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On results of M.Elekes and G.Gruenhage

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1. A perfect set (P.Erdős & S.Kakutani) Choose $\langle\mathcal{I}_n\rangle_{n\ge 1}$ as follows. $\mathcal{I}_1=\{[0,1]\}$. Given that $\mathcal{I}_n$ is a family of $(n-1)!$ non-overlapping closed intervals of length $\frac{1}{n!}$, divide each member of $\mathcal{I}_n$ into $n+1$ closed intervals of equal length, discard one, and keep the rest for $\mathcal{I}_{n+1}$.

Set $K_n=\bigcup\mathcal{I}_n$ for each n, so that $\langle K_n\rangle_{n\ge 1}$ is a non-increasing sequence of compact sets and the Lebesgue measure μK_n of K_n is $\frac{1}{n}$ for each $n\ge 1$. Set $K=\bigcap_{n\ge 1}K_n$, so that K is a compact Lebesgue negligible set.

2. Proposition (Elekes & Steprāns 04, Theorem 1.2) Let E be any uncountable analytic set in $\mathbb{R}$, and K the set of §1. Then there is an $x\in\mathbb{R}$ such that $(E+x)\cap K$ is uncountable.

proof It is enough to consider the case in which E is a non-empty compact set without isolated points. Construct $\langle Q_n\rangle_{n\in\mathbb{N}}$ and $\langle J_n\rangle_{n\in\mathbb{N}}$ as follows. Start with $Q_0=\{q_0\}$ where q_0 is any point of E. Observe that the construction of $\mathcal{I}_5$ kept four out of five subintervals of each interval in $\mathcal{I}_4$, so that $\mathcal{I}_5$ necessarily has a pair of contiguous intervals, and the interior of K_5 has a component of length at least $\frac{2}{5!}$. There is therefore a closed interval J_0, of length $\frac{1}{5!}$, such that $q_0+J_0\subseteq\operatorname{int}K_5$; we may arrange that q_0+a_0 is irrational, where $a_0=\min J_0$.

Now suppose that we have Q_n and J_n, where $Q_n\subseteq E$, $\#(Q_n)=1+\lfloor\frac{n}{4}\rfloor$, J_n is a closed interval of length $\frac{1}{(n+5)!}$, $Q_n+J_n\subseteq\operatorname{int}K_{n+5}$, and $q+a_n$ is irrational for every $q\in Q_n$, where $a_n=\min J_n$. For $q\in Q_n$, let $\mathcal{J}_q$ be the set of subintervals of members of $\mathcal{I}_{n+5}$ which were rejected when constructing $\mathcal{I}_{n+6}$, but meet $q+J_n$. As $q+J_n$ meets just two of the intervals in $\mathcal{I}_{n+5}$ (this is where it is useful to know that $q+a_n$ is irrational, while every interval in $\mathcal{I}_{n+5}$ has rational endpoints), $\#(\mathcal{J}_q)\le 2$ and $\{J_n\cap(I-q):q\in Q,\,I\in\mathcal{J}_q\}$ consists of at most $2\#(Q_n)$ intervals of length at most $\frac{1}{n+6}\mu J_n$. It follows that $H=J_n\setminus\bigcup\{I-q:q\in Q,\,I\in\mathcal{J}_q\}$ has at most $2\#(Q)+1$ components and has measure at least $(1-\frac{2\#(Q_n)}{n+6})\mu J_n$. As $4\#(Q_n)+1\le n+5$, one of the components of H has length greater than $\frac{\mu J_n}{n+6}$ and there must be a closed interval J_{n+1}, of length $\frac{1}{(n+6)!}$, such that $J_{n+1}\subseteq H$ and $q+a_{n+1}$ is irrational for every $q\in Q_n$, where $a_{n+1}=\min J_{n+1}$. Now observe that Q_n+J_{n+1} does not meet any of the subintervals of members of $\mathcal{I}_{n+5}$ which were rejected when forming $\mathcal{I}_{n+6}$, so that $Q_n+J_{n+1}\subseteq\operatorname{int}K_{n+6}$.

