Errata and addenda for Volume 5, 2015 printing

I collect here known errors and omissions, with their discoverers, in Volume 5 of my book *Measure Theory* (see my web page, `http://www1.essex.ac.uk/maths/people/fremlin/mt.htm`).

Part I

p l (511Y) Add new exercise:
(d) Show that, for a set I, (ω_1,ω,ω) is a precaliber triple of $\mathbb{N}^I$ iff I is countable.

p l (514Y) Add new exercise:
(g) Let $\mathfrak{A}$ be a ccc Dedekind complete Boolean algebra with Maharam type κ. Show that there is a σ-ideal $\mathcal{J}$ of the Baire σ-algebra $\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)$ such that $\mathfrak{A}\cong\mathcal{B}\mathfrak{a}(\{0,1\}^\kappa)/\mathcal{J}$.

p l Proposition 515N has been revised, and new results added. The last part of this section now reads
515N Proposition Let I be a set. Write $\mathfrak{G}$ for the regular open algebra $\mathrm{RO}(\{0,1\}^I)$.
(a) $\mathfrak{G}$ is ccc and Dedekind complete and isomorphic to the category algebra of $\{0,1\}^I$. The algebra of open-and-closed subsets of $\{0,1\}^I$ is an order-dense subalgebra of $\mathfrak{G}$.
(b) Let $\mathfrak{A}$ be a Boolean algebra. Then $\mathfrak{A}$ is isomorphic to $\mathfrak{G}$ iff it is Dedekind complete and there is a Boolean-independent family $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}$ such that the subalgebra generated by $\{a_i:i\in I\}$ is order-dense in $\mathfrak{A}$.
(c) If I is infinite, $\mathfrak{G}$ is homogeneous.

515O Proposition (a) A Boolean algebra is isomorphic to $\mathfrak{G}=\mathrm{RO}(\{0,1\}^{\mathbb{N}})$ iff it is Dedekind complete, atomless, has countable π-weight and is not $\{0\}$. In particular, the regular open algebra $\mathrm{RO}(\mathbb{R})$ is isomorphic to $\mathfrak{G}$.
(b) Every atomless order-closed subalgebra of $\mathfrak{G}$ is isomorphic to $\mathfrak{G}$.

515P Proposition A Boolean algebra $\mathfrak{A}$ is isomorphic to $\mathrm{RO}(\{0,1\}^{\omega_1})$ iff
(α) it is non-zero, ccc and Dedekind complete,
(β) every non-zero principal ideal of $\mathfrak{A}$ has π-weight ω_1,
(γ) there is a non-decreasing family $\langle A_\xi\rangle_{\xi<\omega_1}$ of countable subsets of $\mathfrak{A}$ such that
each A_ξ is order-dense in the order-closed subalgebra of $\mathfrak{A}$ which it generates,
$A_\zeta=\bigcup_{\xi<\zeta}A_\xi$ for every non-zero countable limit ordinal ζ,
$\bigcup_{\xi<\omega_1}A_\xi$ is order-dense in $\mathfrak{A}$.

515Q Proposition Let $\mathfrak{A}$ be an atomless order-closed subalgebra of $\mathfrak{G}=\mathrm{RO}(\{0,1\}^{\omega_1})$. Then $\mathfrak{A}$ is isomorphic either to $\mathrm{RO}(\{0,1\}^\omega)$ or to $\mathfrak{G}$ or to the simple product $\mathrm{RO}(\{0,1\}^\omega)\times\mathfrak{G}$.

p l Proposition 516S has been rewritten, and now reads
516S Proposition Let $\mathfrak{A}$ be a Boolean algebra.
(a) If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$ such that $\theta\le\omega$, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{B}$. In particular, every precaliber pair of $\mathfrak{A}$ is a precaliber pair of $\mathfrak{B}$ and $\mathfrak{B}$ will satisfy Knaster's condition if $\mathfrak{A}$ does.
(b) If $\mathfrak{B}$ is a regularly embedded subalgebra of $\mathfrak{A}$, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.
(c) If $\mathfrak{B}$ is a Boolean algebra and $\phi:\mathfrak{A}\to\mathfrak{B}$ is a surjective order-continuous Boolean homomorphism, then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.
(d) If $\mathfrak{B}$ is a principal ideal of $\mathfrak{A}$ then every precaliber triple of $\mathfrak{A}$ is a precaliber triple of $\mathfrak{B}$.
(e) If $\mathfrak{A}$ is the simple product of a family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of Boolean algebras, $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}_i$ for every $i\in I$ and $\mathrm{cf}\,\kappa>\#(I)$, then $(\kappa,\lambda,{<}\theta)$ is a precaliber triple of $\mathfrak{A}$.

