Errata and addenda for Volume 2, 2010 edition

I collect here known errors and omissions, with their discoverers, in Volume 2 of my book *Measure Theory* (`http://www1.essex.ac.uk/maths/people/fremlin/mt2.2010/index.htm`).

p 16 l 11 In Exercise 211Xa, we have to assume that μ is semi-finite. (D.L.Wynter.)

p 16 l 31 Exercise 211Xf is virtually a repetition of 211Xd, and has been dropped. (P.Klinger Monteiro.)

p l Exercise 211Yd now reads

(d) Let (X,Σ,μ) be a strictly localizable measure space. Show that the following are equiveridical: (i) (X,Σ,μ) is atomless; (ii) for every $\epsilon>0$ there is a decomposition of X consisting of sets of measure at most ϵ; (iii) there is a measurable function $f:X\to\mathbb{R}$ such that $\mu f^{-1}[\{t\}]=0$ for every $t\in\mathbb{R}$.

p 22 l 24 (212X) Add new exercise:

(j) Let (X,Σ,μ) be a complete measure space and $f:X\to\mathbb{R}$ a function such that $\overline{\int} f d\mu<\infty$. Show that there is a measurable function $g:X\to\mathbb{R}$ such that $f(x)\le g(x)$ for every $x\in X$ and $\int g\,d\mu=\overline{\int} f d\mu$.

p 22 l 33 (212Y) Add new exercise:

(b) Let (X,Σ,μ) be a strictly localizable measure space. Suppose that for every $n\in\mathbb{N}$ there is a disjoint family $\langle D_i\rangle_{i<n}$ of subsets of full outer measure. Show that there is a disjoint sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets of full outer measure.

p 24 l 26 (part (a) of the proof of 213D): for '$H\cap E=\bigcup_{n\in\mathbb{N}} H\cap H_n$' read '$H\cap E=\bigcup_{n\in\mathbb{N}} H_n\cap E$'. (T.D.Austin)

p 25 l 3 (part (d) of the proof of 213D): for '(X,Σ,μ)' read '$(X,\tilde\Sigma,\tilde\mu)$'. (K.Yates.)

p 25 l 18 Add new fragment to 213Fc:

$\tilde\mu H=\sup\{\mu E:E\in\Sigma,\ \mu E<\infty,\ E\subseteq H\}$ for every $H\in\tilde\Sigma$.

p 27 l 9 In part (b-ii) of the proof of 213H I wrote '$\mu(E\setminus(F\setminus E_0))\le\tilde\mu(H\setminus(F\cap G))$'. This is supposed to be because $E\setminus(F\setminus E_0)\subseteq H\setminus(F\cap G)$ and $\mu(E\setminus(F\setminus E_0))=\tilde\mu(E\setminus(F\setminus E_0))$. The latter assumes the result of (c) just below.

p 30 l 21 Part (v) of Exercise 213Xa is wrong, and should be deleted. (A.Gouberman.)

p 31 l 13 Other exercises in §213 have been rearranged: 213Xh-213Xi are now 213Xj-213Xk, 213Xj-213Xk are now 213Xi-213Xj, 213Ya-213Ye are now 213Yb-213Yf, 213Yf is now 213Ya.

p 31 l 15 (Exercise 213Xi, now 213Xk): for '$\max(\mu^*(E\cap A),\mu^*(E\setminus A))$' read '$\min(\mu^*(E\cap A),\mu^*(E\setminus A))$'.

p 31 l 18 Exercises 213Xj-213Xk (now 213Xh-213Xi) have been rewritten, and are now

(h) Let (X,Σ,μ) be a measure space with locally determined measurable sets. Show that it is semi-finite.

(i) Let (X,Σ,μ) be a measure space, $\hat\mu$ the completion of μ, $\tilde\mu$ the c.l.d. version of μ and $\check\mu$ the measure defined by Carathéodory's method from μ^*. Show that the following are equiveridical: (i) μ has locally determined negligible sets; (ii) μ and $\tilde\mu$ have the same negligible sets; (iii) $\check\mu=\tilde\mu$; (iv) $\hat\mu$ and $\tilde\mu$ have the same sets of finite measure; (v) μ and $\tilde\mu$ have the same integrable functions; (vi) $\tilde\mu^*=\mu^*$; (vii) the outer measure μ^*_{sf} of 213Xg is equal to μ^*.

---

p 32 l 9 (Exercise 213Yd, now 213Ye): for '$\max(\mu^*\{x : x \in D \cap E,\ f(x) \le a\}, \mu^*\{x : x \in D \cap E,\ f(x) \ge b\}) < \mu E$' read '$\min(\mu^*\{x : x \in D \cap E,\ f(x) \le a\}, \mu^*\{x : x \in D \cap E,\ f(x) \ge b\}) < \mu E$'.

p 41 l 21 Part of Exercise 214Xd has been brought into the main exposition, as follows:

***214Q Proposition** Suppose that (X, Σ, μ) is an atomless measure space and Y a subset of X such that the subspace measure μ_Y is semi-finite. Then μ_Y is atomless.

p l (214X) Add new exercise:

(o) Let (X, Σ, μ) be a measure space and $\langle X_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X such that $\bigcup_{n\in\mathbb{N}} X_n$ has full outer measure on X. Suppose that for each $n \in \mathbb{N}$ we have a set $A_n \subseteq X_n$ of full outer measure for the subspace measure on X_n, Show that $\bigcup_{n\in\mathbb{N}} A_n$ has full outer measure in X.

p 42 l 27 (214Y) Add new exercise:

(d) For this exercise only, I will say that a measure μ on a set X is **nowhere all-measuring** if whenever $A \subseteq X$ is not μ-negligible there is a subset of A which is not measured by the subspace measure on A. Show that if X is a set and $\mu_0, \dots , \mu_n$ are nowhere all-measuring complete totally finite measures on X, then there are disjoint $A_0, \dots , A_n \subseteq X$ such that $\mu_i^* A_i = \mu_i X$ for every $i \le n$.

(e) Let (X, Σ, μ) be a measure space and $\mathcal{A}$ a disjoint family of subsets of X. Show that there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

p 43 l 37 (part (b) of the proof of 215A): for 'If G is any measurable set such that $E \setminus F$ is negligible for every $E \in \mathcal{E}$' read 'If G is any measurable set such that $E \setminus G$ is negligible for every $E \in \mathcal{E}$'. (P.K.M.)

p 45 l 21 Add new result:

215E Proposition Let (X, Σ, μ) be an atomless measure space and $x \in X$.

(a) If $\mu^*\{x\}$ is finite then $\{x\}$ is negligible.

(b) If μ has locally determined negligible sets then $\{x\}$ is negligible.

