \subseteq X \setminus E$ and $\mu(E \setminus K_n) + \mu((X \setminus E) \setminus L_n) \leq 2^{-n}$ for every n. Set $F = \bigcup_{n\in\mathbb{N}} K_n$ and $F' = \bigcup_{n\in\mathbb{N}} L_n$; then $F \cup F'$ belongs to $\Sigma \cap \Sigma'$ and

$$\mu'(X \setminus (F \cup F')) = \phi_*(X \setminus (F \cup F')) = \mu(X \setminus (F \cup F'))$$
$$\leq \inf_{n\in\mathbb{N}} \mu(X \setminus (K_n \cup L_n)) = 0.$$

Consequently $E\setminus F\in\Sigma'$ and $E\in\Sigma'$.

The same works with μ and μ' interchanged, so $\Sigma=\Sigma'$ and $\mu'=\phi_*\restriction\Sigma=\mu$.

10E Theorem Let X be a zero-dimensional compact Hausdorff space and $\mathfrak{B}$ the algebra of open-and-closed subsets of X. Let $\nu:\mathfrak{B}\to[0,\infty[$ be an exhaustive submeasure. Then there is a unique Radon submeasure on X extending ν.

proof (a) Let $\mathcal{K}$ be the family of compact subsets of X and for $K\in\mathcal{K}$ set $\phi K=\inf\{\nu E:K\subseteq E\in\mathfrak{B}\}$. Then ϕ satisfies the conditions of Theorem 10D.

P(α) Of course $\phi\emptyset=0$ and $\phi K\le\phi L$ whenever $K\subseteq L$ in $\mathcal{K}$. If $K\subseteq E\in\mathfrak{B}$ and $L\subseteq F\in\mathfrak{B}$, then $K\cup L\subseteq E\cup F\in\mathfrak{B}$ and $\nu(E\cup F)\le\nu E+\nu F$, so ϕ is subadditive. **Q**

(β) The point is that for every $K\in\mathcal{K}$ and $\epsilon>0$ there is an $E\in\mathfrak{B}$ such that $K\subseteq E$ and $\nu F\le\epsilon$ whenever $F\in\mathfrak{B}$ and $F\subseteq E\setminus K$; since otherwise we could find a disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{B}$ with $\nu F_n\ge\epsilon$ for every n. But now $L=X\setminus E$ is compact and disjoint from K, and every compact subset of $X\setminus(K\cup L)=E\setminus K$ is included in a member of $\mathfrak{B}$ included in $E\setminus K$; so $\sup\{\phi K':K'\subseteq X\setminus(K\cup L)$ is compact$\}\le\epsilon$.

(γ) If K and L are compact and $K\subseteq L$ and $\epsilon>0$, take $E\in\mathfrak{B}$ such that $K\subseteq E$ and $\nu E\le\phi K+\epsilon$. Set $K'=L\setminus E$. If $F\in\mathfrak{B}$ and $F\supseteq K'$, then $E\cup F\supseteq L$, so

$$\phi L\le\nu(E\cup F)\le\nu E+\nu F\le\phi K+\epsilon+\nu F.$$

As F is arbitrary, $\phi L\le\phi K+\phi K'+\epsilon$. **Q**

(b) There is therefore a Radon submeasure μ extending ϕ and ν.

(c) If μ' is another Radon submeasure extending ν, then $\mu'\restriction\mathcal{K}=\phi$. **P** Of course $\mu'K\le\phi K$ for every $K\in\mathcal{K}$. **?** If $K\in\mathcal{K}$ and $\epsilon>0$ and $\mu'K+\epsilon<\phi K$, let $E\in\mathfrak{B}$ be such that $K\subseteq E$ and $\phi L\le\epsilon$ whenever $L\subseteq E\setminus K$ is compact, as in (a-β) above. Then

$$\begin{aligned}\mu'(E\setminus K)&=\sup\{\mu'L:L\subseteq E\setminus K\text{ is compact}\}\\&\le\sup\{\phi L:L\subseteq E\setminus K\text{ is compact}\}\le\epsilon\end{aligned}$$

and

$$\nu E=\mu'E\le\epsilon+\mu'K<\mu K\le\mu E=\nu E.\ \textbf{XQ}$$

By the guarantee of uniqueness in 10D, $\mu'=\mu$.

