Errata and addenda for Volume 2, 2003 printing

I collect here known errors and omissions, with their discoverers, in Volume 2 of my book *Measure Theory* (see my web page, `http://www1.essex.ac.uk/maths/people/fremlin/mt.htm`).

p 14 l 32 (part (b) of 211M): for '$\mathbb{R}$' read '$\mathbb{R}^r$'.

p 15 l 38 There is an egregious error in the remark following 211P. I wrote 'there are many methods of constructing non-Borel subsets of the Cantor set [...] and some do not need the axiom of choice at all'. Actually in the absence of the axiom of countable choice the definition of 'Borel set' in 111G becomes seriously problematic, and on a literal interpretation of that definition it is indeed possible for every subset of $\mathbb{R}$ to be a Borel set.

p 16 l 23 The exercises to §211 have been rearranged; 211Xa is now 211Xc, 211Xc is now 211Xe, 211Xd is now split between 234B and 234E, 211Xe is now 211Xa, 211Xf is now 211Xd, 211Ya is now 211Ye, 211Yb is now 211Ya, 211Yc is now 211Yb, 211Yd is now 211Yc, 211Ye is now 211Yd.

p 17 l 11 (211X) Add new exercise:

(g) Let (X, Σ, μ) be a measure space such that $\mu X > 0$. Show that the set of conegligible subsets of X is a filter on X.

p 17 l 32 Exercise 211Yd now reads

(d) Let (X, Σ, μ) be a strictly localizable measure space. Show that the following are equiveridical: (i) (X, Σ, μ) is atomless; (ii) for every $\epsilon > 0$ there is a decomposition of X consisting of sets of measure at most ϵ; (iii) there is a measurable function $f : X \to \mathbb{R}$ such that $\mu f^{-1}[\{t\}] = 0$ for every $t \in \mathbb{R}$.

p 20 l 10 (Proposition 212E): add a new clause to part (b) – 'μ and $\hat\mu$ give rise to the same sets of full outer measure'.

p 23 l 1 (part (c)-(d)(ii) of the proof of 212G): for '$\hat\mu(H \setminus F) = \mu(E \setminus F) = 0$' read '$\hat\mu(H \setminus F) = \mu(E \setminus F') = 0$'.

p 23 l 20 Exercises 212Xe, 212Xh and 212Xj have been incorporated into the new material on sums of measures in §234.

p 23 l 42 (Exercise 212Xk, now 212Xh): for '$\Sigma' = \{E \cap A : E \in \Sigma,\ A \in \mathcal{I}\}$' read '$\Sigma' = \{E \triangle A : E \in \Sigma,\ A \in \mathcal{I}\}$'.

p 23 l 44 Add new exercises:

(i) Let (X, Σ, μ) be a complete measure space such that $\mu X > 0$, Y a set, $f : X \to Y$ a function and μf^{-1} the image measure on Y. Show that if $\mathcal{F}$ is the filter of μ-conegligible subsets of X, then the image filter $f[[\mathcal{F}]]$ (2A1Ib) is the filter of μf^{-1}-conegligible subsets of Y.

(j) Let (X, Σ, μ) be a complete measure space and $f : X \to \mathbb{R}$ a function such that $\overline{\int} f d\mu < \infty$. Show that there is a measurable function $g : X \to \mathbb{R}$ such that $f(x) \le g(x)$ for every $x \in X$ and $\int g\, d\mu = \overline{\int} f d\mu$.

p 24 l 1 Exercise 212Ya has been deleted; 212Yb and 212Yd-212Ye have been incorporated into §234. 212Yc is now 212Ya.

Add new exercise:

(b) Let (X, Σ, μ) be a strictly localizable measure space. Suppose that for every $n \in \mathbb{N}$ there is a disjoint family $\langle D_i \rangle_{i<n}$ of subsets of full outer measure. Show that there is a disjoint sequence $\langle D_n \rangle_{n \in \mathbb{N}}$ of sets of full outer measure.

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p 23 l 34 (Exercise 212Yc, now 212Ya): for 'functional from X to $[0,\infty]$' read 'functional from $\mathcal{P}X$ to $[0,\infty]$'.

p 26 l 25 (part (a) of the proof of 213D): for '$H \cap E = \bigcup_{n\in\mathbb{N}} H \cap H_n$' read '$H \cap E = \bigcup_{n\in\mathbb{N}} H_n \cap E$'. (T.D.Austin.)

p 27 l 3 (part (d) of the proof of 213D): for '(X,Σ,μ)' read '$(X,\tilde{\Sigma},\tilde{\mu})$'. (K.Yates.)

p 27 l 20 Add new fragment to 213Fc:
$\tilde{\mu}H = \sup\{\mu E : E \in \Sigma,\ \mu E < \infty,\ E \subseteq H\}$ for every $H \in \tilde{\Sigma}$.

p 29 l 15 In part (b-ii) of the proof of 213H I wrote '$\mu(E\setminus(F\setminus E_0)) \le \tilde{\mu}(H\setminus(F\cap G))$'. This is supposed to be because $E\setminus(F\setminus E_0) \subseteq H\setminus(F\cap G)$ and $\mu(E\setminus(F\setminus E_0)) = \tilde{\mu}(E\setminus(F\setminus E_0))$. The latter assumes the result of (c) just below.

p 33 l 5 Part (v) of Exercise 213Xa is wrong, and should be deleted. (A.Gouberman.)

p 34 l 1 Other exercises in §213 have been rearranged: 213Xh-213Xi are now 213Xj-213Xk, 213Xj-213Xk are now 213Xi-213Xj, 213Ya-213Ye are now 213Yb-213Yf, 213Yf is now 213Ya.

p 34 l 2 (Exercise 213Xi, now 213Xk): for '$\max(\mu^*(E\cap A),\mu^*(E\setminus A))$' read '$\min(\mu^*(E\cap A),\mu^*(E\setminus A))$'.

p 34 l 4 Exercises 213Xj-213Xk (now 213Xh-213Xi) have been rewritten, and are now

(h) Let (X,Σ,μ) be a measure space with locally determined measurable sets. Show that it is semi-finite.

(i) Let (X,Σ,μ) be a measure space, $\hat{\mu}$ the completion of μ, $\tilde{\mu}$ the c.l.d. version of μ and $\check{\mu}$ the measure defined by Carathéodory's method from μ^*. Show that the following are equiveridical: (i) μ has locally determined negligible sets; (ii) μ and $\tilde{\mu}$ have the same negligible sets; (iii) $\check{\mu} = \tilde{\mu}$; (iv) $\hat{\mu}$ and $\tilde{\mu}$ have the same sets of finite measure; (v) μ and $\tilde{\mu}$ have the same integrable functions; (vi) $\tilde{\mu}^* = \mu^*$; (vii) the outer measure μ^*_{sf} of 213Xg is equal to μ^*.

p 34 l 42 (Exercise 213Yd, now 213Ye): for '$\max(\mu^*\{x : x\in D\cap E,\ f(x)\le a\},\mu^*\{x : x\in D\cap E,\ f(x)\ge b\}) < \mu E$' read '$\min(\mu^*\{x : x\in D\cap E,\ f(x)\le a\},\mu^*\{x : x\in D\cap E,\ f(x)\ge b\}) < \mu E$'.

p 34 l 3 (213Y) Add new exercises:

(a) Let (X,Σ,μ) be a measure space. Show that μ is semi-finite iff there is a family $\mathcal{E}\subseteq\Sigma$ such that $\mu E < \infty$ for every $E \in \mathcal{E}$ and $\mu F = \sum_{E\in\mathcal{E}} \mu(F\cap E)$ for every $F \in \Sigma$.

p 41 l 11 Add new paragraph:

214J Proposition Let (X,Σ,μ) be a measure space and A a subset of X and f a real-valued function defined almost everywhere in X. Then

(a) $\overline{\int}(f\restriction A)d\mu_A \le \overline{\int} f d\mu$ if *either* f is non-negative *or* A has full outer measure in X;

(b) if A has full outer measure in X, $\underline{\int} f d\mu \le \underline{\int}(f\restriction A)d\mu_A$.

214J-214M are now 214K-214N.

p 42 l 20 Add two new paragraphs:

214O Lemma Let (X,Σ,μ) be a measure space, and $\mathcal{I}$ an ideal of subsets of X, that is, a family of subsets of X such that $\emptyset\in\mathcal{I}$, $I\cup J\in\mathcal{I}$ for all I, $J\in\mathcal{I}$, and $I\in\mathcal{I}$ whenever $I\subseteq J\in\mathcal{I}$. Then there is a measure λ on X such that $\Sigma\cup\mathcal{I}\subseteq \operatorname{dom}\lambda$, $\mu E = \lambda E + \sup_{I\in\mathcal{I}}\mu^*(E\cap I)$ for every $E\in\Sigma$, and $\lambda I = 0$ for every $I\in\mathcal{I}$.

214P Theorem Let (X,Σ,μ) be a measure space, and $\mathcal{A}$ a family of subsets of X which is well-ordered by the relation $\subseteq$. Then there is an extension of μ to a measure λ on X such that $\lambda(E\cap A)$ is defined and equal to $\mu^*(E\cap A)$ whenever $E\in\Sigma$ and $A\in\mathcal{A}$.

Part of Exercise 214Xe (now 214Xd) has been brought into the main exposition, as follows:

***214Q Proposition** Suppose that (X,Σ,μ) is an atomless measure space and Y a subset of X such that the subspace measure μ_Y is semi-finite. Then μ_Y is atomless.

p 42 l 23 Exercise 214Xb has been incorporated into 214I. Exercises 214Xc-214Xl are now 214Xb-214Xk. Add new exercises:

(l) Let (X,Σ,μ) be a measure space, Y a subset of X, and $f : X \to [0,\infty]$ a function such that $\int_Y f$ is defined in $[0,\infty]$. Show that $\int_Y f = \overline{\int} f \times \chi Y\, d\mu$.

(m) Write out a proof of 214P in the special case in which $\mathcal{A} = \{A\}$.

(n) Let (X,Σ,μ) be a measure space and $\mathcal{A}$ a finite family of subsets of X. Show that there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

(o) Let (X,Σ,μ) be a measure space and $\langle X_n\rangle_{n\in\mathbb{N}}$ a sequence of subsets of X such that $\bigcup_{n\in\mathbb{N}} X_n$ has full outer measure on X. Suppose that for each $n \in \mathbb{N}$ we have a set $A_n \subseteq X_n$ of full outer measure for the subspace measure on X_n, Show that $\bigcup_{n\in\mathbb{N}} A_n$ has full outer measure in X.

214Y Further exercises (a) Let (X,Σ,μ) be a measure space and C a subset of X such that the subspace measure on C is semi-finite. Set $\alpha = \sup\{\mu E : E \in \Sigma,\ E \subseteq C\}$. Show that if $\alpha \le \gamma \le \mu^* C$ then there is a measure λ on X, extending μ, such that $\lambda C = \gamma$.

(b) Let (X,Σ,μ) be a measure space and $\langle A_n\rangle_{n\in\mathbb{Z}}$ a double-ended sequence of subsets of X such that $A_m \subseteq A_n$ whenever $m \le n$ in $\mathbb{Z}$. Show that there is a measure on X, extending μ, which measures every A_n.

(c) Let X be a set and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical: (i) for every measure μ on X there is a measure on X extending μ and measuring every member of $\mathcal{A}$; (ii) for every totally finite measure μ on X there is a measure on X extending μ and measuring every member of $\mathcal{A}$.

(d) For this exercise only, I will say that a measure μ on a set X is **nowhere all-measuring** if whenever $A \subseteq X$ is not μ-negligible there is a subset of A which is not measured by the subspace measure on A. Show that if X is a set and $\mu_0,\dots,\mu_n$ are nowhere all-measuring complete totally finite measures on X, then there are disjoint $A_0,\dots,A_n \subseteq X$ such that $\mu_i^* A_i = \mu_i X$ for every $i \le n$.

(e) Let (X,Σ,μ) be a measure space and $\mathcal{A}$ a disjoint family of subsets of X. Show that there is a measure on X, extending μ, which measures every member of $\mathcal{A}$.

p 44 l 26 (part (b) of the proof of 215A): for 'If G is any measurable set such that $E \setminus F$ is negligible for every $E \in \mathcal{E}$' read 'If G is any measurable set such that $E \setminus G$ is negligible for every $E \in \mathcal{E}$'. (P.K.)

p 46 l 17 Add new result:

215E Proposition Let (X,Σ,μ) be an atomless measure space and $x \in X$.

(a) If $\mu^*\{x\}$ is finite then $\{x\}$ is negligible.

(b) If μ has locally determined negligible sets then $\{x\}$ is negligible.

(c) If μ is localizable then $\{x\}$ is negligible.

p 52 l 35 (Exercise 216Yb): for '0 otherwise' read '∞ otherwise'.

p 52 l 38 (216Y) Add new exercises:

(d) Set $X = \omega_1 \times \omega_2$. For $E \subseteq X$ set

$$A(E) = \{\zeta : \text{for some } \xi, \text{ just one of } (\xi,\zeta),\ (\xi,\zeta+1) \text{ belongs to } E\},$$

$$B(E) = \{\zeta : \text{ there are } \xi,\ \zeta' \text{ such that } \zeta < \zeta' < \omega_2 \text{ and just one of } (\xi,\zeta),\ (\xi,\zeta') \text{ belongs to } E\},$$

$$W(E) = \{\xi : \#(E[\{\xi\}]) = \omega_2\}.$$

Let Σ be the set of subsets E of X such that $A(E)$ is countable and $\#(B(E)) \le \omega_1$. For $E \in \Sigma$, set $\mu E = \#(W(E))$ if this is finite, ∞ otherwise. (i) Show that (X,Σ,μ) is a measure space. (ii) Show that if $\hat\mu$ is the completion of μ, then its domain is the set of subsets E of X such that $A(E)$ is countable, and $\hat\mu$ is strictly localizable. (iii) Show that μ is not strictly localizable.

