Errata and addenda for Volume 4, 2003 printing

I collect here known errors and omissions, with their discoverers, in Volume 4 of my book *Measure Theory* (see my web page, `http://www1.essex.ac.uk/maths/people/fremlin/mt.htm`).

p 15 l 13 (411Gh): for '$\int_G |f| \le \int_E f \le \|f\|_p \|\chi E\|_q$' read '$\int_G |f| \le \int_E |f| \le \|f\|_p \|\chi E\|_q$'.

p 17 l 34 (411P) Add new subparagraph:

(f) Let $W \subseteq Z$ be the union of all the open subsets of Z with finite measure. W has full outer measure, so $(\mathfrak{A}, \bar{\mu})$ can be identified with the measure algebra of the subspace measure μ_W μ_W is locally finite. If $(\mathfrak{A}, \bar{\mu})$ is localizable, then μ_W is a Radon measure.

p 18 l 20 (411Xe): for '$p \in [0, \infty]$' read '$p \in [1, \infty]$'.

p 18 l 43 (411X) Add new exercise:

(k) Let $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a family of topological spaces, and μ_i a strictly positive probability measure on X_i for each i. Show that the product measure on $\prod_{i \in I} X_i$ is strictly positive.

p 19 l 9 (411Y) Add new exercise:

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a second-countable atomless topological probability space with a strictly positive measure, $\mathcal{E}$ the Jordan algebra of μ as defined in 411Yc, $(\mathfrak{A}, \bar{\mu})$ the measure algebra of μ and $\mathfrak{E}$ the image $\{E^\bullet : E \in \mathcal{E}\} \subseteq \mathfrak{A}$. Let $\mathfrak{B}$ be a Boolean algebra and $\nu : \mathfrak{B} \to [0, 1]$ a finitely additive functional. Show that $(\mathfrak{B}, \nu) \cong (\mathfrak{E}, \bar{\mu} \restriction \mathfrak{E})$ iff (α) ν is strictly positive and properly atomless in the sense of 326F, and $\nu 1 = 1$ (β) there is a countable subalgebra $\mathfrak{B}_0$ of $\mathfrak{B}$ such that $\nu b = \sup\{\nu c : c \in \mathfrak{B}_0, \, c \subseteq b\}$ for every $b \in \mathfrak{B}$ (γ) whenever $A, \, B \subseteq \mathfrak{B}$ are upwards-directed sets such that $a \cap b = 0$ for every $a \in A$ and $b \in B$ and $\sup\{\nu(a \cup b) : a \in A, \, b \in B\} = 1$, then $\sup A$ is defined in $\mathfrak{B}$.

p 20 l 28 The statement of Lemma 412C has been revised, as follows:

Let (X, Σ, μ) be a semi-finite measure space, and suppose that $\mathcal{A} \subseteq \Sigma$ is such that

$\emptyset \in \mathcal{A} \subseteq \Sigma$,

$X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let T be the σ-subalgebra of Σ generated by $\mathcal{A}$. Let $\mathcal{K}$ be a family of subsets of X such that

(†) $K \cup K' \in \mathcal{K}$ whenever $K, \, K' \in \mathcal{K}$,

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$,

whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$.

Then $\mu \restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$.

The proof remains unaltered except for the changed reference and the omission noted below.

p 21 l 4 (proof of 412C): for '316J' read '316H'.

p 21 l 6 (part (b) of the proof of 412C): for '$\subseteq \{L^\bullet : L \in \mathcal{L}, \, L \subseteq H\}$' read '$\subseteq \sup\{L^\bullet : L \in \mathcal{L}, \, L \subseteq H\}$'.

p 22 l 14 Part (b) of Proposition 412H is wrong as stated, and should read

(b) Now suppose that μ is semi-finite and that

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$.

If either $\hat{\mu}$ or $\tilde{\mu}$ is inner regular with respect to $\mathcal{K}$ then μ is inner regular with respect to $\mathcal{K}$.

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p 23 l 32 (part (d) of the proof of 412J): for '$\mu(E\setminus K)\|g\|_\infty\le\gamma-\int g$' read '$\mu(E\setminus K)\|g\|_\infty\le\int g-\gamma$'.

p 24 l 26 Corollary 412L has been elaborated, and is now

412M Proposition Let X be a set and $\mathcal{K}$ a family of subsets of X. Suppose that μ and ν are two complete locally determined measures on X, with domains including $\mathcal{K}$, and both inner regular with respect to $\mathcal{K}$.

(a) If $\mu K\le\nu K$ for every $K\in\mathcal{K}$, then $\mu\le\nu$ in the sense of 234P.

(b) If $\mu K=\nu K$ for every $K\in\mathcal{K}$, then $\mu=\nu$.

412M is now 412L.

p 25 l 19 (part (b) of the proof of 412O): for 'let E be a measurable envelope for F' with respect to μ' read 'let $G\in\Sigma$ be such that $F'=G\cap Y$, and let $E\subseteq G$ be a measurable envelope for F' with respect to μ'.

p 26 l 19 The proof of Lemma 412R should be rewritten, as follows.

Write $\mathcal{A}=\{E\times F:E\in\Sigma\}\cup\{X\times F:F\in\mathrm{T}\}$. Then the σ-algebra of subsets of $X\times Y$ generated by $\mathcal{A}$ is $\Sigma\widehat{\otimes}\mathrm{T}$. If $V\in\mathcal{A}$, $W\in\Lambda$ and $\lambda(W\cap V)>0$, there is an $M\in\mathcal{M}\cap(\Sigma\widehat{\otimes}\mathrm{T})$ such that $M\subseteq W$ and $\lambda(M\cap V)>0$. **P** Suppose that $V=E\times Y$ where $E\in\Sigma$. There must be $E_0\in\Sigma$ and $F_0\in\mathrm{T}$, both of finite measure, such that $\lambda(W\cap V\cap(E_0\times F_0))>0$ (251F). Now there are $K\in\mathcal{K}\cap\Sigma$ and $L\in\mathcal{L}\cap\mathrm{T}$ such that $K\subseteq E\cap E_0$, $L\subseteq F\cap F_0$ and

$$\mu((E\cap E_0)\setminus K)\cdot\nu F_0+\mu E_0\cdot\nu((F\cap F_0)\setminus L)<\lambda(W\cap V\cap(E_0\times F_0));$$

but this means that $M=K\times L$ is included in V and $\mu(W\cap M)>0$. Reversing the roles of the coordinates, the same argument deals with the case in which $V=X\times F$ for some $F\in\mathrm{T}$. **Q**

By 412C, $\lambda\restriction\Sigma\widehat{\otimes}\mathrm{T}$ is inner regular with respect to $\mathcal{M}$. But λ is inner regular with respect to $\Sigma\widehat{\otimes}\mathrm{T}$ (251Ib) so is also inner regular with respect to $\mathcal{M}$ (412Ab).

p 26 l 46 (proof of 412T): for 'is included in W, and meets V' read 'is included in V, and meets W'.

p 26 l 49 (proof of 412T): for 'inner regular with respect to $\mathcal{K}$' read 'inner regular with respect to $\mathcal{M}$'.

p 27 l 37 (proof of 412Wa) The argument given assumes that f is finite-valued. If $E=\{x:f(x)=\infty\}$ is non-empty, it is negligible; choose a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of measurable open sets such that $E\subseteq H_n$ and $\mu H_n\le 2^{-n}\eta$ for every n, and add $\sum_{n=0}^\infty\chi H_n$ to the given formula for g; with an adjustment to the formula for η, this will work.

p 27 l 38 (part (a) of the proof of 412W): for 'measurable cover' read 'measurable envelope'.

p 28 l 9 (part (b)(ii)$\Rightarrow$(iii) of the proof of 412W): for '$H_n\supseteq G_n\cap E_n$', '$\mu(H_n\setminus E_n)$' read '$H_n\supseteq G_n\cap E$', '$\mu(H_n\setminus E)$'.

p 28 l 37 (412X) New exercises have been added:

p 28 l 40 (412X) Add new exercises:

(d) Let X be a set and $\mathcal{K}$ a family of sets. Suppose that μ and ν are two semi-finite measures on X with the same domain and the same null ideal. Show that if one is inner regular with respect to $\mathcal{K}$, so is the other.

(f) X, with sum μ (234G). Suppose that $\mathcal{K}\subseteq\operatorname{dom}\mu$ is a family of sets such $K\cup K'\in\mathcal{K}$ for every K, $K'\in\mathcal{K}$ and every μ_i is inner regular with respect to $\mathcal{K}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(g) measure on X. Suppose that $\mathcal{F}$ is a non-empty downwards-directed family of closed compact subsets of X with intersection F_0, and that $\gamma=\inf_{F\in\mathcal{F}}\mu F$ is finite. Show that $\mu F_0=\gamma$.

(v) Let X be a topological space and μ a measure on X which is effectively locally finite and inner regular with respect to the closed sets. (i) Show that if $\mu E<\infty$ and $\epsilon>0$ there is a measurable open set G such that $\mu(E\triangle G)\le\epsilon$. (ii) Show that if f is a non-negative integrable function and $\epsilon>0$ there is a measurable lower semi-continuous function $g:X\to[0,\infty[$ such that $\int|f-g|\le\epsilon$. (iii) Show that if f is an integrable real-valued function there are measurable lower semi-continuous functions g_1, $g_2:X\to[0,\infty]$ such that $f=_{\text{a.e.}}g_1-g_2$ and $\int g_1+g_2\le\int|f|+\epsilon$. (iv) Now suppose that μ is σ-finite. Show that for every measurable $f:X\to\mathbb{R}$ there are measurable lower semi-continuous functions g_1, $g_2:X\to[0,\infty]$ such that $f=_{\text{a.e.}}g_1-g_2$.

p 30 l 1 Exercise 412Xq (now 412Xu) is wrong as written, and should be

(u) Let X be a topological space and μ a measure on X which is inner regular with respect to the closed sets and outer regular with respect to the open sets. Show that if $f : X \to \mathbb{R}$ is integrable and $\epsilon > 0$ then there is a lower semi-continuous $g : X \to \,]-\infty, \infty]$ such that $f \le g$ and $\int g - f \le \epsilon$.

Other exercises have been moved: 412Xa-412Xb are now 412Xb-412Xc, 412Xc is now 412Xe, 412Xd-412Xp are now 412Xh-412Xt, 412Xr is now 412Xa, 412Xs is now 412Ya.

p 30 l 21 (412Y) Add new exercise:

(d) Give an example of a measure space (X, Σ, μ) and a family $\mathcal{K}$ of sets such that

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$

and the completion of μ is inner regular with respect to $\mathcal{K}$, but μ is not.

412Yd has been moved to 4A3Ye. 412Ya-412Yb are now 412Yb-412Yc, 412Ye-412Yf are now 412Yf-412Yg.

p 32 l 12 (statement of 413B): for '$\phi : X \to [0, \infty]$' read '$\phi : \mathcal{P}X \to [0, \infty]$'.

p 32 l 34 (part (a) of the proof of 413C): for '$\mu A < \infty$' read '$\phi A < \infty$'. (K.Yates.)

p 36 l 24 Add new result:

413H Proposition Let (X, Σ, μ) be a complete totally finite measure space, (Y, T, ν) a measure space, and $\mathfrak{S}$ a Hausdorff topology on Y such that ν is inner regular with respect to the closed sets. Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of inverse-measure-preserving functions from X to Y. If $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in X$, then f is inverse-measure-preserving.

413H-413O are now 413I-413P.

p 36 l 32 In clause (α) of the statement of Lemma 413H (now 413I), read '$\sup\{\phi_0 K' : K' \in \mathcal{K}, K' \subseteq K \setminus L\}$' for '$\sup\{\phi_0 K' : K \in \mathcal{K}, K' \subseteq K \setminus L\}$'.

The error is repeated in the statement of 413I (now 413J).

p 39 l 35 (now 413K): for $F_m = \bigcap_{n\le m} E_{nn}$ read $F_m = \bigcap_{i,j\le m} E_{ji}$. (A.P.Pyshchev.)

p 43 l 30 The following definitions have been brought together as a new paragraph:

413Q Definitions Let P be a lattice and $f : P \to [-\infty, \infty[$ a function.

(a) f is **supermodular** if $f(p \vee q) + f(p \wedge q) \ge f(p) + f(q)$ for all p, $q \in P$.

(b) f is **submodular** if $f(p \vee q) + f(p \wedge q) \le f(p) + f(q)$ for all p, $q \in P$.

(c) f is **modular** if $f(p \vee q) + f(p \wedge q) = f(p) + f(q)$ for all p, $q \in P$.

413P-413S are now 413R-413U.

p 47 l 21 (Exercise 413Xc): in (ii), add 'Now suppose that ϕX is finite'.

p 47 l 25 (Exercise 413Xd): for 'the measure defined from θ by Carathéodory's method extends μ' read 'if μ is semi-finite then the measure defined from θ by Carathéodory's method extends μ'.

p 48 l 42 (413X) Add new exercise:

(h) (i) Let (X, Σ, μ) be a σ-finite measure space and $f : X \to \mathbb{R}$ a function such that $\overline{\int} f d\mu$ is finite. Show that for every $\epsilon > 0$ there is a measure ν on X extending μ such that $\underline{\int} f d\nu \ge \overline{\int} f d\mu - \epsilon$.

(ii) Let (X, Σ, μ) be a totally finite measure space and $f : X \to \mathbb{R}$ a bounded function. Show that there is a finitely additive functional $\nu : \mathcal{P}X \to [0, \infty[$, extending μ, such that $⨍ f d\nu$, interpreted as in 363L, is equal to $\overline{\int} f d\mu$.

413Xh-413Xi are now 413Xi-413Xj. The former 413Xj is wrong as written, and has now been dropped.

p 48 l 22 (Exercise 413Xn) for 'non-decreasing sequence $\langle H_i\rangle_{i\in\mathbb{N}}$' read 'non-increasing sequence $\langle H_i\rangle_{i\in\mathbb{N}}$'.

p 48 l 37 (Exercise 413Xq) delete 'containing K'.

p 48 l 41 (413X) Add new exercise:

(r) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda : \mathcal{K} \to [0, \infty[$ be such that $\lambda K \le \sum_{n=0}^{\infty}(\lambda K_n - \lambda L_n)$ whenever $K \in \mathcal{K}$ and $\langle K_n\rangle_{n\in\mathbb{N}}$, $\langle L_n\rangle_{n\in\mathbb{N}}$ are sequences in $\mathcal{K}$ such that $L_n \subseteq K_n$ for every n and $\langle K_n \setminus L_n\rangle_{n\in\mathbb{N}}$ is a disjoint cover of K. Show that there is a measure on X extending λ.

p 49 l 6 (413Y) Add new exercises:

(a) Let $\mathfrak{A}$ be a Boolean algebra, $(S,+)$ a commutative semigroup with identity e and $\phi:\mathfrak{A}\to S$ a function such that $\phi 0=e$. Show that

$$\mathfrak{B}=\{b:b\in\mathfrak{A},\,\phi a=\phi(a\cap b)+\phi(a\setminus b)\text{ for every }a\in\mathfrak{A}\}$$

is a subalgebra of of $\mathfrak{A}$, and that $\phi(a\cup b)=\phi a+\phi b$ for all disjoint a, $b\in\mathfrak{B}$.

(d)(i) Find a measure space (X,Σ,μ), with $\mu X>0$, and a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of subsets of X, covering X, such that whenever $E\in\Sigma$, $n\in\mathbb{N}$ and $\mu E>0$, there is an $F\in\Sigma$ such that $F\subseteq E\setminus X_n$ and $\mu F=\mu E$. (ii) For $A\subseteq X$ set $\phi A=\sup\{\mu E:E\in\Sigma,\,E\subseteq A\}$. Set $\mathrm{T}=\{G:G\subseteq X,\,\phi A=\phi(A\cap G)+\phi(A\setminus G)\text{ for every }A\subseteq X\}$. Show that $\phi\restriction\mathrm{T}$ is not a measure.

(h) Let $\mathfrak{A}$ be a Boolean algebra and ν is a submeasure on $\mathfrak{A}$ which is *either* supermodular *or* exhaustive and submodular. Show that ν is uniformly exhaustive.

(i) Let X be a set, $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$, and $f:\mathcal{K}\to\mathbb{R}$ a modular functional such that $f(\emptyset)=0$. Show that there is an additive functional $\nu:\mathcal{K}\to\mathbb{R}$ extending f.

(j) Let (X,Σ,μ) be a semi-finite measure space, λ the c.l.d. product measure on $X\times\mathbb{R}$ when $\mathbb{R}$ is given Lebesgue measure, and λ_* the associated inner measure. Show that for any $f:X\to[0,\infty[$,

$$\underline{\int} f\,d\lambda=\lambda_*\{(x,\alpha):x\in X,\,0\le\alpha<f(x)\}=\lambda_*\{(x,\alpha):x\in X,\,0\le\alpha\le f(x)\}.$$

413Ya-413Yb are now 413Yb-413Yc, 413Yc-413Ye are now 413Ye-413Yg.

p 49 l 11 (Exercise 413Yd, now 413Yf): for 'every non-decreasing sequence in $\mathcal{K}$' read 'every non-increasing sequence in $\mathcal{K}$'.

p 52 l 13 (part (b) of the statement of 414E): for '$\sup_{G\in\mathcal{G}}\int f$' read '$\sup_{G\in\mathcal{G}}\int_G f$'.

p 57 l 21 (414X) Add new exercise, formerly part of 416Xe:

(u) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Show that there is a decomposition $\langle X_i\rangle_{i\in I}$ for μ in which every X_i is expressible as the intersection of a closed set with an open set.

p 57 l 33 Exercise 414Yd has been incorporated into 414Xk. 414Ye-414Yh are now 414Yd-414Yg.

p 57 l 40 (Exercise 414Yf, now 414Ye): for '$E\setminus\phi(E)$' read '$E\setminus\underline{\phi}(E)$'.

p 59 l 22 In Proposition 415D, we need to suppose from the beginning that X is regular.

p 64 l 34 (part (a-α) of the proof of 415L): for '$\mu F\ge\mu_0(E_0\setminus E_1)-\epsilon$' read '$\mu_0F\ge\mu_0(E_0\setminus E_1)-\epsilon$'.

p 64 l 39 (part (a-α) of the proof of 415L): for '$\mu_0^*L+\mu_0^*K'>\mu^*K$' read '$\mu_0^*L+\mu_0^*K'>\mu_0^*K$'.

p 70 l 24 (part (b) of the proof of 415R): for '$F^*\cap U\subseteq f^{-1}[A]$' read '$F^*\cap Q\subseteq U\cap Q=f^{-1}[A]$'.

p 71 l 23 Exercise 415Xn has been dropped. Add new exercises:

(h) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space, $(\mathfrak{A},\bar\mu)$ its measure algebra, and $\mathfrak{A}^f$ the ideal $\{a:a\in\mathfrak{A},\,\bar\mu a<\infty\}$. Show that $\{G^\bullet:G\in\mathfrak{T},\,\mu G<\infty\}$ is dense in $\mathfrak{A}^f$ for the strong measure-algebra topology (323Ad).

(j) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a σ-finite quasi-Radon measure space with $\mu X>0$. Show that there is a quasi-Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(p) Find a second-countable Hausdorff topological space X with a τ-additive Borel probability measure which is not inner regular with respect to the closed sets.

(q) Find a second-countable Hausdorff space X, a subset Y and a quasi-Radon probability measure on Y which is not the subspace measure induced by any quasi-Radon measure on X.

Other exercises have been rearranged: 415Xh is now 415Xi, 415Xi-415Xm are now 415Xk-415Xo, 415Xo-415Xq are now 415Xr-415Xt.

p 72 l 9 Exercise 415Yf, rewritten, is now 415Xq. 415Yg-415Ym are now 415Yf-415Yl.

p 74 l 37 (416D) Add new part:

(e) Let X be a Hausdorff space and $\langle\mu_i\rangle_{i\in I}$ a family of Radon measures on X. Let $\mu = \sum_{i\in I}\mu_i$ be their sum. Suppose that μ is locally finite. Then it is a Radon measure.

p 77 l 21 Proposition 416N has been moved to 416K, and re-expressed in the following more general form:

416K Proposition Let X be a Hausdorff space, T a subring of $\mathcal{P}X$ such that $\mathcal{H} = \{G : G \in \mathrm{T}$ is open$\}$ covers X, and $\nu : \mathrm{T} \to [0, \infty[$ a finitely additive functional. Then there is a Radon measure μ on X such that $\mu K \geq \nu K$ for every compact $K \in \mathrm{T}$ and $\mu G \leq \nu G$ for every open $G \in \mathrm{T}$.

Accordingly 416K-416M have been renamed 416L-416N.

p 77 l 26 (condition γ in the statement of 416K, now 416L) For 'open set G containing X' read 'open set G containing x'.

p 78 l 36 (proof of 416L, now 416M) For '$\phi_1'K \leq \phi_0K' \leq \psi G \leq \phi K + \epsilon$' read '$\phi_1'K \leq \phi_0K' \leq \psi G \leq \phi_1K + \epsilon$'.

p 82 l 38 (416S) Add new part:

(b) If ν is a Radon measure on X and $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$, then ν is an indefinite-integral measure over μ.

p 84 l 41 In Exercise 416Xb, we must assume that the topology of X is Hausdorff.

p 85 l 1 Exercises 416Xg, 416Xh, 416Xl and 416Xn have been dropped. Add new exercises:

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite Radon measure space with $\mu X > 0$. Show that there is a Radon probability measure on X with the same measurable sets and the same negligible sets as μ.

(v) Let $(\mathfrak{A}, \bar\mu)$ be a totally finite measure algebra and $(Z, \mathfrak{T}, \Sigma, \mu)$ its Stone space. Show that if ν is a strictly positive Radon measure on Z then μ is an indefinite-integral measure over ν.

Other exercises have been rearranged: 416Xd is now 416Xf, 416Xf is now 416Xg, 416Xi is now 416Xh, 416Xj is now 416Xi 416Xl is now 416Xk, 416Xm is now 416Xj, 416Xo-416Xq are now 416Xn-416Xp, 416Xr-416Xs are now 416Xl-416Xm, 416Xt is now 416Xq, 416Xu-416Xx are now 416Xr-416Xu.

p 85 l 10 Part (ii) of 416Xe (now 416Xf) is now 414Xu.

p 86 l 39 (416Y) Add new exercises:

(f) In 416Qb, show that μ is atomless iff ν is properly atomless in the sense of 326F.

(i) In 245Yh, show that if we start from a continuous inverse-measure-preserving $f : [0,1] \to [0,1]^2$, as in 134Yl, we get a continuous inverse-measure-preserving surjection $g : [0,1] \to [0,1]^{\mathbb{N}}$.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $\mathcal{A} \subseteq \Sigma$ a countable set. Let $\mathfrak{S}$ be the topology generated by $\mathfrak{T} \cup \mathcal{A}$. Show that μ is $\mathfrak{S}$-Radon.

Other exercises have been rearranged: 416Ya-416Yc are now 416Yb-416Yd, 416Yd is now 416Ya, 416Yf-416Yg are now 416Yg-416Yh.

p 88 l 29 Add another part to (i)-(iii) in Lemma 417A:

(iv) whenever $\mathcal{K}$, $\mathcal{G}$ are families of sets such that

(α) μ is inner regular with respect to $\mathcal{K}$,

(β) $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

(γ) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

(δ) for every $A \in \mathcal{A}$ there is a $G \in \mathcal{G}$, including A, such that $G \setminus A \in \Sigma$,

(ϵ) $K \setminus G \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $G \in \mathcal{G}$,

then μ' is inner regular with respect to $\mathcal{K}$.

p l Theorems 417C and 417E have been revised in order to support a definition of 'τ-additive product measure' (417G) without supposing that the factor measures are inner regular with respect to the Borel sets.

p l Clause (v) of the statement of 417C should read 'the support of $\tilde{\lambda}$ is the product of the supports of μ and ν'.

p l Clause (iv) of the statement of 417E should read 'the support of $\tilde{\lambda}$ is the product of the supports of the μ_i'.

p 97 l 30 (part (e-iv) of the proof of 417E): for '$\lambda(W \cap F) = \lambda W = \lambda U = \prod_{i\in J} \mu_i U_i > 0$' read '$\tilde{\lambda}(W \cap F) = \tilde{\lambda} W \geq \lambda U = \prod_{i\in J} \mu_i G_i > 0$'.

p l Corollary 417F has been dropped, and replaced by

417H Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$, and $\tilde{\Lambda}$ its domain. If $A \subseteq X$, $B \subseteq Y$ are non-negligible sets such that $A \times B \in \tilde{\Lambda}$, then $A \in \Sigma$ and $B \in \mathrm{T}$.

417G-417H are now 417F-417G.

p 100 l 38 (proof of 417J): for '$\prod_{j\in J} \prod_{i\in K_j} E_i = \prod_{i\in I} E_i$' read '$\prod_{j\in J} \prod_{i\in K_j} \mu_i E_i = \prod_{i\in I} \mu_i E_i$'.

p l There is a catastrophic error in the proof of 417K, but the result can easily be proved from 417H (now 417G) by the method outlined in 252Xc.

p l Exercise 417Xb is covered by 411Xi, and has been dropped. Exercise 417Xs is wrong, and should be deleted. 417Xt-417Xx are now 417Xs-417Xw.

p 107 l 28 (Exercise 417Xu, now 417Xt) Add new fragments:

(ix) Show that a continuous image of a chargeable space is chargeable. (x) Show that a compact Hausdorff space is chargeable iff it has a strictly positive Radon measure.

p l Add new exercise:

(x) (i) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a finite family of effectively locally finite τ-additive topological measure spaces, and let Λ be the σ-algebra of subsets of $X = \prod_{i\in I} X_i$ generated by $\widehat{\bigotimes}_{i\in I}\Sigma_i$ together with the open subsets of X. Show that there is a unique effectively locally finite τ-additive measure λ with domain Λ such that $\lambda(\prod_{i\in I} E_i) = \prod_{i\in I} \mu_i E_i$ whenever $E_i \in \Sigma_i$ for $i \in I$. (ii) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, and let Λ be the σ-algebra of subsets of $X = \prod_{i\in I} X_i$ generated by $\widehat{\bigotimes}_{i\in I}\Sigma_i$ together with the open subsets of X. Show that there is a unique τ-additive measure λ with domain Λ extending the usual product measure on $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

p 107 l 41 (Exercise 417Ya): for 'λ, λ'' read 'μ, μ''.

p 108 l 25 (Exercise 417Ye): the topologies $\mathfrak{T}$, $\mathfrak{S}$ need to be Hausdorff. Similarly, in 417Yg, the topologies $\mathfrak{T}_n$ need to be Hausdorff.

p 112 l 29 (part (c) of the proof of 418D): for '$\mu E_1 \geq \mu(E \cap f^{-1}[Y_n] \setminus E_1) - \epsilon$' read '$\mu E_1 \geq \mu(E \cap f^{-1}[Y_n]) - \epsilon$'.

p l Add new result:

418V Proposition Let (X, Σ, μ) be a σ-finite measure space, $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the Borel sets, $(Y, \mathfrak{S})$ a topological space and $f : X \to Y$ an almost continuous function. Then there is a Borel measurable function $g : X \to Y$ which is equal almost everywhere to f.

p 113 l 35 (statement of Proposition 418Ha): for 'effectively locally finite τ-additive measure on X' read 'effectively locally finite τ-additive topological measure on X'.

p 116 l 18 (statement of Prokhorov's theorem): add 'if the family $\langle g_i\rangle_{i\in I}$ separates the points of X, then μ is uniquely defined'.

p 118 l 7 Proposition 418O has been re-written, as follows:

Proposition Suppose that $(I, \leq)$, $\langle (X_i, \mathfrak{T}_i, \mu_i, \Sigma_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ are such that

$(I, \le)$ is a non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \mu_i, \Sigma_i)$ is a compact Radon measure space,

$f_{ij} : X_j \to X_i$ is a continuous inverse-measure-preserving function whenever $i \le j$ in I,

$f_{ij} f_{jk} = f_{ik}$ whenever $i \le j \le k$ in I.

Then there are a compact Hausdorff space X and a family $\langle g_i \rangle_{i\in I}$ such that $(I, \langle X_i \rangle_{i\in I}, \langle f_{ij} \rangle_{i\le j\in I}, X, \langle g_i \rangle_{i\in I})$ satisfy all the hypotheses of 418M.

p 119 l 31 (part (d-ii) of the proof of 418P): for

$$\text{‘}F_k = \{x : x \in \prod_{j\in I} K_j^*,\ x(k) = z,\ f_{jk}x(k) = x(j) \text{ whenever } j \le k\}\text{’}$$

read

$$\text{‘}F_k = \{x : x \in \prod_{j\in I} K_j^*,\ x(i) = z,\ f_{jk}x(k) = x(j) \text{ whenever } j \le k\}\text{’}.$$

p 119 l 34 (part (d-ii) of the proof of 418P): for ‘$x(k) = z$’ read ‘$x(i) = z$’.

p 122 l 20 Add new paragraph:

***418U Independent families of measurable functions** In §455 we shall have occasion to look at independent families of random variables taking values in spaces other than $\mathbb{R}$. We can use the same principle as in §272: a family $\langle X_i \rangle_{i\in I}$ of random variables is independent if $\langle \Sigma_i \rangle_{i\in I}$ is independent, where Σ_i is the σ-subalgebra defined by X_i for each i (272D). Of course this depends on agreement about the definition of Σ_i. The natural thing to do, in the context of this section, is to follow 272C, as follows. Let (X, Σ, μ) be a probability space, Y a topological space, and f a Y-valued function defined on a conegligible subset $\operatorname{dom} f$ of X, which is μ-virtually measurable, that is, such that f is measurable with respect to the subspace σ-algebra on $\operatorname{dom} f$ induced by $\hat\Sigma = \operatorname{dom} \hat\mu$, where $\hat\mu$ is the completion of μ. (Note that if Y is not second-countable this may not imply that $f\restriction D$ is Σ-measurable for a conegligible subset D of X.) The ‘σ-algebra defined by f’ will be

$$\{f^{-1}[F] : F \in \mathcal{B}(Y)\} \cup \{(\Omega \setminus \operatorname{dom} f) \cup f^{-1}[F] : F \in \mathcal{B}(Y)\} \subseteq \hat\Sigma,$$

where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y; that is, the σ-algebra of subsets of X generated by $\{f^{-1}[G] : G \subseteq Y \text{ is open}\}$.

Now, given a family $\langle (f_i, Y_i) \rangle_{i\in I}$ where each Y_i is a topological space and each f_i is a $\hat\Sigma$-measurable Y_i-valued function defined on a conegligible subset of X, then I will say that $\langle f_i \rangle_{i\in I}$ is **independent** if $\langle \Sigma_i \rangle_{i\in I}$ is independent (with respect to $\hat\mu$), where Σ_i is the σ-algebra defined by f_i for each i.

Corresponding to 272D, we can use the Monotone Class Theorem to show that $\langle f_i \rangle_{i\in I}$ is independent iff

$$\hat\mu(\bigcap_{j\le n} f_{i_j}^{-1}[G_j]) = \prod_{j\le n} \hat\mu f_{i_j}^{-1}[G_j]$$

whenever $i_0, \dots, i_n \in I$ are distinct and $G_j \subseteq Y_{i_j}$ is open for every $j \le n$.

p 122 l 31 (418X) Add new exercises:

(a) Let (X, Σ, μ) be a measure space, and $(X, \hat\Sigma, \hat\mu)$ its completion. (i) Show that if Y is a second-countable topological space, a function $f : X \to Y$ is $\hat\Sigma$-measurable iff there is a Σ-measurable $g : X \to Y$ such that $f =_{\text{a.e.}} g$. (ii) Show that if X is endowed with a topology, and Y is a topological space, then a function from X to Y is μ-almost continuous iff it is $\hat\mu$-almost continuous.

(m) Let μ be Lebesgue measure on $\mathbb{R}^r$, where $r \ge 1$, X a Hausdorff space and $f : \mathbb{R}^r \to X$ an almost continuous function. Show that for almost every $x \in \mathbb{R}^r$ there is a measurable set $E \subseteq \mathbb{R}^r$ such that x is a density point of E and $\lim_{y\in E, y\to x} f(y) = f(x)$.

(v) Let X be a compact Hausdorff space. Show that there is an atomless Radon probability measure on X iff X is non-scattered.

Other exercises have been renamed: 418Xa-418Xk are now 418Xb-418Xl, 418Xl-418Xn are now 418Xn-418Xp, 418Xo has been deleted, 418Xp-418Xt are now 418Xp-418Xu, 418Xu-418Xx are now 418Xw-418Xz.

p 124 l 24 Exercise 418Yd is wrong, and has been deleted.

p 125 l 14 Exercise 418Yl is wrong, and has been deleted.

p 125 l 19 (418Y) Add new exercises:

(j) Let μ be Lebesgue measure on $[0,1]$, and $A \subseteq [0,1]$ a set with inner measure 0 and outer measure 1; let $\mathfrak{T}$ be the usual topology on $[0,1]$. Let $\mathcal{I}$ be the family of sets $I \subseteq A$ such that every point of A has a neighbourhood containing at most one point of I. Show that $\mathfrak{S} = \{G \setminus I : G \in \mathfrak{T}, I \in \mathcal{I}\}$ is a topology on $[0,1]$ with a countable network. Show that the identity map from $[0,1]$ to itself, regarded as a map from $([0,1],\mathfrak{T},\mu)$ to $([0,1],\mathfrak{S})$, is measurable but not almost continuous.

(k) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y and Z are separable metrizable spaces,and $f : X\times Y \to Z$ is a function such that $x \mapsto f(x,y)$ is measurable for every $y \in Y$, and $y \mapsto f(x,y)$ is continuous for every $x \in X$. Show that μ is inner regular with respect to $\{F : F \subseteq X,\ f{\restriction}F\times Y$ is continuous$\}$.

(m) Let X be a set, Σ a σ-algebra of subsets of X and (Y,T,ν) a σ-finite measure space with countable Maharam type. (i) Let $f : X \to L^1(\nu)$ be a function such that $x \mapsto \int_F f(x)d\nu$ is Σ-measurable for every $F \in \mathrm{T}$. Show that f is Σ-measurable for the norm topology on $L^1(\nu)$. (ii) Let $g : X \times Y \to \mathbb{R}$ be a function such that $\int g(x,y)\nu(dy)$ is defined for every $x \in X$, and $x \mapsto \int_F g(x,y)\nu(dy)$ is Σ-measurable for every $F \in \mathrm{T}$. Show that there is an $h \in \mathcal{L}^0(\Sigma\otimes\mathrm{T})$ such that, for every $x \in X$, $g(x,y) = h(x,y)$ for ν-almost every y.

(n) Use 418M and 418O to prove 328H.

(o) Let X be a set, Σ a σ-algebra of subsets of X, (Y,T,ν) a σ-finite measure space and $W \in \Sigma\widehat{\otimes}\mathrm{T}$. Then there is a $V \subseteq W$ such that $V \in \Sigma\widehat{\otimes}\mathrm{T}$, $W[\{x\}] \setminus V[\{x\}]$ is negligible for every $x \in X$, and $\bigcap_{x\in I} V[\{x\}]$ is either empty or non-negligible for every finite $I \subseteq X$.

(p) Let X be a compact Hausdorff space, Y a Hausdorff space, ν a Radon probability measure on Y and $R \subseteq X\times Y$ a closed set such that $\nu^*R[X] = 1$. Show that there is a Radon probability measure μ on X such that $\mu R^{-1}[F] \ge \nu F$ for every closed set $F \subseteq Y$.

Other exercises have been rearranged: 418Ye-418Yj are now 418Yd-418Yi, 418Yk is now 418Yl.

p 126 l 49 (part (c) of the proof of 419A): for 'there is an $\eta < \kappa$ such that $I_\xi \setminus I_\eta \subseteq^* G$' read 'there is an $\eta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$'.

p 132 l 4 Corollary 419G has been strengthened to read

Let Y be a set of cardinal at most ω_1 and μ a semi-finite measure with domain $\mathcal{P}Y$. Then μ is point-supported; in particular, if μ is σ-finite there is a countable conegligible set $A \subseteq Y$.

There is a fault in part (a) of the proof; I have changed it to

Suppose, if possible, otherwise. Let μ_0 be the point-supported part of μ, that is, $\mu_0 A = \sum_{y\in A}\mu\{y\}$ for every $A \subseteq Y$; then μ_0 is a point-supported measure (112Bd), so is not equal to μ. Let $A \subseteq Y$ be such that $\mu_0 A \ne \mu A$. Then $\mu_0 A < \mu A$; because μ is semi-finite, there is a set $B \subseteq A$ such that $\mu_0 A < \mu B < \infty$. Set $\nu C = \mu(B\cap C) - \mu_0(B\cap C)$ for $C \subseteq Y$; then ν is a non-zero totally finite measure with domain $\mathcal{P}Y$, and is zero on singletons.

p 132 l 19 Example 419H has been strengthened to make it Hausdorff. As the argument now requires the result of the former 419I, 419I-419J have been moved to 419H-419I, and the new example is as follows.

419J Example There is a complete probability space (X,Σ,μ) with a Hausdorff topology $\mathfrak{T}$ on X such that μ is τ-additive and inner regular with respect to the Borel sets, $\mathfrak{T}$ is generated by $\mathfrak{T}\cap\Sigma$, but μ has no extension to a topological measure.

p 134 l 44 (proof of 419K) for '$f_m^{-1}[\pi_{mn}^{-1}[W]] = X_n \cap E$' read '$f_n^{-1}[\pi_{mn}^{-1}[W]] = X_n \cap E$'.

p 136 l 9 (Exercise 419Xd) for 'every straight line meets A in exactly two points' read 'every straight line meets B in exactly two points'.

p 136 l 12 Exercise 419Xe has been moved to 418Yj. In its place is a new exercise

(e) Show that there is a subset A of the Cantor set C (134G) such that $A+A$ is not Lebesgue measurable.

p 136 l 38 (419Y) Add new exercise:

(d) Give an example of a Lebesgue measurable function $\phi:\mathbb{R}^2\to\mathbb{R}$ such that dom $\dfrac{\partial\phi}{\partial\xi_1}$ is not measurable.

p 139 l 36 (421C) Add new part, formerly 421Xn:

(f)(i) I will say that a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S^*}$ is **fully regular** if $E_\sigma\subseteq E_\tau$ whenever σ, $\tau\in S^*$, $\#(\tau)\le\#(\sigma)$ and $\sigma(i)\le\tau(i)$ for every $i<\#(\sigma)$.

(ii) Let $\mathcal{E}$ be a family of sets such that $E\cup F$ and $E\cap F$ belong to $\mathcal{E}$ for all E, $F\in\mathcal{E}$. Then every member of $\mathcal{S}(\mathcal{E})$ can be expressed as the kernel of a regular Souslin scheme in $\mathcal{E}$.

p 144 l 12 My apologies: in §562 I find that a slightly different definition of 'rank' of a tree gives cleaner results. So I have changed the definition of the iterated derivation operator: $\partial^\xi T$ is to be $\bigcap_{\eta<\xi}\partial(\partial^\eta T)$ rather than $\partial(\bigcap_{\eta<\xi}\partial^\eta T)$. Practically no other formulae need changing, but the meanings become slightly different. A change which is necessary is in part (a) of the proof of 421O, where

$$\begin{aligned}\{x:\sigma\in\partial^\xi T_x\}&=\{x:\sigma\in\partial(\bigcap_{\eta<\xi}\partial^\eta T_x)\}=\bigcup_{i\in\mathbb{N}}\{x:\sigma^\frown i\in\bigcap_{\eta<\xi}\partial^\eta T_x\}\\&=\bigcup_{i\in\mathbb{N}}\bigcap_{\eta<\xi}\{x:\sigma^\frown i\in\partial^\eta T_x\}\in\Sigma\end{aligned}$$

becomes

$$\begin{aligned}\{x:\sigma\in\partial^\xi T_x\}&=\{x:\sigma\in\bigcap_{\eta<\xi}\partial(\partial^\eta T_x)\}=\bigcap_{\eta<\xi}\bigcup_{i\in\mathbb{N}}\{x:\sigma^\frown i\in\partial^\eta T_x\}\\&\in\Sigma.\end{aligned}$$

p 145 l 14 (part (b) of the proof of 421O): for '$\partial^{\xi+1}T_x\subseteq\partial^\xi T_x$' read '$\partial^{\xi+1}T_x\supseteq\partial^\xi T_x$'.

p 145 l 17 (part (b) of the proof of 421O): for '$\emptyset\in\partial^\xi T_x$' read '$\partial^\xi T_x\ne\emptyset$'.

p 146 l 7 Exercise 421Xc has been moved to 421Yb; consequently 421Xd-421Xo are now 421Xc-421Xn, 421Yb-421Yd are now 421Yc-421Ye.

p 147 l 5 Exercise 421Ya is wrong as it stands; we must strengthen the hypothesis by adding '$E\cap F\in\mathcal{E}$ for all E, $F\in\mathcal{E}$'.

p 149 l 35 (Lemma 422D) Add new part:

(h) Let Y be a Hausdorff space and $R\subseteq\mathbb{N}^\mathbb{N}\times Y$ an usco-compact relation. Set

$$R'=\{(\alpha,y):\alpha\in\mathbb{N}^\mathbb{N},\ y\in Y\text{ and there is a }\beta\le\alpha\text{ such that }(\beta,y)\in R\}.$$

Then R' is usco-compact.

p 153 l 31 (part (c) of the proof of 422K): for '$\overline{R[I_{\phi\restriction n}]}\cap\overline{S[I_{\psi\restriction n}]}$ is empty' read '$\overline{R[I_{\phi\restriction n}]\cap S[I_{\psi\restriction n}]}$ is empty'.

p 153 l 43 The exercises for §422 have been rearranged: 422Xd is now 422Xf, 422Xe-422Xf are now 422Xd-422Xe, 422Yb-422Ye are now 422Yc-422Yf, 422Yf is now 422Yb.

p 154 l 26 (422Y) Add new exercise:

(g) Let X be a Hausdorff space, $\mathcal{K}$ the family of K-analytic subsets of X, Y a set and $\mathcal{H}$ a family of subsets of Y containing $\emptyset$. Show that $R[X]\in\mathcal{S}(\mathcal{H})$ for every $R\in\mathcal{S}(\{K\times H:K\in\mathcal{K},\ H\in\mathcal{H}\})$.

p 159 l 16 Add new result:

423J Proposition Let $(X,\mathfrak{T})$ be an analytic Hausdorff space, and Σ a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X,\mathfrak{T})$ which separates the points of X. Then $\Sigma=\mathcal{B}(X,\mathfrak{T})$.

p 161 l 31 Add new result:

423O Corollary Let X be an analytic Hausdorff space, Y a set and T a σ-algebra of subsets of Y which is closed under Souslin's operation. Suppose that $W \in \mathcal{S}(\mathcal{B}(X)\widehat{\otimes}\mathrm{T})$ where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

423J-423N are now 423K-423P, 423O-423Q are now 423Q-423S.

p 162 l 16 Corollary 423O (now 423Q) has been revised, and now reads

Let X and Y be analytic Hausdorff spaces, A an analytic subset of X and $f : A \to Y$ a Borel measurable function. Let T be the σ-algebra of subsets of Y generated by the Souslin-F subsets of Y. Then $f[A] \in \mathrm{T}$ and there is a T-measurable function $g : f[A] \to A$ such that fg is the identity on $f[A]$.

p 163 l 1 (Remark 423Qb, now 423Sb) for 'analytic subset of $\mathbb{N}^{\mathbb{N}} \setminus A$' read 'analytic subset of $\mathbb{N}^{\mathbb{N}} \setminus A_0$'.

p 163 l 3 Add new paragraph:

423T Coanalytic and PCA sets Let X be a Polish space.

(a) A subset A of X is **coanalytic** if $X \setminus A$ is analytic, and **PCA** if there is a coanalytic set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) Every PCA set $A \subseteq X$ can be expressed as the union of at most ω_1 Borel sets.

(c) A subset of X is Borel iff it is both analytic and coanalytic. The union and intersection of a sequence of coanalytic subsets of X are coanalytic. If Y is another Polish space and $h : X \to Y$ is continuous, then $h^{-1}[B]$ is coanalytic in X for every coanalytic $B \subseteq Y$. If Y is a G_δ subset of X, and $B \subseteq Y$ is coanalytic in Y then B is coanalytic in X.

(d) If X and Y are Polish spaces, $A \subseteq X$ is PCA and $f : X \to Y$ is a Borel measurable function, then $f[A]$ is a PCA subset of Y.

p 163 l 22 (Exercise 423Xf) for '$f : W[X] \to Y$' read '$f : W[X] \to X$'.

p 170 l 42 (Exercise 424Xh): part (i) ('$f[X] \in \mathrm{T}^*$') is wrong, and should be omitted.

p 171 l 3 Exercise 424Xj has been moved to 424Ye, and 424Xk to 425Xb.

p 171 l 9 (424X) Add new exercise:

(j) Let (X, Σ, μ) be a σ-finite measure space in which Σ is countably generated. Let $\mathcal{A}$ be the set of atoms A of the Boolean algebra Σ such that $\mu A > 0$, and set $H = X \setminus \bigcup \mathcal{A}$. Show that the subspace measure on H is atomless.

p 171 l 17 (Exercise 424Ya) Add new part:

(iii) Show that Σ is the Borel σ-algebra of $\mathcal{C}$ when $\mathcal{C}$ is given its Fell topology.

p 171 l 25 (424Y) Add new exercises:

(e) Let (X, Σ) be a standard Borel space. Show that if X is uncountable, Σ has a countably generated σ-subalgebra not isomorphic either to Σ or to $\mathcal{P}I$ for any set I.

(f) Let (X, Σ, μ) be a σ-finite countably separated perfect measure space. Show that there is a standard Borel space (Y, T) such that $Y \in \Sigma$, $\mathrm{T} \subseteq \Sigma$ and μ is inner regular with respect to T.

p 171 l 45 I have added a new section §425, 'Realization of automorphisms', to cover a theorem of A.Törnquist. The principal results are as follows.

425A Proposition (a) Let (X, Σ) and (Y, T) be non-empty standard Borel spaces, and $\mathcal{I}$, $\mathcal{J}$ σ-subalgebras of Σ, T respectively; write $\mathfrak{A} = \Sigma/\mathcal{I}$ and $\mathfrak{B} = \mathrm{T}/\mathcal{J}$ for the quotient algebras. For $E \in \Sigma$, $F \in \mathrm{T}$ write Σ_E, T_F for the subspace σ-algebras on E, F respectively.

(a) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a sequentially order-continuous Boolean homomorphism, there is a (T, Σ)-measurable $f : Y \to X$ which represents π in the sense that $\pi E^\bullet = f^{-1}[E]^\bullet$ for every $E \in \Sigma$.

(b) If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean isomorphism, there are $G \in \mathcal{I}$, $H \in \mathcal{J}$ and a bijection $h : Y \setminus H \to X \setminus G$ which is an isomorphism between $(Y \setminus H, \mathrm{T}_{Y\setminus H})$ and $(X \setminus G, \Sigma_{X\setminus G})$, and represents π in the sense that $\pi E^\bullet = h^{-1}[E \setminus G]^\bullet$ for every $E \in \Sigma$.

(c) If $\pi : \mathfrak{A} \to \mathfrak{A}$ is a Boolean automorphism, there is a bijection $h : X \to X$ which is an automorphism of (X, Σ) and represents π in the sense of (a).