If $n+1$ is not a multiple of 4, so that $\lfloor\frac{n+1}{4}\rfloor=\lfloor\frac{n}{4}\rfloor$, set $Q_{n+1}=Q_n$. Otherwise, take a loneliest member q of Q_n (that is, one for which the distance from q to $Q_n\setminus\{q\}$ is maximal) and choose $q'\in E\setminus Q$ such that $|q'-q|\le 2^{-n}$ and $q'+J_{n+1}\subseteq\operatorname{int}K_{n+6}$; set $Q_{n+1}=Q_n\cup\{q'\}$. Continue.

At the end of the construction, let x be the single point of $\bigcap_{n\in\mathbb{N}}J_n$, and set $Q=\bigcup_{n\in\mathbb{N}}Q_n$. Then $Q\subseteq E$ has no isolated points and $x+Q\subseteq K$. So $(E+x)\cap K\supseteq\overline{Q}+x$ is uncountable.

3. Proposition (Elekes & Steprāns 04, Theorem 2.1) Suppose that, in the construction of §1, the discarded intervals are always the right-hand ones of each group, so that $K=\{\sum_{n=3}^{\infty}\frac{k_n}{n!}:0\le k_n\le n-2$ for each $n\ge 3\}$[1]. Let $\operatorname{cf}\mathcal{N}$ be the cofinality of the Lebesgue null ideal $\mathcal{N}$. Then there is a set $A\subseteq\mathbb{R}$, with cardinality at most $\operatorname{cf}\mathcal{N}$, such that $A+K=\mathbb{R}$.

[1] See Erdős & Kakutani 57.

proof (Elekes & Steprāns 04) Define $\alpha \in \mathbb{N}^{\mathbb{N}}$ by setting $\alpha(0) = 0$, $\alpha(n) = \lfloor\frac{n-1}{2}\rfloor$ for $n \geq 1$. As in Fremlin 08, 522L, set

$$\mathcal{S}^{(\alpha)} = \{S : S \subseteq \mathbb{N} \times \mathbb{N},\ \#(S[\{n\}]) \leq \alpha(n) \text{ for every } n \in \mathbb{N}\};$$

for $f \in \mathbb{N}^{\mathbb{N}}$ and $S \in \mathcal{S}^{(\alpha)}$, say that $f \subseteq^* S$ if $\{n : n \in \mathbb{N},\ (n, f(n)) \notin S\}$ is finite. Say that $f \subseteq S$ if $(n, f(n)) \in S\}$ for every $n \in \mathbb{N}$. For $S \in \mathcal{S}^{(\alpha)}$ set $S' = \{(n, i) : (n, i) \in S,\ i \leq n - 1\} \cup \{(0, 0)\}$ and

$$Q_S = \{\sum_{n=4}^{\infty} \frac{f(n)}{n!} : f \in \mathbb{N}^{\mathbb{N}},\ f \subseteq S'\}.$$

Then there is an $x_S \in \mathbb{R}$ such that $x_S + Q_S \subseteq K$. **P** For each $n \geq 4$, $S'[\{n\}]$ is a subset of $\{0, \ldots, n-1\}$ with less than $\frac{n}{2}$ members, so there is a $j_n < n$ such that neither j_n nor $j_n - 1$ belongs to $S'[\{n\}]$; allow $j_n = 0$, but only if there is no alternative, in which case n is odd and $S'[\{n\}] = \{1, 3, \ldots, n-2\}$, so that $n - 1 \notin S'[\{n\}]$. Set $j'_n = n - 1 - j_n$. Now set $x_S = \sum_{n=4}^{\infty} \frac{j'_n}{n!}$. In this case, if $f \subseteq S'$, $j'_n + f(n)$ is neither $j'_n + j_n = n - 1$ or $j'_n + j_n - 1 = n - 2$; at the same time, $j'_n + f(n) \leq 2n - 3$, because either $j_n > 0$ or $f(n) < n - 1$. So

