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Four problems in measure theory

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1 The control measure problem

1A The problem Let X be a set and Σ an algebra of subsets of X. Let $\nu : \Sigma \to [0, \infty[$ be a function such that

(i) $\nu\emptyset = 0$

(ii) $\nu E \le \nu F$ whenever E, $F \in \Sigma$ and $E \subseteq F$

(iii) $\nu(E \cup F) \le \nu E + \nu F$ whenever E, $F \in \Sigma$

(ν is a **submeasure**)

(iv) $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ

(ν is **exhaustive**). Does there necessarily exist a functional $\mu : \Sigma \to [0, \infty[$ which is **additive**, that is, $\mu(E \cup F) = \mu E + \mu F$ whenever E, $F \in \Sigma$ are disjoint, and such that ν is **absolutely continuous** with respect to μ, that is, for every $\epsilon > 0$ there is a $\delta > 0$ such that $\nu E \le \epsilon$ whenever $\mu E \le \delta$?

This was for many years one of the classic outstanding problems of measure theory. It was important because it is equivalent to a large number of questions in apparently different topics. I will present one of these in 1I-1K below. Here I will try to give one major theorem, due to Kalton & Roberts 83, and some examples. The (negative, as expected) solution may be found in Talagrand 08, my own note Fremlin n08, or in the web version of Fremlin 02, §394.

1B Definitions We shall need the following ideas.

(a) A submeasure ν on an algebra Σ is **uniformly exhaustive** if for every $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\min_{i\le n} \nu E_i \le \epsilon$ whenever $E_0, \dots , E_n \in \Sigma$ are disjoint.

(b) If $\mathcal{A}$ is a family of sets, its **intersection number** is the greatest number $\delta \ge 0$ such that whenever $\langle A_i\rangle_{i\in I}$ is a finite family in $\mathcal{A}$ (with repetitions allowed), there is a $J \subseteq I$, with $\#(J) \ge \delta\#(I)$, such that $\bigcap_{i\in J} A_i \ne \emptyset$.

1C Kelley's theorem Let X be a non-empty set and $\mathcal{A}$ a family of subsets of X with intersection number δ. Then there is a finitely additive functional $\mu : \mathcal{P}X \to [0, 1]$ such that $\mu X = 1$ and $\mu A \ge \delta$ for every $A \in \mathcal{A}$.

proof If $\mathcal{A}$ has the finite intersection property, let $\mathcal{F}$ be an ultrafilter on X including $\mathcal{A}$, and set $\mu C = 1$ if $C \in \mathcal{F}$, 0 otherwise; this works. Otherwise, on the linear space $\ell^\infty(X)$ of bounded real-valued functions on X define a functional p by setting

$$p(u) = (1 - \delta) \inf\{\sum_{i=0}^{n} \alpha_i : \alpha_0, \dots , \alpha_n \ge 0,\ |u| \le \sum_{i=0}^{n} \alpha_i\chi(X \setminus A_i) \text{ for some } A_0, \dots , A_n \in \mathcal{A}\}.$$

Because $\mathcal{A}$ does not have the finite intersection property, $p(u)$ is well-defined. It is easy to see that p is a seminorm. Also $p(\chi X) \ge 1$. **P?** Otherwise, we have $\alpha_0, \dots , \alpha_k \ge 0$ and $A_0, \dots , A_k \in \mathcal{A}$ such that $(1 - \delta)\sum_{i=0}^{k} \alpha_i < 1$ and $\chi X \le \sum_{i=0}^{k} \alpha_i\chi(X \setminus A_i)$. Increasing the α_i fractionally if necessary, we may suppose that they are all rational; say $\alpha_i = r_i/m$ where $r_i \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{0\}$. Set $n = \sum_{i=0}^{k} r_i$ and set $B_j = A_i$ for $0 \le i \le k$, $\sum_{l<i} r_l \le j < \sum_{l\le i} r_l$. Then $\chi X \le \sum_{j=0}^{n-1} \frac{1}{m}\chi(X \setminus B_j)$, while $(1 - \delta)\sum_{j=0}^{n-1} \frac{1}{m} < 1$. But there must be a set $J \subseteq \{0, \dots , n-1\}$ such that $\#(J) \ge \delta n$ and $\bigcap_{j\in J} B_j \ne \emptyset$. Take $x \in \bigcap_{j\in J} B_j$; then we have

$$1 = \chi X(x) \le \sum_{j=0}^{n} \frac{1}{m}\chi(X \setminus B_j)(x) \le \frac{n-\#(J)}{m} \le \frac{n(1-\delta)}{m} < 1,$$

which is impossible. **XQ**

There is therefore a linear functional $h : \ell^\infty(X) \to \mathbb{R}$ such that $h(\chi X) = 1$ and $h(u) \le p(u)$ for every $u \in \ell^\infty(X)$. Set $\lambda_0 D = h(\chi D)$ for $D \subseteq X$; then $\lambda_0 : \mathcal{P}X \to \mathbb{R}$ is an additive functional, $\lambda_0 X = 1$, $\lambda_0 D \le p(\chi X)$ for every $D \subseteq X$ and $\lambda_0 D \le 1 - \delta$ whenever $A \in \mathcal{A}$ and $D \subseteq X \setminus A$. Set $\lambda_1 C = \sup\{\lambda_0 D : D \subseteq C\}$ for $C \subseteq X$; it is easy to check that $\lambda_1 : \mathcal{P}X \to [0, \infty[$ is additive, and $\lambda_1(X) \ge 1$, $\lambda_1(X \setminus A) \le 1 - \delta$ for every $A \in \mathcal{A}$. Set $\mu C = \frac{1}{\lambda_1 X}\lambda_1 C$ for $C \subseteq X$; then $\mu : \mathcal{P}X \to [0, 1]$ is additive, $\mu X = 1$ and $\mu(X \setminus A) \le 1 - \delta$ for every $A \in \mathcal{A}$. But this means that $\mu A \ge \delta$ for every $A \in \mathcal{A}$.

1D Lemma Suppose that k, l, $m \in \mathbb{N}$ are such that $3 \le k \le l \le m$ and $18mk \le l^2$. Let L, M be sets of sizes l, m respectively. Then there is a set $R \subseteq M \times L$ such that (i) each vertical section of R has just three members (ii) $\#(R[I]) \ge \#(I)$ whenever $I \in [M]^{\le k}$; so that for every $I \in [M]^{\le k}$ there is an injective function $f : I \to L$ such that $(x, f(x)) \in R$ for every $x \in I$.

notation $[M]^{\le k} = \{I : I \subseteq M, \#(I) \le k\}$, $[M]^k = \{I : I \subseteq M, \#(I) = k\}$, $R[I] = \{y : \exists\, x \in I, (x, y) \in R\}$.

proof (a) We need to know that $n! \ge 3^{-n}n^n$ for every $n \in \mathbb{N}$; this is immediate from the inequality

$$\textstyle\sum_{i=2}^{n} \ln i \ge \int_1^n \ln x\,dx = n\ln n - n + 1 \text{ for every } n \ge 2.$$

(b) Let Ω be the set of those $R \subseteq M \times L$ such that each vertical section of R has just three members, so that

$$\#(\Omega) = \#([L]^3)^m = \big(\frac{l!}{3!(l-3)!}\big)^m.$$

Let us regard Ω as a probability space with the uniform probability.

If $J \in [L]^n$, where $3 \le n \le k$, and $x \in M$, then

$$\Pr(R[\{x\}] \subseteq J) = \frac{\#([J]^3)}{\#([L]^3)}$$

(because $R[\{x\}]$ is a random member of $[L]^3$)

$$= \frac{n(n-1)(n-2)}{l(l-1)(l-2)} \le \frac{n^3}{l^3}.$$

So if $I \in [M]^n$ and $J \in [L]^n$, then

$$\Pr(R[I] \subseteq J) = \prod_{x \in I} \Pr(R[\{x\}] \subseteq J)$$

(because the sets $R[\{x\}]$ are chosen independently)

$$\le \frac{n^{3n}}{l^{3n}}.$$

Accordingly

$$\Pr(\text{there is an } I \subseteq M \text{ such that } \#(R[I]) < \#(I) \le k)$$
$$\le \Pr(\text{there is an } I \subseteq M \text{ such that } 3 \le \#(R[I]) \le \#(I) \le k)$$

(because if $I \ne \emptyset$ then $\#(R[I]) \ge 3$)

$$\leq \sum_{n=3}^{k} \sum_{I\in[M]^n} \sum_{J\in[L]^n} \Pr(R[I] \subseteq J) \leq \sum_{n=3}^{k} \#([M]^n)\#([L]^n)\frac{n^{3n}}{l^{3n}}$$

$$= \sum_{n=3}^{k} \frac{m!}{n!(m-n)!}\frac{l!}{n!(l-n)!}\frac{n^{3n}}{l^{3n}} \leq \sum_{n=3}^{k} \frac{m^n l^n n^{3n}}{n!n!l^{3n}} \leq \sum_{n=3}^{k} \frac{m^n n^n 3^{2n}}{l^{2n}}$$

(using (a))

$$= \sum_{n=3}^{k} \Bigl(\frac{9mn}{l^2}\Bigr)^n \leq \sum_{n=3}^{k} \frac{1}{2^n} < 1.$$

There must therefore be some $R \in \Omega$ such that $\#(R[I]) \geq \#(I)$ whenever $I \subseteq M$ and $\#(I) \leq k$.

(c) If now $I \in [M]^{\leq k}$, the restriction $R_I = R \cap (I \times L)$ has the property that $\#(R_I[I']) \geq \#(I')$ for every $I' \subseteq I$. By Hall's Marriage Lemma there is an injective function $f : I \to L$ such that $(x, f(x)) \in R_I \subseteq R$ for every $x \in I$.

1E Lemma Let Σ be an algebra of subsets of X and $\nu : \Sigma \to [0, \infty[$ a uniformly exhaustive submeasure. Then for any $\epsilon \in]0, \nu 1]$ the set $\mathcal{E}_\epsilon = \{E : \nu E \geq \epsilon\}$ has intersection number greater than 0.

proof (a) If $\nu X = 0$ this is trivial, so we may assume that $\nu X > 0$; since neither the hypothesis nor the conclusion is affected if we multiply ν by a positive scalar, we may suppose that $\nu X = 1$. Because ν is uniformly exhaustive, there is an $r \geq 1$ such that whenever $\langle G_i\rangle_{i\in I}$ is a disjoint family in Σ then $\#(\{i : \nu G_i > \frac{1}{5}\epsilon\}) \leq r$, so that $\sum_{i\in I} \nu G_i \leq r + \frac{1}{5}\epsilon\#(I)$. Set $\delta = \epsilon/5r$, $\eta = \frac{1}{74}\delta^2$, so that

$$\delta - \eta \geq \tfrac{1}{18}(\delta - \eta)^2 \geq \tfrac{1}{18}(\delta^2 - 2\eta) = 4\eta.$$

(b) Let $\langle E_i\rangle_{i\in I}$ be a non-empty finite family in $\mathcal{E}_\epsilon$. Let m be any multiple of $\#(I)$ greater than or equal to $1/\eta$. Then there are integers k, l such that

$$3\eta \leq \frac{k}{m} \leq 4\eta \leq \frac{1}{18}(\delta - \eta)^2, \quad \delta - \eta \leq \frac{l}{m} \leq \delta,$$

in which case

$$3 \leq k \leq l \leq m, \quad 18mk \leq m^2(\delta - \eta)^2 \leq l^2.$$

(c) Take a set M of the form $I \times S$ where $\#(S) = m/\#(I)$, so that $\#(M) = m$. For $q = (i, s) \in M$ set $H_q = E_i$. Let L be a set with l members. By Lemma 1D, there is a set $R \subseteq M \times L$ such that every vertical section of R has just three members and whenever $Q \in [M]^{\leq k}$ there is an injective function $f_Q : Q \to L$ such that $(q, f_Q(q)) \in R$ for every $q \in Q$.

For $Q \subseteq M$ set

$$b_Q = \bigcap\nolimits_{q\in Q} H_q \setminus \bigcup\nolimits_{q\in M\setminus Q} H_q,$$

so that $\langle b_Q\rangle_{Q\subseteq M}$ is a partition of X into members of Σ. For $q \in M$, $j \in L$ set

$$G_{qj} = \bigcup\{b_Q : q \in Q \in [M]^{\leq k},\ f_Q(q) = j\}.$$

If p, q are distinct members of M and $j \in L$ then

$$G_{pj} \cap G_{qj} = \bigcup\{b_Q : p, q \in Q \in [M]^{\leq k},\ f_Q(p) = f_Q(q) = j\} = \emptyset,$$

because every f_Q is injective. Set

$$m_j = \#(\{q : q \in M,\ G_{qj} \neq \emptyset\})$$

for each $j \in L$. Note that $G_{qj} = \emptyset$ if $(q, j) \notin R$, so $\sum_{j\in L} m_j \leq \#(R) = 3m$.

We have

$$\sum\nolimits_{q\in M} \nu G_{qj} \leq r + \frac{1}{5}\epsilon m_j$$

for each j, by the choice of r; so

$$\sum_{q\in M,j\in L}\nu G_{qj} \le rl + \frac{1}{5}\epsilon\sum_{j\in L} m_j \le rl + \frac{3}{5}m\epsilon$$

$$\le (r\delta + \frac{3}{5}\epsilon)m = \frac{4}{5}\epsilon m < \epsilon m$$

by the choice of l and δ. There must therefore be some $q \in M$ such that

$$\nu(\bigcup_{j\in L} G_{qj}) \le \sum_{j\in L}\nu G_{qj} < \epsilon \le \nu H_q,$$

and H_q cannot be included in

$$\bigcup_{j\in L} G_{qj} = \bigcup\{b_Q : q \in Q \in [M]^{\le k}\}.$$

But as $\bigcup\{b_Q : q \in Q \subseteq M\}$ is just H_q, there must be an $Q \subseteq M$, of cardinal greater than k, such that $b_Q \ne \emptyset$.