(e) Show that there is a complete atomless semi-finite measure space with a singleton subset which is not negligible.

p 56 l 33 (Exercise 221Yd): for '$\bigcup_{n\ge m} I'_n$' read '$\bigcup_{m\ge n} I'_m$'.

p 62 l 29 (§222) I have added a couple of pages on the Denjoy-Young-Saks theorem, as follows:

***222J Definition** Let f be any real function, and $A\subseteq\mathbb{R}$ its domain. Write

$$\tilde A^+=\{x:x\in A,\,]x,x+\delta[\cap A\ne\emptyset\text{ for every }\delta>0\},$$

$$\tilde A^-=\{x:x\in A,\,]x-\delta,x[\cap A\ne\emptyset\text{ for every }\delta>0\}.$$

Set

$$\overline{D}^+f(x)=\limsup_{y\in A,y\downarrow x}\tfrac{f(y)-f(x)}{y-x}=\inf_{\delta>0}\sup_{y\in A,x<y\le x+\delta}\tfrac{f(y)-f(x)}{y-x},$$

$$\underline{D}^+f(x)=\liminf_{y\in A,y\downarrow x}\tfrac{f(y)-f(x)}{y-x}=\sup_{\delta>0}\inf_{y\in A,x<y\le x+\delta}\tfrac{f(y)-f(x)}{y-x}$$

for $x\in\tilde A^+$, and

$$\overline{D}^-f(x)=\limsup_{y\in A,y\uparrow x}\tfrac{f(y)-f(x)}{y-x}=\inf_{\delta>0}\sup_{y\in A,x-\delta\le y<x}\tfrac{f(y)-f(x)}{y-x},$$

$$\underline{D}^-f(x)=\liminf_{y\in A,y\uparrow x}\tfrac{f(y)-f(x)}{y-x}=\sup_{\delta>0}\inf_{y\in A,x-\delta y<x}\tfrac{f(y)-f(x)}{y-x}$$

for $x\in\tilde A^-$, all defined in $[-\infty,\infty]$. (These are the four **Dini derivates** of f.)

Note that we surely have $(\underline{D}^+f)(x)\le(\overline{D}^+f)(x)$ for every $x\in\tilde A^+$, while $(\underline{D}^-f)(x)\le(\overline{D}^-f)(x)$ for every $x\in\tilde A^-$. The ordinary derivative $f'(x)$ is defined and equal to $c\in\mathbb{R}$ iff (α) x belongs to some open interval included in A (β) $(\overline{D}^+f)(x)=(\underline{D}^+f)(x)=(\overline{D}^-f)(x)=(\underline{D}^+f)(x)=c$.

***222K Lemma** Let A be any subset of $\mathbb{R}$, and define $\tilde A^+$ and $\tilde A^-$ as in 222J. Then $A\setminus\tilde A^*$ and $A\setminus\tilde A^-$ are countable, therefore negligible.

***222L Theorem** Let f be any real function, and A its domain. Then for almost every $x\in A$
either all four Dini derivates of f at x are defined, finite and equal
or $(\overline{D}^+f)(x)=(\underline{D}^-f)(x)$ is finite, $(\underline{D}^+f)(x)=-\infty$ and $(\overline{D}^-f)(x)=\infty$
or $(\underline{D}^+f)(x)=(\overline{D}^-f)(x)$ is finite, $(\overline{D}^+f)(x)=\infty$ and $(\underline{D}^-f)(x)=-\infty$
or $(\overline{D}^+f)(x)=(\overline{D}^-f)(x)=\infty$, $(\underline{D}^+f)(x)=\underline{D}^-f)(x)=-\infty$.

p 62 l 30 The exercises for §222 have been re-shuffled; 222Xa is now 222Xd, 222Xd is now 222Xa.

p 66 l 10 The 'basic exercises' for §223 have been rearranged; 223Xc-223Xh are now 223Xe-223Xj, 223Xi is now 223Xc, 223Xj is now 223Xd.

p 67 l 29 (223Y) Add new exercises:

(h) For each integrable real-valued function f defined almost everywhere in $\mathbb{R}$, let E_f be the Lebesgue set of f. Show that $E_f\cap E_g\subseteq E_{f+g}$, $E_f\subseteq E_{|f|}$ for all integrable f, g.

(i) Let $E\subseteq\mathbb{R}$ be a non-negligible measurable set. Show that 0 belongs to the interior of $E-E=\{x-y:x,\,y\in E\}$.

p 69 l 14 (part (c) of the proof of 224C): for '$\sum_{i=1}^{m+n+1}|f(a_i)-f(a_{i-1})|\le\mathrm{Var}_{[a,b]}(f)$' read '$\sum_{i=1}^{m+n+1}|f(a_i)-f(a_{i-1})|\le\mathrm{Var}_D(f)$'. (K.Y.)

p 70 l 27 (part (ii)⇒(iii) of the proof of 224D): for

$$\text{‘}g_1(b)-g_1(a)=g(b)-g(a)-f(b)+f(a),\quad g_2(b)-g_2(a)=g(b)-g(a)+f(b)-f(a)\text{’}$$

read

$$\text{‘}g_1(b)-g_1(a)=g(b)-g(a)+f(b)-f(a),\quad g_2(b)-g_2(a)=g(b)-g(a)-f(b)+f(a)\text{’}.$$

p 75 l 34 (224X) Add new exercises:

(k) Suppose that $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ is a function. Show that f is expressible as a difference of non-decreasing functions iff $\mathrm{Var}_{D\cap[a,b]}(f)$ is finite whenever $a \le b$ in D.

(l) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a continuous function of bounded variation. Show that f is expressible as the difference of two continuous non-decreasing functions.

(m) Suppose that $D \subseteq \mathbb{R}$ and that $f : D \to \mathbb{R}$ is a function of bounded variation which is continuous on the right, that is, whenever $x \in D$ and $\epsilon > 0$ there is a $\delta > 0$ such that $|f(t) - f(x)| \le \epsilon$ for every $t \in D \cap [x, x+\delta]$. Show that f is expressible as the difference of two non-decreasing functions which are continuous on the right.

p 76 l 16 Exercise 224Ye has been extended and now reads

(e) Let (X, ρ) be a metric space and $f : [a,b] \to X$ a function, where $a \le b$ in $\mathbb{R}$. Set $\mathrm{Var}_{[a,b]}(f) = \sup\{\sum_{i=1}^n \rho(f(a_i), f(a_{i-1})) : a \le a_0 \le \ldots \le a_n \le b\}$. (i) Show that $\mathrm{Var}_{[a,b]}(f) = \mathrm{Var}_{[a,c]}(f) + \mathrm{Var}_{[c,b]}(f)$ for every $c \in [a,b]$. (ii) Show that if $\mathrm{Var}_{[a,b]}(f)$ is finite then f is continuous at all but countably many points of $[a,b]$. (iii) Show that if X is complete and $\mathrm{Var}_{[a,b]}(f) < \infty$ then $\lim_{t\uparrow x} f(t)$ is defined for every $x \in \left]a,b\right]$. (iv) Show that if X is complete then $\mathrm{Var}_{[a,b]}(f)$ is finite iff f is expressible as a composition gh, where $h : [a,b] \to [0,\infty[$ is non-decreasing and $g : [0,\infty[\to X$ is 1-Lipschitz, that is, $\rho(g(c), g(d)) \le |c - d|$ for all c, $d \in [0,\infty[$.

p 76 l 35 (224Y) Add new exercises:

(j) Suppose that $a < b$ in $\mathbb{R}$, and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets covering $[a,b]$. Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that $\mathrm{Var}\, f \le \sum_{n=0}^\infty \mathrm{Var}_{A_n} f$.

(k) Let $f : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}$. Show that the following are equiveridical: (α) $\lim_{n\to\infty} f(t_n)$ is defined for every montonic sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in D; (β) for every $\epsilon > 0$ there is a function $g : D \to \mathbb{R}$ of bounded variation such that $|f(t) - g(t)| \le \epsilon$ for every $t \in D$.

p 79 l 37 (part (iii)⇒(ii) of the proof of 225E: for '$G = F + F(a)$' read '$G = F - F(a)$'.

p 80 l 15 The former exercise 225Xo has been absorbed into 225G, which now reads

Proposition Let $[a,b]$ be a non-empty closed interval in $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ an absolutely continuous function.

(a) $f[A]$ is negligible for every negligible set $A \subseteq \mathbb{R}$.

(b) $f[E]$ is measurable for every measurable set $E \subseteq \mathbb{R}$.

p 82 l 23 (part (b-ii) of the proof of 225J) There is a confusion over the definition of $H(k,p,q,q')$; if we change this to

$$\begin{aligned} H(k,p,q,q') &= \emptyset \text{ if } \left]q,q'\right[\not\subseteq D, \\ &= \{x : x \in E \cap \left]q,q'\right[,\ |f(y) - f(x) - p(y-x)| \le 2^{-k}|y-x| \\ &\qquad\qquad \text{for every } y \in \left]q,q'\right[\} \\ &\qquad \text{if } \left]q,q'\right[\subseteq D, \end{aligned}$$

the argument runs more naturally, the next displayed formula becoming

$$\begin{aligned} |f(y) - f(x) - p(y-x)| &= \lim_{n\to\infty} |f(y) - f(x_n) - p(y - x_n)| \\ &\le 2^{-k} \lim_{n\to\infty} 2^{-k}|y - x_n| = 2^{-k}|y - x|. \end{aligned}$$

Next, the set E need not be itself Borel, but as it is relatively Borel in D we see that $E \cap \left]q,q'\right[$ will be a Borel set whenever $\left]q,q'\right[\subseteq D$, so that $H(k,p,q,q')$ is indeed a Borel set, as claimed.

p 85 l 23 The exercises for 225X have been rearranged: 225Xb is now 225Xl, 225Xc-225Xd are now 225Xb-225Xc, 225Xe is now 225Xn, 225Xf-225Xm are now 225Xd-225Xk, 225Xn is now 225Xm, 225Xo is now 225Gb.

Add a third part to 225Xb (now 225Xl):

(iii) Show that if $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $\inf_{x\in[a,b]} |g(x)| > 0$ then f/g is absolutely continuous.

p 86 l 36 (225Y) Add new exercise:

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is absolutely continuous on every bounded interval. Show that $\operatorname{Var} f \le \frac{1}{2} \operatorname{Var} f' + \int |f|$.

p 90 l 18 (part (d) of 226B): for '$x < t_1 \le t_2 \le y$' read '$x < t_1 \le t_2 < y$'.

p 93 l 21 (proof of 226E): for '$\lim_{n\to\infty} \sum_{i=0}^{n} \int f_i = \sum_{i\in\mathbb{N}} f_i$' read '$\lim_{n\to\infty} \sum_{i=0}^{n} \int f_i = \sum_{i\in\mathbb{N}} \int f_i$'.

p 93 l 22 (226X) Add new exercise:

(g)(i) Show that a continuous bijection $f : [0,1] \to [0,1]$ is either strictly increasing or strictly decreasing, and that its inverse is continuous. (ii) Show that if $f : [0,1] \to [0,1]$ is a continuous bijection, then $f' = 0$ a.e. in $[0,1]$ iff there is a conegligible set $E \subseteq [0,1]$ such that $f[E]$ is negligible, and that in this case f^{-1} has the same property. (iii) Construct a function satisfying the conditions of (ii). (*Hint*: try $f = \sum_{n=0}^{\infty} 2^{-n-1} f_n$ where each f_n is a variation on the Cantor function.) (iv) Repeat (iii) with $f = f^{-1}$.

Other exercises have been rearranged: 226Xa-226Xe are now 226Xb-226Xf, 226Xf is now 226Xa.

p 94 l 3 (226Y) Add new exercise:

(d) Suppose that $a < b$ in $\mathbb{R}$, and that $F : [a,b] \to \mathbb{R}$ is a function of bounded variation; let F_p be its saltus part. Show that $|F(b) - F(a)| \le \mu F[\,[a,b]\,] + \operatorname{Var}_{[a,b]} F_p$, where μ is Lebesgue measure on $\mathbb{R}$.

Other exercises have been rearranged: 226Ya-226Yb are now 226Yb-226Yc, 226Yc is now 226Ya.

p 95 l 39 Part (d) of 231B has been absorbed into part (c), with the addition of the formula $\nu E + \nu F = \nu(E \cup F) + \nu(E \cap F)$.

p 104 l 36 (232H) Part (f) has been amended to give a more general formulation of the term 'Radon-Nikodým derivative':

(f) If (X, Σ, μ) is a measure space and ν is a $[-\infty, \infty]$-valued functional defined on a family of subsets of X, I will say that a $[-\infty, \infty]$-valued function f defined on a subset of X is a **Radon-Nikodým derivative** of ν with respect to μ if $\int_E f d\mu$ is defined (in the sense of 214D) and equal to νE for every $E \in \operatorname{dom} \nu$.

p 105 l 1 (part (a) of the statement of 232I): for 'ν_s is singular with respect to ν' read 'ν_s is singular with respect to μ'.

p 107 l 5 The exercises 232X have been rearranged; 232Xc is now 232Xg, 232Xg is now 232Xh, 232Xh is now 232Xc.

p 107 l 18 (Exercise 232Xf): for 'ν is additive and absolutely continuous with respect to μ_2' read 'ν is additive and absolutely continuous with respect to μ_1'. (K.Y.)

p 107 l 32 The exercises 232Y have been rearranged: 232Yb-232Yd are now 232Yh-232Yj, 232Ye is now 232Yb, 232Yf is now 232Yd, 232Yg is now 232Yk, 232Yh-232Yj are now 232Ye-232Yg.

Add new exercise:

(c) (H.König) Let X be a set and μ, ν two measures on X with the same domain Σ. For $\alpha \ge 0$, $E \in \Sigma$ set $(\alpha\mu \wedge \nu)(E) = \inf\{\alpha\mu(E \cap F) + \nu(E \setminus F) : F \in \Sigma\}$ (cf. 112Ya). Show that the following are equiveridical: (i) $\nu E = 0$ whenever $\mu E = 0$; (ii) $\sup_{\alpha \ge 0} (\alpha\mu \wedge \nu)(E) = \nu E$ for every $E \in \Sigma$.

p 116 l 24 (233Y) Add new exercise:

(e) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of subsets of X, and $f : X \to [0, \infty]$ a Σ-measurable function. Show that (i) there is a T-measurable $g : X \to [0, \infty]$ such that $\int_F g = \int_F f$ for every $F \in \mathrm{T}$ (ii) any two such functions are equal a.e.

