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Errata and addenda for Volume 1, 2004 printing

I collect here known errors and omissions, with their discoverers, in my book *Measure Theory* (see my web page, `http://www1.essex.ac.uk/maths/people/fremlin/mt.htm`).

p 16 l 16 (Notes to §111): for '$\mathcal{A}$ and $\Sigma_{\mathcal{A}}$ belong to $\mathcal{P}X$' read '$\mathcal{A}$ and $\Sigma_{\mathcal{A}}$ belong to $\mathcal{P}(\mathcal{P}X)$'.

p 18 l 28 (112Da) for '$E \subseteq \Sigma$' read '$E \in \Sigma$'. [P.Wallace Thompson]

p 19 l 23 112E-112F (on image measures) have been moved to §234.

p 20 l 5 The exercises to §112 have been rearranged, as follows: 112Xd has been moved to 234E, 112Xe has been deleted (the material is now in §234), 112Xf is now 112Xd, 112Ya has been deleted (the material is now in 234G), 112Yb-112Yf are now 112Ya-112Ye.
There is a new exercise 112Xf:

(f) Let (X, Σ, μ) be a measure space, Y a set, and $\phi : X \to Y$ a function. Set $\mathrm{T} = \{F : F \subseteq Y, \phi^{-1}[F] \in \Sigma\}$ and $\nu F = \mu\phi^{-1}[F]$ for $F \in \mathrm{T}$. Show that ν is a measure on Y.

p 21 l 6 Exercise 112Yd (now 112Yc) is wrong as written, and should be replaced by

(d) Let X be a set and Σ a σ-algebra of subsets of X.

(i) Suppose that $\nu_0, \dots , \nu_n$ are measures on X, all with domain Σ. Set

$$\mu E = \inf\{\sum_{i=0}^n \nu_i F_i : F_0, \dots , F_n \in \Sigma,\, E \subseteq \bigcup_{i\le n} F_i\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(ii) Let N be a non-empty family of measures on X, all with domain Σ. Set

$$\mu E = \inf\{\sum_{n=0}^{\infty} \nu_n F_n : \langle\nu_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in N}, \langle F_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma,\, E \subseteq \bigcup_{n\in\mathbb{N}} F_n\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(iii) Let N be a non-empty family of measures on X, all with domain Σ, and suppose that there is some $\tilde\nu \in \mathrm{N}$ such that $\tilde\nu X < \infty$. Set

$$\mu E = \inf\{\sum_{i=0}^n \nu_i F_i : n \in \mathbb{N},\, \nu_0, \dots , \nu_n \in \mathrm{N},\, F_0, \dots , F_n \in \Sigma, E \subseteq \bigcup_{i\le n} F_i\}$$

for $E \in \Sigma$. Show that μ is a measure on X.

(iv) Suppose, in (iii), that N is downwards-directed, that is, for any ν_1, $\nu_2 \in \mathrm{N}$ there is a $\nu \in \mathrm{N}$ such that $\nu E \le \min(\nu_1 E, \nu_2 E)$ for every $E \in \Sigma$. Show that $\mu E = \inf_{\nu\in\mathrm{N}} \nu E$ for every $E \in \Sigma$.

(v) Show that in all the cases (i)-(iii) the measure μ constructed is the greatest measure with domain Σ such that] $\mu E \le \inf_{\nu\in\mathrm{N}} \nu E$ for every $E \in \Sigma$.

p 21 l 21 (112Y) Add new exercise:

(f) Let X be a set and μ, ν two measures on X, with domains Σ, T respectively. Set $\Lambda = \Sigma\cap\mathrm{T}$ and define $\lambda : \Lambda \to [0, \infty]$ by setting $\lambda E = \mu E + \nu E$ for every $E \in \Lambda$. Show that (X, Λ, λ) is a measure space.

p 25 l 8 (Exercise 113Yd): we need to suppose that $\lambda\emptyset = 0$. [P.W.T.]