Recall now that $M = I \times S$, and that

$$k \ge 3\eta m = 3\eta\#(I)\#(S).$$

The set $J = \{i : \exists\ s,\ (i,s) \in Q\}$ must therefore have more than $3\eta\#(I)$ members, since $Q \subseteq J \times S$. But also $H_{(i,s)} = E_i$ for each $(i,s) \in Q$, so that $\bigcap_{i\in J} E_i \supseteq b_Q \ne \emptyset$.

(d) As $\langle E_i\rangle_{i\in I}$ is arbitrary, the intersection number of $\mathcal{E}_\epsilon$ is at least $3\eta > 0$.

1F The Kalton-Roberts theorem Let Σ be an algebra of subsets of X, and $\nu : \Sigma \to [0,\infty[$ a uniformly exhaustive submeasure. Then there is an additive functional $\mu : \Sigma \to [0,\infty[$ such that ν is absolutely continuous with respect to μ.

proof If $\nu X = 0$ this is trivial. Otherwise, for each n, $\mathcal{E}_n = \{a : \nu a \ge 2^{-n}\nu X\}$ has intersection number $\delta_n > 0$. By 1C, there is a finitely additive functional $\mu_n : \mathcal{P}X \to [0,1]$ such that $\mu_n X = 1$ and $\mu_n E \ge \delta_n$ for every $E \in \mathcal{E}_n$. Set $\mu H = \sum_{n=0}^{\infty} 2^{-n}\mu_n H$ for $H \in \Sigma$; then $\mu : \Sigma \to [0,2]$ is finitely additive and $\nu E < 2^{-n}$ whenever $\mu E < 2^{-n}\delta_n$. So ν is absolutely continuous with respect to μ.

1G Example The following example, taken from Talagrand 80, shows that most of the natural routes to a positive answer to the question in 1A are blocked.

Fix $n \ge 1$, and let I be the set $\{0, 1, \dots, 2n-1\}$, $X = [I]^n$, so that X is a finite set. For each $i \in I$ set $A_i = \{a : i \in a \in X\}$. For $E \subseteq X$ set

$$\nu E = \frac{1}{n+1}\inf\{\#(J) : J \subseteq I,\ E \subseteq \bigcup_{i\in J} A_i\}$$

$$= \frac{1}{n+1}\inf\{\#(J) : a \cap J \ne \emptyset \text{ for every } a \in E\}.$$

It is elementary to check that $\nu : \mathcal{P}X \to [0,\infty[$ is a submeasure, therefore (because $\mathcal{P}X$ is finite) a Maharam submeasure.

The essential properties of ν are twofold: (i) $\nu X = 1$; (ii) for any non-negative additive functional μ such that $\mu E \le \nu E$ for every $E \subseteq X$, $\mu X \le \frac{2}{n+1}$. **P** (i)(α) If $J \subseteq I$ and $\#(J) \le n$, there is an $a \in [I \setminus J]^n$, so that $a \in X \setminus \bigcup_{i\in J} A_i$ and $X \not\subseteq \bigcup_{i\in J} A_i$. This means that X cannot be covered by fewer than $n+1$ of the sets A_i, so that νX must be at least 1. (β) On the other hand, if $J \subseteq I$ is any set of cardinal $n+1$, $a \cap J \ne \emptyset$ for every $a \in X$, so that $X = \bigcup_{i\in J} A_i$ and $\nu X \le 1$. (ii) Every member of X belongs to just n of the sets A_i, so

$$n\mu X = \sum_{i\in I}\mu A_i \le \frac{\#(I)}{n+1} = \frac{2n}{n+1},$$

and $\mu X \le \frac{2}{n+1}$. **Q**

1H Example For the next example, I present a much deeper idea from Roberts 93.

Let $\mathfrak{B}$ be the algebra of open-and-closed subsets of $\{0,1\}^{\mathbb{N}}$. Then for any $\epsilon > 0$ we can find a submeasure $\nu : \mathfrak{B} \to [0,1]$ such that

(i) for every $n \in \mathbb{N}$ there is a disjoint sequence $S_{n0}, \dots, S_{nn}$ in $\mathfrak{B}$ such that $\nu S_{ni} = 1$ for every $i \le n$;

(ii) if $\langle E_n \rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathfrak{B}$ then $\limsup_{n\to\infty} \nu E_n \le \epsilon$.

proof (a) For each $n \in \mathbb{N}$ let I_n be the finite set $\{0, \dots, n\}$, given its discrete topology; set $X = \prod_{n\in\mathbb{N}} I_n$, with the product topology; let $\mathfrak{C}$ be the algebra of subsets of X generated by sets of the form $S_{ij} = \{x : x(i) = j\}$, where $i \in \mathbb{N}$ and $j \le i$. Note that X is compact and Hausdorff and that every member of $\mathfrak{C}$ is open-and-closed (because all the S_{ij} are). Also $\mathfrak{C}$ is atomless, countable and non-zero, so is isomorphic to $\mathfrak{B}$. It will therefore be enough if I can describe a submeasure $\nu : \mathfrak{C} \to [0, 1]$ with the properties (i) and (ii) above, and this is what I will do.

(b) For each $n \in \mathbb{N}$ let $\mathcal{A}_n$ be the set of non-empty members of $\mathfrak{C}$ determined by coordinates in $\{0, \dots, n\}$; note that $\mathcal{A}_n$ is finite. For $k \le l \in \mathbb{N}$, say that $E \in \mathfrak{C}$ is (k, l)**-thin** if for every $A \in \mathcal{A}_k$ there is an $A' \in \mathcal{A}_l$ such that $A' \subseteq A \setminus E$. Note that if $k' \le k \le l \le l'$ then $\mathcal{A}_{k'} \subseteq \mathcal{A}_k$ and $\mathcal{A}_l \subseteq \mathcal{A}_{l'}$, so any (k, l)-thin set is also (k', l')-thin.

Say that every $E \in \mathfrak{C}$ is $(k, 0)$**-small** for every $k \in \mathbb{N}$, and that for k, $r \in \mathbb{N}$ a set $E \in \mathfrak{C}$ is $(k, r+1)$**-small** if there is some $l \ge k$ such that E is (k, l)-thin and (l, r)-small. Observe that E is $(k, 1)$-small iff it is (k, l)-thin for some $l \ge k$, that is, there is no member of $\mathcal{A}_k$ included in E. Observe also that if E is (k, r)-small then it is (k', r)-small for every $k' \le k$.

Write $\mathcal{S} = \{S_{ij} : j \le i \in \mathbb{N}\}$.

(c) Suppose that $E \in \mathfrak{C}$ and $k \le l \le m$ are such that E is both (k, l)-thin and (l, m)-thin. Then whenever $A \in \mathcal{A}_k$, $S \in \mathcal{S}$ and $A \cap S \ne \emptyset$, there is an $A' \in \mathcal{A}_m$ such that $A' \subseteq A \setminus E$ and $A' \cap S \ne \emptyset$. **P** Take $S = S_{ni}$ where $i \le n$. (i) If $n \le l$, then $A \cap S \in \mathcal{A}_l$; because E is (l, m)-thin, there is an $A' \in \mathcal{A}_m$ such that $A' \subseteq (A \cap S) \setminus E$. (ii) If $n > l$, there is an $A' \in \mathcal{A}_l$ such that $A' \subseteq A \setminus E$, because E is (k, l)-thin; now $A' \in \mathcal{A}_m$, and $A' \cap S$ is non-empty because A' is determined by coordinates less than n. **Q**

(d) It follows that if $S \in \mathcal{S}$, $k \in \mathbb{N}$, $A \in \mathcal{A}_k$, $A \cap S \ne \emptyset$, $r \in \mathbb{N}$ and $E_0, \dots, E_{r-1}$ are $(k, 2r)$-small, then $A \cap S$ is not covered by $E_0, \dots, E_{r-1}$. **P** Induce on r. The case $r = 0$ demands only that $A \cap S$ should not be covered by the empty sequence, that is, $A \cap S \ne \emptyset$, which is one of the hypotheses. For the inductive step to $r + 1$, we know that for each $j \le r$ there are l_j, m_j such that $k \le l_j \le m_j$ and E_j is (k, l_j)-thin and (l_j, m_j)-thin and $(m_j, 2r)$-small. Rearranging $E_0, \dots, E_r$ if necessary we may suppose that $m_r \le m_j$ for every $j \le r$; set $m = m_r$. By (c), there is an $A' \in \mathcal{A}_m$ such that $A' \cap S \ne \emptyset$ and $A' \subseteq A \setminus E_r$. Now every E_j, for $j < r$, is $(m_j, 2r)$-small, therefore $(m, 2r)$-small, so by the inductive hypothesis $A' \cap S \not\subseteq \bigcup_{j<r} E_j$. Accordingly $A \cap S \not\subseteq \bigcup_{j\le r} E_j$ and the induction continues. **Q**

(e) Now suppose that $\langle E_n \rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{C}$. Then for any $k \in \mathbb{N}$ there are l, $n^* \in \mathbb{N}$ such that E_n is (k, l)-thin for every $n \ge n^*$. **P** Consider $G_n = \bigcup_{j\ge n} E_j$ for each $n \in \mathbb{N}$. Then every G_n is open and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$. If $A \in \mathcal{A}_k$, then A, with its subspace topology, is compact, so Baire's theorem (3A3G) tells us that there is an n_A such that $G_{n_A} \cap A$ is not dense in A; let l_A be such that $A \setminus G_{n_A}$ includes a member of $\mathcal{A}_{l_A}$. Set $n^* = \max\{n_A : A \in \mathcal{A}_k\}$, $l = \max\{l_A : A \in \mathcal{A}_k\}$. If $n \ge n^*$, $A \in \mathcal{A}_k$ there is an $A' \in \mathcal{A}_{l_A} \subseteq \mathcal{A}_l$ such that

$$A' \subseteq A \setminus G_{n_A} \subseteq A \setminus E_n.$$

As A is arbitrary, E_n is (k, l)-thin. **Q**

It follows at once that for any $r \in \mathbb{N}$ we can find n^*_r, $k_0 < k_1 < \dots < k_r \in \mathbb{N}$ such that E_n is (k_j, k_{j+1})-thin for every $j < r$ and $n \ge n^*_r$; so that E_n is $(0, r)$-small for every $n \ge n^*_r$.

(f) Take an integer $r \ge 1/\epsilon$. Let $\mathcal{U}$ be the set of $(0, 2r)$-small members of $\mathfrak{C}$. Set

$$\nu E = \frac{1}{r} \min\{m : E \subseteq E_1 \cup \dots \cup E_m \text{ for some } E_1, \dots, E_m \in \mathcal{U}\}$$

if E can be covered by r or fewer members of $\mathcal{U}$, 1 otherwise. It is easy to check that $\nu : \mathfrak{C} \to [0, 1]$ is a submeasure. By (d), no member of $\mathcal{S}$ can be covered by r or fewer members of $\mathcal{U}$, so $\nu S_{ni} = 1$ whenever $i \le n \in \mathbb{N}$. By (e), if $\langle E_n \rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathfrak{C}$, E_n belongs to $\mathcal{U}$ for all but finitely many n, so that $\nu E_n \le \frac{1}{r} \le \epsilon$ for all but finitely many n. Thus ν has the required properties.

1I An equivalent problem Consider the question

> Let X be a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a non-zero exhaustive submeasure. Does there necessarily exist a non-zero additive functional $\mu : \Sigma \to [0, \infty[$ such that μ is absolutely continuous with respect to ν?

It is very striking that asking for an additive functional which will dominate ν (in the sense of 1A) should come to the same thing as asking for an additive functional dominated by ν. Here I will briefly sketch what I think are the essential reasons for this coincidence. The details may be found in Fremlin 02, §393.

1J Boolean algebras defined from submeasures (a) If X is a set, Σ an algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a submeasure, then $\mathcal{I} = \{E : \nu E = 0\}$ is an *ideal* of the Boolean algebra Σ, that is,

$\emptyset \in \mathcal{I}$,
$E \cup F \in \mathcal{I}$ for all E, $F \in \mathcal{I}$,
$E \in \mathcal{I}$ whenever $E \in \Sigma$ and $E \subseteq F \in \mathcal{I}$.

We therefore have an equivalence relation

$$E \sim F \iff E \triangle F \in \mathcal{I}$$

on Σ, and can form a quotient Boolean algebra $\mathfrak{A} = \Sigma/\mathcal{I}$ made up of the equivalence classes under Σ with Boolean operations defined by

$$E^\bullet \cup F^\bullet = (E \cup F)^\bullet, \quad E^\bullet \cap F^\bullet = (E \cap F)^\bullet,$$

$$E^\bullet \setminus F^\bullet = (E \setminus F)^\bullet, \quad E^\bullet \triangle F^\bullet = (E \triangle F)^\bullet,$$

$$E^\bullet \subseteq F^\bullet \iff E \setminus F \in \mathcal{I}$$

(Fremlin 02, §312). Moreover, it is easy to see that $\nu E = \nu F$ whenever $E \sim F$, so that we have a functional $\bar{\nu} : \mathfrak{A} \to [0, \infty[$ defined by setting

$$\bar{\nu} E^\bullet = \nu E \text{ for every } E \in \Sigma.$$

This will be a submeasure in the sense that

$$\bar{\nu} 0 = 0, \quad \bar{\nu} a \le \bar{\nu} b \text{ whenever } a \subseteq b, \quad \bar{\nu}(a \cup b) \le \bar{\nu} a + \bar{\nu} b \text{ for all } a,\, b \in \mathfrak{A}.$$

Moreover, $\bar{\nu}$ is exhaustive in the sense that $\lim_{n\to\infty} \bar{\nu} a_n = 0$ whenever $\langle a_n \rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $a_m \cap a_n = 0$ for every m, $n \in \mathbb{N}$, since in this case, if $a_n = E_n^\bullet$ for every n, $\bar{\nu} a_n = \nu(E_n \setminus \bigcup_{i<n} E_i) \to 0$ as $n \to \infty$. The difference between ν and $\bar{\nu}$ is that $\bar{\nu}$ is **strictly positive**, that is, if $a \in \mathfrak{A} \setminus \{0\}$, so that $a = E^\bullet$ for some $E \in \Sigma \setminus \mathcal{I}$, then $\bar{\nu} a = \nu E > 0$.