(f) Suppose that $r \ge 1$ and $C \subseteq \mathbb{R}^r$ is a convex set such that $C \setminus \{0\}$ is convex. Show that there is a non-zero $b \in \mathbb{R}^r$ such that $b\,\text{.}\,z \ge 0$ for every $z \in C$.

(g) Suppose that $r \ge 1$, $C \subseteq \mathbb{R}^r$ is a convex set and $\phi : C \to \mathbb{R}$ is a convex function. Show that there is a function $h : \mathbb{R}^r \to [-\infty, \infty[$ such that $\phi(z) = \sup\{h(y) + z\,\text{.}\,y : y \in \mathbb{R}^r\}$ for every $z \in C$.

(h) Let (X,Σ,μ) be a probability space, $r\ge 1$ an integer and $C\subseteq\mathbb{R}^r$ a convex set. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x : x\in\bigcap_{j\le r}\operatorname{dom}f_j,\ (f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $(\int f_1,\dots,\int f_r)\in C$.

(i) Let (X,Σ,μ) be a probability space, $r\ge 1$ an integer, $C\subseteq\mathbb{R}^r$ a convex set and $\phi : C\to\mathbb{R}^r$ a convex function. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x : x\in\bigcap_{j\le r}\operatorname{dom}f_j,\ (f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $\phi(\int f_1,\dots,\int f_r)\le\int\phi(f_1,\dots,f_r)$.

(j) Let (X,Σ,μ) be a measure space with $\mu X>0$, $r\ge 1$ an integer, $C\subseteq\mathbb{R}^r$ a convex set such that $tz\in C$ whenever $z\in C$ and $t>0$, and $\phi : C\to\mathbb{R}$ a convex function. Let $f_1,\dots,f_r$ be μ-integrable real-valued functions and suppose that $\{x : x\in\bigcap_{j\le r}\operatorname{dom}f_j,\ (f_1(x),\dots,f_r(x))\in C\}$ is a conegligible subset of X. Show that $(\int f_1,\dots,\int f_r)\in C$ and that $\phi(\int f_1,\dots,\int f_r)\le\int\phi(f_1,\dots,f_r)$.

p 117 l 1 Section 234 has been completely re-written, with added material on inverse-measure-preserving functions, image measures, pull-back measures and sums of measures collected from various places in Volumes 1 and 2. The material on indefinite-integral measures is now in the second half of the section: 234A-234G are now 234I-234O, 234Xa is now 234Xh, 234Xb-234Xc are now 234Xj-234Xk, 234Ya-234Yc are now 234Yk-234Ym, 234Yd is now 234Yi, 234Ye is now 234Yn, 234Yf is now 234Yj.

The new material of this section is as follows.

234A (formerly 235G) If (X,Σ,μ) and (Y,T,ν) are measure spaces, a function $\phi : X\to Y$ is **inverse-measure-preserving** if $\phi^{-1}[F]\in\Sigma$ and $\mu(\phi^{-1}[F])=\nu F$ for every $F\in\mathrm{T}$.

234B Proposition Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi : X\to Y$ an inverse-measure-preserving function.

(a) (formerly 235Hc) If $\hat\mu$, $\hat\nu$ are the completions of μ, ν respectively, ϕ is also inverse-measure-preserving for $\hat\mu$ and $\hat\nu$.

(b) (formerly 235Xe) μ is a probability measure iff ν is a probability measure.

(c) (formerly 235Xe) μ is totally finite iff ν is totally finite.

(d) (formerly 235Xe) (i) If ν is σ-finite, then μ is σ-finite.

(ii) If ν is semi-finite and μ is σ-finite, then ν is σ-finite.

(e) (formerly 235Xe) (i) If ν is σ-finite and atomless, then μ is atomless.

(ii) If ν is semi-finite and μ is purely atomic, then ν is purely atomic.

(f)(i) $\mu^*\phi^{-1}[B]\le\nu^*B$ for every $B\subseteq Y$.

(ii) $\mu^*A\le\nu^*\phi[A]$ for every $A\subseteq X$.

(g) (formerly 235Hc) If (Z,Λ,λ) is another measure space, and $\psi : Y\to Z$ is inverse-measure-preserving, then $\psi\phi : X\to Z$ is inverse-measure-preserving.

234C Proposition (formerly 112E) Let (X,Σ,μ) be a measure space, Y any set, and $\phi : X\to Y$ a function. Set

$$\mathrm{T}=\{F : F\subseteq Y,\ \phi^{-1}[F]\in\Sigma\},\quad \nu F=\mu(\phi^{-1}[F])\text{ for every }F\in\mathrm{T}.$$

Then (Y,T,ν) is a measure space.

234D Definition (formerly 112F) In the context of 234C, ν is called the **image measure** or **push-forward measure**; I will denote it $\mu\phi^{-1}$.

234E Proposition Let (X,Σ,μ) be a measure space, Y a set and $\phi : X\to Y$ a function; let $\mu\phi^{-1}$ be the image measure on Y.

(a) ϕ is inverse-measure-preserving for μ and $\mu\phi^{-1}$.

(b) (formerly 212Bd) If μ is complete, so is $\mu\phi^{-1}$.

(c) (formerly 112Xd) If Z is another set, and $\psi : Y\to Z$ a function, then the image measures $\mu(\psi\phi)^{-1}$ and $(\mu\phi^{-1})\psi^{-1}$ on Z are the same.

234F Proposition (formerly 132G) Let X be a set, (Y,T,ν) a measure space, and $\phi : X\to Y$ a function such that $\phi[X]$ has full outer measure in Y. Then there is a measure μ on X, with domain $\Sigma=\{\phi^{-1}[F] : F\in\mathrm{T}\}$, such that ϕ is inverse-measure-preserving for μ and ν.

234G Proposition (formerly 112Xe and 112Ya) Let X be a set, and $\langle\mu_i\rangle_{i\in I}$ a family of measures on X. For each $i \in I$, let Σ_i be the domain of μ_i. Set $\Sigma = \mathcal{P}X \cap \bigcap_{i\in I}\Sigma_i$ and define $\mu : \Sigma \to [0,\infty]$ by setting $\mu E = \sum_{i\in I}\mu_i E$ for every $E \in \Sigma$. Then μ is a measure on X.

234H Proposition Let X be a set and $\langle\mu_i\rangle_{i\in I}$ a family of complete measures on X with sum μ.

(a) μ is complete.

(b)(i) A subset of X is μ-negligible iff it is μ_i-negligible for every $i \in I$.

(ii) A subset of X is μ-conegligible iff it is μ_i-conegligible for every $i \in I$.

(c) Let f be a function from a subset of X to $[-\infty,\infty]$. Then $\int f d\mu$ is defined in $[-\infty,\infty]$ iff $\int f d\mu_i$ is defined in $[-\infty,\infty]$ for every i and one of $\sum_{i\in I} f^+ d\mu_i$, $\sum_{i\in I} f^- d\mu_i$ is finite, and in this case $\int f d\mu = \sum_{i\in I}\int f d\mu_i$.

p 118 l 4 (234C, now 234K) Part (e) has been incorporated into part (b), and replaced with the following:

(e) Because μ and its completion define the same virtually measurable functions, the same null ideals and the same integrals , they define the same indefinite-integral measures.

p 120 l 31 At the end of the section is some further new material.

234P Definition Let μ, ν be two measures on a set X. I will say that $\mu \le \nu$ if μE is defined, and $\mu E \le \nu E$, whenever ν measures E.

234Q Proposition Let X be a set, and write M for the set of all measures on X.

(a) Defining $\le$ as in 234P, $(\mathrm{M}, \le)$ is a partially ordered set.

(b) If μ, $\nu \in \mathrm{M}$, then $\mu \le \nu$ iff there is a $\lambda \in \mathrm{M}$ such that $\mu + \lambda = \nu$.

(c) If $\mu \le \nu$ in M and f is a $[-\infty,\infty]$-valued function, defined on a subset of X, such that $\int f d\nu$ is defined in $[-\infty,\infty]$, then $\int f d\mu$ is defined; if f is non-negative, $\int f d\mu \le \int f d\nu$.

p 121 l 38 In the statement of Theorem 235A, add a final sentence:

Consequently, interpreting $J \times f\phi$ in the same way,

$$\underline{\int} f d\nu \le \underline{\int} J \times f\phi\, d\mu \le \overline{\int} J \times f\phi\, d\mu \le \overline{\int} f d\nu$$

for every $[-\infty,\infty]$-valued function f defined almost everywhere in Y.

p 131 l 31 (235X) Add new exercises:

(l) Let (X,Σ,μ) be a complete measure space, Y a set, $\phi : X \to Y$ a function and $\nu = \mu\phi^{-1}$ the corresponding image measure on Y. Let ν_1 be an indefinite-integral measure over ν. Show that there is an indefinite-integral measure μ_1 over μ such that ν_1 is the image measure $\mu_1\phi^{-1}$.

(m) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Let ν_1 be an indefinite-integral measure over ν. Show that there is an indefinite-integral measure μ_1 over μ such that ϕ is inverse-measure-preserving for μ_1 and ν_1.

(n) Let (X,Σ,μ) and (Y,T,ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that $\overline{\int} h\phi\, d\mu \le \overline{\int} h\, d\nu$ for every real-valued function h defined almost everywhere in Y.

p 139 l 10 The exercises for §241 have been re-shuffled; 241Xd is now 241Xh, 241Xg is now 241Xd, 241Xh is now 241Xg, 241Ya is now 241Yf, 241Yb is now 241Yd, 241Yc is now 241Ya, 241Yd is now 241Ye, 241Ye is now 241Yg, 241Yf is now 241Yh, 241Yg is now 241Yb, 241Yh is now 241Yc.

p 140 l 13 (Exercise 241Yg, formerly 241Ye): *add* 'Show that T is injective iff Y has full outer measure'.

p 146 l 12 (part (e) of 242J): for '$u \in L^0(\mu\restriction\mathrm{T})$' read '$u \in L^1(\mu)$. And add

Finally, P is a bounded linear operator, with norm 1.

p 147 l 31 (part (a-i) of the proof of 242O): for '$\prod_{j=1}^r(\beta_j + 2\delta) \le \epsilon + \prod_{j=1}^r \beta_j$' read '$2^r\prod_{j=1}^r(\beta_j + \delta) \le \epsilon + 2^r\prod_{j=1}^r \beta_j$'.

p 149 l 34 The exercises for §242 have been re-shuffled; 242Xc is now 242Xf, 242Xd is now 242Xc, 242Xe is now 242Xg, 242Xf is now 242Xd, 242Xg is now 242Xe, 242Yb is now 242Yf, 242Yc is now 242Yg, 242Yd

is now 242Yh, 242Ye is now 242Yi, 242Yf is now 242Yb, 242Yg is now 242Yc, 242Yh is now 242Yd, 242Yi is now 242Yj, 242Yj is now 242Yk, 242Yk is now 242Yl, 242Yl is now 242Ye.
In addition, there is a new exercise:

242Xj Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ and $P : L^1(\mu) \to L^1(\mu\restriction\mathrm{T}) \subseteq L^1(\mu)$ the corresponding conditional expectation operator. Show that if u, $v \in L^1(\mu)$ are such that $Pu \times Pv \in L^1(\mu)$, then $\int Pu \times v = \int Pu \times Pv = \int u \times Pv$.

p 150 l 16 (Exercise 242Xi): for 'F is smooth' read 'F_δ is smooth'.

p 158 l 24 (243K) For '$T : L^\infty_{\mathbb{C}} \to L^1_{\mathbb{C}}$' read '$T : L^\infty_{\mathbb{C}} \to (L^1_{\mathbb{C}})^*$'.

p 159 l 37 (Exercise 243Ya): *add* 'iff Y has full outer measure'.

p 161 l 11 (part (a-ii) of the proof of 244B): for '$cu \in \mathcal{L}^p$' read '$cf \in \mathcal{L}^p$'.

p 169 l 1 Add new result:

244O Theorem Suppose that $p \in \,]1,\infty[$ and (X,Σ,μ) is a measure space. Then $L^p = L^p(\mu)$ is uniformly convex.

244O is now 244P.

p 169 l 34 Exercise 244Xb is quite wrong, and must be deleted. Accordingly Exercises 244Xc-244Xk should be renamed 244Xb-244Xj.

p 170 l 23 (244X) Add new exercise:

(k) Let $p \in [1,\infty[$. (i) Show that $|a^p - b^p| \ge |a-b|^p$ for all a, $b \ge 0$. (ii) Let (X,Σ,μ) be a measure space and U a linear subspace of $L^0(\mu)$ such that (α) $|u| \in U$ for every $u \in U$ (β) $u^{1/p} \in U$ for every $u \in U$ (γ) $U \cap L^1$ is dense in $L^1 = L^1(\mu)$. Show that $U \cap L^p$ is dense in $L^p = L^p(\mu)$. (iii) Use this to prove 244H from 242M and 242O.

p 171 l 14 (Exercise 244Yg): for '$I = \{p : p \in [0,\infty[\,,\ u \in L^p(\mu)\}$' read '$I = \{p : p \in [1,\infty[\,,\ u \in L^p(\mu)\}$'.

p 171 l 30 (Exercise 244Yj): the hypothesis should be amended to include '$C \ne \emptyset$'.

p 171 l 40 (244Y) Add new exercises:

(l) Suppose that (X,Σ,μ) is a measure space, and that $p \in \,]0,1[$, $q < 0$ are such that $\frac{1}{p}+\frac{1}{q} = 1$. (i) Show that $ab \ge \frac{1}{p}a^p + \frac{1}{q}b^q$ for all real $a \ge 0$, $b > 0$. (ii) Show that if f, $g \in \mathcal{L}^0(\mu)$ are non-negative and $E = \{x : x \in \operatorname{dom} g,\ g(x) > 0\}$, then

$$(\int_E f^p)^{1/p}(\int_E g^q)^{1/q} \le \int f \times g.$$

(iii) Show that if f, $g \in \mathcal{L}^0(\mu)$ are non-negative, then

$$(\int f^p)^{1/p} + (\int g^p)^{1/p} \le (\int (f+g)^p)^{1/p}.$$

(m) (i) Set $C = [0,\infty[^2 \subseteq \mathbb{R}^2$. Let $\phi : C \to \mathbb{R}$ be a continuous function such that $\phi(tz) = t\phi(z)$ for all $z \in C$. Show that ϕ is convex (definition: 233Xd) iff $t \mapsto \phi(1,t) : [0,\infty[\to \mathbb{R}$ is convex. (ii) Show that if $p \in \,]1,\infty[$ and $q = \frac{p}{p-1}$ then $(t,u) \mapsto -t^{1/p}u^{1/q}$, $(t,u) \mapsto -(t^{1/p}+u^{1/p})^p : C \to \mathbb{R}$ are convex. (iii) Show that if $p \in [1,2]$ then $(t,u) \mapsto |t^{1/p}+u^{1/p}|^p + |t^{1/p}-u^{1/p}|^p$ is convex. (iv) Show that if $p \in [2,\infty[$ then $(t,u) \mapsto -|t^{1/p}+u^{1/p}|^p - |t^{1/p}-u^{1/p}|^p$ is convex. (v) Use (ii) and 233Yj to prove Hölder's and Minkowski's inequalities. (vi) Use (iii) and (iv) to prove Hanner's inequalities. (vii) Adapt the method to answer (ii) and (iii) of 244Yc.