$$x_S + \sum_{n=4}^{\infty} \frac{f(n)}{n!} = \sum_{n=4}^{\infty} \frac{j'_n + f(n)}{n!} = \sum_{n=3}^{\infty} \frac{k_n}{n!},$$

where, for each $n \geq 4$, $k_n < n$ is one of $j'_n + f(n)$, $j'_n + f(n) + 1$, $j'_n + f(n) - n$ or $j'_n + f(n) - n + 1$, while k_3 is either 0 or 1. But this means that $0 \leq k_n \leq n - 2$ for every n, so that $x_S + \sum_{n=4}^{\infty} \frac{f(n)}{n!} \in K$. As f is arbitrary, we have a suitable x_S. **Q**

Observe next that, by 522L and 522M of Fremlin 08, $(\mathbb{N}^{\mathbb{N}}, \subseteq^*, \mathcal{S}^{(\alpha)}) \cong_{\mathrm{GT}} (\mathcal{N}, \subseteq, \mathcal{N})$. There is therefore a set $\mathcal{T} \subseteq \mathcal{S}^{(\alpha)}$, of size $\mathrm{cf}\,\mathcal{N}$, such that for every $f \in \mathbb{N}^{\mathbb{N}}$ there is a $T \in \mathcal{T}$ such that $f \subseteq^* T$. We may suppose that $S \in \mathcal{T}$ whenever $T \in \mathcal{T}$, $S \in \mathcal{S}^{(\alpha)}$ and $S \triangle T$ is finite; in which case, we see that for every $f \in \mathbb{N}^{\mathbb{N}}$ there is a $T \in \mathcal{T}$ such that $f \subseteq T$. Set $A_0 = \{-x_T : T \in \mathcal{T}\}$. If $z \in [0, \frac{1}{6}]$, there is an $f \in \mathbb{N}^{\mathbb{N}}$ such that $f(0) = f(1) = f(2) = f(3) = 0$, $f(n) < n$ for every $n \geq 4$, and $z = \sum_{n=4}^{\infty} \frac{f(n)}{n!}$. Let $T \in \mathcal{T}$ be such that $f \subseteq T$; then $z \in Q_T$ so $x_T + z \in K$ and $z \in A_0 + K$.

Thus $A_0 + K \supseteq [0, \frac{1}{6}]$; setting $A = A_0 + \mathbb{Q}$, $\#(A) \leq \mathrm{cf}\,\mathcal{N}$ and $A + K = \mathbb{R}$.

4. Translates of the Cantor set: Proposition The union of fewer than $\mathfrak{c}$ translates of the Cantor set C always has inner measure 0.

proof (a) Let L be a compact set of positive Lebesgue measure. Write B for $\{3^n k : n,\ k \in \mathbb{Z}\}$. For $n \geq 1$ and $j < 9$ let D_{nj} be the closed set of those $z \in \mathbb{R}$ such that the fractional part of z has a j in the nth place of (one of) its 9-ary expansions. (Take an expansion of a negative number to be of the form $m + 0{\cdot}d_1 d_2 \ldots$ where $m \in \mathbb{Z}$ and $d_1, d_2, \ldots < 9$.) Note that if $t \in \mathbb{R}$ and $n \geq 1$ then $C + t$ does not meet every D_{nj}. **P** Let J be a component of $[0, 1] \setminus C$ of length $3 \cdot 9^{-n}$, then $J + 9 \cdot 9^{-n} k$ does not meet C for any $k \in \mathbb{Z}$. Now there must be a $j \leq 8$ such that $J + t$ covers one of the intervals comprising D_{nj}, in which case $D_{nj} \subseteq \bigcup_{k \in \mathbb{Z}} (J + t + 9 \cdot 9^{-n} k)$ is disjoint from $C + t$. **Q**