10F Theorem Let X be a topological space, $\mathcal{G}$ the family of cozero subsets of X and $\mathcal{B}\mathfrak{a}(X)$ the Baire σ-algebra of X. If $\psi:\mathcal{G}\to[0,\infty[$ is a functional, then ψ can be extended to a Maharam submeasure with domain $\mathcal{B}\mathfrak{a}(X)$ iff

(α) $\psi G\le\psi H$ whenever G, $H\in\mathcal{G}$ and $G\subseteq H$,

(β) $\psi(\bigcup_{n\in\mathbb{N}}G_n)\le\sum_{n=0}^{\infty}\psi G_n$ for every sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$,

(γ) $\lim_{n\to\infty}\psi G_n=0$ for every non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with empty intersection.

In this case, the extension is unique.

proof (a) If ψ can be extended to a Maharam submeasure, then the conditions are surely satisfied, using 1B(a-i) for (β). So for most of the rest of the proof I suppose that the conditions are satisfied and seek to construct a Maharam submeasure on $\mathcal{B}\mathfrak{a}(X)$ extending ψ.

(b) Let $\mathcal{E}$ be the family of those sets $E\subseteq X$ such that for every $\epsilon>0$ there are a cozero set $G\supseteq E$ and a zero set $F\subseteq E$ such that $\psi(G\setminus F)\le\epsilon$.

(i) Zero sets belong to $\mathcal{E}$. **P** If $F\subseteq X$ is a zero set, there is a non-increasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with intersection F; now (γ) tells us that $\inf_{n\in\mathbb{N}}G_n\setminus F=0$. **Q**

(ii) If $E\in\mathcal{E}$ then $X\setminus E\in\mathcal{E}$. **P** If $F\subseteq E\subseteq G$ then $X\setminus G\subseteq X\setminus E\subseteq X\setminus F$. **Q**

(iii) If E_0, $E_1 \in \mathcal{E}$ then $E_0 \cup E_1 \in \mathcal{E}$. **P** Given $\epsilon > 0$, let $F_0 \subseteq E_0$, $F_1 \subseteq E_1$ be zero sets and $G_0 \supseteq E_0$, $G_1 \supseteq E_1$ cozero sets such that $\psi(G_0 \setminus F_0) + \psi(G_1 \setminus F_1) \le \epsilon$; now $G = G_0 \cup G_1$ is a cozero set, $F = F_0 \cup F_1$ is a zero set, $F \subseteq E_0 \cup E_1 \subseteq G$ and $\psi(G \setminus F) \le \epsilon$ (using (α) and (β)). **Q**

Consequently $\mathcal{E}$ is an algebra of subsets of X.

(iv) If $\langle G_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{G}$ then $\lim_{n\to\infty} \psi G_n = 0$ (apply (γ) to $\langle \bigcup_{i\ge n} G_i \rangle_{n\in\mathbb{N}}$). So if $\langle G_n \rangle_{n\in\mathbb{N}}$ is a sequence of cozero sets, $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of zero sets and $G_{n+1} \subseteq F_n \subseteq G_n$ for every n, $\lim_{n\to\infty} \lim_{m\to\infty} \psi(G_n \setminus F_m) = 0$. **P?** Otherwise, we have an $\epsilon > 0$ and a strictly increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $\psi(G_{n_k} \setminus F_{n_{k+1}}) \ge \epsilon$ for every k. But now $\langle G_{n_{2k}} \setminus F_{n_{2k+1}} \rangle_{k\in\mathbb{N}}$ is disjoint sequence of sets on which ψ takes values greater than or equal to ϵ. **XQ**