(n) (i) Show that any inner product space is uniformly convex. (ii) Let U be a uniformly convex Banach space, $C \subseteq U$ a non-empty closed convex set, and $u \in U$. Show that there is a unique $v_0 \in C$ such that $\|u - v_0\| = \inf_{v\in C}\|u-v\|$. (iii) Let U be a uniformly convex Banach space, and $A \subseteq U$ a non-empty bounded set. Set $\delta_0 = \inf\{\delta :$ there is some $u \in U$ such that $A \subseteq B(u,\delta) = \{v : \|v-u\| \le \delta\}\}$. Show that there is a unique $u_0 \in U$ such that $A \subseteq B(u_0,\delta_0)$.

(o) Let (X,Σ,μ) be a measure space, and $u \in L^0(\mu)$. Suppose that $\langle p_n\rangle_{n\in\mathbb{N}}$ is a sequence in $[1,\infty]$ with limit $p \in [1,\infty]$. Show that if $\limsup_{n\to\infty}\|u\|_{p_n}$ is finite then $\lim_{n\to\infty}\|u\|_{p_n}$ is defined and is equal to $\|u\|_p$.

244Yc-244Yk are now 244Yd-244Yl.

p 173 l 16 (part (a) of 245A): for '$\tau_F(f-g)+\tau_F(g-h)=\rho_F(f,g)+\tau_F(g,h)$' read '$\tau_F(f-g)+\tau_F(g-h)=\rho_F(f,g)+\rho_F(g,h)$'.

p 176 l 3 (part (d) of the proof of 245D): for '$\bar\rho_F(v,u)\le 4^{-n}$ but $\bar\rho_F(\bar h(v),\bar h(u))>\epsilon$' read '$\bar\rho_F(v_n,u)\le 4^{-n}$ but $\bar\rho_F(\bar h(v_n),\bar h(u))>\epsilon$'.

p 180 l 21 (part (a-ii) of the proof of 245H: for '$\rho_F(f,f_n)\le\delta$ for every $m\ge n$' read '$\rho_F(f,f_n)\le\delta$ for every $n\ge m$'.

p 183 l 1 The exercises 245X have been re-arranged; 245Xm is now 245Xj, 245Xj is now 245Xk, 245Xk is now 245Xl, 245Xl is now 245Xm.

p 192 l 2 (Exercise 246Yb): for '$\tilde\rho(F,F')=\max(\sup_{z\in F}\inf_{z'\in F'}\rho(z,z'),\sup_{z'\in F'}\inf_{z\in F}\rho(z,z'))$' read '$\tilde\rho(F,F')=\min(1,\max(\sup_{z\in F}\inf_{z'\in F'}\rho(z,z'),\sup_{z'\in F'}\inf_{z\in F}\rho(z,z')))$'.

p 206 l 1 Add new paragraph:

251L Proposition Let (X_1,Σ_1,μ_1), (X_2,Σ_2,μ_2), (Y_1,T_1,ν_1) and (Y_2,T_2,ν_2) be σ-finite measure spaces; let λ_1, λ_2 be the product measures on $X_1\times Y_1$, $X_2\times Y_2$ respectively. Suppose that $f:X_1\to X_2$ and $g:Y_1\to Y_2$ are inverse-measure-preserving functions, and that $h(x,y)=(f(x),g(y))$ for $x\in X_1$, $y\in Y_1$. Then h is inverse-measure-preserving.

p 200 l 31 (part (a) of the proof of 234E): for '$\sum_{n=0}^\infty\mu E_n\cdot\nu E_n\le\theta A+\epsilon$' read '$\sum_{n=0}^\infty\mu E_n\cdot\nu F_n\le\theta A+\epsilon$'. (K.Y.)

p 196 l 4 (part (b-i) of the proof of 247C): for '$\|u\|_1\le 2\sup_{F\in\Sigma}|\int_F u|\le 2(1+M_0\mu(F\cap E_0))$' read '$\|u\|_1\le 2\sup_{F\in\Sigma}|\int_F u|\le 2\sup_{F\in\Sigma}(1+M_0\mu(F\cap E_0))$'.
251L-251T are now 251M-251U.

p 212 l 21 (251W) Add new part:

(p) Now suppose that we have another family $\langle(Y_i,\mathrm{T}_i,\nu_i)\rangle_{i\in I}$ of measure spaces, with product (Y,Λ',λ'), and inverse-measure-preserving functions $f_i:X_i\to Y_i$ for each i; define $f:X\to Y$ by saying that $f(x)(i)=f_i(x(i))$ for $x\in X$ and $i\in I$. If all the ν_i are σ-finite, then f is inverse-measure-preserving for λ and λ'.

p 212 l 22 (251X) Add new exercises:

(a) Let X and Y be sets, $\mathcal{A}\subseteq\mathcal{P}X$ and $\mathcal{B}\subseteq\mathcal{P}Y$. Let Σ be the σ-algebra of subsets of X generated by $\mathcal{A}$ and T the σ-algebra of subsets of Y generated by $\mathcal{B}$. Show that $\Sigma\widehat{\otimes}\mathrm{T}$ is the σ-algebra of subsets of $X\times Y$ generated by $\{A\times B:A\in\mathcal{A},\ B\in\mathcal{B}\}$.

(p) In 251Q, show that $\tilde\lambda$ and $\lambda^\#$ will have the same null ideals, even if none of the conditions of 251Q(ii) are satisfied.

Other exercises have been renamed: 251Xa-251Xn are now 251Xb-251Xo, 251Xo is now 251Xq, 251Xp is now 251Xr, 251Xq is now 251Xt, 251Xr is now 251Xu.

p 214 l 4 (Exercise 251Ya): for '121Yb' read '121Yc'.

p 214 l 6 (251Y) Add new exercise:

(b) Show that there are measure spaces (X_1,Σ_1,μ_1) and (X_2,Σ_2,μ_2), a probability space (Y,T,ν) and an inverse-measure-preserving function $f:X_1\to X_2$ such that $h:X_1\times Y\to X_2\times Y$ is not inverse-measure-preserving for the c.l.d. product measures on $X_1\times Y$ and $X_2\times Y$, where $h(x,y)=(f(x),y)$ for $x\in X_1$ and $y\in Y$.

Exercises 251Yb-251Yc are now 251Yc-251Yd.

p 215 l 45 (252A) For '$g(x)=\int f(x,y)\nu(dy)$ for $y\in D$ read '$g(x)=\int f(x,y)\nu(dy)$ for $x\in D$'. (S.Bianchini.)

p 219 l 40 (part (b*-iii) of the proof of 252B): for 'g is defined $\tilde\mu$-a.e. ... so g is defined $\hat\mu$-a.e.' read 'g_n is defined $\tilde\mu$-a.e. ... so g_n is defined $\hat\mu$-a.e.'.

p 220 l 28 Corollary 252E has been re-written, as follows:

Let (X,Σ,μ) and (Y,T,ν) be measure spaces, with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Suppose that ν is σ-finite and that μ is has locally determined negligible sets. Then if f is a Λ-measurable real-valued function defined on a subset of $X\times Y$, $y\mapsto f(x,y)$ is ν-virtually measurable for μ-almost every $x\in X$.

p 221 l 18 (proof of 252F): for 'μ_E-almost every $x\in X$' read 'μ_E-almost every $x\in E$'.

p 222 l 24 (Corollary 252H) Add new part:

(b) Let f be a Λ-measurable $[0,\infty]$-valued function defined on a member of Λ. Then

$$\int_{X\times Y} f(x,y)\lambda(d(x,y))=\int_Y\int_X f(x,y)\mu(dx)\nu(dy)=\int_X\int_Y f(x,y)\nu(dy)\mu(dx)$$

in the sense that if one of the integrals is defined in $[0,\infty]$ so are the other two, and all three are then equal.

p 225 l 6 (integration through ordinate sets): the result has been fractionally strengthened to

Let (X,Σ,μ) be a measure space, and f a non-negative μ-virtually measurable function defined on a conegligible subset of X. Then

$$\int f d\mu=\int_0^\infty \mu^*\{x:x\in\operatorname{dom}f,\ f(x)\ge t\}dt=\int_0^\infty \mu^*\{x:x\in\operatorname{dom}f,\ f(x)>t\}dt$$

in $[0,\infty]$, where the integrals $\int\ldots dt$ are taken with respect to Lebesgue measure.

p 225 l 14 (proof of 252O): for '$\mu\{x:g_n(x)>t\}=\mu E_{nk}$ if $1\le k\le 4^n$ and $2^{-n}(k-1)<t\le 2^{-n}k$' read '$\mu\{x:g_n(x)>t\}=\mu E_{nk}$ if $1\le k\le 4^n$ and $2^{-n}(k-1)\le t<2^{-n}k$'.

p 225 l 19 (proof of 252O): for 'at any point of C at which h is continuous,

$$\mu\{x:f(x)\ge t\}=\lim_{s\downarrow t}\mu\{x:f(x)>s\}=\mu\{x:f(x)>t\}\text{'}$$

read 'at any point of $C\setminus\{\inf C\}$ at which h is continuous,

$$\mu\{x:f(x)\ge t\}=\lim_{s\uparrow t}\mu\{x:f(x)>s\}=\mu\{x:f(x)>t\}\text{'.}$$

p 228 l 20 The exercises to §252 have been rearranged: 252Xb-252Xd are now 252Xc-252Xe, 252Xe is now 252Xh, 252Xh is now 252Xi, 252Xi is now 252Xb; 252Ye-252Yi are now 252Yj-252Yn, 252Yj-252Ym are now 252Ye-252Yh, 252Yn is now 252Yu, 252Yo is now 252Yt, 252Yp-252Yt are now 252Yo-252Ys, 252Yu is now 252Yi.
There is a new exercise 252Xj:

(j) Let (X,Σ,μ) be a measure space, and $f:X\to[0,\infty[$ a function. Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Show that the following are equiveridical: (α) f is Σ-measurable; (β) $\{(x,a):x\in X,\ 0\le a\le f(x)\}\in\Sigma\widehat{\otimes}\mathcal{B}$; ($\gamma$) $\{(x,a):x\in X,\ 0\le a<f(x)\}\in\Sigma\widehat{\otimes}\mathcal{B}$.

p 231 l 31 (252Y) Add new exercises:

(v) Let (X,Σ,μ) be a complete locally determined measure space and f, g two real-valued, μ-virtually measurable functions defined almost everywhere in X. (i) Let λ be the c.l.d. product of μ and Lebesgue measure on $\mathbb{R}$. Setting $\Omega_f^*=\{(x,a):x\in\operatorname{dom}f,\ a\in\mathbb{R},\ a\le f(x)\}$ and $\Omega_g^*=\{(x,a):x\in\operatorname{dom}g,\ a\in\mathbb{R},\ a\le g(x)\}$, show that $\lambda(\Omega_f^*\setminus\Omega_g^*)=\int(f-g)^+d\mu$ and $\lambda(\Omega^*\triangle\Omega_g^*)=\int|f-g|d\mu$. (ii) Suppose that μ is σ-finite. Show that

$$\int|f-g|d\mu=\int_{-\infty}^\infty \mu(\{x:x\in\operatorname{dom}f\cap\operatorname{dom}g,\ (f(x)-a)(g(x)-a)<0\}da.$$

(w) Let (X,Σ,μ) be a probability space and T a σ-subalgebra of Σ. Suppose that $E\in\Sigma$ and that g is a conditional expectation of χE on T. Show that there is an $F\in$ T such that $\mu(E\triangle F)\le\|\chi E-g\|_1$.

p 239 l 39 (part (d) of the proof of 253I): for '$\langle E_n\rangle_{n\in\mathbb{N}}$' read '$\langle F_n\rangle_{n\in\mathbb{N}}$'.

p 239 l 40 (part (d) of the proof of 253I): for '$\sup_{n\in\mathbb{N}}\theta(W\cap(E_n\times G_n))$' read '$\sup_{n\in\mathbb{N}}\theta(W\cap(F_n\times G_n))$'.

p 243 l 31 (Exercise 253Yl): for '$L^1(\lambda_i)$' read '$L^1(\mu_i)$'.

p 243 l 10 (Exercise 253Yf, part (vii)): add

When $W=V$ and $\phi(u,v)=(\int u)v$ for $u\in L^1$ and $v\in V$, $Tf^\bullet$ is called the **Bochner integral** of f.

p 247 l 21 (part (e) of the proof of 254F): for

$$`\lambda(V \setminus W') + \lambda(V \setminus W) = \lambda V - \lambda W + \lambda(V \setminus W) = \theta V - \theta W + \theta(V \setminus W)'$$

read

$$`\lambda(V \setminus W') + \lambda(V \setminus W) = \lambda V - \lambda W + \lambda(V \setminus W') = \theta V - \theta W + \theta(V \setminus W')'.$$

p 249 l 16 (254J) Add new parts:

(d) Define addition on X by setting $(x+y)(i) = x(i) +_2 y(i)$ for every $i \in I$, x, $y \in X$, where $0 +_2 0 = 1 +_2 1 = 0$, $0 +_2 1 = 1 +_2 0 = 1$. If $y \in X$, the map $x \mapsto x + y : X \to X$ is inverse-measure-preserving.