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p 25 l 27 (Exercise 113Yh): for '$\mu_* : \mathcal{P}X \to [0,\infty[$' read '$\mu_* : \mathcal{P}X \to [0,\infty]$'. [P.W.T.]

p 25 l 31 (Exercise 113Yi): for 'every $A \in \mathcal{A}$' read 'every $E \in \mathcal{A}$'.

p 25 l 40 (Exercise 113Yj): for '$\Sigma = \{E : E \in \mathrm{T},\ \theta E = \theta(E \cap A) + \theta(E \setminus A)$ for every $A \in \mathrm{T}\}$' read '$\Sigma = \{E : E \in \mathrm{T},\ \theta A = \theta(A \cap E) + \theta(A \setminus E)$ for every $A \in \mathrm{T}\}$'.

p 25 l 41 (113Y) Add new exercise:

(k) Let X, $\tau : \mathcal{P}X \to [0,\infty]$ and θ be as in 113Yd; let μ be the measure defined by Carathéodory's method from θ, and Σ the domain of μ. Suppose that $E \subseteq X$ is such that $\theta(C \cap E) + \theta(C \setminus E) \le \lambda C$ whenever $C \subseteq X$ is such that $0 < \lambda C < \infty$. Show that $E \in \Sigma$.

p 27 l 35 (part (a-iv) of the proof of 114D): for '$\lim_{m\to\infty} \sum_{m=0}^{M} \lambda I_{k_m,l_m}$' read '$\lim_{M\to\infty} \sum_{m=0}^{M} \lambda I_{k_m,l_m}$'. [P.W.T.]

p 31 l 8 (Exercise 114Yh): for '$\nu : \mathcal{B} \to [0,\infty[$' read '$\nu : \mathcal{B} \to [0,\infty]$'. [P.W.T.]

p 30 l 19 (114Y) Add new exercise:

(l) Write μ for Lebesgue measure on $\mathbb{R}$. Show that there is a countable family $\mathcal{F}$ of Lebesgue measurable subsets of $\mathbb{R}$ such that whenever μE is defined and finite, and $\epsilon > 0$, there is an $F \in \mathcal{F}$ such that $\mu(E \triangle F) \le \epsilon$.

p 32 ll 33-39 (part (b) of the proof of 115B) In the formulae $\lambda(I_j \cap H_\xi)$ (twice), $\lambda(I_j \cap H_{\alpha_{r+1}})$ and $\lambda(I_j \cap H_\gamma)$, read '$\lambda_{r+1}$' for '$\lambda$'. [P.W.T.]

p 33 l 18 (part (e) of the proof of 115B, mislabeled (d)): for '$(1+\epsilon)\sum_{j=0}^{n} \lambda_{r+1}(I_j \cap H_{\beta_{r+1}})$' read '$(1+\epsilon)\sum_{j=0}^{\infty} \lambda_{r+1}(I_j \cap H_{\beta_{r+1}})$'. [G.Meyer]

p 35 l 3 (statement of Lemma 115F): for '$i \le m$' read '$i \le r$'.

p 32 l 28 (part (d) of the proof of 115G): for '$\inf_{\epsilon>0} \prod_{i=1}^{r}(\beta_i - \alpha_1 + \epsilon)$' read '$\inf_{\epsilon>0} \prod_{i=1}^{r}(\beta_i - \alpha_i + \epsilon)$'. [G.M.]

p 37 l 1 Exercise 115Ya has been re-written, as follows:

(i) Suppose that M is a strictly positive integer and k_i, l_i are integers for $1 \le i \le r$. Set $\alpha_i = k_i/M$ and $\beta_i = l_i/M$ for each i, and $I = [a, b[$. Show that $\lambda I = \#(J)/M^r$, where J is $\{z : z \in \mathbb{Z}^r,\ {}^1Mz \in I\}$. (ii) Show that if a half-open interval $I \subseteq \mathbb{R}^r$ is covered by a *finite* sequence $I_0, \dots, I_m$ of half-open intervals, and all the coordinates involved in specifying the intervals $I, I_0, \dots, I_m$ are rational, then $\lambda I \le \sum_{j=0}^{m} \lambda I_j$. (iii) Assuming the Heine-Borel theorem in the form

whenever $[a, b]$ is a closed interval in $\mathbb{R}^r$ which is covered by a sequence $\langle]a^{(j)}, b^{(j)}[\rangle_{j\in\mathbb{N}}$ of open intervals, there is an $m \in \mathbb{N}$ such that $[a, b] \subseteq \bigcup_{j\le m}]a^{(j)}, b^{(j)}[$,

prove 115B. (*Hint*: if $[a, b[\subseteq \bigcup_{j\in\mathbb{N}} [a^{(j)}, b^{(j)}[$, replace $[a, b[$ by a smaller closed interval and each $[a^{(j)}, b^{(j)}[$ by a larger open interval, changing the volumes by adequately small amounts.)

p 42 l 35 (part (g) of the proof of 121E): for 'is for the form' read 'is of the form'. [P.W.T.]

p 41 l 31 (part (a-i) of the proof of 112K): for '$\mathbb{R} \setminus E \in \mathrm{T}$' read '$\mathbb{R}^r \setminus E \in \mathrm{T}$' [G.M.]

p 47 l 8 Exercise 121Ya is wrong, and should be re-written, as follows:

(a) Let X and Y be sets, Σ a σ-algebra of subsets of X, $\phi : X \to Y$ a function and g a real-valued function defined on a subset of Y. Set $\mathrm{T} = \{F : F \subseteq Y,\ \phi^{-1}[F] \in \Sigma\}$; then T is a σ-algebra of subsets of Y (see 111Xc). (i) Show that if g is T-measurable then $g\phi$ is Σ-measurable. (ii) Give an example in which $g\phi$ is Σ-measurable but g is not T-measurable. (iii) Show that if $g\phi$ is Σ-measurable and *either* ϕ is injective *or* $\mathrm{dom}(g\phi) \in \Sigma$ *or* $\phi[X] \subseteq \mathrm{dom}\, g$, then g is T-measurable. [P.W.T.]

p 46 l 35 (Exercise 121Xc): for '$\limsup_{n\in\mathbb{N}}$' read '$\limsup_{n\to\infty}$'. [P.W.T.]

p 47 l 17 (Exercise 121Yc): add new part

Suppose that $\mathcal{A} \subseteq \mathrm{T}$ is such that T is the σ-algebra of subsets of Y generated by $\mathcal{A}$ (111Gb). Show that $\phi : X \to Y$ is (Σ, T)-measurable iff $\phi^{-1}[A] \in \Sigma$ for every $A \in \mathcal{A}$.

p 49 l 8 (part (a) of the proof of 122C): for '$i \le n$' read '$k \le n$'. [P.W.T.]

p 53 l 2 (part (a) of the proof of 122O): for '$f+g = (f_1+g_1)-(f_2-g_2)$' read '$f+g = (f_1+g_1)-(f_2+g_2)$'. [P.W.T.]

p 55 l 48 Exercise 122Yg is wrong as stated; it works if we assume that f is defined everywhere on X. [T.Helineva]

p 58 l 24 (part (a-iv) of the proof of 123A): in the formula '$g \le f_n + M\chi G_n + M\chi(X \setminus E) + \epsilon\chi H$', the '$\le$' should be '$\le_{\text{a.e.}}$'. An improvement in the whole sentence would be

Then, for any $x \in E$,

$$g(x) \le f_n(x) + \epsilon\chi H(x) + M\chi G_n(x),$$

so

$$g \le_{\text{a.e.}} f_n + M\chi G_n + \epsilon\chi H$$

and

$$\int g \le \int f_n + M\mu G_n + \epsilon\mu H \le c + \epsilon(M + \mu H).$$

[P.W.T.]