(d) If $\#(X) = \#(Y) = \mathfrak{c}$, $\mathfrak{A}$ and $\mathfrak{B}$ are ccc, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a Boolean isomorphism, there is a bijection $h : Y \to X$ which is an isomorphism between (Y, T) and (X, Σ), and represents π in the sense of (a).

425B Lemma Let G be a group, G_0 a subgroup of G, H another group, and X, Z sets; let $\bullet_r$ be the right shift action of H on Z^H. Suppose we are given a group homomorphism $\theta : G \to H$, an injective function $f : \mathbb{N} \times Z^H \to X$ and an action $\bullet_0$ of G_0 on X such that $\pi{\bullet_0}f(n, z) = f(n, \theta(\pi){\bullet_r}z)$ whenever $n \in \mathbb{N}$ and $z \in Z^H$.

(a) If $\#(X \setminus f[\mathbb{N} \times Z^H]) \le \#(Z)$, there is an action $\bullet$ of G on X extending $\bullet_0$.

(b) Suppose that H is countable, X and Z are Polish spaces, and f is Borel measurable when $\mathbb{N} \times Z^H$ is given the product topology. If $x \mapsto \pi{\bullet_0}x$ is Borel measurable for every $\pi \in G_0$, then $\bullet$ can be chosen in such a way that $x \mapsto \psi{\bullet}x$ is Borel measurable for every $\psi \in G$.

425D Törnquist's theorem Let (X, Σ) be a standard Borel space and $\mathcal{I}$ a σ-ideal of Σ containing an uncountable set. Let $\mathfrak{A}$ be the quotient algebra $\Sigma/\mathcal{I}$, and $G \subseteq \operatorname{Aut}\mathfrak{A}$ a subgroup with cardinal at most ω_1. Then there is an action $\bullet$ of G on X which represents G in the sense that $\pi{\bullet}E$ belongs to Σ, and $(\pi{\bullet}E)^\bullet = \pi(E^\bullet)$, for every $E \in \Sigma$ and $\pi \in G$.

p 173 l 19 (Theorem 431D) Add new part, formerly 431Xc:

(b) If $\langle E_\sigma \rangle_{\sigma \in S^*}$ is fully regular, then $\mu A = \sup\{\mu(\bigcap_{n \ge 1} E_{\psi \restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\}$, and if moreover μ is totally finite, $\mu A = \sup\{\inf_{n \ge 1} \mu E_{\psi \restriction n} : \psi \in \mathbb{N}^{\mathbb{N}}\}$.

p 177 l 8 Add new result:

431G Theorem Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

p 175 l 14 Exercise 431Xc is now 431Db; 431Xd-431Xe are now 431Xc-431Xd.

p 175 l 29 (431Y) 431Yc has been moved to 431G; 431Yb is now 431Yc. Add new exercises:

(b) Let (X, Σ, μ) be a totally finite measure space, Y a set and T a σ-algebra of subsets of Y. Suppose that $A \in \mathcal{S}(\Sigma \widehat{\otimes} \mathrm{T})$. Show that $\{y : \mu A^{-1}[\{y\}] > \alpha\}$ belongs to $\mathcal{S}(\mathrm{T})$ for every $\alpha \in \mathbb{R}$.

(d) Let $r \ge 1$ be an integer and $f : \mathbb{R}^r \to \mathbb{R}$ a Borel measurable function. Show that the domain of its first partial derivative $\frac{\partial f}{\partial \xi_1}$ is coanalytic, therefore Lebesgue measurable, but may fail to be Borel.

p 175 l 3 (part (b) of the proof of 431F): for 'negligible' read 'meager'.

p 176 l 8 Add new result:

431G Theorem Let X be a set, Σ a σ-algebra of subsets of X and $\mathcal{I} \subseteq \Sigma$ a σ-ideal of subsets of X. If $\Sigma/\mathcal{I}$ is ccc then Σ is closed under Souslin's operation.

p 175 l 29 (431Y) Add new exercise:

(d) Let $r \ge 1$ be an integer and $f : \mathbb{R}^r \to \mathbb{R}$ a Borel measurable function. Show that the domain of its first partial derivative $\frac{\partial f}{\partial \xi_1}$ is coanalytic, therefore Lebesgue measurable, but may fail to be Borel.

p 178 l 2 (part (a) of the proof of 432G): the reference to '132Yd' was always incorrect; the relevant fact is now in 234F.

p 178 l 21 Add new result:

432I Corollary Let X be a K-analytic Hausdorff space, and $\mathcal{U}$ a subbase for the topology of X. Let (Y, T, ν) be a complete totally finite measure space and $\phi : Y \to X$ a function such that $\phi^{-1}[U] \in \mathrm{T}$ for every $U \in \mathcal{U}$. Then there is a Radon measure μ on X such that $\int f d\mu = \int f\phi\, d\nu$ for every bounded continuous $f : X \to \mathbb{R}$.

432I-432J are now 432J-432K.

p 178 l 28 (432I, now 432J) Add definition:

(b) A Choquet capacity c on X is **outer regular** if $c(A)=\inf\{c(G):G\supseteq A$ is open$\}$ for every $A\subseteq X$.

p 179 l 3 Add new result:

Proposition Let $(X,\mathfrak{T})$ be a topological space.

(a) Let $c_0:\mathfrak{T}\to[0,\infty]$ be a functional such that

$c_0(G)\le c_0(H)$ whenever G, $H\in\mathfrak{T}$ and $G\subseteq H$;

c_0 is submodular;

$c_0(\bigcup_{n\in\mathbb{N}}G_n)=\lim_{n\to\infty}c_0(G_n)$ for every non-decreasing sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{T}$.

Then c_0 has a unique extension to an outer regular Choquet capacity c on X, and c is submodular.

(b) Suppose that X is regular. Let $\mathcal{K}$ be the family of compact subsets of X, and $c_1:\mathcal{K}\to[0,\infty]$ a functional such that

c_1 is submodular;

$c_1(K)=\inf_{G\in\mathfrak{T},G\supseteq K}\sup_{L\in\mathcal{K},L\subseteq G}c_1(L)$ for every $K\in\mathcal{K}$.

Then c_1 has a unique extension to an outer regular Choquet capacity c on X such that

$$c(G)=\sup\{c(K):K\subseteq G\text{ is compact}\}\text{ for every open }G\subseteq X,$$

and c is submodular.

p 179 l 23 (432X) Add new exercises:

(j) Let P be a lattice, and $c:P\to\mathbb{R}$ an order-preserving functional. Show that the following are equiveridical: (i) c is submodular; (ii) $(p,q)\mapsto 2c(p\vee q)-c(p)-c(q)$ is a pseudometric on P; (iii) setting $c_r(p)=c(p\vee r)-c(r)$, $c_r(p\vee q)\le c_r(p)+c_r(q)$ for all p, q, $r\in P$.

(k) Let X be a topological space, $c:X\to[0,\infty]$ a Choquet capacity, and $f:[0,\infty]\to[0,\infty]$ a non-decreasing function. (i) Show that if f is continuous then fc is a Choquet capacity. (ii) Show that if $f{\restriction}[0,\infty[$ is concave and c is submodular, then fc is submodular.

(l) Let X be a Hausdorff space, c a Choquet capacity on X, and $\mathcal{K}$ a non-empty downwards-directed family of compact subsets of X. Show that $c(\bigcap\mathcal{K})=\inf_{K\in\mathcal{K}}c(K)$.

p 179 l 27 (432Y) Add new exercise:

(b) Let X, Y be Hausdorff spaces, $R\subseteq X\times Y$ an usco-compact relation and μ a Radon probability measure on X such that $\mu_*R^{-1}[Y]=1$. Show that there is a Radon probability measure on Y such that $\nu_*R[A]\ge\mu_*A$ for every $A\subseteq X$.

p 180 l 22 (Notes to §432): for '422Ye' read '422Yf'.

p 182 l 1 Add new result:

433H Proposition Let (X,Σ,μ) be a complete locally determined space, and Y an analytic Hausdorff space. Suppose that $W\subseteq X\times Y$ belongs to $\mathcal{S}(\Sigma\widehat{\otimes}\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. Then $W^{-1}[Y]\in\Sigma$ and there is a Σ-measurable function $f:W^{-1}[Y]\to Y$ such that $(x,f(x))\in W$ for every $x\in W^{-1}[X]$.

433H-433K are now 433I-433L.

p 189 l 22 (part (a), $\neg$(i)$\Rightarrow\neg$(v) of the proof of 434H): for '$\mu E=\mu(E\cap F)$' read '$\nu E=\mu(E\cap F)$'.

p 190 l 22 (part (a), (iii)$\Rightarrow$(i) of the proof of 434I): for '$\mu E=\mu(E\cap G^*\setminus G_0^*)$' read '$\nu E=\mu(E\cap G^*\setminus G_0^*)$'.

p 194 l 7 (statement of 434Nb): for 'closed countably compact subsets of X are compact' read 'countably compact subsets of X are compact'.

p 195 l 42 (part (b) of the proof of 434Q): for '$\sup_{n\in\mathbb{N}}\mu F_n=1$' read '$\sup_{n\in\mathbb{N}}\mu F_n(E)=1$'.

p 197 l 7 (part (f), case 2 of the proof of 434Q): when choosing ξ we must ensure that it is so large that J does not meet $J_{\xi+1}\setminus J_\xi$.

p 199 l 6 In Proposition 434T, part (a), the result has been strengthened to

$$\begin{aligned}[\![u\in E]\!]&=\sup\{[\![u\in F]\!]:F\subseteq E\text{ is Borel}\}=\sup\{[\![u\in K]\!]:K\subseteq E\text{ is compact}\}\\&=\inf\{[\![u\in F]\!]:F\supseteq E\text{ is Borel}\}=\inf\{[\![u\in G]\!]:G\supseteq E\text{ is open}\}.\end{aligned}$$

p 199 l 30 Add new result:

434U Proposition Let X and Y be compact Hausdorff spaces and $f : X \to Y$ a continuous open map. If μ is a completion regular topological measure on X, then the image measure μf^{-1} on Y is completion regular.

p 199 l 31 The exercises for §434 have been re-organized, as follows: 434Xk is now 434Xm, 434Xl is now 434Yg, 434Xp-434Xt are now 434Xr-434Xv, 434Xu is now 434Xl, 434Yd is now 434Yf, 434Ye is now 434Yh, 434Yf is now 434Yn, 434Yg is now 434Yj, 434Yi-434Yk are now 434Yk-434Ym, 434Yl is now 434Yo. 434Yc and 434Yh are wrong, and should be deleted.

In addition, there are new exercises:

434Xq Let X be a separable metrizable space. Show that the following are equiveridical: (i) X is a Radon space; (ii) X is a pre-Radon space; (iii) there is a metric on X, defining the topology of X, such that X is universally Radon-measurable in its completion; (iv) whenever Y is a separable metrizable space and X' is a subset of Y such that there is a Borel isomorphism between X and X', then X' is universally measurable in Y; (v) X is a Radon space under any separable metrizable topology giving rise to the same Borel sets as the original topology.

434Xe Let Σ_{um} be the algebra of universally measurable subsets of $\mathbb{R}$, and μ the restriction of Lebesgue measure to Σ_{um}. Show that μ is translation-invariant, but has no translation-invariant lifting.

434Yc If X is a topological space, a set $A \subseteq X$ is **universally capacitable** if $c(A) = \sup\{c(K) : K \subseteq A \text{ is compact}\}$ for every Choquet capacity c on X. (i) Show that if X is a Hausdorff space and π_1, $\pi_2 : X \times X \to X$ are the coordinate maps, then we have a capacity c on $X \times X$ defined by saying that $c(A) = 0$ if $A \subseteq X \times X$ and there is a Borel set $E \subseteq X$ including $\pi_1[A]$ and disjoint from $\pi_2[A]$, and $c(A) = 1$ for other $A \subseteq X \times X$. (ii) Show that there is a universally measurable subset of $\mathbb{R}$ which is not universally capacitable.

434Yd Let X be a Hausdorff space. Let Σ be the family of those subsets E of X such that $f^{-1}[E]$ has the Baire property in Z whenever Z is a compact Hausdorff space and $f : Z \to X$ is continuous. Show that Σ is a σ-algebra of subsets of X closed under Souslin's operation. Show that every member of Σ is universally Radon-measurable.

434Ye Let X be a Hausdorff space such that there is a countable algebra $\mathcal{A}$ of universally Radon-measurable subsets of X which separates the points of X in the sense that whenever $I \in [X]^2$ there is an $A \in \mathcal{A}$ such that $\#(I \cap A) = 1$. Show that two totally finite Radon measures on X which agree on $\mathcal{A}$ are identical.

434Yi Give an example of a Hausdorff uniform space $(X, \mathcal{W})$ with a quasi-Radon probability measure which is not inner regular with respect to the totally bounded sets.

434Yp If X is a topological space and ρ is a metric on X, X is **σ-fragmented** by ρ if for every $\epsilon > 0$ there is a countable cover $\mathcal{A}$ of X such that whenever $\emptyset \ne B \subseteq A \in \mathcal{A}$ there is a non-empty relatively open subset of B of ρ-diameter at most ϵ. Now suppose that $(X, \mathfrak{T})$ is a Hausdorff space which is σ-fragmented by a metric ρ such that (α) X is complete under ρ (β) the topology $\mathfrak{T}_\rho$ defined by ρ is finer than $\mathfrak{T}$. Show that (i) $(X, \mathfrak{T})$ is a pre-Radon space (ii) the totally finite Radon measures for $\mathfrak{T}$ and $\mathfrak{T}_\rho$ are the same.

434Yq Let μ be an atomless strictly positive Radon probability measure on $\mathbb{N}^{\mathbb{N}}$. (i) Show that if $\langle \alpha_n \rangle_{n\in\mathbb{N}}$ is any sequence in $[0, 1]$ such that $\sum_{n=0}^{\infty} \alpha_n = 1$, then there is a partition $\langle U_n \rangle_{n\in\mathbb{N}}$ of $\mathbb{N}^{\mathbb{N}}$ into open sets such that $\mu U_n = \alpha_n$ for every n. (ii) Show that if ν is any other atomless strictly positive Radon probability measure on $\mathbb{N}^{\mathbb{N}}$, there is a homeomorphism $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ such that $\nu = \mu f^{-1}$.

434Yr Let X be a Hausdorff space and μ an atomless strictly localizable tight Borel measure on X. Show that μ is σ-finite.

p 200 l 41 (434Xn): the terms 'Borel-measure-complete' and 'Borel-measure-compact' are reversed, so the first two parts should read '(i) Show that the family of Borel-measure-complete subsets of X is closed under Souslin's operation. (ii) Show that the union of a sequence of Borel-measure-compact subsets of X is Borel-measure-compact.'

p 202 l 13 (Exercise 434Yg, now 434Yj): for 'Borel-measure-compact' read 'Borel-measure-complete'.

p 202 l 22 (Exercise 434Yk): for 'Borel measure' read 'Borel probability measure'.

p 206 l 31 (part (b) of the proof of 435G): for '$\mu A = \mu A' = \mu_* A' \le (\nu_1)_* A' = \nu_1^* A' \le \mu^* A' = \mu A$' read '$\mu A = \mu A' = \mu^* A' \le \nu_1^* A' = (\nu_1)_* A' \le \mu_* A' = \mu A$'.

p 208 l 5 Exercise 435Xk has been rewritten, as follows:

(k) A completely regular space X is **strongly measure-compact** (Moran 69) if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X. (i) Show that a completely regular Hausdorff space X is strongly measure-compact iff every totally finite Baire measure on X has an extension to a Radon measure iff X is measure-compact and pre-Radon. (ii) Show that a Souslin-F subset of a strongly measure-compact completely regular space is strongly measure-compact. (iii) Show that a discrete space with cardinal ω_1 is strongly measure-compact. (iv) Show that a countable product of strongly measure-compact completely regular spaces is strongly measure-compact. (v) Show that $\mathbb{N}^{\omega_1}$ is not strongly measure-compact. (vi) Show that if X and Y are completely regular spaces, X is measure-compact and Y is strongly measure-compact then $X \times Y$ is measure-compact. (vii) Show that a strongly measure-compact completely regular space is Borel-measure-compact.

p 208 l 16 (435X) Add new exercises:

(l) Let X be a topological space and μ an atomless Baire probability measure on X. Show that there is a continuous function $f : X \to [0,1]$ which is inverse-measure-preserving for μ and the completion of Lebesgue measure.

(m) Let X be a normal space and μ a σ-finite topological probability measure on X which is inner regular with respect to the closed sets. Let ν be the restriction of μ to the Baire σ-algebra of X. (i) Show that μ and ν have isomorphic measure algebras. (ii) Show that if μ is an atomless probability measure there is a continuous $f : X \to [0,1]$ which is inverse-measure-preserving for the completion of μ and Lebesgue measure.

(n) Let X be a topological space and $\mathcal{G}$ the family of cozero sets in X. Show that a functional $\psi : \mathcal{G} \to [0,\infty[$ can be extended to a Baire measure on X iff ψ is modular (definition: 413Qc) and $\lim_{n\to\infty} \psi G_n = 0$ whenever $\langle G_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{G}$ with empty intersection.

(o) Let X be a countably compact topological space and μ a totally finite Baire measure on X. Show that μ has an extension to a Borel measure which is inner regular with respect to the closed sets.

p 211 l 20 (part (b) of the proof of 436E): for 'non-decreasing' read 'non-increasing'.

p 216 l 5 Add new result, formerly given as Exercise 436Xr:

***436M Corollary** Let $\mathfrak{A}$ be a Boolean algebra, and $M(\mathfrak{A})$ the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Let $U \subseteq M(\mathfrak{A})$ be a norm-closed linear subspace such that $a \mapsto \nu(a \cap b)$ belongs to U whenever $\nu \in U$ and $b \in \mathfrak{A}$. Then U is a band in $M(\mathfrak{A})$.

p 216 l 25 (Exercise 436Xe) Add two more parts:

(ii) Let X be a regular Lindelöf space. Show that every positive linear functional on $C(X)$ is smooth, so corresponds to a totally finite quasi-Radon measure on X. (iii) Let X be a K-analytic Hausdorff space. Show that every positive linear functional on $C(X)$ corresponds to at least one totally finite Radon measure on X.

p 217 l 31 Exercise 436Xr is now given in 436M, so 436Xs becomes 436Xr.
Add new exercise:

(s) Let X be a locally compact Hausdorff space. (i) Write $M_{\mathrm{R}}^{\infty+}$ for the set of all Radon measures on X. For $\mu \in M_{\mathrm{R}}^{\infty+}$, let $S\mu$ be the corresponding functional on $C_k(X)$, defined by setting $(S\mu)(u) = \int u\,d\mu$ for every $u \in C_k(X)$. Show that $S(\mu+\nu) = S\mu + S\nu$ and $S(\alpha\mu) = \alpha S\mu$ whenever μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ and $\alpha \ge 0$, where addition and scalar multiplication of measures are defined as in 234G[1] and 234Xf[2]. (ii) Write M_{R}^+ for the set of totally finite Radon measures on X. For $\mu \in M_{\mathrm{R}}^+$, let $T\mu$ be the corresponding functional on $C_0(X)$, defined by setting $(T\mu)(u) = \int u\,d\mu$ for every $u \in C_0(X)$. Show that $T(\mu+\nu) = T\mu + T\nu$ and $T(\alpha\mu) = \alpha T\mu$ whenever μ, $\nu \in M_{\mathrm{R}}^+$ and $\alpha \ge 0$.

p 218 l 29 (436Y) Add new exercise:

(g) Let X be any topological space. (i) Let C_k be the set of continuous functions $u : X \to \mathbb{R}$ such that $\overline{\{x : u(x) \ne 0\}}$ is compact, and $f : C_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$. (ii) Let $\tilde{C}_k$ be the set of continuous functions $u : X \to \mathbb{R}$ such that $\{x : u(x) \ne 0\}$ is relatively compact, and $f : \tilde{C}_k \to \mathbb{R}$ a positive linear functional. Show that there is a tight quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k$.

p l Add sentence to 437Jd:

Writing δ_x for the Dirac measure on X concentrated at x, $x \mapsto \delta_x : X \to P_{\mathrm{top}}$ is a homeomorphism between X and $\{\delta_x : x \in X\}$.

p 227 l 19 (437J, vague and narrow topologies): There is an embarrassing blunder in part (e): even for a Hausdorff space X, the narrow topology on the space M_{qR}^+ of totally finite quasi-Radon measures need not be Hausdorff. What is true (and is adequate for the later applications in this book) is that if X is Hausdorff, the narrow topology on the space M_{R}^+ of totally finite Radon measures is Hausdorff.

Add new subparagraphs:

(f) If $u : X \to \mathbb{R}$ is bounded and lower semi-continuous, then $\nu \mapsto \int u\,d\nu : \tilde{M}^+ \to \mathbb{R}$ is lower semi-continuous for the narrow topology.

(g) Let $\tilde{M}_\sigma^+$ be the space of totally finite topological measures on X. If $u : X \to [0,\infty]$ is a lower semi-continuous function, then $\nu \mapsto \int u\,d\nu : \tilde{M}_\sigma^+ \to [0,\infty]$ is lower semi-continuous.

(h) Let X and Y be topological spaces, $\phi : X \to Y$ a continuous function, and $\tilde{M}^+(X)$, $\tilde{M}^+(Y)$ the spaces of functionals described in (d). For a functional ν defined on a subset of $\mathcal{P}X$, define $\nu\phi^{-1}$ by saying that $(\nu\phi^{-1})(F) = \nu(\phi^{-1}[F])$ whenever $F \subseteq Y$ and $\phi^{-1}[F] \in \operatorname{dom}\nu$. Then $\nu\phi^{-1} \in \tilde{M}^+(Y)$ whenever $\nu \in \tilde{M}^+(X)$, and the map $\nu \mapsto \nu\phi^{-1} : \tilde{M}^+(X) \to \tilde{M}^+(Y)$ is continuous for the narrow topologies. If X and Y are Hausdorff spaces, we therefore have a continuous map $\nu \mapsto \nu\phi^{-1}$ from $M_R^+(X)$ to $M_R^+(Y)$.

p 228 l 16 (statement of Corollary 437L) Add new fact: In particular, the narrow topology on M_τ is completely regular.

p 228 l 33 (part (d) of the statement of 437M): we need to suppose that X and Y are Hausdorff.

p 231 l 7he later part of §437 has been substantially revised, with new material in 437R, also incorporating the old 437O. The new statements of the results are

437N Proposition (formerly 437Q) (a) Let X and Y be Hausdorff spaces, and $\phi : X \to Y$ a continuous function. Let $M_{\mathrm{R}}^+(X)$, $M_{\mathrm{R}}^+(Y)$ be the spaces of totally finite Radon measures on X and Y respectively. Write $\tilde{\phi}(\mu)$ for the image measure $\mu\phi^{-1}$ for $\mu \in M_{\mathrm{R}}^+(X)$.

(i) $\tilde{\phi} : M_{\mathrm{R}}^+(X) \to M_{\mathrm{R}}^+(Y)$ is continuous for the narrow topologies on $M_{\mathrm{R}}^+(X)$ and $M_{\mathrm{R}}^+(Y)$.

(ii) $\tilde{\phi}(\mu+\nu) = \tilde{\phi}(\mu) + \tilde{\phi}(\nu)$ and $\tilde{\phi}(\alpha\mu) = \alpha\tilde{\phi}(\mu)$ for all μ, $\nu \in M_{\mathrm{R}}^+(X)$ and $\alpha \ge 0$.

(b) If Y is a Hausdorff space, X a subset of Y, and $\phi : X \to Y$ the identity map, then $\tilde{\phi}$ is a homeomorphism between $M_{\mathrm{R}}^+(X)$ and $\{\nu : \nu \in M_{\mathrm{R}}^+(Y),\ \nu(Y \setminus X) = 0\}$.

437O Uniform tightness (formerly 437T) Let X be a topological space. Let $\tilde{M}$ be the space of bounded additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. I say that a functional $\nu \in \tilde{M}$ is **tight** if $\nu E \in \overline{\{\nu K : K \subseteq E \text{ is closed and compact}\}}$ for every $E \in \operatorname{dom}\nu$, and that a set $A \subseteq \tilde{M}$ is **uniformly tight** if every member of A is tight and for every

[1] Formerly 112Xe.
[2] Later editions only.

$\epsilon > 0$ there is a closed compact set $K \subseteq X$ such that $|\nu E| \le \epsilon$ whenever $\nu \in A$ and $E \in \operatorname{dom}\nu$ is disjoint from K.

437P Proposition (formerly 437U) Let X be a topological space.

(a) Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. Suppose that $A \subseteq M_{\mathrm{qR}}^+$ is uniformly totally finite and for every $\epsilon > 0$ there is a closed compact $K \subseteq X$ such that $\mu(X \setminus K) \le \epsilon$ for every $\mu \in A$. Then A is relatively compact in M_{qR}^+ for the narrow topology.

(b) Suppose now that X is Hausdorff, and that M_{R}^+ is the set of Radon measures on X. If $A \subseteq M_{\mathrm{R}}^+$ is uniformly totally finite and uniformly tight, then it is relatively compact in M_{R}^+ for the narrow topology.

437Q Two metrics (a) If X is a set and μ, ν are bounded additive functionals defined on algebras of subsets of X, set

$$\rho_{\mathrm{tv}}(\mu,\nu) = |\mu-\nu|(X) = \sup_{E,F\in\operatorname{dom}\mu\cap\operatorname{dom}\nu}(\mu-\nu)(E)-(\mu-\nu)(F).$$

In this generality, ρ_{tv} is not even a pseudometric, but when $\rho\restriction M \times M$ is a metric I will call it the **total variation metric** on M.

(b) Suppose that (X,ρ) is a metric space. Write M_{qR}^+ for the set of totally finite quasi-Radon measures on X. For μ, $\nu \in M_{\mathrm{qR}}^+$ set

$$\rho_{\mathrm{KR}}(\mu,\nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1,1] \text{ is 1-Lipschitz}\}.$$

Then ρ_{KR} is a metric on M_{qR}^+.

437R Theorem Let X be a topological space; write $M_{\mathrm{qR}}^+ = M_{\mathrm{qR}}^+(X)$ for the set of totally finite quasi-Radon measures on X, and if X is Hausdorff write $M_{\mathrm{R}}^+ = M_{\mathrm{R}}^+(X)$ for the set of totally finite Radon measures on X, both endowed with their narrow topologies.

(a) If X is regular then M_{qR}^+ is Hausdorff.

(b) If X has a countable network then M_{qR}^+ has a countable network.

(c) Suppose that X is separable.

(i) If X is a T_1 space, then M_{qR}^+ is separable.

(ii) If X is Hausdorff, M_{R}^+ is separable.

(d) If X is a K-analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(e) If X is an analytic Hausdorff space, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

(f)(i) If X is compact, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu X \le \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{qR}}^+,\ \mu X = \gamma\}$ are compact.

(ii) If X is compact and Hausdorff, then for any real $\gamma \ge 0$ the sets $\{\mu : \mu \in M_{\mathrm{R}}^+,\ \mu X \le \gamma\}$ and $\{\mu : \mu \in M_{\mathrm{R}}^+,\ \mu X = \gamma\}$ are compact. In particular, the set P_{R} of Radon probability measures on X is compact.

(g) Suppose that X is metrizable and ρ is a metric on X inducing its topology. For μ, $\nu \in M_{\mathrm{qR}}^+$ set

$$\rho_{\mathrm{KR}}(\mu,\nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1,1] \text{ is 1-Lipschitz}\}.$$

(i) ρ_{KR} is a metric on M_{qR}^+ inducing the narrow topology.

(ii) If (X,ρ) is complete then $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$ is complete under ρ_{KR}.

(h) If X is Polish, so is $M_{\mathrm{qR}}^+ = M_{\mathrm{R}}^+$.

437S Proposition (formerly 437P) Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X. Then the extreme points of P_{R} are just the Dirac measures on X.

Other paragraphs have been rearranged, so that 437S is now 437T, 437V is now 437U and 437W is now 437V.

p 237 l 3 The exercises to §437 have likewise been revised and restructured. 437Xc-437Xf are now 437Xd-437Xg. The former 437Xg(i) has been deleted and 437Xg(ii) is now part of 437Xr. 437Xi is now part of 437Xq, 437Xj is now 437Xr, 437Xk is now 437Xq, 437Xl is now 437Xi, 437Xm is now part of 437Jf, 437Xn is now 437Xl. Part (ii) of 437Xo has been deleted, the rest is now in 437Xt. 437Xp is now part of 437Xt, 437Xq is now part of 436Xn, 437Xs-437Xt are now 437Xw-437Xx.

437Ye, revised, is now 437Ym; 437Yh is now 437Yj. 437Yi, corrected and revised, is now divided between 437Yo and 437Yp. 437Yj, corrected, is now 437Yi, 437Yk-437Yl are now 437Yq-437Yr, 437Ym is now 437Yk, 437Yn-437Yo are now 437Yx-437Yy.

Rewritten and new exercises are

437Xc Let $I \subseteq \mathbb{R}$ be a non-empty interval. (i) Show that if $g : I \to \mathbb{R}$ is of bounded variation on every compact subinterval of I, there is a unique signed tight Borel measure μ_g on I such that $\mu_g[a,b] = \lim_{x\downarrow b} g(x) - \lim_{x\uparrow a} g(x)$ whenever $a \le b$ in I, counting $\lim_{x\uparrow a} g(x)$ as $g(a)$ if $a = \min I$, and $\lim_{x\downarrow b} g(x)$ as $g(b)$ if $b = \max I$. (ii) Show that if $h : I \to \mathbb{R}$ is another function of bounded variation on every compact subinterval, then $\mu_h = \mu_g$ iff $\{x : h(x) \ne g(x)\}$ is countable iff $\{x : h(x) = g(x)\}$ is dense in I. (iii) Show that if ν is any signed Baire measure on I there is a g of bounded variation on every compact subinterval such that $\nu = \mu_g$.

(j) Let X be a topological space, and Σ an algebra of subsets of X containing every open set; let $M(\Sigma)^+$ be the set of non-negative real-valued additive functionals on Σ, endowed with its narrow topology, E a member of Σ, and ∂E its boundary. Show that $\nu \mapsto \nu E : M(\Sigma)^+ \to [0,\infty[$ is continuous at $\nu_0 \in M(\Sigma)^+$ iff $\nu_0(\partial E) = 0$.

(m) Let X be a topological space, Y a regular topological space and $M^+_{\mathrm{qR}}(X)$, $M^+_{\mathrm{qR}}(Y)$ the spaces of totally finite quasi-Radon measures on X, Y respectively. For a continuous $\phi : X \to Y$ define $\tilde\phi : M^+_{\mathrm{qR}}(X) \to M^+_{\mathrm{qR}}(Y)$ by saying that ϕ is inverse-measure-preserving for μ and $\tilde\phi(\mu)$ for every $\mu \in M^+_{\mathrm{qR}}(X)$. Show that $\tilde\phi$ is continuous for the narrow topologies.

(n) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, and Q the set of Radon probability measures μ on $X = \prod_{i\in I} X_i$ such that image of μ under the map $x \mapsto x(i)$ is μ_i for every $i \in I$. Show that Q is uniformly tight and compact for the narrow topology on the set of measures on X.

(o) Let X be any topological space, and M^+_{qR} the space of totally finite quasi-Radon measures on X. Show that M^+_{qR} is complete in the total variation metric.

(p) Let X and Y be topological spaces, and ρ_X, ρ_Y, $\rho_{X\times Y}$ the total variation metrics on the spaces $M^+_{\mathrm{qR}}(X)$, $M^+_{\mathrm{qR}}(Y)$ and $M^+_{\mathrm{qR}}(X\times Y)$ of quasi-Radon measures. Let μ_1, μ_2 be totally finite quasi-Radon measures on X, ν_1, ν_2 totally finite quasi-Radon measures on Y, and $\mu_1 \times \nu_1$, $\mu_2 \times \nu_2$ the quasi-Radon product measures. Show that

$$\rho_{X\times Y}(\mu_1 \times \nu_1, \mu_2 \times \nu_2) \le \rho_X(\mu_1, \mu_2) \cdot \nu_2 Y + \mu_1 X \cdot \rho_Y(\nu_1, \nu_2).$$

(q) (i) Show that the set $M^+_\sigma(\mathcal{B}(X))$ of totally finite Borel probability measures on X is T_0 in its narrow topology for any topological space X. (ii) Give $X = \omega_1 + 1$ its order topology. Show that the narrow topology on $M^+_\sigma(\mathcal{B}(X))$ is not T_1.

(r) Let X be any topological space and $\tilde M^+$ the set of non-negative additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For μ, $\nu \in \tilde M^+$ define $\mu + \nu \in \tilde M^+$ by setting $(\mu + \nu)(E) = \mu E + \nu E$ for $E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu$. (i) Show that addition on $\tilde M^+$ is continuous for the narrow topology. (ii) Show that $(\alpha,\mu) \mapsto \alpha\mu : [0,\infty[\times \tilde M^+ \to \tilde M^+$ is continuous for the narrow topology on $\tilde M^+$. (iii) Writing $\tilde P$ for $\{\mu : \mu \in \tilde M^+, \ \mu X = 1\}$, and δ_x for the Dirac measure concentrated at x for each $x \in X$, show that the convex hull of $\{\delta_x : x \in X\}$ is dense in $\tilde P$ for the narrow topology. (iv) Suppose that A and B are uniformly tight subsets of $\tilde M^+$ and $\gamma \ge 0$. Show that $A \cup B$, $A + B = \{\mu + \nu : \mu \in A, \ \nu \in B\}$ and $\{\alpha\mu : \mu \in A, \ 0 \le \alpha \le \gamma\}$ are uniformly tight.

(t) Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X with its narrow topology. For $x \in X$ let δ_x be the Dirac measure on X concentrated at x. Show that $x \mapsto \delta_x$ is a homeomorphism between X and its image in P_{R}.

(u) Let X be a non-empty compact metrizable space, and $\phi : X \to X$ a continuous function. Show that there is an $x \in X$ such that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(\phi^i(x))$ is defined for every $f \in C(X)$.

(v) Let X be a T_0 topological space, and P_{qR} the set of quasi-Radon probability measures on X. Show that the extreme points of P_{qR} are just the Dirac measures on X concentrated on points x such that $\{x\}$ is closed.

(y) Let X and Y be analytic spaces, and $P = P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology. (i) Let V be an analytic subset of $(P \times Y) \times X$. Show that $\{(\mu, y) : \mu \in P,\, y \in Y,\, \mu V[\{(\mu, y)\}] > \alpha\}$ is analytic for every $\alpha \in \mathbb{R}$. (ii) Let W be a coanalytic subset of $(P \times Y) \times X$. Show that $\{(\mu, y) : \mu \in P,\, y \in Y,\, W[\{(\mu, y)\}]$ is not μ-negligible$\}$ is coanalytic.

(z) Let X be a second-countable topological space, $P = P_\sigma(X)$ the space of topological probability measures on X with its narrow topology and $\mathcal{C}$ the space of closed subsets of X with the Fell topology. For $\mu \in P$ write $\operatorname{supp}\mu$ for the support of μ. (i) Show that $\{(x, C) : C \in \mathcal{C},\, x \in C\}$ is a Borel subset of $X \times \mathcal{C}$. (ii) Show that $\{(\mu, x) : \mu \in P,\, x \in \operatorname{supp}\mu\}$ is a Borel subset of $P \times X$. (iii) Show that $\mu \mapsto \operatorname{supp}\mu : P \to \mathcal{C}$ is Borel measurable. (Cf. 424Ya.)

437Ye Show that in 437Ib the operator $S : \mathcal{L}^\infty(\Sigma_{\mathrm{uRm}}) \to C_0(X)^{**}$ is multiplicative if $C_0(X)^{**}$ is given the Arens multiplication described in 4A6O based on the ordinary multiplication $(u, v) \mapsto u \times v$ on $C_0(X)$.

(h) Let X be a topological space, μ_0 a totally finite τ-additive topological measure on X, and $f : X \to \mathbb{R}$ a bounded function which is continuous μ_0-a.e. Let $\tilde{M}_\sigma^+$ be the set of totally finite topological measures on X, with its narrow topology. Show that $\nu \mapsto \overline{\int} f d\nu : \tilde{M}_\sigma^+ \to \mathbb{R}$ is continuous at μ_0.

(i) Let X be a Hausdorff space, and $M_{\mathrm{R}}^{\infty+}$ the set of all Radon measures on X. Define addition and scalar multiplication (by positive scalars) on $M_{\mathrm{R}}^{\infty+}$ as in 234G, 234Xf and 416De and $\le$ by the formulae of 234P or 416Ea. (i) Show that $M_{\mathrm{R}}^{\infty+}$ is a Dedekind complete lattice. (ii) Show that if $A \subseteq M_{\mathrm{R}}^{\infty+}$ is upwards-directed and non-empty, it is bounded above iff $\{G : G \subseteq X$ is open, $\sup_{\nu\in A} \nu G < \infty\}$ covers X, and in this case $\operatorname{dom}(\sup A) = \bigcap_{\nu\in A} \operatorname{dom}\nu$ and $(\sup A)(E) = \sup_{\nu\in A} \nu E$ for every $E \in \operatorname{dom}(\sup A)$. (iii) Show that if μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ then $\nu = \sup_{n\in\mathbb{N}} \nu \wedge n\mu$ iff every μ-negligible set is ν-negligible. (iv) Show that if μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ then ν is uniquely expressible as $\nu_s + \nu_{ac}$ where ν_s, $\nu_{ac} \in\in M_{\mathrm{R}}^{\infty+}$, $\mu \wedge \nu_s = 0$ and $\nu_{ac} = \sup_{n\in\mathbb{N}} \nu_{ac} \wedge n\mu$. (v) Show that if μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ then $\operatorname{dom}(\mu \vee \nu) = \operatorname{dom}\mu \cap \operatorname{dom}\nu$ and $(\mu \vee \nu)(E) = \sup\{\mu F + \nu(E \setminus F) : F \in \operatorname{dom}\mu \cap \operatorname{dom}\nu,\, F \subseteq E\}$ for every $E \in \operatorname{dom}(\mu \vee \nu)$. (vi) Show that if μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ then $\operatorname{dom}(\mu \wedge \nu) = \{E \cup F : E \in \operatorname{dom}\mu,\, F \in \operatorname{dom}\nu\}$. (vii) Show that if μ, $\nu \in M_{\mathrm{R}}^{\infty+}$ then $\mu \wedge \nu = 0$ iff there is a set $E \subseteq X$ which is μ-negligible and ν-conegligible. (viii) Show that there is a Dedekind complete Riesz space V such that the positive cone of V is isomorphic to $M_{\mathrm{R}}^{\infty+}$.

(l) Let X be a topological space, and $\tilde{M}$ the space of bounded additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For $\nu \in \tilde{M}$, say that $|\nu|(E) = \sup\{\nu F - \nu(E \setminus F) : F \in \operatorname{dom}\nu,\, F \subseteq E\}$ for $E \in \operatorname{dom}\nu$. Show that a set $A \subseteq \tilde{M}$ is uniformly tight in the sense of 437O iff $\{|\nu| : \nu \in A\}$ is uniformly tight.

(m) Let X be a completely regular space and P_{qR} the space of quasi-Radon probability measures on X. Let $B \subseteq P_{\mathrm{qR}}$ be a non-empty set. Show that the following are equiveridical: (i) B is relatively compact in P_{qR} for the narrow topology; (ii) whenever $A \subseteq C_b(X)$ is non-empty and downwards-directed and $\inf_{u\in A} u(x) = 0$ for every $x \in A$, then $\inf_{u\in A} \sup_{\mu\in B} \int u\,d\mu = 0$; (iii) whenever $\mathcal{G}$ is an upwards-directed family of open sets with union X, then $\sup_{G\in\mathcal{G}} \inf_{\mu\in B} \mu G = 1$.

(n) Let X be a topological space, and M_{qR} the set of totally finite quasi-Radon measures on X, with its narrow topology. (i) Show that if X is regular then M_{qR} is regular and Hausdorff. (ii) Find a second-countable Hausdorff space X such that the space $P_{\mathrm{qR}}(X)$ of quasi-Radon probability measures on X is not Hausdorff in its narrow topology.

(o) Let (X, ρ) and (Y, σ) be metric spaces, and give $M_{\mathrm{qR}}^+(X)$ and $M_{\mathrm{qR}}^+(Y)$ the corresponding metrics $\bar\rho_{\mathrm{KR}}$, $\bar\sigma_{\mathrm{KR}}$ as in 437Rg. For a continuous function $\phi : X \to Y$, let $\tilde\phi : M_{\mathrm{qR}}^+(X) \to M_{\mathrm{qR}}^+(Y)$ be the map described in 437Xm. (i) Show that if ϕ is γ-Lipschitz, where $\gamma \ge 0$, then $\tilde\phi$ is γ-Lipschitz. (ii) (J.Pachl) Show that if ϕ is uniformly continuous, then $\tilde\phi$ is uniformly continuous on any uniformly totally finite subset of $M_{\mathrm{qR}}^+(X)$. (iii) Show that if (X, ρ) is $\mathbb{R}$ with its usual metric, then $\bar\rho_{\mathrm{KR}}$ is not uniformly equivalent to Lévy's metric as described in 274Yc. (For a discussion of various metrics related to $\bar\rho_{\mathrm{KR}}$, see Bogachev 07, 8.10.43-8.10.48.)

(p) Let (X,ρ) be a metric space. For $f\in C_b(X)^\sim_\sigma$, set $\|f\|_{\mathrm{KR}}=\sup\{|f(u)|:u\in C_b(X),\ \|u\|_\infty\le 1,\ u$ is 1-Lipschitz$\}$. (i) Show that $\|\ \|_{\mathrm{KR}}$ is a norm on $C_b(X)^\sim_\sigma$. (ii) Let (X',ρ') and (X'',ρ'') be metric spaces, and ρ the ℓ^1-product metric on $X=X'\times X''$ defined by saying that $\rho((x',x''),(y',y''))=\rho'(x',y')+\rho''(x'',y'')$. Identifying the spaces $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ of signed τ-additive Borel measures with subspaces of $C_b(X')^\sim_\sigma$, $C_b(X'')^\sim_\sigma$ and $C_b(X')^\sim_\sigma$, as in 437E-437H, show that the bilinear map $\psi:M_\tau(X')\times M_\tau(X'')\to M_\tau(X)$ described in 437Mc has norm 1 when $M_\tau(X')$, $M_\tau(X'')$ and $M_\tau(X)$ are given the appropriate norms $\|\ \|_{\mathrm{KR}}$.

(q) Let X be a topological space and $\tilde M^+$ the set of non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set, endowed with its narrow topology. Show that $w(\tilde M^+)\le\max(\omega,w(X))$.

(s) Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a non-empty family of probability algebras, and $\mathcal{F}$ an ultrafilter on I. Let $(\mathfrak{A},\bar\mu)=\prod_{i\in I}(\mathfrak{A}_i,\bar\mu_i)|\mathcal{F}$ be the reduced product as defined in 328C[3]. For each $i\in I$, let (Z_i,ν_i) be the Stone space of $(\mathfrak{A}_i,\bar\mu_i)$; give $W=\{(z,i):i\in I,\ z\in Z_i\}$ its disjoint union topology, and let βW be the Stone-Čech compactification of W. For each $i\in I$, define $f_i:Z_i\to W\subseteq\beta W$ by setting $f_i(z)=(z,i)$ for $z\in Z_i$, and let $\nu_if_i^{-1}$ be the image measure on βW. Let ν be the limit $\lim_{i\to\mathcal{F}}\nu_if_i^{-1}$ for the narrow topology on the space of Radon probability measures on W, and Z its support. Show that (Z,ν) can be identified with the Stone space of $(\mathfrak{A},\bar\mu)$.

(t)(i) Show that there are a continuous $\phi:\{0,1\}^{\mathbb{N}}\to[0,1]$ and a positive linear operator $T:C(\{0,1\}^{\mathbb{N}})\to C([0,1])$ such that $T(f\phi)=f$ for every $f\in C([0,1])$. (*Hint*: if $I_\sigma=\{x:\sigma\subseteq x\in\{0,1\}^{\mathbb{N}}$ for $\sigma\in\bigcup_{n\in\mathbb{N}}\{0,1\}^n$, arrange that $\{t:T(\chi I_\sigma)(t)>0\}$ is always an interval of length $(\frac{2}{3})^{\#(\sigma)}$.) (ii) Show that there are a continuous $\tilde\phi:(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}\to[0,1]^{\mathbb{N}}$ and a positive linear operator $\tilde T:C((\{0,1\}^{\mathbb{N}})^{\mathbb{N}})\to C([0,1]^{\mathbb{N}})$ such that $\tilde T(h\tilde\phi)=h$ for every $h\in C([0,1]^{\mathbb{N}})$. (*Hint*: if, in (i), $(Tg)(t)=\int g\,d\nu_t$ for $t\in[0,1]$ and $g\in C(\{0,1\}^{\mathbb{N}})$, take $\nu_{\boldsymbol{t}}$ to be the product measure $\prod_{n\in\mathbb{N}}\nu_{t_n}$ for $\boldsymbol{t}=\langle t_n\rangle_{n\in\mathbb{N}}\in[0,1]^{\mathbb{N}}$.)

(u) Let X be a separable metrizable space and $P=P_{\mathrm{R}}(X)$ the set of Radon probability measures on X, with its narrow topology. Show that there is a family $\langle f_\mu\rangle_{\mu\in P}$ of functions from $[0,1]$ to X such that (i) $(\mu,t)\mapsto f_\mu(t)$ is Borel measurable (ii) writing μ_L for Lebesgue measure on $[0,1]$, $\mu=\mu_Lf_\mu^{-1}$ for every $\mu\in P$ (iii) whenever $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is a sequence in P converging to $\mu\in P$, there is a countable set $A\subseteq[0,1]$ such that $f_\mu(t)=\lim_{n\to\infty}f_{\mu_n}(t)$ for every $t\in[0,1]\setminus A$. (*Hint*: first consider the cases $X=[0,1]$ and $X=\{0,1\}^{\mathbb{N}}$, then use 437Yt to deal with $[0,1]^{\mathbb{N}}$ and its subspaces. See Bogachev 07, §8.5.)

(v) Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\mathfrak{A}^f$ the ideal of elements of $\mathfrak{A}$ of finite measure. For $a\in\mathfrak{A}^f$ and $u\in L^0=L^0(\mathfrak{A})$, let ν_{au} be the totally finite Radon measure on $\mathbb{R}$ defined by saying that $\nu_{au}(E)=\bar\mu(a\cap[\![u\in E]\!])$ (definition: 364G, 434T) for Borel sets $E\subseteq\mathbb{R}$. For $a\in\mathfrak{A}^f$ and u, $v\in L^0$ set $\bar\rho_a(u,v)=\rho_{KR}(\nu_{au},\nu_{av})$, where ρ_{KR} is the metric on $M_{\mathrm{R}}^+=M_{\mathrm{R}}^+(\mathbb{R})$ defined from the usual metric on $\mathbb{R}$. (i) Show that the family $\mathrm{P}=\{\bar\rho_a:a\in\mathfrak{A}^f\}$ of pseudometrics defines the topology of convergence in measure on L^0 (definition: 367L). (ii) Show that if $(\mathfrak{A},\bar\mu)$ is semi-finite then the uniformity $\mathcal{U}$ defined from P is metrizable iff $(\mathfrak{A},\bar\mu)$ is σ-finite and $\mathfrak{A}$ has countable Maharam type. (iii) Show that if $(\mathfrak{A},\bar\mu)$ is semi-finite then L^0 is complete under $\mathcal{U}$ (definition: 3A4F) iff $\mathfrak{A}$ is purely atomic.