(c) If μ is localizable then $\{x\}$ is negligible.

p 51 l 27 (216Y) Add new exercise:

(e) Show that there is a complete atomless semi-finite measure space with a singleton subset which is not negligible.

p l (Definition 222J) For '$\overline{D}^+(x)$', '$\underline{D}^+(x)$', '$\overline{D}^-(x)$', '$\underline{D}^-(x)$' read '$\overline{D}^+f(x)$', '$\underline{D}^+f(x)$', '$\overline{D}^-f(x)$', '$\underline{D}^-f(x)$'. (P.K.M.)

p 60 l 17 (statement of Lemma 222K): for '$A \setminus A^*$' read '$A \setminus A^+$'.

p 60 l 27 (statement of Theorem 222L): for '*or* $(\overline{D}^+f)(x) = (\underline{D}^-f)(x)$ is finite, $(\underline{D}^+f)(x) = -\infty$ and $(\overline{D}^+f)(x) = \infty$' read '*or* $(\overline{D}^+f)(x) = (\underline{D}^-f)(x)$ is finite, $(\underline{D}^+f)(x) = -\infty$ and $(\overline{D}^-f)(x) = \infty$'. (P.K.M.)

p 60 l 30 P.Klinger Monteiro has pointed out that there is a blunder in the proof of Theorem 222L. The following is a corrected version of part (a) of the proof.

(a)(i) Suppose that $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ are such that

$$E_{qn} = \{x : x \in A,\ x < q,\ f(y) \ge f(x) - n(y - x)\} \text{ for every } y \in A \cap [x, q]\}$$

is not empty. Set $\beta_{qn} = \sup E_{qn} \in \,]-\infty, q]$, $\alpha_{qn} = \inf E_{qn} \in [-\infty, \beta_{qn}]$ and $I_{qn} = \,]\alpha_{qn}, \beta_{qn}[$; now for $x \in I_{qn}$ set $f_{qn}(x) = \inf\{f(y) + ny : y \in A \cap [x, q]\}$. Note that if $x \in E_{qn} \setminus \{\alpha_{qn}, \beta_{qn}\}$ then $f_{qn}(x) = f(x) + nx$ is finite; also f_{qn} is non-decreasing, therefore finite everywhere in I_{qn}, and of course $f_{qn}(x) \le f(x) + nx$ for every $x \in A \cap I_{qn}$.

Set $F_{qn} = E_{qn} \cap \operatorname{dom} f'_{qn} \subseteq I_{qn}$, and $g_{qn}(x) = f_{qn}(x) - nx$ for $x \in I_{qn}$; then g_{qn} is differentiable at every point of F_{qn}, while $g_{qn}(x) \le f(x)$ for $x \in A \cap I_{qn}$ and $g_{qn}(x) = f(x)$ for $x \in E_{qn} \cap I_{qn}$.

(ii) Take $x \in \tilde{F}^+_{qn} \cap \tilde{F}^-_{qn}$. Then $x \in I_{qn} \cap \tilde{A}^+ \cap \tilde{A}^-$ so the Dini derivates $(\underline{D}^+f)(x)$ and $(\overline{D}^-f)(x)$ are defined in $[-\infty, \infty]$, while $g_{qn}(x) = f(x)$.

(α) $(\underline{D}^+f)(x)=g'_{qn}(x)$. **P**

$$(\underline{D}^+f)(x)=\sup_{\delta>0}\inf_{y\in A\cap]x,x+\delta]}\tfrac{f(y)-f(x)}{y-x}\le\sup_{\delta>0}\inf_{y\in E_{qn}\cap]x,x+\delta]}\tfrac{f(y)-f(x)}{y-x}$$

$$=\sup_{\delta>0}\inf_{y\in E_{qn}\cap]x,x+\delta]}\tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}=g'_{qn}(x)$$

(because $x\in\tilde{E}^+_{qn}$)

$$=\sup_{0<\delta<\beta_{qn}-x}\inf_{y\in]x,x+\delta]}\tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}$$

$$\le\sup_{0<\delta<\beta_{qn}-x}\inf_{y\in A\cap]x,x+\delta]}\tfrac{g_{qn}(y)-f(x)}{y-x}$$

$$\le\sup_{0<\delta<\beta_{qn}-x}\inf_{y\in A\cap]x,x+\delta]}\tfrac{f(y)-f(x)}{y-x}$$

(because $g_{qn}(y)\le f(y)$ for $y\in A\cap I_{qn}$)

$$\le\sup_{\delta>0}\inf_{y\in A\cap]x,x+\delta]}\tfrac{f(y)-f(x)}{y-x}=(\underline{D}^+f)(x).\ \mathbf{Q}$$

(β) $(\overline{D}^-f)(x)=g'_{qn}(x)$. **P**

$$(\overline{D}^-f)(x)=\inf_{\delta>0}\sup_{y\in A\cap[x-\delta,x[}\tfrac{f(y)-f(x)}{y-x}=\inf_{\delta>0}\sup_{y\in A\cap[x-\delta,x[}\tfrac{f(x)-f(y)}{x-y}$$

$$\le\inf_{0<\delta<x-\alpha_{qn}}\sup_{y\in A\cap[x-\delta,x[}\tfrac{f(x)-f(y)}{x-y}$$

$$\le\inf_{0<\delta<x-\alpha_{qn}}\sup_{y\in A\cap[x-\delta,x[}\tfrac{g_{qn}(x)-g_{qn}(y)}{x-y}$$

(because $g_{qn}(y)\le f(y)$ for $y\in A\cap I_{qn}$)

$$\le\inf_{0<\delta<x-\alpha_{qn}}\sup_{y\in[x-\delta,x[}\tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}=g'_{qn}(x)$$

$$=\inf_{\delta>0}\sup_{y\in E_{qn}\cap[x-\delta,x[}\tfrac{g_{qn}(y)-g_{qn}(x)}{y-x}$$

(because $x\in\tilde{E}^-_{qn}$)

$$=\inf_{\delta>0}\sup_{y\in E_{qn}\cap[x-\delta,x[}\tfrac{f(y)-f(x)}{y-x}$$

$$\le\inf_{\delta>0}\sup_{y\in A\cap[x-\delta,x[}\tfrac{f(y)-f(x)}{y-x}=(\overline{D}^-f)(x).\ \mathbf{Q}$$

(γ) Putting these together, we see that if $x\in\tilde{F}^+_{qn}\cap\tilde{F}^-_{qn}$ then $(\underline{D}^+f)(x)=(\overline{D}^-f)(x)=g'_{qn}(x)$ is finite.