(e) If $\pi : I \to I$ is any permutation, then we have a corresponding measure space automorphism $x \mapsto x\pi : X \to X$.

p 250 l 12 (part (b-ϵ) of the proof of 254K): for '$\phi^{-1}[E] \subseteq V$' read '$\tilde{\phi}^{-1}[E] \subseteq V$'.

p 254 l 40 (part (a-ii) of the proof of 254P): for '$W_z = \{y : y \in X_I, (y,z) \in W\}$' read '$W_z = \{y : y \in X_J, (y,z) \in W\}$'.

p 260 l 25 (254X) Add new exercise:

(i) Show that if $\tilde{\phi} : \{0,1\}^{\mathbb{N}} \to [0,1]$ is any bijection constructed by the method of 254K, then $\{\tilde{\phi}^{-1}[E] : E \subseteq [0,1]$ is a Borel set$\}$ is just the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by the sets $\{x : x(i) = 1\}$ for $i \in \mathbb{N}$.

254Xi-254Xr are now 254Xj-254Xs.

p 261 l 7 Part (ii) of Exercise 254Xp (now 254Xq) is wrong, and has been deleted.

p 261 l 29 (254Y) Add new exercise:

(b) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any family of measure spaces. Set $X = \prod_{i\in I} X_i$ and let $\mathcal{F}$ be a filter on the set $[I]^{<\omega}$ of finite subsets of I such that $\{J : i \in J \in [I]^{<\omega}\} \in \mathcal{F}$ for every $i \in I$. Show that there is a complete locally determined measure λ on X such that $\lambda(\prod_{i\in I} E_i)$ is defined and equal to $\lim_{J\to\mathcal{F}} \prod_{i\in J} \mu_i E_i$ whenever $E_i \in \Sigma_i$ for every $i \in I$ and $\lim_{J\to\mathcal{F}} \prod_{i\in J} \mu_i E_i$ is defined in $[0,\infty[$.

(e) Let $f : [0,1] \to [0,1]^2$ be a function which is inverse-measure-preserving for Lebesgue planar measure on $[0,1]^2$ and Lebesgue linear measure on $[0,1]$, as in 134Yl; let f_1, f_2 be the coordinates of f. Define $g : [0,1] \to [0,1]^{\mathbb{N}}$ by setting $g(t) = \langle f_1 f_2^n(t)\rangle_{n\in\mathbb{N}}$ for $0 \le t \le 1$. Show that g is inverse-measure-preserving.

254Yb-254Yc are now 254Yc-254Yd, 254Yd-254Yf are now 254Yf-254Yh.

p 265 l 2 (part (d-ii) of the proof of 255A): the ideas so far are certainly not enough to ensure that T is equal to $\mathrm{T}' = \{E : E \in \Sigma_2, \phi^{-1}[E] \in \Sigma_2\}$. So what we really need to do here is to repeat the arguments for ϕ^{-1} and T$'$ to see that $\phi^{-1}[E] \in \Sigma_2$ and $\mu_2 E = \mu_2 \phi^{-1}[E]$ for every $E \in \Sigma_2$.

p 265 l 37 (proof of 255D): for '$g(x,y) = (f \otimes \chi\mathbb{R})(\phi(x,y))$' read '$g_1(x,y) = (f \otimes \chi\mathbb{R})(\phi(x,y))$'.

p 267 l 39-40 (proof of 255I): for '$f = \lim_{n\to\infty} f_n$, $g = \lim_{n\to\infty} g_n$' read '$f =_{\text{a.e.}} \lim_{n\to\infty} f_n$, $g =_{\text{a.e.}} \lim_{n\to\infty} g_n$'.

p 271 l 33 Part (c) of 255O has been deleted; 255Od-255Og are now 255Oc-255Of.

p 271 l 37 (255Od, now 255Oc): for '$\int_{]-\pi,\pi]^2} h(x+y)f(x)g(y)d(x,y)$' read '$\int_{]-\pi,\pi]^2} h(x+y)f(x)g(y)d(x,y)$', and similarly on lines 4 and 5 of p. 272.

p 272 l 18 (255X) Add new exercise:

(h) Let μ be Lebesgue measure on $\mathbb{R}$. For u, v, $w \in L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mu)$, say that $u * v = w$ if $f * g$ is defined almost everywhere and $(f * g)^\bullet = w$ whenever f, $g \in \mathcal{L}^0_{\mathbb{C}}(\mu)$, $f^\bullet = u$ and $g^\bullet = w$. (i) Show that $(u_1 + u_2) * v = u_1 * v + u_2 * v$ whenever u_1, u_2, $v \in L^0_{\mathbb{C}}$ and $u_1 * v$ and $u_2 * v$ are defined in this sense. (ii) Show that $u * v = v * u$ whenever u, $v \in L^0(\mathbb{C})$ and either $u * v$ or $v * u$ is defined. (iii) Show that if u, v, $w \ge 0$ in $L^0(\mu)$ and $u*v$ and $v*w$ are defined, then $u*(v*w) = (u*v)*w$ if either is defined.

Other exercises have been rearranged: 255Xc-255Xe are now 255Xi-255Xk, 255Xf-255Xi are now 255Xc-255Xf, 255Ya-255Yg are now 255Yc-255Yi, 255Yh is now 255Ym, 255Yi is now 255Ya, 255Yj is now 255Yb, 255Yk-255Ym are now 255Yj-255Yl.

p 272 l 45 (Exercise 255Xi, now 255Xf): for '$\lim_{\delta\downarrow 0}\sup_{x\in\mathbb{R}}|f(x)-(f*\phi_\delta)(x)|=0$' read '$\lim_{\delta\downarrow 0}\sup_{x\in\mathbb{R}}|f(x)-(f*\psi_\delta)(x)|=0$'.

p 273 l 38 (Exercise 255Yj, now 255Yb): for '$\lim_{a\to\infty}\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{(x-y)^2+a^2}dy$' read '$\lim_{a\to\infty}\frac{a}{\pi}\int_{-\infty}^{\infty}\frac{f(y)}{1+a^2(x-y)^2}dy$'.

p 279 l 36 Part (b) of 256H has been rewritten, and is now

(b) A point-supported measure on $\mathbb{R}^r$ is a Radon measure iff it is locally finite.

p 282 l 22 The second sentence of Exercise 256Xg should read 'Show that in this case ν is an indefinite-integral measure over Lebesgue measure iff the function $x\mapsto\nu[a,x]:[a,b]\to\mathbb{R}$ is absolutely continuous whenever $a\le b$ in $\mathbb{R}$'.

p 282 l 27 Exercise 256Xi has been extended, and now reads

(i) Let ν be a Radon measure on $\mathbb{R}^r$, and ν^* the corresponding outer measure. Show that $\nu A=\inf\{\nu G:G\supseteq A\text{ is open}\}$ for every set $A\subseteq\mathbb{R}^r$.

p 283 l 9 (256Y) Add new exercise:

(d) (i) Let λ be the usual measure on $\{0,1\}^{\mathbb{N}}$. Define $\psi:\{0,1\}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ by setting $\psi(x)(i)=x(i+1)$ for $x\in\{0,1\}^{\mathbb{N}}$ and $j\in\mathbb{N}$. Show that ψ is inverse-measure-preserving. (ii) Define $\theta:\mathbb{R}\to\mathbb{R}$ by setting $\theta(t)={<}3t{>}=3t-\lfloor 3t\rfloor$ for $t\in\mathbb{R}$. Show that θ is inverse-measure-preserving for Cantor measure as defined in 256Hc.

256Yd-256Ye are now 256Ye-256Yf.

p 286 l 11 (Exercise 257Yb) In the definition of 'Radon measure on $]-\pi,\pi]$', I should have said that the measure must be complete.

p 294 l 12 (Exercise 261Yg): add new part:

(ii) Show that in the construction of 261A the residual set $A\setminus\bigcup\mathcal{I}_0$ is always porous.

p 294 l 18 Exercise 261Yj has been elaborated, and now reads

(j)(i) Let $\mathcal{C}$ be the family of those measurable sets $C\subseteq\mathbb{R}^r$ such that $\limsup_{\delta\downarrow 0}\frac{\mu(C\cap B(x,\delta))}{\mu B(x,\delta)}>0$ for every $x\in C$. Show that $\bigcup\mathcal{C}_0\in\mathcal{C}$ for every $\mathcal{C}_0\subseteq\mathcal{C}$. (ii) Show that any union of non-trivial closed balls in $\mathbb{R}^r$ is Lebesgue measurable.

p 294 l 23 (261Y) Add new exercise:

(l) Let $\mathfrak{T}'$ be the family of measurable sets $G\subseteq\mathbb{R}^r$ such that whenever $x\in G$ and $\epsilon>0$ there is a $\delta>0$ such that $\mu(G\cap I)\ge(1-\epsilon)\mu I$ whenever I is an interval containing x and included in $B(x,\delta)$. Show that $\mathfrak{T}'$ is a topology on $\mathbb{R}^r$ intermediate between the density topology (261Yf) and the Euclidean topology.

p 285 l 13 (262A) Add a final remark:

Evidently a Lipschitz function is uniformly continuous.

p 295 l 21 (262X) Add new exercise:

(l) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function, where $a\le b$, and $g:f[\,[a,b]\,]\to\mathbb{R}$ a Lipschitz function. Show that gf is absolutely continuous.

p 306 l 35 (262X) Add new exercise:

(l) Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function, where $a\le b$, and $g:f[\,[a,b]\,]\to\mathbb{R}$ a Lipschitz function. Show that gf is absolutely continuous.

p 306 l 38 Exercise 262Yb seems to require ideas which are not touched on in this volume, so has been moved to 419Yd. Add new exercise:

(d) Let $\phi : D \to \mathbb{R}$ be a function, where $D \subseteq \mathbb{R}^r$. (i) Show that if ϕ is measurable then all its partial derivatives are measurable. (ii) Show that if ϕ is Borel measurable then all its partial derivatives are Borel measurable.

262Ya is now 262Yb, 262Yd-262Yh are now 262Ye-262Yi, 262Yi is now 262Ya.

p 315 l 18 New result added:

263I Theorem Let $D \subseteq \mathbb{R}^r$ be a measurable set, and $\phi : D \to \mathbb{R}^r$ a function differentiable relative to its domain at each point of D. For each $x \in D$ let $T(x)$ be a derivative of ϕ relative to D at x, and set $J(x) = |\det T(x)|$.

(a) Let ν be counting measure on $\mathbb{R}^r$. Then $\int_{\mathbb{R}^r} \nu(\phi^{-1}[\{y\}])dy$ and $\int_D J\,d\mu$ are defined in $[0, \infty]$ and equal.

(b) Let g be a real-valued function defined on a subset of $\phi[D]$ such that $\int_D g(\phi(x)) \det T(x)dx$ is defined in $\mathbb{R}$, interpreting $g(\phi(x)) \det T(x)$ as zero when $\det T(x) = 0$ and $g(\phi(x))$ is undefined. Set

$$C = \{y : y \in \phi[D],\ \phi^{-1}[\{y\}] \text{ is finite}\}, \quad R(y) = \textstyle\sum_{x\in\phi^{-1}[\{y\}]} \operatorname{sgn} \det T(x)$$

for $y \in C$, where $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(\alpha) = \frac{\alpha}{|\alpha|}$ for non-zero α. If we interpret $g(y)R(y)$ as zero when $g(y) = 0$ and $R(y)$ is undefined, then $\int_{\phi[D]} g\times R\,d\mu$ is defined and equal to $\int_D g(\phi(x)) \det T(x)dx$.

263I is now 263J.

p 316 l 11 (part (b) of the proof of 263I, now 263J): for '$= 0$ if $y < v \le v'$ or $v' \le v < y$' read '$= 0$ otherwise'.

p 317 l 27 (263Y) Add new exercises:

(e) Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation, where $a < b$ in $\mathbb{R}$, with Lebesgue decomposition $f = f_p + f_{cs} + f_{ac}$ as in 226Cd; let μ be Lebesgue measure on $\mathbb{R}$. Show that the following are equiveridical: (i) f_{cs} is constant; (ii) $\mu f[\,[c, d]\,] \le \int_c^d |f'|d\mu$ whenever $a \le c \le d \le b$; (iii) $\mu^* f[A] \le \int_A |f'|d\mu$ for every $A \subseteq [a, b]$; (iv) $f[A]$ is negligible for every negligible set $A \subseteq [a, b]$.

(f) Suppose that $r = 2$ and that $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable with non-singular derivative T at $\mathbf{0}$. (i) Show that there is an $\epsilon > 0$ such that whenever Γ is a small circle with centre $\mathbf{0}$ and radius at most ϵ then $\phi\restriction\Gamma$ is a homeomorphism between Γ and a simple closed curve around $\mathbf{0}$. (ii) Show that if $\det T > 0$, then for such circles $\phi(x)$ runs anticlockwise around $\phi[\Gamma]$ as x runs anticlockwise around Γ. (iii) What happens if $\det T < 0$?

p 320 l 3 (part (c) of the proof of 264B): for 'a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of sets, covering A, with $\operatorname{diam} A_{nm} \le \delta$ for every m and $\sum_{m=0}^{\infty}(\operatorname{diam} A_{nm})^r \le \theta_{r\delta} + 2^{-n}\epsilon$'. read 'a sequence $\langle A_{nm}\rangle_{m\in\mathbb{N}}$ of sets, covering A_n, with $\operatorname{diam} A_{nm} \le \delta$ for every m and $\sum_{m=0}^{\infty}(\operatorname{diam} A_{nm})^r \le \theta_{r\delta} A_n + 2^{-n}\epsilon$'. (F.Priuli.)

p 322 l 8 (part (b) of the proof of 264F): for '$\mu_r E$' read '$\mu_{Hr} E$'.

p 324 l 4 (part (c-iii) of the proof of 264H): for '$z' = (y, \xi')$' read '$z' = (y', \xi')$'.

p 326 l 27 (Exercise 264Xa): for 'for and $\delta > 0$' read 'for any $\delta > 0$'.

p 327 l 3 (264X) Add new exercises:

(f) (i) Suppose that $f : [a, b] \to \mathbb{R}$ has graph $\Gamma_f \subseteq \mathbb{R}^2$, where $a \le b$ in $\mathbb{R}$. Show that the outer measure $\mu^*_{H1}(\Gamma_f)$ of Γ for one-dimensional Hausdorff measure on $\mathbb{R}^2$ is at most $b - a + \operatorname{Var}_{[a,b]}(f)$. (ii) Let $f : [0, 1] \to [0, 1]$ be the Cantor function (134H). Show that $\mu_{H1}(\Gamma_f) = 2$.