Choose n_i for $i \in \mathbb{N}$ and $y(\sigma)$, for $\sigma \in 9^i = \prod_{j<i} 9$, as follows. $y(\emptyset)$ is to be any density point of $L \setminus B$. Given that $y(\sigma)$ is a density point of $L \setminus B$ for every $\sigma \in 9^i$, let $n_i \geq 1$ be such that $n_i > n_j$ for every $j < i$ and, setting $A_\sigma = [y(\sigma), y(\sigma) + 10 \cdot 9^{-n_i}]$ for $\sigma \in 9^i$, the 9-ary expansions of any $y \in A_\sigma$ agree with those of $y(\sigma)$ down to the n_jth place for every $j < i$, and moreover $\mu(A_\sigma \setminus L) < \frac{1}{90} \mu A_\sigma$. Now A_σ must include an interval $I_{\sigma j}$ of $D_{n_i j}$ for each $j < 9$, and $\mu(I_{\sigma j} \cap L) > 0$, so we can find a density point $y_{\sigma^\frown<j>}$ of $L \setminus B$ contained in $I_{\sigma j}$ for each j. Continue. Observe that the effect of this construction is that if $i < \#(\sigma)$ then $y_\sigma \in D_{n_i, \sigma(i)}$.

(b) (Compare Gruenhage & Levy 02) There is a family $\mathcal{R}$ of subsets of $\mathbb{N}$, of cardinal $\mathfrak{c}$, which is independent in the sense that $\bigcap_{i \leq n} R_i \setminus \bigcup_{j \leq m} S_j$ is infinite whenever $R_0, \ldots, R_n, S_0, \ldots, S_m$ are distinct elements of $\mathcal{R}$. **P** By Fremlin 03, 491P, we can actually find such a family for which the asymptotic density of $\bigcap_{i \leq n} R_i \setminus \bigcup_{j \leq m} S_j$ is 2^{-m-n-2} whenever $R_0, \ldots, R_n, S_0, \ldots, S_m$ are distinct elements of $\mathcal{R}$. **Q**

Index $\mathcal{R}$ as $\langle R_{j\xi}\rangle_{j<8,\xi<\mathfrak{c}}$. For $\xi<\mathfrak{c}$, define $x_\xi\in 9^{\mathbb{N}}$ by setting $x_\xi(i)=\#(\{j: i\in R_{j\xi}\})$ for each i. Now set $z_\xi=\lim_{n\to\infty} y(x_\xi\restriction n)$ for each ξ. Then $z_\xi\in L\cap D_{n_i,x_\xi(i)}$ for every $\xi<\mathfrak{c}$, $i\in\mathbb{N}$.

If $\xi_0,\dots,\xi_8<\mathfrak{c}$ are distinct, then there is an $i\in\mathbb{N}$ such that, for $j<8$ and $m<9$, $i\in R_{j\xi_m}\iff j<m$; so that $x_{\xi_m}(i)=m$ for each m, and $z_{\xi_m}\in D_{n_i m}$. Thus if $t\in\mathbb{R}$ the translate $C+t$ cannot contain all the z_{ξ_m}. Turning this round, we see that $\{\xi: z_\xi\in C+t\}$ has at most 8 members, for every $t\in\mathbb{R}$. So if we have any set $Q\subseteq\mathbb{R}$ of cardinal less than $\mathfrak{c}$, there is a $\xi<\mathfrak{c}$ such that $z_\xi\in L\setminus(Q+C)$.

As L is arbitrary, the result is proved.

Remark Gruenhage's result that $\mathbb{R}$ is not covered by fewer than $\mathfrak{c}$ translates of C has been strengthened by Darji & Keleti 03. I do not know whether their methods can be applied to the refinement here. See §7 below for a case essentially identical to the one of this proposition.

5. Corollary There is a set $A\subseteq\mathbb{R}$ such that A has full outer measure for μ but $\#(C\cap(A+t))<\mathfrak{c}$ for every $t\in\mathbb{R}$. If the uniformity non$\mathcal{N}$ of Lebesgue measure is $\mathfrak{c}$, then $\nu(C\cap(A+t))=0$ for every $t\in\mathbb{R}$, where ν is the usual measure on the Cantor set C.

proof Enumerate $\mathbb{R}$ as $\langle t_\xi\rangle_{\xi<\mathfrak{c}}$ and the compact sets of non-zero Lebesgue measure as $\langle L_\xi\rangle_{\xi<\mathfrak{c}}$. By Proposition 4, we can choose $a_\xi\in L_\xi\setminus\bigcup_{\eta<\xi}C+t_\eta$ for each ξ; now set $A=\{a_\xi:\xi<\mathfrak{c}\}$.