(iii) Conventionally setting $F_{qn}=\emptyset$ if E_{qn} is empty, the last sentence is true for all $q\in\mathbb{Q}$ and $n\in\mathbb{N}$. Now we know that $A\setminus\tilde{A}^+$ is negligible (222K), as are $F_{qn}\setminus\tilde{F}^+_{qn}$, $F_{qn}\setminus\tilde{F}^-_{qn}$ and $I_{qn}\setminus\operatorname{dom}f'_{qn}$ (222A), whenever $q\in\mathbb{Q}$ and $n\in\mathbb{N}$. So $H=(A\setminus\tilde{A}^+)\cup\bigcup_{q\in\mathbb{Q},n\in\mathbb{N}}((E_{qn}\setminus F_{qn})\cup(F_{qn}\setminus(\tilde{F}^+_{qn}\cap\tilde{F}^-_{qn})))$ is negligible. And if $x\in A\setminus H$ and $(\overline{D}^+f(x))>-\infty$, then $(\underline{D}^+f)(x)=(\overline{D}^-f)(x)\in\mathbb{R}$. **P** As $A\setminus\tilde{A}^+\subseteq H$, $x\in\tilde{A}^+$. Let $n\in\mathbb{N}$ be such that $(\overline{D}^+f)(x)>-n$. Then there is a $\delta>0$ such that $\frac{f(y)-f(x)}{y-x}>-n$ whenever $y\in A$ and $x<y\le x+\delta$. Let $q\in\mathbb{Q}$ be such that $x<q\le x+\delta$; then $f(y)\ge f(x)-n(y-x)$ whenever $y\in A\cap[x,q]$, and $x\in E_{qn}\setminus H$. So $x\in F_{qn}\setminus H\subseteq\tilde{F}^+_{qn}\cap\tilde{F}^-_{qn}$. So (ii) above tells us that $(\underline{D}^+f)(x)=(\overline{D}^-f)(x)=g'_{qn}(x)$ is finite. **Q**

p 66 l 37 (223Y) Add new exercise:

bf (i) Let $E \subseteq \mathbb{R}$ be a non-negligible measurable set. Show that 0 belongs to the interior of $E - E = \{x - y : x, y \in E\}$.

p l (part (c) of the proof of 224C): for '$\sum_{i=1}^{m+n+1} |f(a_i) - f(a_{i-1})| \le \operatorname{Var}_{[a,b]}(f)$' read '$\sum_{i=1}^{m+n+1} |f(a_i) - f(a_{i-1})| \le \operatorname{Var}_D(f)$'. (K.Y.)

p 74 l 6 (224X) Add new exercises:

(k) Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Show that f is expressible as a difference of non-decreasing functions iff $\operatorname{Var}_{D\cap[a,b]}(f)$ is finite whenever $a \le b$ in D.

(l) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a continuous function of bounded variation. Show that f is expressible as the difference of two continuous non-decreasing functions.

(m) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a function of bounded variation which is continuous on the right, that is, whenever $x \in D$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(t) - f(x)| \le \epsilon$ for every $t \in D \cap [x, x+\delta]$. Show that f is expressible as the difference of two non-decreasing functions which are continuous on the right.

p l (224Y) Add new exercise:

(k) Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$. Show that the following are equiveridical: (α) $\lim_{n\to\infty} f(t_n)$ is defined for every montonic sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in D; (β) for every $\epsilon > 0$ there is a function $g : D \to \mathbb{R}$ of bounded variation such that $|f(t) - g(t)| \le \epsilon$ for every $t \in D$.

p 77 l 31 (part (iii)⇒(ii) of the proof of 225E: for '$G = F + F(a)$' read '$G = F - F(a)$'.

p 78 l 7 The former exercise 225Xo has been absorbed into 225G, which now reads

Proposition Let $[a, b]$ be a non-empty closed interval in $\mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ an absolutely continuous function.

(a) $f[A]$ is negligible for every negligible set $A \subseteq \mathbb{R}$.

(b) $f[E]$ is measurable for every measurable set $E \subseteq \mathbb{R}$.

p 80 l 15 (part (b-ii) of the proof of 225J) There is a confusion over the definition of $H(k, p, q, q')$; if we change this to

$$\begin{aligned} H(k,p,q,q') &= \emptyset \text{ if }]q,q'[\not\subseteq D,\\ &= \{x : x \in E\cap\,]q,q'[\,,\ |f(y) - f(x) - p(y-x)| \le 2^{-k}|y-x| \\ &\qquad\qquad \text{for every } y \in\,]q,q'[\} \\ &\qquad \text{if }]q,q'[\,\subseteq D, \end{aligned}$$

the argument runs more naturally, the next displayed formula becoming

$$\begin{aligned} |f(y) - f(x) - p(y-x)| &= \lim_{n\to\infty} |f(y) - f(x_n) - p(y-x_n)| \\ &\le 2^{-k} \lim_{n\to\infty} 2^{-k}|y - x_n| = 2^{-k}|y-x|. \end{aligned}$$

Next, the set E need not be itself Borel, but as it is relatively Borel in D we see that $E \cap\,]q,q'[$ will be a Borel set whenever $]q,q'[\,\subseteq D$, so that $H(k,p,q,q')$ is indeed a Borel set, as claimed.

p 83 l 5 The exercises for §225 have been rearranged: 225Xb is now 225Xl, 225Xc-225Xd are now 225Xb-225Xc, 225Xe is now 225Xn, 225Xf-225Xm are now 225Xd-225Xk, 225Xn is now 225Xm, 225Xo is now 225Gb.

Add a third part to 225Xb (now 225Xl):

(iii) Show that if $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $\inf_{x\in[a,b]} |g(x)| > 0$ then f/g is absolutely continuous.

p 84 l 14 (225Y) Add new exercise:

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is absolutely continuous on every bounded interval. Show that $\operatorname{Var} f \le \frac{1}{2} \operatorname{Var} f' + \int |f|$.

Other exercises have been moved: 225Yd-225Yf are now 225Ye-225Yg.

p 88 l 5 (part (d) of 226B): for '$x < t_1 \le t_2 \le y$' read '$x < t_1 \le t_2 < y$'.

p 90 l 35 (proof of 226E): for '$\lim_{n\to\infty} \sum_{i=0}^n \int f_i = \sum_{i\in\mathbb{N}} f_i$' read '$\lim_{n\to\infty} \sum_{i=0}^n \int f_i = \sum_{i\in\mathbb{N}} \int f_i$'.

p 90 l 36 (226X) Add new exercise:

(g)(i) Show that a continuous bijection $f : [0,1] \to [0,1]$ is either strictly increasing or strictly decreasing, and that its inverse is continuous. (ii) Show that if $f : [0,1] \to [0,1]$ is a continuous bijection, then $f' = 0$ a.e. in $[0,1]$ iff there is a conegligible set $E \subseteq [0,1]$ such that $f[E]$ is negligible, and that in this case f^{-1} has the same property. (iii) Construct a function satisfying the conditions of (ii). (*Hint*: try $f = \sum_{n=0}^\infty 2^{-n-1} f_n$ where each f_n is a variation on the Cantor function.) (iv) Repeat (iii) with $f = f^{-1}$.