(b) This means that if we set

$$\rho(a, b) = \bar{\nu}(a \triangle b) \text{ for } a,\, b \in \mathfrak{A},$$

then ρ is a metric on $\mathfrak{A}$. (For the triangle inequality we need check only that $a \triangle c \subseteq (a \triangle b) \cup (b \triangle c)$ for all a, b, $c \in \mathfrak{A}$.) It is elementary now to confirm that for all four Boolean operations $* = \cup$, $\cap$, $\setminus$ and $\triangle$,

$$\rho(a * a', b * b') \le \rho(a, b) + \rho(a', b')$$

because

$$(a * a') \triangle (b * b') \subseteq (a \triangle b) \cup (a' \triangle b'),$$

while

$$|\bar{\nu} a - \bar{\nu} b| \le \max(\bar{\nu} a - \bar{\nu}(a \cap b), \bar{\nu} b - \bar{\nu}(a \cap b)) \le \rho(a, b)$$

for all a, b, a', $b' \in \mathfrak{A}$. Thus the Boolean operations, and $\bar{\nu}$, are uniformly continuous.

(c) Suppose now we complete $\mathfrak{A}$ under the metric ρ to form a complete metric space $(\widehat{\mathfrak{A}}, \hat{\rho})$. Then the Boolean operations extend to $\widehat{\mathfrak{A}}$, and this is still a Boolean algebra; moreover, $\bar{\nu}$ extends to a functional $\hat{\nu} : \widehat{\mathfrak{A}} \to [0, \infty[$. Because

$$\hat{\nu}(a \cap b) \le \hat{\nu} b, \quad \hat{\nu}(a \cup b) \le \hat{\nu} a + \hat{\nu} b, \quad \hat{\nu} a = \hat{\rho}(a, 0)$$

for all a, $b \in \widehat{\mathfrak{A}}$, $\hat{\nu}$ is still a strictly positive submeasure on $\widehat{\mathfrak{A}}$.

More surprising is the fact that $\hat{\nu}$ is still exhaustive. (This requires a slightly deeper argument; see Fremlin 03, 393B.) It follows that if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\widehat{\mathfrak{A}}$, it is a Cauchy sequence for $\hat{\rho}$, and has a limit. For any $b\in\widehat{\mathfrak{A}}$, the sets

$$\{c:c\subseteq b\}=\{c:\hat{\rho}(c,c\cap b)=0\},\ \{c:b\subseteq c\}=\{c:\hat{\rho}(b,c\cap b)=0\}$$

are closed; it follows that $a=\inf_{n\in\mathbb{N}}a_n$ in the partially ordered set $\widehat{\mathfrak{A}}$. Turning this round, if $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\widehat{\mathfrak{A}}$ with infimum 0, then it converges topologically to 0, so $\lim_{n\to\infty}\bar{\nu}a_n=\lim_{n\to\infty}\rho(a_n,0)=0$; $\hat{\nu}$ is a **Maharam submeasure**. At the same time, we see that $\widehat{\mathfrak{A}}$ is **Dedekind σ-complete**, that is, any countable subset of $\widehat{\mathfrak{A}}$ has a supremum and infimum in $\widehat{\mathfrak{A}}$.

(d) There is more. Because $\widehat{\mathfrak{A}}$ has a strictly positive exhaustive submeasure, it must be **ccc**, that is, every disjoint set is countable. This means that if $A\subseteq\widehat{\mathfrak{A}}$ is a non-empty downwards-directed set with infimum 0, there is a non-increasing sequence in A with infimum 0 (Fremlin 02, 316E), so that $\inf_{a\in A}\hat{\nu}a=0$. Because $\hat{\nu}$ is sequentially order-continuous, we find also that $\widehat{\mathfrak{A}}$ is **weakly (σ,∞)-distributive**, that is,

whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of non-empty downwards-directed sets in $\widehat{\mathfrak{A}}$ all with infimum 0, then

$$B=\{b:\forall\,n\in\mathbb{N}\ \exists\,a\in A_n,\ a\subseteq b\}$$

has infimum 0.

(For given $\epsilon>0$ we can find $a_n\in A_n$ such that $\hat{\nu}a_n\le 2^{-n}\epsilon$; now $b=\sup_{n\in\mathbb{N}}a_n$ belongs to B and $\hat{\nu}b\le 2\epsilon$.)

(e) The remaining facts we need are the following.

(i) Any Boolean algebra $\mathfrak{A}$ is isomorphic to some algebra of sets. (Fremlin 02, §311.)

(ii) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, and μ, ν two Maharam submeasures on $\mathfrak{A}$ such that $\mu a=0$ whenever $\nu a=0$, then μ is absolutely continuous with respect to ν. (Copy the standard proof for the case when $\mathfrak{A}$ is a σ-algebra of sets and μ, ν are both countably additive measures.)

(iii) If $\mathfrak{A}$ is a Boolean algebra, and $\mu:\mathfrak{A}\to[0,\infty[$ is an additive functional, and we set

$$\nu b=\inf\{\sup_{a\in A}\mu a: A\subseteq\mathfrak{A}\text{ is a non-empty upwards-directed set with supremum }b\}$$

for every $b\in\mathfrak{A}$, then ν is an order-continuous additive functional (Fremlin 02, 362Bd); and if $\mathfrak{A}$ is weakly (σ,∞)-distributive and μ is strictly positive, so is ν (Fremlin 02, 391D).

(iv) If $\mathfrak{A}$ is a Boolean algebra, and $a\in\mathfrak{A}$, then the **principal ideal** $\mathfrak{A}_a=\{b:b\subseteq a\}$, with the inherited operations $\cup$, $\cap$ and $\triangle$, is a Boolean algebra in its own right (Fremlin 02, 312D); and if $\mathfrak{A}$ is weakly (σ,∞)-distributive, so is $\mathfrak{A}_a$.

1K We are now ready to look at the statements which I claim are equiveridical. Consider the statements

(†) whenever X is a set, Σ is an algebra of subsets of X, and ν is an exhaustive submeasure on Σ, then there is an additive functional $\mu:\Sigma\to[0,\infty[$ such that ν is absolutely continuous with respect to μ;

(‡) whenever X is a set, Σ is an algebra of subsets of X, and ν is a non-zero exhaustive submeasure on Σ, then there is a non-zero additive functional $\mu:\Sigma\to[0,\infty[$ such that μ is absolutely continuous with respect to ν.

proof that (†) ⇒ (‡) Suppose that (†) is true. Let X be a set, Σ an algebra of subsets of X, and ν a non-zero exhaustive submeasure on Σ. Build $\mathfrak{A}$, $\bar{\mu}$, $\widehat{\mathfrak{A}}$ and $\hat{\nu}$ as in 1J. Now $\hat{\nu}$ is an exhaustive submeasure on $\widehat{\mathfrak{A}}$. $\widehat{\mathfrak{A}}$ is constructed as an abstract Boolean algebra, but it is isomorphic to an algebra of sets (1J(e-i)), so (†) tells us that there is an additive functional $\mu:\widehat{\mathfrak{A}}\to[0,\infty[$ such that $\hat{\nu}$ is absolutely continuous with respect to μ. In particular, because $\hat{\nu}$ is strictly positive, so is μ. Now 1J(e-iii) tells us that there is an order-continuous additive functional $\mu_1:\widehat{\mathfrak{A}}\to[0,\infty[$ which is still strictly positive; as μ_1 is a Maharam submeasure, it is absolutely continuous with respect to $\hat{\nu}$ (1J(e-ii)). We have a natural homomorphism $E\mapsto E^\bullet:\Sigma\to\mathfrak{A}\subseteq\widehat{\mathfrak{A}}$; setting $\mu_2E=\mu_1E^\bullet$ for $E\in\Sigma$, it is easy to check that μ_2 is an additive functional and is absolutely continuous with respect to ν. Also $\mu_2E=0$ iff $\nu E=0$, so μ_2 is non-zero.

proof that (‡) ⇒ (†) Suppose that (‡) is true. Let X be a set, Σ an algebra of subsets of X, and ν a non-zero exhaustive submeasure on Σ. Build $\mathfrak{A}$, $\bar{\mu}$, $\widehat{\mathfrak{A}}$ and $\hat{\nu}$ as in 1J. Now $\hat{\nu}$ is an exhaustive submeasure on $\widehat{\mathfrak{A}}$. It follows that if $\hat{\nu}$ is non-zero then there is a non-zero additive functional absolutely continuous with respect to $\hat{\nu}$; but we need a little more. Let $D \subseteq \widehat{\mathfrak{A}}$ be the set of those $d \in \widehat{\mathfrak{A}}$ such that there is a strictly positive additive functional on the principal ideal $\widehat{\mathfrak{A}}_d$. Then every non-zero $a \in \widehat{\mathfrak{A}}$ includes a non-zero member of D. **P** If $a \in \widehat{\mathfrak{A}} \setminus \{0\}$, then $\hat{\nu}\restriction\widehat{\mathfrak{A}}_a$ is a non-zero exhaustive submeasure on $\widehat{\mathfrak{A}}_a$. By (‡), there is a non-zero additive functional $\lambda : \widehat{\mathfrak{A}}_a \to [0,\infty[$ which is absolutely continuous with respect to $\hat{\nu}$. Now $B = \{b : \lambda b = \lambda a\}$ is a downwards-directed set and $\inf_{b\in B} \hat{\nu}b > 0$, so B cannot have infimum 0 (1Jd); let d be a non-zero lower bound for B. If $0 \neq c \subseteq d$, then $a \setminus c \notin B$ so $\lambda c > 0$; thus $\lambda\restriction\widehat{\mathfrak{A}}_d$ is strictly positive and $d \in D$. **Q**

By Zorn's lemma, there is a maximal disjoint subset D_1 of D. For each $d \in D_1$, let $\lambda_d : \widehat{\mathfrak{A}}_d \to [0,\infty[$ be a strictly positive additive functional. Because $\widehat{\mathfrak{A}}$ is ccc (1Jd), D_1 is countable; let $\langle\epsilon_d\rangle_{d\in D_1}$ be a summable family of strictly positive real numbers. Set

$$\mu a = \sum_{d\in D_1} \frac{\epsilon_d}{1+\lambda_d d}\lambda_d(a \cap d)$$

for every $a \in \widehat{\mathfrak{A}}$. Then $\mu : \widehat{\mathfrak{A}} \to [0,\infty[$ is additive. If $\mu a = 0$ then $a \cap d = 0$ for every $d \cap D_1$, so a cannot include any member of D and must be zero; thus μ is strictly positive. Once again, we have an order-continuous strictly positive additive functional μ_1 on $\widehat{\mathfrak{A}}$, and a corresponding additive functional μ_2 on Σ. This time, we use 1J(e-ii) to see that $\bar{\nu}$ is absolutely continuous with respect to μ_1, so that ν is absolutely continuous with respect to μ_2.

Basic references: Fremlin 02, §§391-394, and Talagrand 08.
Origin of problem: Maharam 47.
Further reading: Kalton 89, Roberts 93.

2 The lifting problem

2A Liftings Let (X,Σ,μ) be a measure space. A **lifting** for μ is a Boolean homomorphism $\phi:\Sigma\to\Sigma$ (that is, a function such that $\phi X=X$, $\phi(E\cap F)=\phi E\cap\phi F$ and $\phi(E\triangle F)=\phi E\triangle\phi F$ for all E, $F\in\Sigma$), such that $\phi E=\emptyset$ whenever $\mu E=0$ and $\mu(E\triangle\phi E)=0$ for every $E\in\Sigma$.

For any measure space (X,Σ,μ) we have an equivalence relation $\sim$ on Σ defined by saying that $E\sim F$ if $\mu(E\triangle F)=0$. The **measure algebra** of (X,Σ,μ) is $(\mathfrak{A},\bar\mu)$ where $\mathfrak{A}$ is the set of equivalence classes in Σ under $\sim$ and $\bar\mu E^\bullet=\mu E$ for every $E\in\Sigma$. We have a Boolean algebra structure on $\mathfrak{A}$ defined by saying that $E^\bullet\cap F^\bullet=(E\cap F)^\bullet$, $E^\bullet\subseteq F^\bullet$ iff $\mu(E\setminus F)=0$, etc. Now if $\phi:\Sigma\to\Sigma$ is a lifting, we have a Boolean homomorphism $\theta:\mathfrak{A}\to\Sigma$ defined by setting $\theta E^\bullet=\phi E$ for every $E\in\Sigma$, and $(\theta a)^\bullet=a$ for every $a\in\mathfrak{A}$; θ is a right inverse of the canonical Boolean homomorphism $E\mapsto E^\bullet$.

2B Theorem (Maharam 58) If (X,Σ,μ) is a probability space which is **complete**, that is, $E\in\Sigma$ whenever $E\subseteq F\in\Sigma$ and $\mu F=0$, then μ has a lifting.

I will not give a proof here; the general argument depends on a thorough understanding of Boolean algebras and the martingale theorem; see Fremlin 02, §341.

2C Maharam's theorem It helps to know the following. First, we say that a probability space (X,Σ,μ) is **Maharam homogeneous**, with **Maharam type** κ (where κ is a cardinal), if there is a stochastically independent family $\langle E_\xi\rangle_{\xi<\kappa}$ of measurable sets of measure $\frac{1}{2}$ such that, writing Σ_0 for the σ-algebra generated by $\{E_\xi:\xi<\kappa\}$, every member of Σ differs by a negligible set from some member of Σ_0. Now Maharam's theorem (Maharam 42, or Fremlin 02, §331) is: for every probability space (X,Σ,μ), there is a partition of X into countably many measurable sets E of non-zero measure such that the normalized subspace measure on E is Maharam homogeneous.