p l Add new result:

516V Proposition Let $\mathfrak{A}$ be an atomless Boolean algebra which satisfies Knaster's condition. Then $\mathfrak{A}$ has an atomless order-closed subalgebra with countable Maharam type.

p l (516X) Add new exercise:

(n) Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras and that there is a surjective Boolean homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Show that if $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{A}$ and θ is countable, then $(\kappa,\lambda,<\theta)$ is a precaliber triple of $\mathfrak{B}$.

p l (516Y) Add new exercise:

(a) Let $\mathfrak{A}$ be an atomless Dedekind complete ccc Boolean algebra such that $(\kappa,\kappa,2)$ is a precaliber triple of $\mathfrak{A}$ for every regular uncountable cardinal κ. Then $\mathfrak{A}$ has an atomless closed subalgebra of countable Maharam type.

p l (517Y) Add new exercise:

(d) (i) Suppose that $\mathcal{A}$, $\mathcal{B}\in[[\mathbb{N}]^\omega]^{<\mathfrak{p}}$, $\mathcal{A}$ is downwards-directed and $A\cap B$ is infinite for all $A\in\mathcal{A}$, $B\in\mathcal{B}$. Show that there is a set $D\subseteq\mathbb{N}$ such that $D\setminus A$ is finite for every $A\in\mathcal{A}$ and $D\cap B$ is infinite for every $B\in\mathcal{B}$. (ii) Show that $\mathfrak{p}$ is regular.

p l In Proposition 521G, add

$\tau(\mu)\le\max(\omega,\sup_{i\in I}\tau(\mu_i),\min\{\lambda:\#(I)\le 2^\lambda\})$.

p l (Proposition 521O) Add new part:

(d) If $\langle A_i\rangle_{i\in I}$ is a disjoint family of subsets of X and $\#(I)>\max(\omega,\mathrm{mag}(\mu))$ then there is an $i\in I$ such that $X\setminus A_i$ has full outer measure.

p l (521X) Add new exercise:

(n) For a measure space (X,Σ,μ) with null ideal $\mathcal{N}(\mu)$, write $\mathrm{hcov}(\mu)$ for $\inf_{E\in\Sigma\setminus\mathcal{N}(\mu)}\mathrm{cov}(E,\mathcal{N}(\mu))$. (Count $\inf\emptyset$ as ∞, as usual.) Show that if (X,Σ,μ) and (Y,T,ν) are semi-finite measure spaces, neither having zero measure, with c.l.d. product $(X\times Y,\Lambda,\lambda)$, then $\mathrm{hcov}(\lambda)=\min(\mathrm{hcov}(\mu),\mathrm{hcov}(\nu))$.

p l Add new part to Lemma 522C:

(iii) $(\mathbb{N}^\mathbb{N},\le^*)\equiv_{\mathrm{T}}(\mathbb{N}^\mathbb{N},\preceq)$.

p l (part (e) of the proof of 522C): for '$(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega})\equiv_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\le^*,\mathbb{N}^\mathbb{N})\equiv_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\le^*,\mathbb{N}^\mathbb{N})$' read '$(\mathbb{N}^\mathbb{N},\le',[\mathbb{N}^\mathbb{N}]^{\le\omega})\equiv_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\le^*,\mathbb{N}^\mathbb{N})\equiv_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\le^*,\mathbb{N}^\mathbb{N})$'.

p 111 l 24 (Theorem 522S) Add new part:

(c) $\mathrm{non}\,\mathcal{M}$ is the least cardinal of any set $A\subseteq\mathbb{N}^\mathbb{N}$ such that for every $g\in\mathbb{N}^\mathbb{N}$ there is an $f\in A$ such that $\{n:f(n)=g(n)\}$ is infinite.

p l (part (b-ii-α) of the proof of 522S): for '$f^*(j+1)=\tilde f(\tilde f(f^*(j)))\ge\tilde f(\tilde f(g(i)))\ge\tilde f(g(i+1))\ge g(i+2)$' read '$f^*(j+1)\ge\tilde f(\tilde f(f^*(j)))\ge\tilde f(\tilde f(g(i)))\ge\tilde f(g(i+1))\ge g(i+2)$'.

p l Exercise 522Xc is wrong, and has been deleted. 522Xe is a copy of 521Xc, and has been dropped. 522Xb is now 522Xc, 522Xd is now 522Xe, 522Xf-522Xg are now 522Xg-522Xh.