Other exercises have been rearranged: 226Xa-226Xe are now 226Xb-226Xf, 226Xf is now 226Xa.
Other exercises have been rearranged: 226Ya-226Yb are now 226Yb-226Yc, 226Yc is now 226Ya.

p 92 l 36 Part (d) of 231B has been absorbed into part (c), with the addition of the formula $\nu E + \nu F = \nu(E \cup F) + \nu(E \cap F)$.

p 101 l 7 (part (a) of the statement of 232I): for 'ν_s is singular with respect to ν' read 'ν_s is singular with respect to μ'.

p 103 l 4 The exercises 232X have been rearranged; 232Xc is now 232Xg, 232Xg is now 232Xh, 232Xh is now 232Xc.

p 103 l 16 (Exercise 232Xf): for 'ν is additive and absolutely continuous with respect to μ_2' read 'ν is additive and absolutely continuous with respect to μ_1'. (K.Y.)

p 103 l 28 The exercises 232Y have been rearranged: 232Yb-232Yd are now 232Yh-232Yj, 232Ye is now 232Yb, 232Yf is now 232Yd, 232Yg is now 232Yk, 232Yh-232Yj are now 232Ye-232Yg, 232Yk is now 232Yc.

p 131 l 46 (235X) Add new exercise:

(n) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that $\overline{\int} h\phi \, d\mu \le \overline{\int} h \, d\nu$ for every real-valued function h defined almost everywhere in Y.

p 193 l 19 (part (b-i) of the proof of 247C): for '$\|u\|_1 \le 2\sup_{F\in\Sigma} |\int_F u| \le 2(1 + M_0\mu(F \cap E_0))$' read '$\|u\|_1 \le 2\sup_{F\in\Sigma} |\int_F u| \le 2\sup_{F\in\Sigma}(1 + M_0\mu(F \cap E_0))$'.

p 197 l 21 (part (a) of the proof of 251E): for '$\sum_{n=0}^\infty \mu E_n \cdot \nu E_n \le \theta A + \epsilon$' read '$\sum_{n=0}^\infty \mu E_n \cdot \nu F_n \le \theta A + \epsilon$'. (K.Y.)

p 218 l 19 (Corollary 252H) Add new part:

(b) Let f be a Λ-measurable $[0,\infty]$-valued function defined on a member of Λ. Then

$$\int_{X\times Y} f(x,y)\lambda(d(x,y)) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$$

in the sense that if one of the integrals is defined in $[0,\infty]$ so are the other two, and all three are then equal.

p 224 l 35 (252X) Add new exercise:

(j) Let (X, Σ, μ) be a measure space, and $f : X \to [0,\infty[$ a function. Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Show that the following are equiveridical: (α) f is Σ-measurable; (β) $\{(x,a) : x \in X,\ 0 \le a \le f(x)\} \in \Sigma\widehat{\otimes}\mathcal{B}$; ($\gamma$) $\{(x,a) : x \in X,\ 0 \le a < f(x)\} \in \Sigma\widehat{\otimes}\mathcal{B}$.

p 241 l 15 (part (e) of the proof of 254F): for

$$\text{`}\lambda(V \setminus W') + \lambda(V \setminus W) = \lambda V - \lambda W + \lambda(V \setminus W) = \theta V - \theta W + \theta(V \setminus W)\text{'}$$

read

$$\text{`}\lambda(V \setminus W') + \lambda(V \setminus W) = \lambda V - \lambda W + \lambda(V \setminus W') = \theta V - \theta W + \theta(V \setminus W')\text{'.}$$

p 242 l 40 (254J) Add new parts:

(d) Define addition on X by setting $(x+y)(i)=x(i)+_2 y(i)$ for every $i \in I$, x, $y \in X$, where $0+_2 0=1+_2 1=0$, $0+_2 1=1+_2 0=1$. If $y \in X$, the map $x \mapsto x+y : X \to X$ is inverse-measure-preserving.

(e) If $\pi : I \to I$ is any permutation, then we have a corresponding measure space automorphism $x \mapsto x\pi : X \to X$.

p 243 l 33 (part (b-ϵ) of the proof of 254K): for '$\phi^{-1}[E] \subseteq V$' read '$\tilde{\phi}^{-1}[E] \subseteq V$'.

p 247 l 40 (part (a-ii) of the proof of 254P): for '$W_z=\{y : y \in X_I, (y,z) \in W\}$' read '$W_z=\{y : y \in X_J, (y,z) \in W\}$'.

p 252 l 44 The exercises for §254 have been rearranged; 254Xi-254Xr are now 254Xj-254Xs, 254Xs is now 254Xi.

p 253 l 23 Part (ii) of Exercise 254Xp (now 254Xq) is wrong, and has been deleted.

p 253 l 43 (254Y) Add new exercises:

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be any family of measure spaces. Set $X=\prod_{i\in I}X_i$ and let $\mathcal{F}$ be a filter on the set $[I]^{<\omega}$ of finite subsets of I such that $\{J : i \in J \in [I]^{<\omega}\} \in \mathcal{F}$ for every $i \in I$. Show that there is a complete locally determined measure λ on X such that $\lambda(\prod_{i\in I}E_i)$ is defined and equal to $\lim_{J\to\mathcal{F}}\prod_{i\in J}\mu_i E_i$ whenever $E_i \in \Sigma_i$ for every $i \in I$ and $\lim_{J\to\mathcal{F}}\prod_{i\in J}\mu_i E_i$ is defined in $[0,\infty[$.

(e) Let $f : [0,1] \to [0,1]^2$ be a function which is inverse-measure-preserving for Lebesgue planar measure on $[0,1]^2$ and Lebesgue linear measure on $[0,1]$, as in 134Yl; let f_1, f_2 be the coordinates of f. Define $g : [0,1] \to [0,1]^{\mathbb{N}}$ by setting $g(t)=\langle f_1 f_2^n(t)\rangle_{n\in\mathbb{N}}$ for $0 \le t \le 1$. Show that g is inverse-measure-preserving.

254Yb-254Yc are now 254Yc-254Yd, 254Yd-254Yf are now 254Yf-254Yh.

p 270 l 32 Part (b) of 256H has been rewritten, and is now

(b) A point-supported measure on $\mathbb{R}^r$ is a Radon measure iff it is locally finite.

p 282 l 10 The second sentence of Exercise 256Xg should read 'Show that in this case ν is an indefinite-integral measure over Lebesgue measure iff the function $x \mapsto \nu[a,x] : [a,b] \to \mathbb{R}$ is absolutely continuous whenever $a \le b$ in $\mathbb{R}$'.

p 273 l 15 Exercise 256Xi has been extended, and now reads

(i) Let ν be a Radon measure on $\mathbb{R}^r$, and ν^* the corresponding outer measure. Show that $\nu A=\inf\{\nu G : G \supseteq A \text{ is open}\}$ for every set $A \subseteq \mathbb{R}^r$.

p 273 l 39 (256Y) Add new exercise:

(d) (i) Let λ be the usual measure on $\{0,1\}^{\mathbb{N}}$. Define $\psi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by setting $\psi(x)(i)=x(i+1)$ for $x \in \{0,1\}^{\mathbb{N}}$ and $j \in \mathbb{N}$. Show that ψ is inverse-measure-preserving. (ii) Define $\theta : \mathbb{R} \to \mathbb{R}$ by setting $\theta(t)=<3t>=3t-\lfloor 3t\rfloor$ for $t \in \mathbb{R}$. Show that θ is inverse-measure-preserving for Cantor measure as defined in 256Hc.