(z) Let X be a regular Hausdorff topological space and C a non-empty narrowly compact set of totally finite topological measures on X, all inner regular with respect to the closed sets. Set $c(A)=\sup_{\mu\in C}\mu^*A$ for $A\subseteq X$. Show that $c:\mathcal{P}X\to[0,\infty[$ is a Choquet capacity.

As for corrections, we have the following.

p 236 l 22 (hint to part (i) of 437Xa): for '$f(v_n\wedge u_{k(n)})\le f(v_n)+2^{-n}$' read '$f(v_n\vee u_{k(n)})\le f(v_n)+2^{-n}$'. (J.Pachl.)

p 239 l 3 J.Pachl has noted that parts (ii) and (iii) of Exercise 437Yi (now 437Yp) are wrong.

p 242 l 29 (part (d) of the proof of 438C): for '$\kappa\setminus\bigcup_{\xi<\kappa}A(\beta,\xi)$' read '$\kappa^+\setminus\bigcup_{\xi<\kappa}A(\beta,\xi)$'.

p 247 l 2 (part (c) of the proof of 438N): for '$\mathcal{D}_n$ is a relatively open family' read '$\mathcal{D}_n$ is an isolated family'.

[3]Later editions only.

p 247 l 35 Proposition 438P has been split into two parts and generalized, as follows:

438P Lemma Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to \mathbb{R}$ such that $\lim_{s\uparrow t}\omega(s)$ and $\lim_{s\downarrow t}\omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) For $A \subseteq B \subseteq \mathbb{R}$ and $f \in X^B$, set

$$\operatorname{jump}_A(f,\epsilon) = \sup\{n : \text{ there is an } I \in [A]^n \text{ such that } \rho(f(s), f(t)) > \epsilon \text{ whenever } s < t \text{ are successive elements of } I\}.$$

Now a function $\omega \in X^{\mathbb{R}}$ belongs to $\tilde{C}^{\Downarrow}$ iff $\operatorname{jump}_{[-n,n]}(\omega,\epsilon)$ is finite for every $n \in \mathbb{N}$ and $\epsilon > 0$.

(b) If $\omega \in \tilde{C}^{\Downarrow}$ then ω is continuous at all but countably many points of $\mathbb{R}$.

(c) If $\omega \in \tilde{C}^{\Downarrow}$ then $\omega[\,[-n,n]\,]$ is relatively compact in X for every $n \in \mathbb{N}$.

438Q Theorem Let X be a Polish space, and $\tilde{C}^{\Downarrow} = \tilde{C}^{\Downarrow}(X)$ the family of functions $\omega : \mathbb{R} \to \mathbb{R}$ such that $\lim_{s\uparrow t}\omega(s)$ and $\lim_{s\downarrow t}\omega(s)$ are defined in X for every $t \in \mathbb{R}$.

(a) $\tilde{C}^{\Downarrow}$, with its topology of pointwise convergence inherited from the product topology of $X^{\mathbb{R}}$, is K-analytic.

(b) $\tilde{C}^{\Downarrow}$ is hereditarily weakly θ-refinable.

Consequently 438Q is now 438R.

p 249 l 4 (case 1 of part (b-iv) of the proof of 438P, now 438Q): for '$H = \bigcup_{\tau\in T^*} H_{\boldsymbol{q}\tau}$ is negligible' read '$H = \bigcup_{\tau\in T^*} H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$'.

p 249 l 42 (part (b-v) of the proof of 438P, now 438Q): for '$x \in X_{\boldsymbol{q}\tau} \subseteq H_{\boldsymbol{q}\tau} \subseteq G$' read '$x \in F_{\boldsymbol{q}\tau} \subseteq H_{\boldsymbol{q}\tau} \subseteq G$'.

p 250 l 12 (proof of 438Q): for '$\mathbf{t} = \langle t_n\rangle_{n\in\mathbb{N}} \in (I^{\|})^L$' read '$\mathbf{t} = \langle t_n\rangle_{n\in L} \in (I^{\|})^L$'.

p 250 l 18 Add a paragraph on càllàl functions:

438S Proposition Let X be a Polish space. Let $C^{\Downarrow}$ be the set of càllàl functions from $[0,\infty[$ to X, with its topology of pointwise convergence inherited from the product topology of $X^{[0,\infty[}$.

(a)(i) If $\omega \in C^{\Downarrow}$, then ω is continuous at all but countably many points of $[0,\infty[$.

(ii) If ω, $\omega' \in C^{\Downarrow}$, D is a dense subset of $[0,\infty[$ containing every point at which ω is discontinuous, and $\omega'\restriction D = \omega\restriction D$, then $\omega' = \omega$.

(b) $C^{\Downarrow}$ is K-analytic.

438R-438S are now 438T-438U.

p 253 l 10 (438X) Add new exercises:

(e) Let (X,Σ,μ) be a complete locally determined measure space, (Y,T,ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν is either finite or a measure-free cardinal. Show that μ is strictly localizable.

(f) Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2), (Y_1,T_1,ν_1) and (Y_2,T_2,ν_2) be measure spaces, and λ_1, λ_2 the product measures on $X_1\times Y_1$, $X_2\times Y_2$ respectively; suppose that $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are inverse-measure-preserving functions, and that $h(x,y) = (f(x), g(y))$ for $x \in X_1$, $y \in Y_1$. Show that if μ_2 and ν_2 are both strictly localizable, with magnitudes which are either finite or measure-free cardinals, then h is inverse-measure-preserving. (Compare 251L[4].)

(n) Let X be a topological space and $\mathcal{G}$ a family of open subsets of X. Show that the following are equiveridical: (i) there is a σ-isolated family $\mathcal{A}$ of sets, refining $\mathcal{G}$, such that $\bigcup\mathcal{A} = \bigcup\mathcal{G}$; (ii) there is a sequencen $\mathcal{H}_n$ of families of open sets, all refining $\bigcup\mathcal{G}$, such that for every $x \in \bigcup\mathcal{G}$ there is an $n \in \mathbb{N}$ such that $\{H : x \in H \in \mathcal{H}_n\}$ is finite and not empty.

Other exercises in 438X have been re-named: 438Xe-438Xk are now 438Xg-438Xm, 438Xl-438Xp are now 438Xo-438Xs.

p 232 l 25 (Exercise 438Xg, now 438Xi): add 'where $h_x(y) = h(x,y)$ for $(x,y) \in \operatorname{dom} h$'.

[4]Later editions only.

p 232 l 29 (Exercise 438Xh, now 438Xj): for 'W_x' read '$W[\{x\}]$'.

p 253 l 27 Exercise 438Yh has been moved to 438P. 438Yi is now 438Yh, 438Yj is now 438Yi.

p 253 l 34 (438Y) Add new exercises:

(j) Suppose that X is a normal metacompact Hausdorff space which is not realcompact. Show that there are a closed discrete subset D of X and a non-principal ω_1-complete ultrafilter on D.

(k) Let Z be a regular Hausdorff space, T a Dedekind complete totally ordered space with least and greatest elements a, b, and $x : T \to Z$ a function such that $\lim_{s\uparrow t} x(s)$ and $\lim_{s\downarrow t} x(s)$ are defined in Z for every $t \in T$ (except $t = a$ in the first case and $t = b$ in the second). Show that $x[T]$ is relatively compact in Z.

(l) Let (X, ρ) be a metric space, and P_{Bor} the set of Borel probability measures on X. For μ, $\nu \in P_{\text{Bor}}$ set $\bar\rho_{\text{KR}}(\mu, \nu) = \sup\{|\int u\,d\mu - \int u\,d\nu| : u : X \to [-1, 1]$ is 1-Lipschitz$\}$. (i) Show that $\bar\rho_{\text{KR}}$ is a metric on P_{Bor}. (ii) Let $\mathfrak{T}_{\text{KR}}$ be the topology it induces on P_{Bor}. Show that $\mathfrak{T}_{\text{KR}}$ is finer than the narrow topology on P_{Bor} (437J), and that the two topologies coincide iff κ is measure-free.

(m) Let (X, Σ, μ) be a complete locally determined measure space, and $\theta = \frac{1}{2}(\mu^* + \mu_*)$ the outer measure described in 413Xd. Show that if the measure μ_θ defined by Carathéodory's method is not equal to μ, then there is a set $A \subseteq X$ such that $0 < \mu^*A < \infty$ and the subspace measure on A induced by μ measures every subset of A.

p 257 l 24 (part (b) of the proof of 439F): for '$f : F \to [0, \nu Z_n]$' read '$f : F \to [0, \nu F]$'.

p 259 l 5 (part (b) of the proof of 439H): for '439Ca' read '439Cb'.

p 259 l 8 (part (c) of the proof of 439H): for '$\mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \cap \Gamma)$' read '$\mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \setminus \Gamma)$', and again in line 10.

p 260 l 46 (part (a-iv) of the proof of 439K): for '$t_m \in I_n \subseteq F_n$' read '$t_m \in J_n \subseteq F_n$'.

p 265 l 2 (proof of 439P): for '$\langle \phi_x(y) \rangle_{x \in C}$' read '$\langle \phi_z(y) \rangle_{z \in Z}$'.

p 265 l 42 (part (b-iii) of the proof of 439Q): for 'measure ν on X' read 'measure ν on X^2'. (J.P.)

p 266 l 5 (statement of 439R): for '$\int f d\lambda W$' read '$\int f d\lambda$'.

p 266 l 9 (proof of 439R): for '$\tilde\nu_Y$' read '$\tilde\mu_Y$'.

p 267 l 11 (part (d) of the proof of 439S): for 'M^+_{P}' read 'M^+_{R}' (six times).

p 267 l 39 (Exercise 439Xc) for 'complete locally determined' read 'with locally determined negligible sets'.

p 268 l 17 Exercise 439Xi is now 439Fb, so has been dropped. 439Xk is now 439Xl, 439Xl is now 439Xo. 439Xm is now 439Xp, and has been revised to read

(p) (i) Suppose that X is a completely regular space and there is a continuous function f from X to a realcompact completely regular space Z such that $f^{-1}[\{z\}]$ is realcompact for every $z \in Z$. Show that X is realcompact (definition: 436Xg). (ii) Show that the spaces X of 439K and X^2 of 439Q are realcompact.

p 268 l 24 (439X) Add new exercises:

(i) Let X be a Polish space, $A \subseteq X$ an analytic set which is not Borel (423Sb, 423Ye), and $\langle E_\xi \rangle_{\xi<\omega_1}$ a family of Borel constituents of $X \setminus A$ (423R). Suppose that $x_\xi \in E_\xi \setminus \bigcup_{\eta<\xi} E_\eta$ for every $\xi < \omega_1$. Show that $\{x_\xi : \xi < \omega_1\}$ is universally negligible. Hence show that any probability measure with domain $\mathcal{P}\omega_1$ is point-supported.

(k) Show that 439Fc, or any of the examples of 439A, can be regarded as an example of a probability space (X, μ) and a function $f : X \to [0, 1]$ such that there is no extension of μ to a measure ν such that f is $\operatorname{dom}\nu$-measurable; and accordingly can provide an example of a probability space (X, μ) with a countable family $\mathcal{A}$ of subsets of X such that there is no extension of μ to a measure measuring every member of $\mathcal{A}$. Contrast with 214P, 214Xm-214Xn and 214Yb-214Yc[5].

(m) Show that the one-point compactification of the space $(Z,\mathfrak{T}_{\mathfrak{c}})$ described in 439K is a scattered compact Hausdorff space with an atomless Borel probability measure.

p 268 l 34 Part (ii) of Exercise 439Ya is wrong, and should be deleted.

p 269 l 3 (439Y) Add new exercises:

(d)(i) A pair $(\langle a_\xi\rangle_{\xi<\omega_1},\langle b_\xi\rangle_{\xi<\omega_1})$ of families of subsets of $\mathbb{N}$ is a **Hausdorff gap** if $a_\xi\setminus a_\eta$, $a_\xi\setminus b_\xi$ and $b_\eta\setminus b_\xi$ are finite whenever $\xi\le\eta<\omega_1$, $a_\eta\setminus a_\xi$ and $b_\xi\setminus b_\eta$ are infinite whenever $\xi<\eta<\omega_1$, and moreover $\{\xi:\xi<\eta,\,a_\xi\subseteq b_\eta\cup n\}$ is finite for every $\eta<\omega_1$. (For a construction of a Hausdorff gap, see Fremlin 84, 21L.) Show that in this case there is no $c\subseteq\mathbb{N}$ such that $a_\xi\setminus c$ and $c\setminus b_\xi$ are finite for every $\xi<\omega_1$, and that $\{a_\xi:\xi<\omega_1\}\cup\{b_\xi:\xi<\omega_1\}$ is universally negligible in $\mathcal{P}\mathbb{N}$. (ii) Let $\phi:(\mathcal{P}\mathbb{N})^{\mathbb{N}}\to\mathcal{P}\mathbb{N}$ be a homeomorphism. For $0<\xi<\omega_1$ let $\langle\theta(\xi,n)\rangle_{n\in\mathbb{N}}$ be a sequence running over ξ. Set $a_0=\emptyset$ and for $0<\xi<\omega_1$ set $a_\xi=\phi(\langle a_{\theta(\xi,n)}\rangle_{n\in\mathbb{N}})$. Show that $\{a_\xi:\xi<\omega_1\}$ is universally negligible.

(g) Show that the space of 439O is locally compact and locally countable, therefore first-countable.

(k) Show that the Sorgenfrey line is not a Prokhorov space.

439Yd-439Ye are now 439Ye-439Yf. 439Yf is now 439Xn and has been re-phrased as

(n) Show that a semi-finite Borel measure on ω_1, with its order topology, must be purely atomic.

439Yg-439Yi are now 439Yh-439Yj. A second part has been added to 439Yi (now 439Yj):

(j)(ii) Show that $\mathbb{R}^{\mathfrak{c}}$ is not a Prokhorov space.

p 272 l 7 (441A) Parts 441Ac and 441Ad have been moved to 4A5Cc-4A5Cd. In their place is a new part 441Ac:

(c) If a group G acts on a set X and a measure μ on X is G-invariant, then $\int f(a\bullet x)\mu(dx)$ is defined and equal to $\int f\,d\mu$ whenever f is a virtually measurable $[-\infty,\infty]$-valued function defined on a conegligible subset of X and $\int f\,d\mu$ is defined in $[-\infty,\infty]$.

p 272 l 19 (proof of 441B): the argument makes better sense if we switch things round, and say

Next, the inequality $\le$ in the hypotheses is an insignificant refinement; since we must also have

$$\mu U=\mu(a^{-1}\bullet a\bullet U)\le\mu(a\bullet U)$$

in (a),

$$\mu K=\mu(a^{-1}\bullet a\bullet K)\le\mu(a\bullet K)$$

in (b), we always have equality here.

(J.Grahl.)

p 273 l 20 (part (c-iii) of the proof of 441C): for '$\lceil A:U\rceil\le\lceil b^{-1}\bullet b\bullet A:U\rceil=\lceil A:U\rceil$' read '$\lceil A:U\rceil=\lceil b^{-1}\bullet b\bullet A:U\rceil\le\lceil b\bullet A:U\rceil$'.

p 273 l 42 (part (f) of the proof of 441C): for '$[0,\lceil A:U_0\rceil]$' read '$[0,\lceil A:V_0\rceil]$'.

p 274 l 33 (proof of 441E): for '$uv^{-1}\in G\setminus M$' read '$u^{-1}v\in G\setminus M$'.

p 277 l 28 New exercises have been added to 441X:

(c) Let $r\ge 1$ be an integer, and $X=[0,1[^r$. Let G be the set of $r\times r$ matrices with integer coefficients and determinant ±1, and for $A\in G$, $x\in X$ say that $A\bullet x=\begin{pmatrix}<\eta_1>\\ \ldots\\ <\eta_r>\end{pmatrix}$ where

[5] Later editions only.

$\begin{pmatrix}\eta_1\\ \ldots\\ \eta_r\end{pmatrix} = Ax$ and $<\alpha>$ is the fractional part of α for each $\alpha \in \mathbb{R}$. (i) Show that $\bullet$ is an action of G on X, and that Lebesgue measure on X is G-invariant. (ii) Show that if X is given the compact Hausdorff topology corresponding to the bijection $\alpha \mapsto (\cos 2\pi\alpha, \sin 2\pi\alpha)$ from X to the unit circle in $\mathbb{R}^2$, and G is given its discrete topology, the action is continuous.

(m) Let X be a non-abelian Hausdorff topological group with a left Haar probability measure μ. Let λ be the quasi-Radon product measure on X^2. Show that $\lambda\{(x,y) : xy = yx\} \le \frac{5}{8}$.

(o) Let (X,ρ) be a locally compact metric space, and $\mathcal{C}$ the set of closed subsets of X with its Fell topology. Show that if G is the isometry group of X with its topology of pointwise convergence, then $(g,C) \mapsto g[C]$ is a continuous action of G on $\mathcal{C}$.

(p) Let (X,ρ) be a metric space and $\mathcal{K}$ the family of compact subsets of X with the topology induced by the Vietoris topology on the space of closed subsets of X. Show that if G is the isometry group of X with its topology of pointwise convergence, then $(g,K) \mapsto g[K]$ is a continuous action of G on $\mathcal{K}$.

(u) Let X be a set with its zero-one metric and G the group of permutations of X with its topology of pointwise convergence. Let $\mathcal{W} \subseteq \mathcal{P}(X^2)$ be the set of total orderings of X. Show that $\mathcal{W}$ is compact for the usual topology of $\mathcal{P}(X^2)$. For $g \in G$ and $W \in \mathcal{W}$ write $g{\bullet}W = \{(g(x), g(y)) : (x,y) \in W\}$; show that $\bullet$ is a continuous action of G on $\mathcal{W}$. Show that there is a unique G-invariant Radon probability measure μ on $\mathcal{W}$ such that $\mu\{W : (x_i, x_{i+1}) \in W$ for every $i < n\} = \frac{1}{(n+1)!}$ whenever $x_0, \ldots, x_n \in X$ are distinct.

((o) and (p) here were formerly in §476.) Other exercises have been rearranged: 441Xc-441Xk are now 441Xd-441Xl, 441Xl is now 441Xn, 441Xm-441Xp are now 441Xq-441Xt.

441Yb is now 441Yc 441Yc is now 441Yk 441Yd-441Yf are now 441Yh-441Yj, 441Yg is now 441Yl, 441Yh-441Yi are now 441Yd-441Ye, 441Yj-441Yk are now 441Ym-441Yn.

p 278 l 39 (Exercise 441Xm, now 441Xq) Add new part:

(v) Show that if (X,ρ) is complete then G is complete under its bilateral uniformity.

p 279 l 10 (441Y) Add new exercises:

(b) (M.Elekes & T.Keleti) Let X be a set, G a group acting on X, Σ a σ-algebra of subsets of X such that $g{\bullet}E \in \Sigma$ whenever $E \in \Sigma$ and $g \in G$ and $H \in \Sigma$. Suppose that μ is a measure, with domain the relative σ-algebra Σ_H, such that $\mu(g{\bullet}E) = \mu E$ whenever $E \in \Sigma_H$ and $g \in G$ are such that $g{\bullet}E \subseteq H$. (i) Show that $\sum_{n=0}^{\infty} \mu E_n = \sum_{n=0}^{\infty} \mu E'_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ and $\langle E'_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ_H for which there are sequences $\langle g_n\rangle_{n\in\mathbb{N}}$, $\langle g'_n\rangle_{n\in\mathbb{N}}$ in G such that $\langle g_n{\bullet}E_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n{\bullet}E'_n\rangle_{n\in\mathbb{N}}$ are partitions of the same subset of X. (ii) Show that there is a G-invariant measure with domain Σ which extends μ.

(f) Let $X = \mathrm{SO}(3)$ be the set of real 3×3 orthogonal matrices with determinant 1. Give X the metric corresponding to its embedding in 9-dimensional Euclidean space. (i) Show that X can be parametrized as the set of matrices

$$\phi\begin{pmatrix} z\\ \alpha\\ \theta\end{pmatrix} = \begin{pmatrix} z & -\cos\theta\sqrt{1-z^2} & \sin\theta\sqrt{1-z^2}\\ \cos\alpha\sqrt{1-z^2} & z\cos\alpha\cos\theta - \sin\alpha\sin\theta & -z\cos\alpha\sin\theta - \sin\alpha\cos\theta\\ \sin\alpha\sqrt{1-z^2} & z\sin\alpha\cos\theta + \cos\alpha\sin\theta & -z\sin\alpha\sin\theta + \cos\alpha\cos\theta\end{pmatrix}$$

with $z \in [-1,1]$, $\alpha \in [-\pi,\pi]$ and $\theta \in [-\pi,\pi]$. (ii) Show that if T is the 9×3 matrix which is the derivative of ϕ at $\begin{pmatrix} z\\ \alpha\\ \theta\end{pmatrix}$, then $T'T = \begin{pmatrix} \frac{2}{\sqrt{1-z^2}} & 0 & 0\\ 0 & 2 & 2z\\ 0 & 2z & 2\end{pmatrix}$ has constant determinant, so that if μ is Lebesgue measure on $[-1,1] \times [-\pi,\pi]^2$ then the image measure $\mu\phi^{-1}$ is a Haar measure on X. (iii) Show that if $A \in X$ corresponds to a rotation through an angle $\gamma(A) \in [0,\pi]$ then its trace $\mathrm{tr}(A)$ (that is, the sum of its diagonal entries) is $1 + 2\cos\gamma(A)$. (iv) Show that if X is given its Haar probability measure then $\cos\gamma(A)$ has expectation $-\frac{1}{2}$.

(g) Let $\mathbb{H}$ be the division ring of the **quaternions**, that is, $\mathbb{R}^4$ with its usual addition and with multiplication defined by the rule

$$(\xi_0,\xi_1,\xi_2,\xi_3)\times(\eta_0,\eta_1,\eta_2,\eta_3)=(\xi_0\eta_0-\xi_1\eta_1-\xi_2\eta_2-\xi_3\eta_3,\xi_0\eta_1+\xi_1\eta_0+\xi_2\eta_3-\xi_3\eta_2,$$
$$\xi_0\eta_2-\xi_1\eta_3+\xi_2\eta_0+\xi_4\eta_1,\xi_0\eta_3+\xi_1\eta_2-\xi_2\eta_1+\xi_3\eta_0).$$

For Lebesgue measurable $E\subseteq\mathbb{H}\setminus\{0\}$, set $\nu E=\int_E\frac{1}{\|x\|^4}dx$. Show that (i) $\|x\times y\|=\|x\|\|y\|$ for all x, $y\in\mathbb{H}$ (ii) $\mathbb{H}\setminus\{0\}$ is a group (iii) ν is a (two-sided) Haar measure on $\mathbb{H}\setminus\{0\}$.

(o) Let $\bullet_X$ be an action of a group G on a set X, μ a G-invariant measure on X, $(\mathfrak{A},\bar\mu)$ its measure algebra and $\bullet_{\mathfrak{A}}$ the induced action on $\mathfrak{A}$. Set $Z=X^G$; define $\phi:X\to Z$ by setting $\phi(x)=\langle g^{-1}\bullet x\rangle_{g\in G}$ for $x\in X$; let ν be the image measure $\mu\phi^{-1}$, and $(\mathfrak{B},\bar\nu)$ its measure algebra. Set $(g\bullet_Z z)(h)=z(g^{-1}h)$ for g, $h\in G$ and $z\in Z$; show that this defines a ν-invariant action on Z, with an induced action $\bullet_{\mathfrak{B}}$ on $\mathfrak{B}$. Show that $(\mathfrak{A},\bar\mu,\bullet_{\mathfrak{A}})$ and $(\mathfrak{B},\bar\nu,\bullet_{\mathfrak{B}})$ are isomorphic.

(p) Let X be a topological space, G a topological group and $\bullet$ a continuous action of G on X. Let M_{qR}^+ be the set of totally finite quasi-Radon measures on X. As in 4A5B-4A5C, write $a\bullet E=\{a\bullet x:x\in E\}$ for $a\in G$ and $E\subseteq X$, and $(a\bullet f)(x)=f(a^{-1}\bullet x)$ for $a\in G$, $x\in X$ and a real-valued function f defined at $a^{-1}\bullet x$. (i) Show that we have an action $\bullet$ of G on M_{qR}^+ defined by saying that $(a\bullet\nu)(E)=\nu(a^{-1}\bullet E)$ whenever $a\in G$, $\nu\in M_{\mathrm{qR}}^+$ and $E\subseteq X$ are such that ν measures $a^{-1}\bullet E$. (ii) Show that this action is continuous if we give M_{qR}^+ its narrow topology. (iii) Show that if $\nu\in M_{\mathrm{qR}}^+$, $f\in\mathcal{L}^1(\nu)$ is non-negative and $f\nu$ is the corresponding indefinite-integral measure, then $a\bullet(f\nu)$ is the indefinite-integral measure $(a\bullet f)(a\bullet\nu)$ for every $a\in G$.

(q) Let X be a set, G a group acting on X and μ a totally finite G-invariant measure on X with domain Σ. Suppose there is a probability measure ν on G, with domain T, such that $(a,x)\mapsto a^{-1}\bullet x:G\times X\to X$ is $(\mathrm{T}\widehat{\otimes}\Sigma,\Sigma)$-measurable and ν is invariant under the left action of G on itself. Let $u\in L^0(\mu)$ be such that $a\bullet u=u$ for every $a\in G$. Show that there is an $f\in\mathcal{L}^0(\mu)$ such that $f^\bullet=u$ and $a\bullet f=f$ for every $a\in G$.

(r) Let X be a topological space, G a compact Hausdorff group, $\bullet$ a continuous action of G on X, and μ a G-invariant quasi-Radon measure on X. Let $u\in L^0(\mu)$ be such that $a\bullet u=u$ for every $a\in G$. Show that there is an $f\in\mathcal{L}^0(\mu)$ such that $f^\bullet=u$ and $a\bullet f=f$ for every $a\in G$.

p 283 l 27 (442H) The sentence 'note that if $A\subseteq X$ is such that $A\cap K\in\Sigma$ for every compact set $K\subseteq X$, then $A\in\Sigma$; if $A\subseteq X$ and $A\cap K\in\mathcal{N}$ for every compact $K\subseteq X$, then $A\in\mathcal{N}$' assumes that we have a locally compact group, and should be deleted.

p 286 l 46 (442Y) Add new exercise:

(d) Let (X,ρ) be a metric space, and μ, ν two non-zero quasi-Radon measures on X such that $\mu B(x,\delta)=\mu B(y,\delta)$ and $\nu B(x,\delta)=\nu B(y,\delta)$ for all $\delta>0$ and x, $y\in X$. Show that μ is a multiple of ν.

p 287 l 2 (Problem 442Z): for 'autohomeomorphisms of Z' read 'autohomeomorphisms of X'.

p 290 l 17 Add a new part to Proposition 443D:

(c) $E^{-1}E$ and EE^{-1} are both neighbourhoods of the identity.

p 291 l 27 I have added another fragment to Theorem 443G, so that part (c) is now

(c) We have shift actions of X on L^0 defined by setting

$$a\bullet_l f^\bullet=(a\bullet_l f)^\bullet,\quad a\bullet_r f^\bullet=(a\bullet_r f)^\bullet,\quad a\bullet_c f^\bullet=(a\bullet_c f)^\bullet$$

for $a\in X$ and $f\in\mathcal{L}^0$. If $^\leftrightarrow$ is the reflection operator on L^0, we have

$$a\bullet_l\overset{\leftrightarrow}{u}=(a\bullet_r u)^\leftrightarrow,\quad a\bullet_c\overset{\leftrightarrow}{u}=(a\bullet_c u)^\leftrightarrow$$

for every $a\in X$ and $u\in L^0$.

(d) If we give L^0 its topology of convergence in measure these three actions, and also the reversal operator $^\leftrightarrow$, are continuous,

and parts (d)-(e) are now parts (e)-(f).

p 294 l 38 (part (b-i) of the proof of 443J: for '$\bigcup_{m\in\mathbb{N}}(E\cap H_{in}\setminus\bigcup_{n\in\mathbb{N}}F_{imn})$' read '$\bigcup_{m\in\mathbb{N}}(E\cap H_{im}\setminus\bigcup_{n\in\mathbb{N}}F_{imn})$'.

p 295 l 13 (443K, part (b) of the proof): for 'because X is locally compact' read 'because $\widehat{X}$ is locally compact'.

p 295 l 15 (443K, part (b) of the proof): for '$G\subseteq X$' read '$G\subseteq\widehat{X}$'.

p 295 l 21 (443K, part (b) of the proof): for '$\tilde{\nu}K=\nu K=0$' read '$\tilde{\mu}K=\nu K=0$'.

p 296 l 3 (part (a) of the proof of 443L): for '$\nu H=\mu F\ge\gamma$' read '$\nu H=\mu E\ge\gamma$'.

p 296 l 6 (part (a) of the proof of 443L): for '$E\cap f^{-1}[F]\in\operatorname{dom}\mu$' read '$E\cap\phi^{-1}[F]\in\operatorname{dom}\mu$'.

p 298 l 3 Add new result:

443O Proposition Let X be a topological group and μ a left Haar measure on X. Then the following are equiveridical:

(i) μ is not purely atomic;
(ii) μ is atomless;
(iii) there is a non-negligible nowhere dense subset of X;
(iv) μ is inner regular with respect to the nowhere dense sets;
(v) there is a conegligible meager subset of X;
(vi) there is a negligible comeager subset of X.

443O-443T are now 443P-443U.

p 298 l 29 (part (a-ii) of the proof of 443O, now 443P): for '$(Tf)(x_0)$' read '$(Tf)(\pi x_0)$'.

p 299 l 7 (part (a-vi) of the proof of 443O, now 443P): for '$h'(z)\in C_k(Z)$' read '$h'\in C_k(Z)$'.

p 301 l 28 A final step in part (c) of the proof of 443P, now 443Q, is missing:

(iv) If $\pi^{-1}[D]$ is Haar negligible, then, in (iii) above, we shall have $\int\nu_z(G\cap\pi^{-1}[D\cap L])\lambda(dz)=0$, so that $\lambda(D\cap L)=0$; as L is arbitrary, $\lambda D=0$, by 412Ib or 412Jc.

p 305 l 28 In part (b) of the proof of 443T, now 443U, every 'ϕ' should be 'π', as in the statement of the result.

p 305 l 42 (part (c) of the proof of 443T, now 443U): for 'if $x\in X$ and $H\subseteq X/Y_z$ is measured by λ' read 'if $x\in X$ and $H\subseteq X/Y_z$ is measured by λ''.

p 306 l 9 (part (c) of the proof of 443T, now 443U): for '$\nu_{xY_z}(G)=\nu(Y\cap x^{-1}G)$' read '$\nu_{xY_z}(G)=\nu(Y_z\cap x^{-1}G)$'.

p 306 l 37 The exercises to §443 have been re-arranged; 443Xa is now 443Xb, 443Xb is now 443Xa, 443Xd and 443Xe are now together as 443Xd, 443Xf is now 443Yb, 443Xg-443Xq are now 443Xf-443Xp, 443Xr is now 443Yk, 443Xs-443Xu are now 443Xq-443Xs, 443Xv-443Xz are now 443Xu-443Xy, 443Yb is now 443Yc, 443Yc is now 443Yh, 443Yd is now 443Yg, 443Ye is now 443Yl, 443Yf is now 443Ym, 443Yg is now 443Yi, 443Yh is now 443Yj, 443Yi is now 443Yn, 443Yj is now 443Yq, 443Yk is now 443Xt, 443Yl-443Yo are now 443Yr-443Yu.

p 307 l 33 Exercise 443Xk (now 443Xj) has been elaborated, and now reads

(j) Let X be a topological group carrying Haar measures. Show that X is totally bounded for its bilateral uniformity iff X is totally bounded for its right uniformity iff its Haar measures are totally finite.

p 307 l 1 Exercise 443Xb (now 443Xa) has been amended, and now reads

(a) Let X be a topological group and μ a left Haar measures on X. (i) Show that the quasi-Radon product measure $\lambda=\mu\times\mu$ on $X\times X$ is a left Haar measure for the product group operation on $X\times X$. (ii) Show that the maps $(x,y)\mapsto(y,x)$, $(x,y)\mapsto(x,xy)$, $(x,y)\mapsto(y^{-1}x,y)$ are automorphisms of the measure space $(X\times X,\lambda)$. (iii) Show that the map $(x,y)\mapsto(x^{-1},yx)$ is an automorphism of the measure space $(X\times X,\lambda)$ (iv) Show that the maps $(x,y)\mapsto(y^{-1},xy)$, $(x,y)\mapsto(yx,x^{-1})$, $(x,y)\mapsto(y^3x,x^{-1}y^{-2})$ are automorphisms of $(X\times X,\lambda)$.

p 308 l 3 Exercise 443Xo (now 443Xn) has been elaborated, and now reads

(n) In 443L, show that (i) $\phi[A]$ is Haar negligible in Z whenever A is Haar negligible in X (ii) $\Delta_X = \Delta_Z\phi$, where Δ_X, Δ_Z are the left modular functions of X, Z respectively (iii) $\phi[X]$ is dense in Z (iv) Z is unimodular iff X is unimodular.

p 308 l 28 Exercise 443Xu(ii) (now 443Xs(ii)) is wrong, and should read

In 443R, suppose that there is an open set $G \subseteq X$ such that GY has finite measure for the left Haar measures of X. Show that Z has an X-invariant Radon probability measure.

p 308 l 32 (part (i) of Exercise 443Xv, now 443Xu): for 'X/Y' read 'X/Y_1' (twice).

p 309 l 7 (443X) Add new exercises:

(e) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that it is isomorphic to the measure algebra of a topological group with a Haar measure iff it is localizable and quasi-homogeneous in the sense of 374G-374H.

(z) Show that 443G is equally valid if we take functions to be complex-valued rather than real-valued, and work with $L^p_{\mathbb{C}}$ rather than L^p.

p 310 l 12 Exercise 443Yo (now 443Yu) is wrong as written; we need an extra hypothesis '$\{x : \alpha < \Delta(x) < \beta\}$ is non-empty'.

p 309 l 15 (443Y) Add new exercises:

(d) Let X be a compact Hausdorff topological group and $\mathfrak{A}$ its Haar measure algebra. Let Y be a subgroup of X; for $y \in Y$, define $\hat y \in \operatorname{Aut}\mathfrak{A}$ by setting $\hat y(a) = y{\bullet_l}a$ for $a \in \mathfrak{A}$. Show that $\{\hat y : y \in Y\}$ is ergodic iff Y is dense in X.

(e) Let $\mathfrak{A}$ be a Boolean algebra, G a group, and $\bullet$ an action of G on $\mathfrak{A}$ such that $a \mapsto g{\bullet}a$ is a Boolean automorphism for every $g \in G$. (i) Show that we have a corresponding action of G on $L^\infty = L^\infty(\mathfrak{A})$ defined by saying that, for every $g \in G$, $g{\bullet}\chi a = \chi(g{\bullet}a)$ for $a \in \mathfrak{A}$ and $u \mapsto g{\bullet}u$ is a positive linear operator on L^∞. (ii) Show that if $\mathfrak{A}$ is Dedekind σ-complete, this action on L^∞ extends to an action on $L^0 = L^0(\mathfrak{A})$ defined by saying that $[\![g{\bullet}u > \alpha]\!] = g{\bullet}[\![u > \alpha]\!]$ for $g \in G$, $u \in L^0$ and $\alpha \in \mathbb{R}$.

(f) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra, G a topological group, and $\bullet$ a continuous action of G on $\mathfrak{A}$ (giving $\mathfrak{A}$ its measure-algebra topology) such that $a \mapsto g{\bullet}a$ is a measure-preserving Boolean automorphism for every $g \in G$. Show that the corresponding action of G on $L^0(\mathfrak{A})$, as defined in 443Ye, is continuous, and induces continuous actions of G on $L^p(\mathfrak{A})$ for $1 \le p < \infty$.

(o) Let X be a locally compact Hausdorff topological group which is not discrete (as topological space). (i) Show that there is a Haar negligible zero set containing the identity of X. (ii) Show that if X is σ-compact, it has a Haar negligible compact normal subgroup Y which is a zero set in X, so that X/Y is metrizable. (iii) Show that there is a Haar negligible set $A \subseteq X$ such that AA is not Haar measurable.

(p) Find a non-discrete locally compact Hausdorff topological group X such that if Y is a normal subgroup of X which is a zero set in X then Y is open.

p 310 l 42 (Notes to §443): for '443Yd' read '443Yg' (now 443Yi).

p 315 l 8 (part (c-iii) of the proof of 444F): for '$\mathfrak{A}^{2n+1}$' read '$\mathfrak{A}^{2n+2}$'.

p 315 l 15 (part (c-iv) of the proof of 444F): for '$\lim_{n\to\infty}(a{\bullet}E)^\bullet$' read '$\lim_{n\to\infty}(a{\bullet}E_n)^\bullet$'. Similarly on line 19.

p 315 l 39 (part (d-ii) of the proof of 444F): for '$\sum_{n=0}^\infty 2^{-n-1}\chi c_i$' read '$\sum_{n=0}^\infty 2^{-n-1}\chi c_n$'.

p 318 l 20 In part (b-ii) of the proof of 444K, I write 'Similarly, $(f * \nu)(x)$ is defined in $\mathbb{R}$ for every x'; this need not be so, since the left modular function Δ may well be unbounded. However, since

$$\iint \Delta(y^{-1})f(xy^{-1})\chi G(x)\nu(dy)\mu(dx) = (f\mu * \nu)(G) < \infty$$

for every open set G of finite μ-measure, $(f * \nu)(x)$ is defined μ-a.e., which is good enough.

p 326 l 24 444Sb has been rewritten, and is now

(b) In particular, $*: L^1 \times L^1 \to L^1$ is associative; L^1 is a Banach algebra. It is commutative if X is abelian.

(c) Let $\mathcal{B}$ be the Borel σ-algebra of X and M_τ the Banach algebra of signed τ-additive Borel measures on X, as in 444E. If, for $f \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $E \in \mathcal{B}$, we write $(f\mu{\restriction}\mathcal{B})(E) = \int_E f d\mu$, then $f\mu{\restriction}\mathcal{B} \in M_\tau$. We have an operator $T : L^1 \to M_\tau$ defined by setting $T(f^\bullet) = f\mu{\restriction}\mathcal{B}$ for $f \in \mathcal{L}^1$. T is a norm-preserving Riesz homomorphism, and is an embedding of L^1 as a subalgebra of M_τ.

p 326 l 37 (part (a) of the proof of 444T): for '$M^p\eta^p$' read '$M^p\eta$'; and again in line 13 of the next page.

p 329 l 25 (part (b-ii) of the proof of 444V): for 'G' read 'X', and again on the next line.

p 330 l 14 The exercises for §444 have been rearranged: 444Xm-444Xu are now 444Xn-444Xv, 444Yf-444Yi are now 444Yg-444Yj, 444Yj-444Yn are now 444Yl-444Yp.

p 334 l 39 (444X) Add new exercises:

(m) Let X be a topological group carrying Haar measures, $E \subseteq X$ a Haar negligible set and ν a σ-finite quasi-Radon measure on X. Show that $\nu(xE) = \nu(Ex) = 0$ for Haar-a.e. $x \in X$.

(w) (i) Let X_1, X_2 be topological groups with totally finite quasi-Radon measures λ_i, ν_i on X_i for each i. Let $\lambda = \lambda_1 \times \lambda_2$, $\nu = \nu_1 \times \nu_2$ be the quasi-Radon product measure on the topological group $X = X_1 \times X_2$. Show that $\lambda * \nu = (\lambda_1 * \nu_1) \times (\lambda_2 * \nu_2)$. (ii) Let $\langle X_i \rangle_{i \in I}$ be a family of topological groups, and λ_i, ν_i quasi-Radon probability measures on X_i for each i. Let $\lambda = \prod_{i \in I} \lambda_i$, $\nu = \prod_{i \in I} \nu_i$ be the quasi-Radon product measures on the topological group $\prod_{i \in I} X_i$. Show that $\lambda * \nu = \prod_{i \in I} \lambda_i * \nu_i$.

(x) Show that 444C, 444O, 444P, 444Qb and 444R-444U remain valid if we work with complex-valued, rather than real-valued, functions, and with $\mathcal{L}^p_{\mathbb{C}}$ and $L^p_{\mathbb{C}}$ rather than $\mathcal{L}^p$ and L^p.

(y) Let X be a topological group with a left Haar measure μ and left modular function Δ. Write $\boldsymbol{\Delta} \in L^0 = L^0(\mu)$ for the equivalence class of the function Δ. For $u \in L^0$ write u^* for $\overleftrightarrow{u} \times \overleftrightarrow{\boldsymbol{\Delta}}$. Show that (i) $(u^*)^* = u$ for every $u \in L^0$ (ii) $u \mapsto u^* : L^0 \to L^0$ is a Riesz space automorphism (iii) $u^* \in L^1$ for every $u \in L^1 = L^1(\mu)$ (iv) $u \mapsto u^* : L^1 \to L^1$ is an L-space automorphism (v) $u^* * v^* = (v * u)^*$ for all u, $v \in L^1$.

p 332 l 18 (Exercise 444Ye): for 'if Y is another Polish group' read 'if X and Y are Polish groups'.

p 332 l 29 (Exercise 444Yg, now 444Yh): part (ii) is wrong, and should be replaced by

(ii) Show that $\|T\nu\| \le \|\nu\|$ for every $\nu \in M_\tau^+$. (iii) Give an example in which $\|T\nu\| < \|\nu\|$.

p 331 l 43 (444X) Add new exercise:

(m) Let X be a topological group carrying Haar measures, $E \subseteq X$ a Haar negligible set and ν a σ-finite quasi-Radon measure on X. Show that $\nu(xE) = \nu(Ex) = 0$ for Haar-a.e. $x \in X$.

p 333 l 25 (444Y) Add new exercises:

(f) Suppose that the continuum hypothesis is true. Let ν be Cantor measure on $\mathbb{R}$ (256Hc). Show that there is a set $A \subseteq \mathbb{R}$ such that $\nu(x + A) = 0$ for every $x \in \mathbb{R}$, but A is not Haar negligible.

(k) Let X be a topological group with a left Haar measure μ and left modular function Δ. (i) Suppose that $f \in \mathcal{L}^0(\mu)$. Show that the following are equiveridical: (α) $f(yx) = \Delta(y)f(xy)$ for $(\mu \times \mu)$-almost every x, $y \in X$; (β) $(a{\bullet_c}f)^\bullet = \Delta(a^{-1})f^\bullet$ for every $a \in X$. (ii) Show that in this case $f(x) = 0$ for almost every x such that $\Delta(x) \ne 1$. (iii) Suppose that $f \in \mathcal{L}^1(\mu)$. Show that the following are equiveridical: (α) $f(yx) = \Delta(y)f(xy)$ for $(\mu \times \mu)$-almost every x, $y \in X$; (β) $(f * g)^\bullet = (g * f)^\bullet$ for every $g \in \mathcal{L}^1(\mu)$.

(q) Let X be a topological group and M_{qR}^+ the set of totally finite quasi-Radon measures on X. For $\nu \in M_{\mathrm{qR}}^+$, define $\overleftrightarrow{\nu} \in M_{\mathrm{qR}}^+$ by saying that $\overleftrightarrow{\nu}(E) = \nu E^{-1}$ whenever $E \subseteq X$ and ν measures E^{-1}. (i) Show that if λ, $\nu \in M_{\mathrm{qR}}^+$ then $\overleftrightarrow{\lambda} * \overleftrightarrow{\nu} = (\nu * \lambda)^{\leftrightarrow}$. (ii) Taking $\bullet_l$, $\bullet_r$ to be the left and right actions of X on itself, and defining corresponding actions of X on M_{qR}^+ as in 441Yp, show that $a{\bullet_l}(\lambda * \nu) = (a{\bullet_l}\lambda) * \nu$ and $a{\bullet_r}(\lambda * \nu) = \lambda * (a{\bullet_r}\nu)$ for λ, $\nu \in M_{\mathrm{qR}}^+$ and $a \in X$.

p 341 l 5 (part (d) of the proof of 445I): for '$f_k = y_k^{-1}\bullet_l f$' read '$f_k = y_k\bullet_l f$'.

p 341 l 15 (part (d) of the proof of 445I): for '$|\theta(x) - \phi(x)| \le \epsilon$' read '$|\theta(x) - \chi(x)| \le \epsilon$'.

p 355 l 36 (445Y) Add new exercise:

(m) Let μ be Lebesgue measure on $[0, \infty[$. (i) For f, $g \in \mathcal{L}^1(\mu)$ define $(f*g)(x) = \int_0^x f(y)g(x-y)\mu(dy)$ whenever the integral is defined. Show that $f * g \in \mathcal{L}^1(\mu)$. (ii) Show that we can define a bilinear operator $*$ on $L^1(\mu)$ by setting $f^\bullet * g^\bullet = (f * g)^\bullet$ for f, $g \in \mathcal{L}^1(\mu)$, and that under this multiplication $L^1(\mu)$ is a Banach algebra. (iii) Show that if $\phi : L^1(\mu) \to \mathbb{R}$ is a multiplicative linear operator then there is some $s \ge 0$ such that $\phi(f^\bullet) = \int_0^\infty f(x)e^{-sx}\mu(dx)$ for every $f \in \mathcal{L}^1(\mu)$.

p 359 l 6 Paragraphs 446E-446G are now 446G, 446E and 446F.

p 360 l 27 (part (d) of the proof of 446G, now 446F): for '$\{n : A_n^{k(n)} \cap W_0w'' \ne \emptyset\}$' read '$\{n : A_n^{k(n)} \cap w''W_0 \ne \emptyset\}$'.

p 360 l 36 (part (e) of the proof of 446G, now 446F): for 'W_0w''' read '$w''W_0$'.

p 364 l 7 In the final clause of the statement of Lemma 446K, replace '$xy \in D_n(U)$' by '$xy \in D_n(V)$'.

p 366 l 10 (part (g) of the proof of 446K): for '$\int\bigl(f_n(y_n^{j(n)}x) - f(x)\bigr)g_n(x)dx$' read '$\int\bigl(f_n(y_n^{j(n)}x) - f_n(x)\bigr)g_n(x)dx$'.

p 368 l 36 (part (a) of the proof of 446N): for '$\|T\|_{HS} = \sum_{i=1}^r \sum_{j=1}^r \tau_{ij}^2$' read '$\|T\|_{HS} = \sqrt{\sum_{i=1}^r \sum_{j=1}^r \tau_{ij}^2}$'.

p 369 l 40 (part (a-ii) of the proof of 446O): for 'Z is compact and Hausdorff' read 'U is compact and Hausdorff'.

p 370 l 6 (part (a-iii) of the proof of 446O): for 'Y' read 'Y_0'.

p 371 l 2 Exercise 446Xa has been dropped. 446Xb-446Xc are now 446Xa-446Xb.

p 371 l 14 (446Y) Add new exercise:

(c) Let κ be an infinite cardinal, and $(\mathfrak{B}_\kappa, \bar\nu_\kappa)$ the measure algebra of the usual measure on $\{0,1\}^\kappa$. Give the group $\mathrm{Aut}_{\bar\nu_\kappa}(\mathfrak{B}_\kappa)$ of measure-preserving automorphisms of $\mathfrak{B}_\kappa$ its topology of pointwise convergence. Let X be a compact Hausdorff topological group of weight at most κ. Show that there is a continuous injective homomorphism from X to $\mathrm{Aut}_{\bar\nu_\kappa}(\mathfrak{B}_\kappa)$.

p 376 l 30 (part (e) of the proof of 447F): add 'Let $\tilde\lambda$ be the invariant Radon measure on X/Y derived from μ and μ_Y as in 447Ea.'

p 378 l 18 (part (i) of the proof of 447F): for 'μ_Y measures $Y \cap x^{-1}E$' read 'μ_Y measures $Y \cap x^{-1}(E_1 \setminus E_2)$'.

p 379 l 23 (part (c) of the proof of 447G): for 'g_{En} is Σ_Y-measurable $\ldots$ $g_{En}(xy) = g_{En}(x)$ whenever $x \in X$, $y \in Y$' read 'g_{En} is Σ_{Y_n}-measurable $\ldots$ $g_{En}(xy) = g_{En}(x)$ whenever $x \in X$, $y \in Y_n$'.

p 379 l 27 (part (c) of the proof of 447G): for '$\langle g_{En}\rangle_{n\in\mathbb{N}} \to g_E$' read '$\langle g_{En}\rangle_{n\in\mathbb{N}} \to \chi E'$.

p 385 l 15 (part (a) of the proof of 448F): I think we need an argument to show that c^* belongs to $\mathfrak{C}$, e.g.,

If $\phi \in G$, then

$$\phi c^* = \sup\{\phi\pi a : \pi \in G\} \subseteq c^*$$

because ϕ is order-continuous and $\phi\pi \in G$ for every $\pi \in G$. Similarly $\phi^{-1}c^* \subseteq c^*$ and $c^* \subseteq \phi c^*$. Thus $\phi c^* = c^*$; as ϕ is arbitrary, $c^* \in \mathfrak{C}$.