6. Proposition (M.Elekes) If the covering number and cofinality of the Lebesgue null ideal are equal, there is a set $A\subseteq\mathbb{R}$ such that A has full outer measure for μ but $\#(C\cap(A+t))\le 8$ for every $t\in\mathbb{R}$.

proof (a) Take the sets D_{nj}, for $n\ge 1$ and $j\le 8$, as in the proof of §4. Set $\kappa=cf\mathcal{N}=\operatorname{cov}\mathcal{N}$, where $\mathcal{N}$ is the Lebesgue null ideal. Let $\langle E_\xi\rangle_{\xi<\kappa}$ enumerate a coinitial subset of $\Sigma\setminus\mathcal{N}$, where Σ is the σ-algebra of Lebesgue measurable sets. (Recall that $\operatorname{ci}(\Sigma\setminus\mathcal{N})=\operatorname{cf}\mathcal{N}$, see Fremlin 08, 524Pb.) Then there is a family $\langle x_\xi\rangle_{\xi<\kappa}$ such that

$x_\xi\in E_\xi$ for every $\xi<\kappa$,

if $\eta_0<\eta_1<\dots<\eta_k<\kappa$ and $j_0,\dots,j_k\le 8$ then there are infinitely many $n\ge 1$ such that $x_{\eta_i}\in D_{nj_i}$ for every $i\le k$.

P Choose x_ξ inductively; the inductive hypothesis will of course be that

if $\eta_0<\eta_1<\dots<\eta_k<\xi$ and $j_0,\dots,j_k\le 8$ then there are infinitely many $n\ge 1$ such that $x_{\eta_i}\in D_{nj_i}$ for every $i\le k$.

Start by taking $x_0\in E_0$ such that $\{n: n\ge 1,\ x_0\in D_{nj}\}$ is infinite for every $j\le 8$; this is possible because $\langle D_{nj}\cap[0,1]\rangle_{n\in\mathbb{N}}$ is stochastically independent for every j, so that for each j the set $\{x: x\in D_{nj}$ for infinitely many $n\}$ is conegligible. When we come to choose ξ, for $\xi>0$, then for each pair $\boldsymbol{\eta}=(\eta_0,\dots,\eta_k)$, $\boldsymbol{j}=(j_0,\dots,j_k)$, where $k\in\mathbb{N}$, $\eta_0<\dots<\eta_k<\xi$ and $j_0,\dots,j_k\le 8$, set

$$I_{\boldsymbol{\eta},\boldsymbol{j}}=\{n: n\ge 1,\ x_{\eta_i}\in D_{nj_i}\text{ for every }i\le k\}.$$

For any $j\le 8$,

$$F_{\boldsymbol{\eta},\boldsymbol{j},j}=\{x:\{n: n\in I_{\boldsymbol{\eta},\boldsymbol{j}},\ x\in D_{nj}\}\text{ is infinite}\}$$

is conegligible. Because $\#(\xi)<\operatorname{cov}\mathcal{N}$, we can therefore find an $x_\xi\in E_\xi$ such that $x_\xi\in F_{\boldsymbol{\eta},\boldsymbol{j},j}$ whenever $\eta_0<\dots<\eta_k<\xi$ and $j_0,\dots,j_k,j\le 8$ (by Fremlin 08, 524Pc, or otherwise, E_ξ cannot be covered by fewer than $\operatorname{cov}\mathcal{N}$ negligible sets), and the induction will proceed. **Q**

(b) Set $A=\{x_\xi:\xi<\kappa\}$. Because A meets every E_ξ, A has full outer measure. If $t\in\mathbb{R}$ and $\eta_0<\dots<\eta_8<\kappa$, then there is an $n\in\mathbb{N}$ such that $x_{\eta_i}\in D_{ni}$ for every $i\le 8$; but there is an $i\le 8$ such that $C+t$ does not meet D_{ni} (see part (a) of the proof of §4), so $x_{\eta_i}\notin C+t$. This shows that $\#(A\cap(C+t))\le 8$ for every $t\in\mathbb{R}$; of course it follows that $\#(C\cap(A+t))\le 8$ for every t.