256Yd-256Ye are now 256Ye-256Yf.

p 276 l 22 (Exercise 257Yb) In the definition of 'Radon measure on $]-\pi,\pi]$', I should have said that the measure must be complete.

p 284 l 4 Exercise 261Yj has been elaborated, and now reads

(j)(i) Let $\mathcal{C}$ be the family of those measurable sets $C \subseteq \mathbb{R}^r$ such that $\limsup_{\delta\downarrow 0}\frac{\mu(C\cap B(x,\delta))}{\mu B(x,\delta)} > 0$ for every $x \in C$. Show that $\bigcup\mathcal{C}_0 \in \mathcal{C}$ for every $\mathcal{C}_0 \subseteq \mathcal{C}$. (ii) Show that any union of non-trivial closed balls in $\mathbb{R}^r$ is Lebesgue measurable.

p 285 l 13 (262A) Add a final remark:

Evidently a Lipschitz function is uniformly continuous.

p 295 l 24 (262Y) Exercise 262Yb seems to require ideas which are not touched on in this volume, so has been moved to 419Yd. Add new exercise:

(d) Let $\phi : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^r$. (i) Show that if ϕ is measurable then all its partial derivatives are measurable. (ii) Show that if ϕ is Borel measurable then all its partial derivatives are Borel measurable.

262Ya is now 262Yb, 262Yd-262Yh are now 262Ye-262Yi, 262Yi is now 262Ya.

p 303 l 21 New result added:

263I Theorem Let $D \subseteq \mathbb{R}^r$ be a measurable set, and $\phi : D \to \mathbb{R}^r$ a function differentiable relative to its domain at each point of D. For each $x \in D$ let $T(x)$ be a derivative of ϕ relative to D at x, and set $J(x) = |\det T(x)|$.

(a) Let ν be counting measure on $\mathbb{R}^r$. Then $\int_{\mathbb{R}^r} \nu(\phi^{-1}[\{y\}])dy$ and $\int_D J\,d\mu$ are defined in $[0, \infty]$ and equal.

(b) Let g be a real-valued function defined on a subset of $\phi[D]$ such that $\int_D g(\phi(x)) \det T(x)dx$ is defined in $\mathbb{R}$, interpreting $g(\phi(x)) \det T(x)$ as zero when $\det T(x) = 0$ and $g(\phi(x))$ is undefined. Set

$$C = \{y : y \in \phi[D],\ \phi^{-1}[\{y\}] \text{ is finite}\}, \qquad R(y) = \textstyle\sum_{x\in\phi^{-1}[\{y\}]} \operatorname{sgn} \det T(x)$$

for $y \in C$, where $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(\alpha) = \dfrac{\alpha}{|\alpha|}$ for non-zero α. If we interpret $g(y)R(y)$ as zero when $g(y) = 0$ and $R(y)$ is undefined, then $\int_{\phi[D]} g\times R\,d\mu$ is defined and equal to $\int_D g(\phi(x)) \det T(x)dx$.

263I is now 263J.

p 304 l 12 (part (b) of the proof of 263I, now 263J): for '$= 0$ if $y < v \le v'$ or $v' \le v < y$' read '$= 0$ otherwise'.

p 305 l 8 (Exercise 263Xf): for '$\det \phi'(\rho, \theta, \alpha) = \rho^2 \sin \theta$' read '$\det \phi'(\rho, \theta, \alpha) = -\rho^2 \sin \theta$'.

p 305 l 25 (Exercise 263Ye): for '(ii) $\mu f[\,[a, b]\,] \le \int_a^b d|f'|d\mu$' read '(ii) $\mu f[\,[c, d]\,] \le \int_c^d |f'|d\mu$ whenever $a \le c \le d \le b$'.

p 305 l 27 (263Y) Add new exercise:

(f) Suppose that $r = 2$ and that $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable with non-singular derivative T at $\mathbf{0}$. (i) Show that there is an $\epsilon > 0$ such that whenever Γ is a small circle with centre $\mathbf{0}$ and radius at most ϵ then $\phi\restriction\Gamma$ is a homeomorphism between Γ and a simple closed curve around $\mathbf{0}$. (ii) Show that if $\det T > 0$, then for such circles $\phi(x)$ runs anticlockwise around $\phi[\Gamma]$ as x runs anticlockwise around Γ. (iii) What happens if $\det T < 0$?

p 307 l 29 (part (c) of the proof of 264B): for 'a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of sets, covering A, with $\operatorname{diam} A_{nm} \le \delta$ for every m and $\sum_{m=0}^{\infty}(\operatorname{diam} A_{nm})^r \le \theta_{r\delta} + 2^{-n}\epsilon$'. read 'a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of sets, covering A_n, with $\operatorname{diam} A_{nm} \le \delta$ for every m and $\sum_{m=0}^{\infty}(\operatorname{diam} A_{nm})^r \le \theta_{r\delta} A_n + 2^{-n}\epsilon$'. (F.Priuli.)

p 319 l 30 (part (b) of the proof of 265E): for '$\sum_{n=0}^{\infty} J\mu_r^* D_n + \epsilon\mu_r^* D_n$' read '$\sum_{n=0}^{\infty} J_n\mu_r^* D_n + \epsilon\mu_r^* D_n$'.

p 320 l 8 (part (c) of the proof of 265E): for '$\nu_r^*\phi[D] = \lim_{k\to\infty} \mu_r^*\phi[D\cap B_k]$' read '$\nu_r^*\phi[D] = \lim_{k\to\infty} \nu_r^*\phi[D\cap B_k]$'.

p 320 l 11 (part (d) of the proof of 265E): for '$\psi(x) = (\phi(x), \eta x)$' read '$\psi_\eta(x) = (\phi(x), \eta x)$'.

p 323 l 30 (265X) Add new exercise:

(f) Suppose that $r \ge 2$. Identifying $\mathbb{R}^r$ with $\mathbb{R}^{r-1} \times \mathbb{R}$, let C_r be the cylinder $B_{r-1} \times [-1, 1] \supseteq B_r$, and $\partial C_r = (B_{r-1} \times \{-1, 1\}) \cup (S_{r-2} \times [-1, 1])$ its boundary. Show that

$$\frac{\mu_r B_r}{\mu_r C_r} = \frac{\nu_{r-1} S_{r-1}}{\nu_{r-1}(\partial C_r)}.$$

p 327 l 31 (266X) Add new exercises:

(d) Show that if $r \ge 1$, μ is Lebesgue measure on $\mathbb{R}^r$ and $A_0, \dots, A_n$ are non-empty subsets of $\mathbb{R}^r$, then $\mu^*(A_0 + \ldots + A_n)^{1/r} \ge \sum_{i=0}^{n}(\mu^* A_i)^{1/r}$.