p l Exercise 522Yi is wrong, and has been revised. Exercise 522Yj is now 522Sc.

p l Add new problem:

522Z Is it the case that $(\mathbb{R},\in,\mathcal{M})\equiv_{\mathrm{GT}}(\mathbb{N}^\mathbb{N},\mathtt{finint},\mathbb{N}^\mathbb{N})$? (See 522S and the proof of 522V.)

p l Add new fragment to 523Ia:

(ii) $\mathrm{non}\,\mathcal{N}_{\kappa^{(+n)}}\le\max(\kappa^{(+n)},\mathrm{non}\,\mathcal{N}_\kappa$ for every $n\in\mathbb{N}$.

523I(a-ii) and 523I(a-iii) are now 523I(a-iii) and 523I(a-iv).

p l (part (c) of the proof of 523N): for '$\mathrm{cov}(\kappa,[\kappa]^{\le\omega})$' read '$\mathrm{cov}(\kappa,\in,[\kappa]^{\le\omega})$'.

p l Part (ii) of Exercise 523Yf is superseded by 524Na below, so has been dropped.

p l Add new result:

524U Lemma Let $(\mathfrak{A},\bar{\mu})$ be a probability algebra. Then there is a Radon probability measure on $\{0,1\}^{\tau(\mathfrak{A})}$ with measure algebra isomorphic to $(\mathfrak{A},\bar{\mu})$.

p l (525X) Add new exercise:

(i) Show that if $(X,\mathfrak{T},\Sigma,\mu)$ is a Radon measure space and $\mu X>0$, then $\operatorname{cov}\mathcal{N}(\mu)\ge\mathfrak{m}_{\mathrm{K}}$.

p l (Exercise 526Xc) For 'non $\mathcal{N}\mathcal{w}\mathcal{d}=\operatorname{non}\mathcal{M}$' and 'cf $\mathcal{N}\mathcal{w}\mathcal{d}=\operatorname{cf}\mathcal{M}$' read 'non $\mathcal{N}\mathcal{w}\mathcal{d}=\omega$' and 'cf $\mathcal{N}\mathcal{w}\mathcal{d}\le\operatorname{cf}\mathcal{M}$'.

p 168 l 2 In Theorem 527C, we have to assume that μ and ν are inner regular with respect to the Borel sets.

p 169 l 18 (part (a) of the proof of 527F) for 'algebra of subsets of $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$' read 'subalgebra of $\mathcal{P}(\mathbb{N}^{\mathbb{N}})$'.

p 169 l 25 (part (a) of the proof of 527F) for 'and again $U_\tau\setminus E\subseteq U_\tau\notin\mathcal{I}$' read 'and again $U_\tau\setminus E\supseteq U_\tau\notin\mathcal{I}$'.

p 173 l 38 (part (c-iii) of the proof of 527J) for '$f^{-1}[F]\subseteq N_V\cup\{x:\phi_0[\{x\}]\notin\mathcal{M}(Y)\}$' read '$f^{-1}[F]\subseteq N_V\cup\{x:\phi_0(V)[\{x\}]\notin\mathcal{M}(Y)\}$'.

p 173 l 45 (part (c-iii) of the proof of 527J) for '$V_0=(N^*\times Y)\cup\tilde{W}$' and '$x\in X\setminus N^*$' read '$V_0=(\tilde{N}\times Y)\cup\tilde{W}$' and '$x\in X\setminus\tilde{N}$'.

p 177 l 4 (part (d) of the proof of 527O): for '$F\in\mathcal{B}(Y)\setminus\mathcal{M}(X)$' read '$F\in\mathcal{B}(Y)\setminus\mathcal{M}(Y)$'.

p 177 l 33 Exercise 527Xh has been moved to 513Xr. Exercise 527Xi has been moved to 527Ye, and corrected, replacing '$\mathcal{N}(\mu)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)\times\mathcal{M}$' with '$\mathcal{N}(\mu)\ltimes_{\mathcal{B}(X\times Y)}\mathcal{M}(Y)\preccurlyeq_{\mathrm{T}}\mathcal{N}(\mu)\times\mathcal{N}$'. Add new exercise:

(h) Show that a measurable algebra is harmless iff it is purely atomic.

p 177 l 39 (527Y) Exercise 527Ye is wrong, and has been dropped. Add new exercise:

(b) Extend the idea of 527Xb to define an ideal $\ltimes_{\xi<\zeta}\mathcal{I}_\xi$ of subsets of $\prod_{\xi<\zeta}X_\zeta$ when ζ is any ordinal and I_ξ is an ideal of subsets of X_ξ for every $\xi<\zeta$.