Later (line 19) we should take the infimum in $\mathfrak{C}$ rather than in $\mathfrak{A}$.

p 386 l 29 (Lemma 448J) To support the proof of 448M, we need the result corresponding to 395Ke, as follows:

(e) If $c \in \mathfrak{C}$ is such that $a \cap c$ is a relative atom over $\mathfrak{C}$, then $c \subseteq [\![\lceil b : a\rceil - \lfloor b : a\rfloor = 0]\!]$.

448Je is now 448Jf.

p 389 l 18 (part (e) of the proof of 448O): for '$\langle\tilde\pi b_n\rangle_{n\in\mathbb{N}}$' read '$\langle\tilde\pi_n b_n\rangle_{n\in\mathbb{N}}$'.

p 389 l 26 (part (e) of the proof of 448O): the definition of the operator ψ is defective, because there is no guarantee that $\{d^*\}\cup\{a_n\setminus d^*:n\in\mathbb{N}\}$ has supremum 1. To remedy this, we can amend the definition of d^*, replacing it by

$$d^*=\operatorname{upr}(d\cup d',\mathfrak{C})$$

where $d'=1\setminus\sup_{n\in\mathbb{N}}a_n$; $d'\in\mathcal{I}$ because $\sup_{n\in\mathbb{N}}a_n^\bullet=1$ in $\mathfrak{B}$.

p 392 l 30 I have added a version of Mackey's theorem on the realization of group actions, as follows:

448Q Lemma Let (X,Σ,μ) be a σ-finite measure space with countable Maharam type. Write $L^0(\Sigma)$ for the set of Σ-measurable functions from X to $\mathbb{R}$. Then there is a function $T:L^0(\mu)\to L^0(\Sigma)$ such that

(i) $u=(Tu)^\bullet$ for every $u\in L^0$,

(ii) $(u,x)\mapsto(Tu)(x):L^0\times X\to[-\infty,\infty]$ is $(\mathcal{B}\widehat{\otimes}\Sigma)$-measurable,

where $\mathcal{B}$ is the Borel σ-algebra of L^0 with the topology of convergence in measure.

448R Lemma Let (X,Σ,μ) be a σ-finite measure space with countable Maharam type.

(a) $L^0=L^0(\mu)$, with its topology of convergence in measure, is a Polish space.

(b) Let $\mathfrak{A}$ be the measure algebra of μ, and $\mathfrak{A}^f$ the set $\{a:a\in\mathfrak{A},\ \bar\mu<\infty\}$. Then the Borel σ-algebra $\mathcal{B}=\mathcal{B}(L^0)$ is the σ-algebra of subsets of L^0 generated by sets of the form $\{u:\bar\mu(a\cap[\![u\in F]\!])>\alpha\}$, where $a\in\mathfrak{A}^f$, $F\subseteq\mathbb{R}$ is Borel, and $\alpha\in\mathbb{R}$.

448S Mackey's theorem Let G be a locally compact Polish group, (X,Σ) a standard Borel space and μ a σ-finite measure with domain Σ. Let $(\mathfrak{A},\bar\mu)$ be the measure algebra of μ with its measure-algebra topology. Let $\circ$ be a Borel measurable action of G on $\mathfrak{A}$ such that $a\mapsto g\circ a$ is a Boolean automorphism for every $g\in G$. Then we have a $(\mathcal{B}(G)\widehat{\otimes}\Sigma,\Sigma)$-measurable action $\bullet$ of G on X such that

$$g\circ E^\bullet=(g\bullet E)^\bullet$$

for every $g\in G$ and $E\in\Sigma$, writing $g\bullet E$ for $\{g\bullet x:x\in E\}$ as usual.

448T Corollary Let G be a σ-compact locally compact Hausdorff group, X a Polish space, μ a σ-finite Borel measure on X, and $(\mathfrak{A},\bar\mu)$ the measure algebra of μ, with its measure-algebra topology. Let $\circ$ be a continuous action of G on $\mathfrak{A}$ such that $a\mapsto g\circ a$ is a Boolean automorphism for every $g\in G$. Then we have a Borel measurable action $\bullet$ of G on X such that

$$g\circ E^\bullet=(g\bullet E)^\bullet$$

for every $g\in G$ and $E\in\mathcal{B}(X)$, writing $g\bullet E$ for $\{g\bullet x:x\in E\}$ as usual.

p 392 l 25 (Exercise 448Xc): for '$\Sigma=\mathfrak{C}$' read '$\Sigma=\mathcal{B}$'.

p 392 l 29 (448X) The former exercise 448Xe has been corrected and strengthened and moved to 448Y; it now reads

(c) Let (X,Σ) be a standard Borel space, Y any set, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of subsets of T. Let $\theta:\Sigma\to L^\infty(\mathrm{T}/\mathcal{J})$ be a non-negative, sequentially order-continuous additive function. Show that there is a non-negative, sequentially order-continuous additive function $\phi:\Sigma\to L^\infty(\mathrm{T})$ such that $\theta E=(\phi E)^\bullet$ for every $E\in\Sigma$.

Other exercises have been renamed: 448Xf-448Xk are now 448Xe-448Xj.

p 394 l 12 (part (a-ii) of the proof of 449B) At several points in this section, $\bullet$ and $\bullet_l$ have got confused. Here, '$\|f-a^{-1}\bullet_l f\|_\infty$' should be '$\|f-a^{-1}\bullet f\|_\infty$'.

p 395 l 2 (part (b-ii) of the proof of 449B): for '$\|a^{-1}\bullet f-a^{-1}\bullet f\|$' read '$\|a^{-1}\bullet f-b^{-1}\bullet f\|$'.

p 397 l 9 (part (c) of the statement of 449D): for '$a\bullet_l\hat b=\widehat{ab}$' read '$a\bullet\hat b=\widehat{ab}$'.

p 397 l 45 (part (c) of the proof of 449D): for '$(a\bullet_l\hat b)(f)$' read '$(a\bullet\hat b)(f)$'. Similarly, in the next line, '$a\bullet_l\hat b=\widehat{ab}$' should be '$a\bullet\hat b=\widehat{ab}$', as in the statement of the result.

p 398 l 10 (part (d) of the proof of 449D): for '$\phi(a\bullet_l\hat{b})$' read '$\phi(a\bullet\hat{b})$'.

p 398 l 11 (part (d) of the proof of 449D): for '$\{\hat{b} : y \in G\}$' read '$\{\hat{b} : b \in G\}$'. (T.D.Austin.)

p 398 l 31 (part (ii)⇒(i) of the proof of 449E): for '$\mu(a\bullet_l\phi^{-1}[F])$' read '$\mu(a\bullet\phi^{-1}[F])$'.

p 399 l 1 (Corollary 449F) Add new parts:

(a)(ii) A dense subgroup of an amenable topological group is amenable.

(b) Let G be a topological group. Suppose that for every sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of neighbourhoods of the identity e of G there is a normal subgroup H of G such that $H \subseteq \bigcap_{n\in\mathbb{N}} V_n$ and G/H is amenable. Then G is amenable.

p 400 l 22 Add a note:

449I Notation If G is any locally compact Hausdorff group, Σ_G will be the algebra of Haar measurable subsets of G and $\mathcal{N}_G$ the ideal of Haar negligible subsets of G, and $\mathcal{B}_G$ the Borel σ-algebra of G.

Theorem 449I has become 449J. Condition (vi) from the list has been dropped, and is now Exercise 449Xk. In their place, four other equiveridical conditions have been added:

(iii) there is an additive functional $\phi : \Sigma_G \to [0,1]$ such that $\phi G = 1$, $\phi(aE) = \phi E$ for every $E \in \Sigma$ and $a \in G$, and $\phi E = 0$ for every $E \in \mathcal{N}_G$;

(iv) there is a finitely additive functional $\phi : \mathcal{B}_G \to [0,1]$ such that $\phi G = 1$, $\phi(aE) = \phi E$ for every $E \in \mathcal{B}$ and $a \in G$, and $\phi E = 0$ for every Haar negligible $E \in \mathcal{B}_G$;

(ix) for any finite set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(L\triangle aL) \le \epsilon\mu L$ for every $a \in K$;

(xii) there is a $q \in [0,\infty[$ such that for any finite set $I \subseteq G$ $\epsilon > 0$, there is a $u \in L^q(\mu)$ such that $\|u\|_q = 1$ and $\|u - a\bullet_l u\|_q \le \epsilon$ for every $a \in I$.

The former (ix) has been sharpened to

(x) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a symmetric compact neighbourhood L of the identity e in G such that $\mu(L\triangle aL) \le \epsilon\mu L$ for every $a \in K$.

The former conditions (iv) and (x) are now (vi) and (xi).

p 400 l 27 (part (ii) of the statement of 449I, now 449J): for 'every $u \in C_b(X)$' read 'every $f \in C_b(X)$'.

p 406 l 23 Add new results:

449K Proposition Let G be an amenable locally compact Hausdorff group, and H a subgroup of G. Then H is amenable.

449J-449M are now 449L-449O.

p 409 l 8 The exercises to §449 have been greatly changed. 449Xg is now 449F(a-ii); 449Xl and 449Xo have been dropped; 449Xm, 449Xn and 449Xq have been absorbed into the new conditions (x), (xii) and (iv) of 449J (formerly 449I).

p 409 l 13 (Exercise 449Xi, now 449Xg): for '$\sup_{x\in X}\sum_{i=0}^n f(x_i) - f(a_i\bullet x_i) \ge 0$' read '$\sup_{x\in X}\sum_{i=0}^n f_i(x) - f_i(a_i\bullet x) \ge 0$'.

New exercises are

(e) Prove 449Cg directly from 441C, without mentioning Haar measure.

(k) Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Show that G is amenable iff for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_k(G)^+$ such that $\int h\,d\mu = 1$ and such that $|\int f(ax)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \le \epsilon$ for every $a \in I$ and $f \in J$.

(m) Let G be a locally compact Hausdorff group and $\mathcal{B}\mathfrak{a}_G$ its Baire σ-algebra. Show that G is amenable iff there is a non-zero finitely additive $\phi : \mathcal{B}\mathfrak{a}_G \to [0,1]$ such that $\phi(aE) = \nu E$ for every $a \in G$ and $E \in \mathcal{B}\mathfrak{a}_G$.

(n) A **symmetric Følner sequence** in a group G is a sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ of non-empty finite symmetric subsets of G such that $\lim_{n\to\infty}\frac{\#(L_n\triangle aL_n)}{\#(L_n)} = 0$ for every $a \in G$. Show that a group G has a symmetric Følner sequence iff it is countable and amenable when given its discrete topology.

p 410 l 26 Exercise 449Yb has been incorporated into the new 449Xj.

p 410 l 36 (449Y) Add new exercises:

(e) Let G be a group with a symmetric Følner sequence $\langle L_n\rangle_{n\in\mathbb{N}}$ (449Xn), and $\bullet$ an action of G on a reflexive Banach space U such that $u\mapsto a\bullet u$ is a linear operator of norm at most 1 for every $a\in G$. For $n\in\mathbb{N}$ set $T_nu=\frac{1}{\#(L_n)}\sum_{a\in L_n}a^{-1}\bullet u$ for $u\in U$. Show that for every $u\in U$ the sequence $\langle T_nu\rangle_{n\in\mathbb{N}}$ is norm-convergent to a $v\in U$ such that $a\bullet v=v$ for every $a\in G$.

(g) Let G be a group and $\bullet$ an action of G on a set X. Let Σ be an algebra of subsets of X such that $g\bullet E\in\Sigma$ for every $E\in\Sigma$ and $g\in G$, and H a member of Σ; write Σ_H for $\{E:E\in\Sigma, E\subseteq H\}$. Let $\nu:\Sigma_H\to[0,\infty]$ be a functional which is additive in the sense that $\nu\emptyset=0$ and $\nu(E\cup F)=\nu E+\nu F$ whenever E, $F\in\Sigma_H$ are disjoint, and locally G-invariant in the sense that $g\bullet E\in\Sigma$ and $\nu(g\bullet E)=\nu E$ whenever $E\in\Sigma_H$, $g\in G$ and $g\bullet E\subseteq H$. Show that there is an extension of ν to a G-invariant additive functional $\tilde\nu:\Sigma\to[0,\infty]$.

Other exercises have been rearranged: 449Xc is now 449Xh, 449Xd-449Xe are now 449Xc-449Xd, 449Xh is now 449Xi, 449Xi is now 449Xg, 449Xk is now 449Xl, 449Xp is now 449Yf, 449Xr is now 449Xp, 449Xs is now 449Xo, 449Xt-449Xu are now 449Xq-449Xr, 449Xv is now 449Xn, 449Ye-449Yg are now 449Yh-449Yj, 449Yh is now 449Yb.

p 411 l 7 Parts (i) and (ii) of Exercise 449Yg (now 449Yj) are wrong, and should read

(j)(i) Show that there is a partition (A,B,C,D) of F_2 such that $aA=A\cup B\cup C$ and $bB=A\cup B\cup D$. (ii) Let S_2 be the unit sphere in $\mathbb{R}^3$. Show that if S, T are the matrices of 449Yi, there is a partition (A,B,C,D,E) of S^2 such that E is countable, $S[A]=A\cup B\cup C$ and $T[B]=A\cup B\cup D$.

p 414 l 14 Definition 451Aa should be rephrased: '$\mathcal{K}$ is a compact class if $\bigcap\mathcal{K}'\neq\emptyset$ whenever $\mathcal{K}'\subseteq\mathcal{K}$ is a non-empty family with the finite intersection property'. (H.König.)

p 418 l 35 Proposition 451K has been elaborated, as follows.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X, and Σ_i a σ-algebra of subsets of X_i for each i. Let λ be a perfect totally finite measure with domain $\Sigma=\widehat{\bigotimes}_{i\in I}\Sigma_i$. Set $\pi_J(x)=x\restriction J$ for $x\in X$ and $J\subseteq I$.

(a) Let $\mathcal{K}$ be the set $\{V:V\subseteq X,\ \pi_J[V]\in\widehat{\bigotimes}_{i\in J}\Sigma_i$ for every $J\subseteq I\}$. Then λ is inner regular with respect to $\mathcal{K}$.

(b) Let $\hat\lambda$ be the completion of λ.

(i) For any $J\subseteq I$, the completion of the image measure $\lambda\pi_J^{-1}$ on $\prod_{i\in J}X_i$ is the image measure $\hat\lambda\pi_J^{-1}$.

(ii) If W is measured by $\hat\lambda$ and W is determined by coordinates in $J\subseteq I$, then there is a $V\in\Sigma$ such that $V\subseteq W$, V is determined by coordinates in J and $W\setminus V$ is λ-negligible.

p 419 l 34 Add new paragraph:

451L Proposition Let (X,Σ,μ) be a strictly localizable space. Let us say that a family $\mathcal{E}\subseteq\Sigma$ is **μ-centered** if $\mu(\bigcap\mathcal{E}_0)>0$ for every non-empty finite $\mathcal{E}_0\subseteq\mathcal{E}$.

(i) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is compact.

(ii) Suppose that μ is inner regular with respect to some $\mathcal{K}\subseteq\Sigma$ such that every countable μ-centered subset of $\mathcal{K}$ has non-empty intersection. Then μ is countably compact.

Consequently 451L-451U are now 451M-451V.

p 420 l 18 (part (b) of the statement of 451O, now 451P): for '$\nu=\mu\phi^{-1}$' read '$\nu=\mu f^{-1}$'.

p 420 l 41 (part (a) of the proof of 451P, now 451Q): for 'assume that μ is totally finite' read 'assume that μ is complete and totally finite'.

In part (e) of the proof, take $\hat\mu_G$ to be the completion of μ_G, so that $\hat\mu_G$ is complete, totally finite and compact, and apply (a)-(d) to $\hat\mu_G$ rather than μ_G.

p 421 l 22 (part (c) of the proof of 451P, now 451Q): for '$K_\xi \in \mathcal{K}_\xi$' read '$K_\xi \in \mathcal{K}$'. (R.M.Solovay.)

p 423 l 25 (part (d) of the proof of 451T, now 451U): for '$\{z : z \in W', h_1(z) < 1\}$' read '$\{z : z \in W', h(z) < 1\}$'.

p 423 l 28 (part (d) of the proof of 451T, now 451U): for '$h\pi_J[X \cap \pi_J^{-1}[W]]$' read '$h\pi_J[X \cap \pi_J^{-1}[W'']]$'.

p 426 l 11 (451X) Add new exercises:

(f) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{K}$ a family of subsets of X such that whenever $E \in \Sigma$ and $\mu E > \gamma$ there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu_* K \geq \gamma$. (i) Show that if $\mathcal{K}$ is a compact class then μ is a compact measure. (ii) Show that if $\mathcal{K}$ is a countably compact class then μ is a countably compact measure.

(j) Let $A \subseteq [0, 1]$ be a set with outer Lebesgue measure 1 and inner measure 0. Show that there is a Borel measure λ on $A \times [0, 1]$ such that λ is not inner regular with respect to sets which have Borel measurable projections on the factor spaces.

(s) In the 'third construction' of 439A, show that ν is countably compact.

p l (451X) There is an error in Exercise 451Xi; part (iv) should be deleted. R.O.Cavalcante
Other exercises have been renamed; 451Xf-451Xh are now 451Xg-451Xi, 451Xi-451Xp are now 451Xk-451Xr.

p 426 l 27 (Exercise 451Ye): for 'metacompact' read 'hereditarily metacompact'.

p 427 l 10 (451Y) Add new exercises:

(g) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Suppose that Y is a separable metrizable space and Z is a metrizable space, and that $f : X \times Y \to Z$ is a function such that $x \mapsto f(x, y)$ is measurable for every $y \in Y$, and $y \mapsto f(x, y)$ is continuous for every $x \in X$. Show that μ is inner regular with respect to $\{F : F \subseteq X, f \restriction F \times Y \text{ is continuous}\}$.

(s) Let X be a set, and $\langle \mu_i \rangle_{i \in I}$ a family of weakly α-favourable measures on X with sum μ. Show that if μ is semi-finite, it is weakly α-favourable.

(t) Let X and Y be locally compact Hausdorff groups and $\phi : X \to Y$ a group homomorphism which is Haar measurable in the sense of 411L, that is, $\phi^{-1}[H]$ is Haar measurable for every open $H \subseteq Y$. Show that ϕ is continuous.

(u) Let μ be a measure on $\mathbb{R}$ which is quasi-Radon for the right-facing Sorgenfrey topology. Show that μ is weakly α-favourable.

Other exercises have been renamed: 451Yg-451Yq are now 451Yh-451Yr.

p 428 l 11 452A-452D have been rephrased in slightly more general form, as follows:

452A Lemma Let (Y, T, ν) be a measure space, X a set, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on X. Let $\mathcal{A}$ be the family of subsets A of X such that $\theta E = \int \mu_y E \, \nu(dy)$ is defined in $\mathbb{R}$. Suppose that $X \in \mathcal{A}$.

(a) $\mathcal{A}$ is a Dynkin class.

(b) If Σ is any σ-subalgebra of $\mathcal{A}$ then $\mu = \theta \restriction \Sigma$ is a measure on X.

(c) Suppose now that every μ_y is complete. If, in (b), $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\hat{\Sigma} \subseteq \mathcal{A}$ and $\hat{\mu} = \theta \restriction \hat{\Sigma}$.

452B Theorem (a) Let X be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on X. Let $\mathcal{E}$ be a family of subsets of X, closed under finite intersections, containing X, such that $\int \mu_y E \, \nu(dy)$ is defined in $\mathbb{R}$ for every $E \in \mathcal{E}$.

(i) If Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, we have a totally finite measure μ on X, with domain Σ, given by the formula $\mu E = \int \mu_y E \, \nu(dy)$ for every $E \in \Sigma$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E \, \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Z be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on Z. Let $\mathcal{H}$ be a family of subsets of Z, closed under finite intersections, containing Z, such that $\int \mu_y H \, \nu(dy)$ is defined in $\mathbb{R}$ for every $H \in \mathcal{H}$.

(i) If Υ is the σ-algebra of subsets of Z generated by $\mathcal{H}$, we have a totally finite measure μ on $Y \times Z$, with domain $\mathrm{T} \widehat{\otimes} \Upsilon$, defined by setting $\mu E = \int \mu_y E[\{y\}] \nu(dy)$ for every $E \in \mathrm{T} \widehat{\otimes} \Upsilon$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

452C Theorem (a) Let Y be a topological space, ν a τ-additive topological measure on Y, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological measures on X such that $\int \mu_y X \nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$.

(i) We can define a τ-additive Borel measure μ on X by writing $\mu E = \int \mu_y E\, \nu(dy)$ for every Borel set $E \subseteq X$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E\, \nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Y be a topological space, ν a τ-additive topological measure on Y, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological measures on Z such that $\int \mu_y Z \nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$.

(i) We can define a τ-additive Borel measure μ on $Y \times Z$ by writing $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}]\nu(dy)$ is defined and equal to $\hat{\mu}E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

452D Theorem (a) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight family of Radon measures on X such that $\int \mu_y X\, \nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$. Then we have a totally finite Radon measure $\tilde{\mu}$ on X such that $\tilde{\mu}E = \int \mu_y E\, \nu(dy)$ whenever $\tilde{\mu}$ measures E.

(b) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight family of Radon measures on Z such that $\int \mu_y Z\, \nu(dy)$ is defined and finite. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$. Then we have a totally finite Radon measure $\tilde{\mu}$ on $X = Y \times Z$ such that $\tilde{\mu}E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde{\mu}$ measures E.

p 428 l 35 (part (a) of the proof of 452B): for '$\mathcal{A} \subseteq \mathcal{P}(Y \times Z)$' read '$\mathcal{A} \subseteq \mathcal{P}X$'.

p 429 l 31 (part (b) of the proof of 452C): for '$\mu'_{y'} G[\{y'\}] > \gamma$' read '$\mu'_{y'} U > \gamma$'.

p 430 l 5 (452E, definition of 'consistent disintegration': for '$\mu_y f^{-1}[F] = 1$ for ν-almost every $y \in Y$' read '$\mu_y f^{-1}[F] = 1$ for ν-almost every $y \in F$'.

p 430 l 10 (452E): delete the clause 'Clearly this disintegration is consistent with the function $(y, z) \mapsto y : X \to Y$'.

p 430 l 28 (part (a) of the proof of 452F): for '$\lim_{n\to\infty} 2^{-n} \sum_{k=1}^{\infty} \mu E_{nk}$' read '$\lim_{n\to\infty} 2^{-n} \sum_{k=1}^{4^n} \mu E_{nk}$'.

p 432 l 24 (part (e) of the proof of 452H): for 'μ_y for $y \in Y \setminus F_0$' read 'μ'_y for $y \in Y \setminus F_0$'.

p 435 l 34 (part (b) of the proof of 452M): the prescription 'just apply the argument of (a) to the completion of ν' calls for a bit of explanation, since μ is declared to have domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and the domain of its completion will not be of this form. The following is a possible patch:

If the original measure ν is not complete, let $\hat{\mu}$ and $\hat{\nu}$ be the completions of μ and ν, and $\hat{\mathrm{T}}$ the domain of $\hat{\nu}$. The projection onto Y is inverse-measure-preserving for μ and ν, so is inverse-measure-preserving for $\hat{\mu}$ and $\hat{\nu}$ (234Ba[6]), and $\hat{\mu}$ measures every member of $\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$; set $\mu' = \hat{\mu}\restriction\hat{\mathrm{T}}\widehat{\otimes}\Upsilon$. Next, the marginal measure of μ' on Z is still λ (since both must have domain Υ). So we can apply (a) to μ' to get the result.

p 439 l 2 Add new result:

452T Theorem Let X be a locally compact Hausdorff space, G a compact Hausdorff topological group and $\bullet$ a continuous action of G on X. Suppose that μ is a G-invariant Radon

[6] Formerly 235Hc.

probability measure on X. For $x \in X$, write $f(x)$ for the corresponding orbit $\{a\bullet x : a \in G\}$ of the action. Let $Y = f[X]$ be the set of orbits, with the topology $\{W : W \subseteq Y,\ f^{-1}[W]$ is open in $X\}$. Write ν for the image measure μf^{-1} on Y.

(a) Y is locally compact and Hausdorff, and ν is a Radon probability measure.

(b) For each $\boldsymbol{y} \in Y$, there is a unique G-invariant Radon probability $\mu_{\boldsymbol{y}}$ on X such that $\mu_{\boldsymbol{y}}(\boldsymbol{y}) = 1$.

(c) $\langle \mu_{\boldsymbol{y}} \rangle_{\boldsymbol{y}\in Y}$ is a disintegration of μ over ν, strongly consistent with f.

p 439 l 19 (Exercise 452Xd): for '$Y \times Z$' and '$\tilde\mu E = \int \mu_y E[\{y\}]\nu(dy)$' read '$X$' and '$\tilde\mu E = \int \mu_y E\nu(dy)$'.

p 440 l 11 (Exercise 452Xp): for '$\hat\mu E = \int \mu_y E^{-1}[\{y\}]\nu(dy)$' read '$\hat\mu E = \int \mu_y E[\{y\}]\nu(dy)$'.

p 440 l 21 (452X) Add new exercises:

(s) Let (X_0, Σ_0, μ_0) and (X_1, Σ_1, μ_1) be σ-finite measure spaces. For each i, let $(Y_i, \mathrm{T}_i, \nu_i)$ be a measure space and $\langle \mu_y^{(i)} \rangle_{y\in Y_i}$ a disintegration of μ_i over ν_i. Show that $\langle \mu_{y_0}^{(0)} \times \mu_{y_1}^{(1)} \rangle_{(y_0,y_1)\in Y_0\times Y_1}$ is a disintegration of $\mu_0 \times \mu_1$ over $\nu_0 \times \nu_1$, where each product here is a c.l.d. product measure.

(t) In 452M, suppose that Z is a metrizable space and $\mathcal{K}$ is the family of compact subsets of Z, and let $(Y, \hat{\mathrm{T}}, \hat\nu)$ be the completion of (Y, T, ν). Show that $y \mapsto \mu_y$ is a $\hat{\mathrm{T}}$-measurable function from Y to the set of Radon probability measures on Z with its narrow topology.

(u) $SU(r)$, for $r \ge 2$, is the set of $r\times r$ matrices T with complex coefficients such that $\det T = 1$ and $TT^* = I$, where T^* is the complex conjugate of the transpose of T. (i) Show that under the natural action $(T, u) \mapsto Tu : SU(r) \times \mathbb{C}^r \to \mathbb{C}^r$ the orbits are the spheres $\{u : u\,\textbf{.}\,\bar u = \gamma\}$, for $\gamma > 0$, together with $\{0\}$. (ii) Show that if a Borel set $C \subseteq \mathbb{C}^r$ is such that $\gamma C \subseteq C$ for every $\gamma > 0$, and μ_0, μ_1 are two $SU(r)$-invariant Radon probability measures on $\mathbb{C}^r$ such that $\mu_0\{0\} = \mu_1\{0\}$, than $\mu_0 C = \mu_1 C$.

(v) Let $\langle X_i \rangle_{i\in I}$ be a family of compact Hausdorff spaces with product X, and μ a completion regular topological measure on X. Show that all the marginal measures of μ are completion regular.

p 440 l 26 (Exercise 452Ya): for '$\mu E = \sup\{\sum_{i=0}^n \int_{F_i} \mu(E[\{y\}]\cap H_i)\nu(dy)$' read '$\mu E = \sup\{\sum_{i=0}^n \int_{F_i} \mu_y(E[\{y\}]\cap H_i)\nu(dy)$'.

p 440 l 39 (452Y) Add new exercises:

(f) Let X be a K-analytic Hausdorff space and μ a totally finite measure which is inner regular with respect to the closed sets. Show that μ is countably compact.

(g) Let X be a set, and $\langle \mu_i \rangle_{i\in I}$ a family of countably compact measures on X with sum μ. Show that if μ is semi-finite, it is countably compact.

(h) Let X be a locally compact Hausdorff space, G a compact Hausdorff group, and $\bullet$ a continuous action of G on X. Let H be another group and $\circ$ a continuous action of H on X which commutes with $\bullet$ in the sense that $g\bullet(h\circ x) = h\circ(g\bullet x)$ for all $g \in G$, $h \in H$ and $x \in X$. (i) Show that $((g, h), x) \to g\bullet(h\circ x) : (G \times H) \times X \to X$ is a continuous action of the product group $G \times H$ on X. (ii) Suppose that the action in (i) is transitive. Show that if μ, μ' are G-invariant Radon probability measures on X and $E \subseteq X$ is a Borel set such that $h\circ E = E$ for every $h \in H$, then $\mu E = \mu' E$.

p 442 l 22 (part (a) of the proof of 453C): for '$T(\chi G_\alpha) = \alpha T(\chi(\phi G_\alpha)) \ge \alpha T(\chi G_\alpha)$' read '$T(\chi G_\alpha) = \alpha\chi(\phi G_\alpha) \ge \alpha\chi G_\alpha$'.

p 444 l 12 (statement of 453Fa): delete 'consisting of measurable sets'. The proof now reads

Let $\mathcal{E}$ be a countable network for $\mathfrak{T}$, and $\phi : \Sigma \to \Sigma$ a lifting. For each $E \in \mathcal{E}$, let $\hat E$ be a measurable envelope of E (213L). Then

$$\begin{aligned}\bigcup_{G\in\mathfrak{T}} G \setminus \phi G &= \bigcup_{G\in\mathfrak{T}, E\in\mathcal{E}, E\subseteq G} E \setminus \phi G \\ &\subseteq \bigcup_{G\in\mathfrak{T}, E\in\mathcal{E}, E\subseteq G} \hat E \setminus \phi \hat E\end{aligned}$$

(because if $E \subseteq G \in \mathfrak{T}$, then $G \in \Sigma$, so $\mu(\hat E \setminus G) = 0$ and $\phi\hat E \subseteq \phi G$)

$$\subseteq \bigcup_{E\in\mathcal{E}} \hat{E} \setminus \phi\hat{E}$$

is negligible, so ϕ is almost strong.

p 445 l 15 (statement of 453I): delete 'consisting of measurable sets'.

The corresponding modification of the proof follows that described for 453F above. In the definition of Q (line 28), we need to take a measurable envelope $\hat{E}$ of each $E \in \mathcal{E}_\xi$, and set

$$Q = \bigcup\{\pi_\xi^{-1}[\hat{E}] \setminus \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) : E \in \mathcal{E}_\xi\}.$$

In line 9 of p 446, the formula becomes

$$x \in \pi_\xi^{-1}[\hat{E}] \setminus Q \subseteq \phi'_\xi(\pi_\xi^{-1}[\hat{E}]) \setminus Q \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[\hat{E}]) \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[G]).$$

p 445 l 37 (part (b-ii) of the proof of 453I): for '$\psi_x : \Sigma \to \{0,1\}$' read '$\psi_x : \Lambda \to \{0,1\}$'.

p 447 l 1 Part (a) of the proof of 453K refers to an extinct version of §452, and should be rewritten:

(i) Suppose first that X is compact, μ is a probability measure and that f is continuous.

Turn back to the proofs of 452H-452I. In part (a) of the proof of 452H, suppose that the lifting $\theta : \mathfrak{B} \to \mathrm{T}$ corresponds to an almost strong lifting $\phi : \mathrm{T} \to \mathrm{T}$ (see 341Ba). Set $B = \bigcup_{H\in\mathfrak{G}} H \setminus \phi H$, so that B is negligible. Take $\mathcal{K}$ to be the family of compact subsets of X. Then all the μ_y, as constructed in 452H, will be Radon probability measures. For every y, $f^{-1}[\{y\}]$ is a closed set, so is necessarily measured by μ_y. But also it is μ_y-conegligible for every $y \in Y \setminus B$. **P** Let $K \subseteq X \setminus f^{-1}[\{y\}]$ be a compact set. Then $f[K]$ is a compact set not containing y. Because Y is Hausdorff, there is an open set H containing y such that $\overline{H} \cap f[K] = \emptyset$ (4A2F(h-i)). Now

$$y \in H \setminus B \subseteq \phi H \subseteq \phi\overline{H}.$$

Let E be the compact set $f^{-1}[\overline{H}]$. Taking $T : L^\infty(\mu) \to L^\infty(\nu)$ as in part (a) of the proof of 452I, $T(\chi E^\bullet) = \chi\overline{H}^\bullet$, so

$$\psi_y E = (ST(\chi E^\bullet))(y) = (S(\chi\overline{H}^\bullet))(y) = (\chi(\phi\overline{H}))(y) = 1.$$

Because $E \in \mathcal{K}$, $\mu_y E \ge \psi_y E$; since we always have $\mu_y X = 1$, E is μ_y-conegligible. But $K \cap E = \emptyset$, so $\mu_y K = 0$. As K is arbitrary, $\mu_y(X \setminus f^{-1}[\{y\}]) = 0$. **Q**

Thus $\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ whenever $y \in Y \setminus B$, which is almost always.

(ii) The result for totally finite μ and ν and continuous f follows at once.

(J.Miller.)

p l (proof of 453Ma): for

$$\text{`} \iff F \subseteq g_\phi^{-1}[F^*] \text{ for every closed set } F \subseteq X$$
$$\iff F \subseteq \phi F \text{ for every closed set } F \subseteq X\text{'}$$

read

$$\text{`} \iff g_\phi^{-1}[F^*] \subseteq F \text{ for every closed set } F \subseteq X$$
$$\iff \phi F \subseteq F \text{ for every closed set } F \subseteq X.\text{'}$$

(M.R.Burke.)

p 453 l 17 Exercise 453Xe ought to read

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space which has an almost strong lifting. Show that any non-zero indefinite-integral measure over μ has an almost strong lifting.

p 456 l 21 (part (c) of the proof of 454D): for 'take $K_{ki} \in \mathcal{K}_i \cap \mathrm{T}_i$' read 'take a closed set $K_{ki} \in \mathrm{T}_i$'.

p 456 l 38 (statement of 454F): for '$\lambda_0 : \Sigma \to [0,\infty[$' read '$\lambda_0 : \bigotimes_{i\in I} \Sigma_i \to [0,\infty[$'.

p 457 l 38 (part (b) of the proof of 454H): for

$$`\int_{Z_n}\int_{X_n} f(z,\xi_{n+1})\nu_z(d\xi_{n+1})\tilde\mu_n(dz)'$$

read

$$`\int_{Z_n}\int_{X_n} f(z,\xi_n)\nu_z(d\xi_n)\tilde\mu_n(dz)'.$$

p 458 l 3 (part (c) of the proof of 454H): for '$\lambda_0\tilde\pi_n^{-1}[W]=\mu_nW$' read '$\lambda_0\tilde\pi_n^{-1}[W]=\tilde\mu_nW$'; two lines later, read '$\tilde\mu_{n+1}$' for 'μ_{n+1}'

p 458 l 41 (part (b) of the proof of 454J): for '$\mu\{\omega:\phi(x)(i_r)\le\alpha_r$ for $r\le n\}$' read '$\mu\{\omega:\phi(\omega)(i_r)\le\alpha_r$ for $r\le n\}$'.

p 459 l 9 Add new results:

454L Theorem Let (Ω,Σ,μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω, with distribution ν on $\mathbb{R}^I$. Then $\langle X_i\rangle_{i\in I}$ is independent iff ν is the c.l.d. product of the marginal measures on $\mathbb{R}$.

454M Proposition Let I be a set, and suppose that for each finite $J\subseteq I$ we are given a Radon probability measure ν_J on $\mathbb{R}^J$ such that whenever K is a finite subset of I and $J\subseteq K$, then the canonical projection from $\mathbb{R}^K$ to $\mathbb{R}^J$ is inverse-measure-preserving. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that the canonical projection from $\mathbb{R}^I$ to $\mathbb{R}^J$ is inverse-measure-preserving for every finite $J\subseteq I$.

454N Proposition Let Ω be a Hausdorff space, μ and ν two Radon probability measures on Ω, and $\langle X_i\rangle_{i\in I}$ a family of continuous functions separating the points of Ω. If μ and ν give $\langle X_i\rangle_{i\in I}$ the same distribution, they are equal.

454L-454P are now 454O-454S.

p 461 l 39 Add new results:

454T Convergence of distributions (a) Let I be a set. Write M for the set of distributions on $\mathbb{R}^I$. I will say that the **vague topology** on M is the topology generated by the functionals $\nu\mapsto\int f d\nu$ as f runs over the space $C_b(\mathbb{R}^I)$ of bounded continuous real-valued functions on $\mathbb{R}^I$.

(b) The vague topology on M is Hausdorff.

454U Theorem Let (Ω,Σ,μ) be a probability space, and I a set. Let M be the set of distributions on $\mathbb{R}^I$; for a family $\boldsymbol{X}=\langle X_i\rangle_{i\in I}$ of real-valued random variables on Ω, let $\nu_{\boldsymbol{X}}$ be its distribution. Then the function $\boldsymbol{X}\mapsto\nu_{\boldsymbol{X}}:\mathcal{L}^0(\mu)^I\to M$ is continuous for the product topology on $\mathcal{L}^0(\mu)^I$ corresponding to the topology of convergence in measure on $\mathcal{L}^0(\mu)$ and the vague topology on M.

454V Distributions of processes in $L^0(\mathfrak{A})^I$ (a)(i) If $\mathfrak{A}$ is a Dedekind σ-complete Boolean algebra, I is a set, and $u\in L^0(\mathfrak{A})^I$, we have a sequentially order-continuous Boolean homomorphism $E\mapsto[\![u\in E]\!]:\mathcal{B}\mathfrak{a}(\mathbb{R}^I)\to\mathfrak{A}$ defined by saying that

$$[\![u\in\{x:x\in\mathbb{R}^I,\,x(i)\le\alpha\}]\!]=[\![u(i)\le\alpha]\!]$$

whenever $i\in I$ and $\alpha\in\mathbb{R}$.

(ii) If $h:\mathbb{R}^I\to\mathbb{R}$ is a Baire measurable function, there is a function $\bar h:L^0(\mathfrak{A})^I\to L^0(\mathfrak{A})$ defined by saying that $[\![\bar h(u)\in E]\!]=[\![u\in h^{-1}[E]]\!]$ for every Borel set $E\subseteq\mathbb{R}$.

(b) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra, I is a set and $u\in L^0(\mathfrak{A})^I$. Then there is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x:x\in\mathbb{R}^I,\,x(i)\in E_i\text{ for every }i\in J\}=\bar\mu(\inf_{i\in J}[\![u(i)\in E_i]\!])$$

whenever $J\subseteq I$ is finite and $E_i\subseteq\mathbb{R}$ is a Borel set for every $i\in J$.

(c) I call ν the **(joint) distribution** of u.

(d) Let $(\mathfrak{A},\bar\mu)$ and $(\mathfrak{A}',\bar\mu')$ be probability algebras, and $u\in L^0(\mathfrak{A})^I$, $u'\in L^0(\mathfrak{A}')^I$ families with the same distribution. Suppose that $\langle h_j\rangle_{j\in J}$ is a family of Baire measurable functions from $\mathbb{R}^I$ to $\mathbb{R}$. Then $\langle\bar h_j(u)\rangle_{j\in J}$ and $\langle\bar h_j(u')\rangle_{j\in J}$ have the same distribution.

(e) If $(\mathfrak{A},\bar\mu)$ is a probability algebra, I a set, and we write ν_u for the distribution of $u\in L^0(\mathfrak{A})^I$, $u\mapsto\nu_u$ is continuous for the product topology on $L^0(\mathfrak{A})^I$ corresponding to the topology of convergence in measure on $L^0(\mathfrak{A})$ and the vague topology on the space of distributions on $\mathbb{R}^I$.

p 462 l 24 Exercise 454Xh is now 454Xe, so that 454Xe-454Xg are now 454Xf-454Xh.

p 462 l 28 Exercise 454Xj is wrong as written, and has been replaced by

(j) Let $\langle X_i\rangle_{i\in I}$ be an independent family of normal random variables. Show that its distribution is a quasi-Radon measure on $\mathbb{R}^I$.

p 462 l 39 (454X) Add new exercise:

(k) Give an example of a metrizable space Ω with a continuous injective function $X:\Omega\to[0,1]$ and two different quasi-Radon probability measures μ, ν on Ω giving the same distribution to the random variable X.

454Xk-454Xm are now 454Xl-454Xn.

p 464 l 1 Section 455 (Markov processes and Brownian motion) has been very substantially expanded, with some material moved to the new §477, and the notes here include only corrections of errors; for the new version see `http://www1.essex.ac.uk/maths/people/fremlin/cont45.htm`.

p 464 l 12 In the statement of Theorem 455A the word 'unique' is put in the wrong sentence. The measure μ is *not* uniquely defined by the formula '$\mu_t=\int\nu_x^{(s,t)}\mu_s(dx)$ when $s<t$'; we need to specify the formulae for $\int fd\tilde\mu_J$, as given afterwards, to fix the joint distributions.

p 468 l 16 (Theorem 455D, now 477B) I have decided that it is more convenient to regard Wiener measure as a Radon probability measure defined on $C([0,\infty[)_0=\{\omega:\omega\in C([0,\infty[),\,\omega(0)=0\}$.

p 469 l 5 (part (c) of the proof of 455D, now 477B): The step '$\Pr(\sup_{t\in D\cap[q,q']}|X_t-X_q|\ge\epsilon)=\lim_{n\to\infty}\Pr(\sup_{t\in I_n}|X_t-X_q|\ge\epsilon$' is illegitimate; it should be

$$\text{'}\Pr(\sup_{t\in D\cap[q,q']}|X_t-X_q|>\epsilon)=\lim_{n\to\infty}\Pr(\sup_{t\in I_n}|X_t-X_q|>\epsilon\text{'}.$$

The simplest fix is to change all the '$\ge\epsilon$' and '$\ge 3\epsilon$', from the beginning of (c) to the end of (e), into '$>\epsilon$' and '$>3\epsilon$'.

p 470 l 2 (part (f) of the proof of 455D, now 477B): for '$C([0,1[)$' read '$C([0,\infty[)$'. (J.G.)

p 470 l 27 (Exercise 455Xe, now 455Xj): for '$\frac{1}{\gamma}\lim_{\gamma\downarrow 0}\lambda_\gamma\,[1,\infty[$' read '$\lim_{\gamma\downarrow 0}\frac{1}{\gamma}\lambda_\gamma\,[1,\infty[$'.

p 470 l 37 (Exercise 455Ya): for '$g(t)+f(s-t)$ if $s\ge t$' read '$g(t)+f(s-t)$ if $s>t$'.

p 475 l 8 In §456 some new results (456C, 456E) have been added, and other material has been rearranged, so that 456C-456L are now 456G-456P, 456M is now 456F and 456N is now 456D.

p 475 l 8 Exercise 456Xa has been moved into the main argument, as follows.

456C Theorem Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Then the following are equiveridical:

(i) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^I$.

(ii) There are a (real) Hilbert space U and a family $\langle u_i\rangle_{i\in I}$ in U such that $(u_i|u_j)=\sigma_{ij}$ for all i, $j\in I$.

(iii) For every finite $J\subseteq I$, $\langle\sigma_{ij}\rangle_{i,j\in J}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^J$.

(iv) $\sigma_{ij}=\sigma_{ji}$ for all i, $j\in I$ and $\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}\ge 0$ whenever $J\subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J}\in\mathbb{R}^J$.

p 475 l 8 Add new result:

456E Proposition (a) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process. Then $\langle X_i\rangle_{i\in I}$ is independent iff $\mathbb{E}(X_i \times X_j) = 0$ for all distinct i, $j \in I$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a centered Gaussian process on a complete probability space (Ω, Σ, μ), and $\mathcal{J}$ a disjoint family of subsets of I; for $J \in \mathcal{J}$ let Σ_J be the σ-algebra of subsets of Ω generated by $\{X_i^{-1}[F] : i \in J, F \subseteq \mathbb{R}$ is Borel$\}$. Suppose that $\mathbb{E}(X_i \times X_j) = 0$ whenever J, J' are distinct members of $\mathcal{J}$, $i \in J$ and $j \in J'$. Then $\langle \Sigma_J\rangle_{J\in\mathcal{J}}$ is independent.

p 476 l 25 (part (b) of the proof of 456D, now 456H): for '$x \in X$' read '$x \in \mathbb{R}^I$'.

p 479 l 5 (statement of 456H, now 456L): for '$F \subseteq \mathbb{R}^I$' read '$F \in \operatorname{dom}\mu$'.

p 479 l 32 (part (c) of the proof of 456H, now 456L): for '$y \in V$ whenever $x \in \mathbb{R}^I$ and ...' read '$x \in V$ whenever $x \in \mathbb{R}^I$ and ...'.

p 481 l 31 (part (a-v) of the proof of 456I, now 456M): for '$B(x, 2\delta)$' read '$B(z, 2\delta)$' (twice).

p 482 l 31 (part (c-i) of the proof of 456I, now 456M): for '$x \mapsto x{\restriction}L : \mathbb{R}^I \to \mathbb{R}^L$' read '$x \mapsto x{\restriction}L \times n : \mathbb{R}^{I\times n} \to \mathbb{R}^{L\times n}$'.

p 485 l 11 (part (f) of the proof of 456J, now 456N): for '$\phi_n(i) \in K \cap \overline{\phi_n[I_{nVk}]}\}$' read '$\phi_n(i) \in \overline{\phi_n[I_{nVk}]}\}$'.

p 487 l 2 Add new result:

456Q Proposition Let I be a set and R the set of functions $\sigma : I \times I \to \mathbb{R}$ which are symmetric and positive semi-definite in the sense of 456C; give R the subspace topology induced by the usual topology of $\mathbb{R}^{I\times I}$. Let $P_{\mathrm{qR}}(\mathbb{R}^I)$ be the space of quasi-Radon probability measures on $\mathbb{R}^I$ with its narrow topology. For $\sigma \in R$, let μ_σ be the centered Gaussian distribution on $\mathbb{R}^I$ with covariance matrix σ, and $\tilde\mu_\sigma$ the quasi-Radon measure extending μ_σ. Then the function $\sigma \mapsto \tilde\mu_\sigma : R \to P_{\mathrm{qR}}(\mathbb{R}^I)$ is continuous.

p 487 l 3 Exercise 456Xa has been moved to 456C.

p 487 l 14 (Exercise 456Xd, now 456Xe): part (i) is wrong, and should be deleted.

p 487 l 33 Exercise 456Xg has been deleted.

p 487 l 35 Exercise 456Xh is now 456Ea.

p 487 l 36 (456X) Add new exercises:

(a) Let I be a set, and $\mu_G^{(I)}$ the standard centered Gaussian distribution on $\mathbb{R}^I$. (i) Show that if $y \in \ell^1(I)$ then $\int \sum_{i\in I} |y(i)x(i)|\mu_G^{(I)}(dx) = \frac{2}{\sqrt{2\pi}}\|y\|_1$. (ii) Show that if $y \in \ell^2(I)$ then $\int \sum_{i\in I} |y(i)x(i)|^2\mu_G^{(I)}(dx) = \|y\|_2^2$.