(v) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ then $E = \bigcap_{n\in\mathbb{N}} E_n$ belongs to $\mathcal{E}$. **P** Let $\epsilon > 0$. For each $n \in \mathbb{N}$ take a zero set $F_n \subseteq E_n$ and a cozero set $G_n \supseteq E_n$ such that $\psi(G_n \setminus F_n) \le 2^{-n}\epsilon$. Choose zero sets F'_n, cozero sets G'_n such that

$$F_{n+1} \subseteq G'_{n+1} \subseteq F'_{n+1} \subseteq G_{n+1} \cap G'_n$$

for every n. Set $F = \bigcap_{m\in\mathbb{N}} F_m$. There is a strictly increasing sequence $\langle n_k \rangle_{k\in\mathbb{N}}$ such that $\psi(G'_{n_k} \setminus F'_m) \le 2^{-k}\epsilon$ for all k, $m \in \mathbb{N}$. Now $F \subseteq E$ is a zero set, $G = G'_{n_0} \supseteq E$ is a cozero set, and

$$G \setminus F \subseteq \bigcup_{k\in\mathbb{N}} G_{n_k} \setminus F_{n_k} \cup \bigcup_{k\in\mathbb{N}} G'_{n_k} \setminus F'_{n_{k+1}},$$

so $\psi(G \setminus F) \le 4\epsilon$, by ($\beta$) in its full strength. As ϵ is arbitrary, $E \in \mathcal{E}$. **Q**

Thus $\mathcal{E}$ is a σ-algebra of subsets of X, and includes $\mathcal{B}\mathfrak{a}(X)$.

(c) For $E \in \mathcal{E}$, set

$$\mu E = \inf\{\psi G : G \text{ is a cozero set including } E\}.$$

(i) μ extends ψ (by (α)); in particular, $\mu\emptyset = 0$ (by (γ)).

(ii) If E_0, $E_1 \in \mathcal{E}$ and $E_0 \subseteq E_1$ then $\mu E_0 \le \mu E_1$.

(iii) If E, $E' \in \mathcal{E}$ then $\mu(E_0 \cup E_1) \le \mu E_0 + \mu E_1$. **P** If $\epsilon > 0$, we have cozero sets $G_0 \supseteq E_0$, $G_1 \supseteq E_1$ such that $\psi G_0 + \psi G_1 \le \mu E_0 + \mu E_1 + \epsilon$; now $G_0 \cup G_1$ is a cozero set including $E_0 \cup E_1$, so

$$\mu(E_0 \cup E_1) \le \psi(G_0 \cup G_1) \le \psi G_0 + \psi G_1 + \epsilon \le \mu E_0 + \mu E_1 + \epsilon. \textbf{ Q}$$

Thus μ is a submeasure.

(iv) If $\langle E_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ with empty intersection, then $\inf_{n\in\mathbb{N}} \mu E_n = 0$. **P** Take any $\epsilon > 0$, and repeat the construction of (b-v) above. At the end, we have a cozero set $G = G_{n_0}$ including E_{n_0}, while F must be empty, so

$$\mu E_{n_0} \le \psi G \le 4\epsilon. \textbf{ Q}$$

Thus μ is a Maharam submeasure, and $\mu\restriction\mathcal{B}\mathfrak{a}(X)$ is an extension of the type we seek.

(d) As for uniqueness, suppose that ν is any Maharam submeasure on $\mathcal{B}\mathfrak{a}(X)$ extending ψ. If $F \subseteq X$ is a zero set, then it is the intersection of a non-increasing sequence of cozero sets, so $\nu F = \mu F$. If $E \in \mathcal{B}\mathfrak{a}(X)$ and $\epsilon > 0$, there are a zero set $F \subseteq E$ and a cozero set $G \supseteq E$ such that $\psi(G \setminus F) \le \epsilon$; now both μE and νE belong to $[\mu F, \mu G]$ and this interval has length at most ϵ, so $|\mu E - \nu E| \le \epsilon$. As E and ϵ are arbitrary, ν agrees with μ on $\mathcal{B}\mathfrak{a}(X)$.