The relevance of this theorem is the following fact.

2D Proposition Let κ be a cardinal such that $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$ has a lifting, where ν_κ is the usual measure on $\{0,1\}^\kappa$ and $\mathcal{B}\mathfrak{a}_\kappa$ is the σ-algebra of Baire subsets of $\{0,1\}^\kappa$, that is, the σ-algebra generated by the sets $H_\xi=\{z:z\in\{0,1\}^\kappa,\ z(\xi)=1\}$. Then every Maharam probability space (X,Σ,μ) of Maharam type κ has a lifting.

proof Let $\langle E_\xi\rangle_{\xi<\kappa}$ and Σ_0 be as in the definition 2C. Define $f:X\to\{0,1\}^\kappa$ by setting $f(x)(\xi)=\chi E_\xi(x)$ for $x\in X$, $\xi<\kappa$; then $E_\xi=f^{-1}[H_\xi]$ for every ξ and $\Sigma_0=\{f^{-1}[H]:H\in\mathcal{B}\mathfrak{a}_\kappa\}$; also $\mu f^{-1}[H]=\nu_\kappa H$ for every $H\in\mathcal{B}\mathfrak{a}_\kappa$. Let $\psi:\mathcal{B}\mathfrak{a}_\kappa\to\mathcal{B}\mathfrak{a}_\kappa$ be a lifting for $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$. Then we can define $\phi:\Sigma\to\Sigma$ by saying that $\phi E=f^{-1}[\psi H]$ whenever $E\in\Sigma$, $H\in\mathcal{B}\mathfrak{a}_\kappa$ and $\mu(E\triangle f^{-1}[H])=0$, and ϕ is a lifting for μ.

2E Corollary The following are equiveridical:
(i) every probability measure has a lifting;
(ii) $\nu_\kappa\restriction\mathcal{B}\mathfrak{a}_\kappa$ has a lifting for every cardinal κ.

proof Proposition 2D tells us that if (ii) is true then every Maharam homogeneous probability measure has a lifting; now Maharam's theorem quickly gives the result.

2F Filtrations The best results known for non-complete probability spaces are based on the following method. First, if P is any partially ordered set, we say that it has **countable cofinality** if there is a countable set $Q\subseteq P$ which is **cofinal** with P, that is, for every $p\in P$ there is a $q\in Q$ such that $p\le q$. Next, if $\mathfrak{A}$ is a Boolean algebra, a **tight ω_1-filtration** for $\mathfrak{A}$ (Geschke 02) is a family $\langle a_\xi\rangle_{\xi<\zeta}$ in $\mathfrak{A}$, for some ordinal ζ, such that, writing $\mathfrak{A}_\alpha$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi:\xi<\alpha\}$, (i) $\mathfrak{A}_\zeta=\mathfrak{A}$ (ii) for every $\alpha<\zeta$ and $a\in\mathfrak{A}_{\alpha+1}$, the set $\{b:b\in\mathfrak{A}_\alpha,\ b\subseteq a\}$ has countable cofinality.

Now the theorem is the following.

2G Theorem Let (X,Σ,μ) be a measure space such that $\mu X>0$, and suppose that its measure algebra has a tight ω_1-filtration. Then μ has a lifting.

proof Let $\langle a_\xi\rangle_{\xi<\zeta}$ be a tight ω_1-filtration in $\mathfrak{A}$; write $\mathfrak{A}_\alpha$ for the subalgebra of $\mathfrak{A}$ generated by $\{a_\xi:\xi<\alpha\}$. Define Boolean homomorphisms $\theta_\alpha:\mathfrak{A}_\alpha\to\Sigma$ inductively, as follows. Start with $\mathfrak{A}_0=\{0,1\}$, $\theta_0 0=\emptyset$,

$\theta_0 1 = X$. (This is where we need to know that $\mu X > 0$, so that $0 \neq 1$ in $\mathfrak{A}$.) Given θ_α, let B, $B' \subseteq \mathfrak{A}_\alpha$ be countable sets such that B is cofinal with $\{b : b \in \mathfrak{A}_\alpha,\, b \subseteq a_\alpha\}$ and B' is cofinal with $\{b : b \in \mathfrak{A}_\alpha,\, b \subseteq 1 \setminus a_\alpha\}$. Choose any $E \in \Sigma$ such that $E^\bullet = a_\alpha$ and set

$$E_\alpha = (E \cup \bigcup_{b\in B} \theta_\alpha b) \setminus \bigcup_{b\in B'} \theta_\alpha b.$$

Because B and B' are both countable, $E_\alpha \in \Sigma$. Because they are cofinal with $\{b : b \in \mathfrak{A}_\alpha,\, b \subseteq a_\alpha\}$ and $\{b : b \in \mathfrak{A}_\alpha,\, b \subseteq 1 \setminus a_\alpha\}$ respectively, $\theta b \subseteq E_\alpha$ whenever $\{b : b \in \mathfrak{A}_\alpha$ and $b \subseteq a_\alpha$, while $\theta b \cap E_\alpha = \emptyset$ whenever $\{b : b \in \mathfrak{A}_\alpha$ and $b \subseteq 1 \setminus a_\alpha$. This means that we can define a Boolean homomorphism $\theta_{\alpha+1} : \mathfrak{A}_{\alpha+1} \to \Sigma$ by setting

$$\theta_{\alpha+1}((b \cap a_\alpha) \cup (c \setminus a_\alpha)) = (\theta_\alpha b \cap E_\alpha) \cup (\theta_\alpha c \setminus E_\alpha$$

for all b, $c \in \mathfrak{A}_\alpha$.

This is the inductive step to a successor ordinal. For the inductive step to a non-zero limit ordinal $\alpha \leq \zeta$, $\mathfrak{A}_\alpha = \bigcup_{\xi<\alpha} \mathfrak{A}_\xi$ and we can define θ_α by setting $\theta_\alpha a = \theta_\xi a$ whenever $\xi < \alpha$ and $a \in \mathfrak{A}_\xi$.

An easy induction now shows that $a = (\theta_\alpha a)^\bullet$ whenever $\alpha \leq \kappa$ and $a \in \mathfrak{A}_\alpha$, so that $\theta_\zeta : \mathfrak{A} \to \Sigma$ is a lifting for μ.

2H Proposition (a) (Mokobodzki 75) Suppose that the continuum hypothesis is true. If (X, Σ, μ) is a probability space with a measure algebra of cardinal at most $\mathfrak{c}^+ = \omega_2$, then its measure algebra is tightly ω_1-filtered, so μ has a lifting.

(b) (T.J.Carlson) If we add ω_2 Cohen reals to a model of the generalized continuum hypothesis, then in the resulting model the Lebesgue measure algebra is tightly ω_1-filtered, so $\mu_L \restriction \mathcal{B}$ has a lifting, where $\mathcal{B}$ is the Borel σ-algebra of $[0, 1]$ and μ_L is Lebesgue measure.

Remark It is easy to prove that $([0, 1], \mathcal{B}, \mu_L \restriction \mathcal{B})$ is isomorphic to $(\{0, 1\}^\omega, \mathcal{B}\mathfrak{a}_\omega, \nu_\omega \restriction \mathcal{B}\mathfrak{a}_\omega)$ (cf. Fremlin 01, 254K).

2I Theorem (Shelah 83) It is relatively consistent with ZFC to suppose that $\mu_L \restriction \mathcal{B}$ has no lifting.

2J Problem Is there a probability space without a lifting?

Remark By 2E, it is enough to consider $(\{0, 1\}^\kappa, \mathcal{B}\mathfrak{a}_\kappa, \nu_\kappa \restriction \mathcal{B}\mathfrak{a}_\kappa)$ where κ is a cardinal. Since Mokobodzki's theorem deals with $\kappa \leq \omega_2$ when $\mathfrak{c} = \omega_1$, the key case to consider seems to be $\kappa = \omega_3$. Here I note that it is a theorem of Geschke 02 that a probability algebra of cardinal ω_3 or more cannot be tightly ω_1-filtered.

2K Further questions. Let $\mathcal{B}$ be the algebra of Borel subsets of $[0, 1]$ and μ_L Lebesgue measure on $\mathbb{R}$.

(a) Shelah's construction of a model in which $\mu_L \restriction \mathcal{B}$ has no lifting is interesting and important. But there are many well-known models in which $\mathfrak{c} > \omega_1$ (e.g., random real models or models of Martin's axiom), and in none of these, except Cohen's original model of not-CH, do we know whether $\mu_L \restriction \mathcal{B}$ has a lifting. For instance, it seems quite possible that *whenever* $\mathfrak{c} > \omega_2$ then $\mu_L \restriction \mathcal{B}$ has no lifting.

(b) A.H.Stone noted that even when $\mathfrak{c} = \omega_1$, so that we have a lifting for $\mu_L \restriction \mathcal{B}$, the methods available give no control over the Borel classes of ϕE, as E runs over $\mathcal{B}$. So he asked: is it possible to have a lifting ϕ of $\mu_L \restriction \mathcal{B}$ such that all the lifted sets ϕE are of bounded Borel class?

(c) Can there be a lifting ϕ for $\mu_L^{(2)} \restriction \mathcal{B}_2$, where $\mathcal{B}_2$ is the algebra of Borel subsets of $[0, 1]^2$ and $\mu_L^{(2)}$ is Lebesgue measure on $\mathbb{R}^2$, which **respects coordinates**, i.e., is such that if E, $F \in \mathcal{B}$ then $\phi(E \times F)$ is of the form $E' \times F'$?

(d) Even for complete probability spaces there remain some puzzles. For instance: let μ be the measure on $\{0, 1\}$ such that $\mu\{0\} = \frac{1}{3}$, $\mu\{1\} = \frac{2}{3}$, and let λ be the (completed) product of countably many copies of μ, so that λ is a probability measure on $\{0, 1\}^{\mathbb{N}}$. Is there a lifting ϕ for λ which respects coordinates in the sense that ϕE is determined by coordinates in J whenever $E \in \operatorname{dom} \lambda$ is determined by coordinates in J?

Basic reference: Fremlin 02, §§341, 344-346.

Further reading: Shelah 83, Just 92, Burke 93, Geschke 02.

3 Radon spaces

3A Definition A Hausdorff space X is **Radon** if every totally finite Borel measure on X is tight (that is, $\mu E = \sup\{\mu K : K \subseteq E$ is closed and compact$\}$ whenever μ measures E).

3B Basic facts (a) A complete metric space X is Radon iff $\kappa = w(X)$ is measure-free (that is, any probability measure on κ with domain $\mathcal{P}\kappa$ gives positive measure to some singleton). In particular, any Polish space is Radon.

(b) If X is a Hausdorff space, the set of subsets of X which are Radon spaces in the subspace topologies is closed under Souslin's operation.

(c) ω_1 and $\omega_1 + 1$, with their order topologies, are not Radon.

(d) For a set I, $[0,1]^I$ is Radon iff $\{0,1\}^I$ is Radon iff I is countable.

(e) A hereditarily Lindelöf K-analytic Hausdorff space is Radon; in particular, the split interval is Radon.

(f) If X is a Banach space with an equivalent Kadec norm (e.g., a weakly K-countably determined Banach space, or a Banach lattice with an order-continuous norm), then X is a Radon space in its weak topology iff $w(X)$ is measure-free.

(g) If X is an Eberlein compact, it is a Radon space iff $w(X)$ is measure-free.

3C Proposition (a) In a Radon Hausdorff space, countably compact sets are compact.
(b) Any Radon compact Hausdorff space is countably tight.

proof (a) Let X be a Radon Hausdorff space and $C \subseteq X$ a countably compact set. Let $\mathcal{E}$ be a non-empty family of relatively closed subsets of C with the finite intersection property; let $\mathcal{E}^* \supseteq \mathcal{E}$ be a maximal family of relatively closed subsets of C with the finite intersection property. Then $\mathcal{E}^*$ is closed under countable intersections. **P?** Otherwise, there is a sequence $\langle E_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{E}^*$ with intersection $E \notin \mathcal{E}^*$. By the maximality of $\mathcal{E}^*$, there must be $F_0, \ldots, F_m \in \mathcal{E}^*$ such that $F_0 \cap \ldots \cap F_m \cap E = \emptyset$. But now $\{F_i : i \leq m\} \cup \{E_i : i \in \mathbb{N}\}$ is a countable family of relatively closed subsets of C which has the finite intersection property and empty intersection; which contradicts the hypothesis that C is countably compact. **XQ**

Set

$$\Sigma = \{A : A \subseteq X, \text{ there is an } E \in \mathcal{E}^* \text{ such that either } E \subseteq A \text{ or } E \cap A = \emptyset\}.$$

Because $\mathcal{E}^*$ is closed under countable intersections, Σ is a σ-algebra of subsets of X and we have a measure μ_0 with domain Σ defined by setting $\mu_0 A = 1$ if A includes some member of $\mathcal{E}^*$, 0 otherwise. If $F \subseteq X$ is closed, then either $F \cap C \in \mathcal{E}^*$ and $\mu_0 F = 1$, or there are $F_0, \ldots, F_m \in \mathcal{E}^*$ such that $F \cap \bigcap_{i\leq m} F_i = \emptyset$ and $\mu_0 F = 0$. So every closed set, therefore every Borel set, belongs to Σ, and the restriction μ of μ_0 to the Borel σ-algebra $\mathcal{B}(X)$ is a Borel measure on X.