(c) Let G be a group, and $h : G \to \mathbb{R}$ a real positive definite function. (i) Show that we have a centered Gaussian distribution μ on $\mathbb{R}^G$ with covariance matrix $\langle h(a^{-1}b)\rangle_{a,b\in G}$. (ii) Show that μ is invariant under the left shift action $\bullet_l$ of G on $\mathbb{R}^G$.

(h) Let I be a set, and let H be a Hilbert space with orthonormal basis $\langle e_i\rangle_{i\in I}$. For $i \in I$, $x \in \mathbb{R}^I$ set $f_i(x) = x(i)$. Show that there is a bounded linear operator $T : H \to L^1(\mu_G^{(I)})$ such that $Te_i = f_i^\bullet$ for every $i \in I$, and that $\|Tu\|_1 = \frac{2}{\sqrt{2\pi}}\|u\|_2$ for every $u \in H$.

Other exercises have been re-named: 456Xc is now 456Xd, 456Xe-456Xf are now 456Xf-456Xg.

p 488 l 7 Exercise 456Ye has been moved to the new 477Yb.

p 488 l 39 In parts (iv) and (v) of the statement of 457A, replace 'whenever $i_0, \dots, i_n \in I$' by 'whenever $i_0, \dots, i_n \in I$ are distinct'. In part (v), for '$\sum_{i=0}^{\infty}$' read '$\sum_{i=0}^{n}$'.

p 489 l 37 (part (b-iii) of the statement of 457C): for '$\nu E_1 \le \nu E_2$' read '$\nu_1 E_1 \le \nu_2 E_2$'.

p 493 l 40 (proof of 457G): for '$E_n \subseteq \bigcup_{i\in J} \Sigma_i$' read '$E_n \in \bigcup_{i\in J} \Sigma_i$'.

p 494 l 34 (part (b) of the proof of 457I): for '$\nu\pi_i^{-1}[E] = \mu E$' read '$\nu\pi_i^{-1}[E] = \mu_L E$'.

p 495 l 26 Add new paragraphs:

457K Definition Let (X,ρ) be a metric space. For quasi-Radon probability measures μ, ν on X, set

$$\rho_{\mathrm{KR}}(\mu,\nu)=\sup\{|\int u\,d\mu-\int u\,d\nu| : u:X\to\mathbb{R} \text{ is bounded and 1-Lipschitz}\}.$$

457L Theorem Let (X,ρ) be a metric space and P_{qR} the set of quasi-Radon probability measures on X; define ρ_{KR} as in 457K.

(a) For all μ, ν and λ in P_{qR},

$$\rho_{\mathrm{KR}}(\mu,\nu)=\rho_{\mathrm{KR}}(\nu,\mu),\quad \rho_{\mathrm{KR}}(\mu,\lambda)\le\rho_{\mathrm{KR}}(\mu,\nu)+\rho_{\mathrm{KR}}(\nu,\lambda),$$

$$\rho_{\mathrm{KR}}(\mu,\nu)=0 \text{ iff } \mu=\nu.$$

(b) If μ, $\nu\in P_{\mathrm{qR}}$, then $\rho_{\mathrm{KR}}(\mu,\nu)=\inf_{\lambda\in Q(\mu,\nu)}\int\rho(x,y)\lambda(d(x,y))$, where $Q(\mu,\nu)$ is the set of quasi-Radon probability measures on $X\times X$ with marginal measures μ and ν.

(c) In (b), if μ and ν are Radon measures, $Q(\mu,\nu)$ is included in $P_{\mathrm{R}}(X\times X)$, the space of Radon probability measures on $X\times X$, and is compact for the narrow topology on $P_{\mathrm{R}}(X\times X)$; and there is a $\lambda\in Q(\mu,\nu)$ such that $\rho_{\mathrm{KR}}(\mu,\nu)=\int\rho(x,y)\lambda(d(x,y))$.

(d) If ρ is bounded, then ρ_{KR} is a metric on P_{qR} inducing the narrow topology.

457M Theorem Let X be a Hausdorff space and $\langle\nu_i\rangle_{i\in I}$ a non-empty finite family of locally finite measures on X all inner regular with respect to the closed sets.

(a) For $A\subseteq X\times[0,\infty[$, set

$$c(A)=\inf\{\sum_{i\in I}\int h_i d\nu_i : h_i:X\to[0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i\in I,$$
$$\alpha\le\sum_{i\in I}h_i(x) \text{ whenever } (x,\alpha)\in A\}.$$

(i) c is a Choquet capacity.

(ii) For every $A\subseteq X\times[0,\infty[$, the infimum in the definition of $c(A)$ is attained.

(b) Let $f:X\to[0,\infty[$ be a function such that $\{x:f(x)\ge\alpha\}$ is K-analytic for every $\alpha>0$. Then

$$\inf\{\sum_{i\in I}\int h_i d\nu_i : h_i:X\to[0,\infty] \text{ is } \operatorname{dom}\nu_i\text{-measurable for each } i\in I,\ f\le\sum_{i\in I}h_i\}$$
$$=\sup\{\int f\,d\mu : \mu \text{ is a Radon measure on } X \text{ and } \mu\le\nu_i \text{ for every } i\in I\}.$$

p 495 l 38 (Exercise 457Xd): for '$\{0,\{i\},X\setminus\{i\},X\}$' read '$\{\emptyset,\{i\},X\setminus\{i\},X\}$'.

p 496 l 9 (457X) Add new exercises:

(g) Suppose that $\mathfrak{A}$ is a Boolean algebra, $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $I\subseteq\mathfrak{A}$ a finite set; let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ generated by $I\cup\mathfrak{B}$ and $\nu:\mathfrak{C}\to[0,\infty[$ a finitely additive functional. (i) Show that if $\nu{\restriction}\mathfrak{B}$ is completely additive then ν is completely additive. (ii) Show that if $\mathfrak{A}$ is Dedekind σ-complete, $\mathfrak{B}$ is a σ-subalgebra and $\nu{\restriction}\mathfrak{B}$ is countably additive then ν is countably additive.

(h) Let (X,Σ,μ) be a probability space, $\mathcal{A}$ a finite family of subsets of X and T the subalgebra of $\mathcal{P}X$ generated by $\Sigma\cup\mathcal{A}$. Show that if $\nu:\mathrm{T}\to[0,1]$ is a finitely additive functional extending μ, then ν is countably additive.

(i) Let (X,Σ,μ) be a probability space, $\langle A_i\rangle_{i\in I}$ a partition of X and $\langle\alpha_i\rangle_{i\in I}$ a family in $[0,1]$ summing to 1. Show that the following are equiveridical: (i) there is a measure ν on X, extending μ, such that $\nu A_i=\alpha_i$ for every $i\in I$; (ii) there is a finitely additive functional $\nu:\mathcal{P}X\to[0,1]$, extending μ, such that $\nu A_i = \alpha_i$ for every $i \in I$; (iii) $\mu_*(\bigcup_{i\in J} A_i) \le \sum_{i\in J} \alpha_i$ for every $J \subseteq I$; (iv) $\mu^*(\bigcup_{i\in J} A_i) \ge \sum_{i\in J} \alpha_i$ for every finite $J \subseteq I$.

(o) Let X be a topological space and P_{qR} the set of quasi-Radon probability measures on X. For μ, $\nu \in P_{\mathrm{qR}}$, write $Q(\mu,\nu)$ for the set of quasi-Radon probability measures on $X \times X$ which have marginal measures μ on the first copy of X, ν on the second. (i) For a bounded continuous pseudometric ρ on X, set $\rho_{\mathrm{KR}}(\mu,\nu) = \inf\{\int \rho(x,y)\mu(d(x,y)) : \mu \in Q(\mu,\nu)\}$. Show that ρ_{KR} is a pseudometric on P_{qR}. (ii) Show that if X is completely regular and P is a family of bounded pseudometrics defining the topology of X, then $\{\rho_{\mathrm{KR}} : \rho \in \mathrm{P}\}$ defines the narrow topology of P_{qR}.

(p) Suppose that X, $\langle \nu_i \rangle_{i\in I}$ and $c : \mathcal{P}(X \times [0,\infty[) \to [0,\infty]$ are as in 457M. (i) Show that c is a submeasure. (ii) Show that if every ν_i is outer regular with respect to the open sets, then c is an outer regular Choquet capacity.

(q) Show that if the metric ρ is bounded, then 457Lc can be deduced from 457Mb and part (b-i) of the proof of 457L.

(r) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i\le n}$ be a finite family of Radon probability spaces, $X = \prod_{i\in I} X_i$, and $f : X \to \mathbb{R}$ a bounded Baire measurable function. Show that

$$\inf\{\int f d\mu : \mu \text{ is a Radon measure on } X \text{ with marginal measure } \mu_i \text{ on each } X_i\}$$

$$= \sup\{\sum_{i=0}^{n} \int h_i d\mu_i : h_i \in \ell^\infty(X_i) \text{ is } \Sigma_i\text{-measurable for each } i,$$

$$\sum_{i=0}^{n} h_i(\xi_i) \le f(x) \text{ whenever } x = (\xi_0, \dots, \xi_n) \in X\}.$$

Other exercises have been renamed: 457Xg-457Xk are now 457Xj-457Xn.

p 496 l 35 (457Y) Add new exercise:

(e) Give an example of a compact Hausdorff space X, a sequence $\langle \nu_n \rangle_{n\in\mathbb{N}}$ of probability measures on X all inner regular with respect to the closed sets, and a K_σ set $E \subseteq X$ such that

$$\inf\{\textstyle\sum_{n=0}^\infty \int h_n d\nu_n : \chi E \le \sum_{n=0}^\infty h_n\} = 1,$$

$$\sup\{\mu E : \mu \text{ is a Radon measure on } X \text{ and } \mu \le \nu_n \text{ for every } n \in \mathbb{N}\} \le \frac{1}{2}.$$

p 497 l 15 Section 458 has been substantially rewritten. I have found an inordinate number of detailed errors, starting with an incorrect definition at the very beginning, and have added new paragraphs 458B-458D, 458F and 458M. Other material has been rearranged: 458B is now 458G, 458C-458E are now 458I-458K, 458F is now 458H, 458G is now 458E, 458H-458P are now 458L-458U.

p 497 l 26 The first part of Definition 458Aa is incorrect, and should be as follows.

I say that a family $\langle E_i \rangle_{i\in I}$ in Σ is **relatively (stochastically) independent** over T if whenever $J \subseteq I$ is finite and not empty, and g_i is a conditional expectation of χE_i on T for each $i \in J$, and $F \in \mathrm{T}$, then $\mu(F \cap \bigcap_{i\in J} E_i) = \int_F \prod_{i\in J} g_i d\mu$.

p 497 l 37 Add new results:

458B Lemma Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle \Sigma_i \rangle_{i\in I}$ a family of σ-subalgebras of Σ at least one of which includes T. Suppose that whenever $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ and g_i is a conditional expectation of χE_i on T for each $i \in J$, then $\mu(\bigcap_{i\in J} E_i) = \int \prod_{i\in J} g_i d\mu$. Then $\langle \Sigma_i \rangle_{i\in I}$ is relatively independent over T.

458C Proposition Let (X,Σ,μ) be a probability space, $\mathbb{T}$ a non-empty upwards-directed family of σ-subalgebras of Σ, and $\langle \Sigma_i \rangle_{i\in I}$ a family of σ-subalgebras of Σ which is relatively independent over T for every $\mathrm{T} \in \mathbb{T}$. Then $\langle \Sigma_i \rangle_{i\in I}$ is relatively independent over the σ-algebra T^* generated by $\bigcup \mathbb{T}$.

458D Proposition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of T which is relatively independent over T.

(a) If $J \subseteq I$ and Σ'_i is a subalgebra of Σ_i for $i \in J$, then $\langle\Sigma'_j\rangle_{i\in J}$ is relatively independent over T.

(b) Let Σ^*_i be the σ-algebra generated by $\Sigma_i \cup \mathrm{T}$ for $i \in I$. Then $\langle\Sigma^*_i\rangle_{i\in I}$ is relatively independent over T.

(c) If $\mathcal{E} \subseteq \bigcup_{i\in I}\Sigma_i$, then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over the σ-algebra generated by $\mathrm{T}\cup\mathcal{E}$.

458F Proposition Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ.

(a) Let $\langle f_i\rangle_{i\in I}$ be a family of non-negative μ-integrable functions on X which is relatively independent over T. For each $i \in I$ let g_i be a conditional expectation of f_i on T. Then for any $F \in \mathrm{T}$ and $i_0, \dots, i_n \in I$,

$$\int_F \prod_{j=0}^n g_{i_j} \le \int_F \prod_{j=0}^n f_{i_j}$$

with equality if all the i_j are distinct.

(b) Suppose that Σ_1, Σ_2 are σ-subalgebras of Σ which are relatively independent over T, and that $f \in \mathcal{L}^1(\mu\restriction\Sigma_1)$. If g is a conditional expectation of f on T, then it is a conditional expectation of f on $\mathrm{T}\vee\Sigma_2$.

458M Proposition Let $(\mathfrak{A}, \bar\mu)$ be a probability algebra and $\mathfrak{B}$, $\mathfrak{C}$ closed subalgebras of $\mathfrak{A}$. Write $P_{\mathfrak{B}}$, $P_{\mathfrak{C}}$ and $P_{\mathfrak{B}\cap\mathfrak{C}}$ for the conditional expectation operators associated with $\mathfrak{B}$, $\mathfrak{C}$ and $\mathfrak{B}\cap\mathfrak{C}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{B}\cap\mathfrak{C}$;

(ii) $P(v \times w) = Pv \times Pw$ whenever $v \in L^\infty(\mathfrak{B})$ and $w \in L^\infty(\mathfrak{C})$;

(iii) $P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{B}\cap\mathfrak{C}}$;

(iv) $P_{\mathfrak{B}}P_{\mathfrak{C}} = P_{\mathfrak{C}}P_{\mathfrak{B}}$;

(v) $P_{\mathfrak{B}}(\chi c) \in L^0(\mathfrak{C})$ for every $c \in \mathfrak{C}$.

p 497 l 39 Lemma 458B (now 458G) must be rephrased in terms of the correct definition of 'relative independence', as follows:

Lemma Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. Let $\mathbb{T}$ be the family of finite subalgebras of T. For $\Lambda \in \mathbb{T}$ write $\mathcal{A}_\Lambda$ for the set of non-negligible atoms in Λ. For non-empty finite $J \subseteq I$, $\langle E_i\rangle_{i\in J} \in \prod_{i\in J}\Sigma_i$ and $F \in \mathrm{T}$, set

$$\phi_\Lambda(F, \langle E_i\rangle_{i\in J}) = \sum_{H\in\mathcal{A}_\Lambda} \mu(H\cap F)\cdot\prod_{i\in J}\frac{\mu(E_i\cap H)}{\mu H}.$$

Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\lim_{\Lambda\in\mathbb{T},\Lambda\uparrow}\phi_\Lambda(F, \langle E_i\rangle_{i\in J}) = \mu(F\cap\bigcap_{i\in J}E_i)$ whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for every $i \in J$.

p 498 l 7 (part (a) of the proof of 458B, now 458G): for 'If H is an atom of Λ and $\mu H > 0$, then there are integers k_i, for $i \in J$, such that $2^{-n}k_i \le g_i(x) < 2^{-n}(k_i + 1)$ for every $i \in J$ and $x \in H$' read 'If H is an atom of Λ and $\mu H > 0$, then there are integers k_{iH}, for $i \in J$, such that $2^{-n}k_{iH} \le g_i(x) < 2^{-n}(k_{iH} + 1)$ for every $i \in J$ and $x \in H$'; and similarly in the next two sentences.

p 498 l 13 (part (a) of the proof of 458B, now 458G): for

$$\begin{aligned}|\phi_\Lambda(F, \langle E_i\rangle_{i\in J}) - \int_F \prod_{i\in J} g_i d\mu| &\le \max(\int |\prod_{i\in J} g'_{in} - \prod_{i\in J} g_i|d\mu, \int |\prod_{i\in J} g''_{in} - \prod_{i\in J} g_i|d\mu)\\ &\le \max(\int \sum_{i\in J}|g'_{in} - g_i|d\mu, \int \sum_{i\in J}|g''_{in} - g_i|d\mu)\end{aligned}$$

read

$$|\phi_\Lambda(F, \langle E_i\rangle_{i\in J}) - \int_F \prod_{i\in J} g_i d\mu| \le \int \prod_{i\in J} g''_{in} - \prod_{i\in J} g'_i d\mu \le \int \sum_{i\in J} g''_{in} - g'_i d\mu.$$

p 498 l 21 (458C, now 458I) There is a confusion in the definition of 'relative distribution' which I have resolved by simply declaring

Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then a **relative distribution** of f over T will be a family $\langle \nu_x \rangle_{x \in X}$ of Radon probability measures on $\mathbb{R}$ such that $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T, for every Borel set $H \subseteq \mathbb{R}$.

p 498 l 33 (part (a) of the proof of 458D, now 458J): for '$\lambda W = \int \lambda_x W^{-1}[\{x\}]\mu_0(dx)$' read '$\lambda W = \int \lambda_x W[\{x\}]\mu_0(dx)$'.

p 499 l 12 In the statement of Theorem 458E (now 458K), add a third equiveridical assertion:

(iii) For any non-negative Baire measurable function $h : \mathbb{R}^I \to \mathbb{R}$ and any $F \in \mathrm{T}$,

$$\int_F h\boldsymbol{f} d\mu = \int_F \int h \, d\lambda_x \mu(dx).$$

p 499 l 18 (part (a) of the proof of 458E, now 458K): for '$\int_F \prod_{i=0}^n \nu_{ix} H_i \mu(dx)$' read '$\int_F \prod_{i \in J} \nu_{ix} H_i \mu(dx)$'.

p 500 l 1 (458F, now 458H): I have added a converse to the result, so that it now reads

Proposition Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle \Sigma_i \rangle_{i \in I}$ be a family of σ-subalgebras of Σ which is relatively independent over T. Let $\langle I_j \rangle_{j \in J}$ be a partition of I, and for each $j \in J$ let $\tilde{\Sigma}_j$ be the σ-algebra of subsets of X generated by $\bigcup_{i \in I_j} \Sigma_i$.

(a) If $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T, then $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T.

(b) Suppose that $\langle \tilde{\Sigma}_j \rangle_{j \in J}$ is relatively independent over T and that $\langle \Sigma_i \rangle_{i \in I_j}$ is relatively independent over T for every $j \in J$. Then $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T.

p 500 l 11 (proof of 458F, now 458Ha): for '$\langle W_j \rangle_{j \in J} \in \boldsymbol{W}$' read '$\langle W_j \rangle_{j \in K} \in \boldsymbol{W}$'.

p 501 l 7 (458Ha, now 458La): add

Corresponding to 458Ab, we can say that a family $\langle w_i \rangle_{i \in I}$ in $L^0(\mathfrak{A})$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is relatively stochastically independent, where $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\{[\![w_i > \alpha]\!] : \alpha \in \mathbb{R}\}$ for each i.

Corresponding to 458C, we see that if $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $\mathfrak{B}_i^*$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{B}_i \cup \mathfrak{C}$ for each i, then $\langle \mathfrak{B}_i^* \rangle_{i \in I}$ is relatively independent over $\mathfrak{C}$.

p 501 l 13 (458H, now 458L): parts (b)-(c) are now parts (f)-(g).

p 501 l 17 (458Hb, now 458Lf): for '$[\![\prod_{i \in J} u_i]\!]$' read '$[\![\prod_{i \in J} u_i > 0]\!]$'.

p 501 l 23 Add a new part to 458H, now 458L:

(g) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra, $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$, and $P : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{C}, \bar{\mu} \restriction \mathfrak{C})$ the conditional expectation operator. Suppose that $\langle \mathfrak{B}_i \rangle_{i \in I}$ is a family of closed subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$. Then

$$\int_c \prod_{j=0}^n P u_j \le \int_c \prod_{j=0}^n u_j$$

whenever $c \in \mathfrak{C}$, $i_0, \dots, i_n \in I$ and $u_j \in L^1(\mathfrak{B}_{i_j}, \bar{\mu} \restriction \mathfrak{B}_{i_j})^+$ for each $j \le n$, with equality if $i_0, \dots, i_n$ are distinct.

p 501 l 36 (part (a-i) of the proof of 458J, now 458O): for '$i \in I$, $a \in \mathfrak{A}$' read '$i \in I$, $a \in \mathfrak{A}_i$'.

p 502 l 20 (part (a-iv) of the proof of 458J, now 458O): following 'and that $d_i \in \phi_i[\mathfrak{A}_i]$ for each $i \in J$', add 'and $d \in \mathfrak{D}$'.

p 502 l 21 (part (a-iv) of the proof of 458J, now 458O): for '$\phi_i a = d_i$' read '$\phi_i a_i = d_i$'.

p 502 l 30 (part (b) of the proof of 458J, now 458O): for '$\mathfrak{D}' = \pi[\mathfrak{C}]$' read '$\mathfrak{D}' = \pi'[\mathfrak{C}]$'.

p 502 l 32 (part (b) of the proof of 458J, now 458O): for '$\bar{\mu}(a \cap \pi_c)$' read '$\bar{\mu}(\phi_i a \cap \pi c)$'.

p 503 l 29 (part (a) of the proof of 458K, now 458P): for

$$\bar{\mu}(a \cap \pi_i c) = \bar{\mu}' \psi_i (a \cap \pi_i c) = \bar{\mu}'(\psi_i a \cap \pi_i' \psi c) = \int_{\psi c} u_{i, \psi_i a}$$

read

$$\bar{\mu}_i(a\cap\pi_i c)=\bar{\mu}'_i\psi_i(a\cap\pi_i c)=\bar{\mu}'_i(\psi_i a\cap\pi'_i\psi c)=\int_{\psi c}u'_{i,\psi_i a}d\bar{\nu}'.$$

p 503 l 30 (part (a) of the proof of 458K, now 458P): for '$Tu_{ia}=u_{i,\psi_i(a)}$' read '$Tu_{ia}=u'_{i,\psi_i(a)}$'.

p 504 l 2 (part (b) of the proof of 458K, now 458P): for '$\lambda=\bar{\nu}\theta'$' read '$\lambda=\bar{\mu}'\theta$'.

p 504 l 7 (part (c) of the proof of 458K, now 458P): for 'measure-preserving on $\phi[\mathfrak{A}]$' read 'measure-preserving on $\phi[\mathfrak{B}]$'.

p 504 l 20 (458L, now 458Q; (†) of the definition of 'relative product measure'): for 'the functional $F\mapsto\mu_i(E\cap\pi_i^{-1}[F]):\mathrm{T}\to[0,1]$' read 'the functional $F\mapsto\mu_i(E_i\cap\pi_i^{-1}[F]):\mathrm{T}\to[0,1]$'.

p 505 l 4 (part (b) of the proof of 458M, now 458R): following 'set $u_{iE}=Tg^\bullet_{iE}\in L^\infty(\mathfrak{D})$', add 'where g_{iE} is a Radon-Nikodým derivative with respect to ν of the functional $F\mapsto\mu_i(E\cap\pi_i^{-1}[F])$'.

p 506 l 24 (part (b) of the proof of 458O, now 458T): for '$W=\prod_{i\in J}\phi_i^{-1}[G_i]$' read '$W=\bigcap_{i\in J}\phi_i^{-1}[G_i]$'. Similarly, a couple of lines later, replace $\prod$ by $\bigcap$ in '$W'=\prod_{i\in J}\phi_i^{-1}[K_i]$', '$\tilde{\lambda}(\prod_{i\in J}\phi_i^{-1}[K_i])=\lambda_0(\prod_{i\in J}\phi_i^{-1}[K_i])$'.

p 506 l 32 (part (b) of the proof of 458O, now 458T): for 'the support Y_0 of Y' read 'the support Y_0 of ν'.

p 506 l 42 (part (c) of the proof of 458O, now 458T): for '$\mu(\bigcap_{i\in J}\phi_i^{-1}[E_i])$' read '$\mu(\Delta\cap\bigcap_{i\in J}\phi_i^{-1}[E_i])$'. Similarly, in the following line, Υ should be the σ-algebra generated by $\{\Delta\cap\phi_i^{-1}[E]:i\in I,\,E\in\Sigma_i\}$.

p 507 l 15 (458P, now 458U) The first sentence of the proof needs expanding, as the definition in 458Lb (now 458Qb) must be applied to $f_i\times\chi(\pi_i^{-1}F)$ rather than to f_i.

p 507 l 30 (458X) Delete Exercise 458Xc. Add new exercises:

(b) Let (X,Σ,μ) be a probability space and T, Σ_1 and Σ_2 σ-subalgebras of Σ. Show that if $\Sigma_1\subseteq\mathrm{T}$ then Σ_1 and Σ_2 are relatively independent over T.

(c) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Let $\langle\mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) each $\mathcal{E}_i$ is closed under finite intersections (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i\in\mathcal{E}_i$ for every i. For each $i\in I$, let Σ_i be the σ-subalgebra of Σ generated by $\mathcal{E}_i$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

(f) Let (X,Σ,μ) be a probability space and Σ_1, Σ_2 and T σ-subalgebras of Σ such that Σ_1 and Σ_2 are relatively independent over T and $\Sigma_2\supseteq\mathrm{T}$. Suppose that g is a conditional expectation on T of $f\in\mathcal{L}^1(\mu\restriction\Sigma_1)$. Show that g is a conditional expectation of f on Σ_2.

(m) (i) Show that there is a set $X\subseteq[0,1]^2$ with outer planar Lebesgue measure 1 and just one point in each vertical section. (ii) Set $X_1=X_2=X$ and $\mu_1=\mu_2$ the subspace measure on X; let (Y,T,ν) be $[0,1]$ with Lebesgue measure, and $\pi_1=\pi_2$ the first-coordinate projection from X to Y. Show that (μ_1,π_1) and (μ_2,π_2) have no relative product measure over ν.

Other exercises have been rearranged: 458Xb is now 458Xe, 458Xd-458Xi are now 458Xg-458Xl, 458Xj-458Xs are now 458Xn-458Xw, 458Xt is now 458Xd.

p 509 l 9 (458Y) Replace 458Ya with the following:

(a) Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of σ-subalgebras of Σ with intersection T, and $\langle\Sigma_i\rangle_{i\in I}$ a family of subalgebras of Σ. Suppose that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T_n for every n. Show that it is relatively independent over T.

p 509 l 24 (458Y) The former 458Yf is now 457Xo. In its place are now

(b) Let X be a set, Σ a σ-algebra of subsets of X, and T a σ-subalgebra of Σ. Let $\langle\mathcal{E}_i\rangle_{i\in I}$ be a family of subsets of Σ such that (i) $E\cap F\in\mathcal{E}_i$ whenever $i\in I$ and E, $F\in\mathcal{E}_i$ (ii) $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i\in\mathcal{E}_i$ for every $i\in I$. For each $i\in I$, let Σ_i be the σ-algebra generated by $\mathcal{E}_i$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.**(g)** Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y,T,ν) a probability space, and $\pi_i:X_i\to Y$ a surjective inverse-measure-preserving function for each $i\in I$. Suppose that $\langle(\mu_i,\pi_i)\rangle_{i\in J}$ has a relative product measure for every countable $J \subseteq I$. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure.

(h) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of perfect probability spaces, (Y, T, ν) a countably separated probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(i) Let $(\mathfrak{A}, \bar{\mu})$ be a probability algebra and $\mathfrak{C}$ a closed subalgebra of $\mathfrak{A}$. Let $\mathfrak{C}_0 \subseteq \mathfrak{C}$ be the core subalgebra described in the canonical form of such structures given in 333N. Show that there is a closed subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, including $\mathfrak{C}_0$, such that $\mathfrak{B}$ and $\mathfrak{C}$ are relatively independent over $\mathfrak{C}_0$, and $\mathfrak{A}$ is the closed subalgebra of itself generated by $\mathfrak{B} \cup \mathfrak{C}$.

Other exercises have been rearranged; 458Yb-458Ye are now 458Yc-458Yf.

p 509 l 33 (Exercise 458Ye, now 458Yf) Delete condition (iii).

p 509 l 38 (Exercise 458Yf, now 457Xo): the pseudometrics considered must all be bounded.

p 515 l 34 Lemma 459F has been replaced by

459F Lemma Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the space of Radon probability measures on X with its narrow topology. If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of compact subsets of X, then $A = \{\mu : \mu \in P_R(X),\ \mu(\bigcup_{n\in\mathbb{N}} K_n) = 1\}$ is a K-analytic subset of $P_R(X)$.

459G Lemma Let X be a topological space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a totally finite quasi-Radon measure space, $y \mapsto \mu_y$ a continuous function from Y to the space $M^+_{\mathrm{qR}}(X)$ of totally finite quasi-Radon measures on X with its narrow topology, and $\mathcal{U}$ a base for the topology of X, containing X and closed under finite intersections. If $\mu \in M^+_{\mathrm{qR}}(X)$ is such that $\mu U = \int \mu_y U\,\nu(dy)$ for every $U \in \mathcal{U}$, then $\langle\mu_y\rangle_{y\in Y}$ is a disintegration of μ over ν.

459G is now 459H.

p 519 l 20 Add new results:

459I Lemma Let (X, Σ, μ) be a probability space and I a set. For a family $\mathbb{T}$ of subalgebras of $\mathcal{P}X$, write $\bigvee\mathbb{T}$ for the σ-algebra generated by $\bigcup\mathbb{T}$. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \neq i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi{\bullet}x$ is inverse-measure-preserving for each $\phi \in G$; set $\phi{\bullet}A = \{\phi{\bullet}x : x \in A\}$ for $\phi \in G$ and $A \subseteq X$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J \text{ is finite}} \Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi{\bullet}E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi{\bullet}E = E$.

Suppose that $\mathcal{J}^*$ is a filter on I not containing any infinite set, and that $K \subseteq I$, $\mathcal{K} \subseteq \mathcal{P}I$ and $\mathcal{J} \subseteq \mathcal{J}^*$ are such that for every $K' \in \mathcal{K}$ there is a $J \in \mathcal{J}$ such that $K \cap K' \subseteq J$. Then Σ_K and $\bigvee_{K'\in\mathcal{K}} \Sigma_{K'}$ are relatively independent over $\bigvee_{J\in\mathcal{J}} \Sigma_J$.

459J Corollary Let (X, Σ, μ) be a probability space and I a set. Let G be the group of permutations ϕ of I such that $\{i : \phi(i) \neq i\}$ is finite. Suppose that $\bullet$ is an action of G on X such that $x \mapsto \phi{\bullet}x$ is inverse-measure-preserving for each $\phi \in G$. Let $\langle\Sigma_J\rangle_{J\subseteq I}$ be a family of σ-subalgebras of Σ such that

(i) for every $J \subseteq I$, Σ_J is the σ-algebra generated by $\bigcup_{K\subseteq J \text{ is finite}} \Sigma_K$;

(ii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$, then $\phi{\bullet}E \in \Sigma_{\phi[J]}$;

(iii) if $J \subseteq I$, $E \in \Sigma_J$ and $\phi \in G$ is such that $\phi(i) = i$ for every $i \in J$, then $\phi{\bullet}E = E$.

Then if $J \subseteq I$ is infinite and $\langle K_\gamma\rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma \cap K_\delta \subseteq J$ for all distinct γ, $\delta \in \Gamma$, $\langle\Sigma_{K_\gamma}\rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J.

459H is now 459K.

p 521 l 16 (459X) Add new exercise:

(f) Let X, I be sets, Σ a σ-algebra of subsets of X and μ a probability measure with domain $\widehat{\bigotimes}_I\Sigma$ which is transposition-invariant in the sense that for every transposition $\tau : I \to I$ the function $x \mapsto x\tau : X^I \to X^I$ is inverse-measure-preserving. For $J \subseteq I$, let Σ_J be the σ-algebra

$W : W \in \widehat{\bigotimes}_I \Sigma$, W is determined by coordinates in $J\}$.

Show that if $J \subseteq I$ is infinite and $\langle K_\gamma \rangle_{\gamma\in\Gamma}$ is a family of subsets of I such that $K_\gamma \cap K_\delta \subseteq J$ for all distinct γ, $\delta \in \Gamma$, $\langle \Sigma_{K_\gamma} \rangle_{\gamma\in\Gamma}$ is relatively independent over Σ_J (i) using 459D (ii) using 459J.

p 523 l 8 §461 has been substantially revised in order to include one of Choquet's uniqueness theorems. First, 461C has been split into two:

461C Lemma Let X be a Hausdorff locally convex linear topological space, C a convex subset of X, and $f : C \to \mathbb{R}$ a lower semi-continuous convex function. If $x \in C$ and $\gamma < f(x)$, there is a $g \in X^*$ such that $g(y) + \gamma - g(x) \le f(y)$ for every $y \in C$.

461D Theorem Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set and μ a probability measure on a subset A of C. Suppose that μ has a barycenter x^* in X which belongs to C. Then $f(x^*) \le \underline{\int}_A f\,d\mu$ for every lower semi-continuous convex function $f : C \to \mathbb{R}$.

p 523 l 24 Consequently, 461D-461I have become 461E-461J.

p 524 l 21 (proof of 461F, now 461G): initialize the proof with 'Set $\phi(g) = \int g\,d\mu$ for $g \in X^*$'.

p 525 l 28 461J and 461L have been elaborated and repackaged:

461K Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Define a relation $\preccurlyeq$ on P by saying that $\mu \preccurlyeq \nu$ if $\int f d\mu \le \int f d\nu$ for every continuous convex function $f : K \to \mathbb{R}$.

(a) $\preccurlyeq$ is a partial order on P.

(b) If $\mu \preccurlyeq \nu$ then $\int f d\mu \le \int f d\nu$ for every lower semi-continuous convex function $f : K \to \mathbb{R}$.

(c) If $\mu \preccurlyeq \nu$ then μ and ν have the same barycenter.

(d) If we give P its narrow topology, then $\preccurlyeq$ is closed in $P \times P$.

(e) For every $\mu \in P$ there is a $\preccurlyeq$-maximal $\nu \in P$ such that $\mu \preccurlyeq \nu$.

461L Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on K. Suppose that $\mu \in P$ is maximal for the partial order $\preccurlyeq$ of 461K.

(a) $\mu(\frac{1}{2}(M_1 + M_2)) = 0$ whenever M_1, M_2 are disjoint closed convex subsets of K.

(b) $\mu F = 0$ whenever $F \subseteq K$ is a Baire set (for the subspace topology of K) not containing any extreme point of K.

p 526 l 25 The former 461K, 461M and 461N are now brought together as

461M Theorem Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X and E the set of extreme points of K. Let $x \in X$. Then there is a probability measure μ on E with barycenter x. If K is metrizable we can take μ to be a Radon measure.

p 528 l 42 The new uniqueness result now appears as

461N Lemma Let X be a Hausdorff locally convex linear topological space, K a compact convex subset of X, and P the set of Radon probability measures on X. Let E be the set of extreme points of X and suppose that $\mu \in P$ and $\mu^* E = 1$. Then μ is maximal in P for the partial order $\preccurlyeq$ of 461K.

461O Lemma Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology. Suppose that $K \subseteq X$ is a compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let P be the set of Radon probability measures on K and $\preccurlyeq$ the partial order described in 461K. If μ, $\nu \in P$ have the same barycenter then they have a common upper bound in P.

461P Theorem Suppose that X is a Riesz space with a Hausdorff locally convex linear space topology. Suppose that $K \subseteq X$ is a metrizable compact convex set such that every non-zero member of the positive cone X^+ is uniquely expressible as αx for some $x \in K$ and $\alpha \ge 0$. Let E be the set of extreme points of K, and x any point of K. Then there is a unique Radon probability measure μ on E such that x is the barycenter of μ.

The former 461O is now

461Q Proposition (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ a sequentially order-continuous Boolean homomorphism. Let M_σ be the L-space of countably additive real-valued functionals on $\mathfrak{A}$, and Q the set

$$\{\nu : \nu \in M_\sigma,\ \nu \ge 0,\ \nu 1 = 1,\ \nu\pi = \nu\}.$$

If $\nu \in Q$, then the following are equiveridical: (i) ν is an extreme point of Q; (ii) $\nu a \in \{0,1\}$ whenever $\pi a = a$; (iii) $\nu a \in \{0,1\}$ whenever $a \in \mathfrak{A}$ is such that $\nu(a \triangle \pi a) = 0$.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\phi : X \to X$ a (Σ, Σ)-measurable function. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which ϕ is inverse-measure-preserving. If $\mu \in Q$, then μ is an extreme point of Q iff ϕ is ergodic with respect to μ.

461P is now 461R.

p 531 l 1 (461X) Add new exercises:

(k) Let G be an abelian group with identity e, and K the set of positive definite functions $h : G \to \mathbb{C}$ such that $h(e) = 1$. (i) Show that K is a compact convex subset of $\mathbb{C}^G$. (ii) Show that the extreme points of K are just the group homomorphisms from G to S^1. (iii) Show that K generates the positive cone of a Riesz space.

(l) Let X be a compact metrizable space and G a subgroup of the group of autohomeomorphisms of X. Let M_τ be the space of signed Borel measures on X with its vague topology, as in 437L, and $K \subseteq M_\tau$ the set of G-invariant Borel probability measures on X. Show that every member of K is uniquely expressible as the barycenter of a Radon measure on the set of extreme points of K.

(q) Let X and Y be Hausdorff locally convex linear topological spaces, $A \subseteq X$ a convex set and $\phi : A \to Y$ a continuous function such that $\phi[A]$ is bounded and $\phi(tx+(1-t)y) = t\phi(x)+(1-t)\phi(y)$ for all x, $y \in A$ and $t \in [0,1]$. Let μ be a topological probability measure on A with a barycenter $x^* \in A$. Show that $\phi(x^*)$ is the barycenter of the image measure $\mu\phi^{-1}$ on Y.

Exercises 461Xk-461Xn are now 461Xm-461Xp.

p 531 l 4 Exercise 461Xl (now 461Xn) has been changed, and now reads:

(n) Let $\mathfrak{A}$ be a Boolean algebra and M the L-space of bounded finitely additive functionals on $\mathfrak{A}$, and $\pi : \mathfrak{A} \to \mathfrak{A}$ a Boolean homomorphism. (i) Show that $U = \{\nu : \nu \in M,\ \nu\pi = \nu\}$ is a closed Riesz subspace of M. (ii) Set $Q = \{\nu : \nu \in U,\ \nu \ge 0,\ \nu 1 = 1\}$. Show that if μ, ν are distinct extreme points of Q then $\mu \wedge \nu = 0$. (iii) Set $Q_\sigma = \{\nu : \nu \in Q,\ \nu$ is countably additive$\}$. Show that any extreme point of Q_σ is an extreme point of Q. (iv) Set $Q_\tau = \{\nu : \nu \in Q,\ \nu$ is countably additive$\}$. Show that any extreme point of Q_τ is an extreme point of Q.

p 531 l 21 (Exercise 461Ya): for 'whenever $M_0, \dots, M_n$ are compact convex sets with empty intersection' read 'whenever $M_0, \dots, M_n$ are compact convex subsets of K with empty intersection'.

p 531 l 27 Exercise 461Yc has been moved to 461Yf. I think we need to suppose that X is locally convex.

p 531 l 27 (461Y) Add new exercise:

(c) Write ν_{ω_1} for the usual measure on $Z = \{0,1\}^{\omega_1}$. Fix any $z_0 \in Z$, and let U be the linear space $\{u : u \in C(Z),\ u(z_0) = \int u\,d\nu_{\omega_1}\}$. Let X be the Riesz space of signed tight Borel measures μ on Z such that $\mu\{z_0\} = 0$, with the topology generated by the functionals $\mu \mapsto \int u\,d\mu$ as u runs over U. Let $K \subseteq X$ be the set of tight Borel probability measures μ on Z such that $\mu\{z_0\} = 0$. (i) Show that K is compact and convex and that every member of $X^+ \setminus \{0\}$ is uniquely expressible as a positive multiple of a member of K. (ii) Show that the set E of extreme points of K can be identified, as topological space, with $Z \setminus \{z_0\}$, so is a Borel subset of K but not a Baire subset. (iii) Show that the restriction of ν_{ω_1} to the Borel σ-algebra of Z is the barycenter of more than one Baire measure on E.

p 531 l 38 Exercise 461Ye: for 'Let Q_ϕ be the set of Radon measures on $X^{\mathbb{N}}$ for which ϕ is inverse-measure-preserving' read 'Let Q_ϕ be the set of Radon probability measures on $X^{\mathbb{N}}$ for which ϕ is inverse-measure-preserving'.

p 531 l 40 (461Y) Add new exercise:

> **(g)** Let G be an amenable topological group, and $\bullet$ an action of G on a reflexive Banach space U, continuous for the given topology on G and the weak topology of U, such that $u \mapsto a{\bullet}u$ is a linear operator of norm at most 1 for every $a \in G$. Set $V = \{v : v \in U,\ a{\bullet}v = v \text{ for every } a \in G\}$. Show that $\{u + v - a{\bullet}u : u \in U,\ v \in V,\ a \in G\}$ is dense in U.

p 533 l 1 I have transposed 462B and 462C.

p 535 l 14 There are several inadequate proofs in §462. In part (d) of the proof of 462E, I say that $\mathfrak{T}_p$, the topology of pointwise convergence on $C_0(X)$ where X is locally compact and Hausdorff, is angelic; this is true, but the reference to 462B (now 462C) is inadequate; I think we need to note that $C_0(X)$ is homeomorphic (for the pointwise topologies) to a subspace of $C(X^*)$, where X^* is the one-point compactification of X. In 462G, I fail to explain why, if μ is a $\mathfrak{T}_p$-Radon measure, it measures every $\mathfrak{T}_\infty$-Borel set. This is easily filled in. More seriously, in 462I, the conclusion claims that '$\phi{\restriction}C$ is continuous for every relatively countably compact set $C \subseteq X$'; but the proof works only for sets C which are actually countably compact.

As it happens, the result as written is true, but I think that it demands a rather more penetrating analysis, starting with a stronger version of 462F. I have therefore re-written this part of the section. The results are now

> **462F Lemma** Let X be a topological space, and Q a relatively countably compact subset of X. Suppose that $K \subseteq C_b(X)$ is $\| \|_\infty$-bounded and $\mathfrak{T}_p$-compact, where $\mathfrak{T}_p$ is the topology of pointwise convergence on $C_b(X)$. Then then map $u \mapsto u{\restriction}Q : K \to C_b(Q)$ is continuous for $\mathfrak{T}_p$ on K and the weak topology of $C_b(Q)$.
>
> **462G Proposition** (formerly 462F) Let X be a countably compact topological space. Then a subset of $C_b(X)$ is weakly compact iff it is norm-bounded and compact for the topology of pointwise convergence.
>
> **462H Lemma** Let X be a topological space, Q a relatively countably compact subset of X, and μ a totally finite measure on $C_b(X)$ which is Radon for the topology $\mathfrak{T}_p$ of pointwise convergence on $C(X)$. Let $T : C_b(X) \to C_b(Q)$ be the restriction map. Then the image measure $\nu = \mu T^{-1}$ on $C_b(Q)$ is Radon for the norm topology of $C_b(Q)$.
>
> **462I Theorem** (formerly 462G) Let X be a countably compact topological space. Then the totally finite Radon measures on $C(X)$ are the same for the topology of pointwise convergence and the norm topology.

So 462H-462J are now 462J-462L.

p 537 l 8 There is a similar blunder in the statement of 462J (now 462L). I state the hypotheses as

> (†) whenever $h \in \mathbb{R}^X$ is such that $h{\restriction}Q$ is continuous for every relatively countably compact $Q \subseteq X$, then h is continuous,
>
> (‡) $\sup_{h\in K,x\in C} |h(x)|$ is finite for any countably compact set $C \subseteq X$.

Of course these don't fit together, and the proof deals with the case in which the 'relatively' in (†) is omitted. In 463H I quote the form in which 'relatively countably compact' appears in both (†) and (‡), so this is the form which I have chosen in the corrected version.

p 538 l 7 (462Y) Add new exercise:

> **(e)** (i) Let X and Y be Polish spaces, and write $B_1(X;Y)$ for the set of functions $f : X \to Y$ such that $f^{-1}[H]$ is G_δ in X for every closed set $H \subseteq Y$ (Kuratowski 66, §31). Show that $B_1(X;Y)$, with the topology of pointwise convergence inherited from Y^X, is angelic. (ii) Let X be a Polish space. Show that the space $\tilde{C}^{\mathbb{1}}(X)$ of 438P-438Q is angelic.

p 540 l 12 In order to be able to use it in §536, I have lifted part (b) of the proof of 463D out, in the form

463D Lemma Let (X,Σ,μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $\mathcal{L}^0$. Suppose that $K\subseteq\mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto $\{0,1\}^{\omega_1}$. If $E\in\Sigma$ has finite measure, then every sequence in K has a subsequence which is convergent almost everywhere in E.