Remark If ψ is a modular functional (that is, $\psi(G \cup H) + \psi(G \cap H) = \psi G + \psi H$ for all G, $H \in \mathcal{G}$), then μ will be a measure; cf. Fremlin 03, 413Xq.

10G Example I refer to Talagrand's example of an exhaustive submeasure ν which is not uniformly exhaustive, as described in Fremlin n06. ν is defined on the algebra of open-and-closed subsets of a compact space $X = \prod_{n\in\mathbb{N}} T_n$, where each T_n is finite, and is invariant under permutations of each T_n; so we can give X a group structure under which it is a compact metrizable abelian group and ν is translation-invariant. Let $\tilde{\nu}$ be the Radon submeasure on X extending ν; then $\tilde{\nu}$ is translation-invariant. Let μ be the Haar probability measure on X.

As noted in §X of Fremlin n06, there is no non-trivial uniformly exhaustive submeasure dominated by ν. Consequently, writing $\mathcal{B}$ for the σ-algebra of Borel subsets of X, $(\mu\restriction\mathcal{B}) \wedge (\nu\restriction\mathcal{B}) = 0$ and there must be a Borel set $E \subseteq X$ such that $\nu E = 0$ and $\mu(X \setminus E) = 0$ (1M). Consider $W = \{(x, y) : x, y \in X, xy \in E\}$. Then $\mu W[\{x\}] = 1$ for every $x \in X$, while $\nu W^{-1}[\{y\}] = 0$ for every $y \in X$. In particular, $W \in (\mathcal{N}(\nu) \rtimes \mathcal{N}(\mu)) \setminus (\mathcal{N}(\nu) \ltimes \mathcal{N}(\mu))$, while (X, μ) is isomorphic, as measure space, to $[0, 1]$ with Lebesgue measure; compare 4Ga.

11 Problems

11A A long-outstanding problem is: is every σ-finite-cc Boolean algebra in fact σ-bounded-cc? It is easy to show that every Maharam algebra is σ-finite-cc, and that every measurable algebra is σ-bounded-cc. But is every Maharam algebra σ-bounded-cc? (See 4D-4E.)

11B Let $\langle \mathfrak{A}_n \rangle_{n\in\mathbb{N}}$ be a sequence of Maharam algebras and μ_n a unital Maharam submeasure on $\mathfrak{A}_n$ for each n. Must there be a Maharam algebra $\mathfrak{A}$ with a Maharam submeasure μ such that $(\mathfrak{A}_n, \mu_n)$ is isometrically isomorphic to a subalgebra of $(\mathfrak{A}, \mu)$ for every n?

11C Is there a strictly positive exhaustive submeasure on Gaifman's algebra, that is, the regular open algebra $\mathrm{RO}(X)$ described in Proposition 6A?

11D(a) Let $\mathfrak{C}$ be a Boolean algebra and $\mathfrak{A}$ a σ-finite-cc Boolean algebra, not $\{0\}$. Suppose that $\Vdash_{\mathfrak{A}}$ '$\check{\mathfrak{C}}$ is σ-finite-cc', in the sense that we have a sequence θ_n of functions from $\mathfrak{C}$ to $\mathfrak{A}$ (interpret $\theta_n(c)$ as $[\![\check{c} \in \dot{S}_n]\!]$) such that

$\sup_{n\in\mathbb{N}} \theta_n(c) = 1$ for every $c \in \mathfrak{C}$;

for any $n \in \mathbb{N}$ and any disjoint sequence $\langle c_k \rangle_{k\in\mathbb{N}}$ in $\mathfrak{C}$, $\langle \theta_n(c_k) \rangle_{k\in\mathbb{N}}$ order*-converges to 0 in $\mathfrak{A}$.