527Yb-527Yc are now 527Yc-527Yd, 527Yd is now 527Yf.

p 210 l 14 Part (d) of Proposition 531E has been strengthened, and now reads

(d) If X is K-analytic (in particular, if X is compact) and Y is a continuous image of X, $\mathrm{Mah}_{\mathrm{R}}(Y)\subseteq\mathrm{Mah}_{\mathrm{R}}(X)$.

p 209 l 14 (part (c) of the proof of 531A): for '$F=\bigcup_{m\in\mathbb{N}}\bigcap_{n\in\mathbb{N}}H_n$' read '$F=\bigcup_{m\in\mathbb{N}}\bigcap_{n\ge m}H_n$'.

p 213 l 1 (case 5 in part (b) of the proof of 531G): for 'becomes $\kappa=\omega$' read 'becomes $\kappa\le\omega$'.

p 214 l 7 Theorem 531L has been restated, and now reads

Theorem Let X be a Hausdorff space.

(a) If $\omega\in\mathrm{Mah}_{\mathrm{R}}(X)$ then $\{0,1\}^\omega$ is a continuous image of a compact subset of X.

(b) If $\kappa\ge\omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$ and $\lambda\le\kappa$ is an infinite cardinal such that (κ,λ) is a measure-precaliber pair of every probability algebra, then $\{0,1\}^\lambda$ is a continuous image of a compact subset of X.

p 214 l 21 (part (b) of the proof of 531L): for

$$\bar{\nu}_\kappa(\inf_{\xi\in I}c_\xi\cap e_\xi\cap\inf_{\eta\in J}c_\eta\setminus e_\eta)=\frac{1}{2^{\#(I\cup J)}}\bar{\nu}_\kappa(\inf_{\xi\in I\cup J}c_\xi)$$

read

$$\bar{\mu}(\inf_{\xi\in I}c_\xi\cap e_\xi\cap\inf_{\eta\in J}c_\eta\setminus e_\eta)=\frac{1}{2^{\#(I\cup J)}}\bar{\mu}(\inf_{\xi\in I\cup J}c_\xi).$$

p l Proposition 531M has been restated, and now reads

Proposition If κ is an infinite cardinal and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X whenever X is a compact Hausdorff space such that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then κ is a measure-precaliber of every probability algebra.

p 215 l 20 Proposition 531N has been moved to 531Vb. 531O-531P are now 531N-531O.

p 217 l 28 (part (b)((i)⇒(ii)) of the proof of 531P, now 531O): for '$\mathfrak{D}_\xi$' read '$\mathfrak{C}_\xi$'.

p 219 l 9 The proof of 531Q has been rewritten, with the first part now presented as follows:

531P Lemma Let Y be a zero-dimensional compact metrizable space, μ an atomless Radon probability measure on Y, $A \subseteq Y$ a μ-negligible set and $\mathcal{Q}$ a countable family of closed subsets of Y. Then there are closed sets K, $L \subseteq Y$, with union Y, such that

$$K \cup L = Y, \quad K \cap L \cap A = \emptyset, \quad \mu(K \cap L) \geq \tfrac{1}{2},$$
$$K \cap Q = \overline{Q \setminus L} \text{ and } L \cap Q = \overline{Q \setminus K} \text{ for every } Q \in \mathcal{Q}.$$

p 219 l 12 A further property of the measure constructed in 531Q is declared, so that the statement of the proposition now reads

Proposition Suppose that $\mathrm{cf}\,\mathcal{N}_\omega = \omega_1$. Then there are a hereditarily separable perfectly normal compact Hausdorff space X, of weight ω_1, with a Radon probability measure of Maharam type ω_1 such that every negligible set is metrizable.

p 221 l 33 (part (g) of the proof of 531Q): for '$\sum_{\delta\in I}\mu_\eta Q'_{\delta\eta} = \mu\pi_\eta[H]$' read '$\sum_{\delta\in I}\mu_\eta Q'_{\delta\eta} = \mu_\eta\pi_\eta[H]$'.

p 224 l 27 Theorem 531T has been revised, and now reads

Theorem Suppose that $\omega \leq \kappa < \mathfrak{m}_{\mathrm{K}}$. If X is a Hausdorff space and $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of X.

p 224 l 39 Add new results:

531U Proposition Let X be a Hausdorff space.