Consequently the paragraphs 463D-463M should be re-named 463E-463N. The former 463N-463O are now covered by results in §536, so have been dropped from the present section.

p 540 l 39 (part (b-ii) of the proof of the old 463D): for '$\phi(f)(x)=\min(h_J(x),\max(f(x),g_J(x)))/(h_J(x)-g_J(x))$' read '$\phi(f)(x)=\frac{\operatorname{med}(0,f(x)-g_J(x),h_J(x)-g_J(x))}{h_J(x)-g_J(x)}$'.

p 541 l 33 (part (c) of the proof of the old 463D): in the choice of x_ξ, require

$$x_\xi\in F,\quad g_\eta(x_\xi)=f_1(x_\xi)\text{ for }\eta\le\xi$$

(in place of '$\eta<\xi$'), and in the definition of K_1 set

$$K_1=\bigcap_{\xi\le\eta<\omega_1}\{f:f\in K,\text{ either }f(x_\xi)=f_0(x_\xi)\text{ or }f(x_\eta)=f_1(x_\eta)\},$$

(in place of '$\xi<\eta<\omega_1$' and '$f(x_\xi)=f(x_\eta)=f_1(x_\eta)$').

p 542 l 11 In part (a) of the proof of 463F, now 463G, delete the clause 'this time all dominated by q', so that $K'\subseteq\mathcal{L}^2(\mu)$'.

p 543 l 24 (part (b-iii) of the proof of 463H, now 463I): for '$\psi(z)(n)=1$ if $z(n)=0$, 1 otherwise' read '$\psi(z)(n)=1$ if $z(n)=0$, 0 otherwise'.

p 544 l 38 (part (b) of the proof of 463J, now 463K): for 'Y' read '$X\times\mathbb{R}$'.

p 546 l 5 Proposition 463L, now 463M, has been re-written in fractionally more general form, and reads

463M Proposition Let $X_0,\dots,X_n$ be countably compact Hausdorff spaces, each carrying a σ-finite perfect strictly positive measure which measures every Baire set. Let X be their product and $\mathcal{B}\mathfrak{a}(X_i)$ the Baire σ-algebra of X_i for each i. Then any separately continuous function $f:X\to\mathbb{R}$ is measurable with respect to the σ-algebra $\widehat{\bigotimes}_{i\le n}\mathcal{B}\mathfrak{a}(X_i)$ generated by $\{\prod_{i\le n}E_i:E_i\in\mathcal{B}\mathfrak{a}(X_i)\text{ for }i\le n\}$.

p 546 l 17 (part (b) of the proof of 463L, now 463M): for '$\mathcal{B}\mathfrak{a}(X_{m+1})$' read '$\mathcal{B}\mathfrak{a}(X_{n+1})$'.

p 546 l 40 (part (a) of the proof of 463M, now 463N): it need not be the case that $C\subseteq Z$; but Z is μ-conegligible, so we can still conclude that f is Σ-measurable.

p 548 l 10 Exercises 463Xc and 463Xm are wrong, and should be deleted; consequently 463Xd-463Xl should be renamed 463Xc-463Xk, and 463Xn should be renamed 463Xl. (For a corrected and extended version of the old 463Xc, see the new 465Ya below.)

p 552 l 36 (remark following 464E): for '463L-463M' read '536D'.

p 553 l 14 (464Fc): for '$\theta\in M^+$' read '$\theta\in M^+\setminus\{0\}$'.

p 556 l 36 (part (e-ii-β) of the proof of 464H): for '$\beta<w<\gamma$ μ-a.e.' read '$\beta\le w\le\gamma$ μ-a.e.'.

p 558 l 15 The statement of Lemma 464K should be amplified, with a new explicit part (a), as follows:

Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Write M_{m} for the set of measurable $\theta\in M$, M_τ for the space of completely additive functionals on $\mathcal{P}I$ and $\Delta(\theta)=\overline{\int}\theta\,d\nu$ for $\theta\in M^+$.

(a) If $\theta\in M_{\mathrm{m}}\cap M_\tau^\perp$ and $b\subseteq I$, then $\theta(a\cap b)=\frac12\theta b$ for ν-almost every $a\subseteq I$.

(b) $|\theta|\in M_{\mathrm{m}}$ for every $\theta\in M_{\mathrm{m}}$.

(c) A functional $\theta\in M^+$ is measurable iff $\Delta(\theta)=\frac12\theta I$.

(d) M_{m} is a solid linear subspace of M.

p 562 l 20 (464Y) Add new exercise:

(c) Let I be any set. Write $\boldsymbol{c}_0(I)$ for the closed linear subspace of $\ell^\infty(I)$ consisting of those $x \in \mathbb{R}^I$ such that $\{t : t \in I, |x(t)| \geq \epsilon\}$ is finite for every $\epsilon > 0$; that is, $C_0(I)$ if I is given its discrete topology. Show that, in 464R, $M_\tau^\perp$ can be identified as Banach lattice with $(\ell^\infty(I)/\boldsymbol{c}_0(I))^*$.

(d)(i) Let $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be a T-measurable finitely additive functional. Show that $\{\theta\{n\} : n \in \mathbb{N}\}$ is bounded. (ii) Let $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ be an additive functional which is universally measurable for the usual topology of $\mathcal{P}\mathbb{N}$. Show that θ is bounded. (iii) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and $\theta : \mathfrak{A} \to \mathbb{R}$ an additive functional which is universally measurable for the order-sequential topology on $\mathfrak{A}$. Show that θ is bounded and that θ^+ is universally measurable.

(e) Show that there is a finitely additive functional $\theta : \mathcal{P}\mathbb{N} \to \mathbb{R}$ which is T-measurable in the sense of 464I, but is not bounded.

p 562 l 42 (464 *notes*) for '$\{a : \alpha\nu b \leq \theta(a \cap b) \leq \gamma\theta b$ for every $b \in K\}$' read '$\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b$ for every $b \in K\}$'.

p 564 l 25 The proof of part (h) of Proposition 465C ('if A is stable, then $\{|f| : f \in A\}$ is stable') is catastrophically wrong. The result is true, and has been moved to 465N. Unfortunately it is quoted in the proofs of 465G and 465M, which need modifications, as described below. With extra fragments, the proposition now reads as follows.

465C Proposition Let (X, Σ, μ) be a semi-finite measure space.

(a) Let $A \subseteq \mathbb{R}^X$ be a stable set.

(i) Any subset of A is stable.

(ii) $\overline{A}$, the closure of A in $\mathbb{R}^X$ for the topology of pointwise convergence, is stable.

(iii) $\gamma A = \{\gamma f : f \in A\}$ is stable, for any $\gamma \in \mathbb{R}$.

(iv) If $g \in \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$, then $A + g = \{f + g : f \in A\}$ is stable.

(v) If $g \in \mathcal{L}^0$, then $A \times g = \{f \times g : f \in A\}$ is stable.

(vi) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous non-decreasing function. Then $\{hf : f \in A\}$ is stable.

(b)(i) Suppose that $A \subseteq \mathbb{R}^X$, $E \in \Sigma$, $n \geq 1$ and $\alpha < \beta$ are such that $0 < \mu E < \infty$ and $(\mu^{2n})^* D_n(A, E, \alpha, \beta) < (\mu E)^{2n}$. Then

$$\lim_{k\to\infty} \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(A, E, \alpha, \beta) = 0.$$

(ii) If A, $B \subseteq \mathbb{R}^X$ are stable, then $A \cup B$ is stable.

(iii) If $A \subseteq \mathcal{L}^0$ is finite it is stable.

(iv) If $A \subseteq \mathbb{R}^X$ is stable, so is $\{f^+ : f \in A\} \cup \{f^- : f \in A\}$.

(c) Let A be a subset of $\mathbb{R}^X$.

(i) If $\hat{\mu}$, $\tilde{\mu}$ are the completion and c.l.d. version of μ, then A is stable with respect to one of the measures μ, $\hat{\mu}$, $\tilde{\mu}$ iff it is stable with respect to the others.

(ii) Let ν be an indefinite-integral measure over μ. If A is stable with respect to μ, it is stable with respect to ν and with respect to $\nu{\restriction}\Sigma$.

(iii) If A is stable, and $Y \subseteq X$ is such that the subspace measure μ_Y is semi-finite, then $A_Y = \{f{\restriction}Y : f \in A\}$ is stable in $\mathbb{R}^Y$ with respect to the measure μ_Y.

(iv) A is stable iff $A_E = \{f{\restriction}E : f \in A\}$ is stable in $\mathbb{R}^E$ with respect to the subspace measure μ_E whenever $E \in \Sigma$ has finite measure.

(v) A is stable iff $A_n = \{\mathrm{med}(-n\chi X, f, n\chi X) : f \in A\}$ is stable for every $n \in \mathbb{N}$.

(d) Suppose that μ is σ-finite, (Y, T, ν) is another measure space and $\phi : Y \to X$ is inverse-measure-preserving. If $A \subseteq \mathbb{R}^X$ is stable with respect to μ, then $B = \{f\phi : f \in A\}$ is stable with respect to ν.

p 566 l 6 (part (i) of the proof of 465Ci, now 465C(a-v)): for 'E_0' read 'E_0^2'.

p 568 l 26 Because we have to avoid reliance on the old 465Ch, part (a) of the proof needs adjustment; I think the following works.

(a) We must have an $f_0 \in A$, a set $F \in \Sigma$ of finite measure, and an $\epsilon > 0$ such that for every $\mathfrak{T}_p$-neighbourhood U of f_0 there is an $f \in A \cap U$ such that $\int \chi F \wedge |f - f_0| d\mu \geq 2\epsilon$. Set

$$B = \{\chi F \wedge (f - f_0)^+ : f \in A\} \cup \{\chi F \wedge (f - f_0)^- : f \in A\}.$$

Then

$$B = \{\chi F - (\chi F - (f - f_0)^+)^+ : f \in A\} \cup \{\chi F - (\chi F - (f - f_0)^-)^+ : f \in A\}$$

is stable, by 465Cf (now 465C(a-iv)), 465Cl (now 465C(a-vi)) and 465Ce (now 465C(a-iii)), used repeatedly. Setting $B' = \{f : f \in B, \int f \ge \epsilon\}$, B' is again stable (465Ca, now 465C(a-i)). Our hypothesis is that f_0 is in the $\mathfrak{T}_p$-closure of

$$\begin{aligned}\{f : f \in A, \int \chi F \wedge |f - f_0| \ge 2\epsilon\} \subseteq \{f : f \in A, &\int \chi F \wedge (f - f_0)^+ \ge \epsilon\} \\ &\cup \{f : f \in A, \int \chi F \wedge (f - f_0)^+ \ge \epsilon\};\end{aligned}$$

since $f \mapsto \chi F \wedge (f - f_0)^+$ and $f \mapsto \chi F \wedge (f - f_0)^-$ are $\mathfrak{T}_p$-continuous, 0 belongs to the $\mathfrak{T}_p$-closure of B'.

p 569 l 17 (part (a) of the proof of 465H): for '$M\bigl(\frac{(k-n)!}{k!} - \frac{1}{k^n}\bigr)$' read '$\frac{Mk!}{(k-n)!}\bigl(\frac{(k-n)!}{k!} - \frac{1}{k^n}\bigr)$'.

p 574 l 37 (part (v) of case 2 in the proof of 465L): for '$W_{IJ} \subseteq W \subseteq E^{2k}$' read '$W_{IJ} \subseteq W$'.

p 575 l 20 (part (a) of the proof of 465M): again, we need to avoid the use of the old 465Ch. At the beginning, in part (α), it will help to take $\eta = \frac{1}{108}\epsilon^2$ rather than $\frac{1}{27}\epsilon^2$. When we come to (γ), we must look at $A'' = \{g^+ : g \in A'\} \cup \{g^- : g \in A'\}$ in place of $\{|g| : g \in A'\}$. The application of 465K now gives us n, W and γ such that $\int h\,d\nu \le 3\sqrt{3\eta} = \frac{1}{2}\epsilon$ whenever $h \in A''$ and $\nu^n W < \gamma$. But now $\int |g| d\nu \le \epsilon$ for all such ν and all $g \in A'$, so we can continue as before.

p 577 l 7 (part (d-β) of the proof of 465M): replace each sum '$\sum_{i=0}^{k}$' by '$\sum_{i=0}^{k-1}$'.

p 578 l 26 (Corollary 465N) We are now ready to prove the former 465Ch. I combine this with a strengthening of part (b) and some new material, as follows.

465N Theorem Let (X, Σ, μ) be a semi-finite measure space.

(a) Let $A \subseteq \mathbb{R}^X$ be a stable set. Suppose that there is a measurable function $g : X \to [0, \infty[$ such that $|f(x)| \le g(x)$ whenever $x \in X$ and $f \in A$. Then the convex hull $\Gamma(A)$ of A in $\mathbb{R}^X$ is stable.

(b) If $A \subseteq \mathbb{R}^X$ is stable, then $|A| = \{|f| : f \in A\}$ is stable.

(c) Let A, $B \subseteq \mathbb{R}^X$ be two stable sets such that $\{f(x) : f \in A \cup B\}$ is bounded for every $x \in X$. Then $A + B = \{f_1 + f_2 : f_1 \in A, f_2 \in B\}$ is stable.

(d) Suppose that μ is complete and locally determined. Let $A \subseteq \mathbb{R}^X$ be a stable set such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$. Then $\Gamma(A)$ is relatively compact in $\mathcal{L}^0(\Sigma)$ for the topology of pointwise convergence.

p 579 l 30 There is another catastrophic error in the proof presented for part (b) of Theorem 465P, and I have no reason to suppose that the claimed result is true. I have therefore rewritten the theorem in the following less general form.

Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$.

(a) Suppose that $A \subseteq \mathcal{L}^0(\Sigma)$ and that $Q = \{f^\bullet : f \in A\} \subseteq L^0(\mu)$, identified with $L^0 = L^0(\mathfrak{A})$. Then Q is stable iff every countable subset of A is stable.

(b) Suppose that μ is complete and strictly localizable and Q is a stable subset of $L^\infty(\mu)$, identified with $L^\infty(\mathfrak{A})$ (363I). Then there is a stable set $B \subseteq \mathcal{L}^\infty(\Sigma)$ such that $Q = \{f^\bullet : f \in B\}$.

p 580 l 27 (part (b-iii) of the proof of 465P): for '$Q = \{f^\bullet : f \in A\}$' read '$Q = \{f^\bullet : f \in B\}$'.

p 580 l 35 (part (b-iii) of the proof of 465P): for '$\pi D_k(\{f\}, E, \alpha', \beta')$' read '$\pi D_k(\{f\}, E, \alpha', \beta')^\bullet$'.

p 581 l 38 (part (a) of the proof of 465R): for '$\mu Y = 1$' read '$\mu X = 1$'.

p 584 l 37 (part (g) of the proof of 465R): for 'B_1 is stable in $L^1(\nu_0)$' read 'B is stable with respect to ν_0'.

p 585 l 24 (proof of 465T): for '$G_k(\dots)$' read '$D_k(\dots)$' (five times).

p 586 l 25 (part (b) of the proof of 465U): for '$\min(j_0, j_1)$' read '$\min(n_{j_0}, n_{j_1})$'.

p 586 l 35 (part (d) of the proof of 465U): for '$\tilde{W} \cap D_m(A^*, Z, 0, 1) > 0$' read '$\tilde{W} \cap D_m(A^*, Z, 0, 1) \neq \emptyset$'.

p 587 l 8 (part (d) of the proof of 465U): for '$\mu(V \cap \prod_{j<m} E'_{kj}) > 0$' read '$\mu^m(V \cap \prod_{j<m} E'_{kj}) > 0$'.

p 588 l 34 465Xh is now covered by 465C(c-v), so has been dropped. Add new exercises:

(h) Let (X, Σ, μ) be a semi-finite measure space and A a subset of $\mathbb{R}^X$. Suppose that for every $\epsilon > 0$ there is a stable set $B \subseteq \mathbb{R}^X$ such that for every $f \in A$ there is a $g \in B$ such that $\|f - g\|_\infty \le \epsilon$. Show that A is stable.

(i) Let (X, Σ, μ) be a semi-finite measure space and $\langle E_n \rangle_{n\in\mathbb{N}}$ a sequence in Σ. (i) Suppose that whenever $F \in \Sigma$ and $\mu F < \infty$ there is a $k \ge 1$ such that $\sum_{n=0}^{\infty} (\mu(F \cap E_n)\, \mu(F \setminus E_n))^k$ is finite. Show that $\langle \chi E_n \rangle_{n\in\mathbb{N}}$ is stable. (ii) Suppose that $\mu X = 1$, that $\langle E_n \rangle_{n\in\mathbb{N}}$ is independent, and that $\sum_{n=0}^{\infty} ((1 - \mu E_n)\mu E_n)^k = \infty$ for every $k \ge 1$. Show that $\langle \chi E_n \rangle_{n\in\mathbb{N}}$ is not stable.

(q) Show that there is a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of functions from $[0,1]$ to $\mathbb{N}$ such that $\{f_n : n \in \mathbb{N}\}$ is stable for Lebesgue measure on $[0,1]$, but $\{f_m - f_n : m, n \in \mathbb{N}\}$ is not.

Other exercises have been renamed; 465Xe is now 465Cd, 465Xi is now 465Xk, 465Xk-465Xo are now 465Xk-465Xp, 465Xp-465Xr are now 465Xr-465Xt.

p 588 l 36 Exercise 465Xi (now 465Xk) is wrong as stated, and has been changed to

(k) Let (X, Σ, μ) be a semi-finite measure space and $A \subseteq \mathbb{R}^X$ a stable set such that $\{f(x) : f \in A\}$ is bounded for every $x \in X$. Let $\overline{A}$ be the closure of A for the topology of pointwise convergence. Show that $\{f^\bullet : f \in \overline{A}\}$ is just the closure of $\{f^\bullet : f \in A\} \subseteq L^0(\mu)$ for the topology of convergence in measure.

p 589 l 2 (Exercise 465Xl, now 465Xm): for '$\mu E > 0$' read '$0 < \mu E < \infty$'.

p 589 l 36 (465Y) Add new exercises:

(a) Let (X, Σ, μ) be a semi-finite measure space. A set $A \subseteq \mathbb{R}^X$ is said to have the **Bourgain property** if whenever $E \in \Sigma$, $\mu E > 0$ and $\epsilon > 0$, there are non-negligible measurable sets $F_0, \dots, F_n \subseteq E$ such that for every $f \in A$ there is an $i \le n$ such that the oscillation $\sup_{x,y\in F_i} |f(x) - f(y)|$ of f on F_i is at most ϵ. Show that in this case A is stable.

(b) Let X be a topological space, and μ a τ-additive effectively locally finite topological measure on X. Show that any equicontinuous subset of $C(X)$ is stable.

(e) Let (X, Σ, μ) be a probability space, and $A \subseteq \mathbb{R}^X$ a uniformly bounded set. Show that A is stable iff for every $\epsilon > 0$ there is a finite subalgebra T of Σ, a sequence $\langle h_k \rangle_{k\in\mathbb{N}}$ of measurable functions on $X^{\mathbb{N}}$, and a family $\langle g_f \rangle_{f\in A}$ of T-measurable functions such that

$$h_k(w) \ge \tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} |f(w(i)) - g_f(w(i))| \text{ for every } w \in X^{\mathbb{N}},\ k \ge 1 \text{ and } f \in A,$$

$$\limsup_{k\to\infty} h_k(w) \le \epsilon \text{ for almost every } w \in X^{\mathbb{N}}.$$

(f) Let (X, Σ, μ) be a probability space, and A, $B \subseteq \mathbb{R}^X$ uniformly bounded stable sets. Show that $\{f \times g : f \in A,\ g \in B\}$ is stable.

(g) Let (X, Σ, μ) and (Y, T, ν) be semi-finite measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Suppose that $A \subseteq \mathbb{R}^X$ and $B \subseteq \mathbb{R}^Y$ are uniformly bounded stable sets. Show that $\{f \otimes g : f \in A,\ g \in B\}$ is stable with respect to λ.

(h) $A \subseteq \mathbb{R}^X$ a uniformly bounded stable set. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{hf : f \in A\}$ is stable.

(k) Set $X = \prod_{n=2}^{\infty} \mathbb{Z}_n$, where each $\mathbb{Z}_n$ is the cyclic group of order n with its discrete topology; let μ be the Haar probability measure on X. For a, $x \in X$ and $f \in \mathbb{R}^X$ write $(a\bullet f)(x) = f(x - a)$. (i) Show that for any $n \in \mathbb{N}$ there is a compact negligible set $K_n \subseteq X$ such that for every $I \in [X]^n$ there are uncountably many $a \in X$ such that $I \subseteq K_n + a$. (ii) Show that there is a negligible set $E \subseteq X$ such that $\{a\bullet\chi E) : a \in X\}$ is dense in $\{0,1\}^X$ for the topology of pointwise convergence. (iii) Show that if $f \in C(X)$ then $\{a\bullet f : a \in X\}$ is stable. (iv) Show that there is a sequence

$\langle f_n\rangle_{n\in\mathbb{N}}$ in $C(X)$ such that $\{f_n : n \in \mathbb{N}\}$ is stable but $\{a\bullet f_n) : a \in X,\ n \in \mathbb{N}\}$ is not. (v) Find expressions of these results when X is replaced by the circle group or by $\mathbb{R}$.

Rename other exercises: 465Ya-465Yb are now 465Yc-465Yd, 465Yc-465Yd are now 465Yi-465Yj.

p 589 l 30 Exercise 465Yc (now 465Yi) has been corrected, and now reads

(i) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space. Show that a countable R-stable subset of $\mathbb{R}^X$ is stable.

p 591 l 13 (part (a) of the proof of 466A): for '463E' read '463G'.

p 592 l 19 (proof of 466D): delete '$\frac{\|x\|}{\|y\|}y \in x + V$. Since'.

p 596 l 30 I have added a fragment on Gaussian measures on linear topological spaces.

466N Definition If X is a linear topological space, a probability measure μ on X is a **centered Gaussian measure** if its domain includes the cylindrical σ-algebra of X and every continuous linear functional on X is either zero almost everywhere or a normal random variable with zero expectation.

466O Proposition Let X be a separable Banach space, and μ a probability measure on X. Suppose that there is a linear subspace W of X^*, separating the points of X, such that every element of W is $\operatorname{dom}\mu$-measurable and either zero a.e. or a normal random variable with zero expectation. Then μ is a centered Gaussian measure with respect to the norm topology of X.

p 596 l 37 Part (ii) of Exercise 466Xb should read

(ii) Write $\mathfrak{T}_s$ for the weak topology of X. Let λ_s be the τ-additive product measure of copies of ν when each copy of ℓ^2 is given its weak topology instead of its norm topology. Show that λ_s is quasi-Radon for $\mathfrak{T}_s$ but does not measure every $\mathfrak{T}$-Borel set.

p 597 l 21 Exercise 466Xl (now 466Xq) has been rewritten, as follows:

(q) Let X be a Banach space, and μ a Radon measure on X. Show that, with respect to μ, the unit ball of X^* is a stable set of functions in the sense of §465.

p 597 l 24 (466X) Add new exercises:

(g) Let K be a scattered compact Hausdorff space. Show that the weak topology and the topology of pointwise convergence on $C(K)$ have the same Borel σ-algebras.

(m) Let X be a linear topological space and μ a centered Gaussian measure on X. (i) Let Y be another linear topological space and $T : X \to Y$ a continuous linear operator. Show that the image measure μT^{-1} is a centered Gaussian measure on Y. (ii) Show that $X^* \subseteq \mathcal{L}^2(\mu)$. (iii) Let us say that the **covariance matrix** of μ is the family $\langle\sigma_{fg}\rangle_{f,g\in X^*}$, where $\sigma_{fg} = \int f \times g\,d\mu$ for f, $g \in X^*$. Suppose that ν is another centered Gaussian measure on X with the same covariance matrix. Show that μ and ν agree on the cylindrical σ-algebra of X.

(n) Let $\langle X_i\rangle_{i\in I}$ be a family of linear topological spaces with product X. Suppose that for each i we have a centered Gaussian measure μ_i on X_i. Show that the product probability measure $\prod_{i\in I}\mu_i$ is a centered Gaussian measure on X.

(o) Let X be a linear topological space. Show that the convolution of two quasi-Radon centered Gaussian measures on X is a centered Gaussian measure.

(p) Let X be a separable Banach space, and μ a complete measure on X. Show that the following are equiveridical: (i) μ is a centered Gaussian measure on X; (ii) μ extends a centered Gaussian Radon measure on X; (iii) there are a set I, an injective continuous linear operator $T : X \to \mathbb{R}^I$ and a centered Gaussian distribution λ on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and λ; (iv) whenever I is a set and $T : X \to \mathbb{R}^I$ is a continuous linear operator there is a centered Gaussian distribution λ on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and λ.

Other exercises have been moved: 466Xg is now 466Xh, 466Xh-466Xi are now 466Xk-466Xl, 466Xj-466Xk are now 466Xi-466Xj, 466Xl-466Xm are now 466Xq-466Xr.

p 597 l 29 Exercise 466Yb has been deleted. Add new exercises:

(b) Let X be a normed space and $\mathfrak{T}$ a linear space topology on X such that the unit ball of X is $\mathfrak{T}$-closed and the topology on the unit sphere S induced by $\mathfrak{T}$ is finer than the norm topology on S. Show that every norm-Borel subset of X is $\mathfrak{T}$-Borel.

(c) (i) Let X be a Banach space. Set $S = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$ and suppose that $\langle K_\sigma\rangle_{\sigma\in S}$ is a family of non-empty weakly compact convex subsets of X such that $K_\sigma \subseteq K_\tau$ whenever σ, $\tau \in S$ and σ extends τ. (α) Show that there is a weakly Radon probability measure on X giving measure at least 2^{-n} to K_σ whenever $n \in \mathbb{N}$ and $\sigma \in \{0,1\}^n$. (β) Show that there are a $\sigma \in S$ and $x \in K_{\sigma^\frown<0>}$, $y \in K_{\sigma^\frown<1>}$ such that $\|x - y\| \le 1$. (ii) Let X be a locally convex Hausdorff linear topological space. and $\langle A_\sigma\rangle_{\sigma\in S}$ a family of non-empty relatively weakly compact subsets of X such that $A_\sigma \subseteq A_\tau$ whenever σ, $\tau \in S$ and σ extends τ. For $\sigma \in S$, set $C_\sigma = A_{\sigma^\frown<1>} - A_{\sigma^\frown<0>}$. Show that $0 \in \overline{\bigcup_{\sigma\in S} C_\sigma}$.

(e) Let X be a complete Hausdorff locally convex linear topological space and μ a Radon probability measure on X. Suppose that there is a linear subspace W of X^*, separating the points of X, such that every member of W is either zero a.e. or a normal random variable with zero expectation. Show that μ is a centered Gaussian measure.

466Yc is now 466Yd.

p 609 l 8 (467X) Add new exercise:

(j) Let K be an Eberlein compactum, and μ a Radon measure on K. Show that μ is completion regular and inner regular with respect to the compact metrizable subsets of K.

p 612 l 39 (part (g) of the proof of 471D): for '$A_i = U(x_i, \frac{1}{2}\rho(x_i, x_{1-i}))$' read '$A_i = A \cap U(x_i, \frac{1}{2}\rho(x_i, x_{1-i}))$'.

p 615 l 4-5 (part (a-vi) of the proof of 471G): for '$\le \lim_{n\in I,n\to\infty} \operatorname{diam} D_{ni} \le \lim_{n\in I,n\to\infty} \operatorname{diam} C_{ni} + 4\alpha_i + 4\zeta_i$' read '$\le \liminf_{n\in I,n\to\infty} \operatorname{diam} D_{ni} \le \liminf_{n\in I,n\to\infty} \operatorname{diam} C_{ni} + 4\alpha_i + 4\zeta_i$'.

p 617 l 3 Corollary 471H has been strengthened, and now ends '$\theta_{r\infty}$ is an outer regular Choquet capacity on X'.

p 617 l 14 (part (c) of the proof of 471H): for '432J' read '432I'.

p 618 l 16-21 (proof of 471J): for 'ϕ' read 'f', throughout.

p 622 l 3 (part (c) of the proof of 471P): for '$\bigcup_{n\in\mathbb{N}} \tilde{A}_{2^{-n}} \setminus A$' read '$\bigcap_{n\in\mathbb{N}} \tilde{A}_{2^{-n}} \setminus A$'.

p 624 l 14 Add new result:

471T Proposition Let (X, ρ) be a metric space, and $r > 0$.

(a) If X is analytic and $\mu_{Hr}X > 0$, then for every $s \in]0, r[$ there is a non-zero Radon measure μ on X such that $\iint \frac{1}{\rho(x,y)^s}\mu(dx)\mu(dy) < \infty$.

(b) If there is a non-zero topological measure μ on X such that $\iint \frac{1}{\rho(x,y)^r}\mu(dx)\mu(dy)$ is finite, then $\mu_{Hr}X = \infty$.

p 624 l 23 (471X) Add new exercises:

(b) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing, and that ν is the corresponding Lebesgue-Stieltjes integral (114Xa). Define $\rho(x, y) = |x - y| + \sqrt{|g(x) - g(y)|}$ for x, $y \in \mathbb{R}$. Show that ρ is a metric on $\mathbb{R}$ defining the usual topology. Show that ν is 2-dimensional Hausdorff measure for the metric ρ.

(c) Let $r \ge 1$ be an integer, and give $\mathbb{R}^r$ the metric $((\xi_1, \dots, \xi_r), (\eta_1, \dots, \eta_r)) \mapsto \max_{i\le r} |\xi_i - \eta_i|$. Show that Lebesgue measure on $\mathbb{R}^r$ is Hausdorff r-dimensional measure for this metric.

(e) Show that all the outer measures $\theta_{r\delta}$ described in 471A are outer regular Choquet capacities.

Other exercises have been renamed: 471Xb is now 471Xd, 471Xc-471Xh are now 471Xf-471Xk.

p 625 l 22 (471Y) Add new exercise:

(f) Let ρ be a metric on $\mathbb{R}$ inducing the usual topology. Show that the corresponding Hausdorff dimension of $\mathbb{R}$ is at least 1.

(i) Let (X,ρ) be a metric space and $0 \le s < t$. Suppose that there is an analytic set $A \subseteq X$ such that $\mu_{Ht}A > 0$. Show that there is a Borel surjection $f : X \to \mathbb{R}$ such that $\mu_{Hs}f^{-1}[\{\alpha\}] \ge 1$ for every $\alpha \in \mathbb{R}$.

p 625 l 38 Exercises 471Yh-471Yi are wrong as stated, and have been replaced by

(j) Let ρ be the metric on $\{0,1\}^{\mathbb{N}}$ defined in 471Xa. Show that for any integer $k \ge 1$ there are a $\gamma_k > 0$ and a bijection $f : [0,1]^k \to \{0,1\}^{\mathbb{N}}$ such that whenever $0 < r \le 1$, $\mu_{H,rk}$ is Hausdorff rk-dimensional measure on $[0,1]^k$ (for its usual metric) and $\tilde\mu_{Hr}$ is Hausdorff r-dimensional measure on $\{0,1\}^{\mathbb{N}}$, then $\mu^*_{H,rk}A \le \gamma_k\tilde\mu^*_{Hr}f[A] \le \gamma_k^2\mu^*_{H,rk}A$ for every $A \subseteq [0,1]^k$.

(k) Let (X,ρ) be a metric space, and $r > 0$. Give $X\times\mathbb{R}$ the metric σ where $\sigma((x,\alpha),(y,\beta)) = \max(\rho(x,y),|\alpha-\beta|)$. Write μ_L, μ_r and μ_{r+1} for Lebesgue measure on $\mathbb{R}$, r-dimensional Hausdorff measure on (X,ρ) and $(r+1)$-dimensional Hausdorff measure on $(X\times\mathbb{R},\sigma)$ respectively. Let λ be the c.l.d. product of μ_r and μ_L. (i) Show that if $W \subseteq X\times\mathbb{R}$ then $\overline{\int}\mu^*_rW^{-1}[\{\alpha\}]d\alpha \le \mu^*_{r+1}W$. (ii) Show that if $I \subseteq \mathbb{R}$ is a bounded interval, $A \subseteq X$ and μ^*_rA is finite, then $\mu^*_{r+1}(A\times I) = \mu^*_rA\cdot\mu_LI$. (iii) Give an example in which there is a compact set $K \subseteq X\times\mathbb{R}$ such that $\mu_{r+1}K = 1$ and $\lambda K = 0$. (iv) Show that if μ_r is σ-finite then $\mu_{r+1} = \lambda$.

Other exercises have been moved: 471Yf-471Yg are now 471Yg-471Yh.

p 630 l 11 (part (a) of the proof of 472D): for 'every point of $A_{nqq'}$ belongs to arbitrarily small members of $\mathcal{I}$' read 'every point of $A_{nqq'}$ is the centre of arbitrarily small members of $\mathcal{I}$'.

p 630 l 15 (part (a) of the proof of 472D): when claiming that $\int_{\bigcup\mathcal{I}_0}fd\lambda \le \int_E fd\lambda + \epsilon$ I seem to have taken it for granted that $G \subseteq B(\mathbf{0},n)$; which demands in turn that $A_{nqq'} \subseteq \operatorname{int}B(\mathbf{0},n)$. The simplest fix seems to be to change the definition of $A_{nqq'}$ to

$$A_{nqq'} = \{y : y \in Z\cap\operatorname{dom}f,\ \|y\| < n,\ f(y) \le q,\ \limsup_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}fd\lambda > q'\}$$

and then to demand that E and G are both included in $\operatorname{int}B(\mathbf{0},n)$.

p 631 l 3 (part (b) of the proof of 472D): for '2ϵ' read '3ϵ'.

p 632 l 18 (472X) Add new exercise:

(d) Let λ be a Radon measure on $\mathbb{R}^r$, and f a locally λ-integrable function. Show that $E = \{y : g(y) = \lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}f\,d\lambda$ is defined in $\mathbb{R}\}$ is a Borel set, and that $g : E \to \mathbb{R}$ is Borel measurable.

p 633 l 1 (Exercise 472Yc) 9^r looks a touch optimistic; $9^r + 1$ is what the easy argument seems to give.

p 633 l 2 (472Y) Add new exercises:

(d) Let $A \subseteq \mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that whenever $x \in A$ and $\epsilon > 0$ there is a ball $B(y,\delta) \in \mathcal{I}$ such that $\|x-y\| \le \epsilon\delta$. Show that there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

(e) Give an example of a strictly positive Radon probability measure μ on a compact metric space (X,ρ) for which there is a Borel set $E \subseteq X$ such that

$$\liminf_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 0, \quad \liminf_{\delta\downarrow 0}\frac{\mu(E\cap B(x,\delta))}{\mu B(x,\delta)} = 1$$

for every $x \in X$.

Other exercises have been moved: 472Yd-472Ye are now 472Yf-472Yg.

p 633 l 4 (Exercise 472Ye, now 472Yg): for '$\frac{\lambda' B_r(x)}{\lambda B_r(x)}$' read '$\frac{\lambda' B(x,\delta)}{\lambda B(x,\delta)}$'.

p 634 l 35 (argument for 473Bc): for '$\eta\|\phi(y)-\phi(x)-S(y-x)\|$' read '$\eta\|\phi(y)-\phi(x)\|$'.

p 636 l 5 (argument for 473Cd): for '$g'(t) = (y-x)\centerdot\operatorname{grad}f(g(t))$' read '$g'(t) = (y-x)\centerdot\operatorname{grad}f((1-t)x+ty)$'.

p 636 l 13 (argument for 473C(f-i)): for '$f_3(x)=\max(0,1-\frac{\|x\|}{\gamma})$' read '$f_3(x)=\min(1,\max(0,1+\gamma-\|x\|))$'.

p 636 l 32 (statement of 473Dd): for '$f*\operatorname{grad}g=\operatorname{grad}(f*g)$ is Lipschitz' read '$f*g$ is Lipschitz'.

p 637 l 21 (part (f) of the proof of 473D): for '$|(f*\frac{\partial g}{\partial\xi})(x)|$' read '$|(f*\frac{\partial g}{\partial\xi_i})(x)|$'.

p 638 l 14 (part (a) of the proof of 473E): for '$h'(t)\ge 0$' read '$h'(t)\le 0$'.

p 639 l 2 (part (e) of the proof of 473E): for '$2(n+1)^2h'((n+1)^2t^2)q(t)$' here, and two lines later, read '$2(n+1)^2th'((n+1)^2t^2)q(t)$'.

p 639 l 26 The second 'Lemma 473F' (the one starting 'Let (X,Σ,μ) be a σ-finite measure space...') should be headed '473G'.

p 639 l 28 (statement of Lemma 473G): for '$h_i(x)=\int S_i(x,t)\lambda(dt)$' read '$h_i(x)=\int h(S_i(x,t))\lambda(dt)$'.

p 641 l 5 (proof of 473H): for '473F' read '473G'.

p 641 l 7 (473I) There seem to be two miscalculations in part (c) of the proof. In the greater scheme of things these are trivial, but in terms of the formulae as written in this book they propagate dramatically, reaching into §474. To begin with, the statement of this lemma ought to read

For any Lipschitz function $f:B(\mathbf{0},1)\to\mathbb{R}$,

$$\int_{B(\mathbf{0},1)}|f|^{r/(r-1)}d\mu\le\Bigl(2^{r+4}\sqrt{r}\int_{B(\mathbf{0},1)}\|\operatorname{grad}f\|+|f|d\mu\Bigr)^{r/(r-1)},$$

'$1+\sqrt{r}$' being replaced by '$4\sqrt{r}$'.

p 641 l 9 (part (a) of the proof of 473I): for '$\|x\|\ne\sqrt{2}$' read '$\|x\|\ne 1/\sqrt{2}$'.

p 641 l 27 (The first significant error, in part (c) of the proof.) For '$\frac{\partial f_1}{\partial\xi_i}(\phi(x))\cdot\frac{1}{\|x\|^2}-\sum_{j=1}^r\frac{\partial f}{\partial\xi_j}(\phi(x))\cdot\frac{\xi_i\xi_j}{\|x\|^4}$' read '$\frac{\partial f_1}{\partial\xi_i}(\phi(x))\cdot\frac{1}{\|x\|^2}-2\sum_{j=1}^r\frac{\partial f_1}{\partial\xi_j}(\phi(x))\cdot\frac{\xi_i\xi_j}{\|x\|^4}$'.

p 641 l 29 (part (c) of the proof of 473I): for '$\{x:\|x\|<\sqrt{2}\}$' read '$\{x:\|x\|<\frac{1}{\sqrt{2}}\}$'.

p 642 l 5 (The second error) The estimate for $\|\operatorname{grad}f_2(x)\|$ seems to be completely off target; correct would I think be

$$|\frac{\partial f_2}{\partial\xi_i}(x)|\le\|\operatorname{grad}f_1(\phi(x))\|\frac{\|x\|+2|\xi_i|}{\|x\|^3}\le 4\|\operatorname{grad}f_1(\phi(x))\|,$$

$$\|\operatorname{grad}f_2(x)\|\le 4\sqrt{r}\|\operatorname{grad}f_1(\phi(x))\|$$

for almost every $x\in F$.

p 642 l 7 The second 'part (c)' of the proof of 473I, introducing 'We are now in a position to estimate', should be headed '(d)'; and the '(d)' on line 23 should be '(e)'.

p 642 l 9 (part (d) of the proof of 473I): for '$\{x:\|x\|<\sqrt{2}\}$' read '$\{x:\|x\|<\frac{1}{\sqrt{2}}\}$'.

p 643 l 17 (part (a) of the proof of 473J): for

$$`=\eta\int_0^1\frac{1}{t^{r-1}}\int_{B(y,\delta)\cap\partial B(z,t\eta)}\|\operatorname{grad}f(w)\|\nu(dw)dt'$$

read

$$`\le\eta\int_0^1\frac{1}{t^{r-1}}\int_{B(y,\delta)\cap\partial B(z,t\eta)}\|\operatorname{grad}f(w)\|\nu(dw)dt'.$$

p 644 l 5 (part (b) of the proof of 473J): to apply 473Dd as stated, we need to assume here that f is bounded. So the proof should end with

(c) Finally, if f is not bounded on the whole of $\mathbb{R}^r$, it is surely bounded on $B(y,\delta)$, so we can apply (b) to the function $x \mapsto \max(-M, \min(f(x), M))$ for a suitable $M \ge 0$ to get the result as stated.

p 644 l 21 (statement of 473K) for '$c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$' read '$c = 2^{r+4}\sqrt{r}(1+2^{r+1})$'.

p 645 l 4 (part (a) of the proof of 473K): for '$\int_0^1 t^{1-r}t^{r-1}\nu(\partial B(\mathbf{0},1))dt$' read '$\int_0^2 t^{1-r}t^{r-1}\nu(\partial B(\mathbf{0},1))dt$.

p 646 l 5 (statement of 473L) for '$c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$' read '$c = 2^{r+4}\sqrt{r}(1+2^{r+1})$'.

p 648 l 3 In 474Bd, we ought to start by confirming that $f \times \phi$ is Lipschitz.

p 648 l 31 (proof of Proposition 474C): for '$\sum_{j=1}^r \delta_{jj} = \operatorname{div}\phi(x)$' read '$\sum_{j=1}^r \delta_{jj} = \operatorname{div}\phi(Tx)$'.

p 650 l 30 In part (d) of the proof of 474E, I think we need to say that $\epsilon = \frac{1}{3}(\lambda_E^\partial(G_0) - \gamma)$ rather than that $\epsilon = \frac{1}{2}(\lambda_E^\partial(G_0) - \gamma)$, so that later we can say that

$$\sum_{i=1}^r \int f_{mi} \times \psi_i d\lambda_E^\partial \ge \lambda_E^\partial(G_0) - 3\epsilon = \gamma.$$

p 652 l 4 (474G): for '$\int_{B(y,\delta)} \|\psi(x) - z\|\, d\lambda_E^\partial$' read '$\int_{B(y,\delta)} \|\psi(x) - z\|\, \lambda_E^\partial(dx)$'.

p 652 l 6 (474G): for '$\partial\lambda_E^\$$' read 'λ_E^∂'.

p 652 l 12 (474G): for

$$\text{`}\lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(x,\delta)} \int_{B(x,\delta)} |\psi_i(x) - \psi_i(y)| \lambda_E^\partial(dx) = 0\text{'}$$

read

$$\text{`}\lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} |\psi_i(x) - \psi_i(y)| \lambda_E^\partial(dx) = 0\text{'}$$

p 653 l 3 (part (c) of the proof of 474H): for 'ψT' read 'ψT^{-1}'.

p 654 l 33 (statement of 474L) for '$c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$' read '$c = 2^{r+4}\sqrt{r}(1+2^{r+1})$'.

p 655 l 22 (part (b-i) of the proof of 474L): for '$\frac{1}{2}(\alpha-\epsilon)^{r/(r-1)}$' read '$\frac{1}{2}(\alpha-\epsilon)^{(r-1)/r}$'.

p 657 l 19 (part (c) of the proof of 474M): for '$\int_{\partial B(y,t)} \frac{x-y}{\eta\|x-y\|} \cdot \phi(x)\nu(dx)$' read '$\int_{E\cap\partial B(y,t)} \frac{x-y}{\eta\|x-y\|} \cdot \phi(x)\nu(dx)$'.

p 658 l 6 (statement of 474N) for '$c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$' read '$c = 2^{r+4}\sqrt{r}(1+2^{r+1})$'.

p 658 l 10 (part (a) of the proof of 474N): for '$\operatorname{div}\phi \times \chi B_1$' read '$\operatorname{div}\phi \times \chi B(y,1)$'.

p 660 l 9 474Q-474S are now 474R-474T. Part of the proof of the former Theorem 474Q has been extracted, in the following form:

474Q Lemma Set $c' = 2^{r+3}\sqrt{r-1}(1+2^r)$. Suppose that c^*, ϵ and δ are such that

$$c^* \ge 0, \quad \delta > 0, \quad 0 < \epsilon < \frac{1}{\sqrt{2}}, \quad c^*\epsilon^3 < \tfrac{1}{4}\beta_{r-1}, \quad 4c'\epsilon \le \tfrac{1}{8}\beta_{r-1}.$$

Set $V_\delta = \{z : z \in \mathbb{R}^{r-1}, \|z\| \le \delta\}$ and $C_\delta = V_\delta \times [-\delta,\delta]$, regarded as a cylinder in $\mathbb{R}^r$. Let $f \in \mathfrak{D}$ be such that

$$\int_{C_\delta} \|\operatorname{grad}_{r-1} f\| + \max(\frac{\partial f}{\partial \xi_r}, 0) d\mu \le c^*\epsilon^3\delta^{r-1},$$

where $\operatorname{grad}_{r-1} f = (\frac{\partial f}{\partial \xi_1}, \dots, \frac{\partial f}{\partial \xi_{r-1}}, 0)$. Set

$$F = \{x : x \in C_\delta,\ f(x) \ge \frac{3}{4}\}, \quad F' = \{x : x \in C_\delta,\ f(x) \le \frac{1}{4}\}.$$

and for $\gamma \in \mathbb{R}$ set $H_\gamma = \{x : x \in \mathbb{R}^r,\ \xi_r \le \gamma\}$. Then there is a $\gamma \in \mathbb{R}$ such that

$$\mu(F \triangle (H_\gamma \cap C_\delta)) \le 9\mu(C_\delta \setminus (F \cup F')) + (c^*\beta_{r-1} + 16c')\epsilon\delta^r.$$

p 660 l 14 (part (a) of the proof of 474Q, now in 474R): for

$$\text{`}c' = 2^{r+1}(1+\sqrt{r-1})(1+2^r),$$

$$c_1 = 1+\max\bigl(4\pi\beta_{r-2}, (3r)^r 2^{r+3}(1+\sqrt{r})(1+2^{r+1})\bigr)\text{'}$$

read

$$\text{`}c' = 2^{r+3}\sqrt{r-1}(1+2^r),$$

$$c_1 = 1+\max\bigl(4\pi\beta_{r-2}, (3r)^r 2^{r+5}\sqrt{r}(1+2^{r+1})\bigr)\text{'}.$$

p 662 l 3 (part (c) of the proof of 474Q): for '$\|\phi\| \le \sqrt{2}\chi C_\delta$' read '$\|\phi(x)\| \le \sqrt{2}\chi C_\delta$ for every x'.

p 666 l 26 (part (b) of the proof of 474R, now 474S): for '$|\lambda^\partial_E B(y,\eta) - \beta_{r-1}\eta^{r-1}| \le \epsilon\delta^{r-1}$' read '$|\lambda^\partial_E B(y,\eta) - \beta_{r-1}\eta^{r-1}| \le \epsilon\eta^{r-1}$'.

p 667 l 4 (part (b) of the proof of 474R, now 474S): for 'almost every $\eta \in \,]0,\delta_0]$' read 'almost every $\eta \in \,]0,(1+\epsilon)\delta_0]$'.

p 667 l 14 (part (b) of the proof of 474R, now 474S): for '$\frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \zeta\delta^r(1+\epsilon)r$' read '$\frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \zeta\delta^r(1+\epsilon)^r$'.

p 668 l 7 (part (b-i) of the proof of 474S) for '$c_1 = 2^{r+3}(1+2^{r+1})(1+\sqrt{r})^{r+1}$' read '$c_1 = 2^{r+5}(1+2^{r+1})\sqrt{r})^r(1+2\sqrt{r})$'.

p 668 l 13 (part (b-i) of the proof of 474S) for '$c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$' read '$c = 2^{r+4}\sqrt{r}(1+2^{r+1})$'.

p 668 l 32 (part (b-ii) of the proof of 474S): for '$(n+2)^{r+1}$' read '$(n+1)^{r+2}$'.

p 669 l 9 (474X) Exercise 474Xa has been deleted. Exercise 474Xb is now 474Xa. A new exercise has been added:

> **(c)** Show that if $E \subseteq \mathbb{R}^r$ has finite perimeter then either E or its complement has finite measure.

p 669 l 12 (Exercise 474Xb, now 474Xa): add 'Show that the canonical-outward normal function of E is Borel measurable'.

p 671 l 20 (part (c) of the proof of 475C): for

$$\text{`cl*}A = \{x : \limsup_{\delta\downarrow 0} \tfrac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} = 1\}\text{'}$$

read

$$\text{`cl*}A = \{x : \limsup_{\delta\downarrow 0} \tfrac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} > 0\}\text{'}.$$

p 674 l 10 (part (a-ii) of the proof of 475F): for

> $H\cap B(\mathbf{0},\delta)$ includes
>
> $$(H_0\cap B(\mathbf{0},\delta)) \cup \{u : u\in\mathbb{R}^{r-1},\ \|u\| \le \frac{1}{2}\delta\} \times [0,\alpha]$$

read

> $H\cap B(\mathbf{0},\delta)$ includes
>
> $$(H_0\cap B(\mathbf{0},\delta)) \cup (\{u : u\in\mathbb{R}^{r-1},\ \|u\| \le \frac{1}{2}\delta\} \times [0,\alpha'])$$
>
> where $\alpha' = \min(|\alpha|, \frac{\sqrt{3}}{2}\delta) > \eta\delta)$.