Since $\mu X > 0$ and X is a Radon space, there is a compact set $K \subseteq X$ such that $\mu K > 0$. Now K cannot be covered by any finite family of negligible open sets, and is therefore not covered by the family of all negligible open sets. Let $x \in K$ be such that every open set containing x has non-zero measure. **?** If $E \in \mathcal{E}^*$ and $x \notin E$, then $\mu\{x\} = 0$ and $\mu(X \setminus \{x\}) > 0$; so there is a compact set $L \subseteq X \setminus \{x\}$ such that $\mu L > 0$. But now there must be an $F \in \mathcal{E}^*$ such that $F \subseteq L$, so that $X \setminus L$ is a negligible open set containing x. **X**
Thus $x \in \bigcap \mathcal{E}^* \subseteq \bigcap \mathcal{E}$ and $\mathcal{E}$ has non-empty intersection. As $\mathcal{E}$ is arbitrary, C is compact.

(b) If X is a Radon compact Hausdorff space and $A \subseteq X$, set $C = \bigcup\{\overline{B} : B \subseteq A$ is countable$\}$. Then $\overline{D} \subseteq C$ for every countable set $D \subseteq C$, so every sequence in C has a cluster point and C is countably compact. By (i), C is compact, therefore closed, and must be the closure of A. As A is arbitrary, X is countably tight.

3D Questions (a) Is the continuous image of a Radon compact Hausdorff space always Radon?
(b)(P.J.Nyikos) Is every Radon compact Hausdorff space sequentially compact? (If $2^{\mathfrak{t}} > \mathfrak{c}$, yes, because in this case countably tight compact Hausdorff spaces are sequentially compact.)
(c) Is there a pair of Radon spaces with a product which is not Radon? (If *either* there is an atomlessly-measurable cardinal *or* $\mathfrak{m} = \mathfrak{c}$, yes; see Wage 80 and Fremlin prW.)

(d) Is there a Banach space with weight ω_1 which is not Radon when given its weak topology? There is certainly a space with weight $\mathfrak{c}$ which is not Radon (Fremlin 03, 466Ia).

(e) If $\mathfrak{c}$ is measure-free, is ℓ^∞ a Radon space in its weak topology?

Basic reference: Fremlin 03, §434; also §§466-467.
Original source: Schwartz 73.

4 The vector-valued McShane integral

4A Vector-valued gauge integrals Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of 'tagged partitions', $\mathcal{F}$ a filter on T and U a normed space. For $\boldsymbol{t} \in T$, set $W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}$, the **footprint** of $\boldsymbol{t}$. If $\delta \subseteq X \times \mathcal{P}X$, say that $\boldsymbol{t} \in T$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. If $\mathcal{R} \subseteq \mathcal{P}X$, say that $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$.

For $f : X \to U$, $\nu : \mathcal{C} \to \mathbb{R}$ and $\boldsymbol{t} \in T$, set $S_{\boldsymbol{t}} = \sum_{(x,C)\in\boldsymbol{t}} \nu C \cdot f(x)$. Now set

$$I_{\nu,\mathcal{F}}(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f)$$

if this is defined in X for the norm topology. Then $I_{\nu,\mathcal{F}}$ is a linear functional defined on a linear subspace of U^X, a **gauge integral** for U-valued functions.

4B Definitions It will be helpful to have some further terminology, following Fremlin 01 and Fremlin 03.

(a) Let (X, Σ, μ) be a measure space.

(i) I say that μ is **inner regular** with respect to a family $\mathcal{K}$ of sets if $\mu E = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma, K \subseteq E\}$ for every $E \in \Sigma$. (Thus a 'tight' measure is one which is inner regular with respect to the closed compact sets.)

Similarly, μ is **outer regular** with respect to a family $\mathcal{H}$ of sets if $\mu E = \inf\{\mu H : H \in \mathcal{H} \cap \Sigma, H \supseteq E\}$ for every $E \in \Sigma$.

(ii) I will write Σ^f for $\{E : E \in \Sigma, \mu E < \infty\}$. Now (X, Σ, μ) is **semi-finite** if μ is inner regular with respect to Σ^f.

(iii) (X, Σ, μ) is **locally determined** if it is semi-finite and $A \in \Sigma$ whenever $A \subseteq X$ is such that $A \cap E \in \Sigma$ for every $E \in \Sigma$ such that $\mu E < \infty$.

(iv) (X, Σ, μ) is **totally finite** if $\mu X < \infty$.

(v) (X, Σ, μ) is **atomless** if whenever $E \in \Sigma$ and $\mu E > 0$ then there is an $F \in \Sigma$ such that $F \subseteq E$, $\mu F > 0$ and $\mu(E \setminus F) > 0$.

(c) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(i) μ is a **topological measure** if it measures every open set (and therefore every Borel set).

(ii) μ is **τ-additive** if $\mu(\bigcup \mathcal{G}) = \sup_{G\in\mathcal{G}} \mu G$ whenever $\mathcal{G}$ is a non-empty upwards-directed family in $\mathfrak{T} \cap \Sigma$ with union in $\mathfrak{T} \cap \Sigma$.

(iii) μ is **effectively locally finite** if $\mu E = \sup\{\mu(G \cap E) : G \in \mathfrak{T} \cap \Sigma, \mu G < \infty\}$ for every $E \in \Sigma$.

(iv) μ is a **quasi-Radon measure** if it is a complete locally determined τ-additive effectively locally finite topological measure on X which is inner regular with respect to the closed sets. (Remark: on a regular topological space, any τ-additive effectively locally finite topological measure which is inner regular with respect to the Borel sets is inner regular with respect to the closed sets; see Fremlin 03, 414Mb.)

(d)(i) If X and Y are topological spaces, μ is a measure on X and $\phi : X \to Y$ is a function, I say that ϕ is **measurable** if μ measures $\phi^{-1}[H]$ for every open set $H \subseteq Y$, and **almost continuous** if μ is inner regular with respect to $\{K : K \subseteq X, \phi\restriction K \text{ is continuous}\}$.

(ii) If (X, Σ, μ) and (Y, T, ν) are measure spaces, a function $\phi : X \to Y$ is **inverse-measure-preserving** if $\mu f^{-1}[F]$ is defined and equal to νF for every $F \in \mathrm{T}$.

4C The McShane integral Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Let $T = T(X, \Sigma^f)$ be the family of finite subsets $\boldsymbol{t}$ of $X \times \Sigma^f$ such that $C \cap C'$ is empty whenever (x, C), (x', C') are distinct members of $\boldsymbol{t}$. Let $\Delta = \Delta(X, \mathfrak{T})$ be the set of all **neighbourhood gauges** on X, that is, families $\delta \subseteq X \times \mathcal{P}X$ such that, for each $x \in X$, $\{C : (x, C) \in \delta\}$ is of the form $\mathcal{P}G$ for some open set G containing x. Note that $\delta_1 \cap \delta_2 \in \Delta$ for all δ_1, $\delta_2 \in \Delta$. For any $H \in \Sigma^f$ and $\eta > 0$, let $\mathcal{R}_{H\eta}$ be the set $\{F : F \in \Sigma, \mu(F \cap H) \leq \eta\}$. Let $\mathcal{F}$ be the filter on T generated by sets of the form

$$\{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine and } \mathcal{R}_{E\eta}\text{-filling}\}$$

where δ runs over Δ, E over Σ^f and η over $]0, \infty[$. (Because μ is an effectively locally finite τ-additive topological measure, there is indeed such a filter; see part (a) of the proof of 4E below.) The corresponding functional $I_{\mu,\mathcal{F}} = \mathrm{M}\!\!\int\!\!\mathrm{S}\, d\mu$ is the **McShane integral** for vector-valued functions defined on X.

4D Remarks The definition here matches that of Bongiorno di Piazza & Musial 00; it differs significantly from that in Fremlin 95, and even more from that in Gordon 90 and Fremlin & Mendoza 94, but coincides with the earlier definitions in the contexts in which they were set out. For the identification of the integral here with that of Fremlin 95, see Bongiorno di Piazza & Musial 00 or Fremlin prET. Rather than appeal to these, I will repeat some of the theorems of Fremlin 95 in the new formulation.

4E Saks-Henstock Lemma (compare Fremlin 03, 482B) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, U a Banach space, and $f : X \to U$ a McShane integrable function. Define T as in 4C. Then we have a unique additive function $F : \Sigma \to U$ such that $F(X) = \mathrm{M}\!\!\int\!\!\mathrm{S}\, f d\mu$ and for every $\epsilon > 0$ there are a neighbourhood gauge δ on X, an $H \in \Sigma^f$ and an $\eta > 0$ such that whenever $\boldsymbol{t} \in T$ is δ-fine, $W_{\boldsymbol{t}} \subseteq E \in \Sigma$ and $\mu(H \cap E \setminus W_{\boldsymbol{t}}) \le \eta$, then $\|S_{\boldsymbol{t}}(f) - F(E)\| \le \epsilon$.

proof (a) For any $E \in \Sigma$, let T_E be the set of those tagged partitions $\boldsymbol{t} \in T$ such that, for $(x, C) \in \boldsymbol{t}$, $C \subseteq E$ whenever $x \in E$ and $C \subseteq X \setminus E$ whenever $x \in X \setminus E$. Set $\mathcal{R}_{H\eta} = \{E : E \in \Sigma,\ \mu(E \cap H) \le \eta\}$, as in 4C. Then for any neighbourhood gauge δ on X, $H \in \Sigma^f$, $\eta > 0$ and finite $\mathcal{E} \subseteq \Sigma$, there is a δ-fine $\mathcal{R}_{H\eta}$-filling $\boldsymbol{t} \in \bigcap_{E \in \mathcal{E}} T_E$. **P** Replacing $\mathcal{E}$ by the set of atoms in the finite algebra of sets which it generates, if necessary, it is enough to consider the case in which $\mathcal{E}$ is a partition of X. For $x \in X$ set $G_x = \bigcup\{C : (x, C) \in \delta\}$, so that G_x is an open set containing x. For each $E \in \mathcal{E}$, $\{G_x : x \in E\}$ is an open cover of $H \cap E$, so there is a finite set $I_E \subseteq E$ such that $\mu(H \cap E \setminus \bigcup_{x \in I_E} G_x) \le \dfrac{\eta}{\#(\mathcal{E})}$ (see Fremlin 03, 414E; I am passing over the trivial case $\mathcal{E} = X = \emptyset$). Enumerate I_E as $\langle x(E, i)\rangle_{i < n_E}$, and set $C(E, i) = H \cap E \cap G_{x(E,i)} \setminus \bigcup_{j<i} G_{x(E,j)}$ for $i < n_E$. Then $\boldsymbol{t} = \{(x(E, i), C(E, i)) : E \in \mathcal{E},\ i < n_E\}$ has the required properties. **Q**

(b) We therefore have a filter $\mathcal{F}^*$ on T generated by the sets

$$\{\boldsymbol{t} : \boldsymbol{t} \text{ is } \delta\text{-fine}\}$$

for neighbourhood gauges δ,

$$\{\boldsymbol{t} : \boldsymbol{t} \text{ is } \mathcal{R}_{H\eta}\text{-filling}\}$$

for $H \in \Sigma^f$ and $\eta > 0$, and the sets T_E for $E \in \Sigma$. For $E \in \Sigma$ and $\boldsymbol{t} \in T$, set $\boldsymbol{t}_E = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in E\}$. Then

$$F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f)$$

is defined in U for every $E \in \Sigma$. **P** Let $\epsilon > 0$. Then there are a neighbourhood gauge δ and $H \in \Sigma^f$, $\eta > 0$ such that $\|S_{\boldsymbol{t}}(f) - \mathrm{M}\!\!\int\!\!\mathrm{S}\, f d\mu\| \le \epsilon$ for every δ-fine, $\mathcal{R}_{H,2\eta}$-filling $\boldsymbol{t} \in T$. Now suppose that $\boldsymbol{s}, \boldsymbol{t} \in T_E$ are δ-fine and $\mathcal{R}_{H\eta}$-filling. In this case, set $\boldsymbol{t}' = \boldsymbol{s}_E \cup \boldsymbol{t}_{X \setminus E}$. Then $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}_{H,2\eta}$-filling. So

$$\|S_{\boldsymbol{s}_E}(f) - S_{\boldsymbol{t}_E}(f)\| = \|S_{\boldsymbol{t}'_E}(f) - S_{\boldsymbol{t}_E}(f)\| = \|S_{\boldsymbol{t}'}(f) - S_{\boldsymbol{t}}(f)\| \le 2\epsilon.$$

As $\{\boldsymbol{t} : \boldsymbol{t} \in T_E \text{ is } \delta\text{-fine and } \mathcal{R}_{H\eta}\text{-filling}\}$ belongs to $\mathcal{F}^*$, and U is complete, this is enough to show that $\lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f)$ is defined in U. **Q**

(c) To see that F is additive, observe that if E, $E' \in \Sigma$ are disjoint, and $\boldsymbol{t} \in T_E \cap T_{E'}$, then $S_{\boldsymbol{t}_{E \cup E'}}(f) = S_{\boldsymbol{t}_E}(f) + S_{\boldsymbol{t}_{E'}}(f)$; taking the limit along $\mathcal{F}^*$, $F(E \cup E') = F(E) + F(E')$.

As for $\mathrm{M}\!\!\int\!\!\mathrm{S}\, f d\mu$, this is just the limit of $S_{\boldsymbol{t}}(f)$ along the smaller filter on T generated by the gauges and the residual families, so must be equal to $F(X)$.