Must $\mathfrak{C}$ be σ-finite-cc?

(b) Repeat (a) for 'σ-bounded-cc'.

11E Is there any general bound for the ordinals $\mathrm{Mhsm}(\mathfrak{A})$ for Maharam algebras $\mathfrak{A}$? Note that Talagrand 06 describes a countable algebra $\mathfrak{B}$ with a strictly positive exhaustive submeasure which is not uniformly exhaustive; for any $\epsilon > 0$, $r_\epsilon(1)$ must be countable; taking the metric completion of $\mathfrak{B}$, we obtain a Maharam algebra $\mathfrak{A}$ such that $\omega^\omega \le \mathrm{Mhsm}(\mathfrak{A}) < \omega_1$, by §§7C-7D and 7J.

11F A measurable algebra of cardinal $\mathfrak{c}$ or less is σ-linked, indeed σ-n-linked for every $n \ge 2$ (Dow & Steprāns 94, or Fremlin 08?, 523Of). Note that the linking number of any Maharam algebra $\mathfrak{A}$ is at most $\max(\omega, \tau(\mathfrak{A}))$; in particular, there is a non-measurable Maharam algebra which is σ-linked, therefore σ-bounded-cc. But is every Maharam algebra of size $\mathfrak{c}$ necessarily σ-linked?

11G Let $\mathfrak{B}_\omega$ be the measure algebra of the usual measure on $\{0, 1\}^\omega$, and $\mathfrak{A}$ a non-measurable Maharam algebra. Must it be true that $\mathfrak{B}_\omega \setminus \{1\} \preccurlyeq_{\mathrm{T}} \mathfrak{A} \setminus \{1\}$?

11H Write S^* for $\bigcup_{n\in\mathbb{N}} \{0, 1\}^n$. For $A \subseteq S^*$, set $E_A = \{x : x \in \{0, 1\}^{\mathbb{N}}, \{n : x\restriction n \in A\}$ is infinite$\}$. For any ideal $\mathcal{I} \lhd \mathcal{P}S^*$, write $\mathcal{E}_{\mathcal{I}}$ for the ideal of the Borel σ-algebra $\mathcal{B}(\{0, 1\}^{\mathbb{N}}$ generated by $\{E_A : A \in \mathcal{I}\}$. Find a combinatorial characterization of those p-ideals $\mathcal{I}$ of $\mathcal{P}S^*$ such that $\mathcal{B}(\{0, 1\}^{\mathbb{N}})/\mathcal{E}_{\mathcal{I}}$ is ccc and weakly (σ, ∞)-distributive.

11I In Theorem 7H, can we improve on the factor $\frac{1}{3}$?

11J Is it possible for a Souslin algebra to be strategically weakly (σ, ∞)-distributive?

11K In Głowczyński's example (see 6B) can $\mathfrak{A}$ be strategically weakly (σ, ∞)-distributive?

11L Let $\mathfrak{A}$ be an atomless Maharam algebra of countable Maharam type, not $\{0\}$. Must we have $\mathrm{wdistr}(\mathfrak{A}) = \mathrm{add}\,\mathcal{N}$ and/or $\pi(\mathfrak{A}) = \mathrm{cf}\,\mathcal{N}$ and/or $d(\mathfrak{A}) = \mathrm{non}\,\mathcal{N}$? (See §9.)

11M Let $\mathfrak{A}$ be a non-zero atomless Maharam algebra. Does it necessarily have an atomless closed subalgebra which is a measurable algebra?

11N Let μ be a non-zero Radon submeasure on an algebra Σ of subsets of $[0, 1]$. Does μ have a lifting? that is, is there a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that (i) $\mu(E \triangle \phi(E)) = 0$ for every $E \in \Sigma$ (ii) $\{E : \phi E = \emptyset\} = \{E : \mu E = 0\}$?

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