(e) In 266C, show that if $p \in [0, 1]$ then (subject to an appropriate interpretation of ∞^0) $\mu^*(pA + (1 - p)B) \ge (\mu^* A)^p(\mu^* B)^{1-p}$.

p 349 l 22 The statement of Lemma 273Cb now reads

Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be such that $\sum_{i=0}^\infty x_i$ is defined in $\mathbb{R}$, and $\langle b_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence in $[0,\infty[$ diverging to ∞. Then $\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=0}^n b_kx_k=0$.

p 360 l 4 (273X) Add new exercise:

(o) Let (X,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ an independent sequence in Σ such that $\alpha=\lim_{n\to\infty}\mu E_n$ is defined. For $x\in X$ set $I_x=\{n:x\in E_n\}$. Show that I_x has asymptotic density α for almost every x.

p 365 l 6 (part (d) of the proof of 274F): for '$Z=Z_0+\ldots+Z_n$' read '$Z=\sigma_0Z_0+\ldots+\sigma_nZ_n$'.

p 369 l 15 (part (a) of the proof of 274M): for '$\int_0^\infty e^{-(x+s)^2/2}ds\ge e^{-x^2/2}\int_0^\infty e^{-xs}ds$' read '$\int_0^\infty e^{-(x+s)^2/2}ds\le e^{-x^2/2}\int_0^\infty e^{-xs}ds$'. (M.R.Burke.)

p 369 l 24 Exercise 274Xc (now 274Xd) seems a bit optimistic as it stands, and I have re-written it in a safer form:

(d) Let $\langle m_k\rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $m_0=0$ and $\lim_{k\to\infty}m_k/m_{k+1}=0$. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of random variables such that $\Pr(X_n=\sqrt{m_k})=\Pr(X_n=-\sqrt{m_k})=1/2m_k$, $\Pr(X_n=0)=1-1/m_k$ whenever $m_{k-1}\le n<m_k$. Show that the Central Limit Theorem is not valid for $\langle X_n\rangle_{n\in\mathbb{N}}$. (*Hint*: setting $W_k=(X_0+\ldots+X_{m_k-1})/\sqrt{m_k}$, show that $\Pr(W_k\in[\epsilon,1-\epsilon])\to 0$ for every $\epsilon>0$.)

Other exercises have been moved: 274Xb-274Xh are now 274Xc-274Xi, 274Xi is now 274Xb, 274Xj is now 274Ya, 274Xk-274Xl are now 274Xj-274Xk.

p 370 l 22 (274Y) Add new exercise:

(b) Show that for any $\epsilon>0$ there is a smooth function $h:\mathbb{R}\to[0,1]$ such that $\chi\,]-\infty,-\epsilon]\le h\le\chi\,[\epsilon,\infty[$.

Other exercises have been moved: 274Ya-274Ye are now 274Yc-274Yg.

p 374 l 5 (part (a) of the proof of 275F): for '$r_u\le k<s_{u+1}$' read '$r_{u+1}\le k<s_{u+1}$'.

p 379 l 36 (Exercise 275Xe) for '$\lim_{n\to\infty}\|P_nu-u\|_p=0$' read '$\lim_{n\to\infty}\|P_nu-P_\infty u\|_p=0$'.

p 380 l 1 (275X) Add new exercise:

(i) Let (Ω,Σ,μ) be a probability space, with completion $(\Omega,\hat\Sigma,\hat\mu)$, and $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of σ-subalgebras of $\hat\Sigma$. Show that if $\langle\tau_i\rangle_{i\in\mathbb{N}}$ is a sequence of stopping times adapted to $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$, and we set $\tau(\omega)=\sup_{i\in\mathbb{N}}\tau_i(\omega)$ for $\omega\in\Omega$, then τ is a stopping time adapted to $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$.

Other exercises have been re-named; 275Xi-275Xk are now 275Xj-275Xl.

p 380 l 11 The exercises 275Y have been rearranged; 275Ya-275Yj are now 275Yb-275Yk, 275Yk-275Yl are now 275Yo-275Yp, 275Ym is now 275Ya 275Yn-275Yo are now 275Yq-275Yr, 275Yp-275Yr are now 275Yl-275Yn.

p 380 l 15 Part (ii) of Exercise 275Ya (now 275Yb) is wrong, and has been replaced by

(ii) Defining X_n/Z_n as in 121E, so that its domain is $\{\omega:\omega\in\operatorname{dom}X_n\cap\operatorname{dom}Z_n,\ Z_n(\omega)\ne0\}$, show that $\langle X_n/Z_n\rangle_{n\in\mathbb{N}}$ is a martingale with respect to the measure ν.

p 380 l 28 Exercise 275Yd (now 275Ye) has been rewritten, and is now

(e) (i) Show that if $a\ge0$ and $b\ge1$ then $a\ln b\le a\ln^+a+\frac{b}{e}$, where $\ln^+a=0$ if $a\le1$, $\ln a$ if $a\ge1$. (ii) Let (Ω,Σ,μ) be a complete probability space and X, Y non-negative random variables on Ω such that $t\mu F_t\le\int_{F_t}X$ for every $t\ge0$, where $F_t=\{\omega:Y(\omega)\ge t\}$. Show that $\int_{F_1}Y\le\int_{F_1}X\times\ln^+Y$, and hence that $\mathbb{E}(Y)\le\frac{e}{e-1}(1+\mathbb{E}(X\times\ln^+X))$. (iii) Show that if $\langle X_n\rangle_{n\in\mathbb{N}}$ is a martingale on Ω, $n\in\mathbb{N}$ and $X^*=\sup_{i\le n}|X_i|$, then $\mathbb{E}(X^*)\le\frac{e}{e-1}(1+\mathbb{E}(|X_n|\times\ln^+|X_n|))$.

p 380 l 37 (Exercise 275Yf, now 275Yg): properly speaking, the phrase 'semi-martingale' means something much more general, and should be deleted at this point.

p 389 l 6 (276X) Add new exercise:

(h) In 276B, show that $\mathbb{E}((\sum_{n=0}^{\infty} X_n)^2) \le \sum_{n=0}^{\infty} \mathbb{E}(X_n^2)$.

p 392 l 7 (part (b) of the proof of 281B): for 'Then there are f_1, $g_1 \in \overline{A}$' read 'Then there are f_1, $g_1 \in A$'. (J.Grahl.)