Two lines later we need $2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha'$ in place of $2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha$. Similarly, when looking at the possibility that $\alpha < -\eta\delta$, we need $2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha'$ rather than $2^{-r+1}\beta_{r-1}\delta^{r-1}|\alpha|$.

p 675 l 2 (part (b) of the proof of 475F): for '$n \ge n_0$' read '$n > n_0$'.

p 675 l 13 (part (c) of the proof of 475F): for '$k \ge n_0$' read '$k > n_0$'.

p 675 l 24 (part (d) of the proof of 475F): Every K_2 in this part of the proof should be replaced by $K_2 \cap B(y, 2^{-n-1})$.

p 675 l 25 (part (d) of the proof of 475F): for '$x,\, z \in F_1$' read '$x,\, z \in K_2 \cap B(y, 2^{-n-1})$'.

p 675 l 28 (part (d) of the proof of 475F): for '$\frac{1}{(1-\eta)^{r-1}}\nu F_1$' read '$(1-\eta)^{r-1}\nu K_2 \cap B(y, 2^{-n-1})$'.

p 676 l 14 (part (b) of the proof of 475G): for '$\lambda^{\partial}_{E} F$' read 'λ^{∂}_{E}'.

p 677 l 4 (part (b) of the proof of 475H): for '$\le \nu A$' read '$\le \nu A + \epsilon$'.

p 677 l 12 (part (a) of the proof of 475I): for '$\frac{\mu(B(x,2^{-m}\sqrt{r})\setminus A)}{\mu B(x,2^{-m}\sqrt{r})}$' read '$\frac{\mu^*(B(x,2^{-m}\sqrt{r})\setminus A)}{\mu B(x,2^{-m}\sqrt{r})}$'.

p 678 l 9 (part (a-i) of the proof of 475J): for '$f : \mathbb{R}^{r-1} \to [-\infty, q]$' read '$f_q : \mathbb{R}^{r-1} \to [-\infty, q]$'.

p 678 l 24 (part (a-ii) of the proof of 475J): for '$f_q(u) < t \le f_q(u) + \delta$' read '$f_q(u) < t < f_q(u) + \delta$'.

p 678 l 33 (part (a-iii) of the proof of 475J): for '$f'_q(u) = \inf(H_u \cap]u, \infty[)$' read '$f'_q(u) = \inf(H_u \cap]q, \infty[)$'.

p 680 l 16 (part (a) of the proof of 475K): for 'D_u' read 'D'_u' (and again in line 18).

p 682 l 37 (part (b) of the proof of 475O): for

$$F_n = F_{q_n} \setminus \bigcup_{m<n} \{u : u \in F_{q_m},\, f_{q_m}(u) < q_n < f'_{q_m}(u)\}$$

read

$$F_n = F'_{q_n} \setminus \bigcup_{m<n} \{u : u \in F'_{q_m},\, f_{q_m}(u) < q_n < f'_{q_m}(u)\}.$$

p 682 l 39 (part (b) of the proof of 75O): for '$u \in F_{q_n}$' read '$u \in F_n$'.

p 686 l 5 Theorem 475Q has been expanded, and now reads

Theorem (a) Let $E \subseteq \mathbb{R}^r$ be a set with finite perimeter. For $v \in S_{r-1}$ write V_v for $\{x : x \centerdot v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\begin{aligned}\operatorname{per} E = \nu(\partial^* E) &= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) \\ &= \lim_{\delta \downarrow 0} \frac{1}{2\beta_{r-1}\delta} \int_{S_{r-1}} \mu(E \triangle (E + \delta v))\nu(dv).\end{aligned}$$

(b) Suppose that $E \subseteq \mathbb{R}^r$ is Lebesgue measurable. Set

$$\gamma = \sup_{x \in \mathbb{R}^r \setminus \{0\}} \frac{1}{\|x\|} \mu(E \triangle (E + x)).$$

Then $\gamma \le \operatorname{per} E \le \frac{r\beta_r\gamma}{2\beta_{r-1}}$.

p 687 l 24 (part (d) of the proof of 475R): for '$\frac{\sqrt{r}}{M}$' read '$\frac{1}{M\sqrt{r}}$'.

p 688 l 1 (part (b) of the proof of 475S): for 'the unique point of C closest to x' read 'the unique point of $\overline{C}$ closest to x'.

p 688 l 3 (part (b) of the proof of 475S): for '$\|x - \phi(x) - \epsilon e\| \ge \|x - \phi(x)\|$' read '$\|x - \phi(x) + \epsilon e\| \ge \|x - \phi(x)\|$'.

p 688 l 26 (part (d) of the proof of 475S): for '$\partial^* E$' read '$\partial^* C$' (and again in the next two lines).

p 699 l 10 The formula in Exercise 475Xc is wrong as written, and should be

$$\partial^*(A \cap B) \triangle ((B \cap \partial^* A) \cup (A \cap \partial B)) \subseteq A \cap \partial B \setminus \partial^* A.$$

p 689 l 30 (475X) Add new exercises:

(l) Let $E\subseteq\mathbb{R}^r$ be a set with finite measure and finite perimeter, and $f:\mathbb{R}^r\to\mathbb{R}$ a Lipschitz function. Show that for any unit vector $e\in\mathbb{R}^r$, $|\int_E e\,.\,\operatorname{grad} f|\le\|f\|_\infty\operatorname{per}E$.

(m) For measurable $E\subseteq\mathbb{R}^r$ set $p(E)=\sup_{x\in\mathbb{R}^r\setminus\{0\}}\frac{1}{\|x\|}\mu(E\triangle(E+x))$. (i) Show that for any measurable E, $p(E)=\limsup_{x\to\mathbf{0}}\frac{1}{\|x\|}\mu(E\triangle(E+x))$. (ii) Show that for every $\epsilon>0$ there is an $E\subseteq\mathbb{R}^r$ such that $\operatorname{per}E=1$ and $p(E)\ge 1-\epsilon$. (iii) Show that if $E\subseteq\mathbb{R}^r$ is a non-trivial ball then $\operatorname{per}E=\frac{r\beta_r}{2\beta_{r-1}}p(E)$. (iv) Show that if $E\subseteq\mathbb{R}^r$ is a cube then $\operatorname{per}E=\sqrt{r}p(E)$.

(n) Suppose that $E\subseteq\mathbb{R}^r$ is a bounded set with finite perimeter, and ϕ, $\psi:\mathbb{R}^r\to\mathbb{R}$ two Lipschitz functions such that $\operatorname{grad}\phi$ and $\operatorname{grad}\psi$ are also Lipschitz. Show that

$$\int_E\phi\times\nabla^2\psi-\psi\times\nabla^2\phi\,d\mu=\int_{\partial^*E}(\phi\times\operatorname{grad}\psi-\psi\times\operatorname{grad}\phi)\,.\,v_x\,\nu(dx)$$

where, for $x\in\partial^*E$, v_x is the Federer exterior normal to E at x. (This is **Green's second identity**.)

p 689 l 31 (475Y) Add new exercises:

(a) Show that if $A\subseteq\mathbb{R}^r$ is negligible, then there is a Borel set $E\subseteq\mathbb{R}^r$ such that $A\subseteq\partial^*E$.

(b) Let (X,ρ) be a metric space and μ a strictly positive locally finite topological measure on X. Show that we can define operations cl*, int* and ∂^* on $\mathcal{P}X$ for which parts (a)-(f) of 475C will be true.

(e) Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence of functions from $\mathbb{R}^r$ to $\mathbb{R}$ which is uniformly bounded and **uniformly Lipschitz** in the sense that there is some $\gamma\ge0$ such that every f_n is γ-Lipschitz. Suppose that $f=\lim_{n\to\infty}f_n$ is defined everywhere in $\mathbb{R}^r$. (i) Show that if $E\subseteq\mathbb{R}^r$ has finite measure and finite perimeter, then $\int_E z\,.\,\operatorname{grad}f d\mu=\lim_{n\to\infty}\int_E z\,.\,\operatorname{grad}f_n d\mu$ for every $z\in\mathbb{R}^r$. (ii) Show that for any convex function $\phi:\mathbb{R}^r\to[0,\infty[$, $\int\phi(\operatorname{grad}f)d\mu\le\liminf_{n\to\infty}\int\phi(\operatorname{grad}f_n)d\mu$.

(f) Let $E\subseteq\mathbb{R}^r$ be a measurable set with locally finite perimeter. Show that

$$\sup_{x\in\mathbb{R}^r\setminus\{0\}}\frac{1}{\|x\|}\mu(E\triangle(E+x))=\sup_{\|v\|=1}\int_{\partial^*E}|v\,.\,v_x|\nu(dx),$$

where v_x is the Federer exterior normal of E at x when this is defined.

(g) Let $E\subseteq\mathbb{R}^r$ be Lebesgue measurable. (i) Show that int*E is an $\mathrm{F}_{\sigma\delta}$ set. (ii) Show that if E is not negligible and cl*E has empty interior, then int*E is not $\mathrm{G}_{\delta\sigma}$.

Exercises 475Ya-475Yb have been re-named 475Yc-475Yd, and 475Yc has been moved to 474Xb.

p 690 l 38 I have re-cast the arguments of 476J, 476K and 476O (now 476F, 476G and 476K) so that they use Fell topologies instead of Hausdorff metrics. The introductory material, formerly in 476A-476C, has been moved to 4A2T; 476D and 476E have been consolidated into 476A, and 476F-476P have become 476B-476L.

p 690 l 42 (Definition 476Aa, now 4A2T(a-iii)): the formula

$$\text{‘}\tilde\rho(E,F)=\max(\sup_{x\in E}\rho(x,F),\sup_{y\in F}\rho(y,E))\text{’}$$

can produce infinite values, and for consistency with the standard definition of ‘metric’ should be changed to

$$\text{‘}\tilde\rho(E,F)=\min(1,\max(\sup_{x\in E}\rho(x,F),\sup_{y\in F}\rho(y,E)))\text{’}.$$

p 692 l 8 (Definition 476Cd, now 4A2T(a-ii)): the ‘k-Vietoris’ topology on the space of closed subsets of a topological space should throughout be called the **Fell topology**.

p 692 l 26 The former 476D-476E are now

476A Proposition Let X be a topological space, $\mathcal{C}$ the family of closed subsets of X, $\mathcal{K}\subseteq\mathcal{C}$ the family of closed compact sets and μ a topological measure on X.

(a) Suppose that μ is inner regular with respect to the closed sets.

(i) If μ is totally finite then $\mu\restriction\mathcal{C}:\mathcal{C}\to[0,\infty[$ is upper semi-continuous with respect to the Vietoris topology on $\mathcal{C}$.

(ii) If μ is locally finite then $\mu\restriction\mathcal{K}$ is upper semi-continuous with respect to the Vietoris topology.

(iii) If f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Vietoris topology.

(b) Suppose that μ is tight.

(i) If μ is totally finite then $\mu\restriction\mathcal{C}$ is upper semi-continuous with respect to the Fell topology on $\mathcal{C}$.

(ii) If f is a non-negative μ-integrable real-valued function then $F \mapsto \int_F f d\mu : \mathcal{C} \to \mathbb{R}$ is upper semi-continuous with respect to the Fell topology.

(c) Suppose that X is metrizable, and that ρ is a metric on X defining its topology; let $\tilde{\rho}$ be the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) If μ is totally finite, then $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(ii) If μ is locally finite, then $\mu\restriction\mathcal{K} \setminus \{\emptyset\}$ is upper semi-continuous with respect to $\tilde{\rho}$.

(iii) If f is a non-negative μ-integrable real-valued function, then $F \mapsto \int_F f d\mu : \mathcal{C} \setminus \{\emptyset\} \to \mathbb{R}$ is upper semi-continuous with respect to $\tilde{\rho}$.

p l Add new fragment to Lemma 476I, now 476E:
(b)(ii) For any $A \subseteq X$, $\nu^*\psi(A) \le \nu^* A \le 2\nu^*\psi(A)$.

p l (part (e-ii) of the proof of 476I, now 476E) For '$\nu^*(B(x,\delta) \cap \psi(A)) = \nu^*(B(x,\delta) \cap A)$' read

$$\text{`}\mu^*(B(x,\delta) \cap \psi(A)) \le \mu^*(B(x,\delta) \cap A) \le 2\mu^*(B(x,\delta) \cap \psi(A))$$

where μ is Lebesgue measure on $\mathbb{R}^r$'.

p l (part (b) of the proof of 476O, now 476K): for '$\int_F (1 + (x|e_0))\nu(dx) \ge \int_{F_1}(1 + (x|e_0))\nu(dx)$ for every $F \in \mathcal{F}$' read '$\int_F (1 + (x|e_0))\nu(dx) \le \int_{F_1}(1 + (x|e_0))\nu(dx)$ for every $F \in \mathcal{F}$'.

p l (proof of 476P, now 476L) for '$\nu^* A_1$' read '$\nu_X^* A_1$'.

p 701 l 31 Exercises 476Xa, 476Xc and 476Xf have been deleted. 476Xb and 476Xd have been moved to 441Xo-441Xp. 476Xg-476Xk are now 476Xb-476Xf. 476Xe has been expanded and re-named, and is now

(a) Let X be a topological space, $\mathcal{C}$ the set of closed subsets of X, μ a topological measure on X and f a μ-integrable real-valued function; set $\phi(F) = \int_F f d\mu$ for $F \in \mathcal{C}$. (i) Show that if *either* μ is inner regular with respect to the closed sets and $\mathcal{C}$ is given its Vietoris topology *or* μ is tight and $\mathcal{C}$ is given its Fell topology, then ϕ is Borel measurable. (ii) Show that if X is metrizable and $\mathcal{C} \setminus \{\emptyset\}$ is given an appropriate Hausdorff metric, then $\phi\restriction\mathcal{C} \setminus \{\emptyset\}$ is Borel measurable.

p 702 l 27 Exercises 476Ya and 476Yb have been deleted. 476Yc is now 476Ya. Add new exercise:

(b) Let $r \ge 1$ be an integer, and $g \in C_0(\mathbb{R}^r)$ a non-negative γ-Lipschitz function, where $\gamma \ge 0$. Let $\phi : \mathbb{R}^r \to [0, \infty[$ be a convex function. Let F be the set of non-negative γ-Lipschitz functions $f \in C_b(\mathbb{R}^r)$ such that f has the same decreasing rearrangement as g with respect to Lebesgue measure μ on $\mathbb{R}^r$ and $\int \phi(\operatorname{grad} f)d\mu \le \int \phi(\operatorname{grad} g)d\mu$. (i) Show that F is compact for the topology of pointwise convergence. (ii) Show that there is a $g^* \in F$ such that $g^*(x) = g^*(y)$ whenever $\|x\| = \|y\|$.

p 703 l 5 Three new sections have been added to Chapter 47:
477 Brownian motion
478 Harmonic functions
479 Newtonian capacity

These aim to give a foundation for the study of Newtonian capacity based on the theory of Brownian motion. For more information see `http://www1.essex.ac.uk/maths/people/fremlin/cont47.htm`.

p 707 l 32 (481H, part (d)): for '$D = \sup_{c\in C_0, c'\in C_1} D_{cc'}$' read '$D = \bigcup_{c\in C_0, c'\in C_1} D_{cc'}$'.

p 708 l 13 (481I, the proper Riemann integral): to match 481G(iv), we need to allow the empty set as a member of $\mathcal{C}$. The same is required in setting up the Henstock integral (481J, 481K) and the McShane integral (481M).

p 708 l 34 (481K): for 'the family of all non-empty intervals' read 'the family of all non-empty bounded intervals'.

p 713 l 2 (Exercise 481Xd): for '$S_t(f,\mu)$' read '$S_{\boldsymbol{t}}(f,\mu)$'.

p 713 l 30 (481X) Add new exercise:

(j) Let X be a set, Σ an algebra of subsets of X, and $\nu:\Sigma\to[0,\infty[$ an additive functional. Set $Q=\{(x,C):x\in C\in\Sigma\}$ and let T be the straightforward set of tagged partitions generated by Q. Let $\mathbb{E}$ be the set of disjoint families $\mathcal{E}\subseteq\Sigma$ such that $\sum_{E\in\mathcal{E}}\nu E=\nu X$, and $\Delta=\{\delta_{\mathcal{E}}:\mathcal{E}\in\mathbb{E}\}$, where

$$\delta_{\mathcal{E}}=\{(x,C):(x,C)\in Q\text{ and there is an }E\in\mathcal{E}\text{ such that }C\subseteq E\}$$

for $\mathcal{E}\in\mathbb{E}$. Set $\mathfrak{R}=\{\mathcal{R}_\epsilon:\epsilon>0\}$ where $\mathcal{R}_\epsilon=\{E:E\in\Sigma,\ \nu E\le\epsilon\}$ for $\epsilon>0$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ.

p 713 l 31 (Exercise 481Ya): for '$[[0,1]\times\mathcal{C}]^{<\omega}$' read '$[[a,b]\times\mathcal{C}]^{<\omega}$'.

p 716 l 23 (part (c) of the proof of 482B) There is a not-quite-trivial slip in the proof here. In the (admittedly unimportant) case in which $\emptyset\in\mathcal{C}$ and $\nu\emptyset\ne0$, it is not necessarily true that

$$S_{\boldsymbol{t}_{E\cup E'}}(f,\nu)=S_{\boldsymbol{t}_E}(f,\nu)+S_{\boldsymbol{t}_{E'}}(f,\nu)$$

for every $\boldsymbol{t}\in T_E\cap T_{E'}$, even when $E\cap E'=\emptyset$; rather, we have

$$S_{\boldsymbol{t}_{E\cup E'}}(f,\nu)+S_{\boldsymbol{t}_{E\cap E'}}(f,\nu)=S_{\boldsymbol{t}_E}(f,\nu)+S_{\boldsymbol{t}_{E'}}(f,\nu)$$

for all E, $E'\in\mathcal{E}$, so that $F(E\cup E')+F(E\cap E')=F(E)+F(E')$. So we need to show that $F(\emptyset)=0$. But this is true, because if $\boldsymbol{t}$ is δ-fine and $\boldsymbol{s}\in T$ is δ-fine and $\mathcal{R}$-filling, then $\boldsymbol{s}'=\boldsymbol{s}\setminus\boldsymbol{t}_\emptyset$, $\boldsymbol{s}''=\boldsymbol{s}\cup\boldsymbol{t}_\emptyset$ are δ-fine and $\mathcal{R}$-filling, and

$$S_{\boldsymbol{t}_\emptyset}(f,\nu)=S_{\boldsymbol{s}''}(f,\nu)-S_{\boldsymbol{s}'}(f,\nu).$$

Accordingly, given $\epsilon>0$, we can find $\delta\in\Delta$ and $\mathcal{R}\in\mathfrak{R}$ such that $|S_{\boldsymbol{s}''}(f,\nu)-S_{\boldsymbol{s}'}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{s}''$, $\boldsymbol{s}'$ are δ-fine and $\mathcal{R}$-filling, and we shall now have $|S_{\boldsymbol{t}_\emptyset}(f,\nu)|\le\epsilon$ whenever $\boldsymbol{t}$ is δ-fine, so that $F(\emptyset)=0$.

p 716 l 33 (part (d-i) of the proof of 482B): for '$\boldsymbol{t}'=\boldsymbol{t}\cup\boldsymbol{s}\setminus\boldsymbol{s}_1$' read '$\boldsymbol{t}'=\boldsymbol{t}\cup(\boldsymbol{s}\setminus\boldsymbol{s}_1)$'.

p 717 l 11 (proof of 482B): the second 'part (d)' should be re-labelled '(e)', and 'part (e)' below should be re-labelled '(f)'.

p 717 l 25 (part (f) of the proof of 482B): for '$S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|\le\epsilon$' read '$|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|\le\epsilon$'.

p 717 l 26 (part (f) of the proof of 482B): for '$|S_{\boldsymbol{t}}-I_\nu(f)|$' read '$|S_{\boldsymbol{t}}(f,\nu)-I_\nu(f)|$'.

p 717 l 38 At the end of the statement of Theorem 482D, add 'In this case, $F(X)=I_\nu(f)$'.

p 717 l 40 (proof of 482D): for 'T_α' read 'T'.

p 718 l 1 (proof of 482D): for '$\mathbb{R}^r$' read 'X' (and again on line 3).

p 718 l 9 To make Theorem 482E apply to the Pfeffer integral (§484), the condition (iv) needs to be relaxed fractionally. Instead of

(iv) Δ is a downwards-directed family of neighbourhood gauges on X containing all the uniform metric gauges,

we need

(iv)(α) Δ is a downwards-directed family of gauges on X containing all the uniform metric gauges;

(β) if $\delta\in\Delta$, there are a negligible set $F\subseteq X$ and a neighbourhood gauge δ_0 on X such that $\delta\supseteq\delta_0\setminus(F\times\mathcal{P}X)$

p 718 l 32 (proof of 482E): for '$\bigcup_{i<m}C_i=W_{\boldsymbol{t}}$ belongs to $\mathcal{R}'$' read '$X\setminus\bigcup_{i<m}C_i=X\setminus W_{\boldsymbol{t}}$ belongs to $\mathcal{R}'$'.

p 718 l 21 (proof of 482E): for '$\delta=\{(x,C):x\in X,\ C\in\mathcal{C},\ C\subseteq G_x\}$' read '$\delta=\{(x,C):x\in X,\ C\subseteq G_x\}$'.

p 719 l 36 (part (b) of the proof of 482F): for 'Because $\bigcup_{i<n} C_i \in \mathcal{R}$' read 'Because $X \setminus \bigcup_{i<n} C_i \in \mathcal{R}$'.

p 720 l 15 (condition (iv) in the statement of Proposition 482G): for '$\{(x,C)\} \in T \cap \delta_0$' read '$(x,C) \in \delta_0$ and $\{(x,C)\} \in T$; and similarly in part (a) of the proof, line 25.

p 720 l 28 (part (a) of the proof of 482G): for '$\{\sum_{i=0}^{n} |\nu C_i| : C_0, \dots , C_n \in \mathcal{C}$ are disjoint, $\bigcup_{i\le n} C_i \cap (F \cup F') = \emptyset\}$' read '$\{\sum_{i=0}^{m} |\nu C_i| : C_0, \dots , C_m \in \mathcal{C}$ are disjoint, $\bigcup_{i\le m} C_i \cap (F_n \cup F'_n) = \emptyset\}$'.

p 721 l 17 (part (c) of the proof of 482G): for 'for each i' read 'for each m'.

p 721 l 18 (part (c) of the proof of 482G): for '$X \subseteq (F_m \cup F'_m)$' read '$X \setminus (F_m \cup F'_m)$'.

p 722 l 25 The definitions of 'full' and 'countably full' families of gauges (482Ia) have been moved to 481Ec. 482Ib is now 482J, and 482J is now 482I.

p 723 l 25 B.Levi's theorem, 482K, has been fractionally strengthened, and now reads

Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, such that Δ is countably full, and $\nu : \mathcal{C} \to [0, \infty[$ a function which is moderated with respect to T and Δ.

Let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of functions from X to $\mathbb{R}$ with supremum $f : X \to \mathbb{R}$. If $\gamma = \lim_{n\to\infty} I_\nu(f_n)$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to γ.

p 723 l 34 (part (a) of the proof of 482K): for '$|I_\nu(f_n) - S_{\boldsymbol{t}}(f, \nu)| \le 2^{-n}\epsilon$' read '$|I_\nu(f_n) - S_{\boldsymbol{t}}(f, \nu)| \le 2^{-n-1}\epsilon$'.

p 724 l 7 (part (b) of the proof of 482K): for '$I_\nu(f) \ge \gamma - \epsilon$' read '$I_\nu(f_n) \ge \gamma - \epsilon$'.

p 724 l 33 (Fubini's theorem, 482M): the hypotheses given are not sufficient to support the argument here. In order to apply Lemma 482L as called for at the end of part (b-i) of the proof, we need a further hypothesis

(vi) whenever $\delta \in \Delta_1$ and $\boldsymbol{s} \in T_1$ is δ-fine, there is a δ-fine $\boldsymbol{s}' \in T_1$, including $\boldsymbol{s}$, such that $W_{\boldsymbol{s}'} = X_1$.

p 725 l 15 (part (b-i) of the proof of 482M): for '$S_{\boldsymbol{t}'_x(f_x,\nu_1}$' read '$S_{\boldsymbol{t}'_x}(f_x, \nu_1)$.

p 725 l 21 (part (b-i) of the proof of 482M): in the displayed formula '$S_{\boldsymbol{t}} = \dots \le 2\epsilon$', every '$\boldsymbol{t}$' should be '$\boldsymbol{s}$'.

p 726 l 18 (Exercise 482Xb): for '$f^-(x) = \max(0, f(x))$' read '$f^-(x) = \max(0, -f(x))$'.

p 727 l 3 (Exercise 482Xg): for '$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}$' read '$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\{\{\emptyset\}\})}$'.

p 727 l 17 (482X) Add new exercises:

(l) Let $r \ge 1$ be an integer, and μ a Radon measure on $\mathbb{R}^r$. Let Q be the set of pairs (x, C) where $x \in \mathbb{R}^r$ and C is a closed ball with centre x, and T the straightforward set of tagged partitions generated by Q. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in 482Xd. (i) Show that T is compatible with Δ and $\mathfrak{R}$. (ii) Show that if I_μ is the associated gauge integral, and $f : \mathbb{R}^r \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(m) Let $(X, T, \Delta, \mathfrak{R})$, Σ and ν be as in 481Xj, so that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by an algebra Σ of subsets of X, and $\nu : \Sigma \to [0, \infty[$ is additive. Let I_ν be the corresponding gauge integral, and $V \subseteq \mathbb{R}^X$ its domain. (i) Show that $\chi E \in V$ and $I_\nu(\chi E) = \nu E$ for every $E \in \Sigma$. (ii) Show that if $f \in \mathbb{R}^X$ then $f \in V$ iff for every $\epsilon > 0$ there is a disjoint family $\mathcal{E} \subseteq \Sigma$ such that $\sum_{E\in\mathcal{E}} \nu E = \nu X$ and $\sum_{E\in\mathcal{E}} \nu E \cdot \sup_{x,y\in E} |f(x)-f(y)| \le \epsilon$. (iii) Show that if $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V with supremum $f \in \mathbb{R}^X$, and $\gamma = \sup_{n\in\mathbb{N}} I_\nu(f_n)$ is finite, then $I_\nu(f)$ is defined and equal to γ. (iv) Show that I_ν extends $⨍ d\nu$ as described in 363L, if we identify $L^\infty(\Sigma)$ with a space $\mathcal{L}^\infty$ of functions as in 363H. (v) Show that if Σ is a σ-algebra of sets then I_ν extends $⨍ d\nu$ as described in 364Xj.

p 727 l 18 (482Y) Add new exercise:

(a) Let X be the interval $[0,1]$, $\mathcal{C}$ the family of subintervals of X, Q the set $\{(x,C) : C \in \mathcal{C}, x \in \overline{\operatorname{int} C}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X, and $\mathfrak{R}$ the singleton $\{[X]^{<\omega}\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$. For $C \in \mathcal{C}$ set $\nu C = 1$ if $0 \in \overline{\operatorname{int} C}$, 0 otherwise, and let f be $\chi\{0\}$. Show that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined and equal to 1. Let F be the Saks-Henstock indefinite integral of f. Show that $F(]0,1]) = 1$.

Exercises 482Ya-482Yc are now 482Yb-482Yd.

p 730 l 15 (part (c-i) of the proof of 483B): for '$\{(x,A) : A \subseteq]x-\eta,x+\eta[$' read '$\{(x,A) : A \subseteq]x-\eta,x+\eta[\}$'.

p 730 l 16 (part (c-i) of the proof of 483B): for '$(x,C) \in T \cap \tilde{\delta}$' read '$(x,C) \in T \cap \delta$'.

p 731 l 4 (part (c-v) of the proof of 483B): for '$\not\!\!\!\int_{\infty}^{c} f$' read '$\not\!\!\!\int_{-\infty}^{c} f$'.

p 731 l 19 (part (c-vii) of the proof of 483B): for '$\not\!\!\!\int_{\infty}^{\beta} f$' read '$\not\!\!\!\int_{-\infty}^{\beta} f$'.

p 731 l 23 (part (c-vii) of the proof of 483B): for '$\not\!\!\!\int_{\infty}^{\beta} f$' read '$\not\!\!\!\int_{-\infty}^{\beta} f$'.

p 736 l 5 (proof of 483J): for

$$\text{`}\lambda_1 C = F_1(\sup C) - F_2(\sup C), \quad \lambda_2 C = F_2(\sup C) - F_2(\sup C)\text{'}$$

read

$$\text{`}\lambda_1 C = F_1(\sup C) - F_1(\inf C), \quad \lambda_2 C = F_2(\sup C) - F_2(\inf C)\text{'}.$$

p 736 l 31 (proof of 483K): for '$(x_{ni},[x_{ni},y_{ni}]) \in \mathcal{C} \cap \delta$' read '$(x_{ni},[x_{ni},y_{ni}]) \in \delta$'.

p 737 l 19 (part (a) of the proof of 483M): for '438Bd' read '483Bd'.

p 738 l 2 (proof of 483N): for 'δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$' read 'δ_1-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$'.

p 739 l 11 (Lemme 483Q): add the requirement '$g(x) = 0$ for $x \in \mathbb{R} \setminus K$' in part (b), so that the lemma reads

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function, and $K \subseteq \mathbb{R}$ a non-empty compact set such that F is AC_* on K. Write $\mathcal{I}$ for the family of non-empty bounded open intervals, disjoint from K, with endpoints in K.

(a) $\sum_{I\in\mathcal{I}} \omega(F{\restriction}I)$ is finite.

(b) Write a^* for $\inf K = \min K$. Then there is a Lebesgue integrable function $g : \mathbb{R} \to \mathbb{R}$, zero off K, such that

$$F(x) - F(a^*) = \int_{a^*}^{x} g + \sum_{J\in\mathcal{I}, J\subseteq[a^*,x]} F(\sup J) - F(\inf J)$$

for every $x \in K$.

p 739 l 16 (part (a) of the proof of 483Q): for '$]j\eta,(j+1)\eta[$' read '$]m\eta,(m+1)\eta[$'.

p 741 l 7 (part (a-i) of the proof of 483R): for

$$\text{`}\sum_{i=0}^{k} \omega(G{\restriction}[a_i,b_i]) \le \sum_{i=0}^{k} G(b_i) - G(a_i) = G(b_k) - G(a_0) \le \omega(G{\restriction}I_{nj})\text{'}$$

read

$$\text{`}\sum_{i=0}^{k} \omega(G{\restriction}[a_i,b_i]) = \sum_{i=0}^{k} G(b_i) - G(a_i) \le G(b_k) - G(a_0) \le \omega(G{\restriction}I_{nj})\text{'}.$$

p 742 l 6 (part (a-vi) of the proof of 483R): for '$\tilde{F}(x) = \int_{a^*}^{x} f \times \chi\overline{A}\,d\mu$' read '$\tilde{F}(x) = \int_{a^*}^{x} |f| \times \chi\overline{A}\,d\mu$'.

p 743 l 12 (part (b-iii) of the proof of 483R): for '483Pc' read '483Pb'.

p 743 l 14 (part (b-iii) of the proof of 483R): on introducing the function $g : \mathbb{R} \to \mathbb{R}$, add the remark that g can be taken to be zero on $\mathbb{R} \setminus K$.

p 743 l 20 (part (b-iii) of the proof of 483R): for '$\sup_{C\in\mathcal{C},C\subseteq I} F(\sup C) - F(\inf C)$' read '$\sup_{C\in\mathcal{C},C\subseteq I} |F(\sup C) - F(\inf C)|$'.

p 744 l 21 Some of the exercises for §483 have been moved: 483Xl is now 483Yh, 483Xm is now 483Xl, 483Xn is now 483Yd, 483Xo is now 483Yi; 483Ye-483Yf are now 483Yf-483Yg, 483Yg is now 483Yj.

p 744 l 22 (part (iii) of Exercise 483Xl, now 483Yh): for 'whenever $f \in U$ and $\mathcal{I}$ is a disjoint family of non-empty open intervals' read 'whenever $f \in \mathbb{R}^{\mathbb{R}}$ and $\mathcal{I}$ is a disjoint family of non-empty open intervals such that $f \times \chi I \in U$ for every $I \in \mathcal{I}$'.

p 744 l 42 (483X) Add new exercise:

(m) For integers $r \ge 1$, write $\mathcal{C}_r$ for the family of subsets of $\mathbb{R}^r$ of the form $\prod_{i<r} C_i$ where $C_i \subseteq \mathbb{R}$ is a bounded interval for each $i < r$. Set $Q_r = \{(x, C) : C \in \mathcal{C}_r,\ x \in \overline{C}\}$; let T_r be the straightforward set of tagged partitions generated by Q_r, Δ_r the set of neighbourhood gauges on $\mathbb{R}^r$, and $\mathfrak{R}_r = \{\mathcal{R}_C : C \in \mathcal{C}_r\}$ where $\mathcal{R}_C = \{\mathbb{R}^r \setminus C' : C \subseteq C' \in \mathcal{C}_r\}$ for $C \in \mathcal{C}_r$. Let ν_r be the restriction of r-dimensional Lebesgue measure to $\mathcal{C}_r$. (i) Show that $(\mathbb{R}^r, T_r, \Delta_r, \mathfrak{R}_r)$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}_r$. (ii) For a function $f : \mathbb{R}^r \to \mathbb{R}$ write $⨎ f(x)dx$ for the gauge integral $I_{\nu_r}(f)$ associated with this structure when it is defined. Show that if r, $s \ge 1$ are integers, $f : \mathbb{R}^{r+s} \to \mathbb{R}$ has compact support and $⨎ f(z)dz$ is defined, then, identifying $\mathbb{R}^{r+s}$ with $\mathbb{R}^r \times \mathbb{R}^s$, $⨎ g(x)dx$ is defined and equal to $⨎ f(x,y)d(x,y)$ whenever $g : \mathbb{R}^r \to \mathbb{R}$ is such that $g(x) = ⨎ f(x,y)dy$ for every $x \in \mathbb{R}^r$ for which this is defined.

p 745 l 3 (Exercise 483Ya): for '$[0, \infty[$' read '$]0, \infty[$'.

p 745 l 31 (483Y) Add new exercises:

(k) Find a function $f : \mathbb{R}^2 \to \mathbb{R}$, with compact support, such that $⨎ f$ is defined in the sense of 483Xm, but $⨎ fT$ is not defined, where $T(x, y) = \frac{1}{\sqrt{2}}(x + y, x - y)$ for x, $y \in \mathbb{R}$.

(l) Show that for a function $g : \mathbb{R} \to \mathbb{R}$ the following are equiveridical: (i) there is a function $h : \mathbb{R} \to \mathbb{R}$, of bounded variation, such that $g =_{\text{a.e.}} h$ (ii) g is a **multiplier** for the Henstock integral, that is, $f \times g$ is Henstock integrable for every Henstock integrable $f : \mathbb{R} \to \mathbb{R}$.

p 746 l 5 (Notes to §483): for '482Bc' read '483Bc'.

p 746 l 20 (484A) For 'Hausdorff outer measure' read 'Hausdorff measure'.

p 746 l 25 (484A) For 'diameter' read 'radius'.

p 747 l 37 - p 748 l 7 The second half of part (c) of the proof of 484B, from 'Suppose, if possible, that $x \notin \mathrm{cl}^*G$', is catastrophically flawed. The following is I hope correct.

Take any $x \in \overline{G}$. For $t > 0$, set

$$g(t) = \mu(G \cap U_t) = \int_0^t \nu(G \cap \partial U_s)ds,$$

so that

$$g'(t) = \nu(G \cap \partial U_t) \ge \frac{1}{2}(g(t)^{(r-1)/r} - \alpha g(t))$$

for almost every t. Because G is self-supporting and U_t is open and $G \cap U_t \ne \emptyset$, $g(t) > 0$ for every $t > 0$; and $\lim_{t\downarrow 0} g(t) = 0$.

Set

$$h(t) = \frac{d}{dt}g(t)^{1/r} = \frac{g'(t)}{rg(t)^{(r-1)/r}}$$

for $t \in \mathrm{dom}\, g'$. Then

$$\liminf_{t\downarrow 0} h(t) \ge \frac{1}{2r} \liminf_{t\downarrow 0, t\in\mathrm{dom}\, g'}(1 - \alpha g(t)^{1/r}) = \frac{1}{2r}.$$

So

$$\liminf_{t\downarrow 0} \frac{g(t)^{1/r}}{t} = \liminf_{t\downarrow 0} \frac{1}{t}\int_0^t h \ge \frac{1}{2r},$$

and

$$\limsup_{t\downarrow 0}\frac{\mu(G\cap B(x,t)))}{\mu B(x,t))}\ge\liminf_{t\downarrow 0}\frac{g(t)}{\beta_r t^r}>0.$$

Thus $x\in\mathrm{cl}^*G$. As x is arbitrary, $\overline{G}\subseteq\mathrm{cl}^*G$. **Q**

p 748 l 20 (part (b) of the proof of 484C): for '$\lfloor 2\nu(D\cap\partial^*E)\rfloor=k$' read '$\lfloor 2\nu(D_0\cap\partial^*E)\rfloor=k$'.

p 748 l 30 (part (b) of the proof of 484C): replace 'Taking $k=2l$, we have the result' by

Now

$$\begin{aligned}\#(\mathcal{L}_1)&=\sum_{D\in\mathcal{L}_0}\#(\{D':D'\in\mathcal{L}_1,\,D'\subseteq D\})\\&\le\sum_{D\in\mathcal{L}_0}\lfloor 2\nu(D\cap\partial^*E)\rfloor^2\\&\le\lfloor\sum_{D\in\mathcal{L}_0}2\nu(D\cap\partial^*E)\rfloor^2\le(2\nu\partial^*E)^2\le 4l^2.\end{aligned}$$

p 749 l 7 (part (d-iii) of the proof of 484C): for '$\nu(D_i\cap\partial^*E)\ge j$' and '$\nu\{x\}\ge j$' read '$\nu(D_i\cap\partial^*E)\ge\frac{j}{2}$' and '$\nu\{x\}\ge\frac{j}{2}$'.

p 750 l 6 (part (d) of the proof of 484E): The claim in (d)(β), line 12, that 'we can cover each A_{ij} by 2^r cubes $D_{ijk}\in\mathcal{D}$ of side length at most $\operatorname{diam}A_{ij}$' is over-optimistic, at least for small r. If we replace '2^r' here by '3^r', the proof works fine, but of course we have to make a string of adjustments, starting with '$3^{-r}/2r$' for '$1/2^{r+1}$ in lines 6, 9 and 11, and replacing '2^r' by '3^r' in lines 14, 19 (twice), 20, 21 and 22; with a final '$3^r\cdot 2r$' for '$2^{r+1}r$' in line 22.

p 751 l 5 (part (b-i) of the proof of 484F): for '475E(b-ii)' read '484E(b-ii)'.

p 751 l 10 (part (b-i) of the proof of 484F): for '475E(a-ii)' read '484E(a-ii)'.

p 751 l 17 (part (b-i) of the proof of 484F): for '$[\mathcal{D}_0]^{\omega}$' read '$[\mathcal{D}_0]^{<\omega}$'.

p 751 l 23 (part (b-ii) of the proof of 484F, condition (β)): for '$\mu(B(x,t)\setminus E)\le\epsilon_1\mu B(x,t)$' read '$\mu(B(x,t)\setminus E)\le\epsilon_1 t^r$'; then on line 26 delete the β_r. (This seems to be the most economical patch.)

p 751 l 34 For '$2r\gamma^{r-1}+\epsilon_1(\operatorname{diam}D)^{r-1}=\gamma^{r-1}(2r+r^{r/2}\epsilon_1)$' read '$2r\gamma^{r-1}+\epsilon_1(\operatorname{diam}D)^{r-1}\le\gamma^{r-1}(2r+r^{r/2}\epsilon_1)$'.

p 751 l 35 For 'ϵ_r' read 'ϵ_1'.

p 752 l 9 (part (b-iii) of the proof of 484F): for '$H\subseteq\mathbb{R}$' read '$H\subseteq\mathbb{R}^r$'.

p 752 l 22 (part (b=v) of the proof of 484F): for $W_{\boldsymbol{t}'}\cup\bigcup\mathcal{D}_1'=W_{\boldsymbol{t}'}\cup\bigcup\mathcal{D}_1$' read $E\subseteq W_{\boldsymbol{t}'}\cup\bigcup\mathcal{D}_1$'.

p 752 l 23 (part (b-v) of the proof of 484F): for

By the choice of $\mathcal{D}_0$, $\bigcup\mathcal{D}_1\in\mathcal{R}''$; by the choice of $\mathcal{R}''$, $E\setminus W_{\boldsymbol{t}}=E\cap\bigcup\mathcal{D}_1$ belongs to $\mathcal{R}'$. But we know also that $C\cap V\setminus E\in\mathcal{R}'$, that is, $C\setminus E\in\mathcal{R}'$, because $\mathcal{R}'=\mathcal{R}_\eta^{(V)}$. By the choice of $\mathcal{R}'$, $C\setminus W_{\boldsymbol{t}}\in\mathcal{R}$. And $\boldsymbol{t}\in T'$ is a δ-fine member of T_α.

read

By the choice of $\mathcal{D}_0$, $\bigcup\mathcal{D}_1'\in\mathcal{R}''$; by the choice of $\mathcal{R}''$, $E\setminus W_{\boldsymbol{t}'}=E\cap\bigcup\mathcal{D}_1'$ belongs to $\mathcal{R}'$. But we know also that $C\cap V\setminus E\in\mathcal{R}'$, that is, $C\setminus E\in\mathcal{R}'$, because $\mathcal{R}'=\mathcal{R}_\eta^{(V)}$. By the choice of $\mathcal{R}'$, $C\setminus W_{\boldsymbol{t}'}\in\mathcal{R}$. And $\boldsymbol{t}'\in T'$ is a δ-fine member of T_α.

p 753 l 1 (part (c) of the statement of 484H): for 'whenever $E\in\mathcal{R}$' read 'whenever $E\in\mathcal{C}\cap\mathcal{R}$'.

p 753 l 19 (part (d) of the proof of 484H): for 'μ, $\mathcal{C}$, T_α, Δ and $\mathfrak{R}$' read 'μ, $\mathcal{C}_\alpha$, T_α, Δ and $\mathfrak{R}$'.

p 753 l 21 (part (d) of the proof of 484H): for '475C(f)' read '475Cg'.

p 755 l 33 (part (d) of the proof of 484L): for 'condition (iii)' read 'condition (ii)'.

p 756 l 18 (part (b) of the proof of 484M) Delete reference to 473Ca.

p 757 l 24 (part (d) of the proof of 484N): for 'F_n', here and on lines 31 and 32 below, read 'H_n'.

p 758 ll 2-5 (part (d) of the proof of 484N): for '$\mu H'_n$', '$\mu\mu(H_n \setminus H'_n)$' read '$\mu_{r-1}H'_n$', '$\mu_{r-1}(H_n \setminus H'_n)$'.

p 758 l 31 (part (a) of the proof of 484O): for 'C_α' read '$\mathcal{C}_\alpha$'.

p 758 l 32 For '$R' \in \mathcal{R}'$' read '$R \in \mathcal{R}'$'.

p 759 l 5 (part (b) of the proof of 484O): for '$\operatorname{diam} C \le \delta$' read '$\operatorname{diam} C \le \zeta$'.

p 759 l 8 (part (b) of the proof of 484O): replace each '$q - q'$' (here and in lines 25, 26, 27 below) by '$q' - q$'.

p 760 l 13 (proof of 484P): replace each '$\int_{C\setminus H}$' by '$\int_{C\cap H}$'.

p 761 l 4 (part (a) of the proof of 484R): for '$\limsup_{\zeta\downarrow 0} \frac{\mu^*(B(\phi(x),\zeta)\cap\phi[A]}{\mu B(\phi(x),\zeta)} \ge \frac{\epsilon}{\gamma^r}$' read '$\limsup_{\zeta\downarrow 0} \frac{\mu^*(B(\phi(x),\zeta)\cap\phi[A]}{\mu B(\phi(x),\zeta)} \ge \frac{\epsilon}{\gamma^{2r}}$'.

p 761 ll 14-15 (part (b) of the proof of 484R): for '$\gamma^{2-2r}\alpha \le \alpha' \le \alpha$' read '$\alpha' \le \min(\gamma^{2-2r}\alpha, \alpha)$'.

p 761 l 17 For '$\frac{1}{\alpha}(\frac{1}{\gamma} \operatorname{diam} \phi[C])^r$' read '$\alpha(\operatorname{diam} \phi[C])^r$'.

p 762 l 13 (part (b-ii) of the proof of 484S): each '$\frac{1}{2^{n+3}}$', here and in lines 25 and 26, should be replaced by '$\frac{\epsilon}{2^{n+2}}$', so that the formulae become

$$\frac{\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}, \quad \frac{(n+1)\epsilon\mu C}{2^{n+2}\beta_r(n+2)^r(n+1)}, \quad \frac{(n+1)\epsilon\mu B(\mathbf{0},n+2)}{2^{n+2}\beta_r(n+2)^r(n+1)}.$$

p 762 l 14 Add 'where $\gamma > 0$ is a Lipschitz constant for ϕ'.

p 762 l 16 For '$\boldsymbol{t} \in T_{\alpha'}$' read '$\boldsymbol{t}' \in T_{\alpha'}$'.

p 762 l 31 (part (b-iii) of the proof of 484S): for '$\phi[C] \in \mathcal{R}$ for every $R \in \mathcal{R}'$' read '$\phi[R] \in \mathcal{R}$ for every $R \in \mathcal{R}'$'.

p 763 l 22 Exercises 484Xb-484Xc should be rewritten, as follows:

(b) For $\alpha > 0$ let $\mathcal{C}'_\alpha$ be the family of bounded Lebesgue measurable sets C such that $\mu C \ge \alpha \operatorname{diam} C \operatorname{per} C$, and T'_α the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C}'_\alpha, x \in \operatorname{cl}^*C\}$. (i) Show that $\mathcal{C}_\alpha \subseteq \mathcal{C}'_{\alpha^2}$ and that $\mathcal{C}'_\alpha \subseteq \mathcal{C}_{\min(\alpha,\alpha^r)}$. (ii) Show that if $0 < \alpha < 1/2r\sqrt{r}$ then T'_α is compatible with Δ and $\mathfrak{R}$.

(c) For $\alpha > 0$, let $\mathcal{C}''_\alpha$ be the family of convex sets C such that $\mu C \ge \alpha(\operatorname{diam} C)^r$. Show that if $0 < \alpha < 1/r^{r/2}$ then $\mathcal{C}''_\alpha \subseteq \mathcal{C}_\alpha$ and $T_\alpha \cap [\mathbb{R}^r \times \mathcal{C}''_\alpha]^{<\omega}$ is compatible with Δ and $\mathfrak{R}$.

p 763 l 28 (484X) Add new exercises:

(f) (Here take $r = 2$.) Let $\langle\delta_n\rangle_{n\in\mathbb{N}}$ be a strictly decreasing summable sequence in $]0, 1]$. Define $f : \mathbb{R}^2 \to \mathbb{R}$ by saying that $f(x) = \frac{(-1)^n}{n(\delta_n^2-\delta_{n+1}^2)}$ if $n \in \mathbb{N}$ and $\delta_{n+1} \le \|x\| < \delta_n$, 0 otherwise. Show that $\lim_{\delta\downarrow 0} \int_{\mathbb{R}^2\setminus B(\mathbf{0},\delta)} f d\mu$ is defined, but that f is not Pfeffer integrable.