(a) Give the space $P_{\mathrm{R}}(X)$ of Radon probability measures on X its narrow topology. If $\kappa \geq \omega_2$ belongs to $\mathrm{Mah}_{\mathrm{R}}(X)$, then $\{0,1\}^\kappa$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$.

(b) Give the space $P_{\mathrm{R}}(X \times X)$ its narrow topology. Then its tightness $t(P_{\mathrm{R}}(X \times X))$ is at least $\sup \mathrm{Mah}_{\mathrm{R}}(X)$.

531V Proposition (a) Suppose that the continuum hypothesis is true. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of $P_{\mathrm{R}}(X)$.

(b) Suppose that there is a family $\langle W_\xi\rangle_{\xi<\omega_1}$ in $\mathcal{N}_{\omega_1}$ such that every closed subset of $\{0,1\}^{\omega_1} \setminus \bigcup_{\xi<\omega_1} W_\xi$ is scattered. Then there is a compact Hausdorff space X such that $\omega_1 \in \mathrm{Mah}_{\mathrm{R}}(X)$ but $\{0,1\}^{\omega_1}$ is not a continuous image of a closed subset of X.

p 225 l 8 (531X) Add new exercises:

(g) Let X be a Hausdorff space such that $\mathrm{Mah}_{\mathrm{R}}(X) \subseteq \{0,\omega\}$, and $\mathcal{N}$ the null ideal of Lebesgue measure on $\mathbb{R}$. Show that the union of fewer than $\mathrm{add}\,\mathcal{N}$ universally Radon-measurable subsets of X is universally Radon-measurable.

>(k) Let X be a Hausdorff space and κ a cardinal. Show that there is a Radon probability measure on X with Maharam type κ iff *either* κ is finite and $2^\kappa \leq 2\#(X)$ *or* $\kappa = \omega \leq \#(X)$ *or* $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ *or* $\mathrm{cf}\,\kappa = \omega$ and $\kappa = \sup \mathrm{Mah}_{\mathrm{R}}(X)$.

(l) Let X be a Hausdorff space and κ an infinite cardinal. (i) Show that $\{0,1\}^\kappa$ is a continuous image of a compact subset of X iff $[0,1]^\kappa$ is a continuous image of a compact subset of X, and that in this case $\{0,1\}^\kappa$ is a continuous image of a compact subset of $P_{\mathrm{R}}(X)$. (ii) Show that if X is normal and $\{0,1\}^\kappa$ is a continuous image of a closed subset of X then $[0,1]^\kappa$ is a continuous image of X. (iii) Show that if X is completely regular and $\{0,1\}^\kappa$ is a continuous image of a compact subset of X then $[0,1]^\kappa$ is a continuous image of X.

531Xg-531Xi are now 531Xh-531Xj, 531Xj is now 531Xq, 531Xk-531Xn are now 531Xm-531Xp,

p 225 l 29 (531Y) Add new exercises:

(d) Let X be a completely regular Hausdorff space and $\kappa \geq \omega_2$ a cardinal. Show that if $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$ then the Banach space $\ell^1(\kappa)$ is isomorphic, as linear topological space, to a subspace of the Banach space $C_b(X)$.

(e) Let X be a locally compact Hausdorff space and κ an infinite cardinal such that $\ell^1(\kappa)$ is isomorphic, as linear topological space, to a subspace of $C_0(X)$ (definition: 436I). Show that $\kappa \in \mathrm{Mah}_{\mathrm{R}}(X)$.

p 243 l 19534 has been thoroughly reorganised, with a few corrections and some supplementary results. In particular, the Galvin-Mycielski-Solovay characterization of strong measure zero in $\mathbb{R}$ (534H) is now 534K.

p l (Proposition 535F) for '$\theta E^\bullet \supseteq \underline{\theta} E$ for every $E \in \Sigma$' read '$\theta a \supseteq \underline{\theta} a$ for every $a \in \mathfrak{A}$'.