7. Proposition (a) Set $K=\{\sum_{i=0}^{\infty}5^{-i-1}\epsilon_i:\epsilon_i\in\{0,1,3,4\}$ for every $i\in\mathbb{N}\}$, the 'middle fifth Cantor set'. Then the union of fewer than $\mathfrak{c}$ translates of K always has inner Lebesgue measure 0.

(b) Set $K'=\{\sum_{i=0}^{\infty}5^{-i-1}\epsilon_i:\epsilon_i\in\{0,4\}$ for every $i\in\mathbb{N}\}$, the 'middle three-fifths Cantor set'. Then there is a set $A\subseteq\mathbb{R}$, of full outer Lebesgue measure, such that K' meets every translate of A in at most one point.

(c) (M.Elekes) There are a Radon probability measure $\tilde{\nu}$ on $\mathbb{R}$ and a set A of full outer Lebesgue measure such that $\nu(A+t)=0$ for every $t\in\mathbb{R}$.

proof (a) A trifling variation on the method used in §4 deals with this case also. Let L be a compact set of positive Lebesgue measure. Write B for $\{5^n k : n,\, k\in\mathbb{Z}\}$. For $n\ge 1$ and $j<25$ let D_{nj} be the closed set of those $z\in\mathbb{R}$ such that the fractional part of z has a j in the nth place of (one of) its 25-ary expansions. Note that if $t\in\mathbb{R}$ and $n\ge 1$ then $K+t$ does not meet every D_{nj}. Choose n_i for $i\in\mathbb{N}$ and $y(\sigma)$, for $\sigma\in 25^i$, as follows. $y(\emptyset)$ is to be any density point of $L\setminus B$. Given that $y(\sigma)$ is a density point of $L\setminus B$ for every $\sigma\in 25^i$, let $n_i\ge 1$ be such that $n_i>n_j$ for every $j<i$ and, setting $A_\sigma=[y(\sigma),y(\sigma)+26\cdot 25^{-n_i}]$ for $\sigma\in 25^i$, the 9-ary expansions of any $y\in A_\sigma$ agree with those of $y(\sigma)$ down to the n_jth place for every $j<i$, and moreover $\mu(A_\sigma\setminus L)<\frac{1}{650}\mu A_\sigma$, for every $\sigma\in 25^i$. Now, for each $\sigma\in 25^i$, A_σ must include an interval $I_{\sigma j}$ of $D_{n_i j}$ for each $j<25$, and $\mu(I_{\sigma j}\cap L)>0$, so we can find a density point $y_{\sigma^\frown<j>}$ of $L\setminus B$ contained in $I_{\sigma j}$ for each j. Continue. Observe that the effect of this construction is that if $i<\#(\sigma)$ then $y_\sigma\in D_{n_i,\sigma(i)}$.

Again take a fully independent family $\mathcal{R}$ of subsets of $\mathbb{N}$ of cardinal $\mathfrak{c}$, and index it as $\langle R_{j\xi}\rangle_{j<24,\xi<\mathfrak{c}}$. For $\xi<\mathfrak{c}$, define $x_\xi\in 25^{\mathbb{N}}$ by setting $x_\xi(i)=\#(\{j: i\in R_{j\xi}\})$ for each i. Now set $z_\xi=\lim_{n\to\infty}y(x_\xi\restriction n)$ for each ξ. Then $z_\xi\in L\cap D_{n_i,x_\xi(i)}$ for every $\xi<\mathfrak{c}$, $i\in\mathbb{N}$.