p 394 l 27 (part (b) of the proof of 281G): for '$\eta = \min(\frac{1}{2}, \frac{\epsilon M}{6M+4})$' read '$\eta = \min(\frac{1}{2}, M, \frac{1}{6}\epsilon)$'.

p 396 l 11 (remark following the statement of 281N): for 'the condition '$\eta_1, \dots, \eta_r$ are linearly independent over $\mathbb{Q}$' read 'the condition '$1, \eta_1, \dots, \eta_r$ are linearly independent over $\mathbb{Q}$'.

p 398 l 19 (part (f) of the proof of 281N): for '$0 \le \epsilon \le \frac{1}{2}$' read '$0 < \epsilon \le \frac{1}{2}$'

p 400 l 28 (Exercise 281Ym) for '$\sum_{k=l}^{k+n} f(t_k)$' read '$\sum_{k=l}^{l+n} f(t_k)$'

p 424 l 7 (proof of 283G): for

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iyx} g(x)dx = \hat{g}(y)$$

read

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} f(x)dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} g(x)dx = \hat{g}(y).$$

p 429 l 8 (part (b) of the the proof of 283N): for '$\frac{1}{\sigma}\hat{\psi}_{1/\sigma}(y)$' read '$\frac{1}{\sigma}\psi_{1/\sigma}(y)$'.

p 430 l 7 The exercises in 283W have been rearranged: 283Wb-283Wc are now 283Wc-283Wd, 283Wd-283Wg are now 283Wh-283Wk, 283Wh-283Wj are now 283We-283Wg, 283Wk is now 283Wl, 283Wl is now 283Wb.

p 432 l 16 (part (i) of Exercise 283Xm): for '$\int_0^a \tilde{f}_x(y)dy$' read '$\int_0^a \hat{f}_x(y)dy$'.

p 432 l 33 Exercise 283Xq has been moved to 283Yb. 283Xr-283Xu are now 283Xq-283Xt, 283Yb-283Yd are now 283Yc-283Ye.

p 433 l 22 (part (vi) of Exercise 283Yb, now 283Yc): for '$2\sqrt{2\pi}f(1)$' read '$2\sqrt{2\pi}f(0)$'.

p 433 l 28 (part (ix) of Exercise 283Yb, now 283Yc): for '$h_2(1) \le \epsilon$' read '$h_2(0) \le \epsilon$'.

p 420 l 16 (part (e) of the statement of 283C): for '$\hat{h}(y) = \frac{1}{c}\hat{f}(cy)$' read '$\hat{h}(y) = \frac{1}{c}\hat{f}(\frac{y}{c})$'.

p 440 l 6 (part (c-ii) of the proof of 284L): for

$$\sigma\sqrt{\frac{2}{\pi}}\Big(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\sigma)}{\sigma} + \int_\sigma^\delta \frac{2\Phi(t)}{t^3}dt\Big)$$

read

$$\sigma\sqrt{\frac{2}{\pi}}\Big(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\sigma)}{\sigma^2} + \int_\sigma^\delta \frac{2\Phi(t)}{t^3}dt\Big).$$

p 446 l 1 (284X) Add new exercise:

(d) For a tempered function f and $\alpha \in \mathbb{R}$, set

$$(S_\alpha f)(x) = f(x+\alpha), \quad (M_\alpha f)(x) = e^{i\alpha x}f(x), \quad (D_\alpha f)(x) = f(\alpha x)$$

whenever these are defined. (i) Show that $S_\alpha f$, $M_\alpha f$ and (if $\alpha \ne 0$) $D_\alpha f$ are tempered functions. (ii) Show that if g is a tempered function which represents the Fourier transform of f, then $M_\alpha g$ represents the Fourier transform of $S_\alpha f$, $S_{-\alpha} g$ represents the Fourier transform of $M_\alpha f$, $\bar{\overleftrightarrow{g}} = \overleftrightarrow{\bar{g}}$ represents the Fourier transform of $\bar{f}$, and if $\alpha \ne 0$ then $\frac{1}{|\alpha|}D_{1/\alpha}g$ represents the Fourier transform of $D_\alpha f$.

Other exercises have been renamed: 284Xd-284Xq are now 284Xe-284Xr.

p 456 l 27 (statement of Lemma 285P): to match the application in 285Q we should include the trivial case $M = 0$ here.

p l Add new result:

285V Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$ such that $\nu * \nu = \nu$. Then ν is the Dirac measure δ_0 concentrated at 0.

p 460 l 24 (285X) Add new exercise:

(d) Let X be a real-valued random variable which is not essentially constant, and φ its characteristic function. Show that $|\varphi(y)| < 1$ for all but countably many $y \in \mathbb{R}$.

Other exercises have been rearranged: 285Xd-285Xe are now 285Xe-285Xf, 285Xf-285Xr are now 285Xi-285Xu, 285Xs is now 285Yt, 285Xt is now 285Xh.

p 461 l 4 (Exercise 285Xh, now 285Xk): for 'whenever $c \le d$ in $\mathbb{R}$' read 'whenever $c < d$ in $\mathbb{R}$'. [J.Pachl.]

p 462 l 14 (part (iii) of Exercise 285Ya) for '$\int \check{h}(x)\nu(dx) = \int h(y)\hat{\nu}(y)dy$' read '$\int h(x)\nu(dx) = \int \check{h}(y)\hat{\nu}(y)dy$'.

p 462 l 29 Other exercises have been rearranged: 285Ye-285Yg are now 285Yf-285Yh, 285Yh is now 285Ye, 285Yj-285Yo are now 285Yl-285Yq, 285Yq-285Yr are now 285Yj-285Yk.

p 463 l 38 (285Y) Add new exercises:

(r) Let ν be a probability measure on $\mathbb{R}$. Show that $|\varphi_\nu(y) - \varphi_\nu(y')|^2 \le 2(1 - \mathcal{R}\mathrm{e}\,\varphi_\nu(y - y'))$ for any y, $y' \in \mathbb{R}$.