(d) Now for the approximating property claimed for F. Given $\epsilon > 0$, take a neighbourhood gauge δ, an $H \in \Sigma^f$ and an $\eta > 0$ such that $\|S_{\boldsymbol{t}}(f) - \mathrm{M}\!\!\int\!\!\mathrm{S}\, f d\mu\| \le \frac{1}{3}\epsilon$ for every δ-fine $\mathcal{R}_{H,2\eta}$-filling $\boldsymbol{t}$. Suppose that $\boldsymbol{t} \in T$ is δ-fine, $E \supseteq W_{\boldsymbol{t}}$ and $\mu(H \cap E \setminus W_{\boldsymbol{t}}) \le \eta$. Then (using the technique in (a)) we can find a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq X \setminus E$ and $\mu((H \setminus E) \setminus W_{\boldsymbol{s}}) \le \eta$. At the same time, there is a δ-fine $\mathcal{R}_{H\eta}$-filling $\boldsymbol{u} \in T_E$ such that $\|F(E) - S_{\boldsymbol{u}_E}(f)\| \le \frac{1}{3}\epsilon$. In this case, both $\boldsymbol{t} \cup \boldsymbol{s}$ and $\boldsymbol{u}_E \cup \boldsymbol{s}$ are δ-fine and $\mathcal{R}_{H,2\eta}$-filling. So

$$\|F(E) - S_{\boldsymbol{t}}(f)\| \le \|F(E) - S_{\boldsymbol{u}_E}(f)\| + \|S_{\boldsymbol{u}_E\cup\boldsymbol{s}}(f) - S_{\boldsymbol{t}\cup\boldsymbol{s}}(f)\|$$
$$\le \frac{1}{3}\epsilon + \|S_{\boldsymbol{u}_E\cup\boldsymbol{s}}(f) - \text{McS}\!\!\int f d\mu\| + \|\text{McS}\!\!\int f d\mu - S_{\boldsymbol{t}\cup\boldsymbol{s}}(f)\| \le \epsilon$$

as required.

(e) Of course this requirement determines F completely. So the proof is complete.

4F Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $f : X \to \mathbb{R}$ a McShane integrable function. Let $F : \Sigma \to \mathbb{R}$ be the Saks-Henstock indefinite integral as defined in 4E. Then the f is integrable in the ordinary sense, and $\int_E f d\mu = F(E)$ for every $E \in \Sigma$.

proof For each $m \in \mathbb{N}$, choose a neighbourhood gauge δ_m on X, an $H_m \in \Sigma^f$ and an $\eta_m \in]0, 2^{-m}]$ such that $\|F(E) - S_{\boldsymbol{t}}(f)\| \le 2^{-m}$ whenever $\boldsymbol{t} \in T$ is δ_m-fine, $W_{\boldsymbol{t}} \subseteq E \in \Sigma$ and $\mu(H_m \cap E \setminus W_{\boldsymbol{t}}) \le \eta_m$.

(a) f is measurable. **P?** If not, then, because μ is complete and locally determined, there are a set $E_0 \in \Sigma^f$ and $\alpha < \beta$ in $\mathbb{R}$ such that $A = \{x : x \in E_0,\ f(x) \le \alpha\}$ and $B = \{x : x \in E_0,\ f(x) \ge \beta\}$ both have outer measure $\mu E_0 > 0$ (see Fremlin 03, 413G). Take any $m \in \mathbb{N}$. Set $G_x = \bigcup\{C : (x, C) \in \delta_m\}$ for each $x \in X$; then $\bigcup_{x\in A} G_x$ is an open set including A, so $\mu(E_0 \cap \bigcup_{x\in A} G_x) = \mu E_0$ and there is a finite sequence $\langle x_i\rangle_{i<n}$ in A such that $\mu(E_0 \setminus \bigcup_{i<n} G_{x_i}) \le \eta_m$. Set $C_i = E_0 \cap G_{x_i} \setminus \bigcup_{j<i} G_{x_j}$ for each $i < n$, $\boldsymbol{t} = \{(x_i, C_i) : i < n\}$. Then $W_{\boldsymbol{t}} = \bigcup_{i<n} C_i \subseteq E_0$ and $\mu(E_0 \setminus W_{\boldsymbol{t}}) \le \eta_m \le 2^{-m}$, while $\boldsymbol{t}$ is δ_m-fine. So

$$F(E_0) \le S_{\boldsymbol{t}} + 2^{-m} \le \alpha\mu W_{\boldsymbol{t}} + 2^{-m} \le \alpha\mu E_0 + 2^{-m}(1 + |\alpha|).$$

Similarly, $F(E_0) \ge \beta\mu E_0 - 2^{-m}(1+|\beta|)$, so $(\beta-\alpha)\mu E_0 \le 2^{-m}(2+|\alpha|+|\beta|)$. As m is arbitrary, $(\beta-\alpha)\mu E_0 = 0$, which is impossible. **XQ**

(b) Now set $V = \{x : f(x) \ge 0\}$. Then $\int_V f d\mu \le F(V)+3$. **P?** Otherwise, we can find a simple function g such that $0 \le g \le f^+$ and $\int g\,d\mu \ge F(V) + 3$. Express g as $\sum_{i=0}^n \alpha_i\chi E_i$ where $E_0, \dots, E_n$ are disjoint measurable sets of finite measure and $\alpha_i \ge 0$ for every i. Let $\eta \in]0, \eta_0]$ be such that $\eta \max_{i\le n} \alpha_i \le 1$. Then we can find a δ_0-fine $\boldsymbol{t} \in \bigcap_{i\le n} T_{E_i} \cap T_V$ (as defined in the proof of 4E) such that $\mu((H_0 \cup \bigcup_{i\le n} E_i) \setminus W_{\boldsymbol{t}}) \le \eta$. In this case, setting $\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in V\}$, $\boldsymbol{s}$ is δ_0-fine and $\mu(V \cap H_0 \setminus W_{\boldsymbol{s}}) \le \eta_0$, so

$$F(V) \ge S_{\boldsymbol{s}}(f) - 1 \ge \sum_{i=0}^n \alpha_i\mu(E_i \cap W_{\boldsymbol{s}}) - 1$$

(because if $(x, C) \in \boldsymbol{s}$, then $f(x) \ge 0$, while if $C \cap E_i \ne \emptyset$ then $C \subseteq E_i$ and $x \in E_i$)

$$\ge \sum_{i=0}^n \alpha_i\mu E_i - 2 > F(V),$$

which is impossible. **XQ**

Similarly, $\int_{X\setminus V} f \ge F(X \setminus V) - 3$, so f is integrable in the usual sense.

(c) For the identification between F and the ordinary indefinite integral of f, take any $m \in \mathbb{N}$. Then we can find $H \in \Sigma^f$ and $\eta > 0$ such that $H \supseteq H_m$, $\eta \le \frac{1}{2}\eta_m$ and $\int_E |f| \le 2^{-m}$ whenever $\mu(E \cap H) \le \eta$. Let $k \in \mathbb{N}$ be such that $\mu\{x : x \in H,\ |f(x)| \ge 2^k\} \le \eta$ and $2^{-k}\mu H \le 2^{-m}$, and for $-4^k \le i \le 4^k$ set $E_i = \{x : x \in H,\ 2^{-k}i \le f(x) < 2^{-k}(i + 1)\}$. Now suppose that $E \in \Sigma$. Then we can find a δ_m-fine $\mathcal{R}_{H\eta}$-filling $\boldsymbol{t} \in \bigcap_{|i|\le 4^k} T_{E\cap E_i}$. Taking $\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t},\ x \in \bigcup_{|i|\le 4^k} E \cap E_i\}$, $\boldsymbol{s}$ is δ_m-fine, $W_{\boldsymbol{s}} \subseteq E$ and

$$\mu(H_m \cap E \setminus W_{\boldsymbol{s}}) \le \mu(H \cap E \setminus \textstyle\bigcup_{|i|\le 4^k} E_i) + \mu(H \cap \bigcup_{|i|\le 4^k} E_i \setminus W_{\boldsymbol{t}}) \le 2\eta \le \eta_m.$$

So $|F(E) - S_{\boldsymbol{s}}(f)| \le 2^{-m}$. On the other hand, whenever $(x, C) \in \boldsymbol{s}$ and $y \in C$, there is some i such that $x \in E_i$ and $C \subseteq E_i$, so that $|f(x) - f(y)| \le 2^{-k}$; accordingly

$$|S_{\boldsymbol{s}}f-\int_{W_{\boldsymbol{s}}}f\,d\mu|\le\sum_{(x,C)\in\boldsymbol{s}}|f(x)\mu C-\int_C f\,d\mu|\le\sum_{(x,C)\in\boldsymbol{s}}2^{-k}\mu C$$
$$=2^{-k}\mu W_{\boldsymbol{s}}\le 2^{-k}\mu H\le 2^{-m}.$$

Finally, because $\mu(H\setminus\bigcup_{|i|\le 4^k}E_i)$ and $\mu(E\cap H\cap\bigcup_{|i|\le 4^k}E_i\setminus W_{\boldsymbol{s}})\le\mu(E\cap H\setminus W_{\boldsymbol{t}})$ are both at most η, $|\int_E f\,d\mu-\int_{W_{\boldsymbol{s}}}f\,d\mu|\le 2^{-m+1}$. Putting these together, $|F(E)-\int_E f\,d\mu|\le 2^{-m+2}$; as m is arbitrary, $F(E)=\int_E f\,d\mu$.

4G Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, U and V Banach spaces, and $P:U\to V$ a continuous linear operator. If $f:X\to U$ is McShane integrable, so is $Pf:X\to V$, and $\text{M}\!\!\oint\!\text{S}\,Pf\,d\mu=P(\text{M}\!\!\oint\!\text{S}\,f\,d\mu)$. Moreover, if $F:\Sigma\to U$ is the Saks-Henstock indefinite integral of f, then $PF:\Sigma\to V$ is the Saks-Henstock indefinite integral of Pf.

proof We have only to note that for every tagged partition $\boldsymbol{t}$ we shall have $S_{\boldsymbol{t}}(Pf)=P(S_{\boldsymbol{t}}(f))$, and take appropriate limits.

4H Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, U a Banach space, and $f:X\to U$ a McShane integrable function. Then f is Pettis integrable, and its Saks-Henstock indefinite integral is its indefinite Pettis integral.

proof Let F be the Saks-Henstock indefinite integral of f. Applying 4G to linear operators $h\in U^*$, we see that hf is McShane integrable, with Saks-Henstock indefinite integral hF. By 4F, $\int_E hf\,d\mu$ is defined and equal to $h(F(E))$ for every $E\in\Sigma$, $h\in U^*$; but this is just the definition of Pettis integrability and indefinite Pettis integral.

4I Theorem Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space in which μ is outer regular with respect to the open sets, and U a Banach space. If $f:X\to U$ is Bochner integrable, then f is McShane integrable.

proof (a) Suppose first that f is of the form $\chi E_0\cdot u$ where $E_0\in\Sigma^f$, so that $f(x)=u$ if $x\in E_0$, 0 otherwise. Let $\epsilon>0$. Because μ is inner regular with respect to the closed sets and outer regular with respect to the open sets, there are a closed set $E_1\subseteq E_0$ and an open set $G\supseteq E_0$ such that $\mu(G\setminus E_1)\le\epsilon$. Let δ be the neighbourhood gauge

$$\{(x,C):x\in X,\,C\subseteq X,\,x\in G\Rightarrow C\subseteq G,\,x\notin E_1\Rightarrow C\cap E_1=\emptyset\}.$$

If $\boldsymbol{t}\in T=T(X,\Sigma^f)$ is δ-fine and $\mathcal{R}_{E_0\epsilon}$-filling, set $\boldsymbol{s}=\{(x,C):(x,C)\in\boldsymbol{t},\,x\in E_0\}$, so that $S_{\boldsymbol{t}}(f)=\mu W_{\boldsymbol{s}}\cdot u$. Also

$$E_0\setminus W_{\boldsymbol{s}}\subseteq(E_0\setminus W_{\boldsymbol{t}})\cup(E_0\setminus E_1)$$

because $C\cap E_1=\emptyset$ if $(x,C)\in\boldsymbol{t}$ and $x\notin E_0$, while

$$W_{\boldsymbol{s}}\setminus E_0\subseteq G\setminus E_0$$

because $C\subseteq G$ if $(x,C)\in\boldsymbol{t}$ and $x\in E_0$. Accordingly

$$\mu(E_0\triangle W_{\boldsymbol{s}})\le\mu(E_0\setminus W_{\boldsymbol{t}})+\mu(E_0\setminus E_1)+\mu(G\setminus E_0)\le 2\epsilon,$$

and

$$\|S_{\boldsymbol{t}}(f)-\text{B}\!\!\oint f\,d\mu\|=\|\mu W_{\boldsymbol{s}}\cdot u-\mu E_0\cdot u\|\le 2\epsilon\|u\|.$$

As ϵ is arbitrary, $\text{M}\!\!\oint\!\text{S}\,f\,d\mu$ is defined and equal to $\text{B}\!\!\oint f\,d\mu$.

(b) It follows at once that $\text{M}\!\!\oint\!\text{S}\,f\,d\mu=\text{B}\!\!\oint f\,d\mu$ for every 'simple' function $f:X\to U$, that is, a measurable function taking finitely many values such that $\{x:f(x)\ne 0\}$ has finite measure.