If $\xi_0,\dots,\xi_{24}<\mathfrak{c}$ are distinct, then there is an $i\in\mathbb{N}$ such that, for $j<24$, $m<25$, $i\in R_{j\xi_m}\iff j<m$; so that $x_{\xi_m}(i)=m$ for each m, and $z_{\xi_m}\in D_{n_i m}$. Thus if $t\in\mathbb{R}$ the translate $K+t$ cannot contain all the z_{ξ_m}, and $\{\xi: z_\xi\in K+t\}$ has at most 24 members, for every $t\in\mathbb{R}$. So if we have any set $Q\subseteq\mathbb{R}$ of cardinal less than $\mathfrak{c}$, there is a $\xi<\mathfrak{c}$ such that $z_\xi\in L\setminus(Q+K)$, and $Q+K$ cannot cover L. As L is arbitrary, the result is proved.

(b) The point is that $K'-K'\subseteq K+(K-1)$. **P** Setting $F_{nj}=\bigcup_{k\in\mathbb{N}}5^{-n-1}[j+5k,j+5k+1]$ for $n\in\mathbb{N}$, we have $K'=[0,1]\cap\bigcap_{n\in\mathbb{N}}(F_{n0}\cup F_{n4})$, while $K=[0,1]\cap\bigcap_{n\in\mathbb{N}}(F_{n0}\cup F_{n1}\cup F_{n3}\cup F_{n4})$ and $K-1=[-1,0]\cap\bigcap_{n\in\mathbb{N}}(F_{n0}\cup F_{n1}\cup F_{n3}\cup F_{n4})$. Since

$$F_{n0}-F_{n0}\subseteq F_{n0}\cup F_{n4},$$

$$F_{n0}-F_{n4}\subseteq F_{n0}\cup F_{n1},$$

$$F_{n4}-F_{n0}\subseteq F_{n3}\cup F_{n4},$$

$$F_{n4}-F_{n4}\subseteq F_{n0}\cup F_{n4}$$

for every n,

$$K'-K'\subseteq[-1,1]\cap\bigcap_{n\in\mathbb{N}}(F_{n0}\cup F_{n1}\cup F_{n3}\cup F_{n4})\subseteq K\cup(K-1).\ \mathbf{Q}$$

Now let $\langle L_\xi\rangle_{\xi<\mathfrak{c}}$ run over the non-negligible compact subsets of $\mathbb{R}$. Choose $\langle x_\xi\rangle_{\xi<\mathfrak{c}}$ such that

$$x_\xi\in L_\xi\setminus\bigcup_{\eta<\xi}((K+x_\eta)\cup(K+x_\eta-1))$$

for every $\xi<\mathfrak{c}$; this is possible by (a). Then $x_\xi\notin(K'-K')+x_\eta$, that is, $x_\xi-K'$ and $x_\eta-K'$ are disjoint, whenever $\eta<\xi$; turning this round, no translate of K' can contain x_ξ for more than one ξ. So we can set $A=\{x_\xi:\xi<\mathfrak{c}\}$.

(c) We have only to take the set A of (b) and the image $\tilde{\nu}$ of the usual measure on $\{0,1\}^{\mathbb{N}}$ under the function $z\mapsto\sum_{n=0}^{\infty}4\cdot 5^{-n-1}z(n)$.

8 Remark Recall that in any Polish group X, a set D is said to be **Haar null** if there are a universally measurable set $E\supseteq D$ and a non-zero Radon measure ν on X such that $\nu(xEy)=0$ for all x, $y\in X$ (Fremlin 03, 444Ye). If X is locally compact, then the Haar null sets are just those which are neglible for the Haar measures on X. Now the set A of §7(b)-(c) above is such that $\tilde{\nu}(x+A+y)=0$ for all x, $y\in\mathbb{R}$, but is not Haar null. If either $\operatorname{non}\mathcal{N}=\mathfrak{c}$ or $\operatorname{cov}\mathcal{N}=\operatorname{cf}\mathcal{N}$, as in §§5-6, then we get an example witnessed by the usual measure on the Cantor set.

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