The result can be proved with a slightly weaker hypothesis: instead of '$\#(\mathfrak{A}) \leq \omega_1$' we can use '$\#(\mathfrak{A}) \leq \operatorname{add}\mu$' and a trifling modification to the proof.

p l (Exercise 535Xa) for 'A any subset of X' read 'A a non-negligible subset of X'.

p l Exercise 535Ya has been moved to 565Yb.

p l Add new result:

536E Proposition Let (X,Σ,μ) be a semi-finite measure space, with null ideal $\mathcal{N}(\mu)$. Suppose that $\pi(\mu) \leq \operatorname{cov}(E,\mathcal{N}(\mu))$ whenever $E \in \Sigma \setminus \mathcal{N}(\mu)$. Then every $\mathfrak{T}_p$-separable $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ is stable.

p l (563X) Add new exercise:

(c) Suppose that $\mathfrak{m}_K = \mathfrak{c}$. Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space such that $\tau(\mu) \leq \mathfrak{c}$. Show that every pointwise compact subset of $L^0(\Sigma)$ is stable.

p 327 l 45 Add two paragraphs:

539V Lemma For $\mathcal{K} \subseteq [\mathbb{N}]^{<\omega}$, set $\partial\mathcal{K} = \{K \setminus \{\max K\} : \emptyset \neq K \in \mathcal{K}\}$. For every $\xi < \omega_1$, there is a PV norm $\| \|$ on $[\mathbb{N}]^{<\omega}$ such that $\emptyset \in \partial^\xi\mathcal{L}$ where $\mathcal{L} = \{L : \|L\| \leq 1\}$.

539W Theorem Let $\mathfrak{C}$ be a countable atomless Boolean algebra, not $\{0\}$. Write M_{sm} for the set of totally finite submeasures on $\mathfrak{C}$, regarded as a subset of $[0,\infty[^{\mathfrak{C}}$, and M_{esm} for the set of exhaustive totally finite submeasures on $\mathfrak{C}$. Then M_{sm} is Polish, and $M_{\mathrm{esm}} \subseteq M_{\mathrm{sm}}$ is coanalytic and not Borel. Setting

$$F_\xi = \{\nu : \nu \in M_{\mathrm{esm}} \text{ has exhaustivity rank at most } \xi\}$$

for $\xi < \omega_1$, every F_ξ is a Borel subset of M_{sm} and every analytic subset of M_{esm} is included in some F_ξ.

p 328 l 24 Exercise 539Yd is now dealt with in 539Tc, so has been dropped.

Part II

p l Add new result:

542K Proposition Let κ be a quasi-measurable cardinal.

(a) For every cardinal $\theta < \kappa$ there is a family $\mathcal{D}_\theta$ of countable sets, with cardinal less than κ, which is stationary over θ.

(b) There is a family $\mathcal{A}$ of countable sets, with cardinal at most κ, which is stationary over κ.

p 30 l 8 (543X) Add new exercise:

(d) Let μ be Lebesgue measure on $\mathbb{R}$, and $\theta = \frac{1}{2}(\mu^* + \mu_*)$ the outer measure described in 413Xd. Show that μ is the measure defined from θ by Carathéodory's method.

p 43 l 23 Sections §§546-547 have been rewritten as §§546-548, incorporating work of A.Kumar and S.Shelah.

p l (555Y) Add new exercises:

(g) Suppose that λ is a two-valued-measurable cardinal, and $\kappa > \lambda$ a cardinal. Show that

$\Vdash_{\mathbb{P}_\kappa}$ there is a probability measure on ω_1 with Maharam type greater than the least atomlessly-measurable cardinal.

(h) In 555C, suppose that $X = \kappa$ and that μ is a $\{0,1\}$-valued measure on κ witnessing that κ is two-valued-measurable. For $J \subseteq \kappa$ let $P_J : L^\infty(\mathfrak{B}_\kappa) \to L^\infty(\mathfrak{B}_\kappa)$ be the corresponding conditional expectation as in part (b) of the proof of 555F. Show that for every $\sigma \in \mathfrak{B}_\kappa^\kappa$ there is a countable set $J \subseteq \kappa$ such that $\mu\{\xi : \xi < \kappa,\, u_\sigma = P_J(\chi\sigma(\xi))\} = 1$.

p l New material has been added, as follows:

5A1C Concatenation Suppose that σ, τ are two functions with domains α, β respectively which are ordinals. Then we can form their **concatenation** $\sigma^\frown\tau$, setting

$$\mathrm{dom}(\sigma^\frown\tau) = \alpha + \beta$$

(the ordinal sum),

$$(\sigma^\frown\tau)(\xi) = \sigma(\xi) \text{ if } \xi < \alpha,$$
$$(\sigma^\frown\tau)(\alpha + \eta) = \tau(\eta) \text{ if } \eta < \beta.$$

The operator $^\frown$ is associative, so we can omit brackets and speak of $\sigma^\frown\tau^\frown\upsilon$. The empty function $\emptyset$ is an identity in the sense that

$$\emptyset^\frown\sigma = \sigma^\frown\emptyset = \sigma$$

whenever $\mathrm{dom}(\sigma)$ is an ordinal.