(g) In 264A, show that

$$\theta_{r\delta} A = \inf\{\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r : \langle A_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of convex sets covering } A,$$
$$\operatorname{diam} A_n \le \delta \text{ for every } n \in \mathbb{N}\}$$

for any $A \subseteq \mathbb{R}^s$.

p 328 l 22 (264Y) Add new exercise:

(q) Let $s \ge 1$ be an integer, and $r \in [1, \infty[$. For x, $y \in \mathbb{R}^s$ set $\rho(x, y) = \|x - y\|^{s/r}$. (i) Show that ρ is a metric on $\mathbb{R}^s$ inducing the Euclidean topology. (ii) Let μ_{Hr} be the associated r-dimensional Hausdorff measure. Show that $\mu_{Hr}B(\mathbf{0}, 1) = 2^s$.

p 332 l 32 (part (b) of the proof of 265E): for '$\sum_{n=0}^{\infty} J\mu_r^* D_n + \epsilon\mu_r^* D_n$' read '$\sum_{n=0}^{\infty} J_n\mu_r^* D_n + \epsilon\mu_r^* D_n$'.

p 333 l 10 (part (c) of the proof of 265E): for '$\nu_r^*\phi[D] = \lim_{k\to\infty} \mu_r^*\phi[D \cap B_k]$' read '$\nu_r^*\phi[D] = \lim_{k\to\infty} \nu_r^*\phi[D \cap B_k]$'.

p 333 l 13 (part (d) of the proof of 265E): for '$\psi(x) = (\phi(x), \eta x)$' read '$\psi_\eta(x) = (\phi(x), \eta x)$'.

p 337 l 12 (265X) Add new exercise:

(f) Suppose that $r \ge 2$. Identifying $\mathbb{R}^r$ with $\mathbb{R}^{r-1} \times \mathbb{R}$, let C_r be the cylinder $B_{r-1} \times [-1, 1] \supseteq B_r$, and $\partial C_r = (B_{r-1} \times \{-1, 1\}) \cup (S_{r-2} \times [-1, 1])$ its boundary. Show that

$$\frac{\mu_r B_r}{\mu_r C_r} = \frac{\nu_{r-1} S_{r-1}}{\nu_{r-1}(\partial C_r)}.$$

p 337 l 19 (265Y) Add new exercise:

(b) Suppose that $a \le b$ in $\mathbb{R}$, and that $f : [a, b] \to \mathbb{R}$ is a continuous function of bounded variation with graph Γ_f. Show that the one-dimensional Hausdorff measure of Γ_f is $\mathrm{Var}_{[a,b]}(f) + \int_a^b (\sqrt{1 + (f')^2} - |f'|)$.

p 337 I have added a new section at this point on the Brunn-Minkowski inequality. The new theorems are

266A Proposition If $u_0, \dots, u_n, p_0, \dots, p_n \in [0, \infty[$ and $\sum_{i=0}^n p_i = 1$, then $\prod_{i=0}^n u_i^{p_i} \le \sum_{i=0}^n p_i u_i$.

266B Proposition For any set $D \subseteq \mathbb{R}^r$ set

$$\mathrm{cl}^* D = \{x : \limsup_{\delta\downarrow 0} \frac{\mu^*(D \cap B(x,\delta))}{\mu B(x,\delta)} > 0\},$$

where μ is Lebesgue measure on $\mathbb{R}^r$.

(a) $D \setminus \mathrm{cl}^* D$ is negligible.
(b) $\mathrm{cl}^* D \subseteq \overline{D}$.
(c) $\mathrm{cl}^* D$ is a Borel set.
(d) $\mu(\mathrm{cl}^* D) = \mu^* D$.
(e) If $C \subseteq \mathbb{R}$ then $\overline{C} + \mathrm{cl}^* D \subseteq \mathrm{cl}^*(C + D)$, writing $C + D$ for $\{x + y : x \in C, y \in D\}$.

Remark In this context, $\mathrm{cl}^* D$ is called the **essential closure** of D.

266C Theorem Let A, $B \subseteq \mathbb{R}^r$ be non-empty sets, where $r \ge 1$ is an integer. If μ is Lebesgue measure on $\mathbb{R}^r$, and $A + B = \{x + y : x \in A, y \in B\}$, then $\mu^*(A + B)^{1/r} \ge (\mu^* A)^{1/r} + (\mu^* B)^{1/r}$.

p 346 l 2 (271Y) Add new exercise:

(e) Let X, Y be non-negative random variables with the same distribution, and $h : [0, \infty[\to [0, \infty[$ a non-decreasing function. Show that $\mathbb{E}(X \times hY) \le \mathbb{E}(Y \times hY)$.

p 347 l 41 In condition (iv) of Proposition 272D, note that $\langle\Sigma_i\rangle_{i\in I}$ is a family of subalgebras of the domain $\hat{\Sigma}$ of the completion of μ rather than the original σ-algebra Σ, so it would be better to say explicitly 'independent with respect to $\hat{\mu}$'.

p 352 l 2 (part (b) of the proof of 272M): for '$\prod_{i\in I, j\in J} \Pr(\tilde{X}_{ij} \in F_{ij})$' read '$\prod_{i\in I, j\in J(i)} \Pr(\tilde{X}_{ij} \in F_{ij})$'. (T.D.A.)

p 354 l 7 (part (b-i) of the proof of 272R): for '$\int (x+y)^2 \nu_{X_1}(dx)\nu_{X_2}(dy)$ read '$\iint (x+y)^2 \nu_{X_1}(dx)\nu_{X_2}(dy)$'. (T.D.A.)

p 353 l 21 Add new result:

272Q Theorem Let (Ω,Σ,μ) be a probability space, and $\langle\Sigma_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ. Let $\mathcal{E}\subseteq\Sigma$ be a family of measurable sets, and T the σ-algebra generated by $\mathcal{E}$. Then there is a set $J\subseteq I$ such that $\#(I\setminus J)\le\max(\omega,\#(\mathcal{E}))$ and T, $\langle\Sigma_j\rangle_{j\in J}$ are independent, in the sense that $\mu(F\cap\bigcap_{r\le n}E_r)=\mu F\cdot\prod_{r=0}^n\mu E_r$ whenever $F\in\mathrm{T}$, $j_0,\dots,j_r$ are distinct members of J and $E_r\in\Sigma_{j_r}$ for each $r\le n$.

272Q-272U are now 272R-272V.

p 355 l 16 Add new result:

272W Theorem Let $X_0,\dots,X_n$ be independent random variables such that $0\le X_i\le 1$ a.e. for every i. Set $S=\frac{1}{n+1}\sum_{i=0}^n X_i$ and $a=\mathbb{E}(S)$. Then

$$\Pr(S-a\ge c)\le\exp(-2(n+1)c^2)$$

for every $c\ge 0$.

p 356 l 4 (272X) Add new exercises:

(k) Let $X_0,\dots,X_n$ be independent random variables such that $d_i\le X_i\le d'_i$ a.e. for every i. (i) Show that if $b\ge 0$ then $\mathbb{E}(e^{bX_i})\le\exp(ba_i+\frac{1}{8}b^2(d'_i-d_i)^2)$ for each i, where $a_i=\mathbb{E}(X_i)$. (ii) Set $S=\frac{1}{n+1}\sum_{i=0}^n X_i$ and $a=\mathbb{E}(S)$. Show that

$$\Pr(S-a\ge c)\le\exp(-\tfrac{2(n+1)^2c^2}{d})$$

for every $c\ge 0$, where $d=\sum_{i=0}^n(d'_i-d_i)^2$.

(l) Suppose that $X_0,\dots,X_n$ are independent random variables, all with expectation 0, such that $\Pr(|X_i|\le 1)=1$ for every i. Set $S=\frac{1}{\sqrt{n+1}}\sum_{i=0}^n X_i$. Show that $\Pr(S\ge c)\le\exp(-c^2/2)$ for every $c\ge 0$.

p 356 l 5 The exercises 272Y have been rearranged: 272Ya is now 272Yh, 272Yb is now 272Yg, 272Yc is now 272Ya, 272Yd is now 272Ye, 272Ye is now 272Yc, 272Yf is now 272Yb, 272Yg is now 272Yf. A new exercise has been added:

(d) Let $\langle X_i\rangle_{i\in\mathbb{N}}$ be an independent sequence of real-valued random variables, and set $S_n=\sum_{i=0}^n X_i$ for each n. Show that if $\langle S_n\rangle_{n\in\mathbb{N}}$ converges to S for the topology of convergence in measure on $\mathcal{L}^0$, then $\langle S_n\rangle_{n\in\mathbb{N}}$ converges to S a.e.

p 358 l 33 The statement of Lemma 273Cb now reads

Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be such that $\sum_{i=0}^\infty x_i$ is defined in $\mathbb{R}$, and $\langle b_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence in $[0,\infty[$ diverging to ∞. Then $\lim_{n\to\infty}\frac{1}{b_n}\sum_{k=0}^n b_kx_k=0$.

p 361 l 15 (part (a) of the proof of 273H): for '$=0$ if $|X_n(\omega)|\ge n$' read '$=0$ if $|X_n(\omega)|>n$'. The same error recurs at the beginning of the proof of 273I.

p 363 l 29 Corollary 273J has been extended, as follows:

Corollary Let (Ω,Σ,μ) be a probability space, and λ the product measure on $\Omega^{\mathbb{N}}$. If f is a real-valued function such that $\int f$ is defined in $[-\infty,\infty]$, then

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^n f(\omega_i)=\int f d\mu$$

for λ-almost every $\boldsymbol{\omega}=\langle\omega_n\rangle_{n\in\mathbb{N}}\in\Omega^{\mathbb{N}}$,

p 364 l 14 (Borel-Cantelli Lemma): the result is now in the stronger form

Let (Ω,Σ,μ) be a probability space and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable subsets of Ω such that $\sum_{n=0}^\infty\mu E_n=\infty$ and $\mu(E_m\cap E_n)\le\mu E_m\cdot\mu E_n$ whenever $m\ne n$. Then almost every point of Ω belongs to infinitely many of the E_n.

p 366 l 19 (proof of 273M): on each occasion on which the subformula '$2p+1$' appears, replace it with '$3p$'.

p 369 l 14 (273X) Add new exercises:

(j) Let (Ω, Σ, μ) be a probability space, and λ the product measure on $\Omega^{\mathbb{N}}$. Let $f : \Omega \to \mathbb{R}$ be a function, and set $f^*(\boldsymbol{\omega}) = \limsup_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\omega_i)$ for $\boldsymbol{\omega} = \langle \omega_n \rangle_{n\in\mathbb{N}} \in \Omega^{\mathbb{N}}$. Show that $\overline{\int} f^* d\lambda = \overline{\int} f d\mu$ whenever the right-hand-side is finite.

(o) Let (X, Σ, μ) be a probability space and $\langle E_n \rangle_{n\in\mathbb{N}}$ an independent sequence in Σ such that $\alpha = \lim_{n\to\infty} \mu E_n$ is defined. For $x \in X$ set $I_x = \{n : x \in E_n\}$. Show that I_x has asymptotic density α for almost every x.

Other exercises have been rearranged: 273Xc-273Xh are now 273Xd-273Xi, 273Xi-273Xl are now 273Xk-273Xn, 273Xm is now 273Xc.

p 369 l 29 Exercise 273Yb has been deleted, and replaced with

(b) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a sequence of random variables with finite variance. Suppose that $\sum_{n=0}^{\infty} \mathbb{E}(X_n) = \infty$ and $\liminf_{n\to\infty} \frac{\sum_{i=0}^{n}\sum_{j=0}^{n} \mathbb{E}(X_i \times X_j)}{(\sum_{i=0}^{n} \mathbb{E}(X_i))^2} \le 1$. Show that $\sum_{i=0}^{\infty} X_i = \infty$ a.e.

p 371 l 5 (part (d) of 274A): add 'the **normal distributions** are the distributions with these density functions' (that is, the functions $x \mapsto \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-a)^2/2\sigma^2}$).

p 374 l 40 (part (d) of the proof of 274F): for '$Z = Z_0 + \ldots + Z_n$' read '$Z = \sigma_0 Z_0 + \ldots + \sigma_n Z_n$'.

p 377 l 8 (part (c) of 274H): for '$Y = X_1 + \ldots + X_n$' read '$Y = X_0 + \ldots + X_n$'.

p 379 l 21 (part (a) of the proof of 274M): for '$\int_0^\infty e^{-(x+s)^2/2} ds \ge e^{-x^2/2} \int_0^\infty e^{-xs} ds$' read '$\int_0^\infty e^{-(x+s)^2/2} ds \le e^{-x^2/2} \int_0^\infty e^{-xs} ds$'. (M.R.Burke.)

p 379 l 26 (part (b) of the proof of 274M): add 'dt' in each of the integrals $\int_x^{x+\frac{1}{x}} e^{-t^2/2}$, $\int_x^{x+\frac{1}{x}} (1 - x(t-x))e^{-x^2/2}$. (T.D.A.)

p 379 l 29 (274X) Add new exercise:

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous on every closed bounded interval, and that $\int_{-\infty}^{\infty} |f'(x)| e^{-ax^2} dx < \infty$ for every $a > 0$. Let X be a normal random variable with zero expectation. Show that $\mathbb{E}(Xf(X))$ and $\mathbb{E}(X^2)\mathbb{E}(f'(X))$ are defined and equal.

(j) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be an independent identically distributed sequence of random variables with non-zero finite variance. Let $\langle t_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\sum_{n=0}^{\infty} t_n^2 = \infty$. Show that $\sum_{n=0}^{\infty} t_n X_n$ is undefined or infinite a.e.

(k) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be an independent sequence of real-valued random variables with zero expectation. Suppose that $M \ge 0$ is such that $|X_n| \le M$ a.e. for every n, and that $\sum_{n=0}^{\infty} \mathrm{Var}(X_n) = \infty$. Set $s_n = \sqrt{\sum_{i=0}^{n} \mathrm{Var}(X_i)}$ for each n, and $S_n = \frac{1}{s_n} \sum_{i=0}^{n} X_i$ when $s_n > 0$. Show that $\lim_{n\to\infty} \Pr(S_n \le a) = \Phi(a)$ for every $a \in \mathbb{R}$.