p 59 l 1 (proof of 123B) The assertion that g_n is integrable needs more support. The sentence now reads

Set $g_n = \inf_{m\ge n} f'_m$; then each g_n is measurable (121Fc), non-negative and defined on the conegligible set $\bigcap_{m\ge n} E_m$, and $g_n \le_{\text{a.e.}} f_n$; by 122Re and 122Ra, $|f_n|$ belongs to U, as defined in 122H, while $|g_n| \le_{\text{a.e.}} |f_n|$, so g_n is integrable (122P) with $\int g_n \le \inf_{m\ge n} \int f_m \le c$.

[A.-P.Fortin]

p ?? l ?? (Exercise 123Xd): for '$\int \limsup_{n\to\infty} f_n \ge \limsup_{n\to\infty} f_n$' read '$\int \limsup_{n\to\infty} f_n \ge \limsup_{n\to\infty} \int f_n$'. T.H.

p ?? l ?? Exercise 123Ya is wrong, and should read

Let (X, Σ, μ) be a measure space, Y any set and $\phi : X \to Y$ any function; let $\mu\phi^{-1}$ be the image measure on Y (112Xf). Show that if $h : Y \to \mathbb{R}$ is $\mu\phi^{-1}$-integrable then $h\phi$ is μ-integrable, and the integrals are then equal.

[T.H.]

p 62 l 29 Definition 131B has been extended, as follows:

When $X = \mathbb{R}^r$, where $r \ge 1$, and μ is Lebesgue measure on $\mathbb{R}^r$, I will call a subspace measure μ_H **Lebesgue measure on** H.

p 63 l 4 (part (a) of the proof of 131E): for '$\sum_{i=0}^n a_i\mu_H E_i = \sum_{i=0}^n \mu E_i$' read '$\sum_{i=0}^n a_i\mu_H E_i = \sum_{i=0}^n a_i\mu E_i$'. [P.W.T.]

p 63 l 18 (part (d) of the proof of 131E): for '$g \le f$ μ_H-a.e.' read '$g \le |f|$ μ_H-a.e.'.

p 64 l 21 Exercise 131Ya is wrong as written; it can be salvaged by adding the hypothesis that $\operatorname{dom} f_n$ is measurable for every n. [P.W.T.]

p 67 l 1 Proposition 132G has been moved to 234F.

p 66 l 11 (132D) For '$E \subseteq \Sigma$' read '$E \in \Sigma$'. [P.W.T.]

p 66 l 34 (part (d) of the proof of 132E): add 'Set $A = \bigcup_{n\in\mathbb{N}} A_n$'. [P.W.T.]

p 68 l 5 The former exercise 132Xk is now covered by 234Xa; it has been replaced by

(k) Let (X, Σ, μ) be a measure space and μ^* the outer measure defined from μ. Show that $\mu^*(A \cup B) + \mu^*(A \cap B) \le \mu^*A + \mu^*B$ for all A, $B \subseteq X$.

p l (132Y) Add new exercise:

(g) Let (X, Σ, μ) be a measure space. Suppose that $A \subseteq B \subseteq C \subseteq X$ and that $\mu^*(B \setminus A) = \mu^*B$. Show that $\mu^*(C \setminus A) = \mu^*C$.

p 68 l 23 Exercise 132Yf has been moved to 234Yd.

p 71 l 1 (Upper and lower integrals) The definition in 133I should be re-written, as follows:

Let (X,Σ,μ) be a measure space and f a real-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f=\inf\{\int g:\int g\text{ is defined in the sense of 133A, } f\le_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\mathbb{R}$. Similarly, the **lower integral** of f is

$$\underline{\int} f=\sup\{\int g:\int g\text{ is defined, } f\ge_{\text{a.e.}} g\},$$

allowing $-\infty$ for $\sup\{-\infty\}$ and ∞ for $\sup\mathbb{R}$.

p 71 l 7 (Proposition 133J) Add new facts:

(a)(i) If If $\overline{\int} f$ is finite, and g is an integrable function such that $f\le_{\text{a.e.}} g$ and $\int g=\overline{\int} f$, then

$$A=\{x:x\in\operatorname{dom}f\cap\operatorname{dom}g,\ g(x)\le f(x)+\epsilon\}$$

has full outer measure for every $\epsilon>0$.