(c) Now observe that if $f:X\to U$ is a measurable function and $\gamma>\int\|f\|d\mu$, there is a neighbourhood gauge δ on X such that $\|S_{\boldsymbol{t}}(f)\|\le\gamma$ for every δ-fine tagged partition $\boldsymbol{t}$. **P** Set $E=\{x:f(x)\ne 0\}$; then E is covered by a sequence of sets of finite measure, so there is a $g_0:X\to[0,\infty[$ such that $g_0(x)>\|f(x)\|$ for every $x\in E$ and $\int g\,d\mu<\gamma$. Because μ is outer regular with respect to the open sets, there is a lower semi-continuous function $g : X \to [0,\infty]$ such that $g_0 \le g$ and $\int g\,d\mu \le \gamma$ (Fremlin 03, 412W). Set $G_x = \{y : g(y) > \|f(x)\|\}$ for $x \in E$, X for $x \in X \setminus E$; then $\delta = \{(x,C) : x \in X,\ C \subseteq G_x\}$ is a neighbourhood gauge on X. Let $\boldsymbol{t} \in T$ be any δ-fine tagged partition. Then for each $(x,C) \in \boldsymbol{t}$ we have either $f(x) = 0$ or $g(y) > \|f(x)\|$ for every $y \in C$, so

$$\|S_{\boldsymbol{t}}(f)\| \le \sum_{(x,C)\in\boldsymbol{t}} \|f(x)\|\mu C \le \sum_{(x,C)\in\boldsymbol{t}} \int_C g\,d\mu \le \gamma. \quad \mathbf{Q}$$

(d) Now let $f : X \to U$ be any Bochner integrable function, and $\epsilon > 0$. Then there is a simple function $f_0 : X \to U$ such that $\int \|f - f_0\| d\mu \le \epsilon$. By (b), there are a neighbourhood gauge δ_1 on X, an $H \in \Sigma^f$ and an $\eta > 0$ such that $\|S_{\boldsymbol{t}}(f_0) - \text{B}\!\!\int f_0 d\mu\| \le \epsilon$ for every δ_1-fine $\mathcal{R}_{H\eta}$-filling $\boldsymbol{t} \in T$. By (c), there is a neighbourhood gauge δ_2 such that $\|S_{\boldsymbol{t}}(f - f_0)\| \le 2\epsilon$ for every δ_2-fine $\boldsymbol{t} \in T$. Now $\delta = \delta_1 \cap \delta_2$ is a neighbourhood gauge on X, and if $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}_{H\eta}$-filling then

$$\|S_{\boldsymbol{t}}(f) - \text{B}\!\!\int f d\mu\| \le \|S_{\boldsymbol{t}}(f - f_0)\| + \|S_{\boldsymbol{t}}(f_0) - \text{B}\!\!\int f_0 d\mu\| + \|\text{B}\!\!\int f_0 d\mu - \text{B}\!\!\int f d\mu\| \le 4\epsilon.$$

As ϵ is arbitrary, $\text{McS}\!\!\int f d\mu$ is defined and equal to $\text{B}\!\!\int f d\mu$.

4J Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space in which μ is outer regular with respect to the open sets, and U a separable Banach space. Then the Pettis, Bochner and McShane integrals coincide for bounded functions from X to U.

proof Because U is separable, the Pettis and Bochner integrals coincide (); now 4I and 4H tell us that (because μ is outer regular) the McShane integral agrees with both.

4K The problem Can the McShane integral be described in terms of the measure space (X, Σ, μ) without reference to the topology? I have been able to show that if $U = \ell^\infty$ and X is the union of a sequence of open sets of finite measure then the McShane integral always coincides with the Birkhoff integral (Fremlin n92), so the topology is irrelevant. Di Piazza & Preiss p03 give examples of spaces U of arbitrarily large weight for which the McShane and Pettis integrals must coincide, so again the McShane integral cannot depend on the topology. But for arbitrary U, I do not know whether we can have two topologies $\mathfrak{S}$, $\mathfrak{T}$ on $[0,1]$ for both of which Lebesgue measure is quasi-Radon but the McShane integrals for U-valued functions are different.

4L Lemma Let (X, Σ, μ) be an atomless totally finite measure space, and $G_0, \dots, G_n \in \Sigma$. Suppose that $\alpha_0, \dots, \alpha_n \ge 0$ are such that $\sum_{i\in I} \alpha_i \le \mu(\bigcup_{i\in I} G_i)$ for every $I \subseteq n+1$. Then there are disjoint $E_0, \dots, E_n \in \Sigma$ such that $E_i \subseteq G_i$ for every i and $\mu E_i = \alpha_i$ for every i.

proof Bollobás & Varopoulos 75, or use the max-flow min-cut theorem.

4M Lemma Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless totally finite quasi-Radon measure space, U a Banach space, $f : X \to U$ a McShane integrable function with $\text{McS}\!\!\int f d\mu = w$ and $\epsilon > 0$. Then there are an $\eta > 0$ and a family $\langle G_x \rangle_{x\in X}$ in $\mathfrak{T}$ such that $x \in G_x$ for every x and $\|w - \sum_{i<n} \alpha_i f(x_i)\| \le \epsilon$ whenever $x_i \in X$, $\alpha_i \ge 0$ are such that $\sum_{i<n} \alpha_i \ge \mu X - \eta$ and $\sum_{i\in J} \alpha_i \le \mu(\bigcup_{i\in J} G_{x_i})$ for every $J \subseteq n$.

proof Let $\delta \in \Delta(X, \mathfrak{T})$, $\eta > 0$ be such that $\|w - S_{\boldsymbol{t}}(f)\| \le \epsilon$ whenever $\boldsymbol{t} \in T = T(X, \Sigma)$ is δ-fine and $\mu W_{\boldsymbol{t}} \ge \mu X - \eta$. Set $G_x = \bigcup\{C : (x,C) \in \delta\}$ for each $x \in X$, so that G_x is open and $\delta = \{(x,C) : x \in X,\ C \subseteq G_x\}$. Suppose that $x_i \in X$, $\alpha_i \ge 0$ are such that $\sum_{i<n} \alpha_i \ge \mu X - \eta$ and $\sum_{i\in J} \alpha_i \le \mu(\bigcup_{i\in J} G_{x_i})$ for every $J \subseteq n$. By Lemma 4L, there are disjoint measurable $F_i \subseteq G_{x_i}$, for $i < n$, such that $\mu F_i = \alpha_i$ for each i. So $\boldsymbol{t} = \{(x_i, F_i) : i < n\}$ belongs to T and is δ-fine, and $\mu W_{\boldsymbol{t}} = \sum_{i<n} \alpha_i \ge \mu X - \eta$. Accordingly

$$\|w - \sum_{i<n} \alpha_i f(x_i)\| = \|w - S_{\boldsymbol{t}}(f)\| \le \epsilon.$$

As x_i and α_i are arbitrary, we have appropriate G_x and η.

4N Theorem Suppose that $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ are atomless quasi-Radon probability spaces, and $\phi : X \to Y$ an almost continuous inverse-measure-preserving function. If U is a Banach space and $f : Y \to U$ is McShane integrable, then $f\phi : X \to Y$ is McShane integrable, with the same integral.

proof Write w for $\text{McS}\!\int f\,d\nu$. Let $\epsilon > 0$. By Lemma 4M, there are an $\eta > 0$ and a family $\langle H_y\rangle_{y\in Y}$ in $\mathfrak{S}$ such that $y \in H_y$ for every y and $\|w-\sum_{i<n}\alpha_i f(y_i)\| \le \epsilon$ whenever $y_i \in Y$, $\alpha_i \ge 0$ are such that $\sum_{i<n}\alpha_i \ge 1-3\eta$ and $\sum_{i\in J}\alpha_i \le \nu(\bigcup_{i\in J}H_{y_i})$ for every $J \subseteq n$.

For each $n \in \mathbb{N}$, let $K_n \subseteq X$ be a closed set such that $\phi{\restriction}K_n$ is continuous and $\mu(X\setminus K_n) \le 2^{-n}\min(\eta,\epsilon)$. For $x \in X$, let $n_x \in \mathbb{N}$ be such that $n_x \le \|f(\phi(x))\| < n_x + 1$ and take $G_x \in \mathfrak{T}$ such that $x \in G_x$ and $K_{n_x} \cap G_x \subseteq \phi^{-1}[H_{\phi(x)}]$. Let δ be the neighbourhood gauge $\{(x, C) : x \in X,\ C \subseteq G_x\}$. Suppose that $\boldsymbol{s} \in T(X,\Sigma)$ is δ-fine and such that $\mu W_{\boldsymbol{s}} \ge 1-\eta$. Set

$$\boldsymbol{s}' = \{(x, E \cap K_{n_x}) : (x, E) \in \boldsymbol{s}\}.$$

so that $\boldsymbol{s}' \in T(X,\Sigma)$ is δ-fine. Then

$$\begin{aligned}\|S_{\boldsymbol{s}}(f\phi) - S_{\boldsymbol{s}'}(f\phi)\| &\le \sum_{(x,E)\in\boldsymbol{s}} \|f(x)\|\mu(E\setminus K_{n_x})\\ &= \sum_{n=0}^{\infty}\sum_{\substack{(x,E)\in\boldsymbol{s}\\ n_x=n}} \|f(x)\|\mu(E\setminus K_{n_x})\\ &\le \sum_{n=0}^{\infty}(n+1)\mu(X\setminus K_n) \le \sum_{n=0}^{\infty}2^{-n}(n+1)\epsilon = 4\epsilon,\\ \mu(W_{\boldsymbol{s}}\setminus W_{\boldsymbol{s}'}) &\le \sum_{(x,E)\in\boldsymbol{s}} \mu(E\setminus K_{n_x})\\ &= \sum_{n=0}^{\infty}\sum_{\substack{(x,E)\in\boldsymbol{s}\\ n_x=n}} \mu(E\setminus K_{n_x})\\ &\le \sum_{n=0}^{\infty}\mu(X\setminus K_n) \le \sum_{n=0}^{\infty}2^{-n}\eta = 2\eta.\end{aligned}$$

Enumerate $\boldsymbol{s}'$ as $\langle(x_i, E_i)\rangle_{i<n}$. Then

$$\textstyle\sum_{i\in J}\mu E_i \le \mu(\bigcup_{i\in J}\phi^{-1}[H_{\phi(x_i)}]) = \nu(\bigcup_{i\in J}H_{\phi(x_i)})$$

for any $J \subseteq n$, while

$$\textstyle\sum_{i<n}\mu E_i = \mu W_{\boldsymbol{s}'} \ge \mu W_{\boldsymbol{s}} - 2\eta \ge 1-3\eta.$$

So

$$\|w - S_{\boldsymbol{s}'}(f\phi)\| = \|w - \textstyle\sum_{i<n}\mu E_i\cdot f(\phi(x_i))\| \le \epsilon,$$

and $\|w - S_{\boldsymbol{s}}(f\phi)\| \le 5\epsilon$. As ϵ is arbitrary, $\text{McS}\!\int f\phi\,d\mu = w$, as required.

remark The hypothesis 'atomless' can be omitted; see Fremlin preT.

4O The following fact will be useful below (in the special case $X = [0,1]$, for which the proof can be shortened a good deal), and seems to be interesting in itself.

Lemma Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite quasi-Radon measure space with countable Maharam type, and $\epsilon > 0$. Then there is an order-preserving function $\psi : \Sigma \to \mathfrak{T}$ such that $\mu(E\setminus\psi E) = 0$ and $\mu(\psi E\setminus E) \le \epsilon\min(1,\mu E)$ for every $E \in \Sigma$, while $\psi E = \psi F$ if $E\triangle F$ is negligible.

Remark Of course the phrase '$\psi E = \psi F$ if $E\triangle F$ is negligible' tells us that really we are dealing with a function defined on the measure algebra.

proof (a) Consider first the case in which $X = \{0,1\}^{\mathbb{N}}$ with its usual topology and μ is a probability measure. Let $\mathcal{C}$ be the family of all sets C of the form $\{x : x \in X,\ x{\restriction}n = z\}$ where $n \in \mathbb{N}$ and $z \in \{0,1\}^n$. For $E \in \Sigma$ set

$$\psi E = \bigcup\{C : C \in \mathcal{C},\ \mu(E\cap C) \ge \tfrac{1}{1+\epsilon}\mu C\}.$$

Then ψE is always open. By Lévy's martingale theorem (Fremlin 01, 275I), applied to the finite σ-algebras $\Sigma_n = \{E : E \in \Sigma$ is determined by coordinates $< n\}$, $E \setminus \psi E$ is negligible for every $E \in \Sigma$. Clearly $\psi E \subseteq \psi F$ whenever $E \subseteq F$, and $\psi E = \psi F$ if $E \triangle F$ is negligible. To estimate $\mu(\psi E)$, note that $\mathcal{C}$ is an inverted tree. So if we set $\mathcal{A} = \{C : C \in \mathcal{C},\, \mu(E \cap C) \geq \frac{1}{1+\epsilon}\mu C\}$ and look at the set $\mathcal{A}_1$ of maximal elements of $\mathcal{A}$, these are disjoint and their union is ψE; so that

$$\mu(\psi E) = \sum_{C \in \mathcal{A}_1} \mu C \leq \sum_{C \in \mathcal{A}_1} (1+\epsilon)\mu(E \cap C) \leq (1+\epsilon)\mu E.$$

So $\mu(\psi E \setminus E) \leq \epsilon \mu E$.

(b) Evidently the result of (a) applies to any totally finite Radon measure on $\{0,1\}^{\mathbb{N}}$. Now let $(X, \mathfrak{T}, \Sigma, \mu)$ be any quasi-Radon probability measure with countable Maharam type. Because μ has countable Maharam type, there is a sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ such that $\{E_n^\bullet : n \in \mathbb{N}\}$ generates the measure algebra $\mathfrak{A}$ of μ; because μ is σ-finite, $\mathfrak{A}$ is ccc and is the σ-subalgebra of itself generated by $\{E_n^\bullet : n \in \mathbb{N}\}$. This means that, writing Σ_0 for the σ-algebra of sets generated by $\{E_n : n \in \mathbb{N}\}$, every member of Σ differs by a negligible set from some member of Σ_0. Define $f : X \to \{0,1\}^{\mathbb{N}}$ by setting $f(x)(n) = (\chi E_n)(x)$ for all $x \in X$, $n \in \mathbb{N}$. Then f is measurable; because $\{0,1\}^{\mathbb{N}}$ is separable and metrizable, and μ is inner regular with respect to the closed sets, f is almost continuous (Fremlin 03, 418J).

Because μ is effectively locally finite, it is inner regular with respect to the family $\mathcal{K}$ of measurable sets K included in open sets of finite measure such that $f \restriction K$ is continuous. Because μ is σ-finite, there is a disjoint sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ such that $\bigcup_{n \in \mathbb{N}} K_n$ is conegligible. For each $n \in \mathbb{N}$, there is an open set $G_n \supseteq K_n$ of finite measure; again because μ is inner regular with respect to the closed sets, we can find a non-increasing sequence $\langle G_{ni} \rangle_{i \in \mathbb{N}}$ of open sets including K_n such that $\mu(G_{ni} \setminus K_n) \leq 2^{-n-i-2}\epsilon$ for each $i \in \mathbb{N}$.