274Xc (now moved to 274Xd) is over-optimistic as it stands, and I have re-written it in a safer form:

(d) Let $\langle m_k \rangle_{k\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $m_0 = 0$ and $\lim_{k\to\infty} m_k/m_{k+1} = 0$. Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be an independent sequence of random variables such that $\Pr(X_n = \sqrt{m_k}) = \Pr(X_n = -\sqrt{m_k}) = 1/2m_k$, $\Pr(X_n = 0) = 1 - 1/m_k$ whenever $m_{k-1} \le n < m_k$. Show that the Central Limit Theorem is not valid for $\langle X_n \rangle_{n\in\mathbb{N}}$. (*Hint*: setting $W_k = (X_0 + \ldots + X_{m_k - 1})/\sqrt{m_k}$, show that $\Pr(W_k \in [\epsilon, 1-\epsilon]) \to 0$ for every $\epsilon > 0$.)

Other exercises have also been moved: 274Xb is now 274Xc, 274Xd-274Xh are now 274Xe-274Xi.

p 380 l 13 (274Y) Add new exercises:

(a) Suppose that $X_0, \ldots, X_n, Y_0, \ldots, Y_n$ are independent random variables such that, for each $i \le n$, X_i and Y_i have the same distribution. Let $h : \mathbb{R}^{n+1} \to \mathbb{R}$ be a Borel measurable function, and set $Z = h(X_0, \ldots, X_n)$, $Z_i = h(X_0, \ldots, X_{i-1}, Y_i, X_{i+1}, \ldots, X_n)$ for each i (with $Z_0 = h(Y_0, X_1, \ldots, X_n)$ and $Z_n = h(X_0, \ldots, X_{n-1}, Y_n)$, of course). Suppose that Z has finite expectation. Show that $\mathrm{Var}(Z) \le \frac{1}{2} \sum_{i=0}^{n} \mathbb{E}(Z_i - Z)^2$.

(b) Show that for any $\epsilon > 0$ there is a smooth function $h : \mathbb{R} \to [0,1]$ such that $\chi\,]-\infty, -\epsilon] \le h \le \chi\,[\epsilon, \infty[$.

(g) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of real-valued random variables. Suppose that there is an $M \ge 0$ such that $|X_n| \le M$ a.e. for every $n \in \mathbb{N}$, and that $\sum_{n=0}^{\infty} X_n$ is defined, as a real number, almost everywhere. Show that $\sum_{n=0}^{\infty} \mathrm{Var}(X_n)$ is finite.

Other exercises have been moved: 274Ya-274Yd are now 274Yc-274Yf.

p 384 l 13 (part (a) of the proof of 275F): for '$r_u \le k < s_{u+1}$' read '$r_{u+1} \le k < s_{u+1}$'.

p 387 l 19 (part (c) of the proof of 275K): for '$\lim_{n\to\infty} X'_n(\omega) = \lim_{n\to\infty} X'_n(\omega)$' read '$\lim_{n\to\infty} X'_n(\omega) = \lim_{n\to\infty} X_n(\omega)$'. (T.D.A.)

p 390 l 17 (Exercise 275Xe) for '$\lim_{n\to\infty} \|P_n u - u\|_p = 0$' read '$\lim_{n\to\infty} \|P_n u - P_\infty u\|_p = 0$'.

p 390 l 27 (275X) Add new exercise:

(i) Let (Ω, Σ, μ) be a probability space, with completion $(\Omega, \hat{\Sigma}, \hat{\mu})$, and $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$ a non-decreasing sequence of σ-subalgebras of $\hat{\Sigma}$. Show that if $\langle\tau_i\rangle_{i\in\mathbb{N}}$ is a sequence of stopping times adapted to $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$, and we set $\tau(\omega) = \sup_{i\in\mathbb{N}} \tau_i(\omega)$ for $\omega \in \Omega$, then τ is a stopping time adapted to $\langle\Sigma_n\rangle_{n\in\mathbb{N}}$.

Other exercises have been re-named; 275Xi-275Xk are now 275Xj-275Xl.

p 390 l 38 The exercises 275Y have been rearranged; 275Ya-275Yj are now 275Yb-275Yk, 275Yk-275Yl are now 275Yo-275Yp, 275Ym is now 275Ya, 275Yn-275Yo are now 275Yq-275Yr, 275Yp is now 275Yl.

p 390 l 43 Part (ii) of Exercise 275Ya (now 275Yb) is wrong, and has been replaced by

(ii) Defining X_n/Z_n as in 121E, so that its domain is $\{\omega : \omega \in \operatorname{dom} X_n \cap \operatorname{dom} Z_n,\ Z_n(\omega) \ne 0\}$, show that $\langle X_n/Z_n\rangle_{n\in\mathbb{N}}$ is a martingale with respect to the measure ν.

p 391 l 10 Exercise 275Yd (now 275Ye) has been rewritten, and is now

(e) (i) Show that if $a \ge 0$ and $b \ge 1$ then $a \ln b \le a \ln^+ a + \frac{b}{e}$, where $\ln^+ a = 0$ if $a \le 1$, $\ln a$ if $a \ge 1$. (ii) Let (Ω, Σ, μ) be a complete probability space and X, Y non-negative random variables on Ω such that $t\mu F_t \le \int_{F_t} X$ for every $t \ge 0$, where $F_t = \{\omega : Y(\omega) \ge t\}$. Show that $\int_{F_1} Y \le \int_{F_1} X \times \ln^+ Y$, and hence that $\mathbb{E}(Y) \le \frac{e}{e-1}(1 + \mathbb{E}(X \times \ln^+ X))$. (iii) Show that if $\langle X_n\rangle_{n\in\mathbb{N}}$ is a martingale on Ω, $n \in \mathbb{N}$ and $X^* = \sup_{i\le n} |X_i|$, then $\mathbb{E}(X^*) \le \frac{e}{e-1}(1 + \mathbb{E}(|X_n| \times \ln^+ |X_n|))$.

p 391 l 21 (Exercise 275Yf, now 275Yg): for '$\int_E X_{n+1} \ge \int_E X_n$ for every $n \in \mathbb{N}$' read '$\int_E X_{n+1} \ge \int_E X_n$ for every $n \in \mathbb{N}$ and every $E \in \Sigma_n$'.

Properly speaking, the phrase 'semi-martingale' means something much more general, and should be deleted at this point.

p 392 l 16 (275Yp, now 275Yl) Add new part:

(i) Find a martingale $\langle X_n\rangle_{n\in\mathbb{N}}$ which is convergent in measure, but is not convergent a.e.

p 392 l 17 (275Y) Add new exercises:

(m) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of real-valued random variables such that $\sum_{n=0}^{\infty} X_n$ is defined in $\mathbb{R}$ almost everywhere. Suppose that there is an $M \ge 0$ such that $|X_n| \le M$ a.e. for every n. Show that $\sum_{n=0}^{\infty} \mathbb{E}(X_n)$ is defined in $\mathbb{R}$.

(n) Let (Ω, Σ, μ) be a probability space and $\langle X_n\rangle_{n\in\mathbb{N}}$ an independent sequence of real-valued random variables on Ω; set $E_n = \{\omega : \omega \in \operatorname{dom} X_n,\ |X_n(\omega)| > 1\}$, $Y_n = X_n \times \chi(\Omega \setminus E_n)$ for each n, and $Z_n(\omega) = \operatorname{med}(-1, X_n(\omega), 1)$ for $n \in \mathbb{N}$ and $\omega \in \operatorname{dom} X_n$. Show that the following are equiveridical: (i) $\sum_{n=0}^{\infty} X_n(\omega)$ is defined in $\mathbb{R}$ for almost every ω; (ii) $\sum_{n=0}^{\infty} \hat{\mu} E_n < \infty$, $\sum_{n=0}^{\infty} \mathbb{E}(Y_n)$ is defined in $\mathbb{R}$, and $\sum_{n=0}^{\infty} \mathrm{Var}(Y_n) < \infty$; (iii) $\sum_{n=0}^{\infty} \hat{\mu} E_n < \infty$, $\sum_{n=0}^{\infty} \mathbb{E}(Z_n)$ is defined in $\mathbb{R}$, and $\sum_{n=0}^{\infty} \mathrm{Var}(Z_n) < \infty$. (This is a version of the **Three Series Theorem**.)

p 400 l 6 (276X) Add new exercises:

(d) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent identically distributed sequence of random variables with zero expectation and non-zero finite variance, and $\langle t_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathbb{R}$. Show that (i) if $\sum_{n=0}^{\infty} t_n^2 < \infty$, then $\sum_{n=0}^{\infty} t_n X_n$ is defined in $\mathbb{R}$ a.e. (ii) if $\sum_{n=0}^{\infty} t_n^2 = \infty$ then $\sum_{n=0}^{\infty} t_n X_n$ is undefined a.e.

(f) Suppose that $\langle X_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded martingale difference sequence and $\langle a_n\rangle_{n\in\mathbb{N}} \in \ell^2$. Show that $\lim_{n\to\infty}\prod_{i=0}^{n}(1+a_iX_i)$ is defined and finite almost everywhere.

(h) In 276B, show that $\mathbb{E}((\sum_{n=0}^{\infty} X_n)^2) \le \sum_{n=0}^{\infty}\mathbb{E}(X_n^2)$.

p 400 l 16 (part (b) of the proof of 281B): for 'Then there are f_1, $g_1 \in \overline{A}$' read 'Then there are f_1, $g_1 \in A$'. (J.Grahl)

p 406 l 4 (part (b) of the proof of 281G): for '$\eta = \min(\frac{1}{2}, \frac{\epsilon M}{6M+4})$' read '$\eta = \min(\frac{1}{2}, M, \frac{1}{6}\epsilon)$'.

p 407 l 38 (remark following the statement of 281N): for 'the condition '$\eta_1,\dots,\eta_r$ are linearly independent over $\mathbb{Q}$' read 'the condition '$1,\eta_1,\dots,\eta_r$ are linearly independent over $\mathbb{Q}$'.

p 410 l 11 (part (f) of the proof of 281N): for '$0 \le \epsilon \le \frac{1}{2}$' read '$0 < \epsilon \le \frac{1}{2}$'

p 412 l 21 (281Y) Add new exercise:

(m) A sequence $\langle t_n\rangle_{n\in\mathbb{N}}$ in $[0,1]$ is **well-distributed** if $\liminf_{n\to\infty}\inf_{l\in\mathbb{N}}\frac{1}{n+1}\#(\{k : l \le k \le l+n,\ t_k \in G\}) \ge \mu G$ for every open set $G \subseteq [0,1]$. (i) Show that $\langle t_n\rangle_{n\in\mathbb{N}}$ is well-distributed iff $\lim_{n\to\infty}\sup_{l\in\mathbb{N}}|\int_0^1 f - \frac{1}{n+1}\sum_{k=l}^{l+n} f(t_k)| = 0$ for every continuous $f : [0,1] \to \mathbb{R}$. (ii) Show that $\langle {<}n\alpha{>}\rangle_{n\in\mathbb{N}}$ is well-distributed for every irrational α.

p 430 l 16 (282Y) Add new exercises:

(f) Let $u : [-\pi,\pi] \to \mathbb{R}$ be an absolutely continuous function such that $u(\pi) = u(-\pi)$ and $\int_{-\pi}^{\pi} u = 0$. Show that $\|u\|_2 \le \|u'\|_2$. (This is **Wirtinger's inequality**.)

(g) (i) Show that $\frac{1}{2\pi}\int_{-\pi}\pi\frac{1-r^2}{1-2r\cos\theta+r^2}d\theta = 1$ for $0 \le r < 1$. (ii) For a real function f which is integrable over $]-\pi,\pi]$, with real Fourier coefficients a_k, b_k, set $S_r(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k(a_k\cos kx + b_k\sin kx)$ for $x \in]-\pi,\pi]$, $r \in [0,1[$. Show that $S_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_r(x -_{2\pi} t)f(t)dt$ for every $x \in]-\pi,\pi]$, where $A_r(t) = \frac{1-r^2}{1-2r\cos t+r^2}$. ($A_r$ is the **Poisson kernel**.) (iii) Show that $\lim_{r\uparrow 1} S_r(x) = f(x)$ for every $x \in]-\pi,\pi[$ which is in the Lebesgue set of f. (iv) Show that $\lim_{r\uparrow 1}\int_{-\pi}^{\pi}|S_r - f| = 0$. (v) Show that if f is defined everywhere on $]-\pi,\pi]$, is continuous, and $f(\pi) = \lim_{x\downarrow -\pi} f(x)$, then $\lim_{r\uparrow 1}\sup_{x\in]-\pi,\pi]}|S_r(x) - f(x)| = 0$.

p 433 l 13 (part (e) of the statement of 283C): for '$\hat{h}(y) = \frac{1}{c}\hat{f}(cy)$' read '$\hat{h}(y) = \frac{1}{c}\hat{f}(\frac{y}{c})$'.

p 437 l 13 (proof of 283G): for

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iyx}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iyx}g(x)dx = \hat{g}(y)$$

read

$$\hat{f}(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}f(x)dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iyx}g(x)dx = \hat{g}(y).$$

p 442 l 25 (part (b) of the the proof of 283N): for '$\frac{1}{\sigma}\hat{\psi}_{1/\sigma}(y)$' read '$\frac{1}{\sigma}\psi_{1/\sigma}(y)$'.

p 443 l 24 The exercises in 283W have been rearranged: 283Wb-283Wc are now 283Wc-283Wd, 283Wd-283Wg are now 283Wh-283Wk, 283Wh-283Wj are now 283We-283Wg, 283Wk is now 283Wl, 283Wl is now 283Wb.

p 446 l 5 (part (i) of Exercise 283Xm): for '$\int_0^a \tilde{f}_x(y)dy$' read '$\int_0^a \hat{f}_x(y)dy$'.

p 446 l 23 Exercise 283Xq has been moved to 283Yb. 283Xr-283Xu are now 283Xq-283Xt, 283Yb-283Yd are now 283Yc-283Ye.

p 447 l 15 (part (vi) of Exercise 283Yb, now 283Yc): for '$2\sqrt{2\pi}f(1)$' read '$2\sqrt{2\pi}f(0)$'.