(s) Let $\langle \nu_n \rangle_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$. Set $E = \{y : y \in \mathbb{R}, \lim_{n\to\infty} \varphi_{\nu_n}(y) = 1\}$. (i) Show that $E - E$ and $E + E$ are included in E. (ii) Show that if E is not Lebesgue negligible it is the whole of $\mathbb{R}$.

p 466 l 1 (part (c) of the proof of 286A): for

$$p\int_0^\infty u^{p-1}\mu G_u du = p^2\int_0^\infty u^{p-2}\Bigl(\int_{-\infty}^\infty (f - \frac{1}{q}u\chi\mathbb{R})^+\Bigr)du$$

read

$$p\int_0^\infty u^{p-1}\mu G_u du \le p^2\int_0^\infty u^{p-2}\Bigl(\int_{-\infty}^\infty (f - \frac{1}{q}u\chi\mathbb{R})^+\Bigr)du.$$

p 470 l 8 (part (e) of the proof of 286G, now part (g)): the case in which $\sigma \ne \tau$ but $I_\sigma = I_\tau$ is not treated; but in this case $J_\sigma \cap J_\tau = \emptyset$ so $(\phi_\sigma|\phi_\tau) = 0$.

p 471 l 7 Lemmas 286I and 286J have been exchanged.

p 478 l 27 (part (h-ii) of the proof of 286L): for '$\phi_\sigma(y) \ne 0$' read '$\hat{\phi}_\sigma(y) \ne 0$'; and similarly two lines later in part (h-iii).

p 479 l 32 (part (i) of the proof of 286L): for '$|x - x'| \le 2^{-l_L+1} \le 2^{-m+1}$' read '$|x - x'| \le 2^{-l_L} \le 2^{-m}$'.

p 482 l 24 (part (b) of the proof of 286O): A.Derighetti has pointed out that the inequality

$$\sum_{\sigma\in Q}\Bigl|\int_{F\cap g^{-1}[J_\sigma^r]}(f|\phi_\sigma)\phi_\sigma\Bigr| \le C_9\|f\|_2\sqrt{\mu F}$$

of 286N has been miscopied as

$$\sum_{\sigma\in Q}\int_{F\cap g^{-1}[J_\sigma^r]}|(f|\phi_\sigma)\phi_\sigma| \le C_9\|f\|_2\sqrt{\mu F}.$$

I do not know whether Lemma 286O, as stated, is true; to correct the error I have re-defined the operator A, so that 286O-286P now read

286O Lemma (a) For $z \in \mathbb{R}$, define $\theta_z : \mathbb{R} \to [0, 1]$ by setting

$$\theta_z(y) = \hat{\phi}(2^{-k}(y - \hat{y}))^2$$

whenever there is a dyadic interval $J \in \mathcal{I}$ of length 2^k such that z belongs to the right-hand half of J and y belongs to the left-hand half of J and $\hat{y}$ is the lower quartile of J, and zero if there is no such J. Then $(y,z)\mapsto\theta_z(y)$ is Borel measurable, $0\le\theta_z(y)\le 1$ for all y, $z\in\mathbb{R}$, and $\theta_z(y)=0$ if $y\ge z$.

(b) For $k\in\mathbb{Z}$, set $Q_k=\{\sigma:\sigma\in Q,\ k_\sigma=k\}$. Let $[Q]^{<\omega}$ be the set of finite subsets of Q, $[\mathbb{Z}]^{<\omega}$ the set of finite subsets of $\mathbb{Z}$ and $\mathcal{L}$ the set of subsets L of Q such that $L\cap Q_k$ is finite for every k. If $K\in[\mathbb{Z}]^{<\omega}$ and $L\in\mathcal{L}$, set

$$\mathcal{P}_{KL}=\{P:P\in[Q]^{<\omega},\ P\cap Q_k\supseteq L\cap Q_k \text{ whenever } k\in\mathbb{Z} \text{ and either } k\in K \text{ or } P\cap Q_k\ne\emptyset\};$$

set

$$\mathcal{F}=\{\mathcal{P}:\mathcal{P}\subseteq[Q]^{<\omega} \text{ and there are } K\in[\mathbb{Z}]^{<\omega},\ L\in\mathcal{L} \text{ such that } \mathcal{P}\supseteq\mathcal{P}_{KL}\}.$$

Then $\mathcal{F}$ is a filter on $[Q]^{<\omega}$ and

$$2\pi\int_F(\hat{h}\times\theta_z)^\vee=\lim_{P\to\mathcal{F}}\sum_{\sigma\in P,z\in J_\sigma^r}\int_F(h|\phi_\sigma)\phi_\sigma$$

for every $z\in\mathbb{R}$ and rapidly decreasing test function h and measurable set $F\subseteq\mathbb{R}$ of finite measure.

286P Lemma Suppose that h is a rapidly decreasing test function. For $x\in\mathbb{R}$, set

$$Ah(x)=\sup_{z\in\mathbb{R}}|2\pi(\hat{h}\times\theta_z)^\vee(x)|.$$

Then $Ah:\mathbb{R}\to[0,\infty]$ is Borel measurable, and $\int_F Ah\le 4C_9\|h\|_2\sqrt{\mu F}$ whenever $\mu F<\infty$.

There are consequent small changes in the proof of 286S, and Lemma 286T now reads

286T Lemma Set $C_{10}=3C_9/\pi\tilde\theta_1(0)$. For $f\in\mathcal{L}^2_{\mathbb{C}}$, define $\hat{A}f:\mathbb{R}\to[0,\infty]$ by setting

$$(\hat{A}f)(y)=\sup_{a\le b}\frac{1}{\sqrt{2\pi}}|\int_a^b e^{-ixy}f(x)dx|$$

for each $y\in\mathbb{R}$. Then $\int_F\hat{A}f\le C_{10}\|f\|_2\sqrt{\mu F}$ whenever $\mu F<\infty$.

p 486 l 3 (part (b-ii) of the proof of 286R): for '$\theta_{z\alpha\beta}(y)$' read '$\theta'_{z\alpha\beta}(y)$'.

p 486 l 7 (part (c) of the proof of 286R): for '$(\alpha,\beta)\mapsto\theta'_{z\alpha\beta}(y)$ is Borel measurable' read '$(\alpha,y)\mapsto\theta'_{z\alpha\beta}(y)$ is Borel measurable'.

p 487 l 27 (part (h) of the proof of 286R): for '$g(\alpha,1,0)$' read '$g(\alpha,0,1)$', and again three lines later.

p 488 l 19 (part (a) of the proof of 286S): for '$g(t)\le 4\gamma/(4+t^2)$ for every t' read '$g(t)\le\frac{\gamma\alpha^2}{4\alpha^2+t^2}$ for every t'.

p 500 l 47 (2A2C) Add new part:

(c) If r, $s\ge 1$, $D\subseteq\mathbb{R}^r$ and $\phi:D\to\mathbb{R}^s$ is a function, we say that ϕ is **uniformly continuous** if for every $\epsilon>0$ there is a $\delta>0$ such that $\|\phi(y)-\phi(x)\|\le\epsilon$ whenever x, $y\in D$ and $\|y-x\|\le\delta$. A uniformly continuous function is continuous.

p 508 l 42 In part (b) of 2A3S, interpolate the following:

If Z is another set, $\mathcal{G}$ is a filter on Z, and $\psi:Z\to X$ is such that $\mathcal{F}=\psi[[\mathcal{G}]]$, then the composition $\phi\psi$ is defined on $\psi^{-1}[D]\in\mathcal{G}$, and if one of the limits $\lim_{x\to\mathcal{F}}\phi(x)$, $\lim_{z\to\mathcal{G}}\phi\psi(z)$ is defined in Y so is the other, and they are then equal.

p 515 l 16 (2A5B) Add definition:

Functionals satisfying the conditions (i)-(iii) of this proposition are called **F-seminorms**.