(g) (Again take $r = 2$.) Show that there are a Lebesgue integrable $f_1 : \mathbb{R} \to \mathbb{R}$ and a Henstock integrable $f_2 : \mathbb{R} \to \mathbb{R}$, both with bounded support, such that $(\xi_1, \xi_2) \mapsto f_1(\xi_1)f_2(\xi_2) : \mathbb{R}^2 \to \mathbb{R}$ is not Pfeffer integrable.

(i) Show that there is a Lipschitz function $f : \mathbb{R}^r \to [0, 1]$ such that $\mathbb{R}^r \setminus \operatorname{dom} f'$ is not thin. (*Hint*: there is a Lipschitz function $f : \mathbb{R} \to [0, 1]$ not differentiable at any point of the Cantor set.)

484Xf is now 484Xh.

p 764 l 10 (Notes to §484): delete the reference to 484Xf.

p 764 l 17 For '475J' read '484J'.

p 766 l 12 (491A) Add new part:

***(e)** If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{Z}$, there is an $I\in\mathcal{Z}$ such that $I_n\setminus I$ is finite for every n.

p 769 l 7 The proof of Theorem 491F is confused, with a potentially catastrophic misquotation of a result from Volume 2, so I have re-written it.

p 771 l 7 (part (a-ii) of the proof of 491H): for the second '$|\int f(yxz)(\lambda*\nu)(dx)-\alpha|\le\epsilon$ for every y, $z\in X$' read '$|\int f(yxz)(\nu*\lambda)(dx)-\alpha|\le\epsilon$ for every y, $z\in X$'.

p 771 l 19 (part (b) of the proof of 491H): for

'Because A is dense, $xV^{-1}\cap A\ne\emptyset$ is not empty for every $x\in X$, that is, $AV=X$; once more because X is compact, there are $x_0,\dots,x_n\in A$ such that $X=\bigcup_{i\le n}x_iV$. Set $E_i=x_iV\setminus\bigcup_{j<i}x_jV$ for each $i\le n$.'

read

'Because A is dense, $V^{-1}x\cap A\ne\emptyset$ for every $x\in X$, that is, $VA=X$; once more because X is compact, there are $x_0,\dots,x_n\in A$ such that $X=\bigcup_{i\le n}Vx_i$. Set $E_i=Vx_i\setminus\bigcup_{j<i}Vx_j$ for each $i\le n$.'

p 775 l 19 (Example 491Ma): for '$\mu L\ge 1-\mu K+\epsilon$' read '$\mu L\ge\mu X-\mu K+\epsilon$'.

p 776 l 36 (part (c) of the proof of 491N): delete all references to 'compactness'; all sets K, L are to be taken to belong to $\mathcal{K}$.

p 778 l 3 (proof of 491O): for 'open G' and 'closed set F', read 'cozero set G' and 'zero set F'.

p 778 l 33 Add new results:

491R Proposition Let X be a topological space, μ an effectively regular topological probability measure on X which has an equidistributed sequence, and ν a probability measure on X which is an indefinite-integral measure over μ. Then ν has an equidistributed sequence.

491S The asymptotic density filter (a) Set

$$\mathcal{F}_d=\{\mathbb{N}\setminus I:I\in\mathcal{Z}\}=\{I:I\subseteq\mathbb{N},\ \lim_{n\to\infty}\frac{1}{n}\#(I\cap n)=1\}.$$

Then $\mathcal{F}_d$ is a filter on $\mathbb{N}$, the **(asymptotic) density filter**.

(b) For a bounded sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{C}$, the following are equiveridical: (i) $\lim_{n\to\mathcal{F}_d}\alpha_n=0$; (ii) $\lim_{n\to\infty}\frac{1}{n+1}\sum_{k=0}^n|\alpha_k|=0$.

(c) For any $m\in\mathbb{N}$ and $A\subseteq\mathbb{N}$, $A+m\in\mathcal{F}_d$ iff $A\in\mathcal{F}_d$.

p 778 l 34 The exercises of §491 have been expanded and consolidated. 491Xe-491Xf are now 491Xf-491Xf, 491Xg is now 491Xi, 491Xh is now 491Ye, 491Xi is now 491Yk, 491Xj is now part of 491Xg, 491Xk-491Xn are now 491Xj-491Xm, 491Xo and 491Xp are now 491Xn, 491Xq-491Xv are now 491Xo-491Xt, 491Xw-491Xx are now 491Yp-491Yq, 491Xy-491Xz are now 491Xu-491Xv.

p 779 l 3 Add new fragments to 491Xc:

(i) Show that there is a $J\subseteq I$ such that $d^*(J)=d^*(I\setminus J)=d^*(I)$. (iii) Show that if $d(I)$ is defined and $0\le\alpha\le d(I)$ there is a $J\subseteq I$ such that $d(J)$ is defined and equal to α.

p 779 l 6 (Exercise 491Xd): for '$\mathbb{N}\setminus I\mathcal{Z}$' read '$\mathbb{N}\setminus I\in\mathcal{Z}$'. Add new fragment:

(i)(α) Show that if I, $K\subseteq\mathbb{N}$ are such that $d(I)$ and $d(K)$ are defined, there is a $J\subseteq\mathbb{N}$ such that $d(J)$ is defined and $d(J)=d^*(J\cap I)=d^*(K\cap I)$.

p 779 l 16 Add new fragment to 491Xf, now 491Xg:

(iv) Show that a sequence $\langle t_i\rangle_{i\in\mathbb{N}}$ in $[0,1]$ is equidistributed iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(t_i)$ is defined and equal to $⨍_0^1 f$ for every Riemann integrable function $f:[0,1]\to\mathbb{R}$.

p 779 l 24 (Exercise 491Xi, now 491Yk): for 'where $g(t)=(t,1)$ for $t\in Z$' read 'where $g(t,0)=g(t,1)=(t,1)$ for $t\in Z$'.

p 779 l 18 (491X) Add new exercises:

(e) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. For $x\in X$, set $I_x=\{n:n\in\mathbb{N},\,x\in E_n\}$. Show that $\int d^*(I_x)\mu(dx)\ge\liminf_{n\to\infty}\mu E_n$.

(h) Let X be a topological space, μ a probability measure on X measuring every zero set, and $\langle x_i\rangle_{i\in\mathbb{N}}$ an equidistributed sequence in X. Show that $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(x_i)$ is defined and equal to $\int f\,d\mu$ for every bounded $f:X\to\mathbb{R}$ which is continuous almost everywhere.

(w) Give an example of a Radon probability space (X,μ) with a closed conegligible set $F\subseteq X$ such that μ has an equidistributed sequence but the subspace measure μ_F does not.

(x) Let X be a topological space. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is called **statistically convergent** to $x\in X$ if $d(\{n:x_n\in G\})=1$ for every open set G containing x. (i) Show that if X is first-countable then $\langle x_n\rangle_{n\in\mathbb{N}}$ is statistically convergent to x iff there is a set $I\subseteq\mathbb{N}$ such that $d(I)=1$ and $\langle x_n\rangle_{n\in I}$ converges to x in the sense that $\{n:n\in I,\,x_n\notin G\}$ is finite for every open set G containing x. (ii) Show that a bounded sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{R}$ is statistically convergent to α iff $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\alpha_i-\alpha|=0$.

p 781 l 1 The exercises in 491Y have been rearranged: 491Yb-491Yf are now 491Yf-491Yj, 491Yg-491Yh are now 491Yl-491Ym, 491Yj-491Yl have been collected together as 491Yo, 491Ym is now 491Yr.

p 781 l 7 Add new exercises:

(c) Let $\mathfrak{A}$ be a Boolean algebra, $A\subseteq\mathfrak{A}\setminus\{0\}$ a non-empty set and $\alpha\in[0,1]$. Show that the following are equiveridical: (i) there is a finitely additive functional $\nu:\mathfrak{A}\to[0,1]$ such that $\nu a\ge\alpha$ for every $a\in A$ (ii) for every sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A there is a set $I\subseteq\mathbb{N}$ such that $d^*(I)\ge\alpha$ and $\inf_{i\in I\cap n}a_i\ne0$ for every $n\in\mathbb{N}$.

(d) Let $\mathfrak{A}$ be a Boolean algebra, and $\nu:\mathfrak{A}\to[0,\infty]$ a submeasure. Show that ν is uniformly exhaustive iff whenever $\langle a_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{A}$ such that $\inf_{n\in\mathbb{N}}\nu a_n>0$, there is a set $I\subseteq\mathbb{N}$ such that $d^*(I)>0$ and $\inf i\in I\cap na_i\ne0$ for every $n\in\mathbb{N}$.

(n) Let $\mathfrak{Z}=\mathcal{P}\mathbb{N}/\mathcal{Z}$ and $\bar d^*:\mathfrak{Z}\to[0,1]$ be as in 491I. Show that $\bar d^*$ is **order-continuous on the left** in the sense that whenever $A\subseteq\mathfrak{Z}$ is non-empty and upwards-directed and has a supremum $c\in\mathfrak{Z}$, then $\bar d^*(c)=\sup_{a\in A}\bar d^*(a)$.

(s) (i) Show that there is a family $\langle a_\xi\rangle_{\xi\in\mathfrak{c}}$ in $\mathfrak{Z}$ such that $\inf_{\xi\in I}a_\xi=0$ and $\sup_{\xi\in I}a_\xi=1$ for every infinite $I\subseteq\mathfrak{c}$. (ii) Show that if $B\subseteq\mathfrak{Z}\setminus\{0\}$ has cardinal less than $\mathfrak{c}$ then there is an $a\in\mathfrak{Z}$ such that $b\cap a$ and $b\setminus a$ are non-zero for every $b\in B$.

(t) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a τ-additive topological probability space. A sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is **completely equidistributed** if, for every $r\ge1$, the sequence $\langle\langle x_{n+i}\rangle_{i<r}\rangle_{n\in\mathbb{N}}$ is equidistributed for some (therefore any) τ-additive extension of the c.l.d. product measure μ^r on X^r. (i) Show that if there is an equidistributed sequence in X, then there is a completely equidistributed sequence in X. (ii) Show that if $\mathfrak{T}$ is second-countable, then $\mu^{\mathbb{N}}$-almost every sequence in X is completely equidistributed. (iii) Show that if X has two disjoint open sets of non-zero measure, then no sequence which is well-distributed in the sense of 281Ym can be completely equidistributed.

(u) Suppose, in 491O, that μ is a topological measure. Show that $T_\pi f^\bullet\le RSf$ for every bounded lower semi-continuous $f:X\to\mathbb{R}$.

p 781 l 26 (Exercise 491Yg, now 491Yl): add second part

(ii) Show that there is a separable compact Hausdorff space with a Radon measure which has no equidistributed sequence.

p 781 l 29 Exercise 491Yi is wrong (unless we find a suitably sophisticated definition of 'non-trivial representation'), and has been deleted.

p 789 l 33 (part (b) of the proof of 493B): the second '$\{x:a\bullet x=x$ for every $a\in I\}$' should be '$\{x:a\bullet x=x$ for every $a\in H_I\}$'.

p 790 l 10 Proposition 493C is now 493Be. 493D-493F are now 493C-493E.

p 790 l 40 (part (c) of the proof of 493D, now 493C): 'Take any non-negative $f \in U$' should be followed by 'and $\epsilon > 0$'.

p 793 l 23 Lemma 493G and Theorem 493H have been moved to 494I-494L. 493I-493K are now 493F-493H.

p 796 l 11 (proof of 493I, now 493F): delete 'orthonormal'.

p 796 l 22 (proof of 493I, now 493F): for '$\lambda'_X Q_1 \ge \frac{1}{2}\epsilon$' read '$\lambda'_X Q'_1 \ge \frac{1}{2}\epsilon$'. In the next line, for '$\lambda_X Q'_2 \ge \frac{1}{2}\epsilon$' read '$\lambda'_X Q'_2 \ge \frac{1}{2}\epsilon$', and for '$X$-invariant' read '$H_X$-invariant'.

p 798 l 39 Exercise 493Xa is now 493Bf, and 493Xe is now 494Be. 493Xb-493Xd are now 493Xa-493Xc, and 493Xf-493Xg are now 493Xe-493Xf.

In addition there are new exercises:

(d) Prove 493G for infinite-dimensional inner product spaces over $\mathbb{C}$.

(g) If X is a (real or complex) Hilbert space, a bounded linear operator $T : X \to X$ is **unitary** if it is an invertible isometry. Show that the set of unitary operators on X, with its strong operator topology, is an extremely amenable topological group.

p 289 l 17 (Exercise 493Ya):

p 799 l 17 (Exercise 493Ya): for '$\{g : a_g \ne e\}$ is finite' read '$\{g : a_g \ne 0\}$ is finite'.

p 799 l 30 Exercises 493Yb and 493Yd are now 494Xl and 494Xf. 493Yc is now split between 494Cc and 494Yf.

p 799 l 42 (493Yd, now 494Xf): for '$\lim_{n\to\infty} \|\frac{1}{n+1}\sum_{i=0}^n \chi(\pi^i d) - \bar\mu d \cdot \chi d\|_1 = 0$' read '$\inf_{n\in\mathbb{N}} \|\frac{1}{n+1}\sum_{i=0}^n \chi(\pi^i d) - \bar\mu d \cdot \chi 1\|_1 = 0$'.

p 800 l 19 A new section §494, 'Groups of measure-preserving automorphisms', has been introduced. The former §494 is now §498.

p 801 l 6 There is a blunder in Proposition 494B, which now reads

498B Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an effectively locally finite τ-additive topological measure space and $\tilde\lambda$ the τ-additive product measure on $X \times Y$. Then if $W \subseteq X \times Y$ is closed and $\tilde\lambda W > 0$ there are a non-scattered compact set $K \subseteq X$ and a closed set $F \subseteq Y$ of positive measure such that $K \times F \subseteq W$.

p 804 l 4 (part (b) of the proof of 495B): for '$j \in I_i$' read '$j \in J_i$'.

p 804 l 9 (part (b) of the proof of 495B): for 'whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a sequence in Σ with union E' read 'whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ with union E'.

p 804 l 22 (part (d) of the proof of 495B): for 'disjoint family in Σ' read 'disjoint family in Σ^f'.

p 805 l 4 (statement of Lemma 495C): for '$n_i \in I$' read '$n_i \in \mathbb{N}$'.

p 805 l 12-13 (proof of 495C): in 'Let $\mathcal{E}$ be the subring of $\mathcal{P}X$ generated by $\operatorname{dom} q$' and '$H_q = \bigcup_{q\subseteq q'\in Q, \operatorname{dom} q'=\mathcal{E}} H_{q'}$' replace $\mathcal{E}$ by some other symbol.

p 805 l 13 (proof of 495C): in '$H_q = \bigcup_{q\subseteq q'\in Q, \operatorname{dom} q'=\mathcal{E}'} H_{q'}$ is the union of a finite disjoint family in T' replace 'finite' by 'countable'.

p 806 l 24 Add new result:

495F Proposition Let (X, Σ, μ) be an atomless measure space, $\langle X_i\rangle_{i\in I}$ a countable partition of X into measurable sets and $\gamma > 0$, Let ν be the Poisson point process of (X, Σ, μ) with intensity γ; for $i \in I$ let ν_i be the Poisson point process of (X_i, Σ_i, μ_i) with intensity γ, where μ_i is the subspace measure on X_i and Σ_i its domain. For $S \subseteq X$ set $\phi(S) = \langle S \cap X_i\rangle_{i\in I} \in \prod_{i\in I} \mathcal{P}X_i$. Then ϕ is an isomorphism between ν and the product measure $\prod_{i\in I} \nu_i$ on $\prod_{i\in I} \mathcal{P}X_i$.

495F is now 495G.

p 807 l 6 Add new results:

495H Lemma Let (X,Σ,μ) be an atomless σ-finite measure space, and $\gamma>0$; let ν be the Poisson point process on X with intensity γ. Suppose that $f:X\to\mathbb{R}$ is a Σ-measurable function such that $\mu f^{-1}[\{\alpha\}]=0$ for every $\alpha\in\mathbb{R}$. Then $\nu\{S:S\subseteq X,\ f\restriction S \text{ is injective}\}=1$.

495I Proposition Let (X,Σ,μ) be an atomless countably separated measure space and $\gamma>0$. Let ν' be a complete probability measure on $\mathcal{P}X$ such that $\nu'\{S:S\subseteq X,\ S\cap E=\emptyset\}$ is defined and equal to $e^{-\gamma\mu E}$ whenever $E\in\Sigma$ has finite measure. Then ν' extends the Poisson point process ν on X with intensity γ.

p 807 l 10 495G-495P are now 495J-495S.

p 809 l 14 (part (b-ii) of the proof of 495H, now 495K) All the formulae written are, I believe, correct; but in the line

$$\nu'H=\sum_{\sigma\in\mathbb{N}^s}\nu'H'_\sigma=\sum_{\sigma\in\mathbb{N}^s}\int\nu'_T(H'_\sigma)\tilde\nu(dT)=\int\nu'_T(H)\tilde\nu(dT)$$

there seems to be a rather large jump. To plug the gap, we can rewrite this bit, as follows:

(ii) For the general case, set $C_{rj}=[0,1]\setminus\bigcup_{i<r}C_{ij}$, $E_{rj}=F_j\times C_{rj}$ for each $j<s$. For $\sigma\in\mathbb{N}^{(r+1)\times s}$, set

$$H_\sigma=\{S:S\subseteq X',\ \#(S\cap E_{ij})=\sigma(i,j)\text{ for every }i\le r\text{ and }j<s\}.$$

By (i), we have $\nu'H_\sigma=\int\nu'_T(H_\sigma)\tilde\nu(dT)$ for every $\sigma\in\mathbb{N}^{(r+1)\times s}$.

Set

$$J=\{\sigma:\sigma\in\mathbb{N}^{(r+1)\times s},\ \sigma(i,j)=n_{ij}\text{ for }i<r,\ j<s\},\quad K=\mathbb{N}^{(r+1)\times s}\setminus J,$$

$$H'_1=\bigcup_{\sigma\in J}H_\sigma,\quad H'_2=\bigcup_{\sigma\in K}H_\sigma.$$

Then $H'_1\subseteq H$, $H'_2\cap H=\emptyset$ and

$$H'_1\cup H'_2=\{S:S\cap E_{ij}\text{ is finite for all }i\le r,\ j<s\}$$

is ν'-conegligible. Accordingly we have

$$\begin{aligned}\underline{\int}(\nu'_T)_*(H)\tilde\nu(dT)&\ge\underline{\int}(\nu'_T)_*(H'_1)\tilde\nu(dT)\\&\ge\sum_{\sigma\in J}\int\nu'_T(H_\sigma)\tilde\nu(dT)=\sum_{\sigma\in J}\nu'H_\sigma,\\\underline{\int}(\nu'_T)_*(\mathcal{P}X'\setminus H)\tilde\nu(dT)&\ge\underline{\int}(\nu'_T)_*(H'_2)\tilde\nu(dT)\\&\ge\sum_{\sigma\in K}\int\nu'_T(H_\sigma)\tilde\nu(dT)=\sum_{\sigma\in K}\nu'H_\sigma,\end{aligned}$$

while

$$\sum_{\sigma\in J}\nu'H_\sigma+\sum_{\sigma\in K}\nu'H_\sigma=1.$$

It follows (because all the ν'_T are complete) that $\int\nu'_T(H)\tilde\nu(dT)$ is defined and equal to $\sum_{\sigma\in J}\nu'H_\sigma=\nu'H$, as required. **Q**

p 810 l 20 (part (e) of the proof of 495H, now 495K): for '$H=\{S:S\subseteq X',\ \#(S\cap\hat E_i)=n_i$ for every $i<r\}$' read '$H=\{S:S\subseteq X',\ \#(S\cap E_i)=n_i$ for every $i<r\}$'.

p 810 l 22 (part (e) of the proof of 495H, now 495K): for 'the σ-algebra T generated by $\mathcal{H}_0$' read 'the σ-algebra T' generated by $\mathcal{H}_0$'.

p 810 l 32 (part (ii) of the statement of 495I, now 495L): for 'if μ is strongly consistent with f' read 'if $\langle\mu_t\rangle_{t\in\tilde X}$ is strongly consistent with f'.

p 811 l 12 (part (a) of the proof of 495I, now 495L): for '$z\restriction T$ is injective' read '$fz\restriction T$ is injective'.

p 811 l 18 (part (b) of the proof of 495I, now 495L): for 'let ν'_T be the image measure $\lambda\psi_T^{-1}$ on $\mathcal{P}X'$' read 'let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$'.

p 811 l 20 (part (b) of the proof of 495I, now 495L): for '$f_i(t) = \sum_{j<i} \mu_t E_j$ for $t \in Y_1$ and $i < r$' read '$f_i(t) = \sum_{j<i} \mu_t E_j$ for $t \in Y_1$ and $i \le r$'.

p 811 l 24 (part (b) of the proof of 495I, now 495L): for '$\#(S' \cap E'_i) = n_i$' read '$\#(S \cap E'_i) = n_i$'.

p 814 l 31 Add new part:

(b) If $f \in \mathcal{L}^1(\mu) \cap \mathcal{L}^2(\mu)$, $\int Q_f^2 d\nu$ is defined and equal to $\gamma \int f^2 d\mu + (\gamma \int f d\mu)^2$.

Parts (b) and (c) are now (c)-(d).

p 816 l 4 Propositions 495N (now 495Q) and 495O (now 495R) have been extended, as follows.

495Q Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space such that μ is outer regular with respect to the open sets, and $\gamma > 0$. Give the space $\mathcal{C}$ of closed subsets of X its Fell topology.

(a) There is a unique quasi-Radon probability measure $\tilde{\nu}$ on $\mathcal{C}$ such that

$$\tilde{\nu}\{C : \#(C \cap E) = \emptyset\} = e^{-\gamma\mu E}$$

whenever $E \subseteq X$ is a measurable set of finite measure.

(b) If $E_0, \dots, E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i \in \mathbb{N}$ for $i \le r$, then

$$\tilde{\nu}\{C : \#(C \cap E_i) = n_i \text{ for every } i \le r\} = \prod_{i=0}^{r} \tfrac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

(c) Suppose that μ is atomless and ν is the Poisson point process on X with density γ.

(i) $\mathcal{C}$ has full outer measure for ν, and $\tilde{\nu}$ extends the subspace measure $\nu_{\mathcal{C}}$.

(ii) If moreover μ is σ-finite, then $\mathcal{C}$ is ν-conegligible.

(d) If X is locally compact then $\tilde{\nu}$ is a Radon measure.

(e) If X is second-countable and μ is atomless then $\tilde{\nu} = \nu_{\mathcal{C}}$.

495R Proposition Let $(X, \mathfrak{T})$ be a σ-compact locally compact Hausdorff space and $M_{\mathrm{R}}^+(X)$ the set of Radon measures on X. Give $M_{\mathrm{R}}^+(X)$ the topology generated by sets of the form $\{\mu : \mu G > \alpha\}$ and $\{\mu : \mu K < \alpha\}$ where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$. Let $\mathcal{C}$ be the space of closed subsets of X with the Fell topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology. For $\mu \in M_{\mathrm{R}}^+(X)$ and $\gamma > 0$ let $\tilde{\nu}_{\mu,\gamma}$ be the Radon measure on $\mathcal{C}$ defined from μ and γ as in 495Q. Then the function $(\mu, \gamma) \mapsto \tilde{\nu}_{\mu,\gamma} : M_{\mathrm{R}}^+(X) \times]0, \infty[\to P_{\mathrm{R}}(\mathcal{C})$ is continuous.

An extra fragment of argument required in 495O is now part (a) of the proof of 495R.

p 817 l 34 (part (a) of the proof of 495O, now part (b) of the proof of 495R) for '$\epsilon > 0$' read '$\eta > 0$'.

p 307 l 36 (part (a) of the proof of 495O, now part (b) of the proof of 495R) for '$\mu(X \setminus \overline{E}) > \mu_0(X \setminus \overline{E}) - \eta$' read '$\mu\overline{E} < \mu_0\overline{E} + \eta$'.

p 817 l 39 (parts (b) and (c) of the proof of 495O, now parts (c) and (d) of the proof of 495R): to allow for the possibility that we have to look at sets of the form $\{C : C \cap K = \emptyset\}$ and therefore need to consider the case $C = \emptyset$, most of the clauses '$i \le r$' should be replaced with '$i < r$'.

p 818 l 14 (part (c) of the proof of 495O, now part (d) of the proof of 495R): for '$\prod_{A\in\mathcal{I}} e^{-\gamma\mu A} \prod_{A\in\mathcal{A}\setminus\mathcal{I}}(1 - e^{-\gamma\mu A})$' read '$\prod_{A\in\mathcal{A}\setminus\mathcal{I}} e^{-\gamma\mu A} \prod_{A\in\mathcal{I}}(1 - e^{-\gamma\mu A})$'.

p 819 l 14 (part (b) of the proof of 495P, now 495S): for '$h_{kn}(S) = g_{k,n+1}(S) - g_{kn}(S)$' read '$h_{kn}(S) = g_{kn}(S) - g_{k,n-1}(S)$'.

p 819 l 30 (part (d), or more properly part (c), of the proof of 495P, now 495S): for '$\inf_{\beta>\alpha} \liminf_{i\to\infty} \Pr(h_{ki} \le \beta)$' read '$\inf_{\beta>\alpha} \liminf_{k\to\infty} \Pr(h_{ki} \le \beta)$'.

p 820 l 8 (495X) These exercises have been rearranged: 495Xa-495Xg are now 495Xb-495Xh, 495Xh is now 495Yb, 495Xm has been deleted, 495Xn-495Xp are now 495Xm-495Xo, 495Yb is now 495Yc, 495Yc is now 495Yd.

p 821 l 35 (Exercise 495Ya): for '$L^1(\mathfrak{A},\bar\mu)$' read '$U$'.

p 821 l 37 (Exercise 495Yb, now 495Yc): for '$L^0_{\mathbb{C}}(\mathfrak{B})$' read '$L^0_{\mathbb{C}}(\mathfrak{A})$'.

p 821 l 42 Exercise 495Yc, now 495Yd, has been amended, and is now

(d) Let (X,ρ) be a totally bounded metric space, μ a Radon measure on X and $\gamma>0$. Let $\mathcal{C}$ be the set of closed subsets of X, and $\tilde\nu$ the quasi-Radon measure of 495Q; let $\tilde\rho$ be the Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$. Show that the subspace measure on $\mathcal{C}\setminus\{\emptyset\}$ induced by $\tilde\nu$ is a Radon measure for the topology induced by $\tilde\rho$.

p 822 l 56 Two new sections have been added: §496 on Maharam submeasures, and §497 on Szemerédi's theorem.

p 823 l 25 (4A1A) Add new fragment:

(c)(ii) If $\#(A)\le\mathfrak{c}$ and D is countable, then $\#(A^D)\le\mathfrak{c}$.

The former (c-ii) is now (c-iii).

p 823 l 43 (4A1B) 4A1Bc and 4A1Bd have been transposed, so are now called 4A1Bd and 4A1Bc.

p 826 l 44 (part (b) of the proof of 4A1N: for '$C_\xi=\bigcup_{i\in\mathbb{N}}f(\theta_\xi(i))$' read '$C_\xi=\bigcup_{i\ge1}f(\theta_\xi(i))$'.

p 828 l 5 (4A2A) Add new definitions:

Baire space A topological space X is a **Baire space** if $\bigcap_{n\in\mathbb{N}}G_n$ is dense in X whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence of dense open subsets of X.

càdlàg If X is a Hausdorff space, a function $x:[0,\infty[\,\to X$ is **càdlàg** ('continue à droit, limites à gauche') or **RCLL** ('right continuous, left limits') if $\lim_{s\downarrow t}x(s)=x(t)$ for every $t\ge0$ and $\lim_{s\uparrow t}x(s)$ is defined in X for every $t>0$.

càllàl If X is a Hausdorff space, a function $f:[0,\infty[\,\to X$ is **càllàl** ('continue à l'une, limite à l'autre') if $f(0)=\lim_{s\downarrow0}f(s)$ and, for every $t>0$, $\lim_{s\downarrow t}f(s)$ and $\lim_{s\uparrow t}f(s)$ are defined in X, and at least one of them is equal to $f(t)$.

irreducible If X and Y are topological spaces, a continuous surjection $f:X\to Y$ is **irreducible** if $f[F]\ne Y$ for any closed proper subset F of X.

perfect A topological space is **perfect** if it is compact and has no isolated points.

p 831 l 2 (4A2B) Add new fragments:

(d)(i) A function $f:X\to\mathbb{R}$ is lower semi-continuous iff $\Omega=\{(x,\alpha):x\in X,\ \alpha\ge f(x)\}$ is closed.

(vii) If $f:X\to[-\infty,\infty]$ is lower semi-continuous, and $\mathcal{F}$ is a filter on X converging to $y\in X$, then $f(y)\le\liminf_{x\to\mathcal{F}}f(x)$.

(viii) If X is compact and $f:X\to[-\infty,\infty]$ is lower semi-continuous then $K=\{x:f(x)=\inf_{y\in X}f(y)\}$ is non-empty and compact.

(ix) If f, $g:X\to[0,\infty]$ are lower semi-continuous and $f+g$ is continuous at x and finite there, then f and g are continuous at x.

(f)(iii) Let X and Y be topological spaces and $f:X\to Y$ a continuous open map. Then $H\mapsto f^{-1}[H]$ is an order-continuous Boolean homomorphism from the regular open algebra of Y to the regular open algebra of X. If f is surjective, then π is injective, and for $H\subseteq Y$, H is a regular open set in Y iff $\pi^{-1}[H]$ is a regular open set in X.

(f)(iv) If X_0, Y_0, X_1, Y_1 are topological spaces, and $f_i:X_i\to Y_i$ is an open map for each i, then $(x_0,x_1)\mapsto(f_0(x_0),f_1(x_1)):X_0\times X_1\to Y_0\times Y_1$ is open.

(j) Let X be a topological space and D a dense subset of X, endowed with its subspace topology.

(i) A set $A\subseteq D$ is nowhere dense in D iff it is nowhere dense in X.

(ii) A set $G\subseteq D$ is a regular open set in D iff it is expressible as $D\cap H$ for some regular open set $H\subseteq X$.

p 835 l 2 In 4A2F(h-vii), read '$\{U : x \in U \in \mathcal{U}_0\} = \{U : y \in U \in \mathcal{U}_0\}$' for '$\{U : x \in U \in \mathcal{U}\} = \{U : y \in U \in \mathcal{U}\}$'.

p 833 l 17 (part (a) of 4A2E): add new fragment

(iv) Any continuous image of a ccc topological space is ccc.

p 835 l 38 (4A2Gc) Add new fragment:

(ii) Let X and Y be compact Hausdorff spaces, $f : X \to Y$ a continuous open map and $Z \subseteq X$ a zero set in X. Then $f[Z]$ is a zero set in Y.

p 836 l 33 Add new parts:

(i) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is a continuous surjection then there is a closed set $K \subseteq X$ such that $f[K] = Y$ and $f{\restriction}K$ is irreducible.

(ii) If X and Y are compact Hausdorff spaces and $f : X \to Y$ is an irreducible continuous surjection, then (α) if $\mathcal{U}$ is a π-base for the topology of Y then $\{f^{-1}[U] : U \in \mathcal{U}\}$ is a π-base for the topology of X (β) if X has a countable π-base so does Y (γ) if x is an isolated point in X then $f(x)$ is an isolated point in Y (δ) if Y has no isolated points, nor does X.

(m) If X is a Hausdorff space, Y is a compact space and $F \subseteq X \times Y$ is closed, then its projection $\{x : (x, y) \in F\}$ is a closed subset of X.

(n) If X is a locally compact topological space, Y is a topological space and $f : X \to Y$ is a continuous open surjection, then Y is locally compact.

Rename 4A2Gi-4A2Gk as 4A2Gj-4A2Gl.

p 838 l 45 Part (b-iv) of 4A2I should be expanded to

(iv) βI is extremally disconnected.

(v) There are no non-trivial convergent sequences in βI.

p 839 l 43 Add new parts:

(b) Let $(X, \mathcal{W})$ be a uniform space with associated topology $\mathfrak{T}$. If $\mathcal{W}$ is countably generated and $\mathfrak{T}$ is Hausdorff, there is a metric ρ on X defining $\mathcal{W}$ and $\mathfrak{T}$.

(i) Let $(X,\mathcal{U})$ and $(Y, \mathcal{V})$ be uniform spaces. If $\mathcal{F}$ is a Cauchy filter on X and $f : X \to Y$ is a uniformly continuous function, then $f[[\mathcal{F}]]$ is a Cauchy filter on Y.

4A2Jb-4A2Jg are now 4A2Jc-4A2Jh.

p 844 l 38 Part (m) of 4A2R should be expanded to

If $F \subseteq X$ and *either* F is an interval *or* F is compact *or* X is Dedekind complete and F is closed, then the subspace topology on F is induced by the inherited order of F.

p 847 l 15 The material on upper limits of families of sets, formerly used in §446, has been deleted; I have replaced it with a paragraph on Vietoris and Fell topologies and Hausdorff metrics, based on the old 476A-476C, as follows.

4A2T Topologies on spaces of subsets Let X be a topological space, and $\mathcal{C} = \mathcal{C}_X$ the family of closed subsets of X.

(a)(i) The **Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\, F \cap G \neq \emptyset\}, \quad \{F : F \in \mathcal{C},\, F \subseteq G\}$$

where $G \subseteq X$ is open.

(ii) The **Fell topology** on $\mathcal{C}$ is the topology generated by sets of the forms

$$\{F : F \in \mathcal{C},\, F \cap G \neq \emptyset\}, \quad \{F : F \in \mathcal{C},\, F \cap K = \emptyset\}$$

where $G \subseteq X$ is open and $K \subseteq X$ is compact. If X is Hausdorff then the Fell topology is coarser than the Vietoris topology. If X is compact and Hausdorff the two topologies agree.

(iii) Suppose X is metrizable, and that ρ is a metric on X inducing its topology. For a non-empty subset A of X, write $\rho(x, A) = \inf_{y \in A} \rho(x, y)$ for every $x \in X$. Note that $x \mapsto \rho(x, A) : X \to \mathbb{R}$ is 1-Lipschitz.

For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde{\rho}(E,F) = \min(1, \max(\sup_{x\in E} \rho(x,F), \sup_{y\in F} \rho(y,E))).$$

If E, $F \in \mathcal{C} \setminus \{\emptyset\}$ and $z \in X$, then $\rho(z,F) \le \rho(z,E) + \sup_{x\in E} \rho(x,F)$; $\tilde{\rho}$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$, the **Hausdorff metric**. $\tilde{\rho}(\{x\},\{y\}) = \min(1,\rho(x,y))$ for all x, $y \in X$.

(b) (i) The Fell topology is T_1.

(ii) The map $(E,F) \mapsto E \cup F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is continuous for the Fell topology.

(iii) $\mathcal{C}$ is compact in the Fell topology.

(c) If X is Hausdorff, $x \mapsto \{x\}$ is continuous for the Fell topology on $\mathcal{C}$.

(d) If X and another topological space Y are regular, and $\mathcal{C}_Y$, $\mathcal{C}_{X\times Y}$ are the families of closed subsets of Y and $X \times Y$ respectively, then $(E,F) \mapsto E \times F : \mathcal{C}_X \times \mathcal{C}_Y \to \mathcal{C}_{X\times Y}$ is continuous when each space is given its Fell topology.

(e) Suppose that X is locally compact and Hausdorff.

(i) The set $\{(E,F) : E, F \in \mathcal{C}, E \subseteq F\}$ is closed in $\mathcal{C} \times \mathcal{C}$ for the product topology defined from the Fell topology on $\mathcal{C}$. $\{E : E \in \mathcal{C}, E \subseteq F\}$ is closed for every $F \in \mathcal{C}$. $\{(x,F) : x \in F\}$ is closed in $X \times \mathcal{C}$ when $\mathcal{C}$ is given its Fell topology.

(ii) The Fell topology on $\mathcal{C}$ is Hausdorff. It follows that if $\langle F_i\rangle_{i\in I}$ is a family in $\mathcal{C}$, and $\mathcal{F}$ is an ultrafilter on I, then we have a well-defined limit $\lim_{i\to\mathcal{F}} F_i$ defined in $\mathcal{C}$ for the Fell topology.

(f) Suppose that X is metrizable, locally compact and separable. Then the Fell topology on $\mathcal{C}$ is metrizable.

(g) Suppose that X is metrizable, and that ρ is a metric inducing the topology of X; let $\tilde{\rho}$ be the corresponding Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$.

(i) The topology $\mathfrak{S}_{\tilde{\rho}}$ defined by $\tilde{\rho}$ is finer than the Fell topology $\mathfrak{S}_F$ on $\mathcal{C} \setminus \emptyset$.

(ii) If X is compact, then $\mathfrak{S}_{\tilde{\rho}}$ and $\mathfrak{S}_F$ are equal, and both are compact.

p l Add new remark to 4A2Ub: $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R} \setminus \mathbb{Q}$.

p 847 l 38 (4A2U) Add new part:

(e) Give the space $C([0,\infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets.

(i) $C([0,\infty[)$ is a Polish locally convex linear topological space.

(ii) Suppose that $A \subseteq C([0,\infty[)$ is such that $\{f(0) : f \in A\}$ is bounded and for every $a \ge 0$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(s) - f(t)| \le \epsilon$ whenever $f \in A$, s, $t \in [0,a]$ and $|s-t| \le \delta$. Then A is relatively compact for $\mathfrak{T}_c$.

p 850 l 9 (4A3G) Add new part:

(b) If X is any topological space, Y is a T_0 second-countable space, and $f : X \to Y$ is Borel measurable, then (the graph of) f is a Borel set in $X \times Y$.

p 850 l 20 To support the proof of 448P, we need to strengthen 4A3I to include the assertion that $\mathfrak{T}'$ can be taken to be zero-dimensional.

p 850 l 34 (part (c) of the proof of 4A3J): for '$\omega_1 \setminus E$ and E belong to Σ' read '$\omega_1 \setminus E$ and E are Borel sets'.

p 852 l 4 (part (a) of the proof of 4A3N): for '$\tilde{\pi}_J$ is $(\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i), \widehat{\bigotimes}_{i\in J}\mathcal{B}(X_j))$-measurable, so f is $\widehat{\bigotimes}_{i\in J}\mathcal{B}(X_j)$-measurable' read '$\tilde{\pi}_J$ is $(\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i), \widehat{\bigotimes}_{j\in J}\mathcal{B}(X_j))$-measurable, so f is $\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$-measurable'.

p 852 l 39 In 4A3O, parts (d) and (e) have been exchanged.

p 853 l 14 Add new paragraph:

4A3Q Càdlàg functions Let X be a Polish space and C_{dlg} the set of càdlàg functions from $[0,\infty[$ to X, with its pointwise topology induced by the product topology of $X^{[0,\infty[}$.

(a) $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(X^{[0,\infty[})$.

(b) $(C_{\text{dlg}}, \mathcal{B}\mathfrak{a}(C_{\text{dlg}}))$ is a standard Borel space.

(c)(i) For any $t \geq 0$, let $\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}})$ be the σ-algebra of subsets of C_{dlg} generated by the functions $\omega \mapsto \omega(s)$ for $s \leq t$. Then $(s,\omega) \mapsto \omega(s) : C_{\text{dlg}} \times [0,t] \to X$ is $\mathcal{B}([0,t])\widehat{\otimes}\mathcal{B}\mathfrak{a}_t(C_{\text{dlg}})$-measurable.

(ii) $(\omega,t) \mapsto \omega(t) : C_{\text{dlg}} \times [0,\infty[\to X$ is $\mathcal{B}([0,\infty[)\widehat{\otimes}\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$-measurable.

(d) The set $C([0,\infty[\,;X)$ of continuous functions from $[0,\infty[$ to X belongs to $\mathcal{B}\mathfrak{a}(C_{\text{dlg}})$.

4A3Q-4A3V are now 4A3R-4A3W.

p 853 l 21 In Proposition 4A3R (now 4A3S), parts (a-i) and (a-ii) have been rewritten, and are now

(a) Let $A \subseteq X$ be any set.

(i) There is a largest open set $G \subseteq X$ such that $A \cap G$ is meager.

(ii) $H = X \setminus \overline{G}$ is the smallest regular open set such that $A \setminus H$ is meager; $H \subseteq \overline{A}$.

p 856 l 25 Part (ii) of Exercise 4A3Xa has been upstaged by the new 4A3Ya, so has been dropped.

p 856 l 40 (4A3X) Add new exercise:

(f) Let X be a ccc completely regular topological space. Show that any nowhere dense set is included in a nowhere dense zero set.

4A3Xf-4A3Xg are now 4A3Xg-4A3Xh.

p 856 l 41 (4A3Y) Add new exercises:

(a) Give an example of a Hausdorff space X with a countable network and a metrizable space Y such that $\mathcal{B}(X \times Y) \neq \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$.

(c) Give an example of compact Hausdorff spaces X, Y and a function $f : X \to Y$ which is $(\mathcal{B}a(X), \mathcal{B}a(Y))$-measurable but not Borel measurable.

4A3Ya is now 4A3Yb, 4A3Yb-4A3Yc are now 4A3Yd-4A3Ye.

p 858 l 4 Parts (f) and (g) of 4A4B ('bounded sets in linear topological spaces') have been moved to 3A5N. Consequently 4A4Bh-4A4Bk are now 4A4Bf-4A4Bi. I have added a new part (j):

(j) If U is a first-countable Hausdorff linear topological space which (regarded as a linear topological space) is complete, then there is a metric ρ on U, defining its topology, under which U is complete.

p 862 l 30 (4A4J) Add new parts:

(h) Let U be an inner product space over $\substack{\mathbb{R}\\\mathbb{C}}$, and $\langle u_i\rangle_{i\in I}$ a countable family in U. Then there is a countable orthonormal family $\langle v_j\rangle_{j\in J}$ in U such that $\{v_j : j \in J\}$ and $\{u_i : i \in I\}$ span the same linear subspace of U.

(i) Let U be an inner product space over $\substack{\mathbb{R}\\\mathbb{C}}$, and $\langle e_i\rangle_{i\in I}$ an orthonormal family in U. Then $\sum_{i\in I} |(u|e_i)|^2 \leq \|u\|^2$ for every $u \in U$.

(j) Let U be an inner product space over $\substack{\mathbb{R}\\\mathbb{C}}$, and $C \subseteq U$ a convex set. Then there is at most one point $u \in C$ such that $\|u\| \leq \|v\|$ for every $v \in C$.

p 863 l 5 (4A4M) for '$T[U]$ is the closed linear span of $\{Tv : v$ is an eigenvector of $T\}$' read '$T[U]$ is included in the closed linear span of $\{Tv : v$ is an eigenvector of $T\}$'.

p 864 l 10 (4A5B) Add new part:

(g) If $\bullet$ is an action of a group X on a set Z, then sets of the form $\{a\bullet z : a \in X\}$ are called **orbits** of the action; they are the equivalence classes under the equivalence relation $\sim$, where $z \sim z'$ if there is an $a \in X$ such that $z' = a\bullet z$.

p 864 l 37 Part (i) of 4A5E should be expanded, as follows:

(i) If $K \subseteq X$ is compact and $G \subseteq X$ is open, Then $W = \{(x,y) : xKy \subseteq G\}$ is open in $X \times X$. It follows that $\{x : xK \subseteq G\}$, $\{x : Kx \subseteq G\}$ and $\{x : xKx^{-1} \subseteq G\}$ are open.

p 865 l 7 (4A5E) Add new part:

(m) If Y is a subgroup of X, its closure $\overline{Y}$ is a subgroup of X.

p 865 l 29 Proposition 4A5J has been expanded, with an initial part dealing with quotients under group actions, and a couple of extra facts about quotient groups; it now reads as follows.

Theorem (a) Let X be a topological space, Y a topological group, and $\bullet$ a continuous action of Y on X. Let Z be the set of orbits of the action, and for $x \in X$ write $\pi(x) \in Z$ for the orbit containing x.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) If Y is compact and X is Hausdorff, then Z is Hausdorff.

(β) If X is locally compact then Z is locally compact.

(b) Let X be a topological group, Y a subgroup of X, and Z the set of left cosets of Y in X. Set $\pi(x) = xY$ for $x \in X$.

(i) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(ii)(α) Z is Hausdorff iff Y is closed.

(β) If X is locally compact, so is Z.

(γ) If X is locally compact and Polish and Y is closed, then Z is Polish.

(δ) If X is locally compact and σ-compact and Y is closed and Z is metrizable, then Z is Polish.

(iii) We have a continuous action of X on Z defined by saying that $x \bullet \pi(x') = \pi(xx')$ for any x, $x' \in X$.

(iv) If Y is a normal subgroup of X, then the group operation on Z renders it a topological group.

p 866 l 28 (part (a) of the statement of 4A5L): for 'with kernel included in Y' read 'with kernel including Y'.

p 868 l 38 (Proposition 4A5Q) Add a further equivalent condition:

(v) the bilateral uniformity of X is metrizable.

p 871 l 20 In Lemmas 4A6M and 4A6N some constants were imperfectly adjusted. In the statement of part (a) of Lemma 4A6M, we need '$\max(\|u\|, \|v\|) \le \frac{2}{3}$' rather than '$\max(\|u\|, \|v\|) \le \frac{1}{2}$'. The proof works unchanged, except that on the last line we need '$\gamma \le \frac{2}{3}$' instead of '$\gamma \le \frac{1}{2}$'. In the proof of 4A6N we can now change the false claim '$\|2v_n\| \le \frac{1}{3}$' into the true statement '$\|2v_n\| \le \frac{2}{3}$' before quoting 4A6Ma.

p 872 l 29 Add new result:

4A6O Proposition Let U be a normed algebra, and U^*, U^{**} its dual and bidual as a normed space. For a bounded linear operator $T : U \to U$ let $T' : U^* \to U^*$ be the adjoint of T and $T'' : U^{**} \to U^{**}$ the adjoint of T'.

(a) We have bilinear operators, all of norm at most 1,

$$(f, x) \mapsto f \circ x : U^* \times U \to U^*,$$
$$(\phi, f) \mapsto \phi \circ f : U^{**} \times U^* \to U^*,$$
$$(\phi, \psi) \mapsto \phi \circ \psi : U^{**} \times U^{**} \to U^{**}$$

defined by the formulae

$$(f \circ x)(y) = f(xy),$$
$$(\phi \circ f)(x) = \phi(f \circ x),$$
$$(\phi \circ \psi)(f) = \phi(\psi \circ f)$$

for all x, $y \in U$, $f \in U^*$ and ϕ, $\psi \in U^{**}$.

(b)(i) Suppose that $S : U \to U$ is a bounded linear operator such that $S(xy) = (Sx)y$ for all x, $y \in U$. Then $S''(\phi \circ \psi) = (S''\phi) \circ \psi$ for all ϕ, $\psi \in U^{**}$.

(ii) Suppose that $T : U \to U$ is a bounded linear operator such that $T(xy) = x(Ty)$ for all x, $y \in U$. Then $T''(\phi \circ \psi) = \phi \circ (T''\psi)$ for all ϕ, $\psi \in U^{**}$.

p 875 l 46 In the reference

Frolík Z. [61] ‘On analytic spaces’, Bull. Acad. Polon. Sci. 8 (1961) 747-750

the volume and page number should be Bull. Acad. Polon. Sci. 9 (1961) 721-726. P. Monteiro