In this context, it will often be helpful to have a special notation for functions with domain the singleton set $\{0\} = 1$; I will write $<t>$ for the function with domain $\{0\}$ and value t.

We can also have infinite concatenations. If $\langle\sigma_n\rangle_{n\in\mathbb{N}}$ is a sequence of functions with ordinal domains, we can form the concatenations

$$\sigma_0^\frown\sigma_1, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2, \quad \sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\sigma_3, \quad \ldots$$

to get a sequence of functions each extending its predecessors. The union will be a function with domain the ordinal $\sup_{n\in\mathbb{N}} \mathrm{dom}(\sigma_0) + \ldots + \mathrm{dom}(\sigma_n)$. I will generally denote it $\sigma_0^\frown\sigma_1^\frown\sigma_2^\frown\ldots$ or in some similar form.

5A1C-5A1O are now 5A1D-5A1P.

p l (5A1C, now 5A1D) Add new part:

(d) If X is a Polish space and $\le$ is a well-founded relation on X such that $\{(x,y) : x < y\}$ is analytic, then the height of $\le$ is countable.

p l (part (e-iv) of 5A1E, now 5A1F): for $\mathrm{cf}[\kappa]^{\le\omega} = \max(\kappa, \mathrm{cf}[\lambda]^{\le\omega})$ read $\mathrm{cf}[\kappa]^{\le\omega} \le \max(\kappa, \mathrm{cf}[\lambda]^{\le\omega})$.

p l (5A1I, now 5A1J) Add new parts:

(d) If $R \subseteq X \times X$ is an equivalence relation on a set X I will say that a set $A \subseteq X$ is ***R*-free** if A meets each equivalence class under R in at most one point.

(e) Let X be a set and R an equivalence relation on X.

(i) For any cardinal κ, there is a partition $\langle X_\xi\rangle_{\xi<\kappa}$ of X into R-free sets iff every R-equivalence class has cardinal at most κ.

(ii) If $A \subseteq X$ is R-free then $R[B] \cap R[C] = \emptyset$ whenever B, $C \subseteq A$ are disjoint.

p l (proof of 5A1J, now 5A1K): delete 'and $S = S_M$'.

p l (part (b) of the proof of 5A1M, now 5A1N) For '$\langle fg_\alpha\rangle_{\alpha\in S}$' read '$\langle fg_\alpha\rangle_{\alpha\in S_1}$'.

p 284 l 8 (part (d) of the proof of 5A1M, now 5A1N) For '$\{x : f_g(x) = g_h(x)\}$' read '$\{x : f_g(x) = f_h(x)\}$'.

p l I have added some new results:

5A1Q Lemma Let I and J be non-empty finite sets, and $R \subseteq I \times J$ a relation such that $R[I] = J$. Set

$$k = \max_{x \in I} \#(R[\{x\}]), \quad l = \min_{y \in J} \#(R^{-1}[\{y\}]).$$

Then there is a $K \subseteq I$ such that $R[K] = J$ and $\#(K) \le \frac{1+\ln k}{l}\#(I)$.

5A1R Definition If I is a set and $\mathcal{A}$ is a family of sets, $\mathcal{A}$ is **stationary over** I if for every function $f : [I]^{<\omega} \to [I]^{\le\omega}$ there is an $A \in \mathcal{A}$ such that $f[J] \subseteq A$ for every $J \in [A \cap I]^{<\omega}$.

5A1S Remarks (a) If $\mathcal{A}$ is stationary over I, then $\{A \cap I : A \in \mathcal{A}\}$ is stationary over I.

(b) If $\mathcal{A}$ is stationary over I, and for every $A \in \mathcal{A}$ we are given a family $\mathcal{B}_A$ which is stationary over A, then $\bigcup_{A \in \mathcal{A}} \mathcal{B}_A$ is stationary over I.

(c) If ζ is an ordinal of uncountable cofinality, and $S \subseteq \zeta$ is stationary in the ordinary sense of 4A1C, then S is stationary over ζ in the sense of 5A1R.

5A1T Theorem (a) There is a family $\langle e_\xi \rangle_{\xi<\omega_1}$ such that $e_\xi : \xi \to \mathbb{N}$ is an injective function for each $\xi < \omega_1$ and $e_\eta \triangle (e_\xi \restriction \eta)$ is finite whenever $\eta < \xi < \omega_1$.