p 447 l 21 (part (ix) of Exercise 283Yb, now 283Yc): for '$h_2(1) \le \epsilon$' read '$h_2(0) \le \epsilon$'.

p 454 l 15 (part (c-ii) of the proof of 284L): for

$$\sigma\sqrt{\frac{2}{\pi}}\Big(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\sigma)}{\sigma} + \int_\sigma^\delta \frac{2\Phi(t)}{t^3}dt\Big)$$

read

$$\sigma\sqrt{\frac{2}{\pi}}\Big(\frac{\Phi(\delta)}{\delta^2} - \frac{\Phi(\sigma)}{\sigma^2} + \int_\sigma^\delta \frac{2\Phi(t)}{t^3}dt\Big).$$

p 460 l 14 (284W) Add new exercise:

(j) Let T be an invertible real $r \times r$ matrix, regarded as a linear map from $\mathbb{R}^r$ to itself. (i) Show that $\hat{\hat{f}} = |\det T|(f\circ T)^\wedge\circ T'$ for every integrable complex-valued function on $\mathbb{R}^r$. (ii) Show that $h\circ T$ is a rapidly decreasing test function for every rapidly decreasing test function h. (iii) Show that if f, g are a tempered functions and g represents the Fourier transform of f, then $\frac{1}{|\det T|}g\circ(T')^{-1}$ represents the Fourier transform of $f\circ T$; so that if T is orthogonal, then $g\circ T$ represents the Fourier transform of $f\circ T$.

p 460 l 24 (284X) Add new exercise:

(d) For a tempered function f and $\alpha \in \mathbb{R}$, set

$$(S_\alpha f)(x) = f(x+\alpha), \quad (M_\alpha f)(x) = e^{i\alpha x}f(x), \quad (D_\alpha f)(x) = f(\alpha x)$$

whenever these are defined. (i) Show that $S_\alpha f$, $M_\alpha f$ and (if $\alpha \ne 0$) $D_\alpha f$ are tempered functions. (ii) Show that if g is a tempered function which represents the Fourier transform of f, then $M_\alpha g$ represents the Fourier transform of $S_\alpha f$, $S_{-\alpha}g$ represents the Fourier transform of $M_\alpha f$, $\overleftrightarrow{\bar{g}} = \bar{\overleftrightarrow{g}}$ represents the Fourier transform of $\bar{f}$, and if $\alpha \ne 0$ then $\frac{1}{|\alpha|}D_{1/\alpha}g$ represents the Fourier transform of $D_\alpha f$.

Other exercises have been renamed: 284Xd-284Xq are now 284Xe-284Xr.

p 475 l 36 Add new result:

285V Proposition Let ν be a Radon probability measure on $\mathbb{R}^r$ such that $\nu * \nu = \nu$. Then ν is the Dirac measure δ_0 concentrated at 0.

p 476 l 26 (Exercise 285Xh): for 'whenever $c \le d$ in $\mathbb{R}$' read 'whenever $c < d$ in $\mathbb{R}$'. (J.Pachl.)

p 472 l 1 (statement of Lemma 285P): to match the application in 285Q we should include the trivial case $M = 0$ here.

p 476 l 6 (285X) Add new exercises:

(d) Let X be a real-valued random variable which is not essentially constant, and φ its characteristic function. Show that $|\varphi(y)| < 1$ for all but countably many $y \in \mathbb{R}$.

(h) Let X be a normal random variable with expectation a and variance σ^2. Show that $\mathbb{E}(e^X) = \exp(a + \frac{1}{2}\sigma^2)$.

Other exercises have been rearranged: 285Xd-285Xe are now 285Xe-285Xf, 285Xf-285Xr are now 285Xi-285Xu, 285Xs is now 285Yt.

p 478 l 7 (part (iii) of Exercise 285Ya) for '$\int \check{h}(x)\nu(dx) = \int h(y)\hat{\nu}(y)dy$' read '$\int h(x)\nu(dx) = \int \check{h}(y)\hat{\nu}(y)dy$'.

p 479 l 28 (285Y) Add new exercises:

(j) Let (Ω, Σ, μ) be a probability space. Suppose that $\langle X_n\rangle_{n\in\mathbb{N}}$ is a sequence of real-valued random variables on Ω, and X another real-valued random variable on Ω; let φ_{X_n}, φ_X be the corresponding characteristic functions. Show that the following are equiveridical: (i) $\lim_{n\to\infty} \mathbb{E}(f(X_n)) =$

$\mathbb{E}(f(X))$ for every bounded continuous function $f:\mathbb{R}\to\mathbb{R}$; (ii) $\lim_{n\to\infty}\varphi_{X_n}(y)=\varphi_X(y)$ for every $y\in\mathbb{R}$. In this case we say that $\langle X_n\rangle_{n\in\mathbb{N}}$ **converges in distribution** to X.

(k) Let (Ω,Σ,μ) be a probability space, and P the set of Radon probability measures on $\mathbb{R}$. (i) Show that we have a function $\psi:L^0(\mu)\to P$ defined by saying that $\psi(X^\bullet)$ is the distribution of X whenever X is a real-valued random variable on Ω. (ii) Show that ψ is continuous for the topology of convergence in measure on $L^0(\mu)$ and the vague topology on P.

(p) Let ν be a Radon probability measure on $\mathbb{R}^r$ with bounded support. Show that its characteristic function is smooth.

(r) Let ν be a probability measure on $\mathbb{R}$. Show that $|\varphi_\nu(y)-\varphi_\nu(y')|^2\le 2(1-\mathcal{R}\mathrm{e}\,\varphi_\nu(y-y'))$ for any y, $y'\in\mathbb{R}$.

(s) Let $\langle\nu_n\rangle_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}$. Set $E=\{y:y\in\mathbb{R},\lim_{n\to\infty}\varphi_{\nu_n}(y)=1\}$. (i) Show that $E-E$ and $E+E$ are included in E. (ii) Show that if E is not Lebesgue negligible it is the whole of $\mathbb{R}$.

Other exercises have been rearranged: 285Ye-285Yg are now 285Yf-285Yh, 285Yh is now 285Ye, 285Yj-285Yo are now 285Yl-285Yq, 285Yp is now 285Xg.

p 482 l 3 (part (c) of the proof of 286A): for

$$p\int_0^\infty u^{p-1}\mu G_u du=p^2\int_0^\infty u^{p-2}\Bigl(\int_{-\infty}^\infty(f-\frac{1}{q}u\chi\mathbb{R})^+\Bigr)du$$

read

$$p\int_0^\infty u^{p-1}\mu G_u du\le p^2\int_0^\infty u^{p-2}\Bigl(\int_{-\infty}^\infty(f-\frac{1}{q}u\chi\mathbb{R})^+\Bigr)du.$$

p 486 l 20 (part (e) of the proof of 286G, now part (g)): the case in which $\sigma\ne\tau$ but $I_\sigma=I_\tau$ is not treated; but in this case $J_\sigma\cap J_\tau=\emptyset$ so $(\phi_\sigma|\phi_\tau)=0$.

p 488 l 26 Lemmas 286I and 286J have been exchanged.

p 495 l 22 (part (h-ii) of the proof of 286L): for '$\phi_\sigma(y)\ne 0$' read '$\hat{\phi}_\sigma(y)\ne 0$'; and similarly two lines later in part (h-iii).

p 496 l 31 (part (i) of the proof of 286L): for '$|x-x'|\le 2^{-l_L+1}\le 2^{-m+1}$' read '$|x-x'|\le 2^{-l_L}\le 2^{-m}$'.

p 499 l 28 (part (b) of the proof of 286O): A.Derighetti has pointed out that the inequality

$$\sum_{\sigma\in Q}\Bigl|\int_{F\cap g^{-1}[J_\sigma^r]}(f|\phi_\sigma)\phi_\sigma\Bigr|\le C_9\|f\|_2\sqrt{\mu F}$$

of 286N has been miscopied as

$$\sum_{\sigma\in Q}\int_{F\cap g^{-1}[J_\sigma^r]}|(f|\phi_\sigma)\phi_\sigma|\le C_9\|f\|_2\sqrt{\mu F}.$$

I do not know whether Lemma 286O, as stated, is true; to correct the error I have re-defined the operator A, so that 286O-286P now read

286O Lemma (a) For $z\in\mathbb{R}$, define $\theta_z:\mathbb{R}\to[0,1]$ by setting

$$\theta_z(y)=\hat{\phi}(2^{-k}(y-\hat{y}))^2$$

whenever there is a dyadic interval $J\in\mathcal{I}$ of length 2^k such that z belongs to the right-hand half of J and y belongs to the left-hand half of J and $\hat{y}$ is the lower quartile of J, and zero if there is no such J. Then $(y,z)\mapsto\theta_z(y)$ is Borel measurable, $0\le\theta_z(y)\le 1$ for all y, $z\in\mathbb{R}$, and $\theta_z(y)=0$ if $y\ge z$.

(b) For $k\in\mathbb{Z}$, set $Q_k=\{\sigma:\sigma\in Q,\ k_\sigma=k\}$. Let $[Q]^{<\omega}$ be the set of finite subsets of Q, $[\mathbb{Z}]^{<\omega}$ the set of finite subsets of $\mathbb{Z}$ and $\mathcal{L}$ the set of subsets L of Q such that $L\cap Q_k$ is finite for every k. If $K\in[\mathbb{Z}]^{<\omega}$ and $L\in\mathcal{L}$, set

$$\mathcal{P}_{KL}=\{P:P\in[Q]^{<\omega},\ P\cap Q_k\supseteq L\cap Q_k \text{ whenever } k\in\mathbb{Z}$$
$$\text{and either } k\in K \text{ or } P\cap Q_k\ne\emptyset\};$$

set

$$\mathcal{F} = \{\mathcal{P} : \mathcal{P} \subseteq [Q]^{<\omega} \text{ and there are } K \in [\mathbb{Z}]^{<\omega},\ L \in \mathcal{L} \text{ such that } \mathcal{P} \supseteq \mathcal{P}_{KL}\}.$$

Then $\mathcal{F}$ is a filter on $[Q]^{<\omega}$ and

$$2\pi\int_F(\hat{h} \times \theta_z)^\vee = \lim_{P\to\mathcal{F}} \sum_{\sigma\in P, z\in J^r_\sigma} \int_F (h|\phi_\sigma)\phi_\sigma$$

for every $z \in \mathbb{R}$ and rapidly decreasing test function h and measurable set $F \subseteq \mathbb{R}$ of finite measure.

286P Lemma Suppose that h is a rapidly decreasing test function. For $x \in \mathbb{R}$, set

$$Ah(x) = \sup_{z\in\mathbb{R}} |2\pi(\hat{h} \times \theta_z)^\vee(x)|.$$

Then $Ah : \mathbb{R} \to [0,\infty]$ is Borel measurable, and $\int_F Ah \le 4C_9\|h\|_2\sqrt{\mu F}$ whenever $\mu F < \infty$.

There are consequent small changes in the proof of 286S, and Lemma 286T now reads

286T Lemma Set $C_{10} = 3C_9/\pi\tilde{\theta}_1(0)$. For $f \in \mathcal{L}^2_{\mathbb{C}}$, define $\hat{A}f : \mathbb{R} \to [0,\infty]$ by setting

$$(\hat{A}f)(y) = \sup_{a\le b} \frac{1}{\sqrt{2\pi}}|\int_a^b e^{-ixy} f(x)dx|$$

for each $y \in \mathbb{R}$. Then $\int_F \hat{A}f \le C_{10}\|f\|_2\sqrt{\mu F}$ whenever $\mu F < \infty$.

p 503 l 19 (part (b-ii) of the proof of 286R): for '$\theta_{z\alpha\beta}(y)$' read '$\theta'_{z\alpha\beta}(y)$'.

p 503 l 23 (part (c) of the proof of 286R): for '$(\alpha,\beta) \mapsto \theta'_{z\alpha\beta}(y)$ is Borel measurable' read '$(\alpha,y) \mapsto \theta'_{z\alpha\beta}(y)$ is Borel measurable'.

p 505 l 17 (part (h) of the proof of 286R): for '$g(\alpha,1,0)$' read '$g(\alpha,0,1)$', and again three lines later.

p 506 l 9 (part (a) of the proof of 286S): for '$g(t) \le 4\gamma/(4+t^2)$ for every t' read '$g(t) \le \frac{\gamma\alpha^2}{4\alpha^2+t^2}$ for every t'.

p 520 l 42 (2A2C) Add new part:

(c) If r, $s \ge 1$, $D \subseteq \mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ is a function, we say that ϕ is **uniformly continuous** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x)\| \le \epsilon$ whenever x, $y \in D$ and $\|y - x\| \le \delta$. A uniformly continuous function is continuous.

p 527 l 6 (part (ii) of the proof of 2A3J): After 'Now suppose that $G \in \mathfrak{T}_D$' add 'Then there is an $H \in \mathfrak{T}$ such that $G = H \cap D$'.

p 529 l 30 In part (b) of 2A3S, interpolate the following:

If Z is another set, $\mathcal{G}$ is a filter on Z, and $\psi : Z \to X$ is such that $\mathcal{F} = \psi[[\mathcal{G}]]$, then the composition $\phi\psi$ is defined on $\psi^{-1}[D] \in \mathcal{G}$, and if one of the limits $\lim_{x\to\mathcal{F}} \phi(x)$, $\lim_{z\to\mathcal{G}} \phi\psi(z)$ is defined in Y so is the other, and they are then equal.

p 530 l 1 In part (e-i) of 2A3S, we need to suppose that $(Y, \mathfrak{S})$ is Hausdorff.

p 535 l 9 Add new definition:

***2A4K Definition** A normed space U is **uniformly convex** if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|u + v\| \le 2 - \delta$ whenever u, $v \in U$, $\|u\| = \|v\| = 1$ and $\|u - v\| \ge \epsilon$.

p 536 l 26 (2A5B) Add definition:

Functionals satisfying the conditions (i)-(iii) of this proposition are called **F-seminorms**.

p 549 l 5 Due to a misplaced bracket in the TEX file, everything from 'ball' to 'cluster point' was omitted in the index of this volume. The missing section is available in a separate file; see `http://www1.essex.ac.uk/maths/people/fremlin/errata.htm`.