(ii) If If $\underline{\int} f$ is finite, and h is an integrable function such that $f\ge_{\text{a.e.}} h$ and $\int h=\underline{\int} f$, then

$$\{x:x\in\operatorname{dom}f\cap\operatorname{dom}h,\ f(x)\le h(x)+\epsilon\}$$

has full outer measure for every $\epsilon>0$.

(e) $\mu^*A=\overline{\int}\chi A$ for every $A\subseteq X$.

p 72 l 31 Add new result:

***133L Proposition** Let (X,Σ,μ) be a measure space and f a real-valued function defined almost everywhere in X. Suppose that h_1, h_2 are non-negative virtually measurable functions defined almost everywhere in X. Then

$$\overline{\int} f\times(h_1+h_2)=\overline{\int} f\times h_1+\overline{\int} f\times h_2.$$

p 73 l 35 (Exercise 133Xf): for '$\underline{\int}\limsup_{n\to\infty}\ge\limsup_{n\to\infty}\underline{\int} f_n$' read '$\underline{\int}\limsup_{n\to\infty} f_n\ge\limsup_{n\to\infty}\underline{\int} f_n$'. [P.W.T.]

p 74 l 1 (Exercise 133Yc): for '$\hat{f}(s)=\frac{1}{(\sqrt{2\pi})^r}\int_{-\infty}^{\infty}e^{-is\,.\,x}f(x)dx$' read '$\hat{f}(s)=\frac{1}{(\sqrt{2\pi})^r}\int e^{-is\,.\,x}f(x)dx$'. [P.W.T.]

p 74 l 15 (133Y) Add new exercise:

(f) Let (X,Σ,μ) be a measure space, $f:X\to[-\infty,\infty]$ a function and $g:X\to[0,\infty]$, $h:X\to[0,\infty]$ measurable functions. Show that $\overline{\int} f\times(g+h)=\overline{\int} f\times g+\overline{\int} f\times h$, where here, for once, we interpret $\infty+(-\infty)$ as ∞.

p 76 l 44 (part (c-i) of the proof of 134D): for '$\mu(F\cap E_n\cap C)+\mu(F\cap E_n\setminus C)$' read '$\mu^*(F\cap E_n\cap C)+\mu^*(F\cap E_n\setminus C)$'. [P.W.T.]

p 84 l 22 (134X) Add new exercise:

(j) Let f be a measurable real function and g a real function such that $\operatorname{dom}g\setminus\operatorname{dom}f$ and $\{x:x\in\operatorname{dom}g\cap\operatorname{dom}f,\ g(x)\ne f(x)\}$ are both negligible. Show that g is measurable.

p 85 l 17 (Hint to exercise 134Yl): for '$f_m^{-1}[D]=I_D$' read '$f_m[I_D]\subseteq D$'. [P.W.T.]

p 89 l 14 Concerning upper and lower integrals of functions taking infinite values, a restatement of the definitions is formally required:

Let (X,Σ,μ) be a measure space and f a $[-\infty,\infty]$-valued function defined almost everywhere in X. Its **upper integral** is

$$\overline{\int} f=\inf\{\int g:\int g\text{ is defined in the sense of 135F and } f\le_{\text{a.e.}} g\},$$

allowing ∞ for $\inf\{\infty\}$ and $-\infty$ for $\inf\,]-\infty,\infty]$ or $\inf[-\infty,\infty]$. Similarly, the **lower integral** of f is

$$\underline{\int} f = \sup\{\int g : \int g \text{ is defined, } f \geq_{\text{a.e.}} g\}.$$

p 89 l 19 Add new paragraph on subspace measures:

135I Proposition Let (X,Σ,μ) be a measure space, and $H \in \Sigma$; write Σ_H for the subspace σ-algebra on H and μ_H for the subspace measure. For any $[-\infty,\infty]$-valued function f defined on a subset of H, write $\tilde{f}$ for the extension of f defined by saying that $\tilde{f}(x) = f(x)$ if $x \in \operatorname{dom} f$, 0 if $x \in X \setminus H$.