(c) For each $n \in \mathbb{N}$, we have a Radon measure ν_n on $\{0,1\}^{\mathbb{N}}$ defined by setting $\nu_n F = \mu(K_n \cap f^{-1}[F])$ for every Borel set $F \subseteq \{0,1\}^{\mathbb{N}}$; let T_n be the domain of ν_n. Write $\mathfrak{S}$ for the topology of $\{0,1\}^{\mathbb{N}}$, and let $\psi_n : \mathrm{T}_n \to \mathfrak{S}$ be an order-preserving function such that

$$\nu_n(F \setminus \psi_n F) = 0, \quad \nu_n(\psi_{ni} F \setminus F) \leq 2^{-n-2}\epsilon \min(1, \nu_n F)$$

for every $F \in \mathrm{T}_n$, and $\psi_n F = \psi_n F'$ whenever $\nu_n(F \triangle F') = 0$. Then we can define $\psi : \Sigma \to \mathfrak{T}$ by setting

$$\begin{aligned} \psi E &= \emptyset \text{ if } \mu E = 0, \\ &= \bigcup_{n \in \mathbb{N}} (G_{ni} \setminus K_n) \cup (K_n \cap f^{-1}[\psi_n F]) \end{aligned}$$

whenever $i = \min\{j : j \in \mathbb{N},\, \mu E \geq 2^{-j}\}$ and F is a Borel subset of $\{0,1\}^{\mathbb{N}}$ such that $\mu(E \triangle f^{-1}[F])) = 0$. Because E is equivalent to the inverse image of a Borel set, we can always find such an F; because $\psi_n F = \psi_n F'$ whenever $\nu_n(F \triangle F') = 0$, it won't matter which we take. Because $\psi_n F$ is open and $f \restriction K_n$ is continuous for every n, ψE will be open. Because $\bigcup_{n \in \mathbb{N}} K_n$ is conegligible and $F \setminus \psi_n F$ is always ν_n-negligible, $E \setminus \psi E$ is μ-negligible.

Consider $\mu(\psi E \setminus E)$. If $\mu E = 0$, this is zero, as $\psi E = 0$. Otherwise, set $i = \min\{j : \mu E \geq 2^{-j}\}$, so that $\min(1, \mu E) \geq 2^{-i}$. Then

$$
\begin{aligned}
\mu(\psi E\setminus E) &\le \sum_{n=0}^{\infty}\mu(G_{ni}\setminus K_n)+\mu(K_n\cap f^{-1}[\psi_n F]\setminus E)\\
&\le \sum_{n=0}^{\infty}2^{-n-i-2}\epsilon+\mu(K_n\cap f^{-1}[\psi_n F]\setminus f^{-1}[F])\\
&= 2^{-i-1}\epsilon+\sum_{n=0}^{\infty}\nu_n(\psi_n F\setminus F)\\
&\le 2^{-i-1}\epsilon+\sum_{n=0}^{\infty}2^{-n-2}\epsilon\min(1,\nu_n F)\\
&\le \frac{1}{2}\epsilon\min(1,\mu E)+\frac{1}{2}\epsilon\sum_{n=0}^{\infty}\min(2^{-n-1},\mu(K_n\cap E))\\
&\le \frac{1}{2}\epsilon\min(1,\mu E)+\frac{1}{2}\epsilon\min(1,\mu E)=\epsilon\min(1,\mu E),
\end{aligned}
$$

which is what we need. Finally, if $E\subseteq E'$ and $\mu E=0$, of course $\psi E\subseteq\psi E'$. If $\mu E>0$, then for any corresponding F and F' we must have $\nu_n(F\setminus F')=0$ for every n, so that $\psi_n F=\psi_n(F\cap F')\subseteq\psi_n F'$; at the same time, if $i=\min\{j:\mu E\ge 2^{-j}\}$ and $i'=\min\{j:\mu E\ge 2^{-j}\}$, then $i\ge i'$ so $G_{ni}\subseteq G_{ni'}$ for every n. Assembling these, we have $\psi E\subseteq\psi E'$, as required.

4P Problem ET Let μ_L be Lebesgue measure on $[0,1]$. Consider the statement

(†) whenever $\langle G_x\rangle_{x\in[0,1]}$ is a family of open subsets of $\mathbb{R}$ such that $\mu_L(A\setminus\bigcup_{x\in A}G_x)=0$ for every $A\subseteq[0,1]$, and $\epsilon>0$, there is a family $\langle H_x\rangle_{x\in[0,1]}$ of open sets such that $x\in H_x$ for every $x\in[0,1]$ and $\mu_L(\bigcup_{x\in A}H_x\setminus\bigcup_{x\in A}G_x)\le\epsilon$ for every $A\subseteq[0,1]$.

If this is true, we have the following result.

4Q Proposition Let $\mathfrak{T}$ be the usual topology on $[0,1]$, and $\mathfrak{S}$ another topology on $[0,1]$ with respect to which Lebesgue measure μ_L on $[0,1]$ is quasi-Radon. Let U be a Banach space, and $I_{\mathfrak{T}}$, $I_{\mathfrak{S}}$ the U-valued McShane integrals. Then $I_{\mathfrak{S}}$ extends $I_{\mathfrak{T}}$, and if (†) in 4P is true, then $I_{\mathfrak{S}}=I_{\mathfrak{T}}$.

proof (a) Write Σ for the σ-algebra of Lebesgue measurable subsets of $[0,1]$. The identity map ι from $([0,1],\Sigma)$ to $([0,1],\mathfrak{T})$ is measurable; because $\mathfrak{T}$ is separable and metrizable, and μ_L is inner regular with respect to the $\mathfrak{S}$-closed sets, ι is almost continuous (Fremlin 03, 418J), so every $I_{\mathfrak{T}}$-integrable function is $I_{\mathfrak{S}}$-integrable with the right integral (Proposition 4N).

(b) Now suppose that (†) is true, U is a Banach space and $f:[0,1]\to U$ is such that $w=I_{\mathfrak{S}}(f)$ is defined. Let $\epsilon>0$. Then there are an $\eta>0$ and a family $\langle W_x\rangle_{x\in[0,1]}$ in $\mathfrak{S}$ such that $\|w-\sum_{i<n}\alpha_i f(x_i)\|\le\epsilon$ whenever $x_i\in[0,1]$, $\alpha_i\ge 0$ are such that $\sum_{i<n}\alpha_i\ge 1-5\eta$ and $\sum_{i\in J}\alpha_i\le\mu_L(\bigcup_{i\in J}W_{x_i})$ for every $J\subseteq n$.

For each $n\in\mathbb{N}$, let $\psi_n:\Sigma\to\mathfrak{T}$ be an order-preserving function such that $E\setminus\psi_n E$ is negligible and $\mu_L(\psi_n E\setminus E)\le 2^{-n}\eta$ for every $E\in\Sigma$ (4O). Set $A_n=\{x:x\in[0,1],\,n\le\|f(x)\|<n+1\}$ for $n\in\mathbb{N}$. For $x\in A_n$, set $G_x=\psi_n W_x$. Then $\mu_L(A\setminus\bigcup_{x\in A}G_x)=0$ for every $A\subseteq[0,1]$. **P** $\{W_x:x\in A\}\subseteq\mathfrak{S}$; because μ_L is a τ-additive topological measure for $\mathfrak{S}$, there is a countable $A'\subseteq A$ such that $\bigcup_{x\in A}W_x\setminus\bigcup_{x\in A'}W_x$ is negligible. But now

$$A\setminus\textstyle\bigcup_{x\in A}G_x\subseteq(\bigcup_{x\in A}W_x\setminus\bigcup_{x\in A'}W_x)\cup\bigcup_{x\in A'}W_x\setminus G_x$$

is negligible. **Q** So (†) tells us that for each $n\in\mathbb{N}$ we have a family $\langle W'_{nx}\rangle_{x\in[0,1]}$ in $\mathfrak{T}$ such that $x\in W'_{nx}$ for every x and $\mu_L(\bigcup_{x\in A}W'_{nx}\setminus\bigcup_{x\in A}G_x)\le 2^{-n}\eta$ for every $A\subseteq[0,1]$. Set

$$\delta'=\textstyle\bigcup_{n\in\mathbb{N}}\{(x,C):x\in A_n,\,C\subseteq W'_{nx}\},$$

so that δ' is a $\mathfrak{T}$-neighbourhood gauge on $[0,1]$.

If $A\subseteq A_n$, then $\mu_L(\bigcup_{x\in A}W'_{nx}\setminus\bigcup_{x\in A}W_x)\le 2^{-n+1}\eta$. **P**

$$\bigcup_{x\in A} W'_{nx}\setminus\bigcup_{x\in A} W_x\subseteq(\bigcup_{x\in A} W'_{nx}\setminus\bigcup_{x\in A} G_x)\cup(\bigcup_{x\in A} G_x\setminus\bigcup_{x\in A} W_x)$$
$$\subseteq(\bigcup_{x\in A} W'_{nx}\setminus\bigcup_{x\in A} G_x)\cup(\psi_n(\bigcup_{x\in A} W_x)\setminus\bigcup_{x\in A} W_x)$$

because ψ_n is order-preserving; but each item in the last expression has measure at most $2^{-n}\eta$. **Q**

Let $\boldsymbol{t}\in T$ be δ'-fine and such that $\mu_L W_{\boldsymbol{t}}\ge 1-\eta$. For each $n\in\mathbb{N}$, set

$$H_n=\bigcup_{(x,E)\in\boldsymbol{t},x\in A_n} W'_{nx}\setminus\bigcup_{(x,E)\in\boldsymbol{t},x\in A_n} W_x,$$

so that $\mu_L H_n\le 2^{-n+1}\eta$. Set $\alpha_{(x,E)}=\mu_L(E\setminus H_n)$ when $(x,E)\in\boldsymbol{t}$ and $x\in A_n$. Then for any $\boldsymbol{s}\subseteq\boldsymbol{t}$,

$$\sum_{(x,E)\in\boldsymbol{s}}\alpha_{(x,E)}=\sum_{n=0}^{\infty}\sum_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}}\mu_L(E\setminus H_n)$$
$$=\mu_L(\bigcup_{n\in\mathbb{N}}(\bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} E\setminus H_n))$$
$$\le\mu_L(\bigcup_{n\in\mathbb{N}}(\bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} W'_{nx}\setminus H_n))$$
$$\le\mu_L(\bigcup_{n\in\mathbb{N}}\bigcup_{\substack{(x,E)\in\boldsymbol{s}\\ x\in A_n}} W_x)=\mu_L(\bigcup_{(x,E)\in\boldsymbol{s}} W_x).$$

Also

$$\sum_{(x,E)\in\boldsymbol{t}}(\mu_L E-\alpha_{(x,E)})=\sum_{n=0}^{\infty}\sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}}\mu_L(E\cap H_n)$$
$$\le\sum_{n=0}^{\infty}\mu_L H_n\le 4\eta,$$

so $\sum_{(x,E)\in\boldsymbol{t}}\alpha_{(x,E)}\ge 1-5\eta$. By the choice of $\langle W_x\rangle_{x\in[0,1]}$ and η, $\|w-\sum_{(x,E)\in\boldsymbol{t}}\alpha_{(x,E)}f(x)\|\le\epsilon$. On the other hand

$$\|S_{\boldsymbol{t}}(f)-\sum_{(x,E)\in\boldsymbol{t}}\alpha_{(x,E)}f(x)\|\le\sum_{n=0}^{\infty}\sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}}(n+1)|\mu_L E-\alpha_{(x,E)}|$$
$$=\sum_{n=0}^{\infty}(n+1)\sum_{\substack{(x,E)\in\boldsymbol{t}\\ x\in A_n}}\mu_L(E\cap H_n)$$
$$\le\sum_{n=0}^{\infty}(n+1)\mu_L H_n$$
$$\le\sum_{n=0}^{\infty}2^{-n+1}(n+1)\eta=8\eta\le 8\epsilon.$$

So $\|w-S_{\boldsymbol{t}}(f)\|\le 9\epsilon$. As ϵ is arbitrary, $I_{\mathfrak{T}}(f)=w$; as f is arbitrary, $I_{\mathfrak{S}}=I_{\mathfrak{T}}$.

4R Remarks (a) It is easy to show that if $\langle G_x\rangle_{x\in[0,1]}$ is any family of relatively open subsets of $[0,1]$ such that $A\setminus\bigcup_{x\in A}G_x$ is negligible for every $A\subseteq[0,1]$, and $\mathfrak{S}$ is the topology on $[0,1]$ generated by the usual topology and $\{\{x\}\cup G_x:x\in[0,1]\}$, then Lebesgue measure is τ-additive for $\mathfrak{S}$, therefore quasi-Radon. So families of this kind can be taken as representative of neighbourhood gauges for topologies on $[0,1]$ which are compatible in this sense with Lebesgue measure.

(b) Suppose that $\langle G_x\rangle_{x\in[0,1]}$ is a family of relatively open subsets of $[0,1]$ such that $\limsup_{\eta\downarrow 0}\frac{1}{2\eta}\mu_L(G_x\cap[x-\eta,x+\eta])>0$ for every $x\in[0,1]$. Then $A'=A\setminus\bigcup_{x\in A}G_x$ is negligible for every $A\subseteq[0,1]$. **P?** If not, then there is an $x\in A'$ such that $\lim_{\eta\downarrow 0}\frac{1}{2\eta}\mu_L^*(A'\cap[x-\eta,x+\eta])=1$ (Fremlin 01, 261D). But as $A'\cap G_x=\emptyset$ this is impossible. **XQ** (Compare Fremlin 03, 453Xk.)

Even for families of this kind I have not been able to determine whether (†) is true.

Basic references: Fremlin 03, §§481-482, Fremlin 95, Fremlin prET.

Further reading: Fremlin n92.

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*Items marked * are obtainable via my home page*

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