(b) There is a sequence $\langle \le_n \rangle_{n\in\mathbb{N}}$ of partial orders on ω_1 such that

$(\omega_1, \le_n)$ is a tree of height at most $n+1$ for each $n \in \mathbb{N}$,

$\eta \le_0 \xi$ iff $\eta = \xi$,

$\le_n \subseteq \le_{n+1}$ for every $n \in \mathbb{N}$,

$\bigcup_{n\in\mathbb{N}} \le_n$ is the usual well-ordering of ω_1.

p 285 l 25 (5A2Ab) for ‘$\mathrm{cf}(\prod_{i\in I} P_i) \le \mathrm{cf}\, P$’ read ‘$\mathrm{cf}(\prod_{i\in F} P_i) \le \mathrm{cf}\, P$’.

p l (part (b-ii) of the proof of 5A2B): for ‘$\mathrm{add}\, P \ge \delta > \#(\xi)$’ read ‘$\mathrm{add}\, P \ge \lambda^+ \#(\xi)$’.

p 287 l 4 (part (a) of the proof of 5A2E): for ‘$\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma', \gamma', \delta) \le \mathrm{cf}[\alpha]^{<\gamma}$’ read ‘$\mathrm{cov}_{\mathrm{Sh}}(\alpha, \gamma', \gamma', \delta) \le \mathrm{cf}[\alpha]^{<\gamma'}$’.

p 292 l 34 (part (e) of the proof of 5A2G): for

$$‘g^*(\eta) = \sup\{f(\xi) : \xi < \gamma_0,\, h^*(\xi) = \eta\}’$$

read

$$‘g^*(\eta) = \sup\{f(\xi) : \xi < \gamma_0,\, f(\xi) < h^*(\xi) = \eta\}.’$$

p l Add ‘$t(Y) \le t(X)$’ to the list in 5A4Bb.

p 309 l 31 Part (d-iii) of 5A4C has been revised, and now reads

So if there is a continuous surjection from a closed subset of X onto $\{0,1\}^\kappa$, there is a non-empty closed $K \subseteq X$ such that $\chi(x, K) \ge \kappa$ for every $x \in K$.

p 311 l 1 5A4Eb is now 4A3S(a-i), so has been deleted. 5A4Ec-5A4Ed are now 5A4Eb-5A4Ec.

p 311 l 30 (5A4E(c-iii), now 5A4E(b-iii)) for ‘$V \in \mathcal{V}_n$’ read ‘$V \in \mathcal{V}'_n$’.

p l (5A4E) Add new fragment:

(c)(iii) Every subset of X with the Baire property is expressible as $G \triangle M$ where G is a cozero set and M is meager.

p l Compact-open topologies I have interpolated a couple of paragraphs on a class of topologies on spaces of functions, and a note on irreducible surjections.

5A4Ia Let X and Y be topological spaces and F a set of functions from X to Y. The **compact-open** topology on F is the topology generated by sets of the form $\{f : f \in F, f[K] \subseteq H\}$ where $K \subseteq X$ is compact and $H \subseteq Y$ is open.

(b) Let X be a compact topological space and I a set; give $Y = \{0,1\}^I$ its usual product topology and $Z = C(X; Y)$ its compact-open topology. Let $\mathcal{E}$ be the algebra of open-and-closed subsets of X. (i) Z is homeomorphic to $\mathcal{E}^I$ with its product topology, where here we give $\mathcal{E}$

its discrete topology. (ii) Set $H_i = \{y : y \in Y,\ y(i) = 1\}$ for $i \in I$. Then $\{\{f : f \in Z, f^{-1}[H_i] = E\} : i \in I,\ E \in \mathcal{E}\}$ is a subbase for the topology of Z.

5A4J Proposition Let X be a set and $\mathcal{A}$ a family of countable sets which is stationary over X. Give X its discrete topology and $X^{\mathbb{N}}$ the product topology; let $\mathcal{M}(X^{\mathbb{N}})$ be the associated ideal of meager sets. Then $\operatorname{non}\mathcal{M}(X^{\mathbb{N}}) \le \max(\#(\mathcal{A}), \operatorname{non}\mathcal{M}(\mathbb{R}))$.

5A4K Lemma Let X be a topological space and K, L closed subsets of X such that $K \cup L = X$. Set $Z = \{(x,1) : x \in K\} \cup \{(x,0) : x \in L\} \subseteq X \times \{0,1\}$, and wwrite $\pi : Z \to X$ for the first-coordinate map. Then π is an irreducible continuous surjection.

The former 5A4I is now 5A4L.