(a) Suppose that f is a $[-\infty,\infty]$-valued function defined on a subset of H.

(i) $\operatorname{dom} f$ is μ_H-conegligible iff $\operatorname{dom} \tilde{f}$ is μ-conegligible.

(ii) f is μ_H-virtually measurable iff $\tilde{f}$ is μ-virtually measurable.

(iii) $\int_H f d\mu_H = \int_X \tilde{f} d\mu$ if either is defined in $[-\infty,\infty]$.

(b) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then $\int_H (h\restriction H) d\mu_H = \int h \times \chi H\, d\mu$ if either is defined in $[-\infty,\infty]$.

(c) If h is a $[-\infty,\infty]$-valued function and $\int_X h\, d\mu$ is defined in $[-\infty,\infty]$, then $\int_H (h\restriction H) d\mu_H$ is defined in $[-\infty,\infty]$.

(d) Suppose that h is a $[-\infty,\infty]$-valued function defined almost everywhere in X. Then

$$\overline{\int}_H (h\restriction H) d\mu_H = \overline{\int}_X h \times \chi H d\mu.$$

p 89 l 30 (Exercise 135Xc): add new part

(iv) a real-valued function h defined on a subset of $[-\infty,\infty]$ is Borel measurable iff $h\phi^{-1}$ is Borel measurable.

p 90 l 7 (135Y) Add new exercise:

(b) Let (X,Σ,μ) be a measure space, and f, g two $[-\infty,\infty]$-valued functions, defined on subsets of X, such that $\int f$ and $\int g$ are both defined in $[-\infty,\infty]$. (i) Show that $\int f \vee g$ and $\int f \wedge g$ are defined in $[-\infty,\infty]$, where $(f \vee g)(x) = \max(f(x), g(x))$, $(f \wedge g)(x) = \min(f(x), g(x))$ for $x \in \operatorname{dom} f \cap \operatorname{dom} g$. (ii) Show that $\int f \vee g + \int f \wedge g = \int f + \int g$ in the sense that if one of the sums is defined in $[-\infty,\infty]$ so is the other, and they are then equal.

p 93 l 15 Add new result:

***136H Proposition** Let (X,Σ,μ) be a measure space such that $\mu X < \infty$, and $\mathcal{E}$ a subalgebra of Σ; let Σ' be the σ-algebra of subsets of X generated by $\mathcal{E}$. If $F \in \Sigma'$ and $\epsilon > 0$, there is an $E \in \mathcal{E}$ such that $\mu(E \triangle F) \leq \epsilon$.

p 93 l 24 (Hint for exercise 136Xc): for '$\mu_n I = \mu(E \cap I_n)$' read '$\mu_n E = \mu(E \cap I_n)$'. [P.W.T.]

p 94 l 10 Exercise 136Xl seems a little harder than the others, and I have moved it to 136Yc.

p 96 l 20 (part (c) of 1A1B); for '(i)-(ii) above' read '(a)-(b) above'. [P.W.T.]

p 101 l 33 (proof of part (a) of 1A3B): for '$\lim_{n\to\infty} \inf_{m\geq n} a_m = \limsup_{n\to\infty} a_n$' read '$\lim_{n\to\infty} \inf_{m\geq n} a_m = \liminf_{n\to\infty} a_n$'. [K.Yates]

p 105 Due to a misplaced bracket in the TEX file, everything from 'Borel' to 'conegligible' was omitted in the index of this volume. The omitted material follows.